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A NEW COMPOUND POISSON PROCESS AND ITS FRACTIONAL VERSIONS

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ABSTRACT. We consider a weighted sum of a series of independent Poisson random variables and show that it results in a new compound Poisson distribution which includes the Poisson distribution and Poisson distribution of order k. An explicit representation for its distribution is obtained in terms of Bell polynomials. We then extend it to a compound Poisson process and time fractional compound Poisson process ($TFCPP$). It is shown that the one-dimensional distributions of the $TFCPP$ exhibit over-dispersion property, are not infinitely divisible and possess the long-range dependence property. Also, their moments and factorial moments are derived. Finally, the fractional differential equation associated with the $TFCPP$ is also obtained.

1. INTRODUCTION

In recent decades, the classical Poisson process, negative binomial process and gamma process have been generalized to various forms of their fractional versions such as fractional Poisson process (FPP), fractional negative binomial processes ($FNBP$) and fractional gamma process (FGP); see, for example, Laskin [21], Begin and Orsingher [5], Begin [3], Mainardi *et. al* [24], Begin and Macci [4], Meerchaert *et al.* [26], Meerchaert *et al.* [25], Vellaisamy and Maheswari [40] and Kataria and Khandakar [18]. These time and space fractional versos have heavy-tailed distributions, non-exponential waiting times and long-range dependence properties; see Biard and Saussereau [9], Kataria and Vellaisamy [14] and Maheshwari and Vellaisamy [31]. These characteristics make the these processes more suitable, than the classical Levy processes (see Applebaum [1]) for modeling various phenomena that arise in many disciplines such as finance, hydrology, atmospheric science, etc. (see Laskin [22]). The time fractional Poisson process which initially derived from certain fractional differential equations (see Laskin [21], Begin and Orshinger [5]) can also be viewed as a Poisson process subordinated to inverse stable subordinator (see Meerchaert *et al.* [25]). This approach initiated the study of various subordinated processes leading to time-fractional and space fractional versions; some references, among others, are Orshinger and Polito [28], Maheswari and Vellaisamy [32], Leonenko *et. al* [23] and Begin and Vellaisamy [6].

In this paper, we look at the sum of a series of independent weighted Poisson random variables (rvs) which leads to a new compound Poisson distribution (CPD). By suitably choosing the associated sequence of parameters, we show that the CPD includes the Poisson distribution, Poisson distribution of order k and infinity and so its study leads a unified approach. Replacing Poisson rvs by Poisson processes, we extend it to a new compound Poisson process (CPP). Later, as a natural extension, we consider its time fractional version and call it $TFCPP$.

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In Section 2, we introduce the notations and required preliminary results and the CPD is introduced and studied in Section 3. In Section 4, the associated CPP is studied and its time fractional version $TFCPP$ is investigated in Section 5. In particular, the one-dimensional distributions of the $TFCPP$ have the over-dispersion property, are not infinitely divisible, and possess the long-range dependence property. The moments and the factorial moments of the $TFCPP$ are derived. Finally, its factional differential equation satisfied by the $TFCPP$ is also derived.

2. Preliminaries

In this section, we introduce the notations and the results that will be used later. We start with some special functions that will be required later.

2.1. **Some special functions.**

Definition 2.1. (i): The one parameter Mittag-Leffler function $M_\beta(z)$ is defined as (see [12])

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \ \alpha, z \in \mathbb{C} \text{ and } \mathrm{Re}(\alpha) > 0. \tag{2.1}$$

(ii): For $z \in \mathbb{C}$, the two parameter Mittag-Leffler function is defined as

$$M_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0, \beta > 0. \tag{2.2}$$

When $\beta = 1, M_{\alpha,1}$ reduces to the one parameter Mittag-Leffler function $M_\alpha(z)$.
(iii): The generalized Mittag-Leffler function (Prabhakar [36]) is defined as

$$M^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{k!\Gamma(\alpha k+\beta)} z^k, \quad \alpha, \beta, \gamma, z \in \mathbb{C} \tag{2.3}$$

where $Re(\alpha), Re(\beta), Re(\gamma) > 0$. Note that $M^{\gamma}_{\alpha,\beta}(0) = 1/\Gamma(\beta)$.

Let $M^{(n)}_{\alpha,\beta}(x)$ denote the n-th derivative of the two parameter Mittag-Leffler function. Then (see Kilbas et al. [19], Eq (1.9.5))

$$M^{(n)}_{\alpha,\beta}(x) = n! M^{n+1}_{\alpha,n\alpha+\beta}(x), \quad n \geq 0. \tag{2.4}$$

It is well known that

$$\alpha M^{(n)}_{\alpha}(-\lambda t^\alpha) = M^{(n-1)}_{\alpha,\alpha}(-\lambda t^\alpha), \quad n \geq 1. \tag{2.5}$$

The Mittag-Leffler $ML(\alpha, \lambda)$ distribution is introduced and studied by Pillai [35]. Its distribution function is given by

$$F(t \mid \alpha, \lambda) = 1 - M_\alpha(-\lambda t^\alpha) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+1)}, \ t > 0, \tag{2.6}$$

where $0 < \alpha \leq 1$ and $\lambda > 0$ are the parameters.
The density of $ML(\alpha, \lambda)$ is given by

$$\begin{aligned} f(t \mid \alpha, \lambda) &= \alpha\lambda t^{\alpha-1} M'_\alpha(-\lambda t^\alpha) \\ &= \lambda t^{\alpha-1} M_{\alpha,\alpha}(-\lambda t^\alpha), \end{aligned} \tag{2.7}$$

using (2.5).

Let $X \sim ML(\alpha, \lambda)$. The its Laplace transform is

$$E(e^{-sX}) = \frac{\lambda}{\lambda + s^{\alpha}}.$$

If $T = X_1 + \cdots + X_n$, where X_i's are IID $ML(\alpha, \lambda)$, then the density of T (see Kataria and Vellaisamy [16]) is

$$\begin{aligned} f(t \mid \alpha, \lambda) =& \frac{\lambda^n}{(n-1)!} t^{\alpha n - 1} M_{\alpha,\alpha}^{n-1}(-\lambda t^{\alpha}) \\ =& \frac{\alpha \lambda^n}{(n-1)!} t^{\alpha n - 1} M_{\alpha}^{(n)}(-\lambda t^{\alpha}). \end{aligned} \tag{2.8}$$

Definition 2.1. (i): The Wright function $W_{\alpha,\ \beta}(z)$ is defined , for $\alpha > -1$ and $\beta \in \mathbb{C}$, as

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{n!\Gamma[\alpha n + \beta]} \tag{2.9}$$

which converges in the whole complex plane.
(ii): A particular case of the Wright function, called the M-Wright function $W_{\beta}(z)$ is defined as

$$\begin{aligned} W_{\beta}(z) =& W_{-\beta, 1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\beta n + (1-\beta)]} \\ =& \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\beta n) \sin(\pi \beta n), \end{aligned} \tag{2.10}$$

which converges for $z \in \mathbb{C}$ and $0 < \beta < 1$.

The following results are well known (see Kilbas et. al [19], Kataria and Vellaismy [15]).

Definition 2.2. (i): Let $D_{\beta}(t)$ be the β-stable subordinator. Then the density of $D_{\beta}(t)$ is

$$g_{\beta}(x; t) = \beta t x^{-(\beta+1)} W_{\beta}(t x^{-\beta}), \quad x > 0. \tag{2.11}$$

(ii): Let $E_{\beta}(t)$ be the inverse β-stable subordinator. Then the density of $E_{\beta}(t)$ is

$$h_{\beta}(x; t) = t^{-\beta} W_{\beta}(t^{-\beta} x), \quad x > 0. \tag{2.12}$$

2.2. **Some fractional derivatives.** Let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$ and

$$AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{R}; f^{(n-1)}(t) \in AC[a, b] \right\},$$

where $AC^1[a, b] = AC[a, b]$.

Henceforth, $\mathbb{Z}_+ = \{0, 1, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$ the set of nonnegative integers and the set of positive integers respectively.

Definition 2.3. Let $n \in \mathbb{N}$, $\beta \geq 0$ and $f(t) \in AC^n[0, T]$. Then
(i): The (left-hand) Riemann-Liouville (R-L) fractional derivative $\mathbb{D}_t^{\beta} f$ of f (see [19, Lemma 2.2]) is

defined by (with $\mathbb{D}^0 f = f$)

$$\mathbb{D}_t^\beta f(t) := \begin{cases} \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds, & \text{if } n-1 < \beta < n, \\ \\ f^{(n)}(t), & \text{if } \beta = n. \end{cases} \tag{2.13}$$

(ii): The (left-hand) Caputo fractional derivative $\partial^\beta f$ of f (see Kilbas *et. al* [19, Theorem 2.1]) is defined by (with $\partial^0 f = f$)

$$\partial_t^\beta f(t) := \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\beta-n+1}} ds, & \text{if } n-1 < \beta < n \\ \\ f^{(n)}(t), & \text{if } \beta = n. \end{cases} \tag{2.14}$$

The relation between the R-L fractional derivative and the Caputo fractional derivative is (see Kilbas *et. al* [19, eq. (2.4.6)])

$$\mathbb{D}_t^\beta f(t) = \partial_t^\beta f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} f^{(k)}(0^+),$$

where $f^{(k)}(0^+) := \lim_{t \to 0^+} \frac{d^k}{dt^k} f(t)$.

2.3. **Poisson Distribution of Order** k**.** Let $\{N_j\}, j \geq 1$, where $N_j \sim Poi(\lambda_j)$, be a sequence of independent Poisson random variables with probability distribution

$$\mathbb{P}\{N_j = n\} = \frac{e^{-\lambda_j} \lambda_j^n}{n!},\ n \in \mathbb{Z}_+,\ j \in \mathbb{N}.$$

It is well known that the Poisson family is stable under convolution, that is,

$$S_k = \sum_{j=1}^k N_j \sim \text{Poi } (\lambda_1 + \cdots + \lambda_k), \quad k \geq 1,$$

the Poisson distribution with mean $\lambda_1 + \cdots + \lambda_k$.
For $k \geq 1$, consider now the random variable $W_k = N_1 + 2N_2 + \cdots + kN_k$, a weighted sum of Poisson rvs. Its distribution is given by, for $n \in \mathbb{Z}_+$,

$$\begin{aligned} \mathbb{P}\{W_k = n\} &= \sum_{\left\{\sum_{j=1}^k jx_j = n\right\}} \mathbb{P}\{N_1 = x_1, \cdots, N_k = x_k\} \\ &= \sum_{\left\{\sum_{j=1}^k jx_j = n\right\}} \prod_{j=1}^k \mathbb{P}\{N_j = x_j\} \\ &= e^{-(\lambda_1 + \cdots + \lambda_k)} \sum_{\left\{\sum_{j=1}^k jx_j = n\right\}} \left(\frac{\lambda_1^{x_1} \cdots \lambda_k^{x_k}}{x_1! \ldots x_k!} \right), \end{aligned} \tag{2.15}$$

where we have used the independence of the X_i's in the second line above. The probability distribution given in (2.15) is called the Poisson distribution of order k (see Philippou *et. al* [34]) and is denoted by Poi $(u_1, \ldots, u_k)$. Thus, it follows that

$$W_k = \sum_{j=1}^{k} jN_j \sim \text{Poi}\,(u_1, \ldots, u_k), \quad k \geq 1.$$

2.4. **Bell Polynomials.** The following definitions and results on Bell polynomials will be required later.

2.4.1. *Ordinary Bell polynomials.* Let c_j's denote the nonnegative integers. For $1 \leq k \leq n$, define

$$\Lambda_{k,n} = \Big\{(c_1, c_2, \ldots, c_n) : c_1 + \cdots + c_n = k;\ c_1 + 2c_2 + \cdots + nc_n = n\Big\} \tag{2.16}$$

and

$$\Lambda^*_{k,n} = \Big\{(c_1, c_2, \ldots, c_{n-k+1}) : c_1 + \cdots + c_{n-k+1} = k;\ c_1 + 2c_2 + \cdots + (n-k+1)c_{n-k+1} = n\Big\}. \tag{2.17}$$

Also, for $n \geq 1$, let

$$\Lambda_n = \Big\{(c_1, c_2, \ldots, c_n) : c_1 + 2c_2 + \cdots + nc_n = n\Big\}. \tag{2.18}$$

The ordinary partial Bell polynomials are defined by

$$\hat{B}_{n,k}(u_1, \ldots, u_{n-k+1}) = \sum_{\Lambda^*_{k,n}} \frac{k!}{c_1! \cdots c_{n-k+1}!} {u_1}^{c_1} \cdots {u_{n-k+1}}^{c_{n-k+1}} \tag{2.19}$$

or equivalently

$$\hat{B}_{n,k}(u_1, \ldots, u_n) = \sum_{\Lambda_{k,n}} \frac{k!}{c_1! \ldots c_n!} {u_1}^{c_1} \ldots {u_n}^{c_n}, \tag{2.20}$$

since, for each fixed k, there can be no nonzero c_j's, for $j \geq (n-k+1)$ and so $c_{n-k+2} = \cdots = c_n = 0$. The ordinary Bell polynomials are defined by

$$\hat{B}_n(u_1, \ldots, u_n) = \sum_{k=1}^{n} \hat{B}_{n,k}(u_1, \ldots, u_n), \quad n \geq 1. \tag{2.21}$$

The following results are well known (see Comtet [10], pp. 133-137):

$$\exp\left(x \sum_{j=1}^{\infty} \frac{u_j t^j}{j!}\right) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left\{\sum_{k=1}^{n} \hat{B}_{n,k}\left(u_1, u_2, \ldots, u_{n-k+1}\right) x^k\right\}$$

and also for the series expansion of the k-th power, for $k \geq 1$,

$$\left(\sum_{j=1}^{\infty} u_j t^j\right)^k = \sum_{n=k}^{\infty} \hat{B}_{n,k}\left(u_1, u_2, \ldots, u_{n-k+1}\right) t^n. \tag{2.22}$$

2.4.2. *Partial exponential Bell polynomials.* The partial or incomplete exponential Bell polynomials are a triangular array of polynomials given by

$$B_{n,k}\left(u_1, u_2, \ldots, u_{n-k+1}\right) = \sum_{\Lambda^*_{k,n}} \frac{n!}{c_1! c_2! \cdots c_{n-k+1}!} \left(\frac{u_1}{1!}\right)^{c_1} \left(\frac{u_2}{2!}\right)^{c_2} \cdots \left(\frac{u_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}$$

or compactly

$$B_{n,k}\left(u_1, u_2, \ldots, u_n\right) = \sum_{\Lambda_{k,n}} \binom{n}{c_1, c_2, \ldots, c_n} \left(\frac{u_1}{1!}\right)^{c_1} \left(\frac{u_2}{2!}\right)^{c_2} \cdots \left(\frac{u_n}{n!}\right)^{c_n},$$

where $B_{0,0} = 1$, $B_{0,k} = 0$, $k \geq 1$. Here $\binom{n}{c_1, c_2, \ldots, c_n}$ denotes the multinomial coefficient.
Let Λ_n be defined as in (2.18). Then the sum

$$\begin{aligned} B_n\left(u_1, \ldots, u_n\right) &= \sum_{k=1}^{n} B_{n,k}\left(u_1, u_2, \ldots, u_n\right) \\ &= \sum_{\Lambda_n} \binom{n}{c_1, c_2, \ldots, c_n} \left(\frac{u_1}{1!}\right)^{c_1} \left(\frac{u_2}{2!}\right)^{c_2} \cdots \left(\frac{u_n}{n!}\right)^{c_n}, \end{aligned} \tag{2.23}$$

with $B_0 = 1$, is called n-th complete exponential Bell polynomial. Henceforth, Bell polynomials always refer to exponential Bell polynomials unless stated otherwise.
The relation between ordinary Bell polynomials and exponential Bell polynomials is

$$\hat{B}_{n,k}\left(u_1, u_2, \ldots, u_n\right) = \frac{k!}{n!} B_{n,k}\left(1! u_1, 2! u_2, \ldots, n! u_n\right). \tag{2.24}$$

The following result from Johnson [37, p. 220] is useful. Let $|A|$ denote the determinant of the matrix A.

Lemma 2.1. If $n \geq 1$, then

$$B_n\left(u_1, u_2, \ldots, u_n\right) = \begin{vmatrix} u_1 & \binom{n-1}{1} u_2 & \binom{n-1}{2} u_3 & \cdots & \binom{n-1}{n-2} u_{n-1} & u_n \\ -1 & u_1 & \binom{n-2}{1} u_2 & \cdots & \binom{n-2}{n-3} u_{n-2} & u_{n-1} \\ 0 & -1 & u_1 & \cdots & \binom{n-3}{n-4} u_{n-3} & u_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_1 & u_2 \\ 0 & 0 & 0 & \cdots & -1 & u_1 \end{vmatrix}.$$

Note in the above matrix, all the entries on the main subdiagonal are -1, and all entries below it are 0 .

Corollary 2.1. Let $u_j = 0$ for $j \geq 2$. Then by using Lemma 2.1

$$
\begin{aligned}
B_n(u_1, 0, 0, \ldots, 0) = &\begin{vmatrix} u_1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & u_1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & u_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & u_1 \end{vmatrix} \\
=& u_1^n.
\end{aligned}
$$

2.4.3. *A Probabilistic Formula.* In this subsection, we present a probabilistic approach to compute the Bell polynomials. Let $X_j \sim Poi(\lambda_j)$ be a sequence of independent Poisson rvs and $u_j = j!\lambda_j, j \geq 1$. As before, let $S_n = X_1 + \cdots + X_n$ and $W_n = X_1 + 2X_2 + \cdots + nX_n$. Then by definition

$$
\begin{aligned}
B_{n,k}(u_1, \ldots, u_n) &= \sum_{\Lambda_{k,n}} \binom{n}{c_1, \ldots, c_n} \left(\frac{u_1}{1!}\right)^{c_1} \cdots \left(\frac{u_n}{n!}\right)^{c_n} \\
&= n! \sum_{\Lambda_{k,n}} \frac{\lambda_1^{c_1}}{c_1!} \cdots \frac{\lambda_n^{c_n}}{c_n!} \\
&= n! e^{\lambda_1 + \cdots + \lambda_n} \sum_{\Lambda_{k,n}} \frac{e^{-\lambda_1} \lambda_1^{c_1}}{c_1!} \cdots \frac{e^{-\lambda_n} \lambda_n^{c_n}}{c_n!} \\
&= \frac{n! P\{S_n = k; W_n = n\}}{P\{S_n = 0\}}.
\end{aligned} \tag{2.25}
$$

Using (2.23), the complete Bell polynomial is for $n \geq 1$,

$$
\begin{aligned}
B_n(u_1, \ldots, u_n) =& \sum_{k=1}^{n} B_{n,k}(u_1, \ldots, u_n) \\
=& n! \sum_{k=1}^{n} \frac{P\{S_n = k; W_n = n\}}{P\{S_n = 0\}} \\
=& \frac{n!}{P\{S_n = 0\}} \sum_{k=0}^{n} P\{S_n = k; W_n = n\} \\
=& \frac{n! P\{W_n = n\}}{P\{S_n = 0\}},
\end{aligned} \tag{2.26}
$$

since $P(S_n = 0; W_n = n) = 0$ for $n \geq 1$.
In other words,

$$
B_n(1!\lambda_1, 2!\lambda_2, \ldots, n!\lambda_n) = \frac{n! P\{W_n = n\}}{P\{S_n = 0\}}, \tag{2.27}
$$

where $X_j \sim Poi(\lambda_j)$.
For some additional details on the probabilistic connections to the Bell polynomials, see Kataria and Vellaisamy [39].

3. A Compound Poisson Distribution

Let $\{N_j\}$ be a sequence of independent Poisson variates, where $\{N_j\} \sim Poi(\lambda_j), j \geq 1$. Henceforth, we define $N_j \equiv 0 \; a.s$ if $\lambda_j = 0$, that is, a Poisson distribution with mean zero is defined to be the degenerate distribution at zero. Let $\boldsymbol{\lambda} = \{\lambda_j\}_{j\geq 1}$ be the sequence of associated parameters such that $\delta = \sum_{j=1}^{\infty} \lambda_j < \infty$ and $\sum_{j=1}^{\infty} t^j \lambda_j < \infty$, for $0 < |t| < M$, for some $M > 0$. We call the distribution of

$$G_P(\boldsymbol{\lambda}) = \sum_{j=1}^{\infty} jN_j, \tag{3.1}$$

a generalized Poisson distribution (GPD). Note that $G_P(\boldsymbol{\lambda}) = W_\infty$. Our generalization of the Poisson distribution is new and different from the ones available in the literature, see for instance, Consul and Jain [11] where a two parameter generalization of the Poisson distribution is obtained as a limiting form of the generalized negative binomial distribution.

Remarks 3.1. (i) Let $\lambda_j = 0$ for $j \geq 2$. Then, obviously, $G_P(\boldsymbol{\lambda}) \stackrel{\mathcal{L}}{=} N_1 \sim Poi(\lambda_1)$, the Poisson distribution with mean λ_1.
(ii) Let $\lambda_j = 0$, for $j \geq k+1$. Then, clearly, $G_P(\boldsymbol{\lambda}) \stackrel{\mathcal{L}}{=} N_1 + 2N_2 + \ldots + kN_k = W_k \sim Poi(u_1, \ldots, u_k)$, the Poisson distribution of order k.

The study of the GPD's provide a unified approach and has not been addressed in the literature. In view of the form of (3.1), the GPD may be viewed as the Poisson distribution of order infinity. First we show that the series in the right-hand side of (3.1) indeed follows a compound Poisson distribution. One could also apply Kolmogorov's three series theorem to check the convergence.

Let now Y be a positive integer-valued random variable with the distribution

$$\mathbb{P}\{Y = \ell\} = \frac{\lambda_\ell}{\delta}, \; \ell \in \mathbb{N}. \tag{3.2}$$

where $\delta = \sum_{\ell=1}^{\infty} \lambda_\ell$. Observe that given a sequence $\boldsymbol{\lambda}$, satisfying the conditions given above, the distributions of $N \sim Poi(\delta)$ and Y can be determined.

Theorem 3.1. *Let $G_P(\boldsymbol{\lambda})$ be the GPD defined in (3.1) and $\{Y_j\}_{j\geq 1}$ be a sequence of IID positive-integer valued rvs with distribution defined in (3.2). Also, let $N \sim Poi(\delta)$ be a Poisson rv with mean δ and is independent of the sequence $\{Y_j\}$. Then*

$$\sum_{j=1}^{\infty} jN_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{N} Y_j. \tag{3.3}$$

Proof. First note that the PGF of N is

$$H_N(z) = \mathbb{E}\left(z^N\right) = e^{-\delta(1-z)}.$$

Consider now the random sum $T_N = \sum_{j=1}^{N} Y_j$. Then the PGF of T_N is

$$\begin{aligned} H_{T_N}(z) &= H_N(H_{Y_1}(z)) \\ &= e^{-\delta(1-H_{Y_1}(z))} \end{aligned}$$

$$= \exp\left(\delta\left(\sum_{j=1}^{\infty} z^j \frac{\lambda_j}{\delta} - 1\right)\right)$$

$$= \exp\left(\sum_{j=1}^{\infty} z^j \lambda_j - \delta\right) \tag{3.4}$$

$$= \exp\left(\sum_{j=1}^{\infty} z^j \lambda_j - \sum_{n=1}^{\infty} \lambda_j\right)$$

$$= \exp\left(\sum_{j=1}^{\infty} \lambda_j (z^j - 1)\right). \tag{3.5}$$

Also, the PGF of $\sum_{j=1}^{\infty} jN_j$ is

$$\mathbb{E}\left(z^{N_1+2N_2+\cdots}\right) = \prod_{j=1}^{\infty} \mathbb{E}\left(z^{jN_j}\right)$$

$$= \prod_{j=1}^{\infty} \mathbb{E}\left((z^j)^{N_j}\right)$$

$$= \prod_{j=1}^{\infty} e^{-\lambda_j(1-z^j)}$$

$$= \exp\left(\sum_{j=1}^{\infty} \lambda_j (z^j - 1)\right). \tag{3.6}$$

which coincides with (3.5). This proves the result. □

The above result motivates the following alternative definition.

Definition 3.1. A Compound Poisson Distribution. Let $\{N_j\}$ be a sequence of Poisson rvs with parameter λ_j and $\delta = \sum\limits_{j=1}^{\infty} \lambda_j$. Let $\{Y_j\}$ be a sequence of iid rvs with $\mathbb{P}(Y_1 = \ell) = \lambda_\ell/\delta$ for $\ell \geq 1$. We call the distribution $\sum_{j=1}^{N} Y_j = T_N$, where $N \sim Poi(\delta)$ and is independent of the Y_j, the compound Poisson distribution (CPD) and denote it by $C_P(\boldsymbol{\lambda})$.

Remarks 3.2. (i) Let $\lambda_j = 0$ for $j \geq 2$. Then, obviously, $\delta = \lambda_1$ and $T_N \stackrel{\mathcal{L}}{=} N_1 \sim Poi(\lambda_1)$, the Poisson distribution.
(ii) Let $\lambda_j = 0$, for $j \geq k+1$. Then, $\delta = \delta_k = \sum_{j=1}^{k} \lambda_j$ and $T_N \stackrel{\mathcal{L}}{=} N_1 + 2N_2 + \ldots + kN_k \sim Poi(u_1, \ldots, u_k)$, the Poisson distribution of order k and has the PGF

$$\exp\left(\sum_{j=1}^{k} -\lambda_j (1 - z^j)\right). \tag{3.7}$$

The mean and variance of the CPD follows easily. Since $E(N) = Var(N)$, we have

$$\mathbb{E}(T_N) = E(Y_1)E(N) = \sum_{j=1}^{\infty} j\lambda_j$$

and similarly

$$\text{Var}(T_N) = E(Y_1^2)E(N) = \sum_{j=1}^{\infty} j^2\lambda_j.$$

Next, we obtain the probability distribution of the CPD. Define $T_0 \equiv 0\ a.s.$ For $1 \le m \le n$, let

$$\Delta_{m,n} =\{(y_1, y_2, \dots y_m) : y_j \ge 1; \sum_{j=1}^{m} y_j = n\}, \tag{3.8}$$

and $\sum_{y_j \in \Delta_{m,n}} \lambda_{y_1} \cdots \lambda_{y_m} = 0$, if the set $\Delta_{m,n}$ is empty.

Theorem 3.2. (i): The PMF of the $C_P(\boldsymbol{\lambda})$ defined in (3.3) is

$$\mathbb{P}\{T_N = n\} = \begin{cases} e^{\delta}, & \text{if } n = 0 \\ e^{-\delta} \sum_{m=1}^{n} \Big\{ \sum_{y_j \in \Delta_{m,n}} \lambda_{y_1} \cdots \lambda_{y_m} \Big\} \frac{1}{m!}, & \text{if } n \ge 1. \end{cases} \tag{3.9}$$

(ii): An explicit expression in terms of Bell polynomials is

$$P(T_N = n) =\frac{e^{-\delta}}{n!} B_n(u_1, u_2, \dots, u_n), \quad n \ge 0, \tag{3.10}$$

where $u_j = j!\lambda_j$, for $1 \le j \le n$.

Proof. (i): First, clearly, $\mathbb{P}\{T_N = 0\} = \mathbb{P}(N = 0) = e^{-\delta}$. For $n \ge 1$, we have

$$\begin{aligned}
\mathbb{P}\{T_N = n\} &= \sum_{m=1}^{n} \mathbb{P}\{Y_1 + Y_2 + \cdots + Y_m = n \mid N = m\} \mathbb{P}\{N = m\} \quad (\because y_j \ge 1) \\
&= \sum_{m=1}^{n} \mathbb{P}\{Y_1 + Y_2 + \cdots + Y_m = n\} \mathbb{P}\{N = m\} \\
&= \sum_{m=1}^{n} \Big\{ \sum_{y_j \in \Delta_{m,n}} \mathbb{P}\{Y_1 = y_1, Y_2 = y_2, \dots, Y_m = y_m\} \Big\} \mathbb{P}\{N = m\} \\
&= \sum_{m=1}^{n} \sum_{y_j \in \Delta_{m,n}} \prod_{i=1}^{m} \mathbb{P}\{Y_i = y_i\} \mathbb{P}\{N = m\} \\
&= \sum_{m=1}^{n} \Big(\sum_{y_j \in \Delta_{m,n}} \frac{\lambda_{y_1} \cdots \lambda_{y_m}}{\delta^m} \Big) \frac{e^{-\delta}\delta^m}{m!} \\
&= e^{-\delta} \sum_{m=1}^{n} \Big\{ \sum_{y_j \in \Delta_{m,n}} \lambda_{y_1} \cdots \lambda_{y_m} \Big\} \frac{1}{m!}, \qquad (3.11)
\end{aligned}$$

where $\delta = \sum_{j=1}^{\infty} \lambda_j < \infty$.

(ii): The probability generating function (PGF) of the $C_P(\boldsymbol{\lambda})$ (see (3.4)) is

$$\begin{aligned}
H_{T_N}(z) =& e^{-\delta(1-H_{Y_1}(z))}\\
=& e^{-\delta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{\infty}\lambda_j z^j\right)^k\\
=& e^{-\delta}\sum_{k=0}^{\infty}\frac{1}{k!}\Big\{\sum_{n=k}^{\infty}\hat{B}_{n,k}\left(\lambda_1,\lambda_2,\ldots,\lambda_{n-k+1}\right)z^n\Big\} \quad \text{(using (2.22))}\\
=& e^{-\delta}\sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\frac{1}{k!}\hat{B}_{n,k}\left(\lambda_1,\lambda_2,\ldots,\lambda_{n-k+1}\right)\Big\}z^n\\
=& e^{-\delta}\sum_{n=0}^{\infty}\Big\{\frac{1}{n!}\sum_{k=0}^{n}B_{n,k}\left(1!\lambda_1,2!\lambda_2,\ldots,(n-k+1)!\lambda_{n-k+1}\right)\Big\}z^n \quad \text{(using (2.24))}\\
=& e^{-\delta}\sum_{n=0}^{\infty}\Big\{\frac{1}{n!}B_n\left(1!\lambda_1,2!\lambda_2,\ldots,n!\lambda_n\right)\Big\}z^n
\end{aligned}$$

which shows that, for $n \geq 0$,

$$\begin{aligned}
P(T_N = n) =& \frac{e^{-\delta}}{n!}B_n\left(1!\lambda_1,2!\lambda_2,\ldots,n!\lambda_n\right)\\
=& \frac{e^{-\delta}}{n!}B_n\left(u_1,u_2,\ldots,u_n\right),
\end{aligned}$$

which proves the result.

□

Corollary 3.1. When $\lambda_j = 0$, for $j \geq 2$, we get $\delta = \lambda_1$ and Y_j's are degenerate at 1. Hence, $\Delta_{m,n}$ is empty for $1 \leq m \leq n-1$. When $m = n$, $\Delta_{n,n} = (1,\ldots,1)$ and so $\sum_{y_j\in\Delta_{n,n}}\lambda_{y_1}\cdots\lambda_{y_1} = \lambda_1\cdots\lambda_1 = \lambda_1^n$. Thus, (3.9) reduces to $Poi(\lambda_1)$, the distribution of N_1, as expected.
(ii) Take now $\lambda_j = 0$ for $j \geq k+1$ so that $\delta = \delta_k$ in (3.9), we get PMF of Poisson distribution of order k. That is,

$$P\{W_k = n\} = e^{-\delta_k}\sum_{m=1}^{n}\Big\{\sum_{y_j\in\Delta_{m,n}}\lambda_{y_1}\cdots\lambda_{y_m}\Big\}\frac{1}{m!}, \tag{3.12}$$

where $\Delta_{m,n} = \{(y_1,y_2,\ldots y_m) : y_i \geq 1; \sum_{j=1}^{m} y_j = n\}$ and $\delta_k = \sum_{j=1}^{k}\lambda_j$, for all $\lambda_j > 0$.

Remark 3.1. It is well known (see Comtet [10]) that the complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\Big(\sum_{n=1}^{\infty}\frac{a_n x^n}{n!}\Big) = \sum_{n=0}^{\infty}\frac{B_n(a_1,a_2,\ldots,a_n)}{n!}x^n$$

which when $x = 1$ leads to

$$\exp\Big(\sum_{n=1}^{\infty}\frac{a_n}{n!}\Big) = \sum_{n=0}^{\infty}\frac{B_n(a_1,a_2,\ldots,a_n)}{n!}.$$

If we take $a_n = n!\lambda_n$, we get

$$\exp\Big(\sum_{n=1}^{\infty}\lambda_n\Big) = \exp(\delta) = \sum_{n=0}^{\infty}\frac{B_n(1!\lambda_1, 2!\lambda_2, \ldots, n!\lambda_n)}{n!},$$

proving that (3.10) is indeed a valid probability distribution.

Remark 3.2. From (2.27) and (3.10),

$$\begin{aligned} P\{T_N = n\} &= \frac{e^{-\delta}}{n!}B_n\left(1!\lambda_1, 2!\lambda_2, \ldots, n!\lambda_n\right) \\ &= e^{-\delta}\frac{P\{W_n = n\}}{P(S_n = 0)} \\ &= e^{-(\delta-\delta_n)}P\{W_n = n\},\ n \in \mathbb{Z}. \end{aligned} \tag{3.13}$$

Let now $\lambda_j = 0$ for $j \geq 2$. Then $\delta = \delta_n = \lambda_1$. and $W_n = X_1$. Hence,

$$P\{T_N = n\} = P\{X_1 = n\},$$

as expected.
Suppose now $\lambda_j = 0$ for $j \geq (k+1)$. Then $\delta = \delta_k$ and $W_n = W_k$ for $n \geq (k+1)$. Hence,

$$P\{T_N = n\} = \begin{cases} e^{-(\delta_k-\delta_n)}P\{W_n = n\}, & \text{if } n \leq k \\ P\{W_k = n\}, & \text{if } n \geq k+1. \end{cases}$$

Note the distribution of W_k is given in (2.15) or in (3.12).

4. A Compound Poisson Processes.

Motivated by the properties of the GPD, we now extend it to processes versions.

Definition 4.1. (i): **A Generalized Poisson Process.** Let $\{N(t,\lambda_j)\}, j \geq 1$, be a sequence of independent Poisson processes with parameters λ_j satisfying $\delta = \sum_{j=1}^{\infty}\lambda_j < \infty$. We call the process

$$G(t,\boldsymbol{\lambda}) = \sum_{j=1}^{\infty} jN(t,\lambda_j), \tag{4.1}$$

the generalized Poisson process (GPP).
(ii): **Poisson Process of order** k**.** For $k \in \mathbb{N}$, let $\boldsymbol{\lambda}^{(k)} = (\lambda_1, \ldots, \lambda_k)$ and

$$G(t,\boldsymbol{\lambda}^{(k)}) = \sum_{j=1}^{k} jN(t,\lambda_j), \tag{4.2}$$

We call the generalized Poisson process $\{G(t,\boldsymbol{\lambda}^{(k)})\}_{t\geq 0}$ the Poisson process of order k.

Note that the process $G(t,\boldsymbol{\lambda})$ reduces to a $G(t,\boldsymbol{\lambda}^{(k)})$, when $\lambda_j = 0$ for $j \geq k+1$. We next show that the GPP is indeed a compound Poisson process (CPP).

Lemma 4.1. Let Y_j's be i.i.d with distribution given in (3.1) and $N(t,\delta)$ be a Poisson process with the parameter $\delta = \sum_{j=1}^{\infty} \lambda_j < \infty$ and is independent of the Y_j. Then the GPP

$$G_P(t, \boldsymbol{\lambda}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{N(t,\delta)} Y_j, \quad t > 0, \tag{4.3}$$

and hence is a compound Poisson process.

Proof. For fixed $t > 0$, the PGF of $G(t, \boldsymbol{\lambda})$ is

$$\begin{aligned} H_{G(t,\boldsymbol{\lambda})}(z) &= \mathbb{E}\left(z^{\sum_{j=1}^{N(t,\delta)} Y_j}\right) \\ &= \exp\Big(-t\delta(1 - F_{Y_1}(z))\Big) \\ &= \exp\Big(-t\delta\big(1 - \sum_{j=1}^{\infty} z^j \frac{\lambda_j}{\delta}\big)\Big) \\ &= \exp\Big(-t\delta + t\sum_{j=1}^{\infty} z^j \lambda_j\Big). \end{aligned} \tag{4.4}$$

Similarly, it follows from (3.6) that

$$\mathbb{E}\left(z^{\sum_{j=1}^{\infty} jN(t,\lambda_j)}\right) = \exp\Big(-t\delta + t\sum_{j=1}^{\infty} z^j \lambda_j\Big), \tag{4.5}$$

The lemma follows from (4.4) and (4.5).

□

Denote henceforth the compound Poisson process (CPP) $\sum_{j=1}^{N(t,\delta)} Y_j$ by $H(t, \boldsymbol{\lambda})$. Taking $\lambda_j = 0$ for $k \geq 1$, the CPP becomes the Poisson process of order k, defined by

$$H(t, \boldsymbol{\lambda}^{(k)}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{N(t,\delta_k)} Y_j, \tag{4.6}$$

where $\boldsymbol{\lambda}^{(k)} = (\lambda_1, \ldots, \lambda_k)$ and $\delta_k = \sum_{j=1}^{k} \lambda_j$.

The one-dimensional distributions of the process $H(t, \boldsymbol{\lambda})$ can be obtained from the CPD, by replacing λ_j by $t\lambda_j$. This leads us to the following result.

Theorem 4.1. The one-dimensional distributions of the CPP are

$$\mathbb{P}\{H(t, \boldsymbol{\lambda}) = n\} = \begin{cases} e^{-t\delta}, \text{ if } n = 0, \\ e^{-t\delta} \sum_{m=1}^{n} \sum_{y_j \in \Delta_{m,n}} \prod_{i=1}^{m} \lambda_{y_i} \frac{t^m}{m!}, \text{ if } n \geq 1, \end{cases} \tag{4.7}$$

where $\Delta_{m,n}$ is defined in (3.8).
Alternatively, in terms of Bell polynomials,

$$\mathbb{P}\{H(t, \boldsymbol{\lambda}) = n\} = \frac{e^{-t\delta}}{n!} B_n\left(1!t\lambda_1, 2!t\lambda_2, \ldots, n!t\lambda_n\right), \quad n \geq 0. \tag{4.8}$$

Corollary 4.1. Taking $\lambda_j = 0$ for $j \geq k+1$ in (4.7), we get the PMF of the Poisson process of order k as

$$\mathbb{P}\left\{H(t, \boldsymbol{\lambda}^{(k)}) = n\right\} = \begin{cases} e^{-t\delta_k}, \ \text{if } \ n = 0, \\ e^{-\delta_k} \sum_{m=1}^{n} \sum_{y_j \in \Delta_{m,n}} \prod_{i=1}^{m} \lambda_{y_i} \frac{t^m}{m!}, \ \text{if } \ n \geq 1, \end{cases} \tag{4.9}$$

4.1. **Mean, Variance and Covariance Functions of the** CPP**.** The mean, variance and covariance functions of $G(t, \boldsymbol{\lambda})$ are as follows. For positive reals s and t,

$$\mathbb{E}\left(H(t, \boldsymbol{\lambda})\right) = \mathbb{E}(Y_1)\mathbb{E}\left(N(t, \delta)\right) = t \sum_{j=1}^{\infty} j\lambda_j;$$

$$\mathrm{Var}(H(t, \boldsymbol{\lambda})) = \mathbb{E}(Y_1^2)\mathbb{E}\left(N(t, \delta)\right) = t \sum_{j=1}^{\infty} j^2\lambda_j,$$

since $\mathbb{E}\left(N(t, \delta)\right) = \mathrm{Var}(N(t, \delta))$.
Using $\mathrm{Cov}(N(s, \delta), N(t, \delta)) = \min(s, t)\delta$, we have for $0 < s \leq t$,

$$\begin{aligned} \mathrm{Cov}(H(s, \boldsymbol{\lambda}), H(t, \boldsymbol{\lambda})) &= \mathrm{Var}(Y_1)\mathbb{E}\left(N(s, \delta)\right) + \mathbb{E}^2(Y_1)\mathbb{E}\Big(\mathrm{Cov}(N(s, \delta), N(t, \delta))\Big) \\ &= \mathrm{Var}(Y_1)s\delta + \mathbb{E}^2(Y_1)s\delta = s\delta\mathbb{E}(Y_1^2) \\ &= s\sum_{j=1}^{\infty} j^2\lambda_j. \end{aligned}$$

The CPP exhibits over-dispersion property. That is, for $t > 0$,

$$\begin{aligned} \mathrm{Var}(H(t, \boldsymbol{\lambda})) - \mathbb{E}\left(H(t, \boldsymbol{\lambda})\right) =& t\sum_{j=1}^{\infty} j^2\lambda_j - t\sum_{j=1}^{\infty} j\lambda_j \\ =& t\sum_{j=1}^{\infty} j(j-1)\lambda_j > 0, \end{aligned}$$

The following definition of long-range property (LRD) and short-range property (SRD) property will be used (see, for e.g., Maheshwari and Vellaisamy [31]):

Definition 4.2. Let $s > 0$ be fixed and $\{X(t)\}_{t\geq 0}$ be a stochastic process such that

$$\mathrm{Corr}(X(s), X(t)) \sim c(s)t^{-\gamma}, \ \text{ as } t \to \infty,$$

for some $c(s) > 0$. The process $\{X(t)\}_{t\geq 0}$ is said to have the LRD property if $\gamma \in (0, 1)$ and SRD property if $\gamma \in (1, 2)$. .

Remark 4.1. For fixed s and $s < t$, the correlation function of CPP is

$$\mathrm{Corr}\left(H(s, \boldsymbol{\lambda}), H(t, \boldsymbol{\lambda})\right) = \sqrt{s/t}$$

and hence it exhibits the LRD property.

Lemma 4.2. The following asymptotic result holds for the CPP :

$$\lim_{t\to\infty} \frac{H(t, \boldsymbol{\lambda})}{t} = \delta\mathbb{E}(Y_1) \ \ a.s. \tag{4.10}$$

Proof. By the renewal theorem of the Poisson process,

$$\lim_{t\to\infty}\frac{N(t,\lambda_j)}{t}=\lambda_j,\ a.s.$$

Hence,

$$\begin{aligned}\lim_{t\to\infty}\frac{H(t,\boldsymbol{\lambda})}{t}&\stackrel{d}{=}\sum_{j=1}^{\infty}j\lim_{t\to\infty}\frac{N(t,\lambda_j)}{t}\\&=\sum_{j=1}^{\infty}j\lambda_j\ \ a.s\\&=\delta\mathbb{E}(Y_1)\ \ a.s.\end{aligned}$$

□

The above result implies also the convergence in law, that is,

$$\frac{H(t,\boldsymbol{\lambda})}{t}\stackrel{\mathcal{L}}{\to}\delta\mathbb{E}(Y_1),\ \ \text{as } t\to\infty. \tag{4.11}$$

Also, replacing δ by δ_k, we obtain the corresponding result for PP_k, the Poisson process of order k.

5. A Time Fractional Compound Poisson Process

In the past few decades, fractional Poisson processes and their extensions have received considerable attention of several researchers, see, for instance, Laskin [22], Begin and Vellaisamy [6] and Kataria and Khandakar [17], Gara *et. al* [13] and the references therein. A multivariate extension of the fractional Poisson process is discussed in Begin and Macci [7].
First, we briefly recall some properties of the time fractional Poisson process, which will be used later.

5.1. Time Fractional Poisson process.

Let $0<\beta<1$ and $\lambda\geq 0$. Unlike the Poisson process, the time fractional Poisson process ($TFPP$) has neither independent nor stationary increments. Also, it is not a Markov process.
Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of IID positive rvs, denoting the inter-arrival times of an event, with the common CDF

$$F_U(u)=\mathbb{P}\{U\leq u\}=1-M_\beta(-\lambda u^\beta), \tag{5.1}$$

where $M_\beta(-t^\beta)$ is the one-parameter Mittag-Leffler function. That is, U_i's follow the Mittag-Leffler distribution with density (see (2.7))

$$f_{U_1}(u)=\lambda u^{\beta-1}M_{\beta,\beta}(-\lambda u^\beta),\ u\geq 0.$$

The sequence of the *epochs*, denoted by $\{V_n\}_{n=1}^{\infty}$, is given by the sums of the inter-arrival times

$$V_n=\sum_{j=1}^{n}U_j,\ n\geq 1. \tag{5.2}$$

The epochs represent the times in which events arrive or occur. The PDF of the n-th epoch V_n, the n-fold convolution of U_j's, is given by (see (2.8))

$$f_{V_n}(v)=f_U^{*n}(v)=\beta\lambda^n\frac{v^{n\beta-1}}{(n-1)!}M_\beta^{(n)}(-\lambda v^\beta). \tag{5.3}$$

The counting process $N_\beta(t,\lambda)$ that counts the number of epochs (events) up to time t, assuming that $U_0=0$, is called the $TFPP$. Note $N_\beta(t,\lambda)$ is given by

$$N_\beta(t,\lambda)=\max\{n:\ V_n\le t\}. \tag{5.4}$$

The PMF $p_{_\beta}(n|t,\lambda)$ of the $TFPP$ is given by (see Laskin [21] or Meerschaert *et. al* [25])

$$\begin{aligned} p_{_\beta}(n|t,\lambda) =&(\lambda t^\beta)^n M^{n+1}_{\beta,n\beta+1}(-\lambda t^\beta)\\ =&\frac{(\lambda t^\beta)^n}{n!}M^{(n)}_\beta(-\lambda t^\beta),\ \text{(using (2.5))} \end{aligned} \tag{5.5}$$

and with $p_{_\beta}(n|0,\lambda)=1$ if $n=0$ and is zero if $n\ge 1$.
Also, the PMF $p_{_\beta}(n|t,\lambda)$ satisfies (see Begin and Orsingher [5])

$$\partial_t^\beta p_{_\beta}(n|t,\lambda)=\begin{cases}-\lambda p_{_\beta}(0|t,\lambda), & \text{if } n=0,\\ -\lambda\left[p_{_\beta}(n|t,\lambda)-p_{_\beta}(n-1|t,\lambda)\right], & \text{if } n\ge 1,\end{cases} \tag{5.6}$$

where ∂_t^β denotes the Caputo fractional derivative defined in (2.14).
The mean and the variance of the $TFPP$ are given by (see [21])

$$\mathbb{E}\left(N_\beta(t,\lambda)\right)=qt^\beta,\quad q=\lambda/\Gamma(1+\beta); \tag{5.7}$$

$$\operatorname{Var}\left(N_\beta(t,\lambda)\right)=qt^\beta\left[1+qt^\beta\left(\frac{\beta B(\beta,1/2)}{2^{2\beta-1}}-1\right)\right], \tag{5.8}$$

where $B(a,b)$ denotes the beta function. An alternative form for Var[$N_\beta(t,\lambda)$] is given in [5, eq. (2.8)] as

$$\operatorname{Var}\left(N_\beta(t,\lambda)\right)=qt^\beta+\lambda^2t^{2\beta}Q(\beta), \tag{5.9}$$

where

$$Q(\beta)=\frac{1}{\beta}\left(\frac{1}{\Gamma(2\beta)}-\frac{1}{\beta\Gamma^2(\beta)}\right). \tag{5.10}$$

Also, the covariance functions (see Leonenko *et. al* [23]) of the $TFPP$ is given by

$$\begin{aligned}\operatorname{Cov}\left(N_\beta(s,\lambda),N_\beta(t,\lambda)\right)&=qs^\beta+\lambda^2\operatorname{Cov}(E_\beta(s),E_\beta(s))\\ &=qs^\beta+ds^{2\beta}+q^2[\beta t^{2\beta}B(\beta,1+\beta;s/t)-(st)^\beta],\end{aligned} \tag{5.11}$$

$0<s\le t$, where $d=\beta q^2B(\beta,1+\beta)$, and $B(a,b;x)=\int_0^x t^{a-1}(1-t)^{b-1}dt,\ 0<x<1$, is the incomplete beta function.
It is also known that (see [25]) when $0<\beta<1$,

$$N_\beta(t,\lambda)\stackrel{\mathcal{L}}{=}N(E_\beta(t),\lambda), \tag{5.12}$$

where $\{E_\beta(t)\}_{t\ge0}$ is the inverse β-stable subordinator and is independent of $\{N(t,\lambda)\}_{t\ge0}$.
The PGF of the $TFPP$ is given by

$$H_{N_\beta(t,\lambda)}(z)=\mathbb{E}(z^{N_\beta(t,\lambda)})=M_\beta(-\lambda t^\beta(1-z)),\ |z|\le1, \tag{5.13}$$

see for example Maheshwari and Vellaisamy[32].

5.2. **Time Fractional Compound Poisson Process.** The study of compound fractional Poisson process has been of recent interests, see, for example Begin and Macci [8] and Kataria and Khandakar [18]. In this section, we introduce a new time-fractional compound Poisson process, motivated by the compound Poisson process discussed in Section 4.

Definition 5.1. (i) Let $\{N_\beta(t,\lambda_j)\}, j \geq 1,$ be a sequence of independent time fractional Poisson processes with parameters β and $\boldsymbol{\lambda} = \{\lambda_j\}_{j\geq 1}$. Let Y_j's be i.i.d with distribution given in (3.1) and is independent of the process $\{N_\beta(t,\delta)\}$, where $\delta = \sum_{j=1}^{\infty} \lambda_j$. We call the process

$$G_\beta(t,\boldsymbol{\lambda}) = \sum_{j=1}^{\infty} jN_\beta(t,\lambda_j), \tag{5.14}$$

a time fractional generalized Poisson process ($TFGPP$).

The next result shows that the $G_\beta(t,\boldsymbol{\lambda})$ is indeed a time fractional compound Poisson processes and is also a subordinated GPP.

Proposition 5.1. Let $G(t,\boldsymbol{\lambda})$ and $G_\beta(t,\boldsymbol{\lambda})$ be the GPP and $TFGPP$ respectively. For fixed $t > 0$,

$$G_\beta(t,\boldsymbol{\lambda}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{N_\beta(t,\delta)} Y_j \stackrel{\mathcal{L}}{=} G(E_\beta(t),\delta), \tag{5.15}$$

where $\{E_\beta(t)\}_{t\geq 0}$ is an idependent inverse β-stable subordinator.

Proof. Note, for fixed $t > 0$,

$$\begin{aligned} G_\beta(t,\boldsymbol{\lambda}) &= \sum_{j=1}^{\infty} jN_\beta(t,\lambda_j) \\ &\stackrel{\mathcal{L}}{=} \sum_{j=1}^{\infty} jN(E_\beta(t),\lambda_j) \ \ \text{(using (5.12))} \\ &\stackrel{\mathcal{L}}{=} \sum_{j=1}^{N(E_\beta(t),\delta)} Y_j \ \ \text{(using (4.3))} \\ &\stackrel{\mathcal{L}}{=} \sum_{j=1}^{N_\beta(t,\delta)} Y_j \ \ \text{(using (4.3))}. \end{aligned}$$

From the above results and (4.3), we have

$$G_\beta(t,\boldsymbol{\lambda}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^{N(E_\beta(t),\delta)} Y_j \stackrel{\mathcal{L}}{=} G(E_\beta(t),\delta), \tag{5.16}$$

showing that it is a subordinated GPP. □

Henceforth, we call the process $\sum_{j=1}^{N_\beta(t,\delta)} Y_j$ the time fractional compound Poisson process ($TFCPP$) and denote it by $H_\beta(t,\boldsymbol{\lambda})$. Similarly, the time fractional compound Poisson process of order k, denoted by $H_\beta(t,\boldsymbol{\lambda}^{(k)})$, can be defined by replacing δ by δ_k in the result given in (5.16).

The PGF of the $TFCPP$ is

$$\begin{aligned} K_{H_\beta(t,\boldsymbol{\lambda})}(z) &= \mathbb{E}\left(z^{H_\beta(t,\boldsymbol{\lambda})}\right) \\ &= K_{N_\beta(t,\delta)}(K_{Y_1}(z)) \\ &= M_\beta\left(-\delta t^\beta\left(1-K_{Y_1}(z)\right)\right) \quad \text{(using (5.13))} \\ &= M_\beta\left(-t^\beta \sum_{j=1}^{\infty} \lambda_j\left(1-z^j\right)\right), \end{aligned} \tag{5.17}$$

where M_β is defined in (2.1).

5.3. **Mean, variance and Covariance Functions of** $TFCPP$**.** Let $H_\beta(t,\boldsymbol{\lambda})$ be the $TFCPP$ defined in (5.15). Then mean, variance and covariance functions of of $H_\beta(t,\boldsymbol{\lambda})$ for $s \leq t$ are as follows. Let $q = \delta/\Gamma(1+\beta)$. Then

$$\mathbb{E}\left(H_\beta(t,\boldsymbol{\lambda})\right) = \mathbb{E}(Y_1)\mathbb{E}\big(N_\beta(t,\delta)\big) = qt^\beta \mathbb{E}(Y_1); \tag{5.18}$$

and

$$\begin{aligned} \operatorname{Var}(H_\beta(t,\boldsymbol{\lambda})) &= \operatorname{Var}(Y_1)\mathbb{E}\big(N_\beta(t,\delta)\big) + \mathbb{E}^2(Y_1)\operatorname{Var}(N_\beta(t,\delta)) \\ &= \operatorname{Var}(Y_1)\mathbb{E}\big(N_\beta(t,\delta)\big) + \mathbb{E}^2(Y_1)\mathbb{E}[N_\beta(t,\delta) + R_\lambda(\beta)] \\ &= \mathbb{E}(Y_1^2)\mathbb{E}\big(N_\beta(t,\delta)\big) + \mathbb{E}^2(Y_1)\lambda^2 t^{2\beta} Q(\beta) \\ &= \mathbb{E}(Y_1^2)qt^\beta + \mathbb{E}^2(Y_1)\lambda^2 t^{2\beta} Q(\beta), \end{aligned} \tag{5.19}$$

where $Q(\beta)$ is defined in (5.10).
Also, for $0 < s \leq t$,

$$\operatorname{Cov}(H_\beta(s,\boldsymbol{\lambda}), H_\beta(t,\boldsymbol{\lambda})) = \operatorname{Var}(Y_1)qs^\beta + \mathbb{E}^2(Y_1)\operatorname{Cov}(N_\beta(s,\delta), N_\beta(t,\delta)), \tag{5.20}$$

where $\delta = \sum_{j=1}^{\infty} \lambda_j$. Note $\operatorname{Var}(N_\beta(t,\delta))$ and $\operatorname{Cov}(N_\beta(s,\delta), N_\beta(t,\delta))$ are given in (5.8) and (5.11) respectively.

Remark 5.1. The $TFCPP$ exhibits over-dispersion property. For $t > 0$,

$$\begin{aligned} \operatorname{Var}(H_\beta(t,\boldsymbol{\lambda})) - \mathbb{E}\left(H_\beta(t,\boldsymbol{\lambda})\right) &= qt^\beta[\mathbb{E}(Y_1^2) - \mathbb{E}(Y_1)] + \mathbb{E}^2(Y_1)\lambda^2 t^{2\beta} Q(\beta) \\ &= \frac{qt^\beta}{\delta} \sum_{j=1}^{\infty} j(j-1)\lambda_j + \mathbb{E}^2(Y_1)\lambda^2 t^{2\beta} Q(\beta) \\ &> 0, \end{aligned}$$

since $Q(\beta) > 0$ for $0 < \beta \in (0,1)$ (see Vellaisamy and Maheshwari [40], p. 87.).

Theorem 5.1. For $1 \leq k \leq n$, let

$$\Delta_{k,n} = \{(y_1, y_2, \ldots y_k) : y_j \geq 1; \sum_{j=1}^{k} y_j = n\}.$$

(i) The PMF of the $TFCPP$ $H_\beta(t,\boldsymbol{\lambda}))$ is

$$\mathbb{P}\{H_\beta(t,\boldsymbol{\lambda})=n\}=\begin{cases}M_\beta(-\delta t^\beta), & \text{if } n=0\\ \frac{1}{n!}\sum_{k=0}^{n}\Big\{\sum_{y_j\in\Delta_{k,n}}\prod_{i=1}^{k}\lambda_{y_i}\Big\}t^{k\beta}M_\beta^{(k)}\left(-\delta t^\beta\right), & \text{if } n\geq 1\end{cases} \tag{5.21}$$

(ii) Alternatively, in terms of Bell polynomials,

$$\mathbb{P}\{H_\beta(t,\boldsymbol{\lambda})=n\}=\frac{1}{n!}\mathbb{E}\Big(e^{-\delta E_\beta(t)}B_n\big(1!\lambda_1E_\beta(t),2!\lambda_2E_\beta(t),\ldots,n!\lambda_nE_\beta(t)\big)\Big) \tag{5.22}$$

for $n\geq 0$.

Proof. (i) From (5.5), we have $\mathbb{P}\{H_\beta(t,\boldsymbol{\lambda})=0\}=\mathbb{P}\{N_\beta(t,\lambda)=0\}=M_\beta(-\delta t^\beta)$. For $n\geq 1$,

$$\begin{aligned}\mathbb{P}\{H_\beta(t,\boldsymbol{\lambda})=n\}&=\sum_{k=1}^{n}\mathbb{P}\{Y_1+Y_2+\cdots+Y_k=n\}\,\mathbb{P}\{N_\beta(t,\delta)=k\}\quad(\because y_j\geq 1)\\&=\sum_{k=1}^{n}\sum_{\Delta_{k,n}}\mathbb{P}\{Y_1=y_1,Y_2=y_2,\ldots,Y_k=y_k\}\,\mathbb{P}\{N_\beta(t,\delta)=k\}\\&=\sum_{k=1}^{n}\sum_{\Delta_{k,n}}\prod_{i=1}^{k}\mathbb{P}\{Y_i=y_i\}\,\mathbb{P}\{N_\beta(t,\delta)=k\}\\&=\frac{1}{n!}\sum_{k=1}^{n}\Big\{\sum_{y_j\in\Delta_{k,n}}\frac{\lambda_{y_1}\cdots\lambda_{y_k}}{\delta^k}\Big\}(\delta t^\beta)^kM_\beta^{(k)}\left(-\delta t^\beta\right)\quad\text{(using (5.5))}\\&=\frac{1}{n!}\sum_{k=1}^{n}\Big\{\sum_{y_j\in\Delta_{k,n}}\lambda_{y_1}\cdots\lambda_{y_k}\Big\}t^{\beta k}M_\beta^{(k)}\left(-\delta t^\beta\right).\end{aligned}$$

(ii) Note that, for fixed t, and $n\geq 0$,

$$\begin{aligned}\mathbb{P}\{H_\beta(t,\boldsymbol{\lambda})=n\}&=\mathbb{P}\left\{\sum_{j=1}^{N(E_\beta(t),\,\delta)}Y_j=n\right\}\\&=\mathbb{E}\left(\mathbb{P}\left\{\sum_{j=1}^{N(E_\beta(t)\delta)}Y_j=n\mid E_\beta(t)\right\}\right)\\&=\frac{1}{n!}\mathbb{E}\Big(e^{-\delta E_\beta(t)}B_n\big(1!\lambda_1E_\beta(t),2!\lambda_2E_\beta(t),\ldots,n!\lambda_nE_\beta(t)\big)\Big),\end{aligned}$$

using (4.8). The density of $E_\beta(t)$ is given in (2.12) □

Theorem 5.2. The one-dimensional distributions of the $H_\beta(t,\boldsymbol{\lambda})$ are not infinitely divisible.

Proof. Using the well-known result (see [27]) that $E_\beta(t)\stackrel{\mathcal{L}}{=}t^\beta E_\beta(1)$, we obtain

$$\begin{aligned}\frac{H_\beta(t,\boldsymbol{\lambda})}{t^\beta}&\stackrel{\mathcal{L}}{=}\frac{G(E_\beta(t),\boldsymbol{\lambda})}{t^\beta}\quad\text{(use (5.15))}\\&\stackrel{\mathcal{L}}{=}\frac{G(t^\beta E_\beta(1),\boldsymbol{\lambda})}{t^\beta}\end{aligned}$$

$$\stackrel{\mathcal{L}}{=} E_\beta(1)\frac{G(t^\beta E_\beta(1),\boldsymbol{\lambda})}{t^\beta E_\beta(1)}$$
$$\stackrel{\mathcal{L}}{\to} E_\beta(1)\delta\mathbb{E}(Y_1),$$

using (4.10).
Suppose $H_\beta(t,\boldsymbol{\lambda}), t>0,$ is infinitely divisible (i.d.). Then this implies $\frac{H_\beta(t,\boldsymbol{\lambda})}{t^\beta}$ is i.d. for every $t>0$ and also its limit $E_\beta(1)\delta\mathbb{E}(Y_1)$ or equivalently $E_\beta(1)$ is also i.d. (see Sato [29]). But this is a contradiction, since $E_\beta(t)$ is not i.d. for any $t>0$ (see Steutel and Van Harn [38]). □

5.4. **Moments and factorial moments of** $TFCPP$**.** For real-valued functions f and g, let $f^{(k)}$ denote its k-th derivative and $g(f)$ denote the composite function. The following two results are from Johnson [37], equation (3.3) and equation (3.6).
Hoppe's formula. If g and f are functions with a sufficient number of derivatives, then

$$(g(f))^{(m)} = \sum_{k=0}^{m}\frac{g^{(k)}(f)}{k!}A_{k,m}(f), \tag{5.23}$$

where $A_{0,0}=1$, $A_{0,m}=0$ for $m\geq 1$ and

$$A_{k,m}(f) = \sum_{j=0}^{k}\binom{k}{j}(-f)^{k-j}(f^j)^{(m)}, \quad 1\leq k\leq m. \tag{5.24}$$

The next lemma is from [37, eq. (3.6)].

Lemma 5.1. (i): If $f_1, f_2, \ldots, f_k$ are functions with a sufficient number of derivatives, then

$$(f_1 f_2\cdots f_k)^{(m)} = \sum_{j_1+\cdots+j_k=m}\binom{m}{j_1,\ldots,j_k}f_1^{(j_1)}\cdots f_k^{(j_k)}. \tag{5.25}$$

(ii): When $f_i=f, 1\leq i\leq k$,

$$(f^k)^{(m)} = \sum_{j_1+\cdots+j_k=m}\binom{m}{j_1,\ldots,j_k}f^{(j_1)}\cdots f^{(j_k)}. \tag{5.26}$$

Note that the moment generating function $K_\beta(s,t)$, $s\geq 0$, of $TFCPP$ is (see (5.17))

$$K_\beta(s,t) = \mathbb{E}\left(e^{-sH_\beta(t,\boldsymbol{\lambda})}\right) = M_\beta\left(t^\beta\sum_{j=1}^{\infty}\lambda_j\left(e^{-sj}-1\right)\right). \tag{5.27}$$

Theorem 5.3. Let Y_j's be IID with distribution given in (3.2) and $T_k=\sum_{j=1}^{k}Y_j$, for $k\geq 1$. Then the r-th raw moment of the $TFCPP$ is given by

$$\mathbb{E}\left(H_\beta^r(t,\boldsymbol{\lambda})\right) = \sum_{k=1}^{r}\frac{t^{k\beta}\delta^k}{\Gamma(k\beta+1)}\mathbb{E}(T_k^r).$$

Proof. Using Hoppe's formula in (5.23), we get

$$\mathbb{E}\left(H_\beta^r(t,\boldsymbol{\lambda})\right) = (-1)^r\left.\frac{\partial^r K_\beta(s,t)}{\partial s^r}\right|_{s=0}$$

$$= \sum_{k=0}^{r} \frac{(-1)^r}{k!} M_{\beta,1}^{(k)} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) A_{k,r} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) \Bigg|_{s=0}, \tag{5.28}$$

where $A_{k,r}$, for $0 \leq k \leq r$, is defined in (5.24).
From (2.4), we obtain

$$\begin{aligned} M_\beta^{(k)} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) \Bigg|_{s=0} &= k! M_{\beta,k\beta+1}^{k+1} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) \Bigg|_{s=0} \\ &= \frac{k!}{\Gamma(k\beta+1)}, \end{aligned} \tag{5.29}$$

using the fact that $M_{\alpha,\beta}^{\gamma}(0) = 1/\Gamma(\beta)$.
Next, for $0 \leq k \leq r$,

$$\begin{aligned} & A_{k,r} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) \Bigg|_{s=0} \\ &= \sum_{m=0}^{k} \binom{k}{m} \left(-t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right)^{k-m} \frac{d^r}{ds^r} \left(t^\beta \sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right)^m \Bigg|_{s=0} \\ &= t^{k\beta} \frac{d^r}{ds^r} \left(\sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right)^k \Bigg|_{s=0}, \end{aligned} \tag{5.30}$$

The last step follows since only the case corresponding to $m = k$ remains in its previous step and also by using $\sum_{j=1}^{\infty} \lambda_j \left(z^j - 1 \right) |_{z=1} = 0$.
Now, by Lemma 5.1, we get

$$\begin{aligned} \frac{d^r}{ds^r} \left(\sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right)^k \Bigg|_{s=0} &= (-1)^r \sum_{n_1+\ldots+n_k=r} \binom{r}{n_1, \ldots, n_k} \prod_{i=1}^{k} \frac{d^{n_i}}{ds^{n_i}} \left(\sum_{j=1}^{\infty} \lambda_j \left(e^{-sj} - 1 \right) \right) \Bigg|_{\rho=0} \\ &= (-1)^r \sum_{n_1+\ldots+n_k=r} \binom{r}{n_1, \ldots, n_k} \prod_{i=1}^{k} \sum_{j=1}^{\infty} \lambda_j j^{n_i} \\ &= (-1)^r \delta^k \sum_{n_1+\ldots+n_k=r} \binom{r}{n_1, \ldots, n_k} \prod_{i=1}^{k} \mathbb{E}(Y_i^{n_i}) \\ &= (-1)^r \delta^k \mathbb{E}(Y_1 + \ldots + Y_k)^r. \end{aligned} \tag{5.31}$$

Substituting (5.29)-(5.31) in (5.28), we get the result. □

For a random variable Z, let $Z^{(r)} = Z(Z-1)\cdots(Z_r+1)$ so that $\mathbb{E}(Z^{(r)})$ denote its r-th factorial moment.

Theorem 5.4. The r-th factorial moment $\mathcal{M}_r^{\beta}(t)$ of the $TFCPP$ is given by

$$\mathbb{E}(H_\beta^{(r)}(t,\boldsymbol{\lambda})) = \sum_{k=1}^{r} \frac{t^{k\beta}\delta^k}{\Gamma(k\beta+1)} \sum_{n_1+\ldots,n_k=r} \binom{r}{n_1,\ldots+n_k} \prod_{i=1}^{k} \mathbb{E}(Y^{(n_i)}).$$

Proof. From (5.17) and using Hope's formula, we get

$$\begin{aligned}
\mathbb{E}(H_\beta^{(r)}(t,\boldsymbol{\lambda})) &= \left.\frac{\partial^r H_{H_\beta(t,\boldsymbol{\lambda})}(z)}{\partial z^r}\right|_{z=1} \\
&= \frac{\partial^r}{\partial z^r}\left\{ M_\beta\left(t^\beta \sum_{j=1}^{\infty} \lambda_j\left(z^j-1\right)\right)\right\}\Bigg|_{z=1} \\
&= \sum_{k=0}^{r} \frac{1}{k!} M_\beta^{(k)}\left(t^\beta \sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right) A_{k,r}\left(t^\beta\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)\Bigg|_{z=1}, \qquad (5.32)
\end{aligned}$$

where, for $0 \le k \le r$,

$$\begin{aligned}
&A_{k,r}\left(t^\beta\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)\Bigg|_{z=1} \\
&\quad= \sum_{m=0}^{k}\binom{k}{m}\left(-t^\beta\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)^{k-m} \frac{d^r}{dz^r}\left(t^\beta\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)^m\Bigg|_{z=1} \\
&\quad= t^{k\beta}\frac{d^r}{dz^r}\left(\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)^k\Bigg|_{z=1} \qquad (5.33) \\
&= t^{k\beta}\sum_{n_1+\ldots+n_k=r}\binom{r}{n_1,\ldots,n_k}\prod_{i=1}^{k}\frac{d^{n_i}}{dz^{n_i}}\left(\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)\Bigg|_{z=1} \\
&= t^{k\beta}\sum_{n_1+\ldots+n_k=r}\binom{r}{n_1,\ldots,n_k}\prod_{i=1}^{k}\sum_{j=1}^{\infty}\lambda_j j(j-1)\cdots(j-n_i+1) \\
&= t^{k\beta}\delta^k\sum_{n_1+\ldots+n_k=r}\binom{r}{n_1,\ldots,n_k}\prod_{i=1}^{k}\mathbb{E}(Y^{(n_i)}). \qquad (5.34)
\end{aligned}$$

Also, as seen in (5.29),

$$M_\beta^{(k)}\left(t^\beta\sum_{j=1}^{\infty}\lambda_j\left(z^j-1\right)\right)\Bigg|_{z=1} = \frac{k!}{\Gamma(k\beta+1)}. \qquad (5.35)$$

The result now follows by using (5.33)-(5.35) in (5.32). □

5.5. **Long Range Dependence Properties.** Here, we discuss about long-range property (LRD) and short-range dependence property (SRD) of the $TFCPP$. The LRD property of the FPP is established by Biard and Saussereau [9]. We use the following definitions (see Maheshwari and Vellaisamy [31], p. 991.) for a non-stationary stochastic process $\{X(t)\}_{t\ge0}$.

Definition 5.2. Let $s>0$ be fixed and $t>s$. Let $\{Z(t)\}_{t\geq 0}$ be a stochastic process whose correlation function satisfies

$$\operatorname{Corr}(Z(s),Z(t))\sim m(s)t^{-d},\ \text{as } t\to\infty,$$

for some function $m(s)>0$. The process $\{X(t)\}_{t\geq 0}$ is said to have the LRD property if $d\in(0,1)$ and the SRD property if $d\in(1,2)$.

The mean and variance of inverse β-stable subordinator are given by (see Leonenko et al., [23], Equation (8) and Equation (11))

$$\mathbb{E}\left(E_\beta(t)\right)=\frac{t^\beta}{\Gamma(\beta+1)}$$

and

$$\operatorname{Var}\left(E_\beta(t)\right)=t^{2\beta}Q(\beta),$$

where $Q(\beta)$ is defined in (5.10).
Also, for $0<s\leq t$, we have from (5.20) and (5.11),

$$\begin{aligned}\operatorname{Cov}(H_\beta(s,\boldsymbol{\lambda}),H_\beta(t,\boldsymbol{\lambda}))&=\operatorname{Var}(Y_1)qs^\beta+\mathbb{E}^2(Y_1)\operatorname{Cov}(N_\beta(s,\delta),N_\beta(t,\delta))\\&=\operatorname{Var}(Y_1)qs^\beta+\mathbb{E}^2(Y_1)\left(qs^\beta+\delta^2\operatorname{Cov}\left(E_\beta(s),E_\beta(t)\right)\right)\\&=qs^\beta\mathbb{E}(Y_1^2)+\delta^2\mathbb{E}^2(Y_1)\operatorname{Cov}\left(E_\beta(s),E_\beta(t)\right).\end{aligned}\tag{5.36}$$

For large t, it is known that (see Leonenko et al. [23])

$$\operatorname{Cov}\left(E_\beta(s),E_\beta(t)\right)\sim\frac{s^{2\beta}}{\Gamma(2\beta+1)}.\tag{5.37}$$

Thus, form (5.37) in (5.36), we get

$$\operatorname{Cov}(H_\beta(s,\boldsymbol{\lambda}),H_\beta(t,\boldsymbol{\lambda}))\sim\mathbb{E}(Y_1^2)+\delta^2\mathbb{E}^2(Y_1)\frac{s^{2\beta}}{\Gamma(2\beta+1)},\tag{5.38}$$

as $t\to\infty$.

Theorem 5.5. The $TFCPP$ $H_\beta(t,\boldsymbol{\lambda})$ possesses the LRD property.

Proof. For fixed $s>0$ and large t, we have, from (5.38) and (5.19),

$$\begin{aligned}&\operatorname{Corr}\left(H_\beta(s,\boldsymbol{\lambda}),\,H_\beta(t,\boldsymbol{\lambda})\right)\\&\quad=\frac{\operatorname{Cov}\left(H_\beta(s,\boldsymbol{\lambda}),H_\beta(t,\boldsymbol{\lambda})\right)}{\sqrt{\operatorname{Var}\left(H_\beta(s,\boldsymbol{\lambda})\right)}\sqrt{\operatorname{Var}\left(H_\beta(t,\boldsymbol{\lambda})\right)}}\\&\quad\sim\frac{\mathbb{E}(Y_1^2)+\delta^2\mathbb{E}^2(Y_1)\frac{s^{2\beta}}{\Gamma(2\beta+1)}}{\left\{H_\beta(s,\boldsymbol{\lambda})\left[\mathbb{E}(Y_1^2)qt^\beta+\delta^2\mathbb{E}^2(Y_1)t^{2\beta}Q(\beta)\right]\right\}^{1/2}}\\&\quad\sim Z(s)t^{-\beta},\ \text{ for }\ \text{large } t,\end{aligned}$$

where

$$Z(s)=\frac{\mathbb{E}(Y_1^2)+\delta^2\mathbb{E}^2(Y_1)\frac{s^{2\beta}}{\Gamma(2\beta+1)}}{\left\{\delta^2H_\beta(s,\boldsymbol{\lambda})\mathbb{E}^2(Y_1)Q(\beta)\right\}^{1/2}}.$$

Since $0<\beta<1$, the $TFCPP$ possesses the LRD property. □

Finally, we obtain the fractional Kolmogorov forward type equations for the PMF of $H_\beta(t,\boldsymbol{\lambda})$. Let

$$\mathbb{P}\left\{H_\beta(t,\boldsymbol{\lambda})=n\right\}=q(n|t,\boldsymbol{\lambda}),\quad n\geq 0,$$
$$\mathbb{P}(S_k=n)=h_k(n),\quad n\geq 1,$$

be respectively the PMF of the $TFCPP$ and $S_k=\sum_{j=1}^{k}Y_j, k\geq 1$.

Theorem 5.6. Let $0<\beta\leq 1$ and ∂_t^β be the Caputao-fractional derivative defined in (2.14). The PMF $q(n|t,\lambda)$ of the $TFCPP$ satisfies the following fractional differential equation:

$$\partial_t^\beta q(n|t,\boldsymbol{\lambda})=\begin{cases}-\delta p_\beta(0|t,\boldsymbol{\lambda}), & \text{if } n=0,\\ -\delta q(n|t,\boldsymbol{\lambda})+\delta\sum_{k=1}^{n}h_k(n)p_\beta(k-1|t,\delta), & \text{if } n\geq 1,\end{cases}\tag{5.39}$$

where $p_\beta(n|t,\lambda)$ denotes the PMF of the fractional Poisson process.

Proof. First note that

$$q(0|t,\boldsymbol{\lambda})=\mathbb{P}\left\{H_\beta(t,\boldsymbol{\lambda})=0\right\}=p_\beta(0|t,\delta)$$

and so

$$\partial_t^\beta q(0|t,\boldsymbol{\lambda})=\partial_t^\beta p_\beta(0|t,\delta)=-\delta p_\beta(0|t,\delta),$$

which follows form (5.6).
When $n\geq 1$,

$$\begin{aligned}q(n|t,\boldsymbol{\lambda})&=\sum_{k=1}^{n}\mathbb{P}(S_k=n)\mathbb{P}(N_\beta(t,\delta)=k)\\&=\sum_{k=1}^{n}h_k(n)p_\beta(k|t,\delta).\end{aligned}$$

This implies, using (5.6),

$$\begin{aligned}\partial_t^\beta q(n|t,\boldsymbol{\lambda})&=\sum_{k=1}^{n}h_k(n)\partial_t^\beta p_\beta(k|t,\delta)\\&=-\delta\sum_{k=1}^{n}h_k(n)\big[p_\beta(k|t,\delta)-p_\beta(k-1|t,\delta)\big]\quad\text{(using (5.6))}\\&=-\delta q(n|t,\boldsymbol{\lambda})+\delta\sum_{k=1}^{n}h_k(n)p_\beta(k-1|t,\delta),\end{aligned}$$

where $p_\beta(n|0,\delta)=1$ if $n=0$ and is zero if $n\geq 1$. This proves the result. □

Similar results for $H_\beta(t,\boldsymbol{\lambda}^{(k)})$ can be obtained by replacing δ by δ_k.

Remark 5.2. Let $\tilde{N}_\beta(t,\delta)=N(D_\beta(t),\delta)$ be the space fractional Poisson process. One could study, similar to $TFCPP$, the space fractional compound Poisson process ($SFCPP$) (see, for example, Orsingher and Polito [28], Begin and Vellaisamy [6])

$$S_\beta(t,\boldsymbol{\lambda})\stackrel{\mathcal{L}}{=}\sum_{j=1}^{\tilde{N}_\beta(t,\delta)}Y_j\stackrel{\mathcal{L}}{=}G(D_\beta(t),\delta),\tag{5.40}$$

by subordinating the $G(t, \boldsymbol{\lambda})$ to stable subordinator $D_\beta(t, \boldsymbol{\lambda})$ and derive the corresponding results. Also, the study of time-changed versions of such subordinated processes also could be of interest.

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