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# HOMOMORPHIC CONDITIONAL EXPECTATIONS AS NONCOMMUTATIVE RETRACTIONS

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ABSTRACT. Let $A$ be a $C^*$-algebra and $\mathcal{E}\colon A \to A$ a conditional expectation. The Kadison-Schwarz inequality for completely positive maps,

$$\mathcal{E}(x)^*\mathcal{E}(x) \le \mathcal{E}(x^*x),$$

implies that

$$\|\mathcal{E}(x)\|^2 \le \|\mathcal{E}(x^*x)\|.$$

In this note we show that $\mathcal{E}$ is homomorphic (in the sense that $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$ for every $x, y$ in $A$) if and only if

$$\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|,$$

for every $x$ in $A$. We also prove that a homomorphic conditional expectation on a commutative $C^*$-algebra $C_0(X)$ is given by composition with a continuous retraction of $X$. One may therefore consider homomorphic conditional expectations as noncommutative retractions.

## 1. INTRODUCTION

It is easy to see that a conditional expectation $\mathcal{E}$ is homomorphic if and only if the kernel of $\mathcal{E}$ is an ideal. Thus, there are no nontrivial homomorphic conditional expectations on simple $C^*$-algebras, but it makes sense to study homomorphic conditional expectations on $C^*$-algebras with rich ideal structure. It follows from [3, Theorem 3.1] that a conditional expectation is homomorphic if and only if

equality holds in the Kadison-Schwarz inequality for every $x$. In our main result, Theorem 3.4 below, we weaken the latter condition to equality of the norms.

A central projection $p$ in a $C^*$-algebra $A$ gives rise to a homomorphic conditional expectation $\mathcal{E}_p\colon A \to A$ given by $\mathcal{E}_p(x) = px$ for all $x$ in $A$. As a bi-product of our main result, we prove a converse in Corollary 3.9.

A retraction of a locally compact Hausdorff space $X$, that is, a continuous map $\tau\colon X \to X$ such that $\tau \circ \tau = \tau$, gives rise to a homomorphic conditional expectation $\mathcal{E}_\tau\colon C_0(X) \to C_0(X)$ given by $\mathcal{E}_\tau(f) = f \circ \tau$ for all functions $f$ in $C_0(X)$. There are expectations on $C(K)$, $K$ compact, which do not come from retractions of $K$, but those expectations are not homomorphic. A unital conditional expectation $\mathcal{E}\colon C(K) \to C(K)$ is homomorphic if and only if it comes from some retraction of $K$ (Theorem 4.2 below), and, in accordance with Theorem 3.4 below, this in turn is equivalent to the requirement that the conditional expectation satisfies $\|\mathcal{E}(f)\|^2 = \|\mathcal{E}(|f|^2)\|$ for every $f$ in $C(K)$. A similar result holds for (not necessarily unital) commutative $C^*$-algebras $C_0(X)$ for a locally compact Hausdorff space $X$. Thus, in the framework of Gelfand duality, we have the equivalence:

$$\left( \begin{array}{c} \text{Retractions } \tau\colon X \to X \text{ of} \\ \text{locally compact spaces } X \end{array} \right) \Leftrightarrow \left( \begin{array}{c} \text{Homomorphic} \\ \text{conditional expectations} \\ \mathcal{E}\colon C_0(X) \to C_0(X) \end{array} \right)$$

$$\Leftrightarrow \left( \begin{array}{c} \text{Conditional expectations} \\ \mathcal{E}\colon C_0(X) \to C_0(X) \\ \text{with } \|\mathcal{E}(f)\|^2 = \|\mathcal{E}(|f|^2)\| \end{array} \right)$$

We believe that this justifies the following definition: A *noncommutative retraction* on a $C^*$-algebra $A$ is a conditional expectation $\mathcal{E}\colon A \to A$ with $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$ for all $x, y \in A$. (By Theorem 3.4 below, this is equivalent to the requirement that the conditional expectation satisfies $\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|$ for $x \in A$.)

## 2. BASIC PROPERTIES OF CONDITIONAL EXPECTATIONS

In this section we review some basic facts and terminology that relate to conditional expectations in a general noncommutative setting of $C^*$-algebras.

**Definition 2.1.** A *conditional expectation* defined on a $C^*$-algebra $A$ is a positive linear map $\mathcal{E}\colon A \to A$ satisfying $\mathcal{E}^2 = \mathcal{E}$ (where $\mathcal{E}^2 = \mathcal{E} \circ \mathcal{E}$) and

$$\mathcal{E}(\mathcal{E}(x)y) = \mathcal{E}(x)\mathcal{E}(y) \quad \text{for every } x, y \text{ in } A.$$

It follows that the range of $\mathcal{E}$ is a $C^*$-subalgebra of $A$. A conditional expectation $\mathcal{E}\colon A \to A$ also satisfies

$$\mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y) \quad \text{for every } x, y \text{ in } A.$$

Thus $\mathcal{E}$ is a bimodule map over its range. Moreover, $\mathcal{E}$ is completely positive and has norm 1 ([2, Corollary II.6.10.3]). The Kadison-Choi-Schwarz inequality is proved in [3, Corollary 2.8].

If $A$ is unital with the identity element 1, then the projection $e = \mathcal{E}(1)$ is an identity element of the range, which is contained in the corner $eAe$, i.e., the largest $C^*$-subalgebra of $A$ containing $e$ as the identity element.

*Remark* 2.2. By a corner of a $C^*$-algebra $A$ we mean a $C^*$-subalgebra $S$ of $A$ with the additional property that there is a norm closed linear subspace $M$ of $A$ such that $A = S \oplus M$ and $M$ is invariant under both left and right multiplication by elements of $S$, i.e. $SM \subseteq M$, $MS \subseteq M$. It follows automatically that $M$ is also invariant under the $*$-operation, i.e. $M^* = M$, so it can be regarded as a (not necesarily unital) involutive Banach $S$-bimodule. If $\mathcal{E} \colon A \to A$ is a conditional expectation, then the range of $\mathcal{E}$ is a corner of $A$. On the other hand, if a $C^*$-subalgebra $S$ is not a corner of $A$, then there is no conditional expectation from $A$ onto $S$.

It is clear that a corner of a unital $C^*$-algebra must be unital, however the identity element of the corner need not be the same as the identity element of the ambient $C^*$-algebra. This observation is useful. It shows, for example, that if $H$ is an infinite-dimensional Hilbert space, then the algebra of compact operators $\mathcal{K}(H)$ is not a corner of $\mathcal{B}(H)$. Consequently, there is no conditional expectation from $\mathcal{B}(H)$ onto $\mathcal{K}(H)$. Exactly the same argument shows that there is no conditional expectation from $\ell^\infty$ onto $c_0$. Of course, these two observations can be strengthened to the assertion that there is no conditional expectation from a unital $C^*$-algebra $A$ onto a non-unital $C^*$-subalgebra of $A$.

Regarding terminology, we will occasionally refer to a conditional expectation simply as an expectation leaving the word "conditional" implicit. The following remark provides us with some basic properties of expectations.

*Remark* 2.3. Let $A$ be a $C^*$-algebra and let $\mathcal{E} \colon A \to A$ be a conditional expectation. The range of $\mathcal{E}$, which we denote by $S$, is the $C^*$-subalgebra of $A$ consisting of all fixed points of $\mathcal{E}$. The kernel of $\mathcal{E}$ is a norm closed linear subspace of $A$ that is closed under the $*$-operation and invariant under left and right multiplication by elements of $S$. In particular, letting $M$ be the kernel of $\mathcal{E}$, one has $M^* = M$, $SM \subseteq M$, $MS \subseteq M$. The space $M$ can be regarded as an involutive Banach $S$-bimodule if one does not require that $1m = m1 = m$, for all $m \in M$, even if $S$ has an identity $1 = 1_S$. With this convention, $A = S \oplus M$ is a direct sum in the category of involutive Banach $S$-bimodules and the following sequence

$$0 \longrightarrow S \xrightarrow{\ 1\ } A \xrightarrow{\ 1-\mathcal{E}\ } M \longrightarrow 0$$

of $*$-preserving $S$-bimodule maps is exact.

There is a link between certain projections and expectations. It has already been observed that every nonzero conditional expectation $\mathcal{E} \colon A \to A$ defined on a $C^*$-algebra $A$ is a projection of norm one. The converse of this observation does not hold in general. For example, the mapping from the matrix algebra $M_2(\mathbb{C})$ into itself that replaces each main diagonal entry of every 2-by-2 matrix with zero is a projection of norm one, yet it is not a conditional expectation because its range is not a subalgebra of $M_2(\mathbb{C})$. However, every projection of norm one whose range is a subalgebra must be a conditional expectation; this is a general

version of the well known theorem of Tomiyama [17], which we mention for the sake of completeness.

## 3. Homomorphic conditional expectations

The main result of this section is Theorem 3.4.

**Definition 3.1.** Let $A$ be a $C^*$-algebra. A conditional expectation $\mathcal{E}\colon A \to A$ is *homomorphic* (or *multiplicative*) if $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$ for every $x, y$ in $A$.

We now give some examples of homomorphic conditional expectations. The first one describes a connection between homomorphic conditional expectations and $C^*$-algebra homomorphisms.

**Example 3.2** (Expectations onto graphs of $C^*$-algebra homomorphisms)**.** Let $A, B$ be $C^*$-algebras. Let $A \oplus B$ be the $C^*$-algebra endowed with the maximum norm, with the summands as ideals, and the algebraic operations performed pointwise. If $\phi\colon A \to B$ is a $*$-homomorphism, then the map

$$\mathcal{E}\colon A \oplus B \to A \oplus B, \qquad \mathcal{E}(x, y) = (x, \phi x), \qquad x \in A, y \in B \qquad (3.1)$$

is a homomorphic conditional expectation of $A \oplus B$ onto the graph of $\phi$. Conversely, if $\phi\colon A \to B$ is a function and $\mathcal{E}$ given by (3.1) is a homomorphic conditional expectation, then $\phi$ is a $*$-homomorphism.

The projection on a direct sum of two $C^*$-algebras onto one of the summands is an example of a homomorphic conditional expectation. In particular, a split extension $E$ of a $C^*$-algebra $A$ by a $C^*$-algebra $B$ gives rise to homomorphic conditional expectations.

**Example 3.3.** In the theory of generalized inductive limits, due to Blackadar and Kirchberg ([2, V.4.3]), $NF$ algebras are not the same as strong $NF$ algebras ([2, V.4.3.24.]). Nevertheless, by [2, Corollary V.4.3.27], any $NF$ algebra $A$ is the range of a homomorphic conditional expectation defined on any split essential extension $B$ of $A$, which is in fact a strong $NF$ algebra. In this corollary, $A$ is called a retraction of $B$, which partially motivated our use of the term *retraction*.

We will establish the following characterization of homomorphic conditional expectations in terms of operator norm and the Kadison–Schwarz inequality. Recall that the Kadison–Schwarz inequality shows that any conditional expectation $\mathcal{E}\colon A \to A$ defined on a $C^*$-algebra $A$ satisfies $\mathcal{E}(x)^*\mathcal{E}(x) \leq \mathcal{E}(x^*x)$ and consequently $\|\mathcal{E}(x)\|^2 \leq \|\mathcal{E}(x^*x)\|$ for every $x$ in $A$.

**Theorem 3.4.** *Let $A$ be a $C^*$-algebra and let $\mathcal{E}\colon A \to A$ be a conditional expectation. Then $\mathcal{E}$ is homomorphic if and only if*

$$\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\| \quad \text{for every } x \text{ in } A. \qquad (3.2)$$

In the proof we will make use of the fact that a closed Jordan ideal (defined in the proof of Lemma 3.6) in a $C^*$-algebras $A$ is a two-sided ideal of $A$. ([5, Theorem 5.3], also see [1, Remark p.188]) and the observations made in Lemmas 3.5 and 3.6.

**Lemma 3.5.** *A conditional expectation* $\mathcal{E}\colon A \to A$ *defined on a* $C^*$-*algebra* $A$ *is homomorphic if and only if the kernel of* $\mathcal{E}$ *is an ideal in* $A$.

*Proof.* This is a straightforward consequence of conditional expectation properties. $\qquad\square$

**Lemma 3.6.** *Let* $A$ *be a* $C^*$-*algebra. If* $\mathcal{E}\colon A \to A$ *is a conditional expectation satisfying* $\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|$ *for all* $x \in A$, *then the kernel of* $\mathcal{E}$ *is a closed Jordan* $*$-*ideal in* $A$.

*Proof.* We use $M$ to denote the kernel of $\mathcal{E}$. It is clear that $M$ is a closed linear subspace of $A$ which is also closed under the $*$-operation. We need only to prove that $M$ is a Jordan ideal in the sense that if $x \in A$ and $y \in M$, then the Jordan product $x \cdot y = \frac{1}{2}(xy + yx)$ is in $M$. The proof of this fact will proceed through several steps.

First, if $y \in M$, then $y^*y \in M$ by the assumption $\|\mathcal{E}(y)\|^2 = \|\mathcal{E}(y^*y)\|$. In particular, $y^2 \in M$ for all self-adjoint elements $y \in M$.

Second, if $y, z$ are self-adjoint elements of $M$, then by the preceding paragraph, both $(y + z)^2$ and $(y - z)^2$ are in $M$, and one has

$$y \cdot z = [(y + z)^2 - (y - z)^2]/4.$$

It follows that $y \cdot z \in M$, whenever $y, z$ are self-adjoint elements of $M$.

Third, if $y, z$ are arbitrary elements of $M$, write $y = y_1 + iy_2$ and $z = z_1 + iz_2$ with $y_i = y_i^*$ and $z_i = z_i^*$ in $M$, and split the Jordan product $y \cdot z$ into real and imaginary parts as

$$y \cdot z = y_1 \cdot z_1 - y_2 \cdot z_2 + i(y_1 \cdot z_2 + y_2 \cdot z_1).$$

By the preceding paragraph, each of the four terms $y_i \cdot z_j$ appearing above is in $M$, thus $y \cdot z \in M$. At this stage, we may conclude that $M$ is closed under the Jordan product and we may indicate this by writing $M \cdot M \subseteq M$.

Fourth, since $M$ is invariant under both left and right multiplication by elements of the range of $\mathcal{E}$, which we denote by $\mathcal{E}(A)$, it follows that $\mathcal{E}(A) \cdot M \subseteq M$. That is, the Jordan product $\mathcal{E}(x) \cdot y$ is in $M$ for all $x \in A$ and all $y \in M$.

Finally, if $x \in A$ and $y \in M$, then the Jordan product

$$x \cdot y = \mathcal{E}(x) \cdot y + (x - \mathcal{E}(x)) \cdot y$$

is in $M$ because, by what we have proved, $\mathcal{E}(x) \cdot y \in \mathcal{E}(A) \cdot M \subseteq M$ and $(x - \mathcal{E}(x)) \cdot y \in M \cdot M \subseteq M$. Thus $M$ is a Jordan ideal (and a Banach $*$-subspace of $A$). $\qquad\square$

We now turn to the proof of Theorem 3.4.

*Proof of Theorem 3.4.* Let $\mathcal{E}\colon A \to A$ be a conditional expectation satisfying $\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|$ for every $x$ in $A$. Then by Lemma 3.6 the kernel of $\mathcal{E}$ is a closed Jordan $*$-ideal in $A$, hence a two-sided ideal. It follows that $\mathcal{E}$ is homomorphic by the observation made in Lemma 3.5. $\qquad\square$

We have already mentioned that if $p$ is a central projection in a $C^*$-algebra $A$, then the map $\mathcal{E}_p\colon A \to A$ defined by $\mathcal{E}_p(x) = px$, for all $x \in A$, is a homomorphic conditional expectation. We prove the converse in Proposition 3.8.

**Lemma 3.7.** *Let $e$ be a projection in a von Neumann algebra $A$, and suppose $\mathcal{E}_e(x) = exe$ is a homomorphism.*
  *(i) If $1 - e$ is subequivalent to $e$, then $e = 1$.*
  *(ii) If $e$ is subequivalent to $1 - e$, then $e = 0$.*

*Proof.* Since $\mathcal{E}_e$ is a homomorphism, we have $exeye = exye$ for every $x, y \in A$. If $1 - e$ is subequivalent to $e$, then by definition, there exists $u \in A$ satisfying $uu^* = 1 - e$ and $u^*u = h \le e$. Then

$$h = ehe = eu^*ue = eu^*eue = eu^*uu^*eue = eu^*(1 - e)eue = 0,$$

which proves (i).

If $e$ is subequivalent to $1 - e$, there exists $u \in A$ satisfy $uu^* = e$ and $u^*u = h \le 1 - e$. Then

$$e = euu^*e = eueu^*e = euu^*ueu^*e = euheu^*e = euh(1 - e)eu^*e = 0,$$

which proves (ii). $\square$

**Proposition 3.8.** *If $e$ is a projection in a $C^*$-algebra $A$, and $\mathcal{E}_e(x) = exe$ is a homomorphism, then $e$ belongs to the center of $A$.*

*Proof.* By passing to the second dual, it suffices to assume that $A$ is a von Neumann algebra. Apply the comparability theorem ([2, III.1.1.10]) to the projections $e$ and $1 - e$ to obtain a central projection $z$ such that $ze$ is subequivalent to $z(1 - e)$ and $(1 - z)(1 - e)$ is subequivalent to $(1 - z)e$. With $A = Az \oplus A(1 - z)$ we have $\mathcal{E}_e = \mathcal{E}_{ez} \oplus \mathcal{E}_{e(1-z)}$. Then by Lemma 3.7, $ez = 0$ and $e(1 - z) = 1 - z$, so that $e = ez + e(1 - z) = 1 - z$ is in the center of $A$. $\square$

**Corollary 3.9.** *If $e$ is a projection in a $C^*$-algebra $A$, and $\|exe\| = \|ex\|$ for every $x \in A$, then $e$ belongs to the center of $A$.*

*Proof.* By Theorem 3.4, the assumption $\|exe\| = \|ex\|$ for every $x \in A$ implies that $\mathcal{E}_e$ is a homomorphism. $\square$

As pointed out to us by Matt Neal, Corollary 3.9 also follows from [10, Lemmas 1.5 and 1.6]. An elegant elementary proof of [10, Lemma 1.5] appears in [12]. Another topological characterization of central projections is given in [11], namely a projection in a von Neumann algebra is central if and only if it is an isolated point in the set of projections with the norm topology.

*Remark* 3.10. After submitting this paper, the authors learned of two alternative arguments that can be used to prove Theorem 3.4, without passing through Jordan theory. However, the method presented in our proof of Theorem 3.4 can be used to deduce a similar operator norm characterization of multiplicative conditional expectations in the context of ternary rings of operators and Jordan triple systems (where the concept of multiplicative domain is not applicable). For example, see Proposition 3.12.

The first alternative argument, due to E. Størmer, shows directly that $\ker \mathcal{E}$ is a two sided ideal. If $x \in \ker \mathcal{E}$ and $a \in A$, then

$$
\begin{aligned}
\|\mathcal{E}(ax)\|^2 &= \|\mathcal{E}((ax)^*ax)\| = \|\mathcal{E}(x^*a^*ax)\| \\
&\leq \|a^*a\|\|\mathcal{E}(x^*x)\| = \|a^*a\|\|\mathcal{E}(x)\|^2 \\
&= 0.
\end{aligned}
$$

so that $\ker \mathcal{E}$ is a left ideal. Since $\ker \mathcal{E}$ is self-adjoint, it is a two sided ideal.

The second alternative argument is due to a referee. Since $x := a - \mathcal{E}(a)$ belongs to $\ker \mathcal{E}$, $\|\mathcal{E}(x^*x)\| = \|\mathcal{E}(x)\|^2 = 0$ immediately implies that $a$ belongs to the multiplicative domain of $\mathcal{E}$ ([3, Theorem 3.1]). This latter argument can be applied to prove two other results (see Propositions 3.13 and 3.14).

The two results which follow, and the tools used in their proofs, are valid for abstract JB*-triples, for which a reference is the monograph [4, Definition 2.5.25]. The principal example of a JB*-triple is a JC*-triple, that is, a norm closed subspace $A$ of a C*-algebra which is closed under the symmetrized triple product $\{xyz\}_A := (xy^*z + zy^*x)/2$. We therefore phrase these two results in this context.

A *triple homomorphism* is a linear mapping $T : A \to B$ between two JC*-triples which preserves the triple product: $T\{xyz\}_A = \{Tx, Ty, Tz\}_B$. A *triple ideal* is a subspace $I$ of a JC*-triple $A$ satisfying $\{IAA\}_A + \{AIA\}_A \subset I$.

Let $A$ be a JC*-triple, with triple product denoted $\{abc\}_A$ (or just $\{abc\}$) and let $P : A \to A$ be a nonzero contractive projection: $P^2 = P$, $\|P\| = 1$. We have the "conditional expectation" formulas ([8, Corollary 1])

$$P\{x, Py, Pz\} = P\{Px, Py, Pz\} = P\{Px, y, Pz\} \text{ for all } x, y, z \in A. \qquad (3.3)$$

We recall ([9, Theorem 2], [4, Theorem 3.3.1]) that $P(A)$ is isometric to a JC*-triple under the norm of $A$ and the triple product

$$\{Px, Py, Pz\}_{P(A)} := P(\{Px, Py, Pz\}_A). \qquad (3.4)$$

**Lemma 3.11.** *A contractive projection $P\colon A \to A$ defined on a JC*-triple $A$ is a triple homomorphism of $A$ into $P(A)$, that is, for all $a, b, c \in A$,*

$$P\{abc\}_A = \{Pa, Pb, Pc\}_{P(A)}, \qquad (3.5)$$

*if and only if the kernel of $P$ is a triple ideal in $A$.*

*Proof.* Assume (3.5), let $a \in \ker P$, and let $b, c \in A$. Then

$$P\{abc\}_A = \{Pa, Pb, Pc\}_{P(A)} = P\{Pa, Pb, Pc\}_A = 0$$

and similarly, $P\{bac\}_A = 0$.

Conversely, suppose $\ker P$ is an ideal. For $x \in A$, with $x = Px + P'x$, where $P' = I - P$, we have (noting that $\{Px, Px, P'x\} = \{P'x, Px, Px\}$)

$$\{xxx\}_A = \{Px + P'x, Px + P'x, Px + P'x\} = \{Px, Px, Px\} + y,$$

where $y \in \ker P$. Thus $P\{xxx\}_A = P\{Px, Px, Px\}_A = \{Px, Px, Px\}_{P(A)}$ and by the polarization identity,

$$\{xyz\} = \frac{1}{8} \sum_{\alpha^4=1, \beta^2=1} \alpha\beta\{x + \alpha y + \beta z, x + \alpha y + \beta z, x + \alpha y + \beta z\},$$

$P$ is a triple homomorphism. $\qquad\square$

**Proposition 3.12.** *Let $A$ be a $JC^*$-triple and let $P\colon A \to A$ be a contractive projection. Then $P$ is a triple homomorphism of $A$ onto $P(A)$ if and only if $P$ satisfies*

$$\{\ker P, \ker P, ran P\}_A \subset \ker P, \tag{3.6}$$

$$\{\ker P, ran P, \ker P\}_A \subset \ker P, \tag{3.7}$$

*and*

$$\|P(x)\|^3 = \|P\{xxx\}_A\| \quad \text{for every } x \in A. \tag{3.8}$$

*Proof.* If $P$ is a triple homomorphism, it is obvious that (3.6) and (3.7) hold, and if $x \in A$, then

$$P\{xxx\}_A = \{Px, Px, Px\}_{P(A)},$$

so that

$$\|P\{xxx\}_A\| = \|\{Px, Px, Px\}_{P(A)}\| = \|Px\|^3_{P(A)} = \|Px\|^3_A.$$

Conversely, assume (3.6)-(3.8) hold. We shall show that $\ker P$ is an ideal, so that Lemma 3.11 is applicable.

For $x \in \ker P$ and $y, z \in A$, it is required to show that $P\{xyz\}_A = 0$ and $P\{yxz\}_A = 0$. Write $y = Py + P'y$, and $z = Pz + P'z$. Then

$$\begin{aligned}
\{xyz\}_A &= \{P'x, Py + P'y, Pz + P'z\} \\
&= \{P'x, Py, Pz\} + \{P'x, P'y, Pz\} \\
&\quad + \{P'x, Py, P'z\} + \{P'x, P'y, P'z\}.
\end{aligned}$$

By (3.8), $\ker P$ is closed under $x \mapsto \{xxx\}_A$, so by the polarization identity, it is a subtriple of $A$, and therefore $P\{P'x, P'y, P'z\} = 0$. By (3.6) and (3.7), $P(\{P'x, P'y, Pz\} + \{P'x, Py, P'z\}) = 0$. By (3.3), $P\{P'x, Py, Pz\} = 0$. Thus $P\{xyz\}_A = 0$ and a similar proof shows $P\{yxz\}_A = 0$. $\qquad\square$

As noted in Remark 3.10, the technique mentioned there can be used to show the following two results, which are responses to a question posed to the authors independently by C. Akemann and by the referee.

A *$JC^*$-algebra* is a norm closed subspace $A$ of a C*-algebra which is closed under the Jordan product $x \circ y := (xy + yx)/2$ and the involution. A *Jordan homomorphism* is a linear mapping $T : A \to B$ between two JC*-algebras which preserves the Jordan product: $T(x \circ y) = Tx \circ Ty$, equivalently, $T(a^2) = T(a)^2$ for all $a = a^*$.

**Proposition 3.13.** *Let $A$ be a C*-algebra and let $\mathcal{E}\colon A \to A$ be a conditional expectation. Then $\mathcal{E}$ is a Jordan homomorphism if and only if*

$$\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^2)\| \quad \text{for every } x = x^* \text{ in } A. \tag{3.9}$$

*Proof.* If $\mathcal{E}$ is a Jordan homomorphism, then $\mathcal{E}(a^2) = \mathcal{E}(a)^2$ so (3.9) holds. Conversely, if $a = a^* \in A$, then $x = x^* = a - \mathcal{E}(a) \in \ker \mathcal{E}$, and

$$0 = \mathcal{E}(x^2) = \mathcal{E}(a^2 - a\mathcal{E}(a) - \mathcal{E}(a)a + \mathcal{E}(a)^2) = \mathcal{E}(a^2) - \mathcal{E}(a)^2,$$

so $\mathcal{E}$ is a Jordan homomorphism. $\qquad\square$

Let $A$ be a unital JC*-algebra, with Jordan product denoted $a \circ b$, and let $P : A \to A$ be a nonzero positive unital projection. The conditional expectation formulas (3.3) reduce to

$$P(x \circ Py) = P(Px \circ Py), \qquad (3.10)$$

and by (3.4), $P(A)$ is isometric to a JC*-algebra under the norm of $A$ and the Jordan product $(a, b) \mapsto a * b := P(a \circ b)$, for $a, b \in P(A)$ (see [7, Theorem 1.4] for the original proof of the latter statement and [7, Lemma 1.1] for the original proof of (3.10)). Note that $P(a) * P(a) = P(P(a)^2)$.

**Proposition 3.14.** *Let $A$ be a unital JC*-algebra and let $P : A \to A$ be a positive unital projection. Then $P$ is a Jordan homomorphism, that is, $P(a^2) = P(P(a)^2)$ if and only if*

$$\|Px\|^2 = \|P(x^2)\| \quad \textit{for every } x = x^* \textit{ in } A. \qquad (3.11)$$

*Proof.* If $P$ is a Jordan homomorphism, so that $P(a^2) = P(P(a)^2)$, then $\|P(a^2)\| \leq \|P(a)^2\| = \|P(a)\|^2$. However, since $P$ is positive, $P(a^2) \geq P(a)^2$ ([15, Theorem 1.2]), so that (3.11) holds.

Conversely, if $a = a^* \in A$, then $x = x^* = a - \mathcal{E}(a) \in \ker \mathcal{E}$, and

$$
\begin{aligned}
0 &= P(x^2) = P(a^2 - P(a)a - aP(a) + P(a)^2) \\
&= P(a^2) - 2P(P(a) \circ a) + P(P(a)^2) \\
&= P(a^2) - P(P(a)^2) \text{ (by (3.10))},
\end{aligned}
$$

so $P$ is a Jordan homomorphism.                                      □

## 4. Homomorphic conditional expectations on $C_0(X)$

This section is based on [13, 5.1]. We discuss the relationship between homomorphic conditional expectations on commutative C*-algebras $C_0(X)$ and retractions on $X$, for compact and locally compact Hausdorff spaces $X$. When we deal specifically with a compact Hausdorff space we usually use $K$ in place of $X$.

If $K$ is a compact Hausdorff space, we use $C(K)$ to denote the unital C*-algebra (with pointwise operations and the supremum norm) of all complex-valued continuous functions on $K$. If $X$ is a locally compact Hausdorff space, we use $C_0(X)$ to denote the C*-algebra of all complex-valued continuous functions on $X$ which vanish at infinity. If $K$ is compact, then $C_0(K) = C(K)$.

**Example 4.1.** Retractions $\tau \colon K \to K$ on a (locally) compact Hausdorff space $K$ give rise to homomorphic conditional expectations $\mathcal{E}_\tau \colon C(K) \to C(K)$ via $\mathcal{E}_\tau(f) = f \circ \tau$. But there are expectations on $C(K)$ which do not come from any retraction of $K$ (those expectations are not homomorphic). For instance, let $K = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and define

$$\mathcal{E} \colon C(K) \to C(K) \text{ by } \mathcal{E}(f)(\zeta) = \frac{f(\zeta) + f(-\zeta)}{2}.$$

Then $\mathcal{E}$ is a (not homomorphic) conditional expectation on $C(K)$ and there is no retraction $\tau \colon K \to K$ with $\mathcal{E} = \mathcal{E}_\tau$; see [13, Proposition 5.1.6].

**Theorem 4.2.** *Let $K$ be a compact Hausdorff space. If $\tau\colon K \to K$ is a retraction (i.e., a continuous function with $\tau \circ \tau = \tau$), then the map*

$$\mathcal{E}_\tau\colon C(K) \to C(K), \qquad \mathcal{E}_\tau(f) = f \circ \tau, \qquad \text{for every } f \text{ in } C(K) \qquad (4.1)$$

*is a unital homomorphic conditional expectation.*

*Conversely, if $\mathcal{E}\colon C(K) \to C(K)$ is a unital homomorphic conditional expectation, then there is a retraction $\tau\colon K \to K$ such that $\mathcal{E} = \mathcal{E}_\tau$, where $\mathcal{E}_\tau$ is given by formula (4.1).*

*Proof.* The first implication is a straightforward verification. For the second implication, let $\mathcal{E}\colon C(K) \to C(K)$ be a unital homomorphic conditional expectation. Then the kernel of $\mathcal{E}$, which will be denoted by $\ker \mathcal{E}$, is a closed ideal and hence there is a closed set $K_1 \subseteq K$ such that $\ker \mathcal{E} = \{f \in C(K) : f\,|_{K_1} = 0\}$; (see, for example, [14, Theorem 4.2.4] or [16, Theorem 85]). If we let $S$ denote the range of $\mathcal{E}$, then $S$ is a closed subalgebra of $C(K)$ (containing the constants) and $\mathcal{E}$ induces an algebra isomorphism

$$\tilde{\mathcal{E}}\colon C(K)/\ker \mathcal{E} \to S, \qquad \tilde{\mathcal{E}}(f + \ker \mathcal{E}) = \mathcal{E}(f), \qquad \text{for every } f \text{ in } C(K).$$

We also have an isomorphism

$$\pi\colon C(K)/\ker \mathcal{E} \to C(K_1), \qquad \pi(f + \ker \mathcal{E}) = f\,|_{K_1}, \qquad \text{for every } f \text{ in } C(K);$$

([14, Theorem 4.2.4], or [16, Theorem 85]). Now $\tilde{\mathcal{E}} \circ \pi^{-1}\colon C(K_1) \to S \subseteq C(K)$ is a unital algebra homomorphism and so there exists a continuous function $\phi\colon K \to K_1$ such that $(\tilde{\mathcal{E}} \circ \pi^{-1})(h) = h \circ \phi$, for $h \in C(K_1)$; see [2, II.2.2.5]. Let $\tau\colon K \to K$ be given by $\tau(t) = \phi(t)$, for $t \in K$, so that $\tau$ has the same values as $\phi$ but with a different co-domain. Note that $\tau$ is continuous (since $\phi$ is). We claim that $\mathcal{E}(f) = f \circ \tau$ for all $f \in C(K)$. Indeed, if $f \in C(K)$, then $\pi(f + \ker \mathcal{E}) = f\,|_{K_1}$ thus $f + \ker \mathcal{E} = \pi^{-1}(f\,|_{K_1})$, and this implies that $(\tilde{\mathcal{E}} \circ \pi^{-1})(f\,|_{K_1}) = \mathcal{E}(f)$. But also $(\tilde{\mathcal{E}} \circ \pi^{-1})(f\,|_{K_1}) = (f\,|_{K_1}) \circ \phi = f \circ \tau$. Hence we have $\mathcal{E}(f) = f \circ \tau$ for all $f \in C(K)$, as claimed. Since $\mathcal{E}(\mathcal{E}(f)) = \mathcal{E}(f)$ we must have $f \circ \tau \circ \tau = f \circ \tau$ for each function $f \in C(K)$. Since the functions in $C(K)$ separate the points of $K$, it follows that $\tau \circ \tau = \tau$ so that $\tau$ is a retraction. $\qquad \square$

**Corollary 4.3.** *Let $K$ be a compact Hausdorff space and let $\mathcal{E}\colon C(K) \to C(K)$ be a homomorphic conditional expectation. Then there is a clopen set $L \subseteq K$ and a retraction $\tau\colon L \to L$ such that $\mathcal{E}$ is given by*

$$\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{for } t \in K \setminus L \end{cases}$$

*for $f \in C(K)$, $t \in K$.*

*Proof.* Let $1_K$ denote the constant function $1$ in $C(K)$. Then $\mathcal{E}(1_K)^2 = \mathcal{E}(1_K)$ and so there is $L = \{t \in K : \mathcal{E}(1_K)(t) = 1\}$ so that $\mathcal{E}(1_K) = 1_L$ (the characteristic function of $L$). Moreover, since $1_L \in C(K)$, $L \subseteq K$ is a clopen subset. Since $\mathcal{E}(1_K - 1_L) = \mathcal{E}(1_K) - \mathcal{E}(1_L) = \mathcal{E}(1_K) - \mathcal{E}(\mathcal{E}(1_K)) = 0$, we have that $1_K - 1_L \in \ker \mathcal{E}$ (which is an ideal). So if $f \in C(K)$, then

$$\mathcal{E}(f) = \mathcal{E}(1_L f + (1_K - 1_L)f) = \mathcal{E}(1_L f) = 1_L \mathcal{E}(1_L f). \qquad (4.2)$$

Identifying $C(L)$ with $\{f \in C(K) : f = 1_L f\}$ via $g \in C(L) \mapsto \tilde{g} \in C(K)$ (where $\tilde{g}\mid_L = g$ and $\tilde{g}(t) = 0$ for $t \in K \setminus L$), we see that $\mathcal{E}$ induces a homomorphic unital conditional expectation $\mathcal{E}_L \colon C(L) \to C(L)$ by $\mathcal{E}_L(g) = \mathcal{E}(\tilde{g})\mid_L$. The result follows by applying Theorem 4.2 to $\mathcal{E}_L$ and using (4.2). $\square$

Let $X$ be a locally compact Hausdorff space and $X^* = X \cup \{\omega\}$ the one point compactification. We use this notation here even when $X$ is already compact, in which case $\{\omega\}$ is open (and closed) in $X^*$. Subsets $U$ of $X^*$ are open if $U \cap X$ is open in $X$ and if $\omega \in U$ we insist that $X^* \setminus U$ be a compact subset of $X$.

We consider $C_0(X)$ as embedded in $C(X^*)$ via

$$f \mapsto \tilde{f} \colon C_0(X) \to C(X^*),$$

where

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in X \\ 0 & \text{for } t = \omega. \end{cases}$$

Note that this identifies $C_0(X)$ with $\{g \in C(X^*) : g(\omega) = 0\}$ (the maximal ideal of $C(X^*)$ consisting of functions which take the value zero at $\omega$) and $f \mapsto \tilde{f}$ is a *-algebra isomorphism onto its range.

If $\tau \colon X^* \to X^*$ is a retraction such that $\tau(\omega) = \omega$, then we can define a conditional expectation $\mathcal{E}_{\tau,*} \colon C_0(X) \to C_0(X)$ by $\mathcal{E}_{\tau,*}(f) = (\tilde{f} \circ \tau)\mid_X$.

**Corollary 4.4.** *If $X$ is a locally compact Hausdorff space and $\mathcal{E} \colon C_0(X) \to C_0(X)$ is a homomorphic conditional expectation, then there is a retraction $\tau \colon X^* \to X^*$ ($X^* = X \cup \{\omega\}$) with $\tau(\omega) = \omega$ such that $\mathcal{E} = \mathcal{E}_{\tau,*}$.*

*Proof.* First consider the case when $X$ is compact. We apply Corollary 4.3 above to get $L \subseteq X$ compact and clopen and $\tau \colon L \to L$ a retraction with

$$\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{if } t \in X \setminus L. \end{cases}$$

Define a retraction $\rho \colon X^* \to X^*$ by $\rho(t) = \tau(t)$ for $t \in L$ and $\rho(t) = \omega$ for $t \in (X \setminus L) \cup \{\omega\}$. Since $L$ is clopen and so is $\{\omega\}$, $\rho$ is continuous. We can verify that $\rho \circ \rho = \rho$ and $\mathcal{E} = \mathcal{E}_{\rho,*}$.

In the case that $X$ is not compact, note that $C(X^*)$ is isomorphic as a *-algebra to the unitisation $C_0(X)^\sharp$, where $C_0(X)^\sharp$ is defined as in [6, Definition 1.3.3]. The isomorphism is given by $g \mapsto \phi(g) := (g\mid_X - g(\omega), g(\omega))$. Indeed, if $\phi(g_1) = \phi(g_2)$, then $g_1(\omega) = g_2(\omega)$ and $g_1\mid_X = g_2\mid_X$, thus $g_1 = g_2$. On the other hand, if $(h, \alpha) \in C_0(X)^\sharp$, then $\phi(g) = (h, \alpha)$, where

$$g(x) = \begin{cases} h(x) + \alpha & \text{if } x \in X \\ \alpha & \text{if } x = \omega. \end{cases}$$

Regard $\mathcal{E}^\sharp \colon C_0(X) \oplus \mathbb{C} \to C_0(X) \oplus \mathbb{C}$ as $\mathcal{E}^\sharp \colon C(X^*) \to C(X^*)$, where $\mathcal{E}^\sharp(h, \alpha) = (\mathcal{E}(h), \alpha)$ for $h \in C_0(X)$, $\alpha \in \mathbb{C}$. Then $\mathcal{E}^\sharp$ is an algebra homomorphism and a conditional expectation.

We apply Corollary 4.3 to get $L \subset X^*$ clopen and $\tau \colon L \to L$ a retraction so that

$$\mathcal{E}^\sharp(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{if } t \in X^* \setminus L, \end{cases} \tag{4.3}$$

for $f \in C(X^*)$. Since $C_0(X)$ can be identified with the maximal ideal of $C(X^*)$ consisting of functions which take the value zero at $\omega$, i.e., $C_0(X) = \{g \in C(X^*) : g(\omega) = 0\}$, we have

$$\mathcal{E}^\sharp(C_0(X)) = \mathcal{E}^\sharp(\{g \in C(X^*) : g(\omega) = 0\}) \subset C_0(X),$$

and, therefore, if $\omega \in L$, then $\tau(\omega) = \omega$. (Indeed, if $\omega \in L$ and $\tau(\omega) = t \in X$, there is $f \in C_0(X)$ with $f(t) = 1$ and we would have a contradiction from $0 = \mathcal{E}^\sharp(f)(\omega) = \tilde{f}(\tau(\omega)) = f(t) \neq 0$). If $\omega \notin L$, then $\omega \in X^* \setminus L$, $L \subseteq X$ is compact, and $\mathcal{E}^\sharp$ is given by (4.3).

Thus we can extend $\tau$ to a retraction $\rho \colon X^* \to X^*$ by $\rho(t) = \tau(t)$ for $t \in L$ and $\rho(t) = \omega$ for $t \in X^* \setminus L$. Since $L$ is clopen, $\rho$ is continuous, and we can verify that $\rho \circ \rho = \rho$ and $\mathcal{E} = \mathcal{E}_{\rho,*}$. $\qquad\square$

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