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# Moments for General Quadratic Densities in $n$ Dimensions*

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**Abstract**

We present the calculation of the generating functions and the $r$th-order moments for densities of the form $\rho(\mathbf{x}) \propto g(s(\mathbf{x}))$ where $g(s)$ is a non-negative function of the quadratic "action" $s(\mathbf{x}) = \sum_{i,j} H_{ij} x_i x_j$, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a real $n$-dimensional vector and $H$ is a real, symmetric $n \times n$ matrix whose eigenvalues are strictly positive. In particular, we find the connection between the $(r+2)$th-order and $r$th-order moments, which constitutes a generalization of the Gaussian moment theorem [1], which corresponds to the particular choice $g(s) = e^{-s/2}$. We present several examples for specific choices for $g(s)$, including the explicit expression for the generating function for each case and the subspace projection of $\rho(\mathbf{x})$ in a few cases. We also provide the straightforward generalizations to: 1) the case where $g = g(s(\mathbf{x}) + \mathbf{a} \cdot \mathbf{x})$, where $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ is an arbitrary real $n$-dimensional vector, and 2) the complex case, in which the action is of the form $s(\mathbf{z}) = \sum_{i,j} H_{ij} z_i^* z_j$ where $\mathbf{z} = (z_1, z_2, \cdots, z_n)$ is an $n$-dimensional complex vector and $H$ is a Hermitian $n \times n$ matrix whose eigenvalues are strictly positive.



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# 1 Introduction.

The Gaussian density plays an ubiquitous role in many areas of physics. For example, consider a system of $n$ degrees of freedom characterized by the real $n$-dimensional vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and a quadratic "action" of the form $s(\mathbf{x}) = \sum_{i,j} H_{ij} x_i x_j$ where $H$ is a real, symmetric, positive-definite matrix[1]. A Gaussian density, given by $\rho(\mathbf{x}) \propto \exp(-s(\mathbf{x})/2)$, yields a 2nd-order moment $\langle x_i x_j \rangle = H^{-1}_{ij}$. Since the entire physical contents of the model is embodied in the matrix $H$, all higher-order moments are recursively expressed in terms of the 2nd-order moments by means of the well-known recursion formula

$$\langle \underbrace{x_i x_j x_k \cdots x_m}_{r\ x\text{'s}} \rangle = \langle x_i x_j \rangle \langle \underbrace{x_k \cdots x_m}_{r-2\ x\text{'s}} \rangle + \text{permutations} \tag{1.1}$$

where the "permutations" are such as to make the right-hand side a fully symmetric function of the $r$ indices $i$, $j$, $k$, $\cdots$, $m$. Eq. (1.1) is isomorphic with Wick's expansion of the vacuum expectation value of a product of bosonic operators [2], and with Mayer's cluster expansion for a gas with two-body potentials [3]. By successive application of this recursion, one arrives at the Gaussian moment theorem [1], which expresses any even-order Gaussian moment purely in terms of 2nd-order moments. For example, the 4th-order moment is given by the fully symmetrized product of 2nd-order moments

$$\langle x_i x_j x_k x_m \rangle = \langle x_i x_j \rangle \langle x_k x_m \rangle + \langle x_i x_k \rangle \langle x_j x_m \rangle + \langle x_i x_m \rangle \langle x_j x_k \rangle \tag{1.2}$$

In beam physics it is sometimes the case that the particle distribution in phase space is not Gaussian but it is, nevertheless, matched to the lattice optics in linear order, and one may want to study the time evolution of the moments of the canonical coordinates $(q_i, p_i)$ [4, 5]. The linearly-matched condition implies that the density is a function of the quadratic part of the Hamiltonian, hence it is of the form described above. Motivated by such cases, we consider in this article a more general density $\rho(\mathbf{x}) \propto g(s)$, where $g(s)$ is any real, non-negative function of the quadratic action $s(\mathbf{x})$. As in the Gaussian case, the entire information of the model is contained in $H$, hence all moments must be expressible in terms of the 2nd-order moments $\langle x_i x_j \rangle$. Furthermore, since the $r$th-order moment is a fully symmetric tensor of $r$ indices, and the only ingredients to construct it are the $\langle x_i x_j \rangle$'s, it must be the case that a higher-order moment must proportional to the fully symmetrized product of these. Indeed, in this article we show that, in this more general case, the main expressions are given by

$$\langle x_i x_j \rangle = a H^{-1}_{ij} \tag{1.3a}$$

$$\langle x_i x_j x_k \cdots x_m \rangle = b \left[ \langle x_i x_j \rangle \langle x_k \cdots x_m \rangle + \text{permutations} \right] \tag{1.3b}$$

$$= c \left[ \langle x_i x_j \rangle \langle x_k x_m \rangle \langle \cdots \rangle \cdots + \text{permutations} \right] \tag{1.3c}$$

The main purpose of this article is the calculation of the coefficients $a$, $b$, and $c$ given the dimensionality $n$, the order of the moment $r$, and the function $g(s)$ (the Gaussian density is unique in that $a = b = c = 1$ for all $n$ and $r$). Although in beam physics [4, 5] the only interesting cases are $1 \leq n \leq 6$, the calculation is just as simple for any value of $n$, hence we allow it to be an arbitrary number whose only requirement is $n > 0$. We also compute the generating functions for the moments, and present a discussion of the subspace projection of $\rho(\mathbf{x})$, which is useful when averaging a function that depends only on a subset of the components of $\mathbf{x}$.

In Sec. 2 we set up the general formalism. In Sec. 3 we consider the strictly quadratic action $s(\mathbf{x}) = \sum_{i,j} H_{ij} x_i x_j$. In Sec. 4 we extend the analysis to the general quadratic action of the form $\sum_{i,j} H_{ij} x_i x_j + \sum_i a_i x_i$, where $(a_1, a_2, \cdots, a_n)$ is an arbitrary real $n$-dimensional vector. Section 5 describes the subspace projection of a distribution. Section 6 contains results for specific choices of the function $g(s)$. Section 7 contains the generalization of the above results to densities $\rho(\mathbf{z})$ defined over the complex plane whose points are described by the complex $n$-dimensional vector $\mathbf{z}$. Section 8 summarizes our conclusions. Appendix A contains the definition and basic properties of spherical coordinates in $n$ dimensions, along with some other mathematical details, and Appendix B extends some of these results to the complex case.

[1] A positive-definite matrix is, by definition, one whose all eigenvalues are real and strictly $> 0$.

We use throughout Dirac's "bra-ket" notation [6] interchangeably with the conventional notation for the matrix and scalar product. Thus, for example, if $\mathbf{a}$ and $\mathbf{b}$ represent two real $n$-dimensional vectors with components $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$, respectively, then the scalar product $\langle \mathbf{a}|M|\mathbf{b}\rangle$ stands for $\sum_{i,j} M_{ij} a_i b_j$, where $M$ is an arbitrary real $n \times n$ matrix. If $\mathbf{u}$ and $\mathbf{v}$ represent complex vectors, then $\langle \mathbf{u}|M|\mathbf{v}\rangle$ stands for $\sum_{i,j} M_{ij} u_i^* v_j$ where $M$ is an arbitrary complex $n \times n$ matrix (we use interchangeably the asterisk and the overbar to denote complex conjugation).

## 2 Averages, Moments and Generating Functions.

Let $\rho(\mathbf{x})$ be a real function of the $n$-dimensional vector $\mathbf{x}$ such that $\rho(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. We assume that $\rho(\mathbf{x})$ is normalizable and we chose, without any loss of generality, unit normalization, *i.e.*

$$\int d^n\mathbf{x}\, \rho(\mathbf{x}) = 1 \tag{2.1}$$

The average of any function $f(\mathbf{x})$ with respect to the density $\rho(\mathbf{x})$ is given by

$$\langle f(\mathbf{x})\rangle = \int d^n\mathbf{x}\, \rho(\mathbf{x}) f(\mathbf{x}) \tag{2.2}$$

The $r$th-order moment is, by definition, the average of the product of $r$ components of $\mathbf{x}$,

$$C^{(r)}_{ij\cdots} \equiv \langle x_i x_j \cdots\rangle = \int d^n\mathbf{x}\, \rho(\mathbf{x})(x_i x_j \cdots) \qquad (r \text{ indices}) \tag{2.3}$$

with $r = 1,\ 2,\ \cdots$. If $\rho(\mathbf{x})$ is nonzero only over a finite region of $\mathbf{x}$, then obviously the moments exist for all $r$. If, on the other hand, $\rho(\mathbf{x})$ extends over all space and vanishes as $x \to \infty$ as $\rho(\mathbf{x}) \sim x^{-2p}$, where $x = |\mathbf{x}|$ and $p > 0$, then the $r$th-order moment exists only if $r < 2p - n$. In particular, the normalizability property of $\rho(\mathbf{x})$ requires $2p > n$. If $\rho(\mathbf{x})$ vanishes at infinity faster than any power of $x$, the moments exist for all $r$. If $\rho(\mathbf{x})$ is of even parity, *i.e.*, $\rho(-\mathbf{x}) = \rho(\mathbf{x})$, then all odd-order ($r$ = odd) moments vanish.

The $C^{(r)}$'s are fully symmetric tensors whose generating function is the Fourier transform of $\rho(\mathbf{x})$,

$$\mathcal{G}(\mathbf{k}) = \int d^n\mathbf{x}\, \rho(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} = \left\langle e^{-i\mathbf{k}\cdot\mathbf{x}}\right\rangle \tag{2.4}$$

hence

$$C^{(r)}_{ij\cdots} = i^r \left.\frac{\partial^r \mathcal{G}(\mathbf{k})}{\partial k_i \partial k_j \cdots}\right|_{\mathbf{k}=0} \qquad (r \text{ indices}) \tag{2.5}$$

Note that $\mathcal{G}(\mathbf{k})$ exists for all $\mathbf{k}$ provided only that $\rho(\mathbf{x})$ vanishes as $x \to \infty$ faster than $x^{-n}$. However, as seen above, the existence of the moments requires a faster falloff of $\rho(\mathbf{x})$ at $\infty$. In other words, although $\mathcal{G}(\mathbf{k})$ itself exists, its $r$th derivative at $\mathbf{k} = 0$ is finite only if $r < 2p - n$.

The normalization condition (2.1) implies $\mathcal{G}(0) = 1$. In case that all moments exist, we can expand $e^{-i\mathbf{k}\cdot\mathbf{x}}$ in (2.4) and integrate term by term to obtain

$$\begin{aligned}
\mathcal{G}(\mathbf{k}) &= \int d^n\mathbf{x}\rho(\mathbf{x}) \left(1 - i\sum_i k_i x_i + \frac{(-i)^2}{2!}\sum_{i,j} k_i k_j x_i x_j + \cdots\right) \\
&= 1 - i\sum_i k_i C^{(1)}_i + \frac{(-i)^2}{2!}\sum_{i,j} k_i k_j C^{(2)}_{ij} + \cdots
\end{aligned} \tag{2.6}$$

If we express $f(\mathbf{x})$ in terms of its Fourier transform,

$$f(\mathbf{x}) = \int \frac{d^n\mathbf{k}}{(2\pi)^n}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k}) \tag{2.7}$$

then we also have

$$\langle f(\mathbf{x})\rangle = \int \frac{d^n\mathbf{k}}{(2\pi)^n} \left\langle e^{i\mathbf{k}\cdot\mathbf{x}}\right\rangle \tilde{f}(\mathbf{k}) = \int \frac{d^n\mathbf{k}}{(2\pi)^n} \mathcal{G}(-\mathbf{k})\tilde{f}(\mathbf{k}) \tag{2.8}$$

# 3 Strictly Quadratic Density and Moments.

Given a real symmetric $n \times n$ matrix $H$, we consider the quadratic form $s(\mathbf{x})$

$$s(\mathbf{x}) = \langle \mathbf{x}|H|\mathbf{x}\rangle \equiv \sum_{i,j=1}^{n} H_{ij}x_i x_j \tag{3.1}$$

where $\mathbf{x}$ is an arbitrary real $n$-dimensional vector. Then we can state the following theorem [7, 8]:

**Theorem 3.1** *$s(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ if and only if all eigenvalues of $H$ are strictly $> 0$ (i.e., $H$ is a positive-definite matrix). Furthermore, $s(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.*

**Proof.** Since $H$ is real and symmetric, it has $n$ real eigenvalues and $n$ nontrivial orthogonal eigenvectors, called $h_k$ and $\mathbf{v}_k$, respectively, with $k = 1, 2, \cdots, n$. Choosing $\mathbf{x} = \mathbf{v}_k$ yields $s(\mathbf{x}) = h_k\mathbf{v}_k^2$. Since $s(\mathbf{x}) > 0$ and $\mathbf{v}_k$ is nontrivial, one obtains $h_k > 0$ for any $k$. For the proof of the converse theorem, we assume $h_k > 0$ for all $k$ and use the fact that the $\mathbf{v}_k$'s form a complete orthogonal set to express $\mathbf{x}$ as a linear combination $\mathbf{x} = \sum_i a_i\mathbf{v}_i$, hence

$$s(\mathbf{x}) = \sum_{i,j=1}^{n} a_i a_j \langle \mathbf{v}_i|H|\mathbf{v}_j\rangle = \sum_{i=1}^{n} a_i^2 h_i \mathbf{v}_i^2 \tag{3.2}$$

which is $> 0$ provided at least one of the $a_i$'s is $\neq 0$. Setting $s(\mathbf{x}) = 0$ in this equation yields $a_i = 0$ for all $i$, which corresponds to $\mathbf{x} = 0$.

**Corollary 1**: Define a "submatrix" $H'$ of $H$ to be the matrix obtained from $H$ by eliminating an arbitrary subset of columns and the corresponding subset of rows. By construction, $H'$ is a symmetric matrix hence the above theorem applies to $H'$ as well. This corollary is proven by choosing vectors $\mathbf{x}$ whose components corresponding to the eliminated rows and columns are 0. This implies that any submatrix of a positive-definite matrix is itself a positive-definite matrix. In particular, by eliminating all but one of the columns and the corresponding rows, this corollary implies that the diagonal elements of $H$ are $> 0$.

**Corollary 2**: The above theorem and corollary apply to $H^{-1}$ as well. This follows from the obvious fact that $H^{-1}$ exists (because $\det H > 0$) and is symmetric. In particular, this implies that the inverse of a positive-definite matrix is itself a positive-definite matrix.

We now consider a density function $\rho(\mathbf{x})$ that depends on $\mathbf{x}$ only through $s(\mathbf{x})$, *i.e.*,

$$\rho(\mathbf{x}) = Ng(s(\mathbf{x})) \tag{3.3}$$

where $g(s)$ is a real function of $s$ such that $g(s) \geq 0$ for $0 \leq s \leq \infty$, and $N$ is a normalization constant determined by (2.1),

$$N^{-1} = \int d^n\mathbf{x}\, g(\langle \mathbf{x}|H|\mathbf{x}\rangle) \tag{3.4}$$

Now $H$ can be diagonalized by a matrix $U$, $H = U^{-1}H_D U$ where $H_D = \mathrm{diag}(h_1, h_2, \cdots, h_n)$. Since $H$ is symmetric, $U$ is orthogonal, *i.e.* $U^{-1} = U^T$. Since $h_i > 0$ for all $i$, the matrix $H_D^{1/2}$ exists unambiguously hence we can define the change of variables $\mathbf{v} = H_D^{1/2}U\mathbf{x}$ yielding $s(\mathbf{x}) = \langle \mathbf{v}|\mathbf{v}\rangle = v^2$. Using the integrals in Appendix A one obtains

$$N^{-1} = \frac{1}{\sqrt{\det H}} \int d^n\mathbf{v}\, g(v^2) = \frac{\Omega_n g_0}{\sqrt{\det H}} \tag{3.5}$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the $n$-dimensional unit sphere and

$$g_0 \equiv \int_0^\infty dv\, g(v^2) v^{n-1} = \frac{1}{2}\int_0^\infty ds\, g(s) s^{n/2-1} \tag{3.6}$$

Using the positivity of $s(\mathbf{x})$ (Theorem 3.1) one can write (3.3) in the form

$$\rho(\mathbf{x}) = N \int_0^\infty ds\, g(s)\, \delta(\langle \mathbf{x}|H|\mathbf{x}\rangle - s) \tag{3.7}$$

Using again the results in Appendix A and the change of variable $s = v^2$ one obtains

$$\begin{aligned} \mathcal{G}(\mathbf{k}) &= N \int_0^\infty ds\, g(s) \int d^n\mathbf{x}\, \delta(\langle \mathbf{x}|H|\mathbf{x}\rangle - s) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{\Gamma(n/2)}{g_0}\left(\frac{2}{q}\right)^\nu \int_0^\infty dv\, g(v^2) v^{n/2} J_\nu(qv)\,, \qquad \nu = n/2 - 1 \end{aligned} \tag{3.8}$$

where $J_\nu(z)$ is the ordinary Bessel function of order $\nu$, and $\mathbf{q}$ is a vector defined by $\mathbf{q} = H_D^{-1/2} U\mathbf{k}$ whose magnitude is given by

$$q^2 = \langle \mathbf{q}|\mathbf{q}\rangle = \langle \mathbf{k}|H^{-1}|\mathbf{k}\rangle \tag{3.9}$$

If $\rho(\mathbf{x})$ falls off as $x \to \infty$ faster than any power of $x$, then $g(v^2)$ falls off faster than any power of $v$, hence we can expand the Bessel function in Eq. (3.8) in its Taylor series and integrate term by term to yield

$$\mathcal{G}(\mathbf{k}) = \sum_{\ell=0}^\infty A(n;\ell) \frac{(-q^2/2)^\ell}{\ell!} \tag{3.10}$$

where $A(n;\ell)$ is given by

$$A(n;\ell) = \frac{g_\ell/g_0}{2^\ell (n/2)_\ell} \tag{3.11}$$

where the symbol $(z)_\ell$ is defined by [9]

$$(z)_\ell = \frac{\Gamma(z+\ell)}{\Gamma(z)} = \begin{cases} 1, & \ell = 0 \\ z(z+1)\cdots(z+\ell-1), & \ell \geq 1 \end{cases} \tag{3.12}$$

and where $g_\ell$ is a moment of the function $g(s)$,

$$g_\ell \equiv \int_0^\infty dv\, g(v^2) v^{n+2\ell-1} = \frac{1}{2}\int_0^\infty ds\, g(s) s^{n/2+\ell-1} \tag{3.13}$$

Defining $G \equiv H^{-1}$ and writing $q^2$ explicitly in the right-hand side of (3.10) and comparing with (2.6) taking into account the symmetry[2] of $G$ yields

$$C^{(r)}_{ij\cdots} = \begin{cases} A(n;\ell) S^{(r)}_{ij\cdots} & \text{for } r = 2\ell \text{ indices} \\ 0 & \text{for } r = \text{odd} \end{cases} \tag{3.14}$$

[2] $G$ is symmetric because $H$ is symmetric.

where $S^{(r)}_{ij\cdots}$ is, by definition, the fully symmetric tensor of rank $r$ formed out of products of $G$'s. This tensor can be defined recursively by

$$S^{(r)}_{ijkm\cdots} = G_{ij} S^{(r-2)}_{km\cdots} + \text{permutations} \tag{3.15}$$

with $S^{(0)} = 1$. For example,

$$S^{(2)}_{ij} = G_{ij} \tag{3.16a}$$

$$S^{(4)}_{ijkm} = G_{ij}G_{km} + G_{ik}G_{jm} + G_{im}G_{jk} \tag{3.16b}$$

etc. The tensor $S^{(r)}$, with $r = 2\ell$ indices, has $(2\ell)!/2^\ell \ell! = 1 \cdot 3 \cdot 5 \cdots (2\ell - 1)$ monomials, each formed out of a product of $\ell$ $G$'s. Alternatively, it is straightforward to prove that

$$\left. \frac{\partial^r \langle \mathbf{k}|G|\mathbf{k}\rangle^\ell}{\underbrace{\partial k_i \partial k_j \cdots}_{r\ k\text{'s}}} \right|_{\mathbf{k}=0} = \begin{cases} 2^\ell \ell!\, S^{(r)}_{ij\cdots} & \text{for } r = 2\ell \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

which, when combined with definition (2.5) applied to the formula (3.10), yields (3.14).

If $\rho(\mathbf{x})$ vanishes as $x \to \infty$ as $x^{-2p}$ then $C^{(r)}$ exists and is given by expression (3.14) only as long as $g_\ell$ exists, *i.e.*, only for $r < 2p - n$.

## 3.1 Recursion Formulas for the Moments.

Note that Eqs. (3.14) and (3.16) imply that all moments with $\ell > 1$ are determined by the second-order ($\ell = 1$) moments as long as $g_\ell$ exists. Thus Eq. (3.14) implies the fundamental recursion relation

$$C^{(r)}_{ijkm\cdots} = B(n;\ell) \left[ C^{(2)}_{ij} C^{(r-2)}_{km\cdots} + \text{permutations} \right] \tag{3.18}$$

with $C^{(0)} = 1$ or, equivalently,

$$\langle x_i x_j x_k x_m \cdots \rangle = B(n;\ell) \left[ \langle x_i x_j \rangle \langle x_k x_m \cdots \rangle + \text{permutations} \right] \tag{3.19}$$

where

$$B(n;\ell) = \frac{A(n;\ell)}{A(n;\ell-1)A(n;1)} = \frac{g_\ell g_0}{g_1 g_{\ell-1}} \cdot \frac{n}{n + 2(\ell - 1)} \tag{3.20}$$

By recursive applications of Eq. (3.18) to its right-hand side, one can reach an expression for $C^{(r)}$ purely in terms of $C^{(2)}$'s,

$$C^{(r)}_{ijkm\cdots} = D(n;\ell) \Big[ \underbrace{C^{(2)}_{ij} C^{(2)}_{km} \cdots}_{\ell\ C^{(2)}\text{'s}} + \text{permutations} \Big] \tag{3.21}$$

or, equivalently,

$$\langle x_i x_j x_k x_m \cdots \rangle = D(n;\ell) \Big[ \underbrace{\langle x_i x_j \rangle \langle x_k x_m \rangle \cdots}_{\ell \text{ factors}} + \text{permutations} \Big] \tag{3.22}$$

where

$$D(n;\ell) = \frac{A(n;\ell)}{(A(n;1))^\ell} = \frac{g_\ell g_0^{\ell-1} (n/2)^\ell}{g_1^\ell (n/2)_\ell} \tag{3.23}$$

It is straightforward to verify that $A(n;0) = B(n;1) = D(n;1) = 1$, as consistency demands. As an example we consider the 4th-order moments by setting $\ell = 2$. In this case we have

$$D(n;2) = B(n;2) = \frac{n\, g_2 g_0}{(n+2)\, g_1^2} \tag{3.24}$$

hence

$$C^{(4)}_{ijkm} = \frac{n\, g_2 g_0}{(n+2)\, g_1^2} \left[ C^{(2)}_{ij} C^{(2)}_{km} + C^{(2)}_{ik} C^{(2)}_{jm} + C^{(2)}_{im} C^{(2)}_{jk} \right] \tag{3.25}$$

where

$$C^{(2)}_{ij} = \frac{g_1}{n g_0} G_{ij} \tag{3.26}$$

Equations (3.10–3.16), complemented by Eqs. (3.18–3.23), constitute the central results of this article.

#### 3.1.1 Special Case: 2nd-order Moments.

Eq. (3.26) and the definition of $G$ can be cast in matrix form,

$$G = H^{-1} = \frac{n g_0}{g_1} C^{(2)} \tag{3.27}$$

where $C^{(2)}$ is the "correlation matrix"

$$C^{(2)} = \begin{pmatrix} \langle x_1^2 \rangle & \langle x_1 x_2 \rangle & \cdots & \langle x_1 x_n \rangle \\ \langle x_1 x_2 \rangle & \langle x_2^2 \rangle & \cdots & \langle x_2 x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1 x_n \rangle & \langle x_2 x_n \rangle & \cdots & \langle x_n^2 \rangle \end{pmatrix} \tag{3.28}$$

which is, obviously, symmetric. Since $H$ is positive-definite, Corollary 2 implies that so is $C^{(2)}$.

For the specific case $n = 2$ we obtain

$$\det G = \left( \frac{2 g_0 \epsilon}{g_1} \right)^2 \tag{3.29}$$

where we have defined

$$\epsilon^2 = \langle x_1^2 \rangle \langle x_2^2 \rangle - \langle x_1 x_2 \rangle^2 \tag{3.30}$$

which is guaranteed to be positive by Schwarz's inequality. The matrix $H$ is, therefore,

$$H = G^{-1} = \frac{g_1}{2 g_0 \epsilon^2} \begin{pmatrix} \langle x_2^2 \rangle & -\langle x_1 x_2 \rangle \\ -\langle x_1 x_2 \rangle & \langle x_1^2 \rangle \end{pmatrix} \tag{3.31}$$

with eigenvalues

$$h_\pm = \frac{g_1}{4 g_0 \epsilon^2} \left[ \langle x_1^2 \rangle + \langle x_2^2 \rangle \pm \sqrt{(\langle x_1^2 \rangle - \langle x_2^2 \rangle)^2 + 4 \langle x_1 x_2 \rangle^2} \right] \tag{3.32}$$

Note that $h_\pm > 0$, as it should be the case.

# 4 General Quadratic Density and Moments.

We now consider a density function $\rho(\mathbf{x})$ that depends on $\mathbf{x}$ through the combination

$$\rho(\mathbf{x}) = N g(\langle \mathbf{x} | H | \mathbf{x} \rangle + \langle \mathbf{x} | \mathbf{a} \rangle) \tag{4.1}$$

where $H$ is a positive-definite symmetric matrix, $\mathbf{a}$ is a constant vector and $g \geq 0$ for all $\mathbf{x}$. The normalization constant $N$ is determined by

$$N^{-1} = \int d^n\mathbf{x}\, g(\langle \mathbf{x} | H | \mathbf{x} \rangle + \langle \mathbf{x} | \mathbf{a} \rangle) \tag{4.2}$$

The linear term can be eliminated by a shift of origin in $\mathbf{x}$ defined by $\mathbf{x} = \mathbf{y} + \mathbf{h}$ where $\mathbf{h}$ is a constant vector to be determined. Using the symmetry property of $H$ we obtain

$$\langle \mathbf{x} | H | \mathbf{x} \rangle + \langle \mathbf{x} | \mathbf{a} \rangle = \langle \mathbf{y} | H | \mathbf{y} \rangle + 2 \langle \mathbf{y} | H | \mathbf{h} \rangle + \langle \mathbf{h} | H | \mathbf{h} \rangle + \langle \mathbf{y} | \mathbf{a} \rangle + \langle \mathbf{h} | \mathbf{a} \rangle \tag{4.3}$$

It is clear that the term linear in $\mathbf{y}$ is eliminated by the choice $2H\mathbf{h} + \mathbf{a} = 0$, thus $\mathbf{h} = -H^{-1}\mathbf{a}/2$ hence

$$\langle \mathbf{x}|H|\mathbf{x}\rangle + \langle \mathbf{x}|\mathbf{a}\rangle = \langle \mathbf{y}|H|\mathbf{y}\rangle + \Delta \tag{4.4}$$

where $\Delta = -\frac{1}{4}\langle \mathbf{a}|G|\mathbf{a}\rangle$ and $G \equiv H^{-1}$, hence

$$N^{-1} = \int d^n\mathbf{y}\, g(\langle \mathbf{y}|H|\mathbf{y}\rangle + \Delta) \tag{4.5}$$

Following the same steps as in Sec. 3, using the results from Appendix A and defining the change of variables $\mathbf{y} = U^{-1}H_D^{-1/2}\mathbf{v}$, one obtains

$$N^{-1} = \frac{1}{\sqrt{\det H}} \int d^n\mathbf{v}\, g(v^2 + \Delta) = \frac{\Omega_n g_0(\Delta)}{\sqrt{\det H}} \tag{4.6}$$

where

$$g_0(\Delta) = \int_0^\infty dv\, g(v^2 + \Delta)\, v^{n-1} \tag{4.7}$$

The generating function is given by (2.4). Following the same procedure as above one obtains

$$\mathcal{G}(\mathbf{k}) = \int d^n\mathbf{y}\, \rho(\mathbf{y} + \mathbf{h})\, e^{-i\mathbf{k}\cdot(\mathbf{y}+\mathbf{h})} \tag{4.8a}$$

$$= \frac{N\, e^{-i\mathbf{k}\cdot\mathbf{h}}}{\sqrt{\det H}} \int d^n\mathbf{v}\, g(v^2 + \Delta)\, e^{-i\mathbf{q}\cdot\mathbf{v}} \tag{4.8b}$$

$$= \frac{\Gamma(n/2)\, e^{-i\mathbf{k}\cdot\mathbf{h}}}{g_0(\Delta)} \int_0^\infty dv\, g(v^2 + \Delta) v^{n-1} \left(\frac{2}{qv}\right)^\nu J_\nu(qv)\,, \quad \nu = n/2 - 1 \tag{4.8c}$$

where $\mathbf{q} = H_D^{-1/2}U\mathbf{k}$. If $g(s) \to 0$ as $s \to \infty$ faster than any power of $s$ we can Taylor-expand the Bessel function above and integrate term by term to obtain

$$\mathcal{G}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{h}} \sum_{\ell=0}^\infty A(n;\ell) \frac{(-q^2/2)^\ell}{\ell!} \tag{4.9}$$

where $A(n;\ell)$ is formally identical to Eq. (3.11) except that $g_\ell$ is now given by

$$g_\ell(\Delta) = \int_0^\infty dv\, g(v^2 + \Delta)\, v^{n+2\ell-1} \tag{4.10}$$

rather than by (3.13).

The exponential term $e^{-i\mathbf{k}\cdot\mathbf{h}}$ in Eq. (4.9) leads to nonvanishing odd-order moments. An easy way to compute the moments is obtained by setting $\mathbf{x} = \mathbf{y} + \mathbf{h}$ so that, from Eq. (4.8a), one gets

$$\langle x_i x_j \cdots \rangle = \langle (y_i + h_i)(y_j + h_j) \cdots \rangle \tag{4.11}$$

Now Eq. (4.8a) is suggestively written in the form

$$\mathcal{G}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{h}}\, \mathcal{G}_y(\mathbf{k}) \tag{4.12}$$

where $\mathcal{G}_y(\mathbf{k})$ is the generating function for the $y$-moments. Eq. (4.9) implies that

$$\mathcal{G}_y(\mathbf{k}) = \int d^n\mathbf{y}\, \rho(\mathbf{y} + \mathbf{h})\, e^{-i\mathbf{k}\cdot\mathbf{y}} = \sum_{\ell=0}^\infty A(n;\ell) \frac{(-q^2/2)^\ell}{\ell!} \tag{4.13}$$

which, in turn, implies that the even-order moments for the $y$'s are given by (3.14). For example,

$$\langle x_i \rangle = \langle y_i + h_i \rangle = h_i \tag{4.14a}$$

$$\langle x_i x_j \rangle = \langle (y_i + h_i)(y_j + h_j) \rangle = \langle y_i y_j \rangle + h_i h_j = (g_1/ng_0)G_{ij} + h_i h_j \tag{4.14b}$$

$$\begin{aligned}\langle x_i x_j x_k \rangle &= \langle (y_i + h_i)(y_j + h_j)(y_k + h_k) \rangle = \langle y_i y_j \rangle h_k + \langle y_j y_k \rangle h_i + \langle y_k y_i \rangle h_j + h_i h_j h_k \\ &= (g_1/ng_0)\left(G_{ij}h_k + G_{jk}h_i + G_{ki}h_j\right) + h_i h_j h_k\end{aligned} \tag{4.14c}$$

etc.

# 5 Subspace Projection.

Suppose now that the arbitrary function $f(\mathbf{x})$ depends only on a subset of components of $\mathbf{x}$, namely $f = f(\mathbf{x}')$ where $\mathbf{x}'$ is the $m$-dimensional vector $\mathbf{x}' = (x_1, x_2, \cdots, x_m)$ such that $1 \le m \le n-1$. Then a projected density $\rho'(\mathbf{x}')$ can be defined such that

$$\langle f(\mathbf{x}') \rangle = \int d^m \mathbf{x}' \rho'(\mathbf{x}') f(\mathbf{x}') \tag{5.1}$$

This implies that

$$\rho'(\mathbf{x}') = \int d^{n-m}\mathbf{y}\, \rho(\mathbf{x}', \mathbf{y}) \tag{5.2}$$

where $\mathbf{y} = (x_{m+1}, x_{m+2}, \cdots, x_n)$.

For the general quadratic density, $\rho(\mathbf{x}) = Ng(\langle \mathbf{x}|H|\mathbf{x}\rangle + \langle \mathbf{x}|\mathbf{a}\rangle)$, we start by partitioning $H$, $\mathbf{x}$ and $\mathbf{a}$ in the form

$$H = \left(\begin{array}{c|c} S & E \\ \hline F & K \end{array}\right), \qquad \mathbf{x} = \left(\begin{array}{c} \mathbf{x}' \\ \hline \mathbf{y} \end{array}\right), \qquad \mathbf{a} = \left(\begin{array}{c} \mathbf{c} \\ \hline \mathbf{d} \end{array}\right) \tag{5.3}$$

where the vectors $\mathbf{c}$ and $\mathbf{d}$ are of dimension $m$ and $n-m$, respectively, and where $S$ is $m \times m$, $E$ is $m \times (n-m)$, $F$ is $(n-m) \times m$ and $K$ is $(n-m) \times (n-m)$, with $S^T = S$, $K^T = K$ and $E^T = F$. This partition leads to

$$\langle \mathbf{x}|H|\mathbf{x}\rangle + \langle \mathbf{x}|\mathbf{a}\rangle = \langle \mathbf{y}|K|\mathbf{y}\rangle + 2\langle \mathbf{y}|F|\mathbf{x}'\rangle + \langle \mathbf{y}|\mathbf{d}\rangle + \langle \mathbf{x}'|S|\mathbf{x}'\rangle + \langle \mathbf{x}'|\mathbf{c}\rangle \tag{5.4}$$

Next we shift the origin of $\mathbf{y}$ by setting $\mathbf{y} = \mathbf{y}' + \mathbf{h}$, substitute this in Eq. (5.4), and choose $\mathbf{h}$ such that the term linear in $\mathbf{y}'$ vanishes. A straightforward examination of the resultant expression shows that that the choice $\mathbf{h} = -K^{-1}(E\mathbf{x}' + \mathbf{d}/2)$ accomplishes the desired cancellation, hence

$$\langle \mathbf{x}|H|\mathbf{x}\rangle + \langle \mathbf{x}|\mathbf{a}\rangle = \langle \mathbf{y}'|K|\mathbf{y}'\rangle + \Delta'(\mathbf{x}') \tag{5.5}$$

where

$$\Delta'(\mathbf{x}') = \langle \mathbf{x}'|H'|\mathbf{x}'\rangle - \langle \mathbf{x}'|EK^{-1}|\mathbf{d}\rangle + \langle \mathbf{x}'|\mathbf{c}\rangle - \tfrac{1}{4}\langle \mathbf{d}|K^{-1}|\mathbf{d}\rangle \tag{5.6}$$

where we have defined

$$H' \equiv S - EK^{-1}F \tag{5.7}$$

In the above formulas, $K^{-1}$ is guaranteed to exists on account of Corollary 1, which implies that $K$ is a symmetric positive-definite matrix. For this reason, we can set $\mathbf{y}' = V^{-1}K_D^{-1/2}\mathbf{v}$ where $V$ is the orthogonal matrix that diagonalizes $K$, namely $VKV^{-1} = K_D$. Since $V^T = V^{-1}$ we obtain $\langle \mathbf{y}'|K|\mathbf{y}'\rangle = \langle \mathbf{v}|K_D^{-1/2}VKV^{-1}K_D^{-1/2}|\mathbf{v}\rangle = v^2$ hence the projected density is

$$\rho'(\mathbf{x}') = \int d^{n-m}\mathbf{y}' \rho(\mathbf{x}', \mathbf{y}' + \mathbf{h})$$

$$= \frac{N}{\sqrt{\det K}} \int d^{n-m}\mathbf{v}\, g(v^2 + \Delta'(\mathbf{x}'))$$

$$= \frac{N\Omega_{n-m}}{\sqrt{\det K}} \int_0^\infty dv\, g(v^2 + \Delta'(\mathbf{x}'))v^{n-m-1} \tag{5.8}$$

where $N$ is given by Eqs. (4.6-4.7), hence the full expression for the projected density is

$$\rho'(\mathbf{x}') = \frac{\Omega_{n-m}}{\Omega_n g_0(\Delta)} \sqrt{\frac{\det H}{\det K}} \int_0^\infty dv\, g(v^2 + \Delta'(\mathbf{x}'))v^{n-m-1} \tag{5.9}$$

where $g_0(\Delta)$ is given by Eq. (4.7).

In Fourier space, subspace projection is straightforward. If $\tilde{\rho}(\mathbf{k})$ if the Fourier transform of $\rho(\mathbf{x})$, then the Fourier transform $\tilde{\rho}'(\mathbf{k}')$ of $\rho'(\mathbf{x}')$ is simply

$$\tilde{\rho}'(\mathbf{k}') = \tilde{\rho}(k_1, k_2, \cdots, k_m, 0, 0, \cdots, 0) \tag{5.10}$$

or, equivalently, the generating function $\mathcal{G}'(\mathbf{k}')$ is simply given by

$$\mathcal{G}'(\mathbf{k}') = \mathcal{G}(k_1, k_2, \cdots, k_m, 0, 0, \cdots, 0) \tag{5.11}$$

where $\mathbf{k}'$ is the vector $\mathbf{k}' = (k_1, k_2, \cdots, k_m)$. As a result,

$$\langle f(\mathbf{x}')\rangle = \int \frac{d^m\mathbf{k}'}{(2\pi)^m} \left\langle e^{i\mathbf{k}'\cdot\mathbf{x}'}\right\rangle \tilde{f}(\mathbf{k}') = \int \frac{d^m\mathbf{k}'}{(2\pi)^m}\, \mathcal{G}'(-\mathbf{k}')\tilde{f}(\mathbf{k}') \tag{5.12}$$

This result confirms the obvious fact, which is true by the construction of $\rho'(\mathbf{x}')$, that the moments $C^{(r)\prime}_{ij\cdots}$, derived from $\rho'(\mathbf{x}')$, are exactly the same as the $C^{(r)}_{ij\cdots}$ derived from $\rho(\mathbf{x})$ provided the indices $i, j, \cdots$ are constrained to the range $1 \le i, j, \cdots \le m$. In other words, for the specific case of $r = 2$, we can state that the correlation matrix $C^{(2)\prime}$ is the $m \times m$ upper-left submatrix of $C^{(2)}$.

Consider now the case of a strictly quadratic action, namely $\mathbf{a} = 0$. In this case $\Delta = 0$ and Eq. (5.9) yields

$$\rho'(\mathbf{x}') = \frac{\Gamma(n/2)}{g_0\pi^{m/2}\Gamma((n-m)/2)} \sqrt{\frac{\det H}{\det K}} \int_0^\infty dv\, g(v^2 + s'(\mathbf{x}'))v^{n-m-1} \tag{5.13}$$

where $g_0$ is given by Eq. (3.6) and $s'(\mathbf{x}')$ is the "reduced action," given by

$$s'(\mathbf{x}') = \langle \mathbf{x}'|(S - EK^{-1}F)|\mathbf{x}'\rangle \tag{5.14}$$

Defining $\rho'(\mathbf{x}') = N'g'(s'(\mathbf{x}'))$ we obtain, up to an irrelevant multiplicative ambiguity,

$$g'(s'(\mathbf{x}')) = \int_0^\infty dv\, g(v^2 + s'(\mathbf{x}'))v^{n-m-1} \tag{5.15}$$

We can now apply the general result (3.26) to compute the 2nd-order moment for the $m$-dimensional density (5.13) to obtain

$$\langle x'_i x'_j\rangle = C'^{(2)}_{ij} = \frac{g'_1}{m g'_0} G'_{ij} \tag{5.16}$$

where $G'^{-1} = H'$.

According to the general formula (3.13), the moment $g'_\ell$ obtained from (5.15) is

$$g'_\ell = \int_0^\infty du\, u^{m+2\ell-1} \int_0^\infty dv\, g(v^2+u^2)v^{n-m-1} \tag{5.17}$$

where we made the change of variables $s' = u^2$. Using now the change of variables $(u, v) = (t\cos\phi, t\sin\phi)$ where $0 \le t < \infty$ and $0 \le \phi \le \pi/2$ yields

$$g'_\ell = \frac{g_\ell}{2} B\left(\frac{n-m}{2}, \frac{m+2\ell}{2}\right) \tag{5.18}$$

where $B(\mu, \nu)$ is Euler's beta function. This result implies $g'_1/g'_0 = mg_1/ng_0$ so that Eq. (5.16) yields

$$C'^{(2)} = \frac{g_1}{ng_0} G' = \frac{g_1}{ng_0} H'^{-1} \tag{5.19}$$

Since $\rho'(\mathbf{x}')$ is normalized to unity we conclude, following the same integration procedure applied to Eq. (5.13), that

$$\det H' = \det(S - EK^{-1}F) = \frac{\det H}{\det K} \tag{5.20}$$

To prove this, we integrate Eq. (5.13) to obtain

$$\begin{aligned} \int d^m\mathbf{x}' \rho'(\mathbf{x}') &= \frac{\Omega_{n-m}}{\Omega_n g_0} \sqrt{\frac{\det H}{\det K}} \times \int_0^\infty dv\, v^{n-m-1} \int d^m\mathbf{x}' g(v^2 + \Delta'(\mathbf{x}')) \\ &= \frac{\Omega_{n-m}\Omega_m g'_0}{\Omega_n g_0} \sqrt{\frac{\det H}{\det K \det H'}} \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} g'_0 &= \int_0^\infty du\, u^{m-1} \int_0^\infty dv\, g(v^2+u^2)v^{n-m-1} \\ &= \frac{g_0}{2} B\left(\frac{n-m}{2}, \frac{m}{2}\right) \end{aligned} \tag{5.22}$$

Substituting the definition of $B(\mu, \nu)$ into (5.21) we obtain

$$1 = \sqrt{\frac{\det H}{\det K \det H'}} \tag{5.23}$$

which completes the proof.

Comparing Eq. (5.19) with $C^{(2)} = (g_1/ng_0)H^{-1}$ we arrive at the following theorem:

**Theorem 5.1** *Consider a real, symmetric, positive-definite, $n \times n$ matrix $G$ whose inverse is $G^{-1} = H$. Consider the upper-left $m \times m$ submatrix $G'$ of $G$, with $1 \le m \le n-1$. Then $G'^{-1} = S - EK^{-1}F$, where $S$, $E$, $F$ and $K$ are the matrices that result from partitioning $H$ in the form shown in Eq. (5.3). Furthermore, $\det(S - EK^{-1}F) = \det H/\det K$.*

**Corollary 3**: The above theorem, combined with Corollary 2, implies that $S - EK^{-1}F$ is a positive-definite matrix since $G'$ is a positive-definite matrix.

A simplified proof of this theorem follows from computing the subspace projection of a Gaussian density of a strictly quadratic action, as shown in Sec. 6.1.1 below.

# 6 Examples.

In this section we provide several explicit examples of strictly-quadratic densities, and compute the normalization $N$ according to Eq. (3.5), their moments $g_\ell$ according to Eq. (3.13), and their generating function $\mathcal{G}(\mathbf{k})$ according to Eq. (3.8). Table 1 summarizes the results.

## 6.1 Gaussian Density.

In this case $g(s) = e^{-s/2}$ and the explicit form of $\rho(\mathbf{x})$ is

$$\rho(\mathbf{x}) = \frac{\sqrt{\det H}}{(2\pi)^{n/2}} \exp\left\{-\tfrac{1}{2}\langle \mathbf{x}|H|\mathbf{x}\rangle\right\} \tag{6.1}$$

from which it follows that

$$g_\ell = 2^{n/2+\ell-1}\Gamma(n/2+\ell) \tag{6.2}$$

$$A(n;\ell) = 1 \tag{6.3}$$

The fact that the $g_\ell$'s exist for all $\ell$ is a consequence of the fact that $g(s) \to 0$ as $s \to \infty$ faster than any power of $s$, hence all moments exist. The fact that $A(n;\ell) = 1$ for all $n$ and $\ell$ implies that $B(n;\ell) = D(n;\ell) = 1$ hence we recover the well-known results for the recursion formulas for the higher-order moments, Eq. (1.1).

The generating function is

$$\mathcal{G}(\mathbf{k}) = \sum_{\ell=0}^{\infty} \frac{(-q^2/2)^\ell}{\ell!} = \exp(-q^2/2) \tag{6.4}$$

which can also be obtained from the familiar Gaussian integration formula

$$\mathcal{G}(\mathbf{k}) = \frac{\sqrt{\det H}}{(2\pi)^{n/2}} \int d^n\mathbf{x} \exp\left\{-\tfrac{1}{2}\langle \mathbf{x}|H|\mathbf{x}\rangle - i\langle \mathbf{k}|\mathbf{x}\rangle\right\} = \exp\left\{-\tfrac{1}{2}\langle \mathbf{k}|G|\mathbf{k}\rangle\right\} \tag{6.5}$$

The 2nd-order moment $\langle x_i x_j\rangle = H^{-1}_{ij}$ or, indeed, any even-order moment, can also be obtained by differentiating both sides of the equation

$$\int d^n\mathbf{x} \exp\{-\tfrac{1}{2}\langle \mathbf{x}|H|\mathbf{x}\rangle\} = \frac{(2\pi)^{n/2}}{\sqrt{\det H}} \tag{6.6}$$

with respect to $H_{ij}$ and using the formula $\partial \det M/\partial M_{ij} = (\det M) M^{-1}_{ji}$, valid for any nonsingular matrix $M$.

Eq. (6.3) implies that

$$G = H^{-1} = C^{(2)} = \begin{pmatrix} \langle x_1^2\rangle & \langle x_1x_2\rangle & \cdots & \langle x_1x_n\rangle \\ \langle x_1x_2\rangle & \langle x_2^2\rangle & \cdots & \langle x_2x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1x_n\rangle & \langle x_2x_n\rangle & \cdots & \langle x_n^2\rangle \end{pmatrix} \tag{6.7}$$

### 6.1.1 Subspace Projection for the Gaussian Density.

Using Eqs. (5.13-5.14) we obtain

$$\rho'(\mathbf{x}') = \frac{1}{(2\pi)^{m/2}}\sqrt{\frac{\det H}{\det K}} \exp\left\{-\tfrac{1}{2}\langle \mathbf{x}'|H'|\mathbf{x}'\rangle\right\} \tag{6.8}$$

where $\mathbf{x}' = (x_1, x_2, \cdots, x_m)$ and $H' = S - EK^{-1}F$. Here the matrices $S$, $E$, $F$ and $K$ are defined by the partition of $H$ shown in Eq. (5.3). This result states that the dimensionally-projected Gaussian density is also Gaussian. This unique property of the Gaussian density is a direct consequence of the composition rule $\exp(x)\exp(y) = \exp(x+y)$ of the exponential function.

Since $\rho'(\mathbf{x}')$ is normalized to unity we conclude, comparing Eq. (6.8) with (6.1), that

$$\det H' = \frac{\det H}{\det K} \tag{6.9}$$

in accordance with the general Theorem 5.1.

The generating function for the dimensionally-projected Gaussian density is

$$\mathcal{G}'(\mathbf{k}') = \exp\left\{-\tfrac{1}{2}\langle \mathbf{k}'|G'|\mathbf{k}'\rangle\right\} \tag{6.10}$$

where $\mathbf{k}' = (k_1, k_2, \cdots, k_m)$ and $G'$ is the $m \times m$ upper-left submatrix of $G$, obtained from $G$ by eliminating all rows and all columns higher than $m$.

## 6.2 Gamma-Function Density.

This is a slight generalization of the Gaussian density, defined by $g(s) = s^\alpha e^{-s/2}$, for which all moments exist provided $\alpha$ is large enough, as explained below. We find

$$g_\ell = 2^{n/2+\alpha+\ell-1}\Gamma(n/2+\alpha+\ell) \tag{6.11}$$

$$A(n;\ell) = \frac{(n/2+\alpha)_\ell}{(n/2)_\ell} \tag{6.12}$$

$$\mathcal{G}(\mathbf{k}) = {}_1F_1(n/2+\alpha; n/2; -q^2/2) \tag{6.13}$$

where ${}_1F_1$ is the confluent hypergeometric function [10]. Clearly, if $\alpha$ is sufficiently negative the density will not be normalizable as a consequence of the divergence of $\rho(\mathbf{x})$ in the neighborhood of $\mathbf{x} = 0$. The normalization exists (*i.e.*, $g_0$ is finite) if $\alpha > -n/2$. This condition ensures the existence of all the $g_\ell$'s.

## 6.3 Exponential Density.

This case is defined by the choice $g(s) = e^{-\sqrt{s}}$; defining $\nu = (n+1)/2$ we obtain

$$g_\ell = \Gamma(n+2\ell) \tag{6.14}$$

$$A(n;\ell) = 2^\ell(\nu)_\ell \tag{6.15}$$

$$\mathcal{G}(\mathbf{k}) = (1+q^2)^{-\nu} \tag{6.16}$$

## 6.4 Generalized Lorentzian Density.

Here we choose $g(s) = (s+1)^{-p}$. This density extends over all space, but $\rho(\mathbf{x})$ falls off as $|\mathbf{x}|^{-2p}$ at infinity. Defining $\mu = p - n/2$ we obtain

$$g_\ell = \frac{\Gamma(n/2+\ell)\Gamma(\mu-\ell)}{2\Gamma(p)}\,, \qquad \ell < \mu \tag{6.17}$$

$$A(n;\ell) = \frac{\Gamma(\mu-\ell)}{2^\ell\Gamma(\mu)}\,, \qquad \ell < \mu \tag{6.18}$$

(the $g_\ell$'s and $A(n;\ell)$'s do not exist for $\ell \geq \mu$).

In this case it is not legitimate to obtain $\mathcal{G}(\mathbf{k})$ by term-by-term integration of the Taylor expansion of the Bessel function in Eq. (3.8). Instead, we have :

$$\mathcal{G}(\mathbf{k}) = \frac{2\Gamma(p)}{\Gamma(\mu)} \left(\frac{2}{q}\right)^{\nu} \int_0^{\infty} dv \, \frac{v^{n/2} J_\nu(qv)}{(v^2+1)^p}\,, \qquad \nu = n/2-1 \tag{6.19a}$$

$$= \frac{2}{\Gamma(\mu)} \left(\frac{q}{2}\right)^{\mu} K_\mu(q) \tag{6.19b}$$

$$\text{“}=\text{”} \frac{1}{\Gamma(\mu)} \sum_{0\le \ell<\mu} \Gamma(\mu-\ell) \frac{(-q^2/4)^\ell}{\ell!} + \text{infinite terms} \tag{6.19c}$$

where $K_\mu(z)$ is the usual modified Bessel function [10] of order $\mu = p - n/2$. For the distribution to exist, *i.e.*, to be normalizable, the condition $p > n/2$, or $\mu > 0$, must be satisfied. Eq. (6.18) shows that the moments exist only for values of $\ell$ such that $\ell < \mu$, or $r < 2p - n$, in consequence of the behavior $\rho(\mathbf{x}) \sim |\mathbf{x}|^{-2p}$ as $x \to \infty$. Although the generating function $\mathcal{G}(\mathbf{k})$, given by Eq. (6.19b), exists for all $\mathbf{k}$ when $\mu > 0$, its Taylor expansion at $\mathbf{k} = 0$ does not formally exist because its $r$th derivative at $\mathbf{k} = 0$ is finite only as long as $r < 2\mu$, in full agreement with the discussion on densities with a power-law fall-off at infinity following Eq. (2.3). This is the meaning of the symbol "=" in Eq. (6.19c).

### 6.4.1 Subspace Projection for the Lorentzian Density.

For the strictly quadratic action ($\mathbf{a} = 0$), Eq. (5.13) yields

$$\rho'(\mathbf{x}') = \frac{2\Gamma(p)\sqrt{\det H'}}{\pi^{m/2}\Gamma(\mu)\Gamma\left((n-m)/2\right)} \int_0^{\infty} dv \, \frac{v^{n-m-1}}{(v^2+s'(\mathbf{x}'))^p}$$

$$= \frac{\sqrt{\det H'}}{\pi^{m/2}\Gamma(\mu)} \frac{\Gamma(p')}{(s'(\mathbf{x}')+1)^{p'}} \tag{6.20}$$

where $p' = p - (n-m)/2$ and where $s'(\mathbf{x}')$ is given by Eq. (5.14). The projected density is to be compared with the original,

$$\rho(\mathbf{x}) = \frac{\sqrt{\det H}}{\pi^{n/2}\Gamma(\mu)} \frac{\Gamma(p)}{(s(\mathbf{x})+1)^p} \tag{6.21}$$

which shows that the form is maintained (the projected density is also a Lorentzian), although the power $p$ is shifted by $(n-m)/2$. Note that $\mu = p - n/2$ can be rewritten in the suggestive form $\mu = p' - m/2$, hence Eq. (6.20) can be obtained from (6.21) by the replacements $s(\mathbf{x}) \to s'(\mathbf{x}')$, $\det H \to \det H'$, $n \to m$ and $p \to p'$, as it should be expected.

## 6.5 Microcanonical Density.

This case corresponds to the choice $g(s) = \delta(s-1)$ in the general formula (3.7). We call it "microcanonical" because it corresponds to the microcanonical distribution in statistical mechanics for quadratic Hamiltonians. It also corresponds to the "airbag," or KV, distribution in beam physics [11]. We obtain

$$g_\ell = 1/2 \tag{6.22}$$

$$A(n;\ell) = \frac{1}{2^\ell (n/2)_\ell} \tag{6.23}$$

$$\mathcal{G}(\mathbf{k}) = \Gamma(\nu+1)\left(\frac{2}{q}\right)^{\nu} J_\nu(q)\,, \qquad \nu = n/2-1 \tag{6.24}$$

### 6.5.1 Subspace Projection for the Microcanonical Density.

Setting $\mathbf{a} = 0$ in Eq. (5.13) yields, for this particular example,

$$\rho'(\mathbf{x}') = \frac{2\Gamma(n/2)\sqrt{\det H'}}{\pi^{m/2}\Gamma\left((n-m)/2\right)} \int_0^\infty dv\, \delta(v^2 + s'(\mathbf{x}') - 1) v^{n-m-1}$$
$$= \frac{\Gamma(n/2)\sqrt{\det H'}}{\pi^{m/2}\Gamma(\beta+1)} (1 - s'(\mathbf{x}'))^\beta \theta(1 - s'(\mathbf{x}')) \tag{6.25}$$

where $\beta = (n-m)/2 - 1$ and where $s'(\mathbf{x}')$ is given by Eq. (5.14). The projected density is a generalized waterbag density (see Sec. 6.6.2) of dimension $m$ and power $\beta = (n-m)/2 - 1$.

## 6.6 Beta-Function Density.

By "beta-function density" we mean the choice $g(s) = s^\alpha(1-s)^\beta\theta(1-s)$. This density is of finite extent, as it vanishes, by definition, for $s > 1$, hence all moments exist provided $\alpha$ and $\beta$ are appropriately restricted, as spelled out below. We obtain:

$$g_\ell = \frac{\Gamma(n/2+\alpha+\ell)\Gamma(\beta+1)}{2\Gamma(n/2+\alpha+\beta+\ell+1)} \tag{6.26}$$

$$A(n;\ell) = \frac{(n/2+\alpha)_\ell}{2^\ell (n/2)_\ell (n/2+\alpha+\beta+1)_\ell} \tag{6.27}$$

$$\mathcal{G}(\mathbf{k}) = {}_1F_2(n/2+\alpha; n/2, n/2+\alpha+\beta+1; -q^2/4) \tag{6.28}$$

where $\nu = n/2 - 1$ and ${}_1F_2$ is a generalized hypergeometric function (not to be confused with the ordinary hypergeometric function ${}_2F_1$) [10]. The normalizability condition of $\rho(\mathbf{x})$ demands $\alpha > -n/2$ and $\beta > -1$.

### 6.6.1 Waterbag Density.

This case corresponds to the choice $\alpha = \beta = 0$ in the beta-function density, *i.e.*, $g(s) = \theta(1-s)$, so that $\rho(\mathbf{x})$ is uniform for $0 \le s \le 1$. We obtain from the general formulas above

$$g_\ell = \frac{1}{n+2\ell} \tag{6.29}$$

$$A(n;\ell) = \frac{1}{2^\ell (n/2+1)_\ell} \tag{6.30}$$

$$\mathcal{G}(\mathbf{k}) = \Gamma(\nu+1)\left(\frac{2}{q}\right)^\nu J_\nu(q)\,, \qquad \nu = n/2 \tag{6.31}$$

### 6.6.2 Generalized Waterbag Density.

This corresponds to setting $\alpha = 0$ in the beta-function density, *i.e.*, $g(s) = (1-s)^\beta\theta(1-s)$, and we obtain

$$g_\ell = \frac{\Gamma(\beta+1)\Gamma(n/2+\ell)}{2\Gamma(n/2+\ell+\beta+1)} \tag{6.32}$$

$$A(n;\ell) = \frac{1}{2^\ell (n/2+\beta+1)_\ell} \tag{6.33}$$

$$\mathcal{G}(\mathbf{k}) = \Gamma(\nu+1)\left(\frac{2}{q}\right)^\nu J_\nu(q)\,, \qquad \nu = n/2+\beta \tag{6.34}$$

The normalizability condition of $\rho(\mathbf{x})$ demands $\beta > -1$.

### 6.6.3 Power Density.

This corresponds to setting $\beta = 0$ in the beta-function density, *i.e.*, $g(s) = s^\alpha \theta(1-s)$ so that

$$g_\ell = \frac{1}{n + 2\alpha + 2\ell} \tag{6.35}$$

$$\mathcal{G}(\mathbf{k}) = (n + 2\alpha) \sum_{\ell=0}^{\infty} \frac{(-q^2/4)^\ell}{(n + 2\alpha + 2\ell)(n/2)_\ell \, \ell!} \tag{6.36}$$

The normalizability condition of $\rho(\mathbf{x})$ demands $\alpha > -n/2$.

# 7 Extension to the Complex Space.

In this section we will consider an action of the form

$$s(\mathbf{z}) = \langle \mathbf{z} | H | \mathbf{z} \rangle \equiv \sum_{ij=1}^{n} H_{ij} \bar{z}_i z_j \tag{7.1}$$

where $\mathbf{z}$ is a complex $n$-dimensional vector and $H$ is an $n \times n$ positive-definite Hermitian matrix ($H^* = H^T$), so that $s(\mathbf{z})$ is real. In this case the $n$-dimensional complex vector $\mathbf{z}$ is $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, the bra $\langle \mathbf{z}|$ and ket $|\mathbf{z}\rangle$ are defined by $|\mathbf{z}\rangle = \mathrm{col}(z_1, z_2, \cdots, z_n)$ and $\langle \mathbf{z}| = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$ (we use interchangeably the notations $z^*$ and $\bar{z}$ to denote complex conjugation). Although we could add a linear term to the action, of the form $\langle \mathbf{a}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{a}\rangle$, we shall restrict the discussion to strictly quadratic action of the form (7.1).

A real density is defined by

$$\rho(\mathbf{z}) = N g(s(\mathbf{z})) \tag{7.2}$$

where $g(s)$ is a real non-negative function of the real variable $s$. It is straightforward to prove that Theorems 3.1 and 5.1, along with Corollaries 1, 2 and 3 carry over to the complex case with the replacement "real symmetric matrix" → "complex Hermitian matrix" and "orthogonal matrix" → "unitary matrix." In particular, we conclude that $s(\mathbf{z}) \geq 0$, the equality holding if and only if $\mathbf{z} = 0$.

The normalization constant $N$, as well as any statistical average of a function of $\mathbf{z}$, is determined by an $n$-fold surface integral, one for each of the components $z_i$ of $\mathbf{z}$, *i.e.*

$$N^{-1} = \int d^{2n}\mathbf{z} \, g(s(\mathbf{z})) \tag{7.3}$$

where

$$d^{2n}\mathbf{z} = \prod_{i=1}^{n} dx_i dy_i \tag{7.4}$$

where $x_i$ and $y_i$ are the real and imaginary parts of $z_i$.

The most straightforward way to extend the results from the real to the complex case is by casting the latter in terms of a real $2n \times 2n$ matrix. This is accomplished by splitting $H$ and $\mathbf{z}$ into their real and imaginary parts, $H = A + iB$ and $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $A$, $B$, $\mathbf{x}$ and $\mathbf{y}$ are real, so that

$$A = \frac{1}{2}(H + H^*) = A^T \tag{7.5a}$$

$$B = \frac{1}{2i}(H - H^*) = -B^T \tag{7.5b}$$

hence

$$\begin{aligned} s(\mathbf{z}) &= \sum_{i,j=1}^{n} (A + iB)_{ij}(x_i - iy_i)(x_j + iy_j) \\ &= \sum_{i,j=1}^{n} (A_{ij}(x_i x_j + y_i y_j) - B_{ij}(x_i y_j - x_j y_i)) \end{aligned} \tag{7.6}$$

This expression can be cast in the form $s(\mathbf{z}) = \mathbf{X}^T \cdot M\mathbf{X} = \langle \mathbf{X}\,|M|\,\mathbf{X}\rangle$ where $M$ is the real symmetric $2n\times 2n$ matrix

$$M = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{7.7}$$

and $\mathbf{X}$ is the real $2n$-dimensional vector

$$\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \tag{7.8}$$

It is now straightforward to prove that the eigenvalues of $H$ are the same as those of $M$: let $\mathbf{z}$ be an eigenvector of $H$ with eigenvalue $h$, that is $H\mathbf{z} = h\mathbf{z}$ or

$$(A + iB)(\mathbf{x} + i\mathbf{y}) = h(\mathbf{x} + i\mathbf{y}) \tag{7.9}$$

Separating the real and imaginary parts and using the fact that $h$ is real, this equation is written in the form $M\mathbf{X} = h\mathbf{X}$ which shows that $h$ is an eigenvalue of $M$. Now this equation, combined with Eq. (7.7), implies that $M\widetilde{\mathbf{X}} = h\widetilde{\mathbf{X}}$ where $\widetilde{\mathbf{X}} = \mathrm{col}(\mathbf{y}, -\mathbf{x})$, *i.e.* the vector $\widetilde{\mathbf{X}}$ is also an eigenvector of $M$ with eigenvalue $h$. Since $\widetilde{\mathbf{X}}$ is orthogonal to (and therefore linearly independent of) $\mathbf{X}$, we conclude that each eigenvalue $h$ of $H$ is a doubly degenerate eigenvalue of $M$. A corollary of this result is that, if $H$ is positive-definite, so is $M$. Another corollary is that $\det M = (\det H)^2$. With these observations, the results of all previous sections carry over to this case with the replacements $n \to 2n$, $H \to M$ and $\det H \to (\det H)^2$. All moments are expressed in terms of the matrix elements of $M^{-1}$, which is given, in terms of $A$ and $B$, by

$$M^{-1} = \begin{pmatrix} (A + BA^{-1}B)^{-1} & (B + AB^{-1}A)^{-1} \\ -(B + AB^{-1}A)^{-1} & (A + BA^{-1}B)^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(H^{-1}) & -\mathrm{Im}(H^{-1}) \\ \mathrm{Im}(H^{-1}) & \mathrm{Re}(H^{-1}) \end{pmatrix} \tag{7.10}$$

where we have used the decomposition of $H^{-1}$ into its real and imaginary parts,

$$H^{-1} = (A + iB)^{-1} = (A + BA^{-1}B)^{-1} - i(B + AB^{-1}A)^{-1} \tag{7.11}$$

This result implies that $\mathrm{Re}(H^{-1})$ is symmetric while $\mathrm{Im}(H^{-1})$ is antisymmetric, as it should be the case.

As an example, consider a Gaussian density $\rho(\mathbf{z}) = N \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle\} = N \exp\{-\langle \mathbf{X}|M|\mathbf{X}\rangle\}$. The 2nd-order moments are given by the results of Sec. 6.1,

$$\langle x_i x_j\rangle = \langle y_i y_j\rangle = \tfrac{1}{2}(A + BA^{-1}B)^{-1}_{ij} = \tfrac{1}{2}\mathrm{Re}(H^{-1})_{ij} \tag{7.12a}$$

$$\langle x_i y_j\rangle = -\langle y_i x_j\rangle = \tfrac{1}{2}(B + AB^{-1}A)^{-1}_{ij} = -\tfrac{1}{2}\mathrm{Im}(H^{-1})_{ij} \tag{7.12b}$$

where the factors 1/2 arise from the absence of the factor 1/2 in the exponent of $\exp\{-\langle \mathbf{X}|M|\mathbf{X}\rangle\}$ (*cf.* compare with Eq. (6.1)). In terms of the $z$'s, this implies

$$\langle z_i z_j\rangle = \langle (x_i + iy_i)(x_j + iy_j)\rangle = 0 \tag{7.13a}$$

$$\langle z_i \bar{z}_j\rangle = \langle (x_i + iy_i)(x_j - iy_j)\rangle = H^{-1}_{ij} \tag{7.13b}$$

As an equivalent alternative to the real $2n$-dimensional space of the $\mathbf{x}$'s and $\mathbf{y}$'s, one may deal with the action (7.1) directly in the $n$-dimensional complex space of the $\mathbf{z}$'s. For example, we compute the normalization constant $N$, given by

$$N^{-1} = \int d^{2n}\mathbf{z}\, g(\langle \mathbf{z}|H|\mathbf{z}\rangle) \tag{7.14}$$

by noting that $H$, being Hermitian, is diagonalized by a unitary matrix $U$, namely $H = U^{-1}H_D U$ where $H_D = \mathrm{diag}(h_1, h_2, \cdots, h_n)$ and $U^{-1} = U^\dagger$. Since $h_i > 0$ for all $i$, the matrix $H_D^{1/2}$ exists unambiguously hence we can define the change of variables $|\mathbf{v}\rangle = H_D^{1/2}U|\mathbf{z}\rangle$, or $\langle \mathbf{v}| = \langle \mathbf{z}|U^\dagger H_D^{1/2}$, yielding $s(\mathbf{z}) = \langle \mathbf{z}|H|\mathbf{z}\rangle = \langle \mathbf{v}|\mathbf{v}\rangle = v^2$ hence

$$N^{-1} = \frac{1}{\det H}\int d^{2n}\mathbf{v}\, g(v^2) = \frac{\Omega_{2n}\, g_0}{\det H} = \frac{2\pi^n g_0}{\Gamma(n)\det H} \tag{7.15}$$

where now

$$g_0 = \int_0^\infty dv\, g(v^2) v^{2n-1} = \frac{1}{2}\int_0^\infty ds\, g(s) s^{n-1} \tag{7.16}$$

The moment of the $z_i$'s and $\bar{z}_i$'s is defined by

$$C^{(\ell_1,\ell_2)}_{ij\cdots,km\cdots} \equiv \langle \underbrace{z_i z_j \cdots}_{\ell_1\ z\text{'s}} \underbrace{\bar{z}_k \bar{z}_m \cdots}_{\ell_2\ \bar{z}\text{'s}} \rangle = \int d^{2n}\mathbf{z}\, \rho(\mathbf{z}) ( \underbrace{z_i z_j \cdots}_{\ell_1\ z\text{'s}} \underbrace{\bar{z}_k \bar{z}_m \cdots}_{\ell_2\ \bar{z}\text{'s}} ) \tag{7.17}$$

This expression implies that, if $\rho(\mathbf{z})$ is of finite extent, or if it falls off as $|\mathbf{z}| \to \infty$ faster than any power of $z$, then the moments exist for any $\ell_1,\ \ell_2$. If, on the other hand, $\rho(\mathbf{z}) \sim |\mathbf{z}|^{-2p}$ as $|\mathbf{z}| \to \infty$, then the moments exist only if $\ell_1 + \ell_2 < 2(p-n)$. Eq. (7.17) also implies that the moment $C^{(\ell_1,\ell_2)}_{ij\cdots,km\cdots}$ is fully symmetric under any permutation of the first group of indices $(i, j, \cdots)$ and/or the second $(k, m, \cdots)$. Furthermore, these sets of indices are exchanged under complex conjugation, namely

$$C^{(\ell_1,\ell_2)*}_{ij\cdots,km\cdots} = C^{(\ell_2,\ell_1)}_{km\cdots,ij\cdots} \tag{7.18}$$

The generating function for the moments is defined by

$$\mathcal{G}(\mathbf{k},\mathbf{q}) = \int d^{2n}\mathbf{z}\, \rho(\mathbf{z}) e^{-i(\langle \mathbf{k}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{q}\rangle)} \tag{7.19}$$

where $\mathbf{k}$ and $\mathbf{q}$ are two real,[3] independent, $n$-dimensional vectors, so that

$$C^{(\ell_1,\ell_2)}_{ij\cdots,km\cdots} = i^{\ell_1+\ell_2} \frac{\partial^{\ell_1+\ell_2}\mathcal{G}(\mathbf{k},\mathbf{q})}{\underbrace{\partial k_i \partial k_j \cdots}_{\ell_1\ k\text{'s}} \underbrace{\partial q_k \partial q_m \cdots}_{\ell_2\ q\text{'s}}} \Bigg|_{\mathbf{k}=\mathbf{q}=0} \tag{7.20}$$

In case that $\rho(\mathbf{z})$ is of finite extent, or if it falls off as $|\mathbf{z}| \to \infty$ faster than any power of $|\mathbf{z}|$, then all moments exist and we can Taylor-expand the exponential factor in (7.19) about $\mathbf{k} = \mathbf{q} = 0$ to obtain

$$\begin{aligned} \mathcal{G}(\mathbf{k},\mathbf{q}) = 1 &- i\sum_{i=1}^{n} (k_i\langle z_i\rangle + q_i\langle \bar{z}_i\rangle) \\ &+ \frac{(-i)^2}{2!}\sum_{i,j=1}^{n} (k_i k_j\langle z_i z_j\rangle + q_i q_j\langle \bar{z}_i \bar{z}_j\rangle + 2k_i q_j\langle z_i \bar{z}_j\rangle) + \cdots \quad (7.21\text{a}) \\ = 1 &- i\sum_{i=1}^{n} \left(k_i C^{(1,0)}_{i,} + q_i C^{(0,1)}_{,i}\right) \\ &+ \frac{(-i)^2}{2!}\sum_{i,j=1}^{n} \left(k_i k_j C^{(2,0)}_{ij,} + q_i q_j C^{(0,2)}_{,ij} + 2k_i q_j C^{(1,1)}_{i,j}\right) + \cdots \quad (7.21\text{b}) \end{aligned}$$

In what remains of this section we confine our attention to strictly quadratic actions, of the form (7.1). In this case we note that the moment $C^{(\ell_1,\ell_2)}$ vanishes if $\ell_1 \neq \ell_2$. To prove this assertion, we make the change of variable $z_i \to z_i e^{i\alpha}$ in the right-hand side of (7.17), where $\alpha$ is a real constant. Since the integral over $\mathbf{z}$ extends over all space, and the Jacobian of this transformation is unity, and $s(\mathbf{z})$ is invariant under this transformation, we obtain the identity

$$C^{(\ell_1,\ell_2)}_{ij\cdots,km\cdots} = e^{i(\ell_1-\ell_2)\alpha}\, C^{(\ell_1,\ell_2)}_{ij\cdots,km\cdots} \tag{7.22}$$

[3]The analytic extension to complex $\mathbf{k}$ and $\mathbf{q}$ is straightforward but not necessary for our discussion.

which implies $C^{(\ell_1,\ell_2)} = 0$ if $\ell_1 \neq \ell_2$ in consequence of the arbitrariness of $\alpha$. Using now the positivity of $s$ one can write, in complete analogy with Eq. (3.7),

$$\rho(\mathbf{z}) = N \int_0^\infty ds\, g(s)\, \delta(\langle \mathbf{z}|H|\mathbf{z}\rangle - s) \tag{7.23}$$

hence

$$\begin{aligned}\mathcal{G}(\mathbf{k},\mathbf{q}) &= N \int_0^\infty ds\, g(s) \int d^{2n}\mathbf{z}\, \delta(\langle \mathbf{z}|H|\mathbf{z}\rangle - s) e^{-i(\langle \mathbf{k}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{q}\rangle)} \\ &= \frac{\Gamma(n)}{g_0 u^{n-1}} \int_0^\infty dv\, g(v^2) v^n J_{n-1}(2vu)\end{aligned} \tag{7.24}$$

where we have used Eq. (B.10), made the change of variable $s = v^2$, and defined $u = \sqrt{\langle \mathbf{k}|G|\mathbf{q}\rangle}$ and $G = H^{-1}$. Although $\langle \mathbf{k}|G|\mathbf{q}\rangle$ is not, in general, positive, no ambiguity arises in the definition of its square root since only even powers of $u$ appear in Eq. (7.24).

If $g(s)$ falls off as $s \to \infty$ faster than any power of $s$, we can Taylor-expand the Bessel function in Eq. (7.24) and integrate term by term yielding

$$\mathcal{G}(\mathbf{k},\mathbf{q}) = \sum_{\ell=0}^{\infty} A(n;\ell) \frac{(-u^2)^\ell}{\ell!} \tag{7.25}$$

where $A(n;\ell)$ is defined by

$$A(n;\ell) = \frac{g_\ell}{g_0 (n)_\ell} \tag{7.26}$$

and $g_\ell$ is given by

$$g_\ell = \int_0^\infty dv\, g(v^2) v^{2(n+\ell)-1} = \frac{1}{2} \int_0^\infty ds\, g(s) s^{n+\ell-1} \tag{7.27}$$

In consequence of the vanishing of $C^{(\ell_1,\ell_2)}$ for $\ell_1 \neq \ell_2$, only the even terms contribute to the Taylor expansion (7.21), and of these, only those for which $\ell_1 = \ell_2$, thus

$$\mathcal{G}(\mathbf{k},\mathbf{q}) = 1 + \frac{(-i)^2}{2!}\binom{2}{1} \sum_{i,j=1}^{n} (k_i q_j) C^{(1,1)}_{i,j} + \frac{(-i)^4}{4!}\binom{4}{2} \sum_{i,j,k,m=1}^{n} (k_i k_j q_k q_m) C^{(2,2)}_{ij,km} + \cdots \tag{7.28a}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!^2} \sum_{\substack{i,j,\cdots=1 \\ k,m,\cdots=1}}^{n} (k_i k_j \cdots q_k q_m \cdots) C^{(\ell,\ell)}_{ij\cdots,km\cdots} \tag{7.28b}$$

By using the formula

$$\frac{\partial^\ell (x_{i_1} x_{i_2} \cdots x_{i_\ell})}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_\ell}} = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_\ell j_\ell} + \text{ permutations} \qquad \text{(total of } 2\ell \text{ indices)} \tag{7.29}$$

where the $x$'s represent here either the $k$'s or the $q$'s, and where the "permutations" are such as to make the right-hand side a fully symmetric tensor of rank $2\ell$ (total of $\ell!$ terms), one recovers the result (7.20) (the derivatives with respect to the $k$'s in (7.20) will yield a factor $\ell!$, as will the derivatives with respect to the $q$'s; these two factors of $\ell!$ will just cancel the $\ell!^2$ in the denominator of (7.28b)).

The moment $C^{(\ell,\ell)}_{ij\cdots,km\cdots}$ is obtained from Eqs. (7.20) and (7.25) by setting $\ell_1 = \ell_2$ and using the easily-proven result

$$\frac{\partial^{2\ell}\langle \mathbf{k}|G|\mathbf{q}\rangle^{\ell}}{\underbrace{\partial k_i \partial k_j \cdots}_{\ell\ k\text{'s}} \underbrace{\partial q_k \partial q_m \cdots}_{\ell\ q\text{'s}}} = \ell! S^{(\ell,\ell)}_{ij\cdots,km\cdots} \tag{7.30}$$

where $S^{(\ell,\ell)}_{ij\cdots,km\cdots}$ is defined to be

$$S^{(\ell,\ell)}_{ij\cdots,km\cdots} = \Big[ \underbrace{G_{ik} G_{jm} \cdots}_{\ell\ G\text{'s}} + \text{permutations} \Big] \qquad (\ell!\ \text{terms, } 2\ell \text{ indices}) \tag{7.31}$$

where the "permutations" are such as to make the right-hand side fully symmetric[4] under any permutation of the first set of indices $(i, j, \cdots)$ and simultaneously under any permutation of the second set $(k, m, \cdots)$. When this result is applied to either (7.24) or (7.25) we obtain the fundamental result

$$C^{(\ell,\ell)}_{ij\cdots,km\cdots} = \langle \underbrace{z_i z_j \cdots}_{\ell\ z\text{'s}} \underbrace{\bar{z}_k \bar{z}_m \cdots}_{\ell\ \bar{z}\text{'s}} \rangle = A(n;\ell) S^{(\ell,\ell)}_{ij\cdots,km\cdots} \tag{7.32}$$

A few examples are:

$$C^{(1,1)}_{i,j} = \langle z_i \bar{z}_j \rangle = A(n;1) G_{ij} \tag{7.33a}$$

$$C^{(2,2)}_{ij,km} = \langle z_i z_j \bar{z}_k \bar{z}_m \rangle = A(n;2) \left[ G_{ik} G_{jm} + G_{im} G_{jk} \right] \tag{7.33b}$$

$$C^{(3,3)}_{ijk,\alpha\beta\gamma} = \langle z_i z_j z_k \bar{z}_\alpha \bar{z}_\beta \bar{z}_\gamma \rangle = A(n;3) \left[ G_{i\alpha} G_{j\beta} G_{k\gamma} + \cdots \right] \qquad (6\ \text{terms}) \tag{7.33c}$$

If $g(s) \sim s^{-p}$ as $s \to \infty$, corresponding to $\rho(\mathbf{z}) \sim |\mathbf{z}|^{-2p}$ as $\mathbf{z} \to \infty$, the moment $C^{(\ell,\ell)}_{ij\cdots,km\cdots}$ exists only if $g_\ell$ exists, and is given by expression (7.32), only if $\ell < p - n$, in agreement with the discussion following Eq. (7.17).

## 7.1 Recursion Formulas for the Moments.

Eq. (7.32) can be cast in the recursive form

$$C^{(\ell,\ell)}_{ijk\cdots,\alpha\beta\gamma\cdots} = B(n;\ell) \Big[ C^{(1,1)}_{i,\alpha} C^{(\ell-1,\ell-1)}_{jk\cdots,\beta\gamma\cdots} + \text{permutations} \Big] \tag{7.34}$$

or, equivalently,

$$\langle z_i z_j \cdots \bar{z}_k \bar{z}_m \cdots \rangle = B(n;\ell) \Big[ \langle z_i \bar{z}_k \rangle \langle z_j \cdots \bar{z}_m \cdots \rangle + \text{permutations} \Big] \tag{7.35}$$

where consistency demands the definition $C^{(0,0)} = 1$, and where

$$B(n;\ell) = \frac{A(n;\ell)}{A(n;\ell-1) A(n;1)} = \frac{n g_0 g_\ell}{(n+\ell-1) g_1 g_{\ell-1}} \tag{7.36}$$

From Eq. (7.32), the moment $C^{(\ell,\ell)}$ can also be expressed purely in terms of the $C^{(1,1)}$'s,

$$C^{(\ell,\ell)}_{ijk\cdots,\alpha\beta\gamma\cdots} = D(n;\ell) \Big[ \underbrace{C^{(1,1)}_{i,\alpha} C^{(1,1)}_{j,\beta} \cdots}_{\ell\ C^{(1,1)}\text{'s}} + \text{permutations} \Big] \tag{7.37}$$

or, equivalently,

$$\langle z_i z_j \cdots \bar{z}_k \bar{z}_m \cdots \rangle = D(n;\ell) \Big[ \underbrace{\langle z_i \bar{z}_k \rangle \langle z_j \bar{z}_m \rangle \cdots}_{\ell\ \text{terms}} + \text{permutations} \Big] \tag{7.38}$$

where

$$D(n;\ell) = \frac{A(n;\ell)}{A^\ell(n;1)} = \frac{g_\ell g_0^{\ell-1}}{g_1^\ell (n)_\ell} \tag{7.39}$$

[4]Note that $G$ is Hermitian rather than symmetric, hence $G_{ji} = G^*_{ij}$.

## 7.2 Example: the Complex Gaussian Density.

For the case of the complex Gaussian density we choose $g(s) = e^{-s}$ (it is convenient not to include the factor 1/2 in the exponent in this complex case), thus

$$\rho(\mathbf{z}) = N \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle\} \tag{7.40}$$

The normalization constant $N$ is determined by Eqs. (7.15-7.16), yielding

$$N^{-1} = \int d^{2n}\mathbf{z}\, \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle\} = \frac{\pi^n}{\det H} \tag{7.41}$$

The generating function follows from the integrals computed in Appendix B,

$$\mathcal{G}(\mathbf{k},\mathbf{q}) = N \int d^{2n}\mathbf{z}\, \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle - i\langle \mathbf{k}|\mathbf{z}\rangle - i\langle \mathbf{z}|\mathbf{q}\rangle\} = \exp\{-\langle \mathbf{k}|H^{-1}|\mathbf{q}\rangle\} \tag{7.42}$$

and the $g_\ell$'s are

$$g_\ell = \frac{1}{2}\int_0^\infty ds\, e^{-s} s^{n+\ell-1} = \frac{1}{2}\Gamma(n+\ell) \tag{7.43}$$

which implies $A(n;\ell) = 1$ for all $n$ and $\ell$, as expected from the comparison of Eqs. (7.25) and (7.42).

Taking $\partial/\partial k_i \partial q_j$ or $\partial/\partial k_i \partial k_j$ or $\partial/\partial q_i \partial q_j$ of (7.42), then setting $\mathbf{k} = \mathbf{q} = 0$ yields

$$\langle z_i \bar{z}_j\rangle = H^{-1}_{ij} \tag{7.44}$$

and $\langle z_i z_j\rangle = \langle \bar{z}_i \bar{z}_j\rangle = 0$, in agreement with Eqs. (7.13). The moment $\langle z_i \bar{z}_j\rangle$ or, indeed, any even-order moment, can also be obtained by differentiating both sides of Eq. (7.41) with respect to $H_{ij}$ and using the formula $\partial \det M/\partial M_{ij} = (\det M) M^{-1}_{ji}$, valid for any nonsingular matrix $M$.

## 7.3 Subspace Projection.

The subspace projection of $\rho(\mathbf{z})$ is obtained in a similar fashion as in the real case, thus

$$\rho'(\mathbf{z}') = \int d^{2(n-m)}\mathbf{y}\, \rho(\mathbf{z}', \mathbf{y}) \tag{7.45}$$

where $\mathbf{y} = (z_{m+1}, z_{m+2}, \cdots, z_n)$. For the strictly quadratic density, $\rho(\mathbf{z}) = Ng(\langle \mathbf{z}|H|\mathbf{z}\rangle)$, we partition $H$ and $\mathbf{z}$ in the form

$$H = \left(\begin{array}{c|c} S & E \\ \hline F & K \end{array}\right), \qquad \mathbf{z} = \left(\begin{array}{c} \mathbf{z}' \\ \hline \mathbf{y} \end{array}\right), \tag{7.46}$$

where $S$ is $m \times m$, $E$ is $m \times (n-m)$, $F$ is $(n-m) \times m$ and $K$ is $(n-m)\times(n-m)$, with $S^\dagger = S$, $K^\dagger = K$ and $E^\dagger = F$. This partition leads to

$$\langle \mathbf{z}|H|\mathbf{z}\rangle = \langle \mathbf{y}|K|\mathbf{y}\rangle + \langle \mathbf{y}|E|\mathbf{z}'\rangle + \langle \mathbf{z}'|F|\mathbf{y}\rangle + \langle \mathbf{z}'|S|\mathbf{z}'\rangle \tag{7.47}$$

By shifting $\mathbf{y}$ by a constant vector $\mathbf{h}$ in the form $\mathbf{y} = \mathbf{y}' + \mathbf{h}$ and choosing $|\mathbf{h}\rangle = -K^{-1}F|\mathbf{z}'\rangle$ or $\langle \mathbf{h}| = -\langle \mathbf{z}'|EK^{-1}$, we cancel the terms linear in $\mathbf{y}'$ hence

$$\langle \mathbf{z}|H|\mathbf{z}\rangle = \langle \mathbf{y}'|K|\mathbf{y}'\rangle + s'(\mathbf{z}') \tag{7.48}$$

where we have defined the reduced action

$$s'(\mathbf{z}') = \langle \mathbf{z}'|(S - EK^{-1}F)|\mathbf{z}'\rangle \tag{7.49}$$

hence

$$\rho'(\mathbf{z}') = N \int d^{2(n-m)}\mathbf{y}'\, g(\langle \mathbf{y}'|K|\mathbf{y}'\rangle + s'(\mathbf{z}')) \tag{7.50}$$

In the above formulas, $K^{-1}$ is guaranteed to exist on account of Corollary 1, which implies that $K$ is a Hermitian positive-definite matrix. For this reason we can set $\mathbf{y}' = V^{-1}K_D^{-1/2}\mathbf{v}$ where $V$ is the unitary matrix that diagonalizes $K$, namely $VKV^{-1} = K_D$. Using $V^\dagger = V^{-1}$ we obtain $\langle \mathbf{y}'|K|\mathbf{y}'\rangle = \langle \mathbf{v}|\mathbf{v}\rangle \equiv v^2$ hence the projected density is

$$\begin{aligned}\rho'(\mathbf{z}') &= \frac{N}{\det K} \int d^{2(n-m)}\mathbf{v}\, g(v^2 + s'(\mathbf{z}')) \\ &= \frac{\Gamma(n)\det H}{g_0\pi^m\Gamma(n-m)\det K} \int_0^\infty dv\, g(v^2 + s'(\mathbf{z}'))v^{2(n-m)-1} \end{aligned} \tag{7.51}$$

where $g_0$ is given by Eq. (7.16). This result should be compared with (5.13).

# 8 Conclusions.

We have presented the computation of the moments and their generating functions for $n$-dimensional densities of the form $\rho(\mathbf{x}) \propto g(s(\mathbf{x}))$, where $s(\mathbf{x})$ is a positive quadratic form of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $g(s)$ is a non-negative function of $s$. In particular, we have found the recursion relation that relates the $r$th-order moment with the $(r-2)$th-order moment. This result constitutes a generalization of the Gaussian moment theorem, which corresponds to the specific choice $g(s) = e^{-s/2}$. We have presented the extension of the above analysis to densities $\rho(\mathbf{z})$ defined over the $n$-dimensional complex space of vectors $\mathbf{z}$.


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# A Certain $n$-Dimensional Integrals.

Consider the $n$-dimensional vector $\mathbf{x}$ with Cartesian coordinates $(x_1, x_2, \cdots, x_n)$. The vector $\mathbf{x}$ can also be fully specified by its length $x = |\mathbf{x}|$ plus $n-1$ angles $\theta_1,\ \theta_2, \cdots,\ \theta_{n-1}$ such that [12]

$$\left.\begin{aligned} x_1 &= x\cos\theta_1 \\ x_2 &= x\sin\theta_1\cos\theta_2 \\ x_3 &= x\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ x_{n-1} &= x\sin\theta_1\sin\theta_2\cdots\cos\theta_{n-1} \\ x_n &= x\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1} \end{aligned}\right\} \quad \text{where} \begin{cases} 0 \le \theta_i \le \pi\,, & i = 1,\ 2,\ \cdots,\ n-2 \\ 0 \le \theta_i \le 2\pi\,, & i = n-1 \end{cases} \tag{A.1}$$

The generalized coordinates $(q_i) = (x, \theta_1, \theta_2, \cdots, \theta_{n-1})$ are orthogonal in the sense that

$$\frac{\partial\mathbf{x}}{\partial q_i}\cdot\frac{\partial\mathbf{x}}{\partial q_j} = 0 \qquad \text{for } i \neq j \tag{A.2}$$

The volume element is $d^n\mathbf{x} = x^{n-1}dx d\Omega_n$ where

$$d\Omega_n = (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-1})^0 d\theta_1 d\theta_2\cdots d\theta_{n-1} \tag{A.3}$$

The surface of the unit sphere in $n$ dimensions, $\Omega_n$, is

$$\begin{aligned} \Omega_n &= \int d^n\mathbf{x}\,\delta(|\mathbf{x}|-1) = \int d\Omega_n \\ &= \int_0^\pi d\theta_1\,(\sin\theta_1)^{n-2}\int_0^\pi d\theta_2\,(\sin\theta_2)^{n-3}\cdots\int_0^{2\pi} d\theta_{n-1} \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \end{aligned} \tag{A.4}$$

This result follows from repeated application of the formula

$$\int_0^\pi d\theta\,(\sin\theta)^\nu = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)} \tag{A.5}$$

which is valid for general complex $\nu$ provided only that $\mathrm{Re}\,\nu > -1$. A shortcut method to compute $\Omega_n$ is as follows: consider the integral

$$I_n = \int d^n\mathbf{x}\,e^{-\mathbf{x}^2} \tag{A.6}$$

By making use of the factorization property of the exponential function it is straightforward to carry out this integral in Cartesian coordinates, yielding $I_n = \pi^{n/2}$. On the other hand, going to $n$-dimensional spherical coordinates, we obtain

$$I_n = \int d\Omega_n \int_0^\infty dx\,x^{n-1}e^{-x^2} = \tfrac{1}{2}\Omega_n\int_0^\infty dt\,t^{n/2-1}e^{-t} = \tfrac{1}{2}\Omega_n\Gamma(n/2) \tag{A.7}$$

By equating the two results we obtain (A.4).

The volume of the unit sphere is

$$V_n = \int d^n\mathbf{x}\,\theta(1-|\mathbf{x}|) = \int\limits_0^1 dx\,x^{n-1}\int d\Omega_n = \frac{\Omega_n}{n} = \frac{\pi^{n/2}}{\Gamma\left(n/2+1\right)} \tag{A.8}$$

If a function $f(\mathbf{x})$ depends only on $|\mathbf{x}|$ and/or on a few components of $\mathbf{x}$, the integral $\int d^n\mathbf{x}\,f(\mathbf{x})$ can be reduced by integrating out the components on which $f(\mathbf{x})$ does not depend, *e.g.*,

$$\int d^n\mathbf{x}\,f(x) = \Omega_n \int\limits_0^\infty dx\,x^{n-1}f(x) \tag{A.9}$$

$$\int d^n\mathbf{x}\,f(x,\mathbf{k}\cdot\mathbf{x}) = \Omega_{n-1}\int\limits_0^\infty dx\,x^{n-1}\int\limits_0^\pi d\theta_1\,(\sin\theta_1)^{n-2}\,f(x,kx\cos\theta_1) \tag{A.10}$$

etc. (in this last integral we chose the “1” axis of $\mathbf{x}$ aligned with the vector $\mathbf{k}$).

With these results, the computation of the integral found in Eq. (3.8),

$$I(\mathbf{k}) \equiv \int d^n\mathbf{x}\,\delta(\langle\mathbf{x}|H|\mathbf{x}\rangle - s)e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{A.11}$$

proceeds as follows: we first make the change of variables $\mathbf{x} = U^{-1}H_D^{-1/2}U\mathbf{v}$ where $U$ is a real orthogonal matrix that diagonalizes $H$, namely $H = U^{-1}H_D U$, hence

$$I(\mathbf{k}) = \frac{1}{\sqrt{\det H}}\int d^n\mathbf{v}\,\delta(v^2-s)e^{-i\mathbf{q}\cdot\mathbf{v}} \tag{A.12}$$

where we used $\langle\mathbf{x}|H|\mathbf{x}\rangle = \mathbf{v}^2$, defined the vector $\mathbf{q} = H_D^{-1/2}U\mathbf{k}$, and where the prefactor $(\det H)^{-1/2}$ is the Jacobian of the transformation $\mathbf{x}\to\mathbf{v}$. This integral is now in the form of Eq. (A.10), hence

$$I(\mathbf{k}) = \frac{\Omega_{n-1}}{\sqrt{\det H}}\int\limits_0^\infty dv\,v^{n-1}\delta(v^2-s)\int\limits_0^\pi d\theta_1\,(\sin\theta_1)^{n-2}e^{-iqv\cos\theta_1} \tag{A.13a}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det H}}\left(\frac{2\sqrt{s}}{q}\right)^\nu J_\nu(q\sqrt{s}) \tag{A.13b}$$

$$= \frac{\pi^{n/2}s^\nu}{\sqrt{\det H}}\sum_{\ell=0}^\infty \frac{(-q^2s/4)^\ell}{\ell!\Gamma(n/2+\ell)} \tag{A.13c}$$

where $\nu = n/2-1$, $q = |\mathbf{q}| = \sqrt{\langle\mathbf{k}|H^{-1}|\mathbf{k}\rangle}$, and we used a standard definition [10] of the ordinary Bessel function $J_\nu(z)$.

# B Complex Integrals.

In this section the area element $d^2z$ for a surface integral over the complex plane of the variable $z = x+iy$ is defined as $d^2z = dxdy$. The $n$-dimensional complex vector $\mathbf{z}$ is $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, the ket $|\mathbf{z}\rangle$ and bra $\langle\mathbf{z}|$ are defined by $|\mathbf{z}\rangle = \mathrm{col}(z_1, z_2, \cdots, z_n)$ and $\langle\mathbf{z}| = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$, respectively, and $H$ is a Hermitian positive-definite matrix. We denote complex conjugation by either a bar or an asterisk.

The two basic integrals are

$$\int d^2z\,e^{-a|z|^2} = \frac{\pi}{a}\,, \quad \mathrm{Re}\,a > 0 \tag{B.1}$$

and

$$\int d^2z\, e^{-a|z|^2+uz+v\bar{z}} = \frac{\pi}{a} e^{uv/a}\,, \quad \text{Re}\, a > 0;\ u, v \text{ complex} \tag{B.2}$$

which can be proven by elementary methods.

The gaussian integral

$$\int d^{2n}\mathbf{z}\, \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle\} \tag{B.3}$$

is carried out by first noting that $H$, being Hermitian, is diagonalized by a unitary matrix $U$, namely $H = U^\dagger H_D U$, where $U^\dagger = U^{-1}$. We then perform the change of variables

$$|\mathbf{z}\rangle = U^{-1} H_D^{-1/2} |\mathbf{v}\rangle \tag{B.4}$$

so that $\langle \mathbf{z}|H|\mathbf{z}\rangle = \langle \mathbf{v}|\mathbf{v}\rangle = |\mathbf{v}|^2$, hence

$$\int d^{2n}\mathbf{z}\, \exp\{-\langle \mathbf{z}|H|\mathbf{z}\rangle\} = \frac{1}{\det H} \int d^{2n}\mathbf{v}\, e^{-|\mathbf{v}|^2} = \frac{\pi^n}{\det H} \tag{B.5}$$

The integral

$$L(\mathbf{k}, \mathbf{q}) \equiv \int d^{2n}\mathbf{z}\, \delta(\langle \mathbf{z}|H|\mathbf{z}\rangle - s) e^{-i(\langle \mathbf{k}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{q}\rangle)} \tag{B.6}$$

is carried out in similar fashion. Although in Sec. 7 we are only interested in real $\mathbf{k}$ and $\mathbf{q}$, we can allow them to be, in general, complex vectors. The change of variables (B.4) yields

$$L(\mathbf{k}, \mathbf{q}) = \frac{1}{\det H} \int d^{2n}\mathbf{v}\, \delta(|\mathbf{v}|^2 - s) e^{-i(\langle \mathbf{k}'|\mathbf{v}\rangle + \langle \mathbf{v}|\mathbf{q}'\rangle)} \tag{B.7}$$

where $\langle \mathbf{k}'| = \langle \mathbf{k}|U^{-1}H_D^{-1/2}$ and $|\mathbf{q}'\rangle = H_D^{-1/2} U|\mathbf{q}\rangle$. Defining the $2n$-dimensional real vector $\mathbf{V}$ and the $2n$-dimensional complex vector $\mathbf{Q}$ as

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{k}'^* + \mathbf{q}' \\ i(\mathbf{k}'^* - \mathbf{q}') \end{pmatrix} \tag{B.8}$$

we obtain

$$L(\mathbf{k}, \mathbf{q}) = \frac{1}{\det H} \int d^{2n}\mathbf{V}\, \delta(V^2 - s) e^{-i\mathbf{V}\cdot\mathbf{Q}} \tag{B.9}$$

which is of the form (A.12), except that now $\mathbf{Q}$ is complex. However, Eq. (A.13c) shows that $I(\mathbf{k})$ is an analytic function of $q^2$, hence it is valid for complex $q^2$ as well. Therefore we obtain

$$L(\mathbf{k}, \mathbf{q}) = \frac{1}{\det H} \int d^{2n}\mathbf{V}\, \delta(V^2 - s) e^{-i\mathbf{V}\cdot\mathbf{Q}} \tag{B.10a}$$

$$= \frac{\pi^n}{\det H} \left(\frac{2\sqrt{s}}{Q}\right)^{n-1} J_{n-1}(Q\sqrt{s}) \tag{B.10b}$$

$$= \frac{\pi^n s^{n-1}}{\det H} \sum_{\ell=0}^{\infty} \frac{(-s\langle \mathbf{k}|H^{-1}|\mathbf{q}\rangle)^\ell}{\ell!\,\Gamma(n+\ell)} \tag{B.10c}$$

where $Q$ is interpreted as $\sqrt{\mathbf{Q}^2}$, *not* as $|\mathbf{Q}|$. The final result, Eq. (B.10c) was obtained by substituting $\mathbf{k}'$ and $\mathbf{q}'$ in terms of $\mathbf{k}$ and $\mathbf{q}$, *i.e.*,

$$Q^2 = 4\mathbf{k}'^* \cdot \mathbf{q}' = 4\langle \mathbf{k}'|\mathbf{q}'\rangle = 4\langle \mathbf{k}|H^{-1}|\mathbf{q}\rangle \tag{B.11}$$

No ambiguity should arise in the definition of $Q$ in (B.10b) because it involves only even powers of $Q$.

Table 1: Summary of examples.

| $g(s)$ | $g_\ell$ | $A(n;\ell)$ | $\mathcal{G}(\mathbf{k})$ | comments |
|---|---|---|---|---|
| $e^{-s/2}$ | $2^{n/2+\ell-1}\Gamma(n/2+\ell)$ | $1$ | $\exp(-q^2/2)$ | (none) |
| $s^\alpha e^{-s/2}$ | $2^{n/2+\alpha+\ell-1}\Gamma(n/2+\alpha+\ell)$ | $\dfrac{(n/2+\alpha)_\ell}{(n/2)_\ell}$ | ${}_1F_1(n/2+\alpha;n/2;-q^2/2)$ | $\alpha > -n/2$ |
| $e^{-\sqrt{s}}$ | $\Gamma(n+2\ell)$ | $2^\ell(\nu)_\ell$ | $(1+q^2)^{-\nu}$ | $\nu = (n+1)/2$ |
| $\dfrac{1}{(s+1)^p}$ | $\dfrac{\Gamma(n/2+\ell)\Gamma(\mu-\ell)}{2\Gamma(p)}$ | $\dfrac{\Gamma(\mu-\ell)}{2^\ell\Gamma(\mu)}$ | $\dfrac{2}{\Gamma(\mu)}\left(\dfrac{q}{2}\right)^\mu K_\mu(q)$ | $\ell < \mu = p - n/2 > 0$ |
| $\delta(1-s)$ | $\dfrac{1}{2}$ | $\dfrac{1}{2^\ell(n/2)_\ell}$ | $\Gamma(\nu+1)\left(\dfrac{2}{q}\right)^\nu J_\nu(q)$ | $\nu = n/2-1$ |
| $s^\alpha(1-s)^\beta\theta(1-s)$ | $\dfrac{\Gamma(n/2+\alpha+\ell)\Gamma(\beta+1)}{2\Gamma(n/2+\alpha+\beta+\ell+1)}$ | $\dfrac{(n/2+\alpha)_\ell}{2^\ell(n/2)_\ell(n/2+\alpha+\beta+1)_\ell}$ | ${}_1F_2(n/2+\alpha;n/2,n/2+\alpha+\beta+1;-q^2/4)$ | $\alpha > -n/2$, $\beta > -1$ |
| $\theta(1-s)$ | $\dfrac{1}{n+2\ell}$ | $\dfrac{1}{2^\ell(n/2+1)_\ell}$ | $\Gamma(\nu+1)\left(\dfrac{2}{q}\right)^\nu J_\nu(q)$ | $\nu = n/2$ |
| $(1-s)^\beta\theta(1-s)$ | $\dfrac{\Gamma(n/2+\ell)\Gamma(\beta+1)}{2\Gamma(n/2+\beta+\ell+1)}$ | $\dfrac{1}{2^\ell(n/2+\beta+1)_\ell}$ | $\Gamma(\nu+1)\left(\dfrac{2}{q}\right)^\nu J_\nu(q)$ | $\beta > -1$; $\nu = n/2+\beta$ |
| $s^\alpha\theta(1-s)$ | $\dfrac{1}{n+2\alpha+2\ell}$ | $\dfrac{n+2\alpha}{2^\ell(n+2\alpha+2\ell)(n/2)_\ell}$ | ${}_1F_2(n/2+\alpha;n/2,n/2+\alpha+1;-q^2/4)$ | $\alpha > -n/2$ |