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Authors
Danielski, Daniel
Kapovich, Michael
Świątkowski, Jacek

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Complete characterizations of hyperbolic Coxeter groups with Sierpiński curve boundary and with Menger curve boundary

by

Daniel Danielski (Wrocław), **Michael Kapovich** (Davis, CA) and
Jacek Świątkowski (Wrocław)

Abstract. We give complete characterizations (in terms of nerves) of those word hyperbolic Coxeter groups whose Gromov boundary is homeomorphic to the Sierpiński curve and to the Menger curve, respectively. The justification is mostly an appropriate combination of various results from the literature.

0. Introduction

0.1. Overview and context. It is a classical and widely open problem to characterize those word hyperbolic groups whose Gromov boundary is homeomorphic to a given topological space. The complete answers (for non-elementary hyperbolic groups) are known only for the Cantor set (virtually free groups) and for the circle S^1 (cocompact Fuchsian groups). For the sphere S^2 the expected answer is known as Cannon's conjecture, and it remains open. Some partial answers are known in restricted frameworks. For example, Cannon's conjecture is known to be true for Coxeter groups (we discuss this issue in more detail in Subsection 1.4). In this paper we deal with spaces known as the Sierpiński curve and the Menger curve, providing complete characterizations of word hyperbolic Coxeter groups for which these spaces appear as Gromov boundaries.

Some partial results in this direction have been presented quite recently by several authors. For example, P. Dani, M. Haulmark and G. Walsh [6] have shown that for a word hyperbolic right-angled Coxeter group W whose

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nerve L is 1-dimensional, ∂W is homeomorphic to the Menger curve iff L is *unseparable* (i.e. connected, with no separating vertex and no separating pair of non-adjacent vertices) and non-planar. The third author of the present paper, in [15], characterized those word hyperbolic Coxeter groups with Sierpiński curve boundary whose nerves are planar complexes. The first author [7] provided a sufficient condition for the nerve of a word hyperbolic right-angled Coxeter group W, which can be applied to nerves of arbitrary dimension, under which the Gromov boundary ∂W is the Menger curve.

The current paper resulted from an observation (by the second author) that some results of M. Bourdon and B. Kleiner [4] can be applied to obtain complete characterizations, as presented below.

0.2. Results. Before presenting our main result we need to recall some terminology and notation appearing in its statement. The *nerve* of a Coxeter system (W, S) is the simplicial complex $L = L(W, S)$ whose vertex set is identified with S and whose simplices correspond to those subsets $T \subset S$ for which the special subgroup W_T is finite. The *labelled nerve* $L^\bullet$ of (W, S) is the nerve L in which the edges are equipped with labels in such a way that any edge $[s, t]$ has label equal to the exponent m_{st} from the standard presentation associated to (W, S) (equivalently, m_{st} is the appropriate entry of the Coxeter matrix of the system (W, S)). Obviously, the labelled nerve of a Coxeter system carries the same information as its Coxeter matrix. Note that the labelled nerve of the direct product of two Coxeter systems is the simplicial join of the nerves of the two factors, where the labels at the edges of the joined complexes are preserved, and the labels at all "connecting" edges (i.e. edges having endpoints in both joined complexes) are equal to 2. We call such a labelled nerve the *labelled join* of the labelled nerves of the two factors. A Coxeter system is called *indecomposable* if it cannot be expressed as a direct product of non-trivial Coxeter systems. Observe that a Coxeter system is indecomposable iff its labelled nerve cannot be expressed as a labelled join of two non-trivial labelled complexes.

We use the convention of speaking of topological or simplicial properties of labelled nerves as of the properties of the corresponding underlying unlabelled nerves. The labelled nerve of a Coxeter system is *unseparable* if it is connected, has no separating simplex, no separating pair of non-adjacent vertices, and no separating *labelled suspension* (i.e. a full subcomplex which is the labelled join of a simplex and a doubleton). The concept of unseparability is useful because of the following characterization of non-existence of a splitting along a finite or a 2-ended subgroup in a Coxeter group, due to Mihalik and Tschantz [13]: *the group W in a Coxeter system (W, S) has no non-trivial splitting along a finite or a 2-ended subgroup iff its labelled nerve is unseparable* (see Subsection 1.2 for more details).

Given a finite simplicial complex K we define its *puncture-respecting cohomological dimension* by the formula

$$\operatorname{pcd}(K) := \max\{n : \overline{H}^n(K) \neq 0 \text{ or } \overline{H}^n(K \setminus \sigma) \neq 0 \text{ for some } \sigma \in \mathcal{S}(K)\},$$

where $\mathcal{S}(K)$ is the family of all closed simplices of K. This concept is useful for us due to its role in a formula (by M. Davis) for the virtual cohomological dimension of a Coxeter group; see Proposition 1.3 below and its proof.

A 3-*cycle* is a triangulation of the circle S^1 consisting of precisely 3 edges.

Our main result is the following.

THEOREM 0.1. *Let (W, S) be an indecomposable Coxeter system such that W is infinite word hyperbolic, and let $L^\bullet$ be its labelled nerve.*

(1) *The Gromov boundary ∂W is homeomorphic to the Sierpiński curve iff $L^\bullet$ is unseparable, planar* (*in particular, not a triangulation of S^2*), *and not a* 3-*cycle.*
(2) *The Gromov boundary ∂W is homeomorphic to the Menger curve iff $L^\bullet$ is unseparable,* $\operatorname{pcd}(L^\bullet) = 1$, *and $L^\bullet$ is not planar.*

REMARKS 0.2. (1) Recall that W is infinite iff its nerve is not a simplex. Recall also that word hyperbolicity of W has been characterized by G. Moussong (see [14], or [9, Theorem 12.6.1]) as follows: W is word hyperbolic iff it has no affine special subgroup of rank ≥ 3, and no special subgroup which decomposes as the direct product of two infinite special subgroups.

(2) One of the consequences of the above Moussong's characterization of word hyperbolicity is as follows: A word hyperbolic infinite Coxeter group decomposes (uniquely) into the direct product of an infinite indecomposable special subgroup (which is also word hyperbolic) and a finite special subgroup (possibly trivial). This allows one to extend Theorem 0.1 in the obvious way to Coxeter systems (W, S) which are not necessarily indecomposable. Namely, the conditions for the nerve $L^\bullet$ have to be satisfied up to the labelled join with a simplex.

(3) The above two remarks show that Theorem 0.1 actually yields a complete characterization (in terms of Coxeter matrices or labelled nerves) of those Coxeter systems (W, S) for which W is word hyperbolic and its Gromov boundary ∂W is homeomorphic to the Sierpiński curve or to the Menger curve. We skip the straightforward details of such characterizations.

0.3. Plan of the paper. In Section 1 we collect various (rather numerous) preparatory results, and in Section 2 we provide the main line of the proof of Theorem 0.1 (which is relatively short).

More precisely, here is the structure of Section 1. In Subsection 1.1 we recall the famous topological characterizations of the Sierpiński curve and of the Menger curve, due to Whyburn [16] and to Anderson [1], respectively. In Subsection 1.2 we present a complete characterization (in terms of labelled nerves) of those word hyperbolic Coxeter groups whose Gromov boundary is connected and has no local cut points. As we explain, this characterization is a more or less direct consequence of the results of Bowditch [5], Davis [8, 9], and Mihalik and Tschantz [13]. In Subsection 1.3 we present a useful formula for the topological dimension of the Gromov boundary of a word hyperbolic Coxeter group, which is due to Davis [8] and Bestvina and Mess [3]. In Subsection 1.4 we recall a result of Bourdon and Kleiner [4] which confirms Cannon's conjecture in the framework of word hyperbolic Coxeter groups. In Subsection 1.6 we discuss another result which is implicit in [4], namely that if the Gromov boundary of an indecomposable word hyperbolic Coxeter group is the Sierpiński curve then the nerve of the corresponding Coxeter system is a planar simplicial complex. Since the arguments for this fact provided in [4] are extremely sketchy, we include an extended exposition of its proof. In particular, in this exposition we refer to some auxiliary result from combinatorial group theory, which we state and prove in Subsection 1.5, and for which we could not find an appropriate reference in the literature.

The proof of Theorem 0.1 provided in Section 2 is split into separate parts concerning the Sierpiński curve and the Menger curve. It uses all the preparatory results from Section 1.

1. Preliminaries and preparations. In this section we collect various useful results from the literature (or some more or less direct consequences of such results), and a few other preparatory observations. We will refer to all these in our main arguments in Section 2.

1.1. Characterizations of the Sierpiński curve and of the Menger curve. By a result of Whyburn [16], the Sierpiński curve is the unique metrizable topological space which is compact, connected, locally connected, 1-dimensional, without local cut points and planar. A somewhat similar result of Anderson [1] characterizes the Menger curve as the unique compact metrizable space which is connected, locally connected, 1-dimensional, has no local cut points, and is nowhere planar (nowhere planarity means that no open subset of the space is planar).

By referring to the above characterizations, the second author and B. Kleiner [12] made the following observation.

Proposition 1.1 (M. Kapovich and B. Kleiner [12]). *Let G be a word hyperbolic group, and suppose that its Gromov boundary ∂G is connected, 1-dimensional, and has no local cut points. Then ∂G is homeomorphic either to the Sierpiński curve or to the Menger curve.*

1.2. Connectedness and non-existence of local cut points in the Gromov boundary ∂W. It is a well known fact that if a hyperbolic group is 1-ended then its Gromov boundary is not only connected, but also locally connected (see e.g. [11, Theorem 7.2]). This allows us to discuss existence of local cut points in the boundary. As far as this issue is concerned, we have the following observation, which probably belongs to folklore.

PROPOSITION 1.2. *Let (W, S) be a Coxeter system, and let $L^\bullet$ be its labelled nerve. Suppose also that the group W is infinite and word hyperbolic. Then the Gromov boundary ∂W is connected and has no local cut points iff $L^\bullet$ is unseparable and not a* 3*-cycle.*

Proof. STEP 1. Since connectedness of the boundary ∂W is equivalent to 1-endedness of W, by [9, Theorem 8.7.2] we find that ∂W is connected iff the nerve L is connected and has no separating simplex.

STEP 2. By [9, Theorem 8.7.3], a Coxeter group is 2-ended iff it decomposes as the direct product of its infinite dihedral special subgroup and its finite (possibly trivial) special subgroup. Equivalently, a Coxeter group is 2-ended iff its labelled nerve is either a doubleton or a labelled suspension (as defined in the introduction).

As a consequence of the above, if the group W is 1-ended, non-existence of a separating pair of non-adjacent vertices and of a separating labelled suspension (in $L^\bullet$) means exactly that W does not visually split (in the sense of Mihalik and Tschantz [13]) over a 2-ended subgroup. More precisely, this means that W cannot be expressed as an essential free product of its two special subgroups, amalgamated along a 2-ended special subgroup. It follows from the main result of [13] that non-existence of a separating pair of non-adjacent vertices and of a separating labelled suspension in $L^\bullet$ is equivalent to the fact that W does not split along any 2-ended subgroup.

STEP 3. By a result of Bowditch [5], the Gromov boundary ∂G of a 1-ended hyperbolic group G has no local cut point iff G has no splitting along a 2-ended subgroup and is not a cocompact Fuchsian group. By a result of Davis (see [8, Theorem B] or [9, Theorem 10.9.2]), a Coxeter group is a cocompact Fuchsian group iff either its nerve is a triangulation of S^1 or the group splits as the direct product of a special subgroup with the nerve S^1 and another special subgroup which is finite. It follows from these two results, and from the conclusion of Step 2, that the Gromov boundary ∂W of a 1-ended word hyperbolic Coxeter group W has no local cut point iff its labelled nerve $L^\bullet$ has no separating pair of non-adjacent vertices, no separating labelled suspension, and is not a 3-cycle.

STEP 4. Proposition 1.2 follows by combining the observations of Steps 1 and 3. ∎

1.3. Topological dimension of the Gromov boundary ∂W. Recall that given a finite simplicial complex K we have defined (in the introduction) its *puncture-respecting cohomological dimension* by the formula

$$\operatorname{pcd}(K) := \max\{n : \overline{H}^n(K) \neq 0 \text{ or } \overline{H}^n(K \setminus \sigma) \neq 0 \text{ for some } \sigma \in \mathcal{S}(K)\},$$

where $\mathcal{S}(K)$ is the family of all closed simplices of K. The role of this concept for our considerations is explained by the following observation.

PROPOSITION 1.3. *Let (W, S) be a Coxeter system, and let L be its nerve. Suppose also that the group W is word hyperbolic. Then*

$$\dim \partial W = \operatorname{pcd}(L).$$

Proof. Denote by $\operatorname{vcd}(W)$ the virtual cohomological dimension of W. It follows from results of Mike Davis that $\operatorname{vcd}(W) = \operatorname{pcd}(L) + 1$ (see [9, Corollary 8.5.5]). On the other hand, by a result of M. Bestvina and G. Mess [3], we have $\operatorname{vcd}(W) = \dim \partial W + 1$, hence the proposition. ∎

1.4. Cannon's conjecture for Coxeter groups. The following result has been proved using quite advanced methods by M. Bourdon and B. Kleiner [4], and its short proof as presented below (indicated by M. Davis) has also been outlined in the same paper. We include this short proof for completeness (since our statement, being convenient for our applications, is not identical to that in [4]), and for the reader's convenience.

PROPOSITION 1.4. *Let (W, S) be an indecomposable Coxeter system, and let L be its nerve. Suppose also that the group W is word hyperbolic. Then the following conditions are equivalent:*

(1) $\partial W \cong S^2$,
(2) *L is a triangulation of S^2,*
(3) *W acts properly discontinuously and cocompactly, by isometries, as a reflection group, on the hyperbolic space $\mathbb{H}^3$.*

Proof. We justify the implications (1)⇒(2)⇒(3)⇒(1).

(1)⇒(2) By the result of M. Bestvina and G. Mess [3, Corollary 1.3(c)], if $\partial W \cong S^2$ then W is a virtual Poincaré duality group of dimension 3. By the result of M. Davis [9, Theorem 10.9.2], the nerve L is then a triangulation of S^2 (here we use the assumption of indecomposability).

(2)⇒(3) This implication follows by applying Andreev's theorem (see [2], or [9, Theorem 6.10.2]) to the dual polyhedron of the triangulation.

(3)⇒(1) By the assumptions on W in condition (3), we obviously have $\partial W = \partial \mathbb{H}^3$, and the implication follows from the fact that $\partial \mathbb{H}^3 \cong S^2$. ∎

For the later arguments we only need the implication (1)⇒(2).

1.5. An observation from combinatorial group theory. Let Γ be an arbitrary group and H_i for $1 \leq i \leq n$ be a collection of (not necessarily pairwise distinct) subgroups of Γ. In this subsection we describe two group operations associated to this data, and discuss the relationship between the groups obtained by these operations. This observation (Lemma 1.7 below) will be useful in the argument in Subsection 1.6.

In the next definition we describe the first of the two operations, which the second author and B. Kleiner call the *double* of Γ with respect to the tuple (H_i) (see [12]).

DEFINITION 1.5. Given a group Γ and a finite tuple (H_i) of its subgroups, the *double* $\Gamma \circledast \Gamma$ is the fundamental group $\pi_1\mathcal{G}$ of the graph of groups $\mathcal{G}$ described as follows. The underlying graph of $\mathcal{G}$ consists of two vertices v and v' and n edges $e_1, \ldots, e_n$ each of which has both v and v' as its endpoints. The vertex groups at v and v' are both identified with Γ while the edge group at any edge e_i is identified with H_i. The structure homomorphisms are all taken to be the inclusions.

Let $\Gamma = \langle S \mid R\rangle$, and let $\Gamma' = \langle S' \mid R'\rangle$ be a second copy of Γ (given by the same presentation). Denote by $\mathcal{W}_{H_i}$ the set of words over $S \cup S^{-1}$ that represent elements of the subgroup H_i, and for a word w over $S \cup S^{-1}$ let w' be the word over $S' \cup S'^{-1}$ obtained from w by replacing each letter with its counterpart from $S' \cup S'^{-1}$. Note that (e.g. by [10, Definition 7.3]), the double $\Gamma \circledast \Gamma$ can also be described as follows. Consider an auxilliary group $P = P(\Gamma, (H_i))$ given by the presentation

$$\langle S \sqcup S' \sqcup \{u_i : 1 \le i \le n\} \mid R \cup R' \cup \{h_i u_i = u_i h_i' : 1 \le i \le n,\ h_i \in \mathcal{W}_{H_i}\}\rangle.$$

Then $\Gamma \circledast \Gamma$ is the subgroup of P consisting of all elements p such that there exists an expression $p = w_0 u_{i_1} w_1 u_{i_2}^{-1} w_2 \cdot \ldots \cdot w_{2m-1} u_{i_{2m}}^{-1} w_{2m}$ for some $m \ge 0$ and $1 \le i_k \le n$ and words w_k over $S \cup S^{-1}$ and $S' \cup S'^{-1}$ for even and odd k respectively.

The second of the group operations is given in the following.

DEFINITION 1.6. Given a group $\Gamma = \langle S \mid R\rangle$ and a finite tuple (H_i) of its subgroups, the *mirror double* $\widetilde{\Gamma}$ of the group Γ with respect to (H_i) is the group given by the presentation

$$\widetilde{\Gamma} := \langle S \sqcup \{s_i : 1 \le i \le n\} \mid R \cup \{s_i^2 = 1 : 1 \le i \le n\} \cup \{h_i s_i = s_i h_i : 1 \le i \le n, h_i \in \mathcal{W}_{H_i}\}\rangle.$$

Observe that the mirror double is (up to isomorphism) independent of the presentation of Γ used in the definition above.

LEMMA 1.7. *For each group Γ and any finite tuple (H_i) of its subgroups the double $\Gamma \circledast \Gamma$ is isomorphic to an index 2 subgroup of the mirror double $\widetilde{\Gamma}$.*

REMARK. The concepts of a double $\Gamma \circledast \Gamma$ and a mirror double $\widetilde{\Gamma}$ are well known e.g. in the context of compact hyperbolic manifolds, M, with non-empty totally geodesic boundary ∂M. If we take $\Gamma = \pi_1 M$, and if subgroups $H_i < \Gamma$ correspond to the fundamental groups of boundary components, the double $\Gamma \circledast \Gamma$ is the fundamental group of the double DM of the manifold

M along ∂M. In the same situation, the mirror double $\widetilde{\Gamma}$ corresponds to the fundamental group of the orbifold $\mathcal{O}_M$ with the underlying space M, in which the local groups at the boundary are the groups of order 2 representing geometrically local reflections. Since DM is obviously a degree 2 covering of $\mathcal{O}_M$ (in the orbifold sense), the assertion of Lemma 1.7 is obvious in this situation. The full statement of Lemma 1.7 is just a group-theoretic extension of that observation (which could also be given a geometrical meaning).

Proof of Lemma 1.7. Consider the homomorphism $\rho\colon P \to \widetilde{\Gamma}$ given by $\rho(s) = \rho(s') = s$ for each $s \in S$, and $\rho(u_i) = s_i$ for each $1 \le i \le n$. Consider also the homomorphism $\sigma\colon \widetilde{\Gamma} \to \mathbb{Z}_2$ given by $\sigma(s) = 0$ for $s \in S$, and $\sigma(s_i) = 1$ for $1 \le i \le n$. It suffices to show that ρ restricts to an isomorphism between $\Gamma \circledast \Gamma$ and $\ker\sigma$, which is an index 2 subgroup of $\widetilde{\Gamma}$. It is easy to check that $\rho(\Gamma \circledast \Gamma) = \ker\sigma$, so it remains to show that $\rho|_{\Gamma\circledast\Gamma}$ is injective. To this end we introduce the following lift function $\ell\colon \ker\sigma \to \Gamma \circledast \Gamma$. For an element $\xi \in \ker\sigma$ and any of its expressions of the form

$$w_0 s_{i_1}^{\epsilon_1} w_1 s_{i_2}^{\epsilon_2} \cdot \ldots \cdot w_{2m-1} s_{i_{2m}}^{\epsilon_{2m}} w_{2m}$$

for some (possibly empty) words w_i over the alphabet $S \cup S^{-1}$, $\epsilon_j \in \{-1, 1\}$ and $1 \le i_j \le n$, put $\ell(\xi) := w_0 u_{i_1} w_1' u_{i_2}^{-1} \cdot \ldots \cdot w_{2m-1}' u_{i_{2m}}^{-1} w_{2m}$. The map ℓ is well defined, since it is easy to check that for each word U and each elementary operation consisting in inserting at an arbitrary place in U (or deleting) a subword of the form $a^{-1}a$ for some letter a, or a relator (in $\widetilde{\Gamma}$) or inverse of such, resulting in a word $\hat{U}$, the words representing $\ell(U)$ and $\ell(\hat{U})$ in the definition of ℓ differ by an analogous elementary operation (in P). Moreover, since we then clearly have $\ell \circ \rho|_{\Gamma\circledast\Gamma} = \mathrm{id}_{\Gamma\circledast\Gamma}$, we conclude that $\rho|_{\Gamma\circledast\Gamma}$ is injective, hence the lemma. ∎

1.6. Planarity of nerves. We recall the following rather easy observation from [15].

LEMMA 1.8 (J. Świątkowski, [15, Lemma 1.3]). *If the nerve L of a word hyperbolic Coxeter group W is a planar complex then the Gromov boundary ∂W is a planar topological space.*

The converse implication is not true in general [6], but it does hold in an important special case. This is the contents of the next result, which appears implicitly as Corollary 7.5 in [4]. The proof given below is an expansion of the rather sketchy argument provided in [4].

PROPOSITION 1.9. *Let (W, S) be an indecomposable Coxeter system such that the group W is word hyperbolic. If the Gromov boundary ∂W is homeomorphic to the Sierpiński curve then the nerve L of (W, S) is a planar simplicial complex.*

Proof. We will embed the group W, as a special subgroup, in some larger indecomposable and word hyperbolic Coxeter group $\widetilde{W}$ such that $\partial\widetilde{W} \cong S^2$. The assertion will then follow from the implication (1)⇒(2) in Proposition 1.4.

We start by recalling some facts established in the paper by the second author and B. Kleiner [12]. First, the Sierpiński curve contains the family of topologically well distinguished pairwise disjoint subsets homeomorphic to S^1, called *peripheral circles*. Moreover, in its action on ∂W the group W maps peripheral circles to peripheral circles. A setwise stabilizer of each peripheral circle in ∂W, called a *peripheral subgroup* of W, is a quasi-convex subgroup of W for which the circle is its limit set, and consequently each such stabilizer is a cocompact Fuchsian group. The action of W on the family of peripheral circles in ∂W has finitely many orbits, and thus we have finitely many conjugacy classes of peripheral subgroups in W.

Claim. *Each peripheral subgroup of W is a conjugate of some special subgroup of W.*

To prove the Claim we need some terminology and notation of [4, Section 5.1]. For a generator $s \in S$, the *wall* M_s is the set of setwise s-stabilized open edges of $\mathrm{Cay}(W,S)$ (the Cayley graph of W with respect to the set of generators S). Then $\mathrm{Cay}(W,S) \setminus M_s$ consists of two connected components $H_-(M_s)$ and $H_+(M_s)$. For a generator $s \in S$ and for an arbitrary element $g \in W$ we consider the reflection $r := gsg^{-1}$, its wall $M_r := gM_s$ and its components $H_-(M_r)$ and $H_+(M_r)$ of $\mathrm{Cay}(W,S) \setminus M_r$. The components are closed and convex subsets of $\mathrm{Cay}(W,S)$ and $\partial H_-(M_r) \cup \partial H_+(M_r) = \partial W$, $\partial H_-(M_r) \cap \partial H_+(M_r) = \partial M_r$ and r pointwise stabilizes ∂M_r.

Proof of Claim. In view of [4, Definition 5.4 and Theorem 5.5] it suffices to show that for each peripheral circle F and each reflection r such that $\partial H_-(M_r) \cap F$ and $\partial H_+(M_r) \cap F$ are non-empty, F is setwise stabilized by r. Since $(\partial H_-(M_r) \cap F) \cup (\partial H_+(M_r) \cap F) = \partial W \cap F = F$, by connectedness of $F \cong S^1$, we have $\emptyset \neq (H_-(\partial M_r) \cap F) \cap (H_+(\partial M_r) \cap F) = \partial M_r \cap F$. Since ∂M_r is pointwise stabilized by r, we have $rF \cap F \neq \emptyset$, and finally $rF = F$ by the fact that each element of W maps peripheral circles to peripheral circles.

Coming back to the proof of Proposition 1.9, denote by H_i, $1 \leq i \leq n$, a set of representatives of the conjugacy classes of peripheral subgroups of W consisting of special subgroups of W. For each $1 \leq i \leq n$, denote by L_i the nerve of H_i, and view it as a subcomplex of the nerve L of W. We will discuss below the double $W \circledast W$ and the mirror double $\widetilde{W}$ of W with respect to the tuple (H_i) (see Subsection 1.5). As shown in [12], the double $W \circledast W$ is a hyperbolic group and its Gromov boundary is homeomorphic to S^2. Observe also that the mirror double $\widetilde{W}$ is (isomorphic to) a Coxeter group with nerve $\widetilde{L}$ obtained from the nerve L of W by adding a simplicial cone over each of the subcomplexes L_i. Moreover, since each H_i is a proper special subgroup of W, indecomposability of W implies indecomposability of $\widetilde{W}$. By Lemma 1.7, the group $\widetilde{W}$ contains $W \circledast W$ as a subgroup of index 2, and hence it is also word hyperbolic and its Gromov boundary is homeomorphic to S^2. By Proposition 1.4, $\widetilde{L}$ is then a triangulation of S^2. Since L is clearly a proper subcomplex of $\widetilde{L}$, it is necessarily planar, which completes the proof of Proposition 1.9. ∎

2. Proof of the main theorem

2.1. Sierpiński curve boundary. In this subsection we prove part (1) of Theorem 0.1.

Proof of the implication $\Rightarrow$. Suppose that ∂W is homeomorphic to the Sierpiński curve. Then, in view of the fact that the Sierpiński curve is connected and has no local cut points, it follows from Proposition 1.2 that $L^\bullet$ is unseparable and not a 3-cycle. Moreover, by Proposition 1.9, L is then a planar simplicial complex, which completes the proof.

Proof of the implication $\Leftarrow$. As any Gromov boundary of a hyperbolic group, ∂W is a compact metrizable space. Since L is planar, it follows from Lemma 1.8 that ∂W is a planar space. Since $L^\bullet$ is unseparable and not a 3-cycle, it follows from Proposition 1.2 that ∂W is connected, locally connected, and has no local cut point. Finally, it is not hard to see that since L is planar, connected, has no separating simplex, and does not coincide with a single simplex, its puncture-respecting cohomological dimension $\mathrm{pcd}(L)$ is equal to 1. Consequently, due to Proposition 1.3, ∂W has topological dimension 1. Thus, by Whyburn's characterization recalled in Subsection 1.1, ∂W is homeomorphic to the Sierpiński curve, as required.

2.2. Menger curve boundary. We now pass to the proof of part (2) of Theorem 0.1.

Proof of the implication $\Rightarrow$. Suppose that ∂W is homeomorphic to the Menger curve. Then, in view of the fact that the Menger curve is connected and has no local cut points, it follows from Proposition 1.2 that $L^\bullet$ is unseparable. Since the Menger curve has topological dimension 1, it follows from Proposition 1.3 that $\mathrm{pcd}(L) = 1$. Since the Menger curve is not planar, it follows from Lemma 1.8 that L is not planar either and this completes the proof of the first implication.

Proof of the implication $\Leftarrow$. The boundary ∂W is obviously a compact metrizable space. Since $L^\bullet$ is not planar, is not a 3-cycle, and is unseparable, it follows from Proposition 1.2 that ∂W is connected, locally connected, and has no local cut point. Since $\mathrm{pcd}(L) = 1$, it follows that ∂W has topological dimension 1. In view of the above properties, it follows from Proposition 1.1 that ∂W is homeomorphic either to the Sierpiński curve or to the Menger curve. However, since L is not planar, Proposition 1.9 shows that ∂W cannot be homeomorphic to the Sierpiński curve. Consequently, it must be homeomorphic to the Menger curve, as required.

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Daniel Danielski, Jacek Świątkowski
Instytut Matematyczny
Uniwersytet Wrocławski
50-384 Wrocław, Poland
E-mail: danielski@math.uni.wroc.pl
swiatkow@math.uni.wroc.pl

Michael Kapovich
Department of Mathematics
University of California, Davis
Davis, CA 95616, USA
E-mail: kapovich@math.ucdavis.edu