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# SYNCHROTRON RADIATION AND RING FORMATION IN THE ELECTRON RING ACCELERATOR*

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## ABSTRACT

We discuss the possibility of using synchrotron radiation to form electron rings having a very high electric field to hold the ions inside the ring. The formulas describing how the energy and the dimension of the ring change under the effect of synchrotron radiation are derived, and a numerical example is given.

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## I. Introduction

In electron ring accelerators (ERA) it is required to have a high electric field, $\mathcal{E}_H$, holding the ions inside the ring, and a large rate of acceleration of ions, $\frac{dE_i}{dz}$. For a ring having a cylindrical cross section of radius "a", and a uniform electron distribution inside this cylinder, the holding field can be written as

$$e\mathcal{E}_H = \frac{N_e r_e mc^2}{\pi a R} , \tag{I-1}$$

where $N_e$ is the number of electrons, e, $r_e$ and $mc^2$ are the charge, classical radius and rest energy of the electron and R is the ring major radius. The rate of energy gain can be written as

$$\frac{dE_i}{dz} = e \; \mathcal{E}_{ex} \frac{1-fZ}{[f+E_e/E_i]} , \tag{I-2}$$

where $\mathcal{E}_{ex}$ is the external field accelerating the ring, eZ and M are the charge and rest mass of the ion, $E_i$ and $E_e$ are the ion and electron energies and f is the ratio of ion number to electron number. In order not to lose the ions from the ring during the acceleration process one must also satisfy the condition

$$Ze\mathcal{E}_H > \frac{dE_i}{dz} \tag{I-3}$$

In order to increase the holding field for a given number of electrons one can reduce the ring major and minor radius. If the ring is formed by magnetic compression of a circular electron beam, as has been done in all the work on ERAs carried on up to now, the quantities R and a are related approximately to their initial values at injection by[1]

$$\begin{aligned} R &= R_i/B^{\frac{1}{2}} , \\ a &= a_i/B^{\frac{1}{2}} , \end{aligned} \tag{I-4}$$

where the subscript "i" indicates the initial value and B is the ratio of the final to the initial value of the magnetic field. In the same process the electron energy transforms as[1]

$$E = E_i B^{\frac{1}{2}} \tag{I-5}$$

so that when reducing R, and increasing $\mathcal{E}_H$, one increases $E_e$ and reduces $dE_i/dz$. It is possible to inject an electron beam of low energy in order to have a low final energy and hence a high $dE_i/dz$, but because space-charge effects and beam instabilities are strong functions of beam energy, the resulting limit on the electron number makes satisfying (I-3) difficult.

It is interesting to consider the possibility of using other processes, to form electron rings in order that the transformation laws (I-4), (I-5) might be broken and, hopefully, ERA performance improved. One such possibility is to compress the ring in a magnetic field such that the field value at the electron orbit and the magnetic flux enclosed by the ring can be changed independently[2]. An example of this class of compressors is the static compressor[3] in which the electron energy remains constant while ring radius decreases. In this paper we want to call attention to another possibility, which employs the process of synchrotron radiation.

Electrons moving in a magnetic field emit synchrotron radiation; and as a consequence both the energy and the radius of the electron ring decreases and $\mathcal{E}_H$ and $dE_i/dz$ increases. The rate of change of energy and major and minor radius is evaluated in sections II and III. A numerical example is given in section IV. It is interesting to note that the energy spread in the ring can either decrease or increase because of synchrotron radiation, depending on the choice of the magnetic field gradient in the region where the radiation occurs. On the contrary, the betatron amplitudes are almost unaffected by the radiation process in the case when the energy spread decreases. Hence, in order to have a small ring minor radius, ring compression by synchrotron radiation alone probably does not suffice. We suggest that a combination of different compression techniques, for instance the use of a static compressor plus an additional compression by synchrotron radiation, could well lead to the formation of rings with very high holding fields. Holding fields in the range of 1 GeV/m appear, in this way, to be attainable.

One should also notice that the time necessary to achieve a significant reduction in ring radius, by using synchrotron radiation, can in some cases be very long, of the order of ten milliseconds or more. This introduces an additional problem in connection with the process of ion loading of the ring, since, in order to keep the contamination of the ring by unwanted ions within tolerable limits, one requires a vacuum better than $10^{-11}$ torr.

Let us assume, for instance, that one wants to charge the ring with one per cent of protons in ten milliseconds. The pressure required, in torr, is given by

$$P = \frac{1.41 \times 10^{17} f}{\sigma c t} \qquad \text{(I-6)}$$

where $\sigma$ is the ionization cross section ($\sigma = 10^{-19} \mathrm{cm}^2$ for $H_2$), $c$ the velocity of light and $t$ is the time. From (I-6) one obtains $P \approx 5 \times 10^{-9}$ torr. If we want to keep the number of unwanted ions -- including $H_2O^+$ or $CO^+$ -- at a level ten times smaller than that of protons we need a vacuum of the order $5 \times 10^{-11}$ torr, since the ionization cross section for $H_2O$ or CO is about ten times as large as that for hydrogen.

## II. Equations of Motion

The equation of motion of a particle in a magnetic field $\underline{H}$, can be written as

$$\frac{d}{dt}(E\dot{\underline{r}}) = ec\, \dot{\underline{r}} \times \underline{H} + c^2\underline{R} + c^2\underline{G}, \qquad \text{(II-1)}$$

where $\underline{r}(t)$ describes the position of the particle, $E$ is the energy, $\underline{R}$ is the average value of the reaction force due to photon emission, and $\underline{G}$ is the fluctuation in this force. If $\underline{q}$ is the change in the electron momentum due to the emission of a photon and $P(\underline{q},t)$ the rate of photon emission, one has

$$\underline{R}(t) = \int \underline{q}\, P(\underline{q},t) d\underline{q}\ , \qquad \text{(II-2)}$$

and

$$\underline{G}(t) = \sum_j \underline{q}_j\, \delta(t-t_j) - \underline{R}(t), \qquad \text{(II-3)}$$

where $\underline{q}_j$ is the electron momentum change upon emitting a photon at time $t_j$.

The force $\underline{R}$ can be written as

$$\underline{R} = - \dot{\underline{r}}\, P_\gamma . \qquad \text{(II-4)}$$

Using (II-4) and multiplying equation (II-1) by $\dot{\underline{r}}$, one obtains an equation for the energy variation of the particle

$$\dot{E} = - \dot{r}^2 P_\gamma + \dot{\underline{r}} \cdot \underline{G} . \qquad \text{(II-5)}$$

Substituting in (II-1), the equation of motion becomes

$$E\, \ddot{\underline{r}} = ec\, \dot{\underline{r}} \times \underline{H} - \dot{\underline{r}} \frac{c^2 P_\gamma}{\gamma^2} + c^2 \underline{G} - \dot{\underline{r}}(\dot{\underline{r}} \cdot \underline{G}), \qquad \text{(II-6)}$$

where $\gamma$ is the ratio of $E$ to the rest mass energy. It is interesting to note that the radiation reaction terms are multiplied by a factor of $\frac{1}{\gamma^2}$ in equation (II-6), and can be neglected -- to a good approximation -- for relativistic particles.

Since the rate of change of energy, described by (II-5), is very slow compared to the cyclotron period, we can solve equations (II-5), (II-6) assuming in first approximation $E$ = constant. As a second step we will consider the effect of the change in $E$. For $E$ = constant we can introduce a reference trajectory (RT) defined by

$$\ddot{\underline{r}}_s = \frac{ec}{E_s} \left[ \dot{\underline{r}}_s \cdot \underline{H}(\underline{r}_s) \right] . \qquad \text{(II-7)}$$

We can now study small displacements around the RT, by assuming

$$\underline{r} = \underline{r}_s + \delta \underline{r}, \quad E = E_s(1 + p), \qquad \text{(II-8)}$$

and linearizing equation (II-6) with respect to $\delta\underline{r}$ and $p$. Following the usual procedure we introduce the derivative with respect to the arc length, $s$, on the RT and consider a reference frame defined on the RT by the orthonormal tangent, normal and binormal vectors $\underline{\alpha}(s)$, $\underline{\beta}(s)$, $\underline{\gamma}(s)$, such that

$$\dot{\underline{r}}_s = v_s\, \underline{\alpha} , \qquad \underline{\alpha}' = K(s)\, \underline{\beta} ,$$

$$\underline{\beta}' = - K(s)\ \underline{\alpha} + T(s)\ \underline{\gamma}, \qquad \text{(II-9)}$$
$$\underline{\gamma}' = - T(s)\ \underline{\beta},$$

where $K(s)$ and $T(s)$ are the curvature and torsion of the RT, and a prime denotes a derivative with respect to $s$. Any vector $\underline{u}$ can be written as

$$u = u_1\underline{\alpha} + u_2\underline{\beta} + u_3\underline{\gamma}\ .$$

In particular, we chose

$$\delta\underline{r} = x(s)\underline{\beta} + z(s)\underline{\gamma}. \qquad \text{(II-10)}$$

In the following we will only consider a planar RT so that $T(s) = 0$.

From the definition of the RT, Eqn. (II-7), we have

$$H_1(\underline{r}_s) = 0, \quad H_2(\underline{r}_s) = 0,$$
$$\text{(II-11)}$$
$$e\ c\ H_3(\underline{r}_s) = - v_s E_s K(s)\ .$$

We assume that $H_1$ is everywhere zero and that $H_2$ and $H_3$ can be written, near the RT, as

$$e\ c\ H_2 = - K^2 n\ v_s E_s z, \qquad \text{(II-12)}$$

$$e\ c\ H_3 = - v_s E_s K(1 + nKx). \qquad \text{(II-13)}$$

We can now write equation (II-6) in the familiar form for betatron oscillations:

$$\ddot{x} + K^2(1 - n)x = - Kp,$$
$$\text{(II-14)}$$
$$\ddot{z} + K^2 nz = 0.$$

We can now consider equation (II-5). Writing

$$P_\gamma = P_{\gamma s} + \delta P_\gamma$$

we obtain from (II-5)

$$\dot{E}_s = - \dot{r}_s{}^2 P_{\gamma s} \qquad \text{(II-15)}$$

and, to first order

$$E_s \dot{p} = \dot{r}_s^2 P_{\gamma s} p - 2\dot{\underline{r}}_s \cdot \delta\dot{\underline{r}}\, P_{\gamma s} - \dot{r}_s^2 \delta P_\gamma + \dot{\underline{r}}_s \cdot \underline{G}_s \ . \qquad \text{(II-16)}$$

The last term is assumed to be very small so that it is evaluated directly on the RT.

Using the conditions $\beta_s$ = constant and assuming also $\beta_s = 1$, whenever possible, one has[4]

$$P_\gamma = \frac{2}{3}\frac{e^2}{c^5}\gamma^4\left\{\ddot{\underline{r}}^2 + \frac{\gamma^2}{c^2}(\dot{\underline{r}}\cdot\ddot{\underline{r}})^2\right\}$$

$$= \frac{2}{3}\frac{r_e}{(m_o c^2)^3}\frac{1}{c}E_s^4 K^2\left\{1 + 2p + 2nK\mathbf{x}\right\}. \qquad \text{(II-17)}$$

Defining

$$C = \frac{2}{3}\frac{r_e c}{(m_o c^2)^3} \approx 4.5 \times 10^{-2}\frac{\text{cm}^2}{(M_eV)^3\text{sec}} \ , \qquad \text{(II-18)}$$

$$g = \underline{r}_s \cdot \underline{G}/E_s, \qquad \text{(II-19)}$$

equations (II-15), (II-16) become

$$\dot{E}_s = - C E_s^4 K^2 \ , \qquad \text{(II-20)}$$

$$\dot{p} = + C E_s^3 K^2 \frac{3n-1}{1-n} p + g \ . \qquad \text{(II-21)}$$

## III. Solution of the Equations

We assume, for the remainder of this paper, that the particles are moving in a constant gradient magnetic field so that

$$H_3(\rho, z=0) = H_{30}\left(\frac{R}{\rho}\right)^n \ . \qquad \text{(III-1)}$$

From (III-1) and (II-11) we obtain (writing K as $1/\rho$)

$$E_s = -\frac{ec}{v_s}\rho H_3 = -\frac{ec}{v_s}H_{30} R \frac{R^{n-1}}{\rho^{n-1}}$$

or

$$E_s = E_{so}\left(\frac{R}{\rho}\right)^{n-1} , \tag{III-2}$$

where $E_{so}$ and $R$ are respectively the injection energy and radius.

From (III-2), (II-20) we obtain

$$\dot{E}_s = - C \frac{E_{so}^{2/(1-n)}}{R^2} E_s^{(2-4n)/(1-n)}$$

or

$$\left(\frac{E_s}{E_o}\right) = \left\{1 - C \frac{3n-1}{1-n} \frac{E_o^3}{R^2} t \right\}^{\frac{1-n}{3n-1}} , \tag{III-3}$$

and

$$\frac{\rho}{R} = \left\{1 - C \frac{3n-1}{1-n} \frac{E_o^3}{R^2} t \right\}^{\frac{1}{3n-1}} . \tag{III-4}$$

In the special case $n = \frac{1}{3}$ one has

$$E = E_o \, e^{- C \frac{E_o^3}{R^2} t} \tag{III-5}$$

$$\frac{\rho}{R} = \left(\frac{E_s}{E_o}\right)^{3/2} = e^{- \frac{3}{2} C \frac{E_o^3}{R^2} t} \tag{III-6}$$

The solution of (II-21) is given by

$$p(t) = e^{\int_0^t \alpha(t')dt'} \left\{ p_o + \int_0^t g(t') e^{- \int_0^{t'} \alpha(t'')dt''} dt' \right\} \tag{III-7}$$

where, for $n \neq 1/3$,

$$\alpha(t) = + C \frac{3n-1}{1-n} \frac{E_s{}^3}{\rho^2}$$

$$= C \frac{3n-1}{1-n} E_{so}{}^3 R^{3(n-1)} \rho^{1-3n} \quad . \qquad \text{(III-8)}$$

Using (III-8) one has

$$e^{\int_0^t \alpha(t')dt'} = \left\{ 1 - C \frac{3n-1}{1-n} \frac{E_o{}^3}{R^2} t \right\}^{-1} , \qquad \text{(III-9)}$$

so that the solution of the homogenous part of equation (II-21) is

$$p = p_o \{1 - C \frac{3n-1}{1-n} \frac{E_o{}^3}{R^2} t\}^{-1} = p_o \left(\frac{\rho}{R}\right)^{1-3n} \qquad \text{(III-10)}$$

To find the complete solution of (II-21) we assume that the emission of a photon is a random process and that considering the averages over the distribution of the random variable appearing in $g(t)$ one has

$$< g(t) > = 0,$$

$$< g(t)\, g(t') > = \epsilon(t)\, \delta(t-t') \; . \qquad \text{(III-11)}$$

The quantity $\epsilon$ can be obtained from the definition of $g$ and is given by

$$\epsilon = \frac{55}{24\sqrt{3}} \lambda\!\!\!^{-}_c \, r_e \, c \, \gamma^5 K^3 \qquad \text{(III-12)}$$

where $\lambda\!\!\!^{-}_c$ is the electron Compton wavelength.

From (III-7), (III-11) we now obtain

$$< p^2(t) > = \left[ 1 - C \frac{3n-1}{1-n} \frac{E_o{}^3}{R^2} t \right]^{-2} .$$

$$\cdot \left\{ p_o^{\ 2} + \frac{D}{C} \frac{1-n}{4n-1} \frac{E_o^{\ 2}}{R} \left[ 1 - \left( 1 - C \frac{3n-1}{1-n} \frac{E_o^{\ 3}}{R^2} t \right)^{3n-1} \right] \right\}$$

$$= \left(\frac{\rho}{R}\right)^{2(1-3n)} \left\{ p_o^{\ 2} + \frac{D}{C} \frac{1-n}{4n-1} \frac{E_o^{\ 2}}{R} \left[ 1 - \left(\frac{\rho}{R}\right)^{4n-1} \right] \right\} \qquad \text{(III-13)}$$

where

$$D = \frac{55}{24\sqrt{3}} \frac{\lambda\!\!\!^{-}_c r_e C}{(m_o c^2)^5} \approx 1.3 \times 10^{-11} \frac{\text{cm}^3}{(\text{MeV})^5 \text{ sec}} \ . \qquad \text{(III-14)}$$

The condition

$$\frac{D}{C} \frac{E_o^{\ 2}}{R} \ll p_o^{\ 2} \qquad \text{(III-15)}$$

is usually satisfied, and in this case it is possible to neglect the contribution to $< p^2 >$ from quantum fluctuations. When this is the case it follows from (III-13) that $< p^2 >$ decreases or increases with time according to whether $n < \frac{1}{3}$ or $n > \frac{1}{3}$.

Equations (III-3), (III-4), and (III-13) describe how the energy, major radius and energy spread of the ring change in time under the effect of synchrotron radiation. Let us consider the betatron oscillations. We already noticed that it was possible to neglect the terms proportional to $P_\gamma$ and $\underline{G}$ in equation (II-6), since they are smaller by a factor $\gamma^{-2}$ than the corresponding terms in equation (II-5). However, we must consider the effect of the change in the ring energy and major radius on the betatron oscillation amplitude, a, and this can easily be done by introducing the adiabatic invariant

$$\frac{E}{\rho}\, \nu\, a^2 = \text{constant} \tag{III-16}$$

where $\nu$ is the betatron wave number, $\nu = (1-n)^{\frac{1}{2}}$ or $\nu = n^{\frac{1}{2}}$. Using (III-3), (III-4) and assuming that $n$ is constant, (III-16) can also be written as

$$a = a_o \left(\frac{\rho}{R}\right)^{n/2} \tag{III-17}$$

where $a_o$ is the initial betatron amplitude. The relationships (III-13), (III-17) determine the behavior of the ring minor radius during radiation compression. It is clear that if we require that the synchrotron amplitude be damped, we require $n < \frac{1}{3}$, and in this case the betatron amplitude changes only slightly with time.

## IV. Numerical Examples

An example of how the energy, major radius and the quantity $R(a+b)$ change with time under the effect of synchrotron radiation, is given, for different $n$ values, in Fig. 1. The quantity $b$ is defined as the ring radial dimension and is assumed to be related to the betatron amplitude, $a$, and to the synchrotron amplitude $\frac{R}{1-n}\, p$, by

$$b = \left\{ a^2 + \left(\frac{R}{1-n}\, p\right)^2 \right\}^{\frac{1}{2}}$$

This quantity is of interest since, for a ring with elliptical cross section and semi-axis a,b, the holding power is inversely proportional to $a + b$.

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## Figure Caption

Figure 1. Energy, radius and $R(a+b)$ (dashed lines) versus time for different $n$ values and for initial values $E_o = 30$ MeV, $R = 5$ cm, $p_o = 10^{-2}$, and $a_o = 0.1$ cm.

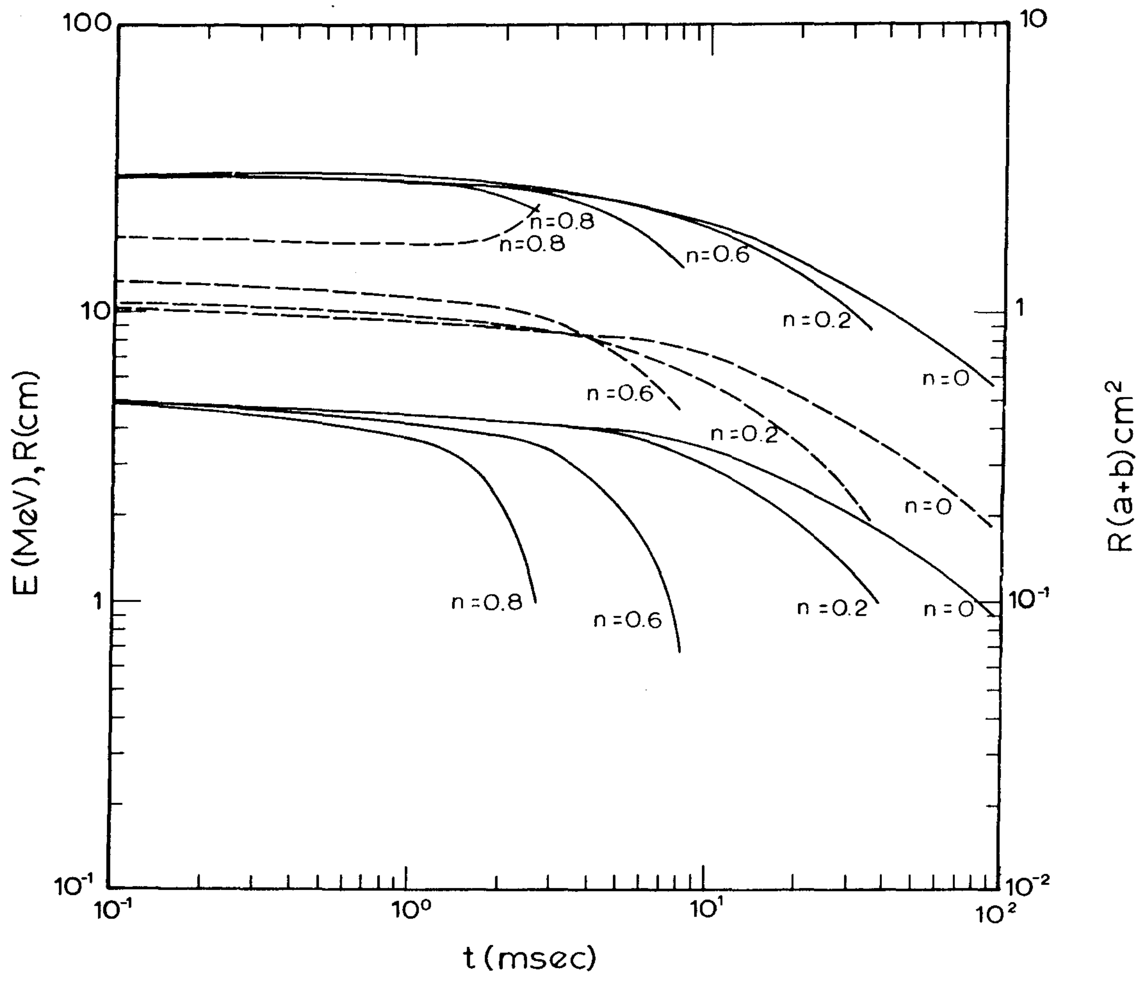
100
10
1
$10^{-1}$
$10^{-1}$
$10^{0}$
$10^{1}$
$10^{2}$
10
1
$10^{-1}$
$10^{-2}$
E (MeV), R(cm)
R(a+b)cm$^2$
t (msec)
n=0.8
n=0.8
n=0.6
n=0.2
n=0
n=0.6
n=0.2
n=0
n=0.8
n=0.6
n=0.2
n=0
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Fig. 1