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Journal

Bulletin of Symbolic Logic, 21(3)

ISSN

1079-8986

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Publication Date

2015-09-01

DOI

10.1017/bsl.2015.19

The Bulletin of Symbolic Logic
Volume 21, Number 3, September 2015

CHANG'S CONJECTURE, GENERIC ELEMENTARY EMBEDDINGS AND INNER MODELS FOR HUGE CARDINALS

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Abstract. We introduce a natural principle *Strong Chang Reflection* strengthening the classical Chang Conjectures. This principle is between a huge and a two huge cardinal in consistency strength. In this note we prove that it implies the existence of an inner model with a huge cardinal. The technique we explore for building inner models with huge cardinals adapts to show that *decisive* ideals imply the existence of inner models with supercompact cardinals. Proofs for all of these claims can be found in [10].[1,2]

Much of 20th century logic in general and model theory in particular was tied up with understanding the *expressive power* of first and second order logic (and their variants). Of particular interest is the role of the downwards Lowenheim–Skolem theorem; indeed in some ways it is the distinguishing feature of first order logic ([19]).

The Downward Lowenheim–Skolem theorem states that if $\mathfrak{A}$ is a structure in a countable language then for all infinite cardinals κ less than the cardinality of $\mathfrak{A}$ there is a $\mathfrak{B} \prec \mathfrak{A}$ of cardinality κ. Tremendous effort was put into generalizing the downwards Lowenheim–Skolem theorem in an attempt to show that elementary substructure $\mathfrak{B}$ can be taken to have some second order properties.

The coarsest second order properties have to do with cardinality. In this paper we consider various more subtle second order properties. Among them is being correct for the nonstationary ideal. By demanding that $\mathfrak{B}$ have these properties we are able to formulate a version of the downwards Lowenheim–Skolem theorem with very strong large cardinal strength.[3]

§1. Two cardinal transfer theorems. Let $\mathcal{L}$ be a countable language with a distinguished unary predicate R. Then

$$\mathfrak{A} = \langle A, R^{\mathfrak{A}}, f_i, R_j, c_k \ldots \rangle_{i,j,k \in \omega}$$

is said to have *type* (κ, λ) if and only if $|A| = \kappa$ and $|R^{\mathfrak{A}}| = \lambda$.

Received January 10, 2011.
Key words and phrases. Chang's Conjecture, inner models, huge cardinals.

[1]This research was partially supported by NSF grant DMS 0701030.
[2]For definitions of "Huge," "Supercompact," and "Measurable" cardinals, see Appendix A.
[3]Basic definitions of large cardinals are provided in the Appendix.

1079-8986/15/2103-0001
DOI:10.1017/bsl.2015.19

In this definition, we are considering two structures simultaneously—the whole structure $\mathfrak{A}$ and a related structure whose universe is $R^{\mathfrak{A}}$. We often write $\mathfrak{A} = \langle \kappa; \lambda, f_i \dots \rangle$ to mean a structure of type (κ, λ).

The earliest investigations did not ask for elementary substructures, merely for elementarily equivalent structures. This weakening had the virtue of allowing simultaneous generalization of both the downwards and upwards Lowenheim–Skolem theorems. (The reader is referred to Section 7.2 of [1] for background information.)

We write

$$(\kappa, \lambda) \to (\kappa', \lambda')$$

to mean that if $\mathfrak{A}$ is an $\mathcal{L}$-structure of type (κ, λ) then there is a $\mathfrak{B} \equiv \mathfrak{A}$ of type (κ', λ').

We state here two classical results that exemplify the type of theorems that were proven. The first is Vaught's "gap-one" two cardinal theorem ([4]).

Theorem 1.1 (Vaught's Two Cardinal Theorem). *$(\kappa, \rho) \to (\omega_1, \omega)$ for all $\kappa > \rho \geq \omega$.*

The gist of this theorem is that there is no first order definable way to describe exactly how large κ is, even by mentioning a smaller cardinal ρ. For example, it is impossible to say that κ is a singular cardinal, even if one explicitly has a parameter for the cofinality of κ.

In a positive direction, it is not difficult to see how to express the property that $\kappa \leq \lambda^{+n}$. The next theorem ([26]) says that "infinite gaps" are not expressible.[4]

Theorem 1.2 (The infinite gap two cardinal theorem:). *$(\kappa^{+\gamma}, \kappa) \to (\rho^{+\delta}, \rho)$ for all infinite cardinals κ, ρ and infinite ordinals γ, δ.*

Further progress was both hindered and abetted by the intrusion of Set Theory. Indeed Jensen ([4]) developed his remarkable combinatorial principles, called *Morasses* to prove:

Theorem 1.3 (Jensen's Gap-n two cardinal theorem).

$$L \models (\forall n \in \omega)(\forall \textit{ infinite } \kappa, \lambda)((\kappa^{+n}, \kappa) \to (\lambda^{+n}, \lambda)).$$

In counterpoint, there is a first order sentence σ such that for any regular cardinal κ there is a model of σ of type (κ^{+2}, κ) just in case there is a κ-Kurepa tree. Thus, for example, the principal $(\omega_3, \omega_1) \to (\omega_2, \omega)$ fails in a model of Silver where there is a Kurepa tree on ω_2, but no ω_1-Kurepa tree. (See e.g. [23])

In the spirit of the Lowenheim–Skolem theorem, one can ask for elementary substructures:

Definition 1.4. For $\kappa \geq \kappa'$ and $\lambda \geq \lambda'$, we say $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ if and only if for all $\mathfrak{A}$ of type (κ, λ) in a countable language there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ of type (κ', λ').

The first nontrivial instance of this became known as *Chang's Conjecture*:

$$(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega).$$

[4]Here and elsewhere, we write $\kappa^{+\alpha}$ for the α^{th} successor of κ.

To modern eyes, Chang's Conjecture is obviously set theoretical: If you apply it to the structure $\langle L_{(\omega_2)^V}, (\omega_1)^V, \in\rangle$ the result is an elementary substructure $N \prec L_{\omega_2}$. If $\bar{N}$ is the transitive collapse of N and j is the inverse of the transitive collapse map, then the embedding j yields an L-ultrafilter on $crit(j)$ and hence the existence of $O^{\#}$. As a consequence, it is clear that Chang's Conjecture cannot be a theorem of ZFC.

On the other hand, variations on the classical Chang's Conjecture have become a proving ground for new set theoretic techniques, such as those related to semiproper forcing. An early example of this is Silver's theorem, which contained the first instance of the use of *master conditions* in a forcing argument.

THEOREM 1.5 (Silver ([16])). *Con(ZFC + there is an ω_1-Erdös cardinal) implies Con(ZFC + GCH + $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$).*

In fact the exact consistency strength of Chang's Conjecture has been shown to be an ω_1-Erdös cardinal ([16]).

Quite surprisingly, Silver's technique is inherently related to the cardinal ω_1, and does not generalize to ω_n for $n > 1$. Kunen developed a technique for proving the consistency of saturated ideals on ω_1 and showed that his technique gave another model of $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$. Laver remarked that Kunen's construction worked at larger cardinals as well, and hence the following holds:

THEOREM 1.6 (Kunen/Laver). *Con(ZFC + there is a huge cardinal) implies that for all $n \geq 1$, Con(ZFC + GCH + $(\omega_{n+1}, \omega_n) \twoheadrightarrow (\omega_n, \omega_{n-1})$).*

Getting an analogous gap-one result for finitely many cardinals was not difficult using the Kunen technique, but more global results needed new ideas ([7]):

THEOREM 1.7. *Con(ZFC + there is a 2-huge cardinal) implies Con(ZFC + GCH + $(\forall m < n)(\omega_{n+1}, \omega_n) \twoheadrightarrow (\omega_{m+1}, \omega_m)$.)*

There rest yet many open problems in the area. For example, while Levinsky, Magidor, and Shelah ([18]) showed that the following is consistent:

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega),$$

it is still not known if the property:

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_2, \omega_1)$$

is consistent. Cummings ([2]) has results indicating that this is a difficult problem.

One reason for the attention that Chang's Conjecture properties receive is that they tend to show up in a wide variety of contexts. They are incompatible with $\square$-type principles and are therefore useful in showing the negation of square.

Another reason is the connection with Hungarian partition theory. It is well-known that

$$\kappa \to [\mu]^{<\omega}_{\rho^+}$$

is equivalent to

$$(\kappa, \rho^+) \twoheadrightarrow (\mu, \rho).$$

A lesser-known, but deeper family of results are partition relations with infinite exponents. For example:

THEOREM 1.8 ([11]). *Assume the CH. Then*:

$$(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$$

is equivalent to

$$\omega_3 \to [\omega_2]^{\omega}_{\omega_2}.$$

We refer the reader to [5] for details.

§2. **More refined second order reflection properties.** To investigate other possible second order reflection properties, we begin by reformulating Chang's Conjecture in modern language. We write $\lambda \gg \kappa$ to mean that λ is a regular cardinal bigger than 2^{2^κ} and $\triangle$ is a well-ordering of $H(\theta)$ in order-type $|H(\theta)|$. Since $\triangle$ is in the language, $\langle H(\theta), \in, \Delta\rangle$ has canonically definable Skolem functions for every structure that belongs to $H(\theta)$.

PROPOSITION 2.1. $(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$ *if and only if there is a* $\theta \gg \omega_{n+2}$ *and an* $N \prec \langle H(\theta), \in \Delta\rangle$ *such that*

if $\pi : N \to \bar{N}$ *is the transitive collapse then*

- $\pi \restriction \omega_n = id$ *and*
- $\pi(\omega_{n+2}) = \omega_{n+1}$.

The relationship between $N, \bar{N}$, and π_N is illustrated in figure 1.

$\vdash$ (Proposition 2.1) ($\Leftarrow$) Suppose that the Chang's Conjecture fails. Let $\mathfrak{A} = \langle \omega_{n+2}, \omega_{n+1}, f_i, R_j, c_k\rangle_{i,j,k\in\omega}$ be the Δ-least counterexample. Suppose

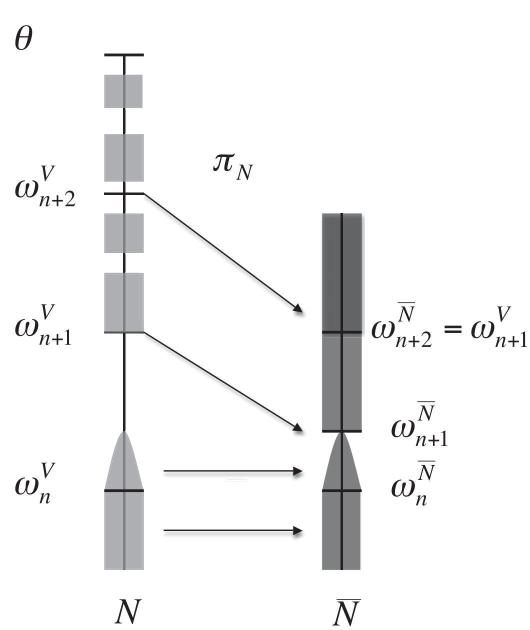

FIGURE 1. Proposition 2.1.

that $N \prec \mathfrak{A}$ is as in the hypothesis. Then $\mathfrak{A} \in N$ so $N \cap \omega_{n+2}$ is closed under Skolem functions for $\mathfrak{A}$. In particular $N \cap \omega_{n+2} \prec \mathfrak{A}$. But $|N \cap \omega_{n+2}| = \omega_{n+1}$ and $|N \cap \omega_{n+1}| = \omega_n$. This is a contradiction.

($\Rightarrow$) Let $\mathfrak{A} = \langle \omega_{n+2}, \in, f_i \rangle$ be a fully Skolemized structure such that for all $z \prec \mathfrak{A}$ we know that $sk^{H(\theta)}(z) \cap \omega_{n+2} = z$. By the Chang's Conjecture we know there is a $z \prec \mathfrak{A}$ such that the type of z is (ω_{n+1}, ω_n). Let $N_0 = sk^{H(\theta)}(z)$. Then $N_0 \prec H(\theta)$.

Let $N = sk^{H(\theta)}(N_0 \cup \omega_n)$. We claim that:

- $\sup(N \cap \omega_{n+1}) = \sup(N_0 \cap \omega_{n+1})$,
- $\sup(N \cap \omega_{n+2}) = \sup(N_0 \cap \omega_{n+2})$.

To see this, let $\tau : N_0 \times \omega_n \to \omega_{n+1}$ be a Skolem function. Since Δ is in the language, for $\vec{x} \in N$, the function $\tau(\vec{x}, \cdot) : \omega_n \to \omega_{n+1}$ is definable in N_0. In particular, $\sup(\tau(\vec{x}, \cdot)``\omega_n) \in N_0$. Since

$$N \cap \omega_{n+1} = \bigcup \{\tau(\vec{x}, \cdot)``\omega_n : \tau \text{ is a Skolem function and } \vec{x} \in N_0\},$$

we see that $\sup(N_0 \cap \omega_{n+1}) = \sup(N \cap \omega_{n+1})$. The result for ω_{n+2} is seen similarly.

Let $\pi : N \to \bar{N}$ be the transitive collapse. Since $\omega_n \subseteq N$, $\pi \upharpoonright \omega_n$ is the identity map. To see that $\pi(\omega_{n+2}) = \omega_{n+1}$ we need to see that $N \cap \omega_{n+2}$ has order type ω_{n+1}. Note that the order type is at least ω_{n+1} by the choice of z. Let $\alpha \in N \cap \omega_{n+2}$. Then there is a bijection $f : \omega_{n+1} \to \alpha$ that lies in N. Hence $f : N \cap \omega_{n+1} \to N \cap \alpha$ is a bijection. In particular, $|N \cap \alpha| = |N \cap \omega_{n+1}|$; thus $|N \cap \alpha| < \omega_{n+1}$. $\dashv$

Standard Skolem function arguments combined with Proposition 2.1 show that the existence of a single $\theta \gg \omega_{n+2}$ with an $N \prec \langle H(\theta), \in, \triangle \rangle$ satisfying the conditions about where π moves ordinals implies that for all $\theta \gg \omega_{n+2}$ there is an N satisfying the conditions. Moreover, the existence of a single N satisfying the hypothesis implies the existence of stationarily many such N. We will say more about stationary sets later.

For the rest of the paper we will write $\bar{N}$ for the transitive collapse of N and π_N for the transitive collapse map.

In [10], the following more involved extension of Proposition 2.1 is proved:

Proposition 2.2. *Let $\lambda \leq \kappa \ll \theta$ be cardinals with λ and θ regular and with $cf(\kappa) \geq \lambda$. Let $\mathfrak{A}$ be a structure expanding $\langle H(\theta), \in, \Delta, \{\kappa, \lambda\} \rangle$ and $N_0 \prec \mathfrak{A}$. Let $N_1 = sk^{\mathfrak{A}}(N_0 \cup \sup(N_0 \cap \lambda))$ and let $\rho = |\sup(N_0 \cap \lambda)|$. Suppose that either*:

1. *The GCH holds or*
2. *there is a $\xi \subseteq N_0$ and $\kappa \leq \lambda^{+\xi}$.*

Then $N_1 \cap \lambda = \sup(N_0 \cap \lambda)$, $\sup(N_1 \cap \kappa) = \sup(N_0 \cap \kappa)$ and $|N_1 \cap \kappa| = |N_0 \cap \kappa| \cdot \rho$.

The GCH is only used for computing the cardinality of $N_1 \cap \kappa$. It seems to be open whether the GCH is necessary or even relevant.

2.1. Adding reflection properties: The first new property. Restating Chang's Conjectures in terms of the transitive collapse of an $N \prec H(\theta)$ opens the door to asking additional properties of N than that the collapsing map send ordinals to the right places. Our first new property is quite easy, we ask for an $N \prec H(\theta)$ as in Proposition 2.1 with the additional requirement that:

$$N \cap \omega_{n+2} \in \bar{N}. \tag{1}$$

This requirement is clearly in the spirit of the usual Chang's Conjecture. It asks that you can find an element a of N (necessarily lying above ω_{n+2} in some sense) such that $\pi_N(a) = N \cap \omega_{n+2}$. One might hope to construct such an N by adding an a with these properties to an N_0 so that the collapse of a becomes $N_0 \cap \omega_{n+2}$. But this must be done in such a manner that $N_0 \cap \omega_{n+2}$ is not changed. This type of self-referential obstacle, reminiscent of the arguments involved in analyzing semiproper forcing, is typical of Chang's Conjecture.

2.2. A digression on nonstationary ideals. We remind the reader of the modern notion of stationarity.

Definition 2.3. Let $S \subseteq P(X)$. Then S is *stationary* if and only if for all $\mathfrak{A} = \langle X, f_i \rangle_{i \in \omega}$, there is a $z \prec \mathfrak{A}$ such that $z \in S$. We write NS_X for the nonstationary ideal on X.

This notion was called "weakly stationary" in [13]. It gives a uniformly-defined normal and fine ideal on $P(X)$ for arbitrary sets X that generalizes older notions of stationary set. Examples include:

- If κ is a regular cardinal then NS_κ restricted to κ (as a set of ordinals) coincides with the classical notion of nonstationary.
- If λ is cardinal and $\kappa < \lambda$ is regular, then $NS_\lambda \restriction \{z \in [\lambda]^{<\kappa} : z \cap \kappa \in \kappa\}$ is the nonstationary ideal on $P_\kappa(\lambda)$ in the sense of Jech [15].

With this language we see that all "Chang Conjectures" are simply statements that certain sets are stationary. For example

$$(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$$

is equivalent to

$$\{z \in [\omega_{n+2}]^{\omega_{n+1}} : |z \cap \omega_{n+1}| = \omega_n\} \text{ is stationary.}$$

2.3. A second new reflection property. Let us see what happens to the image of the nonstationary ideal under the transitive collapse map of an elementary substructure. Suppose that $N \prec H(\theta)$ and $|N \cap \omega_{n+i}| = \omega_{n+i-1}$ for $i = 1, 2, 3$, and assume that $\omega_n \subseteq N$. Let $A' \subseteq [\omega_{n+3}]^{\omega_{n+2}}$ be an element of N. Let $\pi : N \to \bar{N}$ be the transitive collapse.

Then $\pi(A') = A$ for some $A \subseteq [\omega_{n+2}]^{\omega_{n+1}}$. Moreover, since being nonstationary is witnessed in an absolute way by a structure $\mathfrak{A}$ on ω_{n+2}^V, we see that

$$\pi(NS \restriction A') \subseteq (NS \restriction A) \cap \bar{N}. \tag{2}$$

It is clearly a closure or "thickness property" of N to ask for equality in equation 2. We do not need a property as strong as equality for our results (though it is consistent), so we give the following weaker definition:

DEFINITION 2.4. N is correct for $NS \restriction A$ if and only if $A \in N$ and there are $A', I' \in N$ such that

$$\pi_N(I' \restriction A') = (NS \restriction A) \cap \bar{N}.$$

(The set A' is mentioned only to make sure that there is a set A' belonging to N such that $\pi_N(A') = A$.)

2.4. Strong Chang Reflection.

DEFINITION 2.5. We will say that *Strong Chang reflection* holds for (ω_{n+3}, ω_n) if and only if for all large enough θ there is an $A \subseteq \{N \in [\omega_{n+2}]^{\omega_{n+1}} : N \cap \omega_{n+1} \in \omega_{n+1}\}$ such that for some

$$N \prec \langle H(\theta), \in, \Delta, A\rangle$$

we have:

1. $N \cap \omega_{n+2} \in A$ and $|N \cap \omega_{n+3}| = \omega_{n+2}$,
2. $N \cap \omega_{n+2} \in \bar{N}$,
3. N is correct for $NS \restriction A$.

The relationship between $N, \bar{N}, NS \restriction A$ and $NS \restriction A'$ is illustrated in figure 2.

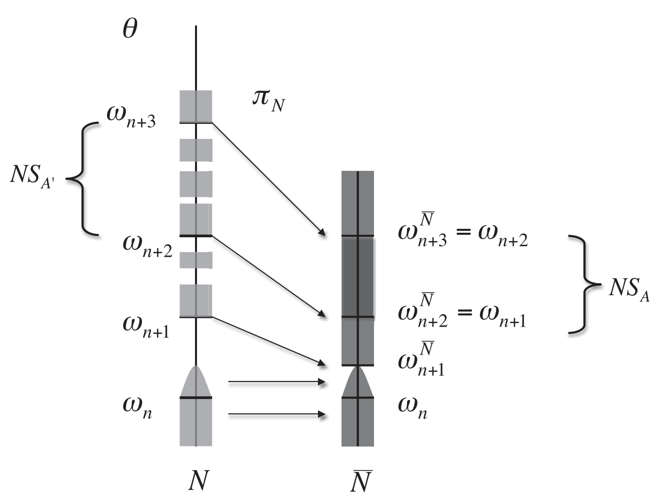

FIGURE 2. Strong Chang Reflection.

As we remarked earlier, saying that Strong Chang's Conjecture holds for a single $\theta \gg \omega_{n+3}$ implies that it holds for all large θ, and the existence of a single $N \prec H(\theta)$ where θ is large implies the existence of a stationary set of such N. Thus, using regressive function arguments, it follows that we can find fixed A', I', and O' such that for stationarily many $N \prec H(\theta)$, 1–3 hold, with I', A' witnessing correctness, and $N \cap \omega_{n+2} = \pi_N(O')$.

Informally Strong Chang Reflection says that the collection of N that are correct about $NS \restriction A$ and whose transitive collapse contains $N \cap \omega_{n+2}$ is stationary and canonically well-ordered.

2.5. Analogous Large Cardinal statements. We now note that large cardinal statements imply stationarity properties analogous to Strong Chang Reflection. Example 2.6 illustrates the first new reflection property. Example 2.7 is analogous to the second new reflection property.

EXAMPLE 2.6. The requirement in Section 2.1 is in direct analogy to a 2-huge cardinal. Suppose that j is a 2-huge embedding with critical point κ, $j(\kappa) = \lambda$ and $j(\lambda) = \mu$. Let $\theta \gg \kappa$. Then the collection of $N \prec H(\theta)$ such that if $\pi : N \to \bar{N}$ is the transitive collapse:

1. $\pi \restriction N \cap \kappa = id$ and $\pi(\lambda) = \kappa$.
2. $\pi(\mu) = \lambda$.
3. $N \cap \lambda \in \bar{N}$.
4. N is correct for the nonstationary ideal on $P([\lambda]^\kappa)$

is stationary. (See fig 3.)

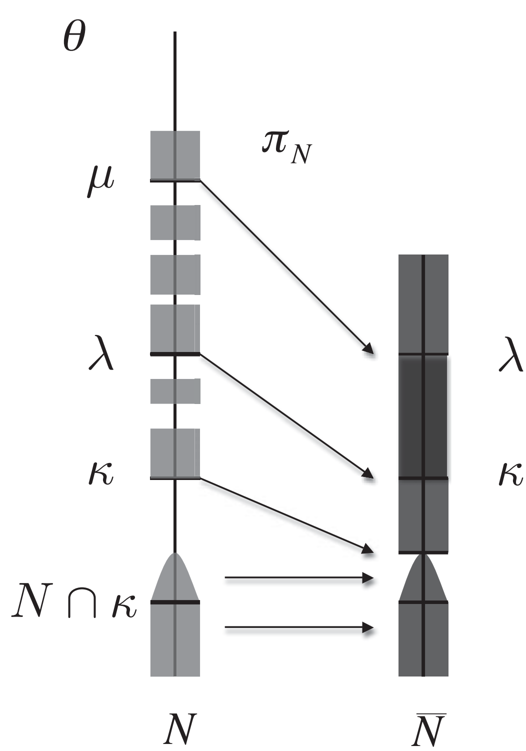

FIGURE 3. Example 2.6.

EXAMPLE 2.7 (Magidor). Suppose that $j : V \to M$ is an elementary embedding with M a transitive class. Let κ be the critical point of j and $\lambda > \kappa$

be regular. Suppose that $j(\kappa) > \lambda$ and M is correct for $NS \restriction (\lambda \cap \mathrm{cof}(\omega))$. Then there is a λ-supercompact embedding.

To see this it suffices to show that $j``\lambda \in M$. Let $\langle A_\alpha : \alpha < \lambda\rangle$ be a partition of $\lambda \cap \mathrm{cof}(\omega)$ into stationary sets, and $\langle A^j_\alpha : \alpha < j(\lambda)\rangle = j(\langle A_\alpha : \alpha < \lambda\rangle)$. If $\gamma = \sup(j``\lambda)$, then:

$$\alpha \in j``\lambda$$
$$\text{iff}$$
$$A^j_\alpha \cap \gamma \text{ is stationary in } \gamma.$$

2.6. The main theorems.

THEOREM 2.8. *Suppose Strong Chang Reflection holds for (ω_{n+3}, ω_n). Then there is a transitive inner model for "ZFC + there is a huge cardinal."*

This theorem would not be interesting except for the accompanying theorem:

THEOREM 2.9. *Suppose there is a 2-huge cardinal. Then for each n there is a forcing extension in which Strong Chang Reflection holds for (ω_{n+3}, ω_n).*

A proof of Theorem 2.9 appears in [10]. It is much harder than Theorem 2.8, but less novel. The rest of this note outlines a proof of Theorem 2.8.

§3. Constructing models with very large cardinals in them. One of the most central and important themes in set theory has been the exploration of large cardinal axioms by building canonical fine-structural models. This remarkable program has succeeded in establishing an exact correspondence between descriptive set theoretic properties and various large cardinal axioms. The success of the program has reinforced the widespread belief that large cardinal axioms provide a unifying, linearly ordered framework for establishing the consistency strength of virtually every set theoretic statement.

However, despite intense sustained effort of the best minds in the subject, to date there has not been a satisfactory inner model theory of *large* large cardinals. Attempts to prove the existence of fine structural inner models of supercompact cardinals have all foundered on the rocks of the "iterability problem." (We refer readers to [20–22] for information on the current state of the art.)

The author does not claim to have solved this problem. Rather the results in this paper represent an *ad hoc* way of constructing models of ZFC that contain very large cardinals. As an example of the author's ignorance, it is not known if the GCH holds in the models constructed in this paper.

In this section we give a short description of a way of building inner models for supercompact and larger cardinals. We begin by giving a superficial review of the obstacles for building inner models for large large cardinals.

The basic method lying behind the constructions of inner models for large cardinals was first studied at length by Silver ([24]).[5] This method begins

[5]While extender models are of more complicated form, at root they start with constructions from ultrafilters on a set of ordinals.

with a κ-complete ultrafilter and builds a model of the form $L[U]$. Silver showed the GCH holds in these models; later Solovay developed a fine structure theory for $L[U]$.

The initial problem for building an inner model for a supercompact or huge cardinal is that if U is a normal, fine ultrafilter on $[\lambda]^{<\kappa}$ or $[\lambda]^{\kappa}$, then $L[U] = L$. This is because L contains no set that lies in U, hence one sees inductively that for all $\alpha \in OR$, $L_\alpha[U] = L_\alpha[\emptyset] = L_\alpha$.

An immediate response to this problem is take a set $S \subseteq [\lambda]^{\leq\kappa}$ that belongs to U and construct $L[S, U]$. However, typically, $[\lambda]^{\leq\kappa} \cap L[S] \notin U$, and hence we do not get a model of a very large cardinal in this manner.

We could also put S into a model of set theory by *force majeure* (e.g. building $L(S)[U]$), but this risks constructing a model without choice. We take a variation of this route, making sure that the set S has a canonical and absolute well-ordering.

Let $A \subseteq P(\lambda)$ and $W = \langle a_\alpha : \alpha < \gamma \rangle$ be an enumeration of A.

$$A^* = \{(\beta, \alpha) : \beta \in a_\alpha\}.$$

Then A, A^*, W are all elements of $L[A^*]$ and $L[A^*] \models ZFC$. In particular if U is a normal ultrafilter on $[\lambda]^{\leq\kappa}$ and $A \in U$, then $L[A^*, U] \models$ U is a normal ultrafilter.

Solovay ([25]) in a precursor to Proposition 3.2, showed that if U is a supercompact ultrafilter then there is a set S in U such that the function $x \mapsto \sup(x)$ is one to one on S. Hence this set S has a natural candidate for a well-ordering: the ordering induced by the supremum. We use a variation on this idea, the notion of an *ordinary* set. If A is ordinary then there is a canonical candidate for W–the lexicographical ordering on the pairs of critical ordinals.

Thus, if we are presented with a normal, fine ultrafilter U on $[\lambda]^{\leq\kappa}$ we can build a model of the form $L[A^*, U]$. However this presupposes that we have a normal, fine ultrafilter in hand to begin with and hence is useless for relative consistency results. To build models of the appropriate large cardinal we need to figure out what filter to use in the role of U. For this we use a generalization of some classical ideas. Our models will be of the form:

$$L[A^*, \breve{I}],$$

where $\breve{I}$ is the dual of the nonstationary ideal on $[\lambda]^{\leq\kappa}$.

Since the dual to the nonstationary ideal on $[\lambda]^{\leq\kappa}$ is already normal, fine and κ-complete and these properties are downwards absolute, to get a model with a supercompact or huge cardinal we need only show that $L[A^*, \breve{I}] \models$ "$\breve{I}$ is an ultrafilter." We record this as a proposition:

PROPOSITION 3.1. *Suppose that I is a normal, fine ideal on $[\lambda]^{\leq\kappa}$ and that $L[A^*, I] \models$ "$\breve{I}$ is an ultrafilter." Then $L[A^*, I] \models$ "$\breve{I}$ is a normal, fine ultrafilter on $[\lambda]^{\leq\kappa}$.*

In particular,

1. *If $[\lambda]^{<\kappa} \in \breve{I}$, then $L[A^*, I] \models \kappa$ is λ-supercompact.*
2. *If $[\lambda]^{\kappa} \in \breve{I}$, then $L[A^*, I] \models \kappa$ is huge.*

In [10], it is shown that if κ is a supercompact cardinal implies that for each λ there is a stationary subset $A \subseteq [(2^\lambda)^+]$ such that

$$L[A, NS \restriction A] \models ZFC + \kappa \text{ is } \lambda\text{-supercompact.}$$

Similar results for small cardinals seem out of reach at the moment, however it is not inconceivable that, assuming Martin's Maximum, one might find a similar set $A \subseteq [\omega_4]^{<\omega_2}$.

3.1. Canonically well-ordered stationary sets. We now show that the sets A that we will use to construct large cardinals are naturally well-ordered. For such A, the set A^* described above is absolutely definable.

PROPOSITION 3.2 (Baumgartner). *Let $M, N \prec H(\theta)$. Suppose that*

- $\sup(M \cap \omega_{n+2}) = \sup(N \cap \omega_{n+2}) \in cof(> \omega)$,
- $N \cap \omega_{n+1} = M \cap \omega_{n+1}$ *and* $\sup(N \cap \omega_{n+1}) \in cof(> \omega)$.

Then $M \cap \omega_{n+2} = N \cap \omega_{n+2}$.

To use the Baumgartner proposition, we need the following result ([12]):

PROPOSITION 3.3 (Foreman–Magidor). *For $n \in \omega$ and $N \prec H(\theta)$: if $N \in [\omega_{n+2}]^{\omega_{n+1}}$, $N \cap \omega_{n+1} \in \omega_{n+1}$, then $cof(N \cap \omega_{n+1}) = \omega_n$.*

Putting these two propositions together we get the following:

COROLLARY 3.4. *Let $\theta > \omega_{n+2}$ be a regular cardinal and $n > 0$.*

$$p : [\omega_{n+2}]^{\omega_{n+1}} \to \omega_{n+1} \times \omega_{n+2}$$

be defined by

$$z \mapsto (\sup(z \cap \omega_{n+1}), \sup(z \cap \omega_{n+2})).$$

Then p is 1-1 *on the collection of z such that*:

1. $sk^{H(\theta)}(z) \cap \omega_{n+2} = z$,
2. $z \cap \omega_{n+1} \in \omega_{n+1}$.

Upshot: Relative to a closed unbounded set, p restricted to $\{z \in [\omega_{n+2}]^{\omega_{n+1}} : z \cap \omega_{n+1} \in \omega_{n+1}\}$ is 1-1.

DEFINITION 3.5. A set $A \subseteq \{z \in [\omega_{n+2}]^{\omega_{n+1}} : z \cap \omega_{n+1} \in \omega_{n+1}\}$ is *ordinary* if p is 1-1 on A.

COROLLARY 3.6. *If Strong Chang Reflection holds then we can assume that the witnessing set A is ordinary.*

We note that ordinary sets have a canonical and absolute well-ordering, the pullback of the lexicographical ordering on $\omega_{n+1} \times \omega_{n+2}$.

The assertion that for $n > 0$ there is an ordinary stationary set is equivalent to the assertion of the Chang's conjecture $(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$. The best known upper bound on the consistency strength of this property is a huge cardinal.

3.2. Decisive ideals. In this section we will be using the technique of generic ultrapowers. We begin by describing how to project ideals from a large set to a smaller subset.

Suppose that $X' \subseteq X$. We get a map $\Pi : P(X) \to P(X')$ by setting $\Pi(z) = z \cap X'$. Then Π induces a Boolean algebra homomorphism $\iota : PP(X') \to PP(X)$ by setting $\iota(A) = \{z : \Pi(z) \in A\}$.

If $J \subseteq PP(X)$ is an ideal on $P(X)$ we get an ideal $I = \pi_{X'}(J) \subseteq PP(X')$ on $P(X')$ by setting $A \in I$ if and only if $\iota(A) \in J$ (or equivalently $\Pi^{-1}(A) \in J$).[6]

Definition 3.7. Let $Z \subseteq P(X)$ and J be an ideal on Z. Let $X' \subseteq X$ and $I = \pi_{X'}(J)$. Then J *decides* I if and only if there is a set $A \in \breve{I}$ and a well-ordering W of A, and A', W', O', I' such that for all generic $G \subseteq P(X)/J$:

1. an initial segment of the ordinals of V^Z/G is well-founded and isomorphic to $(|A'|^+)^V$ and
2. if $j : V \to M \cong V^Z/G$ is the canonical embedding, where M is transitive up to $(|A'|^+)^V$, then
   (a) $j(A) = A'$.
   (b) $j(W) = W'$.
   (c) $j``|A| = O'$.
   (d) $I' = j(I) \cap V$.

We will say that J is *decisive* if and only if J decides J.

Remark 3.8. Since the definition is complicated we make some remarks explaining it.

- Usually A will be ordinary, in which case the canonical well-ordering is absolute and so clause (b) is vacuous.
- Surprisingly, precipitousness is not part of the definition. In fact, it will follow from the next lemma, that well-foundedness is not a major issue for the applications.
- There seem to be two kinds of ideals yielding good generic elementary embeddings–those ideals that are the remnants of collapsed large cardinals and those ideals that have "antichain catching" properties.[7] This definition is an attempt to characterize the former.
- The notational convention is that the "primed" objects (A', W', O', I') are on the "j-side" of the definition.

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[6]Alternatively: Let B be the set $P(X)$ and C be the set $P(X')$. Then $\Pi : B \to C$ is a many-to-one map between two sets. It induces the map $\iota : P(C) \to P(B)$ as described. An ideal $J \subseteq PP(X)$ is an ideal on the Boolean algebra $P(B)$. We can compose the map $\iota : P(C) \to P(B)$ with the canonical projection map $p : P(B) \to P(B)/J$. Then $p \circ \iota : P(C) \to P(B)/J$. The kernal of $p \circ \iota$ is the ideal I.

[7]We are referring to arguments about the nonstationary ideal on ω_1 and the various nonstationary towers.

The following lemma is easy (see [9] for a proof).

LEMMA 3.9. *Suppose that J is a normal and fine ideal on $Z \subseteq P(X)$. Let $G \subseteq P(Z)/J$ be generic and $j : V \to V^Z/G = M$ be the canonical elementary embedding,*

1. *If $id : Z \to Z$ is defined by $id(z) = z$, then $[id]_G = j``X$.*[8]
2. *V^Z/G is well-founded up to $(|X|^+)^V$.*

Lemma 3.9 implies that if we have a "local" property of the $\aleph_n$'s, then by taking generic ultrapowers using the index set $H(\theta)$ for large enough θ we can get all the well-foundedness we want.

The next result is our main tool for building inner models with very large cardinals:[9]

THEOREM 3.10. *Let $\mu \leq \lambda$ be cardinals. Suppose that $J \subseteq P(Z)$ is a normal, fine ideal on a set $Z \subseteq P(\lambda)$ that decides a countably complete ideal $I \subseteq P(Z')$ for some $Z' \subseteq P(\mu)$. Suppose that A, A', W, W', I', O' witness that J decides I. Then either*:

a. *$L[A^*, I] \models \check{I}$ is an ultrafilter on A*
   or for some generic $G \subseteq P(Z)/J$, if $j : V \to V^Z/G$ is the ultrapower embedding, then
b. *$L[j(A^*), I'] \models \check{I}'$ is an ultrafilter on $j(A)$.*

Note:

1. The model $L[j(A^*), j(I)]$ is not the same as $(L[j(A^*), j(I)])^{V^Z/G}$ if the ultrapower is ill-founded.
2. In light of the fact that the nonstationary ideal on $P_\kappa(\lambda)$ is normal, fine, and κ-complete, if Z' is $P_\kappa(\lambda)$ this says that there is an inner model with a cardinal κ that is λ-supercompact. Similarly if $Z' = [\lambda]^\kappa$, then this gives an inner model with a huge cardinal.

$\vdash$ We begin the proof of Theorem 3.10 with an easy claim:

Claim: For $S \subseteq P(Z')$:

$$S \in I \text{ iff for all generic } G \subseteq P(Z)/J,\ j``\mu \notin j(S)$$

$S \in I$ iff $\pi_\mu^{-1}(S) \in J$, so it suffices to show that for all $T \subseteq P(Z)$, we have that $T \in J$ iff for all generic G, $j``\lambda \notin j(T)$. We compute:

$$\begin{aligned}(Z \setminus T) \in \check{J} &\text{ iff } \{z : z \in Z \setminus T\} \in G \text{ for all generic } G \\ &\text{ iff } \{z : id(z) \in Z \setminus T\} \in G \text{ for all generic } G \\ &\text{ iff } [id]^M \in j(Z \setminus T) \text{ for all generic } G \\ &\text{ iff } j``\lambda \in j(Z) \setminus j(T) \text{ for all generic } G \\ &\text{ iff } j``\lambda \notin j(T) \text{ for all generic } G.\end{aligned}$$

This finishes the proof of the claim.

[8] More precisely, for all $f : Z \to V$ with $f \in V$ if $\{z : f(z) \in id(z)\} \in G$, then there is an $x \in X$ such that $\{z : f(z) = x\} \in G$. Equivalently, if E is the relation coming from $\in$ in the ultrapower, then for all $[f]$, $[f]E[id]$ iff $[f] = j(x)$ some $x \in X$.

[9] Schindler remarked that in many situations items **a.** and **b.** are equivalent. See Appendix B.

We now prove the theorem. Suppose that **a.** fails. Let δ be the least ordinal such that $L_{\delta+1}[A^*, I] \models$ "$\check{I}$ is not an ultrafilter." Then δ is definable in $L[A^*, I]$ and δ is less than $|A|^+$ as computed in $L[A^*, I]$. Moreover, there is a formula $\phi(w, u, v)$ such that

$$B = \{z' \in A : L_{\delta+\omega}[A^*, I] \models \phi(z', A^*, I)\} \notin I \cup \check{I}.$$

Let $G \subseteq P(Z)/J$ be generic and $j : V \to M$ be the canonical elementary embedding where $M \cong V^Z/G$ and M is well-founded up to $(|A'|^+)^V$. Then

$$j \restriction L[A^*, I] : L[A^*, I] \to L^M[j(A^*), j(I)].$$

Since $j(A) = A', I' = j(I) \cap V, j(W) = W'$ we know that $j(A^*) \in V$. An inductive argument shows that if $\xi \in OR^M$ is well-founded then:

$$\begin{aligned} L_\xi^M[j(A^*), j(I)] &= L_\xi[j(A)^*, j(I)] \\ &= L_\xi[(A')^*, I'] \end{aligned}$$

Case 1: For some $G \subseteq P(Z)/I$, $j(\delta)$ is not in the well-founded part of M.
In this case

$$L_{|A'|^+}[j(A)^*, I'] \models \check{I}' \text{ is an ultrafilter.}$$

Since $|A'|^+ = |j(A^*)|^+ \geq (|j(A^*)|^+)^{L[j(A)^*, I']}$,

$$L[j(A)^*, I'] \models \check{I}' \text{ is an ultrafilter.}$$

Case 2: For all generic $G \subseteq P(Z)/I$, $j(\delta)$ belongs to the well-founded part of M.

In this case we let δ' be the least ordinal in M such that

$$L_{\delta'+1}[j(A^*), j(I)] \models j(\check{I}) \text{ is not an ultrafilter.}$$

Then $j(\delta) = \delta'$ and δ' is in the well-founded part of M. It follows that

$$L_{\delta'+1}[j(A^*), j(I)]^M = L_{\delta'+1}[j(A^*), j(I)]^V.$$

Since δ' is definable in $L[j(A^*), j(I)]$ and $L[j(A^*), j(I)]$ is independent of G, it follows that for all generic G, $j(\delta) = \delta'$. Moreover,

$$j(B) = \{z' \in j(A) : L_{\delta'+\omega}[j(A)^*, I'] \models \phi(z', j(A)^*, I')\}$$

is independent of G.

By hypothesis j"μ is independent of G (as it is determined by O') and thus we see that either:

- for all G, j"$\mu \in j(B)$ or
- for all G, j"$\mu \notin j(B)$.

By the claim this says that either $B \in \check{I}$ or $B \in I$, contradicting the definition of B. $\dashv$

THE PROOF OF THEOREM 2.8.

We will prove a more general theorem:

THEOREM 3.11. *Suppose that $\kappa_2 > \kappa_1 > \kappa_0$ are cardinals and that there is a regular $\theta \gg \kappa_2$ and a stationary set $S \subseteq P(H(\theta))$ and an ordinary $A \subseteq [\kappa_1]^{\kappa_0}$ such that for all $N \in S$:*

1. *$N \cap \kappa_1 \in A$, $|N \cap \kappa_2| = \kappa_1$.*
2. *$N \cap \kappa_1 \in \bar{N}$.*
3. *N is correct for $NS \restriction A$.*

Then there is an inner model with a huge cardinal.

Remarks

- The usual "proper forcing tricks" show that having a single large θ where the hypothesis holds is equivalent to the hypothesis holding for all large θ. Moreover the existence of a single N satisfying the hypothesis 1.-3. implies the existence of a stationary set of such N.
- If we take $\kappa_0 = \omega_{n+1}, \kappa_1 = \omega_{n+2}$, and $\kappa_2 = \omega_{n+3}$ then, using Corollary 3.6 this is a restatement of Strong Chang Reflection.
- If $\kappa_0, \kappa_1, \kappa_2$ is the cardinal sequence of a 2-huge cardinal then it is not difficult to verify that the hypothesis of this theorem hold.

$\vdash$ Since A contains the projection of S, A is stationary. A regressive function argument shows that we can find a stationary set $S' \subset S$, an ideal I' and sets A', O' such that for all $N \in S'$:

1. both A' and $I' \in N$ and $\pi_N(I' \restriction A') = (NS \restriction A) \cap \bar{N}$, and
2. $\pi_N(O') = N \cap \kappa_1$.

Without loss of generality we assume S has this property. We also assume that θ is bigger than $(2^{2^{|A'|}})^+$.

By Theorem 3.10 will be done if we can show the following:

Claim: $NS \restriction S$ decides $NS \restriction A$.

Let $G \subseteq P(S)/NS$ be generic and $j : V \to M$ be the canonical embedding. Then:

- Since $\theta \gg \kappa_2$, V^S/G is well-founded up to θ, which we have assumed is at least $(2^{2^{|A'|}})^+$.
- $crit(j) = \kappa_0$, $j(\kappa_0) = \kappa_1$ and $j(\kappa_1) = \kappa_2$.

We need to verify items a–d of clause 2 of the definition of *decisiveness*. We use the easy fact that if $N = j``H^V((2^{2^{|A'|}})^+)$ then $\pi_N = j^{-1}$.

(a) In M:

$$\pi_{j``H((2^{2^{|A'|}})^+)}(j(A')) = A'.$$

because $\pi_{j``H((2^{2^{|A'|}})^+)} = j^{-1}$. Moreover, for all $N \in S, \pi_N(A') = A$. Thus

$$\pi_{j``H((2^{2^{|A'|}})^+)}(j(A')) = j(A),$$

in particular, $j(A) = A'$.

(b) Since A is ordinary, there is a canonical absolute well-ordering W of A that gets sent by j to a canonical absolute well-ordering W'.

(c) We do a calculation similar to that of (a). For all $N \in S$

$$\pi_N(O') = N \cap \kappa_1.$$

Hence:

$$\begin{aligned} O' &= \pi_{j``H((2^{2^{|A'|}})^+)}(j(O')) \\ &= j``H((2^{2^{|A'|}})^+) \cap j(\kappa_1) \\ &= j``\kappa_1. \end{aligned}$$

(d) Similarly, since the transitive collapse of $j``H((2^{2^{|A'|}})^+)$ is $H((2^{2^{|A'|}})^+)$:

$$\begin{aligned} I' &= \pi_{j``H((2^{2^{|A'|}})^+)}(j(I')) \\ &= (NS \restriction j(A))^M \cap (H((2^{2^{|A'|}})^+))^V \\ &= j(NS \restriction A) \cap V. \end{aligned}$$

This finishes the proof of Theorem 3.11. ⊣

We have shown that a certain natural second order reflection property of ω_4 lies between a huge cardinal and a 2-huge cardinal in consistency strength. While *ad hoc*, the technique does provide a way of distinguishing between ideals yielding generic embeddings that are the remnants of large cardinal embeddings and those that are not. One could perhaps hope that similar *ad hoc* techniques might show that the consistency strength of Proper Force Axiom or Martin's Maximum is that of a supercompact cardinal.

§4. **Acknowledgments.** This note is a summary of lectures given at the workshop "The interplay between large cardinals and small cardinals" which took place at the Research Institute for Mathematics, in Kyoto, Japan. The author would like to thank the organizers for their hospitality.

§A. **Appendix.** Large cardinals are usually formulated in terms of the existence of nontrivial elementary embeddings of V into transitive subclasses. The embedding is required to be definable over V using a set parameter. If j is a nontrivial elementary embedding it must move an ordinal and the *critical point* of j is the least such ordinal moved. Here are the four definitions most relevant to this paper:

Measurable Cardinal There is measurable cardinal if and only there is a nontrivial elementary embedding $j : V \to M$, for some transitive M. The *measurable cardinal* is the critical point of j.

Supercompact Cardinal There is a λ-supercompact cardinal if and only if there is a $j : V \to M$ such that

1. The first ordinal moved by j is below λ,
2. M is closed under λ-sequences.

The *supercompact cardinal* is the critical point of j.

Huge Cardinal There is a huge cardinal if and only if there is a nontrivial elementary embedding $j : V \to M$ such that if κ is the critical point of j, then M is closed under $j(\kappa)$ sequences.
The critical point of j is called a *Huge Cardinal*

2-Huge Cardinal There is a nontrivial elementary embedding $j : V \to M$ such that if κ is the critical point of j, then M is closed under $j^2(\kappa)$-sequences.
The critical point of j is called a *2-Huge Cardinal*.

These definitions are formulated inside ZFC by stating an appropriate type of ultrafilter exists. Here are the equivalences:

- κ is measurable iff there is a normal, fine, κ-complete ultrafilter on $P(\kappa)$.
- κ is λ-supercompact iff there is a normal, fine, κ-complete ultrafilter on $P([\lambda]^{<\kappa})$.
- κ is huge iff there is a normal, fine, κ-complete ultrafilter on $P([\lambda]^{\kappa})$ for some $\lambda > \kappa$.
- κ is 2-huge iff there are $\lambda < \mu$ and a normal, fine, κ-complete ultrafilter on $P([\mu]^{\lambda})$ containing $\{x : x \cap \lambda \in [\lambda]^{\kappa}\}$.

The ultrafilter definitions are downwards absolute in the sense that if $M \subseteq V$ is a subclass containing a set A belonging to the relevant ultrafilter U and $U \cap M \in M$, then $U \cap M$ retains the relevant property in M.

§B. Appendix. In Theorem 3.11 we showed that if Strong Chang Reflection holds, then for arbitrarily large θ we can find a stationary set $S \subseteq H(\theta)$ such that $NS \restriction S$ decides $NS \restriction A$. It then follows from Theorem 3.10 that either:

a. $L[A^*, I] \models \check{I}$ is an ultrafilter on A
   or for some generic $G \subseteq P(S)/NS$, if $j : V \to V^S/G$ is the ultrapower embedding, then
b. $L[j(A^*), I'] \models \check{I}'$ is an ultrafilter on $j(A)$.

Ralf Schindler remarked that under mild large cardinal hypothesis, the two conclusions are equivalent provided that θ is large enough. (In fact, **a.** is equivalent to the statement that for *all* generic G, **b.** holds.) To see this, let A' be as in the definition of "decides" (Definition 3.7) and $\theta \gg |A'|$.

Assuming mild large cardinal assumptions, the "sharp" for $L[A^*, I]$ exists and is coded by a set Σ of size $|A^*|$. We can view Σ as a collection of formulas in the language built by expanding usual language of set theory by adding constants for A^*, all elements of A^* and for I. The model $L[A^*, I]$ can be constructed by using the skeleton coded by this sharp. By standard theory, the sharp Σ can be recognized in an absolute way in models of set theory that are well-founded up to any $\lambda \gg |A^*|$ and contain $A^* \cup \{A^*\}$ and $\{I \cap L[A^*, I]\}$ as well as Σ.

Let $G \subset P(S)/NS$ be generic. Then $j(A^*) = (A')^*$. Replacing V^Z/G by an isomorphic class M that is transitive through its well-founded part (and viewing $j : V \to M$), $M \models$ "$j(\Sigma)$ is the sharp of the model $L^M[j(A^*), I']$". Because M is well-founded to at least θ^+ and $\theta \gg |A'|$, $j(\Sigma)$ is the actual sharp of a well-founded model. Thus the skeleton determined by $j(\Sigma)$ can be expanded to a well-founded class model of "$V = L[A', I']$" by using a well-founded class of indiscernibles.

Because Σ and $j(\Sigma)$ have the same formulas in the language $\mathcal{L} = \{\varepsilon\}$ and $j(\Sigma)$ gives the theory of $L[(A')^*, I']$ we see that

$$L[A^*, I] \equiv L[(A')^*, I'].$$

This establishes Schindler's claim.

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