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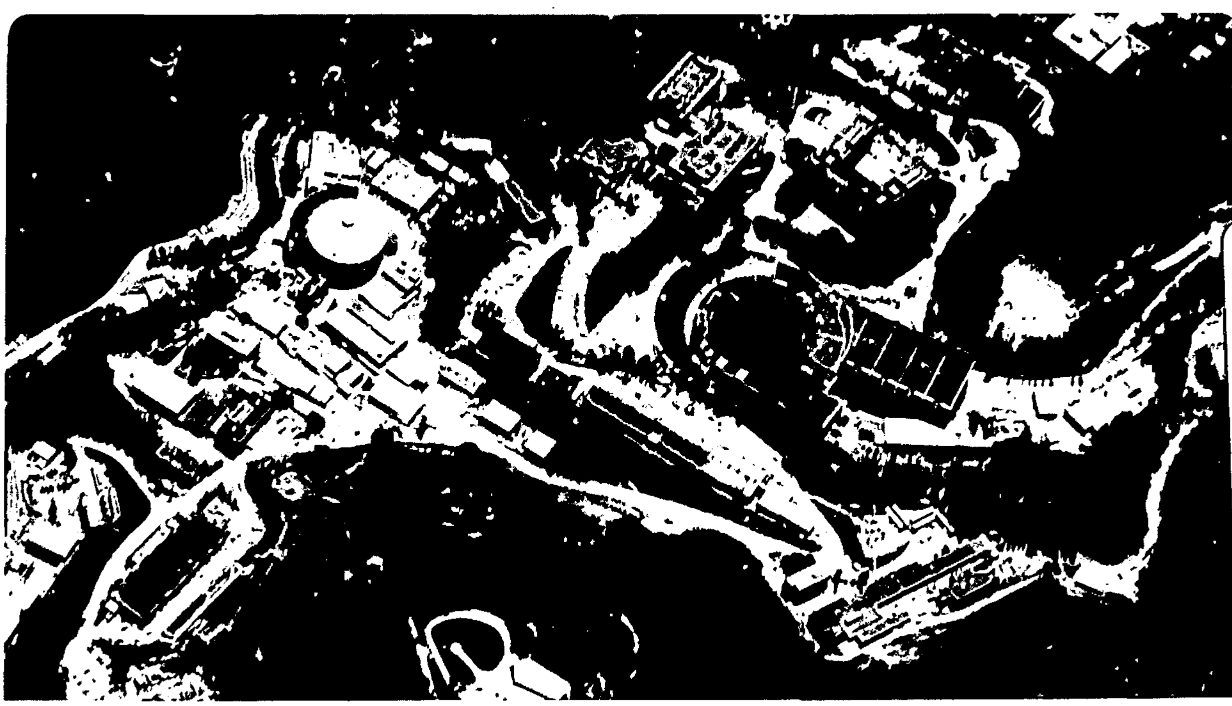

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Geometric Continuum Regularization of Quantum Field Theory[*][†]

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Abstract

An overview of the continuum regularization program is given. The program is traced from its roots in stochastic quantization, with emphasis on the examples of regularized gauge theory, the regularized general nonlinear sigma model and regularized quantum gravity. In its coordinate-invariant form, the regularization is seen as entirely geometric: only the supermetric on field deformations is regularized, and the prescription provides universal nonperturbative invariant continuum regularization across all quantum field theory.

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Contents

1. Introduction

The continuum regularization program [1–13] is now complete, so this conference provides an ideal opportunity to summarize our results and put the program in some perspective. Earlier partial reviews of the program are found in [14,15].

The starting point of the program is the observation that stochastic quantization [16-18] sees to the heart of the ultraviolet problem. The result of the program is a universal geometric prescription for nonperturbative invariant continuum regularization of all quantum field theory. In spirit, the regulator might be compared to lattice gauge theory and lattice gravity, except that we preserve all relevant continuum symmetries including coordinate invariance. In fact, the regulator is interpreted as an invariant all–order covariant derivative or proper–time regularization.

In its final coordinate–invariant form, the prescription is seen as entirely geometric, with all regularization contained in regularized DeWitt superstructures [19] on the space of field deformations. Indeed, as will become clear in these lectures, completion of the program has given birth to regularized (infinite–dimensional) supergeometry.

After a brief review of relevant background in Section 2, I will follow the historical development of the program, beginning with the regularization of the scalar prototype [3] (Section 3), gauge theory [1,2,4,5, 12] (Section 4), gauge theory with fermions [6] and superfield supersymmetry [7]. Section 4 also reviews the regularized Migdal–Makeenko equation [13] which is the projection of regularized large N gauge theory onto loop space.

In Section 5, I introduce the universal geometric form of the regularization [8–10], which begins in phase space [9,11]. The previous regularizations of the program [1–7] are special cases in flat space and flat superspace. As examples, I discuss the following topics.

1. Nonperturbative geometric characterization of the Weyl anomaly in the presence of the regulator for the general two dimensional non–linear sigma model (Section 5.5). The anomaly is the invariant trace of the regularized supermetric [9].

2. Regularized integration of the momenta [9] (Section 6), which is the rigorous passage from regularized phase-space to regularized coordinate-space formulations. Regularized coordinate-space supergeometry is generated automatically in the passage.

3. Regularized Euclidean quantum gravity [8–10] is discussed as an explicit example in Section 7. Here I also review the Schwinger–Dyson stabilization mechanism [8–10] and its stochastic counterpart [20, 8–10], which allows the treatment of bottomless actions such as Euclidean gravity.

2. Background

2.1 Stochastic quantization

The first stochastic process was studied by Langevin [16] in 1908. Reviews of the field through 1981 are found in [17]. The subject was brought to the attention of particle physicists in 1981 by Parisi and Wu [18], whose proposal of fifth-time stochastic quantization is an elegant covariant generalization of the real-time stochastic quantization studied earlier by Nelson [21].

The formal (unregularized) coordinate-invariant stochastic processes [8–10] invented for use in the regularization program are generalizations of the following early work: processes on group manifolds [22], finite-dimensional coordinate-invariant processes on arbitrary manifolds [23–25], and scalar phase-space processes [26] on flat space and flat superspace.

2.2 Advantages of stochastic quantization

Being a change of variable to a Gaussian noise η, stochastic quantization is equivalent to action and Hamiltonian formulations. The important physical question has always been to find the advantages of the stochastic methods. In fact, the approach has given birth to a number of new ideas which are by no means obvious in more conventional quantizations. Among these, I mention Zwanziger's nonperturbative gauge-fixing [27, 28], large N quenching [29, 30], large N master fields [30], stochastic stabilization [31], the QCD_4 maps [32] which run in ordinary Euclidean time, and numerical applications in lattice gauge theory [33]. The regularization program [1–15] itself stands high on the list since stochastic quantization sees into the ultraviolet problem (see Section

3) in a way that the conventional quantizations cannot.

2.3. Zwanziger's gauge fixing and the flow gauges

I remark in particular on Zwanziger's stochastic ghostless d-dimensional gauge-fixing [27–28], since we employ it naturally in the program. The procedure is somewhat mysterious from the d-dimensional action point of view because it is in fact a Faddeev–Popov "flow gauge" fixing [28]

$$A^a_{d+1} = \alpha^{-1}\partial \cdot A^a_d, \qquad a = 1, \ldots, dim\ g \tag{2.3.1}$$

in a $(d+1)$–dimensional action formulation of the d–dimensional gauge theory. Not surprisingly, the flow gauges are also ghostless and infrared soft. The flow–gauge equivalence automatically [28] provides ordinary $(d+1)$–dimensional Slavnov–Taylor [34] and BRST identities [35] for Zwanziger's gauge–fixing of a d-dimensional theory. The flow gauges were independently rediscovered in [36], without discussion of the stochastic connection.

2.4. Early regulator proposals

Early ideas about stochastic regularization are found in [37–42]. I note in particular that the regulator of Niemi and Wijewardhana [38] is a Parisi–Sourlas [43] analogue of our scalar prototype in Section 3. The non-Markovian regularization of [39] is incompatible with Zwanziger's gauge–fixing [40]. The regularization of Doering [42] is identical to the scalar prototype.

2.5. Five formulations of quantum field theory.

A quantum theory is defined by an action (S) and a supermetric ($\mathcal{G}$). For example

$$Z = \int (dx)\sqrt{\mathcal{G}}e^{-S} \tag{2.5.1}$$

is the action formulation. The variable changes of stochastic quantization have explored the following system of five equivalent formulations

$$\begin{array}{ccccc} & \text{Action} & \leftrightarrow & \text{Schwinger–Dyson} & \\ \frac{d\uparrow}{(d+1)\downarrow} & \updownarrow & & \updownarrow & \updownarrow \\ & \text{stochastic} & \leftrightarrow & \text{Fokker–Planck} \leftrightarrow & \text{Fifth action} \end{array} \tag{2.5.2}$$

where the d-dimensional formulations above the dashed line correspond to the $(d+1)$-dimensional formulations below, which include the extra Markov time t (also called fictitious or fifth time) as the $(d+1)st$ dimension The stochastic process is the Nicolai map of the fifth action formulation.* Early work on Nicolai maps as stochastic processes include [43-46, 24, 32]. The interpretation in (2.5.2) is d=number of space-time dimensions. To obtain d=number of spatial dimensions and t = ordinary time, read instead

$$\begin{aligned} e^{-S} &\rightarrow \psi_0^2 = (\text{ground state wave function})^2 \\ \text{action} &\rightarrow \text{Hamiltonian} \\ \text{fifth action} &\rightarrow \text{action} \end{aligned} \tag{2.5.3}$$

so that the real time stochastic process is the Nicolai map of the ordinary d-dimensional Euclidean action. This is the older interpretation of Feynman and of Nelson [21].

Given a d-dimensional theory, our scheme regularizes only [1-3]

1. the stochastic formulation in $(d+1)$-dimensions
2. the Schwinger-Dyson formulation in d-dimensions.

This is a consequence of the no-go theorem of Lee and Zinn-Justin [47], who showed in 1972 that covariant-derivative regularization of the action fails at the one-loop level (gauge-invariant improvement of the propagator cancels against growth of the vertices): although no divergences are present in the regularized stochastic or Schwinger-Dyson formulations, any attempt to force the scheme into an action formulation, including Fokker-Planck and fifth-actions, will require divergent action counterterms at fixed cutoff [2]. The final interpretation of the no-go theorem is that action formulations are an unnatural language for covariant derivative regularization.

*The question of supersymmetry and stochastic processes arises at the fifth-action level (or the action level for real time processes). In fact, there is no finite-time supersymmetry (or fermions) associated to a single stochastic process [45-46, 24, 32] since the determinants are defined with retarded boundary condition: Finite-time supersymmetries are generally associated to a *family* of distinct stochastic processes [24], although an *infinite* time supersymmetry may be present for a single equilibrating process.

3. The Scalar Prototype

The formal stochastic process in $(d+1)$-dimensions

$$\dot{\phi}_\eta(x,t) = -\frac{\delta S}{\delta\phi}(x,t) + \sqrt{\hbar}\eta(x,t) \tag{3.1a}$$

$$\langle\eta(x,t)\eta(y,\tau)\rangle = 2\delta^d(x-y)\delta(t-\tau) \tag{3.1b}$$

corresponds to the scalar theory $Z = \int \mathcal{D}\phi \exp(-S)$ in d-dimensions. We consider instead the Markovian-regularized process [2]

$$\dot{\phi}_\eta(xt) = -\frac{\delta S}{\delta\phi}(xt) + \int (dx)\sqrt{\hbar}R(\Box)_{xy}\eta(y,t) \tag{3.2}$$

where the regulator R may be taken as [2,5]

$$R = \begin{cases} R_n = (1-\frac{\Box}{\Lambda^2})^{-n} & \text{(power law)} \quad (3.3a) \\ R_H = \exp(\Box/\Lambda^2) & \text{(heat kernel)} \quad (3.3b) \end{cases}$$

and $\Box = \partial_\mu\partial_\mu$ is ordinary laplacian. The basic idea of the regularization scheme [1,2,14] is seen by choosing an interaction and expanding ϕ_η into the usual Langevin tree graphs, each leg of which now ends in a regularized noise factor $\sqrt{\hbar}R\eta$. The loops of the theory are formed by contracting the white noise η according to (3.1b), so that every closed loop contains at least one power of the regulator squared in the form $\hbar R^2$. It is clear that the stochastic quantization has gone to the heart of the ultraviolet problem, and that our regularization of the noise will render the theory finite for proper choices of the regulator R.

Detailed results [2] are as follows. For any polynomial interaction in d-dimensions, the finite-time Green functions of the theory are ultraviolet-finite to all orders when we choose power law regularization

$$R_n: \qquad n \geq \left[\frac{d+2}{4}\right] \tag{3.4}$$

where $[x]$ is the greatest integer $\leq x$. For the heat-kernel regulator R_H, all finite-derivative composite operators are regularized to all orders for any interaction in arbitrary dimension.

The regularized d-dimensional Schwinger-Dyson(SD) system [2]

$$0 = \langle LF[\phi]\rangle \tag{3.5a}$$

$$L = -\int (dx) \frac{\delta S}{\delta\phi(x)} \frac{\delta}{\delta\phi(x)} + \hbar\mathbf{\Delta} \tag{3.5b}$$

$$\mathbf{\Delta} \equiv \int (dx)(dy) R^2_{xy}(\Box) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \tag{3.5c}$$

is equivalent to the stochastic formulation (3.2) under assumption of equilibration . Here F is an arbitrary functional of d-dimensional ϕ and the structure $\mathbf{\Delta}$ (in the Schwinger–Dyson operator L) is a regularized super–Laplacian. In fact, we generally prefer the SD formulation, especially since it is often well-defined when the action is bottomless (see section 7).

The regularized Langevin diagrams of (3.2), and the regularized Schwinger–Dyson diagrams of (3.5), add to Feynman diagrams in the formal regulator limit

$$R \underset{\Lambda\to\infty}{\longrightarrow} 1 \tag{3.6}$$

but the Langevin and Schwinger–Dyson diagrams are quite different in the presence of the regulator [2], and in no sense do they correspond to "regularized Feynman diagrams". This is another aspect of the non-action character of the regularization.

4. Gauge Theory

4.1. Regularized Parisi–Wu equation

The regularized form of the Parisi–Wu process is [1,3,5]

$$\dot{A}^a_\mu(x,t) = -\frac{\delta S_{YM}}{\delta A^a_\mu}(x,t) + D^{ab}_\mu Z^b(x,t) + \int (dy) R^{ab}_{xy}(\Delta)\eta^b_\mu(y,t) \tag{4.1.1a}$$

$$\langle \eta^a_\mu(x,t)\eta^b_\nu(y,\tau)\rangle = 2\delta^{ab}\delta_{\mu\nu}\delta^d(x-y)\delta(t-\tau) \tag{4.1.1b}$$

where S_{YM} is the Euclidean Yang–Mills action in d–dimensions and $Z^a = \alpha^{-1}\partial\cdot A^a, a = 1\ldots \dim g$, is Zwanziger's gauge–fixing. To maintain gauge covariance, the regulator [1,3,5]

$$R(\Delta) = \begin{cases} R_n & = (1 - \frac{\Delta}{\Lambda^2})^{-n} \quad (4.1.2a) \\ R_H & = \exp(\Delta/\Lambda^2) \quad (4.1.2b) \end{cases}$$

is taken as a function of the gauge–covariant Laplacian $\Delta = D_\mu(A)D_\mu(A)$, with D the gauge–covariant derivative.

The field-dependence of the regulator introduces two new features beyond the scalar prototype. In the first place, expansion of the regulator in powers of the field is easily organized into two new regulator vertices [1,3,5] corresponding to one – and two – gluon emission from regulator strings . The second feature is the need to choose a stochastic calculus [17] which corresponds to the value of the contractions

$$\overbrace{R(\Delta(A))\eta} = \begin{cases} & 0, \ \text{Ito calculus} \\ \neq & 0, \ \text{otherwise} \end{cases} \tag{4.1.3}$$

between the noise and its regulator prefactor.

In fact, the regularization is gauge–invariant for any choice of stochastic calculus, so we work with an arbitrary choice. This generates the one–parameter γ–family of invariant regularized Schwinger–Dyson systems [1, 3,5]

$$0 = \langle LF[A]\rangle \tag{4.1.4a}$$

$$L = \int (dx) \left[-\frac{\delta S_{YM}}{\delta A^a_\mu(x)} + D^{ab}_\mu Z^b(x) \right] \frac{\delta}{\delta A^a_\mu(x)} + \mathbf{\Delta}(\gamma) \tag{4.1.4b}$$

$$\begin{aligned} \mathbf{\Delta}(\gamma) &= \int (dx)(dy) [R^2(\Delta)]^{ab}_{xy} \frac{\delta^2}{\delta A^b_\mu(y)\delta A^a_\mu(x)} \\ &\quad + \gamma \int (dx)(dy)(dz) R^{bc}_{yz}(\Delta) \frac{\delta R^{ba}_{yx}(\Delta)}{\delta A^c_\mu(z)} \frac{\delta}{\delta A^a_\mu(x)} \end{aligned} \tag{4.1.4.c}$$

which exhibits the stochastic ambiguity in the regularized gauge-invariant super-Laplacian $\mathbf{\Delta}(\gamma)$ ($\gamma = 0$ is Ito and $\gamma = 1$ is Stratonovich). Detailed analysis [1,3] of the Langevin and Schwinger–Dyson diagrams verifies finiteness of the gauge-field Green functions to all orders when

$$n \geq \left[\frac{d+1}{2}\right], \quad \gamma \neq 0 \tag{4.1.5a}$$

$$n \geq \left[\frac{d+3}{4}\right], \quad \gamma = 0 \tag{4.1.5b}$$

is satisfied for the power-law regulator R_n. The result (4.1.5) shows that the contractions (4.1.3) are the most singular diagrams in the theory. The heat-kernel regulator R_H regularizes all finite-derivative composite operators uniformly across all (d, γ) [5].

The simplest regularization chooses Ito calculus ($\gamma = 0$) and the heat kernel regulator R_H. Then only three diagrams, one of which contains a regulator vertex, contribute to the photon mass at the one-loop level [5]. The cancellation of these contributions, so that the photon remains massless, is a satisfying check of gauge invariance of the regulator in all dimensions at once.

Perturbative renormalization of the regularized scalar prototype and gauge theory are discussed in [1,4]. Only the standard action counterterms, and a Zwanziger counterterm, are required. Regularized gauge theory with fermions is studied in [6], and the standard background gauge-field anomalies are obtained. Regularized superfield supersymmetry is discussed in [7].

4.2. Regularized Migdal–Makeenko equation.

The Migdal–Makeenko equation [48, 49] is a description of large N gauge theory as the invariant dynamics of Wilson loops. In fact, the first-order [48] and second-order [49] versions of this equation correspond respectively to the first- and second- order unregularized Schwinger-Dyson systems

$$0 = \left\langle \left(-\frac{\delta S}{\delta A_\mu^a(x)} + \frac{\delta}{\delta A_\mu^a(x)} \right) W[A;C] \right\rangle \tag{4.2.1a}$$

$$0 = \left\langle \int (dx) \left(-\frac{\delta S}{\delta A_\mu^a(x)} + \frac{\delta}{\delta A_\mu^a(x)} \right) \frac{\delta}{\delta A_\mu^a(x)} W[A;C] \right\rangle \tag{4.2.1b}$$

where $W[A;C]$ is any Wilson loop. The program regularizes only the second-order system (4.2.1b), which is the formal $\Lambda \to \infty$ limit of (4.1.4) with $F = W$. It follows that we will obtain a regularized form of the second-order Migdal-Makeenko equation [13] by projecting the regularized Schwinger–Dyson equations (4.1.4) of gauge theory onto loop space. We choose the heat kernel regulator and the Ito form

$$\mathbf{\Delta}(\gamma = 0) = \int (dx)(dy) [R_H^2(\Delta)]_{xy}^{ab} \frac{\delta^2}{\delta A_\mu^b(y) \delta A_\mu^a(x)} \tag{4.2.2}$$

of the regularized super-Laplacian for simplicity.

The crucial identity is the representation of the $SU(N)$ heat-kernel regulator $R_H = \exp(\Delta/\Lambda^2)$ as a Gaussian integral over particle variables [50, 48]

$$(R_H^2)_{xy}^{ab} = \int_{r(0)=x}^{r(\epsilon)=y} \mathcal{D}r \; e^{-1/2 \int_0^\epsilon d\tau \; \dot{r}^2} Tr[t^a U(r_{xy}) t^b U(r_{yx})] \tag{4.2.3a}$$

$$U(r_{xy}) = P e^{ig \int_0^\epsilon d\tau \dot{r}_\mu(\tau) A_\mu(r)} \tag{4.2.3b}$$

where $r_\mu(\tau)$, the regulator path, moves from x to y in regulator proper time $\epsilon(\Lambda) = 4/\Lambda^2$. The conventional measure is implied in (4.2.3) so that, for example

$$\int_{r(0)=x}^{r(\epsilon)=y} \mathcal{D}r \; e^{-1/2 \int_0^\epsilon d\tau \; \dot{r}^2} Tr(t^a t^b) = \delta^{ab} (e^{2\Box/\Lambda^2})_{xy} \tag{4.2.4}$$

reproduces the zeroth-order regulator. The general form (4.2.3) emphasizes that the full regulator may be interpreted as an invariant nonperturbative generalization of conventional one-loop proper-time regularization.

The regularized second-order Schwinger–Dyson system (4.1.4) is first written in terms of two Laplacians,

$$0 = \langle (\Delta_L + \mathbf{\Delta}) W[A;C] \rangle \tag{4.2.5}$$

where $\mathbf{\Delta}$ is the regularized super-Laplacian (4.2.2) on the space of gauge fields and

$$\Delta_L \equiv \int_0^1 d\sigma \int_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta^2}{\delta x_\mu(\sigma') \delta x_\mu(\sigma)} \tag{4.2.6}$$

is the Laplacian on loop-space [51, 49]. Then standard large N manipulations of $\mathbf{\Delta} W$ with (4.2.3) result in the form [13]

$$\Delta_L W[C] = \lambda \oint_C dx_\mu \oint_C dy_\mu \int_{r(0)=x}^{r(\epsilon)=y} \mathcal{D}r \; e^{-1/2 \int_0^\epsilon d\tau \; \dot{r}^2(\tau)} W[C_{xy} r_{yx}] W[C_{yx} r_{xy}] \tag{4.2.7a}$$

$$W[C] \equiv \langle W[A;C] \rangle, \ \lambda \equiv g^2 N \tag{4.2.7b}$$

which is an ultraviolet regularized version of the second-order Migdal-Makeenko equation. The infamous loop-crossing at $x = y$ is smoothed over a domain $\epsilon(\Lambda)$ of regulator proper time. This form of regularized gauge theory may be useful for nonperturbative analysis.

5. The Geometry of Continuum Regularization

5.1 Phase space

The advance, coordinate-invariant regularization [8-10], which I am going to describe in this section, is a giant step from the previous examples since it allows us to see that the regularization is entirely geometric, and in fact universal across all quantum field theory. I should say before beginning that my original attempts to extend the program in this direction were frustrated by certain divergences of formal (unregularized) coordinate-invariant stochastic processes in coordinate-space. It was to avoid these divergences that I turned to phase space [9-11], which offers the following advantages.

1. The formal coordinate-invariant phase-space processes [9], being free of such divergences, are easily regularized as above, thereby extending the regularization prescription to all theories with Liouville measure [9].

2. Regularized integration of the momenta [9,11] provides a rigorous path back to regularized coordinate-invariant coordinate-space formulations. [8-11], thus resolving the original difficulties in coordinate space.

3. The regularized phase-space processes require a minimum of regularized supergeometry, most of which emerges automatically during the transition to regularized coordinate space.

Another point of note is that, in contrast to coordinate-space processes, the phase-space processes are in fact stochastically unambiguous [9, 11], that is independent of the choice of stochastic calculus.

5.2. Coordinate-invariant phase-space processes

The formal theories we wish to regularize are phase-space functional integrals of the form

$$Z = \int \mathcal{D}\omega e^{-H[\phi,\pi]}, \quad \mathcal{D}\omega = \prod_{\xi} d\pi_M(\xi) \wedge d\phi^M(\xi) \tag{5.2.1}$$

where H is a general phase-space action and $\mathcal{D}\omega$ is Liouville measure on a set of generic (field) coordinates $\phi^M(\xi)$ and conjugate momenta $\pi_M(\xi)$. Here ξ^m are d-dimensional spacetime coordinates and $\{M\}$ may include tensor indices; for example, $\phi^M(\xi) = g_{mn}(\xi)$ is the metric on spacetime when we study regularized gravity.

As a matter of orientation, the class of theories (5.2.1) contains at least two important categories:

1. Non-covariant formulations, including real-time Hamiltonian and constrained Hamiltonian systems.

2. Covariant formulations with DeWitt supermetric.

In fact, the program regularizes both [9], but explicit examples have been studied only in the simpler second category, for which

$$H = \frac{1}{2}\int (d\xi)\pi_M \mathcal{G}^{MN}(\phi)\pi_N + S[\phi] \tag{5.2.2}$$

where $\mathcal{G}_{MN}$ is the DeWitt supermetric [19]. These theories correspond to the coordinate-space functional integrals

$$Z = \int \mathcal{D}\phi\mathcal{E}[\phi]\, e^{-S[\phi]}, \quad \mathcal{E}[\phi] = \prod_\xi \det{}^{\frac{1}{2}}\mathcal{G}_{MN}(\phi(\xi)) \tag{5.2.3}$$

on formal integration of the momenta. The action $S[\phi]$ may be chosen, for example, as

$$S = \frac{1}{2}\int (d\xi)\mathcal{G}_{MN} g^{mn}\partial_m\phi^M\partial_n\phi^N + \ldots \tag{5.2.4}$$

for the general non-linear sigma model, or the Einstein-Hilbert action for gravity as in section 7.

In what follows, I assume the existence, for all theories of the form (5.2.1), of an inner product on field deformations

$$\|\delta\phi\|^2 = \int (d\xi)\mathcal{G}_{MN}(\phi(\xi))\delta\phi^N(\xi)\delta\phi^M(\xi) \tag{5.2.5a}$$

$$\mathcal{G}_{MN}(\phi(\xi)) = \mathcal{E}_{MA}(\phi(\xi))\mathcal{E}_{NA}(\phi(\xi)) \tag{5.2.5b}$$

where $\mathcal{G}_{MN}$ is the DeWitt supermetric and $\mathcal{E}_{MA}$ is its supervielbein. In particular, I will assume that the superstructures are covariant under the two types of coordinate-invariance

$$\xi \to \bar{\xi}(\xi) \quad \text{(Einstein invariance)} \tag{5.2.6a}$$

$$\phi(\xi) \to \bar{\phi}(\phi(\xi)) \quad \text{(reparametrization invariance)} \tag{5.2.6b}$$

that I wish to preserve in the regularization. In fact the superstructures are not uniquely determined by this requirement, so that, for example,

$$\mathcal{G}_{MN} = \mathcal{G}^{mn;rs} = e\left[\frac{1}{2}(g^{mr}g^{ns} + g^{ms}g^{nr}) + \gamma g^{mn}g^{rs}\right] \tag{5.2.7b}$$

is a one-parameter γ-family of supermetrics on deformations of the metric in the reparametrization frame with g_{mn} a tensor.

The general formal coordinate-invariant phase-space processes which correspond to the theories (5.2.1) are then [9,11]

$$\dot{\pi}_M = -\frac{\delta H}{\delta\phi^M} - \beta\mathcal{G}_{MN}\frac{\delta H}{\delta\pi_N} + \sqrt{\beta}\mathcal{E}_{MA}\eta_A \tag{5.2.8a}$$

$$\dot{\phi}^M = \frac{\delta H}{\delta\pi_M} \tag{5.2.8b}$$

$$\langle\eta_A(\xi,t)\eta_B(\xi',t')\rangle = 2\delta_{AB}\delta^d(\xi-\xi')\delta(t-t') \tag{5.2.8c}$$

under assumption of equilibration. In fact, the processes equilibrate to (5.2.1) as expected for bounded H, the rate of equilibration being controlled by the positive parameter β. The DeWitt superstructures $\mathcal{G}_{MN}$ and $\mathcal{E}_{MA}$ appear in (5.2.8) as convenient auxiliary quantities or covariant kernels, independent of the specific structure of the phase-space action H.

5.3 Uniqueness of the phase-space stochastic calculus.

A remarkable property of the phase-space (second-order) processes (5.2.8) is that they are *stochastically unambiguous* [9,11]

$$\overbrace{\mathcal{E}_{MA}(\phi)\eta_A} = 0, \tag{5.3.1}$$

that is, independent of the choice of stochastic calculus. This phenomenon, and the contrasting ambiguity of first-order (Parisi-Wu) stochastic processes, has been understood diagrammatically [11] in terms of the response of first and second order retarded ($\phi(t) = 0$ for $t < 0$) mechanical systems

$$\dot{\phi}_I(t) = \delta(t) \quad \text{(Parisi-Wu)} \tag{5.3.2a}$$

$$\ddot{\phi}_{II}(t) = \delta(t) \quad \text{(phase-space)} \tag{5.3.2b}$$

to an impulse at the origin. When interpreted as the stochastic Green functions of first and second order processes, the undetermined value of ϕ_I at $t = 0$ is precisely the first-order stochastic ambiguity. Correspondingly, the softer unambiguous response $\phi_{II}(0) = 0$ of the Newtonian system (5.3.2b) guarantees the uniqueness of the phase-space stochastic calculus.

5.4 Coordinate-invariant phase-space regularization

The invariant-regularized form of the general phase-space process (5.2.8) is [9, 11]

$$\dot{\pi}_M(\xi,t) = -\frac{\delta H}{\delta\phi^M}(\xi,t) - \beta\mathcal{G}_{MN}\frac{\delta H}{\delta\pi_N}(\xi,t) + \sqrt{\beta}\int(d\xi')\mathcal{E}^{\Lambda}_{M\xi;A\xi'}\eta_A(\xi',t) \tag{5.4.1a}$$

$$\dot{\phi}^M(\xi,t) = \frac{\delta H}{\delta\pi_M}(\xi,t) \tag{5.4.1b}$$

$$\langle\eta_A(\xi,t)\eta_B(\xi',t')\rangle = 2\delta_{AB}\delta^d(\xi-\xi')\delta(t-t') \tag{5.4.1c}$$

where the regulator $R(\tilde{\Delta}) = \exp(\tilde{\Delta}/\Lambda^2)$ appears only in the regularized super-vielbein

$$\mathcal{E}^{\Lambda}_{M\xi;A\xi'} = R(\tilde{\Delta})_{M\xi}{}^{;N\xi'}\mathcal{E}_{NA}(\phi(\xi')) \tag{5.4.2}$$

which multiplies the noise. This geometric regularization is universal across all quantum field theories at once since the regularized noise controls the closure of all loops as discussed above. Diagrammatic expansion of simple theories are studied in [11].

The regularization is invariant under the covariances of the spacetime Laplacian $\tilde{\Delta}$. For example, the Laplacians of general relativity suffice to maintain Einstein invariance of the regularization, in analogy with the gauge-covariant Laplacian for gauge theory. I call such regularization provisional, since it would be preferable to maintain manifest covariance under field reparametrizations $\overline{\phi}(\phi)$ as well. A spacetime Laplacian which respects *both* Einstein and reparametrization invariance has been constructed for the general non-linear sigma model [9], but the construction of such doubly-invariant Laplacians for gauge theory and gravity remains an open problem.

The corresponding phase-space Schwinger-Dyson system [9,11]

$$0 = \left\langle \{F,H\} + \beta\Big[\int(d\xi)\mathcal{G}_{MN}\frac{\delta H}{\delta\pi_N}\frac{\delta}{\delta\pi_M} + \Delta_\pi\Big]F[\phi,\pi]\right\rangle \tag{5.4.3a}$$

$$\Delta_\pi = \int(d\xi)(d\xi')\mathcal{G}^{\Lambda}_{M\xi;N\xi'}\frac{\delta^2}{\delta\pi_N(\xi')\delta\pi_M(\xi)} \tag{5.4.3b}$$

$$\mathcal{G}^{\Lambda}_{M\xi;N\xi'} = \int(d\xi'')\mathcal{E}^{\Lambda}_{M\xi;A\xi''}\mathcal{E}^{\Lambda}_{N\xi',A\xi''} \tag{5.4.3c}$$

$$= \int(d\xi'')R(\tilde{\Delta})_{M\xi}{}^{;P\xi''}R(\tilde{\Delta})_{N\xi'}{}^{;Q\xi''}\mathcal{G}_{PQ}(\phi(\xi'')) \tag{5.4.3d}$$

provides the equivalent d-dimensional form of the regularization. Here $\{F,H\}$ is functional Poisson bracket and $\mathcal{G}^{\Lambda}_{M\xi;N\xi'}$ is the regularized supermetric, which appears in the regularized phase-space super-Laplacian Δ_π.

Although its formal large Λ limit is independent of the equilibration parameter β, the regularized system (5.4.1) or (5.4.3) describes a β-family of regularizations, analogous to latticeization ambiguities, among which the case $\beta \to \infty$ is diagrammatically the simplest [11]. Similar regularization families were encountered earlier in the study of regularized gauge theory with fermions [6].

5.5. Geometric characterization of the general Weyl anomaly

As a first application of the phase-space regularization, I remark on the Einstein and reparametrization invariant regularization of the general non-linear sigma model (5.2.4) in two dimensions. With θ the invariant trace of the stress tensor, the exact result [9]

$$\left\langle\int(d\xi)e\theta\right\rangle = \left\langle\int(d\xi)\mathcal{G}^{\Lambda}_{M\xi;}{}^{M\xi}\right\rangle \tag{5.5.1}$$

is obtained from (5.4.3) for the all-order Weyl anomaly in the presence of the regulator. According to (5.5.1), the general nonperturbative anomaly is the invariant trace of the regularized supermetric.

To compare (5.5.1) with known background field results, consider flat D-dimensional superspace (target space) with $\mathcal{G}_{MN} = e\delta_{MN}$. In this frame, the Einstein and reparametrization-invariant spacetime Laplacian [9] of the sigma model reduces to the ordinary Einstein-invariant Laplacian on scalars, and the result

$$\left\langle \int (d\xi) e\theta \right\rangle = \int (d\xi) e \left[\frac{\Lambda^2 D}{8\pi} + \frac{D}{24\pi} R(g) + 0(\Lambda^{-1}) \right] . \tag{5.5.2}$$

is obtained by heat-kernel expansion as expected [52]. The general non-perturbative result (5.5.1) invites further analysis on non-trivial target manifolds $\mathcal{G}_{MN}$.

6. Invariant Coordinate–Space Regularization

I now discuss regularized integration [9] of the momenta at large β to obtain the corresponding form of coordinate-space regularization in the case of the general theory (5.2.2-3) with DeWitt measure. As in the case of regularized Grassmann integration [6], integration of the momenta at finite β seems prohibitively complex.

Analysis of the regularized phase–space Schwinger–Dyson system (5.4.3) in this case implies that the large β momentum integration is still Gaussian, with the contraction rule

$$\overbrace{\pi_M(\xi)\pi_N}(\xi') = \mathcal{G}^{\Lambda}_{M\xi;N\xi'} \tag{6.1}$$

inside any average. As a result, the regularized coordinate–space Schwinger–Dyson system [8,9]

$$0 = \langle LF[\phi] \rangle \tag{6.2a}$$

$$L = -\int (d\xi) \frac{\delta S}{\delta \phi^M} \mathcal{G}^{MN} \frac{\delta}{\delta \phi^N} + \mathbf{\Delta} \tag{6.2a}$$

$$\mathbf{\Delta} = \int (d\xi)(d\xi') \mathcal{G}_{\Lambda}^{M\xi;N\xi'} \frac{D}{D\phi^N(\xi')} \frac{\delta}{\delta \phi^M(\xi)} \tag{6.2c}$$

is obtained from the phase space equations (5.4.3). This system is the invariant–regularized form of the general coordinate–space theory (5.2.3) with action $S[\phi]$ and DeWitt measure $\mathcal{E} = \sqrt{\mathcal{G}}$. The regularization now appears in the regularized inverse supermetric

$$\mathcal{G}_{\Lambda}^{M\xi;N\xi'} = \int (d\xi'') R(\Delta)^{M\xi}{}_{;P\xi''} R(\Delta)^{N\xi'}{}_{;Q\xi''} \mathcal{G}^{PQ}(\phi(\xi'')) \tag{6.3}$$

of the regularized coordinate–space super–Laplacian $\mathbf{\Delta}$. Here $D/D\phi^M$ is super-covariant derivative

$$\frac{D}{D\phi^Q(\xi')} \frac{\delta}{\delta\phi^P(\xi)} = \left[\frac{\delta}{\delta\phi^Q(\xi')} \delta^R_P - \delta^d(\xi - \xi') \Gamma^R_{QP}(\mathcal{G}) \right] \frac{\delta}{\delta\phi^R(\xi)} \tag{6.4}$$

in terms of the superconnection $\Gamma^R_{QP}(\mathcal{G})$ of the supermetric $\mathcal{G}_{MN}$.

I remind the reader that I began in phase space in part to avoid certain divergences in formal coordinate–invariant coordinate–space formulations. Now that we have obtained the regularized coordinate–space Schwinger–Dyson system (6.2), we can see that the problem in the unregularized formulation was

$$\mathcal{G}^{M\xi;N\xi'}\frac{D}{D\phi^N(\xi')}\,\frac{\delta}{\delta\phi^M(\xi)}\underset{\Lambda}{\rightarrow}(\mathcal{G}^{MN}(\phi(\xi))\delta^d(\xi-\xi'))(-\delta^d(\xi-\xi')\Gamma^Q_{MN})\,\frac{\delta}{\delta\phi^Q(\xi)}. \tag{6.5}$$

Moreover, with the hindsight of the result, we can give the universal coordinate–space regularization rule [9]

$$\mathcal{G}^{MN}(\phi(\xi))\delta^d(\xi-\xi') \rightarrow \mathcal{G}_\Lambda^{M\xi;N\xi'} \tag{6.6a}$$

$$\mathcal{G}^{MN}(\phi(\xi))\delta^d(0) \rightarrow \mathcal{G}_\Lambda^{M\xi;N\xi} \tag{6.6b}$$

$$\mathcal{E}^{MA}(\phi(\xi))\delta^d(\xi-\xi') \rightarrow \mathcal{E}_\Lambda^{M\xi;A\xi'} \tag{6.6c}$$

which may be used to attain regularization of more general coordinate–space Schwinger–Dyson systems and stochastic processes. In particular, Chan and I have used this rule to regularize gravity with an arbitrary power–law Euclidean measure [10] (see Section 7).

The regularized Schwinger–Dyson system (6.2) also has stochastic equivalents. For example, the corresponding regularized Ito process is [9]

$$\dot{\phi}^M(\xi)+\mathcal{G}_\Lambda^{P\xi;Q\xi}\Gamma^M_{PQ}(\mathcal{G}(\phi(\xi)) = -\mathcal{G}^{MN}(\phi(\xi))\,\frac{\delta S}{\delta\phi^N(\xi)}+\int(d\xi')\mathcal{E}_\Lambda^{M\xi;A\xi'}\eta_A(\xi') \tag{6.7a}$$

$$\langle\eta_A(\xi,t)\eta_B(\xi',t')\rangle = 2\delta_{AB}\delta^d(\xi-\xi')\delta(t-t') \tag{6.7b}$$

and regularized Stratonovich equivalents are given in [9]. After this derivation from regularized phase–space, it was called to my attention that (6.7) is a regularization, according to the rule (6.6), of an explicitly divergent formal process ($\mathcal{G}_\Lambda^{P\xi,Q\xi}\underset{\Lambda}{\rightarrow}\delta^d(0)\mathcal{G}^{PQ}, \mathcal{E}_\Lambda^{M\xi;A\xi'}\underset{\Lambda}{\rightarrow}\mathcal{E}^{MA}\delta^d(\xi-\xi')$) independently noted by Rumpf[20].

The previous regularizations of the program are special cases of the universal geometric coordinate–space regularization (6.2) and (6.7). For example, the choice

$$\phi_M = A^a_\mu,\ \eta_A = \eta^b_\nu \tag{6.8a}$$

$$\mathcal{E}^{MA} = \mathcal{G}^{MN} = \delta_{\mu\nu}\delta^{ab} \tag{6.8b}$$

$$\mathcal{E}_\Lambda^{M\xi;A\xi'} = R(\Delta))^{ab}_{\xi\xi'}\delta_{\mu\nu} \tag{6.8c}$$

$$\mathcal{G}_\Lambda^{M\xi;N\xi'} = (R^2(\Delta))^{ab}_{\xi\xi'}\delta_{\mu\nu} \tag{6.8d}$$

in (6.2) and (6.7) results in the original (Ito) regularization (4.1.1) and (4.1.4) of gauge theory [1,3,5]. The choice of flat superspace in (6.8b) is a choice of superspace coordinate system which corresponds to the gauge–invariant inner product

$$\|\delta A\|^2 = \int(d\xi)\delta A^a_\mu\delta A^a_\mu \tag{6.9}$$

on the space of gauge–field deformations. This coordinate system and the parallel choice of gauge–covariant Laplacian Δ in $R(\Delta)$ mark the regularization (4.1.1) and (4.1.4) as gauge–invariant but provisional, since field reparametrization $\overline{A}(A)$ is not a manifest covariance of the system.

7. Regularized Quantum Gravity

7.1. Schwinger–Dyson regularization

As a non-trivial example of coordinate-invariant regularization, we have studied the case of quantum gravity [8–10] with $\phi^M = g_{mn}$ the spacetime metric in d-dimensions. The explicit form of the regularized coordinate-space Schwinger-Dyson system (6.2) for gravity is [8–10]

$$0 = \langle LF[g_{mn}]\rangle \tag{7.1.1a}$$

$$L = \int (d\xi)\Big[\mathcal{L}_Z g_{mn} - \mathcal{G}_{mn;rs}\frac{\delta S}{\delta g_{rs}}\Big]\frac{\delta}{\delta g_{mn}} + \Delta \tag{7.1.1b}$$

$$\Delta = \int (d\xi)(d\xi')\mathcal{G}^{\Lambda}_{mn\xi;rs\xi'}\frac{D}{Dg_{rs}(\xi')}\frac{\delta}{\delta g_{mn}(\xi)} \tag{7.1.1c}$$

where $\mathcal{G}_{mn;rs}$ is the inverse of the supermetric (5.2.7b) on deformations of the metric and $\mathcal{G}^{\Lambda}_{mn;rs}$ is its regularized form (6.3). We choose the heat kernel regulator $R = \exp(\Delta/\Lambda^2)$ and the (provisional) spacetime Laplacian $\Delta = \mathcal{G}^{mn}D_m D_n$ of general relativity on symmetric tensors of rank 2, which maintains Einstein invariance to all orders in the regularized theory. I have also included a Zwanziger gauge-fixing term

$$\mathcal{L}_Z g_{mn} = D_m Z_n + D_n Z_m, \tag{7.1.2}$$

whose generic form [9] is always a gauge transformation, to fix the Einstein invariance.

To complete the theory, we specify the Euclidean Einstein-Hilbert action and a convenient gauge choice

$$S = \frac{1}{\kappa^2}\int (d\xi) eR \tag{7.1.3a}$$

$$Z_m = \frac{1}{2\kappa}(\partial_n h_{nm} - \frac{1}{2}\partial_m h_{nn}) \tag{7.1.3b}$$

$$g_{mn} = \delta_{mn} + \kappa h_{mn} \tag{7.1.3c}$$

so that the system (7.1.1) describes regularized and gauge-fixed Euclidean Einstein gravity with DeWitt measure in d-dimensions.

Chan and I [10] have further generalized the theory (7.1.1-3) to include

regularization of arbitrary power-law measure

$$\prod_{\xi}\sqrt{g(\xi)}^{\nu}, \quad \nu_{\rm DeWitt} = \frac{1}{4}(d+1)(d-4) \tag{7.1.4}$$

and studied the expansion to all orders about flat space. Some highlights of the analysis are as follows.

1. The theory is regularized to all orders as expected. A geometrization of previously-obtained Schwinger-Dyson rules is obtained, including superconnection vertices and regularized inverse supermetric vertices.

2. The free regularized graviton propagator is

$$\langle h_{mn}(\xi)h_{rs}(\xi')\rangle^{(0)} = \Big[(1 + \frac{T}{2-d})_{mn;rs} - \frac{1+2\gamma}{3-d+2\gamma}(\delta_{mn}\frac{\partial_r\partial_s}{\Box} + \delta_{rs}\frac{\partial_m\partial_n}{\Box})\Big](\frac{2e^{2\Box/\Lambda^2}}{-\Box})_{\xi\xi'} \tag{7.1.5a}$$

$$1_{mn;rs} = \frac{1}{2}(\delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}), \quad T_{mn;rs} = \delta_{mn}\delta_{rs} \tag{7.1.5b}$$

which is gauge-equivalent for all γ (the supermetric parameter in (5.2.7b)) to the usual Euclidean Feynman gauge (first term). The result (7.1.5) also reflects the fact that the perturbation expansion is much simpler for the supermetric parameter choice $\gamma = -\frac{1}{2}$, which we adopt for the explicit one-loop computations.

3. A one-loop cosmological counterterm

$$\lambda_C(d,\nu) = \frac{\kappa^2\Lambda^d}{(8\pi)^{d/2}}\Big[\nu - \frac{d^2-7d-2}{4}\Big] \tag{7.1.6}$$

is required to stabilize the expansion about flat space.

4. Including the contribution of the cosmological counterterm, we compute the one-loop graviton mass

$$m_{\rm graviton}(d,\nu) = 0 \tag{7.1.7}$$

as it should be in an Einstein-invariant regularization.

7.2. Schwinger–Dyson stabilization of Euclidean gravity

As indicated in the results above, the differential Schwinger–Dyson formulation has bypassed the question of integration contour for the gauge-invariant [10] unstable conformal mode of Euclidean gravity, giving directly the correct results of Gibbons, Hawking and Perry [53]. This Schwinger–Dyson stabilization mechanism [8–10] should be considered as a variant of the original stochastic stabilization [31]. It is instructive to see how the Schwinger–Dyson formulation manages the stablization in a toy model whose portrayal, minus the tensor indices, is completely accurate.

The simple second-order Schwinger–Dyson equations

$$0 = \langle LF(x) \rangle, \qquad L = -\frac{dS}{dx}\frac{d}{dx} + \frac{d^2}{dx^2} \tag{7.2.1}$$

correspond to the one-dimensional Boltzmann factor $\exp(-S(x))$, according to the identities $0 = \int dx (\exp(-S)F')'$. We adopt the Schwinger–Dyson description (7.2.1) as fundamental for any action $S(x)$, whether or not the action is bounded, which leaves the question of integration contour for a later stage. The simple choices $S(x) = ax^2/2$, $F(x) = x^2/2$ in (7.2.1) give

$$0 = \langle -ax^2 + 1 \rangle \tag{7.2.2}$$

which is the prescription of Gibbons, Hawking and Perry [53] when $a < 0$. In fact, this is the mechanism by which the Schwinger–Dyson equations (7.1.1) produce the correct results for Euclidean gravity, although the action is unbounded and the stochastic formulation generally fails to equilibrate.

7.3. Langevin regularization and the stabilization window

Although the Schwinger–Dyson regularization and stabilization above is completely general across the (d, ν, γ) parameter space of Euclidean gravity, there is a window in parameter space within which the same results can be obtained from the stochastic formulation.

The regularized stochastic process (6.7) for DeWitt measure Euclidean

gravity† [8–10]

$$\begin{aligned} \frac{1}{\kappa^2}\dot{g}_{mn}(\xi,t) \; &+ \mathcal{G}^{\Lambda}_{pq\xi;rs\xi}\Gamma^{pq;rs}_{mn}(\xi,t) \\ &= -\mathcal{G}_{mn;rs}\frac{\delta S}{\delta g_{rs}}(\xi,t) + \mathcal{L}_Z g_{mn}(\xi,t) \\ &+ \frac{1}{\kappa}\int (d\xi')\mathcal{E}^{\Lambda}_{mn\xi;ab\xi'}\eta_{ab}(\xi',t) \end{aligned} \tag{7.3.1}$$

is equivalent to the Schwinger–Dyson system (7.1.1) on assumption of equilibration. In fact, equilibration of this process is observed in the negative supermetric ($\det \mathcal{G} < 0$) window [8–10]

$$d > 2, \qquad -1 < \gamma < -\frac{1}{d} \tag{7.3.2}$$

because the effective drift term $-\mathcal{G}_{mn;rs}\delta S/\delta g_{mn}$ in (7.3.1) is stabilized by the negative eigenvalue of its inverse supermetric prefactor. This result was independently obtained by Rümpf [20] for the explicitly divergent unregularized form of the process.

Within this window, it follows that the regularized stochastic process (7.3.1) and the regularized Schwinger–Dyson system (7.1.1) are equivalent descriptions of regularized quantum gravity, although the Schwinger–Dyson stabilization is more general.

† The explicitly divergent ($\mathcal{G}^{\Lambda}_{pq\xi;rs\xi} \underset{\Lambda}{\rightarrow} \delta^d(0)\mathcal{G}_{pq;rs}(\xi)$) unregularized form of the process (7.3.1) was noted independently in [20, 54].

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