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# THE FORGOTTEN MONOID

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ABSTRACT. We study properties of the forgotten monoid which appeared in work of Lascoux and Schützenberger and recently resurfaced in the construction of dual equivalence graphs by Assaf. In particular, we provide an explicit characterization of the forgotten classes in terms of inversion numbers and show that there are  $n^2 - 3n + 4$  forgotten classes in the symmetric group  $S_n$ . Each forgotten class contains a canonical element that can be characterized by pattern avoidance. We also show that the sum of Gessel’s quasi-symmetric functions over a forgotten class is a 0-1 sum of ribbon-Schur functions.

## 1. INTRODUCTION

The forgotten monoid appeared in a paper by Lascoux and Schützenberger [9] in their construction of the plactic monoid and was subsequently forgotten. The current research has been motivated by the recent resurface of the forgotten monoid in the definition of dual equivalence graphs in Assaf’s study of LLT polynomials and Macdonald theory [1].

In their seminal paper [9], Lascoux and Schützenberger introduced the well-known *plactic monoid* as a monoid that allows noncommutative symmetric elementary symmetric functions to commute. Their main goal was to obtain a combinatorial proof that the product of two Schur functions decomposes as a sum of Schur functions with nonnegative integer coefficients, also known as the Littlewood–Richardson coefficients.

Their idea can be rephrased by starting with a *noncommutative* analog of the algebra of symmetric functions. The usual commutative symmetric functions are obtained by imposing certain *minimal commuting conditions* on the variables. Working with this noncommutative algebra on partially commuting variables makes the computation of products of functions easier and more precise. This approach led to a very simple proof of the Littlewood–Richardson rule with the help of Hopf algebras [3].

Elementary symmetric functions generate the algebra of symmetric functions. Hence, in order to obtain a partially commuting algebra isomorphic to its commutative version, one only needs to define noncommutative analogues of the elementary symmetric functions and ask that their products commute. The usual definition is as follows: given a (possibly infinite) ordered alphabet  $A = \{a_1 < \dots < a_n\}$ , let

$$(1.1) \quad e_k(A) := \sum_{n \geq i_1 > \dots > i_k \geq 1} a_{i_1} a_{i_2} \dots a_{i_k}.$$

The first nontrivial condition that is imposed by commutativity comes from the relation  $e_1 e_2 = e_2 e_1$ . Since this relation needs to hold for an alphabet with two

letters, it requires

$$(1.2) \quad (a_1 + a_2)a_2a_1 = a_2a_1(a_1 + a_2).$$

But since commutativity is also required to be true for an alphabet with three letters, the relation

$$(1.3) \quad a_2a_3a_1 + a_1a_3a_2 = a_3a_1a_2 + a_2a_1a_3,$$

must hold. All the other terms of the expansion simplify, either by appearing on both sides of the equality, or thanks to relation (1.2).

At this point, Lascoux and Schützenberger make the choice of what are now known as the *Knuth relations*, introduced by Knuth [8] in his study of the Schensted algorithm [10]:

$$(1.4) \quad \begin{cases} acb \equiv cab & \text{for all } a \leq b < c, \\ bac \equiv bca & \text{for all } a < b \leq c. \end{cases}$$

It is then easy to show that the quotient of the noncommutative algebra generated by the  $e_k$  by the Knuth (or, plactic) relations on the variables makes the  $e_k$  commute with one another, and hence provides an algebra isomorphic to the usual commutative symmetric functions. These relations have been weakened later by Fomin and Greene [5].

But, as we shall see in the sequel, there is another choice of relations that imposes commutativity of the  $e_k$ :

$$(1.5) \quad \begin{cases} aba \equiv baa & \text{for all } a < b, \\ bab \equiv bba & \text{for all } a < b, \\ acb \equiv bac & \text{for all } a < b < c, \\ bca \equiv cab & \text{for all } a < b < c. \end{cases}$$

To the best of our knowledge, these relations have not been studied in the literature and hence are called the *forgotten relations* in the folklore. Considering these relations as equivalence relations on words, it is then legitimate to make use of the term *forgotten equivalence classes*, or forgotten classes for short. In the sequel, we shall generally consider the forgotten classes on permutations, that is, the classes generated only by the two relations:

$$(1.6) \quad \begin{cases} acb \equiv bac & \text{for all } a < b < c, \\ bca \equiv cab & \text{for all } a < b < c. \end{cases}$$

For example, here is a complete forgotten class at  $n = 5$ :

$$(1.7) \quad \{12543, 13452, 13524, 14253, 14325, 15234, 21453, \\ 21534, 23154, 23415, 24135, 31254, 31425, 32145, 41235\}.$$

Since the forgotten relations provide some rewritings on consecutive letters, the quotient of the free algebra on  $A$  has naturally a structure of monoid which we shall call the *forgotten monoid*.

Our initial motivation to study the forgotten monoid is the appearance of the forgotten relations in the construction of dual equivalence graphs by Assaf [1], based on ideas of Haiman [7]. Assaf considers two sets of relations on permutations:

$$(1.8) \quad \begin{cases} \dots i - 1 \dots i + 1 \dots i \dots & \equiv^* \dots i \dots i + 1 \dots i - 1 \dots \\ \dots i \dots i - 1 \dots i + 1 \dots & \equiv^* \dots i + 1 \dots i - 1 \dots i \dots \end{cases}$$

$$(1.9) \quad \begin{cases} \dots i + 1 \dots i - 1 \dots i \dots & \equiv^* \dots i \dots i + 1 \dots i - 1 \dots \\ \dots i \dots i - 1 \dots i + 1 \dots & \equiv^* \dots i - 1 \dots i + 1 \dots i \dots \end{cases}$$

The first set consists of the plactic relations on the inverses of the permutations (also known as the coplactic relations) and the second set consists of the forgotten relations on the inverse permutation, hence called the *coforgotten relations*.

In this paper, we show in Theorem 1 that the number of forgotten classes of the symmetric group on  $n$  letters  $S_n$  is  $n^2 - 3n + 4$ . More precisely, two permutations are forgotten-equivalent if and only if they have the same number of inversions and the letters 1 and  $n$  appear in the same order in both permutations. In addition, we characterize canonical elements in each forgotten class by pattern avoidance. It turns out that these canonical elements are the lexicographically minimal elements in each class. Theorem 2 shows that the quotient of the algebra generated by the  $e_k$  by the forgotten relations is indeed isomorphic to the algebra of commutative symmetric functions.

Using our characterization of the forgotten classes in terms of inversions we also prove a conjecture by Zabrocki [11], independently proved by Assaf [2], namely that the sum of Gessel's quasi-symmetric functions over a forgotten class is 0-1 sum of ribbon Schur functions and hence is a symmetric function. In addition we give a precise combinatorial description of the ribbons that appear in the sum. The precise result is stated in Theorem 3,

The paper is organized as follows. Section 2 provides *invariants* on the forgotten classes that take at least  $n^2 - 3n + 4$  values, hence showing that the number of forgotten classes in  $S_n$  is  $\text{classes}(n) \geq n^2 - 3n + 4$ . In Section 3 we provide a set of so-called *canonical elements* using pattern avoidance and shows that each permutation can be rewritten as one of these. Since there are  $n^2 - 3n + 4$  canonical elements, this shows that  $\text{classes}(n) \leq n^2 - 3n + 4$ . In Section 4 we state Theorem 1 about the characterization of the forgotten classes in terms of the inversion number and give many properties which are consequences of this theorem. In Section 5 an iterative algorithm for the forgotten relations is presented, analogous to the Schensted insertion algorithm for the plactic monoid (see [10] for the original algorithm). Section 6 uses standard properties to prove Theorem 2. The last section is devoted to the proof of Theorem 3 which gives the expansion of sums of Gessel's quasi-symmetric functions in terms of ribbon Schur functions.

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## 2. INVARIANTS

Let  $S_n$  denote the set of permutations on  $n$  letters. An inversion of a permutation  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in S_n$  is a pair  $\{\sigma_i, \sigma_j\}$  such that  $\sigma_i > \sigma_j$  but  $i < j$ . The inversion number  $\text{inv}(\sigma)$  is the number of inversions in  $\sigma$ . In this section, we shall only consider the *standard* forgotten classes, that is the forgotten classes on permutations.

**Remark 1.** The forgotten elementary equivalence relations (1.6) preserve the inversion number of the permutations. Hence the inversion number is an invariant of each class.

**Remark 2.** Directly from the forgotten relations, we have that in each forgotten class of  $S_n$ , either  $\{1, n\}$  forms an inversion or not.

**Example 1.** Here is a list of the forgotten classes for  $n = 2, 3, 4$  together with the inversion numbers

(2.1)

$n$	class	inv
2	12	0
	21	1
3	123	0
	132, 213	1
	231, 312	2
	321	3
4	1234	0
	1243, 1324, 2134	1
	1342, 1423, 2143, 2314, 3124	2
	1432, 3142, 3214	3
	2341, 2413, 4123	3
	2431, 3241, 3412, 4132, 4213	4
	3421, 4231, 4312	5
	4321	6

The classes for which  $\{1, n\}$  does not form an inversion are in one-to-one correspondence with the classes for which  $\{1, n\}$  forms an inversion by the operation  $\text{rev}$  reading the word from right to left. It is clear from (1.6) that  $\text{rev}(\sigma) \equiv \text{rev}(\sigma')$  if  $\sigma \equiv \sigma'$ .

**Proposition 1.** *The number of forgotten classes satisfies  $\text{classes}(n) \geq n^2 - 3n + 4$ .*

*Proof.* To get a lower bound on  $\text{classes}(n)$ , we have to find what number of inversions have permutations where 1 is before  $n$ , and permutations where  $n$  is before 1.

The number of inversions of a permutation in  $S_n$  can be any number from 0 to  $n(n-1)/2$ . If we restrict ourselves to permutations where 1 is before  $n$ , then those permutations can have at most  $n(n-1)/2 - (n-1) = (n-1)(n-2)/2$  inversions since each of the  $n-2$  remaining numbers is either to the right of 1 or to the left of  $n$ , hence having no inversion with at least one of these, and  $\{1, n\}$  is not an inversion. It is easy to see that  $1n(n-1)\dots 2$  has inversion number  $(n-1)(n-2)/2$  so that permutations where 1 is before  $n$  can have any number of inversions between 0 and  $(n-1)(n-2)/2$ . This amounts to  $1 + (n-2)(n-1)/2$  different sets of permutations.

By the symmetry  $\text{rev}$  that reads the words from right to left, the same reasoning works for permutations having  $n$  before 1, so that we finally get  $2 + (n-2)(n-1) = n^2 - 3n + 4$  sets of permutations such that each forgotten class belongs entirely to one set. Hence  $\text{classes}(n) \geq (n-1)(n-2) + 2 = n^2 - 3n + 4$ .  $\square$

### 3. CANONICAL ELEMENTS AND PATTERN AVOIDANCE

We consider the set  $\mathbf{Lex}(n)$  of permutations defined by pattern avoidance: they are the elements of  $S_n$  avoiding the four patterns 213, 312, 13452, and 34521. A permutation  $\sigma = \sigma_1\sigma_2\cdots\sigma_n$  avoids the patterns 213 if there is no subword  $\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}$

with  $i_1 < i_2 < i_3$  such that the relative values of  $\sigma_{i_1}$ ,  $\sigma_{i_2}$ , and  $\sigma_{i_3}$  are the same as in 213, that is  $\sigma_{i_2} < \sigma_{i_1} < \sigma_{i_3}$ . Avoidance of the patterns 312, 13452, and 34521 are defined in same way.

**Example 2.** Here are the elements of  $\mathbf{Lex}(n)$  for all  $n \leq 5$ .

$$(3.1) \quad 1; \quad 12, 21; \quad 123, 132, 231, 321$$

$$(3.2) \quad 1234, 1243, 1342, 1432, 2341, 2431, 3421, 4321$$

$$(3.3) \quad 12345, 12354, 12453, 12543, 13542, 14532, 15432, \\ 23451, 23541, 24531, 25431, 35421, 45321, 54321.$$

The elements of  $\mathbf{Lex}(n)$  can be characterized as follows. Since they avoid the patterns 213 and 312, they have to be of  $\Lambda$ -shape, meaning that they first increase and then decrease. More precisely

$$(3.4) \quad \sigma_1 < \sigma_2 < \cdots < \sigma_k > \sigma_{k+1} > \cdots > \sigma_n$$

for some  $k$ , where  $\sigma = \sigma_1 \cdots \sigma_n$ . To simplify the presentation, from now on, we will only write the increasing part of those elements, being understood that the remaining numbers are in decreasing order to the right of  $n$ . Since in addition they avoid the patterns 13452 and 34521 they cannot have a gap in consecutive numbers in a position greater than 1 followed by three increasing numbers. Hence the elements of  $\mathbf{Lex}(n)$  are of one of the following forms:

$$(3.5) \quad \begin{array}{l} 12 \dots k n \\ 12 \dots k a n \\ 23 \dots k n \\ 23 \dots k a n \end{array}$$

where  $1 \leq k < n$  in cases one and three, and  $1 \leq k < n - 2$  and  $k + 1 < a < n$  in cases two and four.

**Proposition 2.** *The set  $\mathbf{Lex}(n)$  contains  $n^2 - 3n + 4$  elements.*

*Proof.* Let us use the characterization in (3.5). For case one in (3.5), there are  $n - 1$  choices for  $k$ . For case two in (3.5), there are  $n - k - 2$  choices for each  $a$  for a given  $k$ , and hence  $\sum_{k=1}^{n-2} (n - k - 2) = (n - 2)^2 - (n - 1)(n - 2)/2$  choices in total. Hence altogether there are  $(n^2 - 3n + 4)/2$  permutations of form one and two. The permutations belonging to cases three and four are just the permutations of form one and two where 1 has been cycled to the end of the word. Hence the total number of permutations in  $\mathbf{Lex}(n)$  is  $n^2 - 3n + 4$  as claimed.  $\square$

**Lemma 1.** *Let  $F$  be a standard forgotten class of  $S_n$ . Then  $F$  contains either a word that begins with 1, or with  $n$ . Both cases cannot appear simultaneously inside  $F$ . Moreover, if there is a word beginning with 1 (resp.  $n$ ) in  $F$ , then there is a word ending with  $n$  (resp. 1).*

*Proof.* We know that in each forgotten class either  $\{1, n\}$  is an inversion or not. Hence  $F$  cannot have both words starting with 1 and starting with  $n$ . Moreover, the last statement is a consequence of the first statement since the forgotten elementary rewritings (1.6) are invariant under the reversal  $\text{rev}$  of words. This shows that every forgotten class must have an element ending with 1 or  $n$ . Because of the property

that in each forgotten class  $\{1, n\}$  is either an inversion or not, the third statement follows.

We now prove the first statement. Let  $\sigma$  be a permutation where 1 is before  $n$ . Write  $\sigma$  as  $x.w.y$  where  $x$  (resp.  $y$ ) is its first (resp. last) letter, and  $w$  the remaining part of  $\sigma$ . Then, if  $y \neq n$ , we are done by induction on the length of  $\sigma$  since the desired property holds for  $x.w$ . If  $y = n$ , then by induction, depending on the respective order of 1 and  $n - 1$  in  $x.w$ , the 1 can be put either to the beginning of  $x.w$  (we are then done) or to the end of  $x.w$ . The last situation means there would be a word congruent to  $\sigma$  that ends with  $1n$ . But in this case  $1n$  can be moved to the left past all other letters via the forgotten relations. Hence in all cases, there exists a word congruent to  $w$  that starts with 1.  $\square$

**Proposition 3.** *The minimal lexicographic element of each forgotten class is an element of  $\mathbf{Lex}(n)$ .*

*Proof.* Let  $\sigma$  be the minimal lexicographic word of a forgotten class of  $S_n$ . We first prove that  $\sigma$  avoids the patterns 312 and 213. Consider a word that contains the patterns 312 or 213 and, among all subwords having one of these patterns, consider one subword with minimal distance between the two extreme letters. Let us assume that this subword is a 312 pattern, which we shall write as  $x_3x_1x_2$ .

There cannot be any letter in  $\sigma$  between  $x_3$  and  $x_1$ : if this letter were greater than  $x_2$ , we would have a shorter pattern 312, which contradicts the minimal distance assumption. The letter also cannot be smaller than  $x_1$  for the same reason and it cannot be in the interval of values between  $x_1$  and  $x_2$  since we would have a shorter 213 pattern. Therefore  $x_3$  and  $x_1$  are consecutive in  $\sigma$ . For similar reasons,  $x_1$  and  $x_2$  are consecutive in  $\sigma$ . Analogous arguments work if the minimal subword is a 231 pattern.

This shows that if  $\sigma$  contains the pattern 312 or 213, there are three consecutive letters in  $\sigma$  having one of those patterns. Hence  $\sigma$  can be rewritten as a lexicographic smaller element via the forgotten relations (1.6), which contradicts our assumption that  $\sigma$  was minimal. Hence  $\sigma$  must avoid the patterns 312 and 213 and we conclude that it is a  $\Lambda$ -word.

Let us now write  $\sigma$  as  $x.w.y$ , where  $x$  and  $y$  are letters and  $w$  is a word. Since  $\sigma$  is the smallest element of its forgotten class,  $w.y$  and  $x.w$  are the smallest elements in their forgotten classes as well. By induction, they are in  $\mathbf{Lex}(n)$  and hence avoid the patterns 13452 and 34521. So if  $\sigma$  has one of the two patterns, then both letters  $x$  and  $y$  have to be part of it. Let us first assume that  $\sigma$  contains pattern 13452. Then  $x = 1$  since  $x < y$  and  $\sigma$  is a  $\Lambda$ -word. In addition  $y = 2$ , since otherwise the letter next to  $x$  would be smaller than  $y$ , and  $w.y$  would have a 13452 pattern as well. Also by induction hypothesis, we have that the first letter of  $w$  is 3. Putting together everything, we have that  $\sigma$  is of the form  $13z * n * 2$  where  $*$  represents any sequence of numbers, possibly empty. We claim that there is a word in the forgotten class of  $3z * n * 2$  of the form  $abn2*$  with  $a < b$ . Since the word  $1abn2$  is of the form 13452, it can be rewritten into  $12nba$ , the claim shows that  $13z * n * 2$  is not minimal in its class, yielding a contradiction.

Now let us prove the claim. By induction on  $n$  of this proposition, using Corollary 1 below and Remarks 1 and 2, the elements of a forgotten class are characterized by their inversion number and the property that either 1 appears before  $n$  or  $n$  appears before 1. Among all words of the form  $abn2*$ , the one with minimal number of inversions is  $34n256 \dots n - 1$  (which has  $n - 2$  inversions) and the one with

maximal number of inversions is  $n-2n-1n2n-3n-4\dots3$  (which has  $\binom{n-2}{2}$  inversions). Now, among all words of the form  $abn2*$  with  $a < b$ , there are permutations with any number of inversions between those two values. The number of inversions of  $3z * n * 2$  is necessarily inside this interval. Indeed, among all lexicographically minimal words of size  $n-1$ , the one with the smallest number of inversions of the form  $2z * n-1 * 1$  is  $23456\dots n-11$  which has  $n-2$  inversions and the one with the greatest number of inversions is  $2n-2n-1n-3\dots431$  which has  $\binom{n-2}{2}$  inversions. Hence  $\sigma$  cannot contain the pattern 13452. A similar argument works for pattern 34521.  $\square$

**Corollary 1.** *The number of forgotten classes satisfies  $\text{classes}(n) \leq n^2 - 3n + 4$ .*

*Proof.* By Proposition 3, there is at least one permutation of  $\mathbf{Lex}(n)$  in each forgotten class. Combining this result with Proposition 2 yields the claim.  $\square$

#### 4. CHARACTERIZATION AND PROPERTIES

In this section we provide an explicit characterization of the forgotten classes and derive several properties. The main result of this section is the following theorem.

**Theorem 1.** *The number of forgotten classes of  $S_n$  is  $\text{classes}(n) = n^2 - 3n + 4$ . More precisely, two permutations are forgotten-equivalent if and only if they have same number of inversions and the letters 1 and  $n$  appear in the same order in both permutations.*

*Proof.* The equality  $\text{classes}(n) = n^2 - 3n + 4$  is a consequence of Corollary 1 and Proposition 1. The characterization follows from Remarks 1 and 2.  $\square$

**Corollary 2.** *Let  $\sigma$  be in  $\mathbf{Lex}(n)$ . Then  $\sigma^{-1}$  is in the same forgotten class as  $\sigma$ , namely  $\sigma \equiv \sigma^{-1}$ .*

*Proof.* We have that  $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ . In addition, the letters 1 and  $n$  are in the same order in  $\sigma$  and  $\sigma^{-1}$  for  $\sigma \in \mathbf{Lex}(n)$  as can be easily seen from their characterization (3.5). Hence, by Theorem 1, they must be in the same class.  $\square$

**Corollary 3.** *The elements in  $\mathbf{Lex}(n)$  and their inverses constitute canonical elements for the coforgotten classes corresponding to the relations (1.9). In addition, the number of standard coforgotten classes is  $n^2 - 3n + 4$ .*

The Schützenberger involution  $\#$  on a permutation  $\sigma \in S_n$  is the composition of the two maps

- rev which reverses the order of all letters in  $\sigma$ ;
- complementation which changes letter  $x$  to  $n+1-x$  for all letters  $x$  in  $\sigma$ .

For example,

$$(4.1) \quad \#842956137 = 379451862.$$

**Corollary 4.** *For any  $\sigma \in S_n$ , we have  $\#\sigma \equiv \sigma$ .*

*Proof.* The composition of reversal and complementation does not change the inversion number of a permutation. In addition, the letters 1 and  $n$  stay in the same order. Hence the corollary follows from Theorem 1.  $\square$



Thanks to the characterization of forgotten classes as given in Theorem 1, it is easy to find which  $\Lambda$ -shaped elements belong to the same forgotten class since the number of inversions of a permutation of  $\Lambda$ -shape is easy to compute: it is equal to the maximal number of inversions minus the difference between  $n$  and all the numbers to its left.

Let  $\sigma \in S_n$  be a  $\Lambda$ -shape permutation such that 1 is to the left of  $n$ . If there are two letters  $(b, c)$  such that  $1 < b < c < n$  that belong to the increasing part of  $\sigma$  and such that  $b - 1$  is not in this part, so that  $\sigma$  is not in  $\mathbf{Lex}(n)$ , then there is a smaller  $\Lambda$ -word with the same number of inversions: look for the smallest number  $d$  greater than  $c$  that is not in the increasing part of  $\sigma$ . If it exists, then replace  $(b, d - 1)$  with  $(b - 1, d)$ . Otherwise, replace  $b$  by  $b - 1$  and remove  $n - 1$  from the increasing part. The case when  $n$  is to the left of 1 is the same, replacing the condition  $1 < b < c < n$  by  $2 < b < c < n$ .

**Example 3.** The following example illustrates the last construction. The permutation 13567842 is forgotten-equivalent to 13468752: in that case,  $b = 5$ ,  $c = 6$ , and  $d$  does not exist. This last permutation is equivalent to 12568743, with  $b = 3$ ,  $c = 4$ , and  $d = 5$ . This last permutation is equivalent to 12478653, with  $b = 5$ ,  $c = 6$ , and  $d = 7$ . This last permutation is in turn equivalent to 12387654, with  $b = 4$ ,  $c = 7$ , and  $d$  does not exist. All of these permutations have inversion number 10.

## 5. THE INSERTION ALGORITHM

Let us now write the elements in (3.5) more compactly as

$$(5.1) \quad \begin{aligned} \sigma(k, a) &= 12 \dots k a n \\ \tau(k, a) &= 23 \dots k a n \end{aligned}$$

for  $1 \leq k < n$  and  $k < a \leq n$ , where we identify  $\sigma(k, n) = \sigma(k - 1, k)$  and  $\tau(k, n) = \tau(k - 1, k)$ .

**Lemma 2.** *We have*

$$(5.2) \quad \begin{aligned} \text{inv}(\sigma(k, a)) &= \binom{n-k}{2} + a - n \\ \text{inv}(\tau(k, a)) &= \binom{n-k}{2} + a - 1. \end{aligned}$$

*Proof.* These formulas can easily be checked explicitly.  $\square$

Let us define the two subsets of  $\mathbf{Lex}(n)$  depending on whether  $\{1, n\}$  forms an inversion or not

$$(5.3) \quad \begin{aligned} \mathbf{Lex}(n, \sigma) &:= \{\sigma(k, a) \mid 1 \leq k < n, k < a \leq n\} \\ \mathbf{Lex}(n, \tau) &:= \{\tau(k, a) \mid 1 \leq k < n, k < a \leq n\}. \end{aligned}$$

By Lemma 2, there exist two bijections

$$(5.4) \quad \begin{aligned} \text{inv}_\sigma : \mathbf{Lex}(n, \sigma) &\rightarrow \left\{ 0, 1, \dots, \binom{n-1}{2} \right\} \\ \text{inv}_\tau : \mathbf{Lex}(n, \tau) &\rightarrow \left\{ n-1, n, \dots, \binom{n}{2} \right\}, \end{aligned}$$

where  $\text{inv}_\sigma$  (resp.  $\text{inv}_\tau$ ) is just the inversion number restricted to the subset of permutations  $\mathbf{Lex}(n, \sigma)$  (resp.  $\mathbf{Lex}(n, \tau)$ ).

The inverses of the maps  $\text{inv}_\sigma$  and  $\text{inv}_\tau$  can also be stated explicitly. Let  $i \in \{0, 1, \dots, (n-1)(n-2)/2\}$ . Pick the largest integer  $m$  such that  $\binom{m}{2} \leq i$  and write

$$(5.5) \quad i = \binom{m}{2} + b \quad \text{for } m \geq 1 \text{ and } 0 \leq b < m.$$

Then  $\text{inv}_\sigma^{-1}(i) = \sigma(k, a)$  where  $k = n - m - 1$  and  $a = n + b - m$ .

Similarly, if  $i \in \{n-1, n, \dots, n(n-1)/2\}$ , pick  $m$  and  $b$  as before. Then  $\text{inv}_\tau^{-1}(i) = \tau(k, a)$  where  $k = n - m$  and  $a = b + 1$ .

**Example 4.** Take  $n = 7$  and  $i = 13$ . Then  $m = 5$  and  $b = 3$  and

$$(5.6) \quad \text{inv}_\sigma^{-1}(13) = 1576542 \quad \text{and} \quad \text{inv}_\tau^{-1}(13) = 2476531.$$

The bijections  $\text{inv}_\sigma$  and  $\text{inv}_\tau$  can be used to define an insertion algorithm  $\omega' \leftarrow i$  of a letter  $i \in \{0, 1, \dots, n-1\}$  into a permutation  $\omega' \in \mathbf{Lex}(n-1)$ . Call  $\omega \in \mathbf{Lex}(n)$  the result of the insertion algorithm  $\omega' \leftarrow i$ . Then

$$(5.7) \quad \omega = \omega' \leftarrow i = \begin{cases} \text{inv}_\sigma^{-1}(\text{inv}(\omega') + n - 1 - i) & \text{if } i \neq 0 \text{ and } \omega' = \sigma(k', a') \\ \text{inv}_\tau^{-1}(\text{inv}(\omega') + n - 1 - i) & \text{if } i = 0 \text{ and } \omega' = \sigma(k', a') \\ \text{inv}_\tau^{-1}(\text{inv}(\omega') + n - 1 - i) & \text{if } i \neq n-1 \text{ and } \omega' = \tau(k', a') \\ \text{inv}_\sigma^{-1}(\text{inv}(\omega') + n - 1 - i) & \text{if } i = n-1 \text{ and } \omega' = \tau(k', a'). \end{cases}$$

**Example 5.** Take  $n = 7$ ,  $\omega' = \sigma(1, 3) = 136542$  and  $i = 0$ . Then  $\text{inv}(\omega') + n - 1 - i = 7 + 7 - 1 = 13$ , so that  $\omega = \text{inv}_\tau^{-1}(13) = 2476531$ . More generally, varying  $i$  we obtain

$i$	$\omega' \leftarrow i$
0	2476531
1	1476532
2	1376542
3	1276543
4	1267643
5	1257643
6	1247653

**Remark 3.** Let  $\omega' \in \mathbf{Lex}(n-1)$  and  $i \in \{0, 1, \dots, n-1\}$ . Then, by construction, the standardization of the word  $\omega' i$  is in the same forgotten class as  $\omega' \leftarrow i$ .

## 6. NONCOMMUTATIVE FORGOTTEN ELEMENTARY SYMMETRIC FUNCTIONS

Let us consider the quotient of the noncommutative elementary symmetric functions

$$(6.1) \quad e_k(A) := \sum_{i_1 > \dots > i_k \geq 1} a_{i_1} a_{i_2} \dots a_{i_k}.$$

by the general forgotten relations (1.5).

**Theorem 2.** *The quotient of the algebra generated by the  $e_k$  by the forgotten relations is isomorphic to the algebra of commutative symmetric functions.*

*Sketch of proof.* As stated in the introduction, it suffices to prove that all  $e_k$  commute. The forgotten relations were chosen such that  $e_1$  and  $e_2$  commute.

Now, consider the general case: let us evaluate  $e_i(A)e_j(A) - e_j(A)e_i(A)$  in the forgotten monoid. The noncommutative words appearing in this expression are of two sorts: words with repetition and words without repetition.

Let us begin with the words without repetition. Since the congruence only takes into account the mutual ordering of the words, we can restrict ourselves to permutations. Then there is a simple bijection between the noncommutative words appearing in  $e_i e_j$  and the words appearing in  $e_j e_i$ , that is the Schützenberger involution. By Lemma 4, those two words are equivalent and hence their difference is 0 in the quotient of the algebra by the forgotten monoid.

Recall that the descent composition of a word is the sequence of lengths of all maximal nondecreasing factors of  $w$ . The property of equivalence of a permutation and its image by the Schützenberger involution proves indeed a more general fact: since the descent composition of  $\#\sigma$  is equal to the reversal of the descent composition of  $\sigma$ , the set of all descent composition of a forgotten class is invariant by the operation consisting in reading compositions from right to left. The same property holds on non-standard forgotten classes, so that the case of words with repetition is solved in the same way.  $\square$

## 7. APPLICATIONS TO QUASI-SYMMETRIC FUNCTIONS

Let

$$(7.1) \quad F_{n,D}(x) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin D}} x_{i_1} \cdots x_{i_n}$$

be the quasi-symmetric functions first introduced by Gessel [6], defined for any subset  $D$  of  $\{1, \dots, n-1\}$ . Recall that there is a simple bijection between these subsets and compositions of  $n$  (that is, all positive sequences of sum  $n$ ): send a composition  $I = [i_1, \dots, i_p]$  to  $D := \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{p-1}\}$ . We then say that  $I$  is the descent composition of  $D$ .

One of the crucial observations in Assaf's thesis [1] is the fact that the sum of Gessel's quasi-symmetric functions for the descents of the vertices of so-called dual equivalence graphs yield Schur functions. Hence as a corollary, it follows [1, Corollary 3.12] that the generating function of dual equivalence graphs with a constant statistics on connected components is a symmetric function and Schur positive.

In this section we prove a similar result conjectured by Zabrocki [11] and independently proved by Assaf [2], stating the expansion of Gessel's quasi-symmetric functions over a forgotten class as a nonnegative multiplicity-free sum of skew-ribbon Schur functions.

Let  $\mathcal{M}_{k,+}$  (resp.  $\mathcal{M}_{k,-}$ ) be the set of permutations  $\sigma$  such that  $\text{maj}(\sigma^{-1}) = k$  and such that  $n-1$  comes before (resp. after)  $n$ . Here  $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$  is the major index and  $\text{Des}(\sigma) = \{i \mid 1 \leq i < n, \sigma_i > \sigma_{i+1}\}$  is the descent set of  $\sigma$ . Let us also define the set  $\mathcal{C}_{k,+}$  (resp.  $\mathcal{C}_{k,-}$ ) of compositions  $c = (c_1, \dots, c_\ell)$  with major index  $\text{maj}(c) = \sum_{i=1}^{\ell} (\ell - i)c_i = k$  that either end in 1 (resp. do not end in 1). Then we have

$$(7.2) \quad \mathcal{M}_{k,\pm} = \bigcup_{I \in \mathcal{C}_{k,\pm}} \{\sigma \mid \text{Recoil}(\sigma) = I\},$$

where  $\text{Recoil}(\sigma)$  is the recoil composition of  $\sigma$ , or equivalently the descent composition of  $\sigma^{-1}$ .

**Theorem 3.** *Let  $S$  be a forgotten class. Then*

$$(7.3) \quad \sum_{\sigma \in S} F_{n, \text{Des}(\sigma)}(x) = \sum_{I \in \mathcal{C}_{k, \text{sign}(S)}} r_I,$$

where  $r_I$  is the ribbon Schur function and  $\text{sign}(S) = +$  if 1 comes before  $n$  in  $S$  and  $\text{sign}(S) = -$  otherwise. In particular,  $\sum_{\sigma \in S} F_{n, \text{Des}(\sigma)}(x)$  is a symmetric function.

We provide a combinatorial proof of Theorem 3.

*Proof.* Let  $\Phi$  be the second fundamental transformation of Foata [4]. It is well-known that  $\Phi$  is a bijection, and that

$$(7.4) \quad \text{Recoil}(\Phi(\sigma)) = \text{Recoil}(\sigma) \quad \text{and} \quad \text{inv}\Phi(\sigma) = \text{maj}(\sigma).$$

We shall make use of a very trivial fact about  $\Phi$ :

$$(7.5) \quad \Phi(\sigma)_1 < \Phi(\sigma)_n \iff \sigma_{n-1} < \sigma_n.$$

Let us now consider the map

$$(7.6) \quad \text{NS}(\sigma) := (\Phi(\sigma^{-1}))^{-1}.$$

The inverse map interchanges descents and recoils. Hence, since  $\Phi$  preserves recoils, NS preserves the descent composition

$$(7.7) \quad \text{Des}(\text{NS}(\sigma)) = \text{Des}(\sigma).$$

Moreover,  $\text{inv}(\sigma^{-1}) = \text{inv}(\sigma)$ , so that

$$(7.8) \quad \text{invNS}(\sigma) = \text{maj}(\sigma^{-1}).$$

Finally, Equation (7.5) becomes

$$(7.9) \quad 1 \text{ comes before } n \text{ in } \text{NS}(\sigma) \iff n-1 \text{ comes before } n \text{ in } \sigma.$$

Now, let  $k$  be the number of inversions of any element of  $S$  and let us consider that 1 is before  $n$  in any element of  $S$ , the other case being similar.

Recall the set  $\mathcal{M}_{k,+}$  of permutations  $\sigma$  such that  $\text{maj}(\sigma^{-1}) = k$  and such that  $n-1$  is before  $n$ . Then this set is the disjoint union of the recoil classes  $\mathcal{C}_{k,+}$  of the compositions  $I$  with major index  $\text{maj}(I)$  equal to  $k$ . Thanks to the previous properties, the image of this set under NS is exactly the set of permutations with  $k$  inversions and such that 1 is before  $n$ , that is, the forgotten class  $S$ .

Since the bijection NS preserves the descents of the permutations, we have

$$(7.10) \quad \sum_{\sigma \in S} F_{n, \text{Des}(\sigma)} = \sum_{\tau \in \text{NS}^{-1}(S)} F_{n, \text{Des}(\tau)} = \sum_{\tau \in \mathcal{M}_{k,+}} F_{n, \text{Des}(\tau)}.$$

But this sum is easy to compute: consider

$$(7.11) \quad \sum_{\substack{\sigma \\ \text{Recoil}(\sigma)=I}} \mathbf{F}_\sigma,$$

in the Hopf algebra  $\mathbf{FQSym}$  of Free Quasi-Symmetric functions realized as non-commutative polynomials in the free algebra. Then taking the commutative image of this expression gives

$$(7.12) \quad \sum_{\substack{\sigma \\ \text{Recoil}(\sigma)=I}} F_{n, \text{Des}(\sigma)}.$$

But this sum is also equal to its realization in terms of the noncommutative ribbon Schur function  $\mathbf{R}_I$  whose commutative image is the usual ribbon Schur function  $r_I$ . Hence

$$(7.13) \quad \sum_{\substack{\sigma \\ \text{Recoil}(\sigma)=I}} F_{n, \text{Des}(\sigma)} = r_I.$$

This shows indeed that

$$(7.14) \quad \sum_{\sigma \in S} F_{n, \text{Des}(\sigma)} = \sum_{I \in \mathcal{C}_{k,+}} r_I$$

is a multiplicity-free sum of ribbon Schur functions.  $\square$

There is a very simple way to compute directly from a given forgotten class the set of compositions appearing in its evaluation on  $QSym$ : consider the set of words defined by applying the inverses to  $V$ -permutations, that is permutations satisfying  $\sigma_1 > \sigma_2 > \dots > \sigma_i < \sigma_{i+1} < \dots < \sigma_n$  for some  $1 \leq i \leq n$ . Those are exactly the words such that each prefix ends by either its smallest letter or its greatest letter. Then, by definition of the bijection  $\Phi$ , those elements are fixed points under  $\Phi$ , so that the  $V$ -permutations are fixed-points under the NS bijection. Now, since the  $V$ -permutations have all different recoil compositions, the sum over the whole forgotten class of  $F_{n, \text{Des}(\sigma)}$  is equal to the sum of the  $r_I$ , where  $I$  are the recoil compositions of the  $V$ -permutations of this forgotten class. Alternatively, conjugating by the Schützenberger involution  $\#$  using Corollary 4, the sum over the whole forgotten class of  $F_{n, \text{Des}(\sigma)}$  is equal to the sum of the  $r_I$ , where  $I$  runs over the set of reversed recoil compositions of the  $\Lambda$ -permutations of this forgotten class.

**Example 6.** Let us compute the evaluation of the two forgotten classes of  $S_8$  with 10 inversions. The two canonical words are 12387654 and 23458761.

Using Example 3, one can check that the  $\Lambda$ -permutations belonging to the forgotten class of 12387654 are

$$(7.15) \quad 12387654, 12478653, 12568743, 13468752, 13567842.$$

Their recoil composition are (41111), (3212), (3131), (2321), (224) respectively, so that the evaluation of the forgotten class of 12387654 is

$$(7.16) \quad r_{11114} + r_{2123} + r_{1313} + r_{1232} + r_{422}.$$

One also checks that the  $\Lambda$ -permutations belonging to the forgotten class of 23458761 are

$$(7.17) \quad 23458761, 23467851.$$

Since their recoil composition are (1511) and (143) respectively, the evaluation of the forgotten class of 23458761 is

$$(7.18) \quad r_{1151} + r_{341}.$$

One can easily check that the seven ribbons listed here exactly are the seven compositions with major index equal to 10.

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