Title

Quantum W3 Gravity

Permalink

https://escholarship.org/uc/item/29j4x6zd

Authors

Schoutens, K.
Sevrin, A.
Nieuwenhuizen, P. van

Publication Date

1991-11-01

Lawrence Berkeley Laboratory
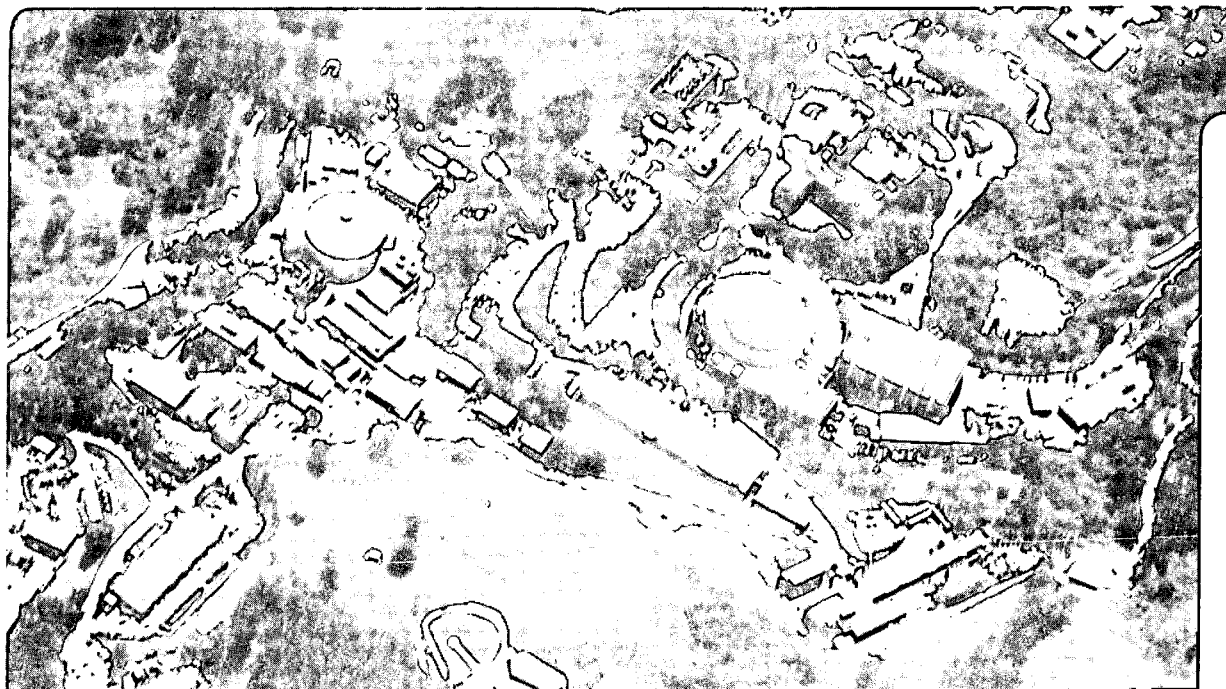## UNIVERSITY OF CALIFORNIA

Physics Division

Presented at the Trieste Summer School on High Energy Physics and
Cosmology, Trieste, Italy, August 8–9, 1991, and to be
published in the Proceedings

Quantum W_3 Gravity

K. Schoutens, A. Sevrin, and P. van Nieuwenhuizen

November 1991

DISCLAIMER

DISCLAIMER

QUANTUM W_3 GRAVITY[*†‡]

K. Schoutens[1], A. Sevrin[2] and P. van Nieuwenhuizen[3,1]

1. Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840, U.S.A.

2. Department of Physics
University of California at Berkeley
and
Theoretical Physics group
Lawrence berkeley Laboratory
Berkeley, CA 94720, U.S.A.

3. Theory Division, CERN
CH-1211 Geneva 23
Switzerland

Abstract

We briefly review some results in the theory of quantum W_3 gravity in the chiral gauge. We compare them with similar results in the analogous but simpler cases of $d = 2$ induced gauge theories and $d = 2$ induced gravity.

Gravity in two dimensions has the exceptional property that it allows for higher spin extensions: the W gravity theories (reviews of the classical formulations of these theories can be found in [1]). The basic structure behind these theories are W algebras, which are higher-spin extensions of the Virasoro algebra.

In a recent project, we studied the quantum theory for the particular theory of chiral gauge W_3 gravity in considerable detail. Extensive accounts of this work can be found in the research papers [2]-[4] and in the review papers [5, 6]. In our contribution to these proceedings we will recall the main results that were obtained and give some comments; for detailed derivations and proofs we refer to the papers just cited.

The first step in the construction of classical W_3 gravity was made by Hull [7], who constructed a classical theory of free scalars coupled to W_3 gravity in the chiral lightcone gauge. Subsequently, more covariant formulations were given in [8, 9]. A similar program for classical w_∞ gravity was carried out in [10].

These developments were purely classical. In W gravities, the classical theory (in the gauge sector) is in some sense trivial as there are as many gauge field components as local symmetries. This is in fact a phenomenon which occurs often in two-dimensional conformally invariant theories: a classical theory with local invariances has as many invariances as there are gauge field components. Classically, the gauge sector is then pure gauge (up to moduli). However, at the quantum level, some of the symmetries may become anomalous and thereby some gauge degrees of freedom may become propagating.

Typical examples for this 'anomalous quantization' are induced gauge theories and the theory of $d = 2$ induced gravity. Let us focuss on the latter and consider free massless scalar fields coupled to gravity. The gauge multiplet consists of four components: the zweibein. The theory has also four gauge invariances: diffeomorphisms, local Lorentz and Weyl invariance. Quantizing the theory, one finds that one of these symmetries becomes anomalous. At that point two strategies are available: either one cancels the anomaly through a suitable choice of background (which would be 26 scalar fields in this example) so that the over-all coefficient of the integrated anomaly vanishes (the so-called *critical* approach), or one learns how to solve the quantum theory in the presence of propagating gauge degrees of freedom (the *non-critical* approach). In the latter case, integrating out the matter fields will result in a quantum action for the gauge fields: the induced action. The action obtained from there by performing a path-integral over the (propagating) gauge fields

2

is called the effective action.

W_3 gravity is another example of a theory with a non-trivial quantum induced action. (The same is of course true for W gravity theories in general.) The structure of the anomalies in the chiral light-cone gauge has been studied in [11]-[15], some results departing from the covariant formulation were obtained in [16]. Contrary to what happens for induced gauge theories or for pure gravity, one finds that the induced action is not proportional to the central charge c. This phenomenon, which is due to the non-linearities in the W_3 algebra, leads to considerable complications which we will address below.

Since $d = 2$ induced gauge theories and the theories of $d = 2$ gravity and W gravity are similar, we think it worthwhile to discuss them in parallel. We first focuss on induced gauge theories in two dimensions (see for example [17]). Let us suppose that we are given a matter system which has rigid symmetries that generate an affine Kac-Moody algebra. In the language of current algebra this would mean that we can define currents $J_a(x, \bar{x})$, which are conserved on-shell ($\bar{\partial} J_a(x, \bar{x}) = 0$)[§] and which satisfy the following Operator Product Expansions (OPE's)

$$J_a(x)J_b(y) = -\frac{k}{2}g_{ab}(x-y)^{-2} + (x-y)^{-1}f_{ab}{}^c J_c(y) + \cdots .\qquad(1)$$

The c-number k is an integer and is called the level of the affine algebra. A typical example would be a theory of free fermions, taking values in the adjoint representation of a Lie algebra g, which would give rise to the current algebra (1) for the affine algebra $g^{(1)}$. In that case, k would be equal to $\tilde{h}$, which is the dual Coxeter number of the Lie algebra g.

We can now consider the coupling of this matter system to gauge fields $A^a(x, \bar{x})$. This will promote the rigid affine symmetries to local gauge symmetries. The gauge-field sector of the coupled theory is trivial in the sense described above: both in the left and the right moving sectors we would have $\dim g$ gauge-field components and $\dim g$ local symmetries. However, at the quantum level this balance of degrees of freedom can work out differently due to the occurence of anomalies. One can study this phenomenon by considering the so-called quantum induced action $\Gamma_{\text{ind}}[A]$ for the gauge fields, which is obtained by integrating out the matter degrees of freedom in a path-integral. In the chiral gauge, where we keep only one chirality of the

---

[§]We work in two-dimensional euclidean space with complex coordinates denoted by x, $\bar{x}$

3

gauge-fields (say, the left-movers), we have the following relation[¶]

$$e^{-\Gamma_{\rm ind}[A]} = \left\langle \exp -\frac{1}{\pi x} \int d^2x \, tr \left\{ J(x)A(x) \right\} \right\rangle. \qquad (2)$$

An important observation is that the leading term in the induced action $\Gamma_{\rm ind}[A]$ is a quadratic term, which can be viewed as a kinetic term for the gauge fields $A^a(x,\bar{x})$. Roughly speaking, one could say that the presence of quantized matter fields 'induces' propagation of the gauge fields in the theory. Of course, this phenomenon violates what one would expect on the basis of gauge invariance, which, as we saw, precisely balanced the gauge-field degrees of freedom at the classical level. Indeed, the kinetic term in the induced action is closely related to an anomaly in the gauge invariance, the strength of which is proportional to the level k.

The induced action for chiral gauge fields can formally be written as

$$\Gamma_{\rm ind}[A] = \frac{k}{2\pi x} \int d^2x \, tr \left\{ A \sum_{n\geq 0} \frac{1}{n+2} \left(\left[\frac{1}{\bar{\partial}}A, \cdot \right] \right)^n \frac{\partial}{\bar{\partial}} A \right\}. \qquad (3)$$

We now write

$$\Gamma_{\rm ind}[A] = -k\,\Gamma_{WZW}[A], \qquad (4)$$

which defines a k-independent reference functional $\Gamma_{WZW}[A]$. This functional is related to the well-known Wess-Zumino-Witten action as follows [18, 19]. If we parametrize A as $A \equiv \bar{\partial}gg^{-1}$ we find that $\Gamma_{WZW}[A(g)]$ is precisely equal to the WZW action with $k = 1$. Thus we see that the change of variables from A to g makes it possible to write the induced action in a form which avoids non-localities. We will later find the analogous 'local' formulations of the induced actions for gravity and W_3 gravity.

We would now like to 'quantize' the gauge-field degrees of freedom, which were up to now treated as external fields. To this end, we consider the following functional integral

$$e^{-W[t]} = \int \mathcal{D}A \, e^{-\Gamma_{\rm ind}[A] + \frac{k}{2\pi x} \int d^2x \, tr\{tA\}}. \qquad (5)$$

[¶]We normalize such that if $[T_a, T_b] = f_{ab}{}^c T_c$ then $f_{ac}{}^d f_{bd}{}^c = -\tilde{h}g_{ab}$, where $\tilde{h}$ is the dual Coxeter number. In a representation R we have $tr(T_a T_b) = -xg_{ab}$, where x is the index of the representation ($x = \tilde{h}$ for the adjoint representation).

4

In here, $W[t]$ is the generating functional of connected diagrams with propagating A fields. The Legendre transform of this functional yields the generating functional for 1PI diagrams, which is usually called the effective action. It can be argued [6] that the effective action takes the following form

$$\Gamma_{\text{eff}}[A] = -(k + 2\tilde{h}) \, \Gamma_{WZW} \left[\frac{k}{k + \tilde{h}} A \right] . \tag{6}$$

According to this, the effective action is again given in terms of the reference functional Γ_{WZW}, which is now 'dressed' through finite multiplicative renormalizations. We remark that the quantity $1/k$ plays the role of 'Planck's constant' for the quantization of the gauge-fields, which is clear from the fact that the induced action is proportional to k. As such, the loop-expansion of the path-integral (5) is equivalent to a $1/k$ expansion. [The result 6 has been checked through order $1/k$ by an explicit 1-loop calculation.] For k large, the effective action reduces to the 'classical' induced action (4).

Before turning to W_3 gravity we will first look at induced pure gravity, which shares many characteristics with W gravities but avoids the problems associated with the non-linearities inherent to W symmetries.

By $d = 2$ gravity one always means conformal gravity, and the matter systems that can be coupled are thus Conformal Field Theories (CFT's). As is well-known, these give rise to conserved currents $T(x)$ and $\overline{T}(\bar{x})$, with short distance expansion

$$T(x)T(y) = \frac{c}{2}(x - y)^{-4} + 2(x - y)^{-2}T(y) + (x - y)^{-1}\partial T(y) + \cdots , \tag{7}$$

where c denotes the central charge. (This relation is equivalent to the Virasoro algebra.) One can for example think about free scalar fields ϕ^i, $i = 1, 2, \cdots, n$, which would give a central charge $c = n$.

In the chiral gauge, the only non-trivial component of the metric field is h_{--}. Coupling this to a scalar field theory gives the classical action

$$S[\phi^i, h] = \frac{1}{2\pi} \int d^2x \left(\partial \phi^i \bar{\partial} \phi^i + 2hT \right) , \tag{8}$$

where

$$T = -\frac{1}{2}\partial \phi^i \partial \phi^i . \tag{9}$$

Classically, this action has a local gauge invariance

$$\delta\phi = \epsilon\partial\phi$$

$$\delta h = \bar{\partial}\epsilon + \epsilon\partial h - \partial\epsilon h , \tag{10}$$

which balances the degree of freedom represented by h.

The induced action $\Gamma_{\text{ind}}[h]$ for gravity coupled to a general CFT is defined by

$$e^{-\Gamma_{\text{ind}}[h]} = \langle \exp -\frac{1}{\pi}\int d^2x \, T(x)h(x) \rangle . \tag{11}$$

The induced action can be given in a closed (albeit non-local) form

$$\Gamma_{\text{ind}}[h] = \frac{c}{12}\Gamma_L[h] \tag{12}$$

with

$$\Gamma_L[h] = \frac{1}{2\pi}\int d^2x \, \partial^2 h \frac{1}{\bar{\partial}} \frac{1}{1-h\frac{\partial}{\bar{\partial}}} \frac{1}{\partial}\partial^2 h . \tag{13}$$

In the scalar field theory, one can think of this result as the sum of an infinite set of contributions, that correspond to Feynman diagrams containing a ϕ loop and $2, 3, \cdots$ external h lines. The expression (13) is easily recognized as the reduction to the chiral gauge of the covariant result $\int \sqrt{g}R \,\Box^{-1}R$. The local expression for this action was first given by Polyakov in [20]. It is found by introducing a coordinate f which is related to h through $h = \bar{\partial}f/(\partial f)$. In terms of f, the induced action reads

$$\Gamma_L[h(f)] = \frac{1}{2\pi}\int d^2x \, \frac{\bar{\partial}f}{\partial f}\partial^2 (\ln \partial f) . \tag{14}$$

It is important for us to understand the systematics of the change of variables from h to f, since we will need to make a similar but much more complicated step in the case of W_3 gravity. It has been found that the introduction of f follows naturally if one approaches the induced action from the point of view of its 'hidden' $Sl(2,\mathbf{R})$ invariance. It was shown in [21] (see also [22, 23]), that the Ward Identities and the induced action for induced gravity can be obtained from the similar quantities in the induced gauge theory for $Sl(2,\mathbf{R})$, by applying a reduction procedure. We will here not go into the details of this procedure, but we would like to stress once more that it introduces the variable f in a natural way.

In the quantization of the h field, the quantity $1/c$ plays the role of Planck's constant. We define

$$e^{-W[t]} = \int \mathcal{D}h \, e^{-\Gamma_{\mathrm{ind}}[h] + \frac{1}{\pi} \int d^2x \, ht} \tag{15}$$

and we define $\Gamma_{\mathrm{eff}}[h]$ as the Legendre transform of the generating functional $W[t]$. The saddle-point approximation to the path-integral (15) gives the leading terms in $\Gamma_{\mathrm{eff}}[h]$, which simply coincide with the induced action. The 1-loop corrections to the saddle-point result can be computed by standard determinant techniques, as was demonstrated by Polyakov (unpublished, see [24]). The relevant determinant is the determinant of the second derivative of the induced action $\Gamma_{\mathrm{ind}}[h]$ with respect to $h(x)$ and $h(y)$, evaluated at the saddle-point. The 1-loop corrections lead to order $1/c$ corrections in $\Gamma_{\mathrm{eff}}[h]$. By using the results of KPZ [25], one can then extend these perturbative results to an exact expression for the effective action, which reads

$$\Gamma_{\mathrm{eff}}[h] = \frac{k}{2} \Gamma_L \left[\frac{k+2}{k} h \right], \tag{16}$$

where k is given by

$$k = -\frac{1}{12}\left(25 - c + \sqrt{(c-1)(c-25)}\right) - 2. \tag{17}$$

Let us now turn to a discussion of the case of induced W_3 gravity. We start by choosing an appropriate matter system, which should be a CFT that has so-called W_3 symmetry. This means that it posseses conserved (chiral) currents $T(x)$ and $W(x)$ (plus their antichiral counterparts), which obey the following OPE's

$$
\begin{aligned}
T(x)T(y) &= \frac{c}{2}(x-y)^{-4} + 2(x-y)^{-2}T(y) + (x-y)^{-1}\partial T(y) + \cdots \\
T(x)W(y) &= 3(x-y)^{-2}W(y) + (x-y)^{-1}\partial W(y) + \cdots \\
W(x)W(y) &= \frac{c}{3}(x-y)^{-6} + 2(x-y)^{-4}T(y) + (x-y)^{-3}\partial T(y) \\
&\quad + (x-y)^{-2}\left[2\beta\Lambda(y) + \frac{3}{10}\partial^2 T(y)\right] \\
&\quad + (x-y)^{-1}\left[\beta\partial\Lambda(y) + \frac{1}{15}\partial^3 T(y)\right] + \cdots,
\end{aligned} \tag{18}
$$

where

$$\Lambda(x) = (TT)(x) - \frac{3}{10}\partial^2 T(x) \tag{19}$$

and

$$\beta = \frac{16}{22 + 5c}. \tag{20}$$

For the purpose of extracting a string-theory interpretation, one might wish to represent the matter system in terms of scalar fields in the presence of so-called background charges (these are needed if the number n of scalar fields is different from 2). We will later make some comments on this.

In analogy to our treatment of induced gauge theories and pure gravity, we can now define the induced action of chiral W_3 gravity by the following formula

$$e^{-\Gamma_{\text{ind}}[h, b]} = \langle \exp -\frac{1}{\pi} \int d^2 x \, [h(x)T(x) + b(x)W(x)] \rangle. \tag{21}$$

In contrast with what we found for pure gravity, we find here that the induced action is not simply proportional to the central charge c. Rather, it can be given by a $1/c$ expansion

$$\Gamma_{\text{ind}}[h, b] = \frac{1}{12} \sum_{i \geq 0} c^{1-i} \, \Gamma^{(i)}[h, b]. \tag{22}$$

This phenomenon is due to the non-linearities in the W_3 algebra. More precisely, it arises because the expectation value of the quantity $\beta\Lambda(z)$ has a non-trivial c-dependence.

Explicit expressions for all the terms $\Gamma^{(i)}[h, b]$ in the $1/c$ expansion of the induced action seem to be beyond reach. However, it is possible to explicitly evaluate the leading term $\Gamma^{(0)}[h, b]$, which dominates the induced action in the limit $c \to \pm\infty$. As in the case of pure gravity, this is done by reducing an induced gauge theory, which for the case at hand is based on $Sl(3, \mathbf{R})$.

To explain the reduction procedure, we first mention that the action $\Gamma^{(0)}[h, b]$ is completely characterized by the following Ward Identities, which take the form of local differential equations

$$\begin{aligned}
\bar{\partial}u &= D_1 h + \left[\frac{1}{10}v\,\partial + \frac{1}{15}(\partial v)\right] b, \\
\bar{\partial}v &= [3v\,\partial + (\partial v)]h + D_2 b,
\end{aligned} \tag{23}$$

8

where

$$u(x) = \pi \frac{\delta \Gamma^{(0)}[h, b]}{\delta h(x)}, \qquad v(x) = 30\pi \frac{\delta \Gamma^{(0)}[h, b]}{\delta b(x)} \qquad (24)$$

and D_1 and D_2 are the 3rd and 5th order Gelfand-Dickey operators given by (the primes denote ∂)

$$D_1 = \partial^3 + 2u\partial + u',$$

$$D_2 = \partial^5 + 10\,u\,\partial^3 + 15\,u'\,\partial^2 + 9\,u''\,\partial + 2\,u''' + 16\,u^2\,\partial + 16\,uu'. \qquad (25)$$

The important observation is now that the differential equations (23) can be extracted from the Ward Identities defining the $Sl(3, \mathbf{R})$ WZW functional, by imposing certain constraints on the currents [21, 22, 26, 27]. We refer to [3] for a detailed discussion of this procedure. As a result of this connection, it is possible to write an explicit formula for the leading term $\Gamma^{(0)}[h, b]$ in the induced action. Furthermore, the procedure automatically leads to a choice of fundamental variables f_1 and f_3 (instead of h and b), which are such that the action $\Gamma^{(0)}[f_1, f_3]$ takes a local form. The expressions for h and b in terms of f_1 and f_3 read

$$h = \frac{\bar{\partial}\left(\frac{\partial f_3}{\partial f_1}\right)}{\partial\left(\frac{\partial f_3}{\partial f_1}\right)} - \frac{\gamma}{3}\frac{\partial^3 f_3 \partial f_1 - \partial f_3 \partial^3 f_1}{\partial^2 f_3 \partial f_1 - \partial f_3 \partial^2 f_1}\,b - \frac{\gamma}{2}\,\partial b\,,$$

$$b = \gamma^{-1}\frac{(\bar{\partial}f_3\partial f_1 - \bar{\partial}f_1\partial f_3)}{(\partial^2 f_3 \partial f_1 - \partial^2 f_1 \partial f_3)}\,. \qquad (26)$$

The final result for the leading term in the induced action reads

$$\Gamma^{(0)}[h(f_1, f_3), b(f_1, f_3)] =$$

$$-\frac{1}{4\pi}\int d^2x \left\{ \frac{\bar{\partial}\left(\frac{\partial f_3}{\partial f_1}\right)}{\partial\left(\frac{\partial f_3}{\partial f_1}\right)}\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right) + \frac{\bar{\partial}f_1}{\partial f_1}\partial\left(\frac{\partial\lambda_1}{\lambda_1} + \frac{\partial\lambda_2}{\lambda_2}\right) \right.$$

$$+\frac{\bar{\partial}f_3\partial f_1 - \bar{\partial}f_1\partial f_3}{\partial^2 f_3 \partial f_1 - \partial^2 f_1 \partial f_3}\left[\frac{\partial\lambda_2}{\lambda_2}\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right) - \frac{\partial\lambda_1}{\lambda_1}\partial\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right.$$

$$\left.\left. -\left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\left(\frac{\partial\lambda_1}{\lambda_1} + \frac{\partial\lambda_2}{\lambda_2}\right)\right]\right\}, \qquad (27)$$

where

$$(\lambda_1)^3 = \left[\partial\left(\frac{\partial f_3}{\partial f_1}\right)\right]^{-2}(\partial f_1)^{-1}$$

9

$$(\lambda_2)^3 = \partial\left(\frac{\partial f_3}{\partial f_1}\right)(\partial f_1)^{-1} \tag{28}$$

and $\gamma^2 = -2/5$.

The effective currents u and v can be written as

$$u = -\frac{1}{2}\left[\partial\left(2\frac{\partial\lambda_1}{\lambda_1} + \frac{\partial\lambda_2}{\lambda_2}\right) + \left(\frac{\partial\lambda_1}{\lambda_1}\right)^2 + \left(\frac{\partial\lambda_2}{\lambda_2}\right)^2 + \left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right]$$

$$v = -15\gamma\left[\frac{1}{2}\partial^2\left(\frac{\partial\lambda_2}{\lambda_2}\right) + \left(\frac{3}{2}\frac{\partial\lambda_1}{\lambda_1} + \frac{\partial\lambda_2}{\lambda_2}\right)\partial\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{\partial\lambda_2}{\lambda_2}\right)\partial\left(\frac{\partial\lambda_1}{\lambda_1}\right) + \left(\frac{\partial\lambda_1}{\lambda_1} + \frac{\partial\lambda_2}{\lambda_2}\right)\left(\frac{\partial\lambda_1}{\lambda_1}\right)\left(\frac{\partial\lambda_2}{\lambda_2}\right)\right]. \tag{29}$$

These expressions, when written in terms of f_1 and f_3, can be viewed as 'W_3 Schwarzian derivatives'. In [3], we also discussed an alternative parametrization, which stays closer to Polyakov's f variable for gravity, and which makes clear how the truncation from W_3 gravity to pure gravity can be performed.

Let us now turn to the quantization of the W_3 gravity fields h and b. The generating functional $W[t,w]$ of connected Green's functions is defined by

$$e^{-W[t,w]} = \int \mathcal{D}h\mathcal{D}b\, e^{-\Gamma_{\text{ind}}[h,b] + \frac{1}{\pi}\int d^2x\,(ht + bw)}. \tag{30}$$

The functional $W[t,w]$ can be analyzed as follows in terms of a $1/c$ expansion for large c (which is the weak coupling regime). We first approximate the path integral (30) by the saddle-point contribution. This leads to the leading term in $W[t,w]$, which is simply the Legendre transform of the induced action, which by itself was given as a $1/c$ expansion. The saddle-point result should then be corrected by further terms coming from diagrams with h and b loops. As in the case of pure gravity, we can see that $1/c$ plays the role of Planck's constant so that the the loop-corrections to the saddle-point result are suppressed by strictly positive powers of $1/c$.

In [4], we computed the functional $W[t,w]$ through the first nontrivial order in $1/c$. In doing so, we observed remarkable cancellations of certain terms, which we view as a sign of the integrability of this quantum field theory. The final outcome of our computations through order $1/c$ can be summarized by the following formula

$$W[t,w] = \frac{c}{12}\left(1 - \frac{122}{c} + \dots\right)W_L\left[\frac{12}{c}\left(1 + \frac{50}{c} + \dots\right)t, \frac{360}{c}\left(1 + \frac{386}{5c} + \dots\right)w\right], \tag{31}$$

10

where the functional $W_L[t, w]$ is related to $\Gamma^{(0)}[h, b]$ by

$$W_L[u, v] = \Gamma^{(0)}[h(u, v), b(u, v)] - \frac{1}{\pi} \int d^2x \left(hu + \frac{1}{30} bv \right), \qquad (32)$$

where $h(u, v)$ and $b(u, v)$ are determined through the relations (24).

We now propose that the exact, all-order result for the functional $W[t, w]$ can be gotten by simply completing the $1/c$ expansions indicated by the dots in (31). This leads to the formula

$$W[t, w] = 2 k W_L \left[Z^{(t)} t, Z^{(w)} w \right], \qquad (33)$$

where k, $Z^{(t)}$ and $Z^{(w)}$ are functions of c that allow the $1/c$ expansions

$$k = \frac{c}{24} \left(1 - \frac{122}{c} + \dots \right)$$

$$Z^{(t)} = \frac{12}{c} \left(1 + \frac{50}{c} + \dots \right)$$

$$Z^{(w)} = \frac{360}{c} \left(1 + \frac{386}{5c} + \dots \right). \qquad (34)$$

The result for k is consistent (in the classical limit $c \to -\infty$) with the formula

$$k = -\frac{1}{48} \left(50 - c + \sqrt{(c-2)(c-98)} \right) - 3, \qquad (35)$$

which is the *conjectured* outcome of a KPZ type analysis of constraints in a more covariant formulation of W_3 gravity [22, 11]. Recently, the following all-order results for the Z factors have been proposed [28]:

$$Z^{(t)} = \frac{1}{2(k+3)}, \qquad Z^{(w)} = \frac{\sqrt{30}}{\sqrt{\beta}(k+3)^{3/2}}. \qquad (36)$$

They correctly reproduce the singularity structure that one expects, and are in agreement with the expansions eq. (34). The all-order result for the effective action, which is defined to be the Legendre transform of $W[t, w]$, follows from (33):

$$\Gamma_{\text{eff}}[h, b] = 2 k \Gamma^{(0)} \left[\frac{1}{2kZ^{(t)}} h, \frac{30}{2kZ^{(w)}} b \right]. \qquad (37)$$

As we mentioned before, another approach to W gravity, which avoids studying the gauge sector altogether, is to use critical theories, *i.e.*, theories which are such

11

that, by a specific choice of the background, all anomalies are made to cancel. For the purpose of applications in string theory, one would like to write critical matter systems in terms of scalar 'string-coordinate' fields. The construction of W strings along these lines has turned out to be much harder than the construction of ordinary bosonic strings. The reason for this is not hard to see. The bosonic string requires a $c = 26$ contribution to the central charge from the matter sector for cancellation of the anomaly. The basic Virasoro multiplet consists of one scalar field. Taking 26 copies of this theory and coupling them to gravity indeed yields a viable string theory. The cancellation of the W_3 anomalies requires a matter sector which provides an *exact* realization of the W_3 algebra with central charge $c = 100$ [29]. The basic W_3 multiplet consists of two scalar fields ($c = 2$). A priori one would expect that taking 50 copies of this theory would save the day. However, the resulting theory is not anymore W_3 invariant. Its symmetry algebra contains besides dimension 2 and 3 operators also higher dimensional ones. One way to obtain matter sectors with $c = 100$ is through the introduction of background charges in the scalar matter sector [7, 30]. This was further analyzed in [31, 32]. However, the presence of background charges leads to shifts in the mass formulas [33], so that the existence of massless states in the string-spectrum is in danger (indeed, see [32]). It remains an elusive problem to find a non-trivial W_3 background with $c = 100$ and which leads to massless spin-2 and and possible higher spin states in the string spectrum.

We would like to thank the organizers of the Summerschool for giving us the opportunity to present these results.

References

[1] C.M. Hull, Nucl. Phys. **B353** (1991) 707; E. Bergshoeff, A. Bilal and K. S. Stelle, preprint CERN-TH.5924/90; K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, in Proceedings of the Trieste conference on *Topological methods in field theory*, Int. Jour. Mod. Phys. **A6** (1991) 2891

[2] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B364** (1991) 584

[3] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, preprint ITP-SB-91-16

[4] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, preprint ITP-SB-91-21, Nucl. Phys. B in print

[5] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, contribution to the *Festschrift on the occasion of M. Veltmans 60th birthday*, preprint ITP-SB-91-34

[6] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, contribution to the Proceedings of *Strings & Symmetries 1991*, Stony Brook, May 1991, preprint ITP-SB-91-54

[7] C.M. Hull, Phys. Lett. **240B** (1990) 110

[8] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. **243B** (1990) 245

[9] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B349** (1991) 791

[10] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen and K.S. Stelle, Phys. Lett. **243B** (1990) 350; K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. **251B** (1990) 355

[11] Y. Matsuo, Phys. Lett. **227B** (1989) 222; in proceedings of the meeting *Geometry and Physics*, Lake Tahoe, July 1989

[12] C.M. Hull, preprint QMW/PH/91/2; Phys. Lett. **265B** (1991) 347; preprint QMW/PH/91/14

[13] K. Li and C. Pope, in Proceedings of the *Trieste Summer High-Energy Workshop*, July 1990

[14] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, in Proceedings of the Jan. 1991 Miami Workshop on *Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology* (Plenum, New York, 1991)

[15] C.M. Hull and L. Palacios, preprint QMW/PH/91/15

[16] A. Ceresole, M. Frau, J. McCarthy and A. Lerda, Phys. Lett. **265B** (1991) 72

[17] A.M. Polyakov, in Proceedings of the Les Houches 1988 meeting on *Fields, Strings and Critical Phenomena* (North-Holland, Amsterdam, 1989)

[18] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B** (1983) 121; Phys. Lett. **141B** (1984) 223

[19] O. Alvarez, Nucl. Phys. **B238** (1984) 61

[20] A. M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893

[21] A. M. Polyakov, Int. Jour. Mod. Phys. **A5** (1990) 833

[22] M. Bershadsky and H. Ooguri, Comm. Math. Phys. **126** (1989) 49

[23] A. Alekseev and S. Shatashvili, Nucl. Phys. **B323** (1989) 719

[24] Al.B. Zamolodchikov, preprint ITEP 84-89 (1989)

[25] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819

[26] A. Das, W.-J. Huang and S. Roy, preprint UR 1197, ER 13065-652

[27] A. Bilal, V.V. Fock and I.I. Kogan, Nucl. Phys. **B359** (1991) 635

[28] J. de Boer, private communication

[29] J. Thierry-Mieg, Phys. Lett. **197B** (1987) 368; K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. **124** (1989) 87

[30] L. Romans, Nucl. Phys. **B352** (1991) 829

[31] C.M. Hull, preprint QMW-PH-91-14; C.N. Pope, L.J. Romans and K.S. Stelle, preprints CTP-TAMU-52-91 and CERN-TH-6171-91

[32] C.N. Pope, L.J. Romans, E. Sezgin and K.S. Stelle, preprint CTP-TAMU-68-91

[33] S.R. Das, A. Dhar and K.S. Rama, preprints TIFR/TH/91-11, TIFR/TH/91-20

LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
INFORMATION RESOURCES DEPARTMENT
BERKELEY, CALIFORNIA 94720