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Reframing learning:
How small shifts in teacher talk can support new patterns of mathematical
engagement by nondominant students

By

Alyssa Manogue Dray Sayavedra

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Science and Mathematics Education

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Andrea diSessa, Co-chair
Professor Alan Schoenfeld, Co-chair
Professor Judith Warren Little

Spring 2018

Reframing learning: How small shifts in teacher talk can support new patterns of
mathematical engagement by nondominant students

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by

Alyssa Manogue Dray Sayavedra

Abstract

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Doctor of Philosophy in Science and Mathematics Education

University of California, Berkeley

Professor Andrea diSessa, Co-Chair

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This dissertation documents and analyzes the successful efforts of four experienced Black teachers in a high poverty urban school to develop and sustain ambitious teaching practices. These four focal teachers were part of a larger department learning community that also included three new teachers and a full-time instructional coach, of various ethnic backgrounds, and a White researcher in a participant observer role (myself). The work of these focal teachers was consistent with, and can in turn inform and enrich, research on culturally and mathematically responsive classroom discourse.

The first contribution of the dissertation is to propose and illustrate an integrated model of culturally and mathematically responsive classroom pedagogy. Mathematically responsive pedagogy is conceptualized in terms of the shift from a knowledge transmission (KT) frame, consistent with traditional and typical mathematics instruction in the U.S., toward a productive disciplinary engagement (PDE) frame. This shift is conceptualized and analyzed in terms of three aspects of framing: (1) mathematics content, (2) classroom discourse moves, and (3) teacher and student roles. Culturally and mathematically responsive pedagogy is conceptualized in terms of explicit attention to the negotiation of equitable power relationships that (1) recognize the funds of knowledge of nondominant communities, (2) welcome the discourse practices of nondominant communities, and (3) position nondominant students in positions of mathematical authority. Illustrative cases are provided in which the focal teachers leverage Black cultural practices to support nondominant students' co-development of racial and mathematical identities, as the teacher and students engage together in new types of classroom activity that center these students' mathematical ideas.

The second contribution of the dissertation is to conceptualize and describe changes in classroom discourse from a continuity perspective (Russ, Sherin & Sherin, 2016). A continuity perspective holds that learning is rarely a “gestalt shift,” and instead looks for ways that prior knowledge and practices are adapted and repurposed to create new knowledge and practice. Two examples are analyzed in detail. The first example concerns Initiation-Response-Evaluation (IRE) sequences (Mehan, 1979), which are typically associated in the literature with a KT frame. My analysis illustrates how IRE sequences can serve important functions in responsive classrooms. Specifically, when combined with more open forms of talk such as student presentations, IRE sequences can help elaborate and refine student ideas while ascribing ownership for mathematical ideas to students. The second example concerns student presentations, which are often associated in the literature with a PDE frame. I analyze the learning trajectory of one teacher who implemented and adapted a new student presentation practice over the course of the school year. Initially, the practice was implemented in a way that was consistent with a KT frame, but over the course of the year it shifted substantially and became increasingly consistent with a PDE frame. My analysis details these shifts and their affordances. Both examples provide evidence that points of continuity between the KT and PDE frames can be identified and leveraged to understand and support teacher learning.

To my parents, Corinne Manogue and Tevian Dray.

You showed me what it can look like for work, particularly teaching and research, to be carried out in a spirit of service to humanity.

“Say: no man can attain his true station except through his justice.
No power can exist except through unity.
No welfare and no well-being can be attained except through consultation.”
– Bahá’u’lláh

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1 Introduction

This dissertation documents and analyzes the successful efforts of four experienced Black teachers in a high poverty urban school to develop and sustain ambitious teaching practices. These four focal teachers were part of a larger department learning community that also included three new teachers and a full-time instructional coach, of various ethnic backgrounds, and a White researcher in a participant observer role (myself). The focus of the learning community was on mathematically responsive classroom discourse. Teachers set individual and collective goals around shifts in classroom talk patterns, such as having students present at the front of the classroom or asking more open-ended questions, that they believed would significantly enrich the learning opportunities available to students. Although it was less explicit as a focus of collaboration, teachers also did substantial work to make their teaching culturally relevant for their students, almost all of whom were Latinx or Black. As the teachers tried new teaching practices, they constantly adapted them to fit with the strengths and needs of their students. In this sense, the teachers worked individually and collectively toward an ambitious vision of culturally and mathematically responsive teaching. To do so, the focal teachers drew on a substantial repertoire of prior knowledge about teaching, including some resources that were specific to their cultural experiences as Black teachers as well as other resources that were shared by teachers of various ethnic backgrounds across the United States.

The work of these focal teachers was consistent with, and can in turn inform and enrich, research on culturally and mathematically responsive classroom discourse. Studies of classroom discourse reveal how seemingly small teacher moves can have major implications for student learning. Early classroom discourse research revealed that typical U.S. classrooms provide very few opportunities for students to explain their mathematical ideas in detail or to explore, critique, or build on each other's ideas (Mehan, 1979; Hiebert et al., 2003). Subsequent research provided several exemplars of teaching that made students' mathematical thinking central and gradually refined students' emerging ideas into powerful disciplinary understandings (Lampert, 1990; O'Connor & Michaels, 1993; Ball, Lewis & Thames, 2008; Stein, Engle, Smith & Hughes, 2008). These papers demonstrated that it was both possible and powerful to make students' mathematical ideas central to classroom activity, often through a significant reorganization of the purposes and roles available to students in classroom discourse. Furthermore, researchers have argued that centering student sensemaking, particularly the sensemaking of students from nondominant backgrounds, is powerful for equity reasons. Traditional U.S. mathematics instruction has valued a very narrow set of competencies (Schoenfeld, 2002), and has excluded many students – including a disproportionate number of students from nondominant backgrounds (Schoenfeld, 2002, 2004; Diversity in Mathematics Education Center for Teaching and Learning [DiME], 2007). Expanding the types of sensemaking that are valued in the

classroom has the potential both to make mathematics classrooms more inclusive and to enrich the mathematics being taught.

Substantial literatures document the potential of teaching practices that center student sensemaking (*mathematically responsive teaching*) and/or support the central participation of nondominant students (*culturally responsive teaching*). A growing body of research, to which this dissertation aims to contribute, argues that mathematically responsive teaching and culturally responsive teaching are deeply intertwined. Centering student thinking has the potential to create powerful and equitable learning opportunities for all students, but does not guarantee it. Attention also needs to be paid to addressing issues of power and positioning that too often restrict the learning opportunities available to nondominant students (e.g., Nasir et al., 2012). Although well-documented examples of both mathematically responsive teaching and culturally responsive teaching exist in the literature, helping teachers advance concerning both types of teaching has proven difficult (González et al., 1995; Ladson-Billings, 2000; Visnovska & Cobb, 2009).

This dissertation takes a *continuity approach* to teacher learning: Everyone, including teachers, learns by building on, adapting and/or modifying what they already know (see, e.g., Russ, Sherin & Sherin, 2016). In the case of teachers learning to teach in more culturally and/or mathematically responsive ways, this means that we expect teachers will adapt and modify their prior knowledge and pedagogical practices to create new, more powerful practices. In order to support this type of teacher learning, it is important to have an understanding of teaching practice that goes beyond the documentation of extremes – very responsive and very unresponsive discourse practices – that dominates the existing literature. A more thorough research understanding of hybrid discourse practices (e.g., Gutiérrez, Rymes & Larson, 1995; Hudicourt-Barnes, 2003) can be helpful. It is also important to document sources of overlooked but potentially productive prior knowledge and practices that teachers already possess, that can be leveraged to support their creation of culturally and mathematically responsive discourse spaces.

In the remainder of this introduction, I will: (1) define *mathematically responsive teaching* and *culturally responsive teaching*; (2) introduce the idea of *framing* as a central conceptual tool of the dissertation and provide a more detailed definition of *culturally and mathematically responsive teaching* in terms of framing; (3) outline a *continuity approach* to studying teacher learning that emphasizes points of continuity between typical U.S. teaching practices and ambitious, culturally and/or mathematically responsive teaching; (4) situate the dissertation in the conceptual framework outlined in Sections 1.1–1.3; and (5) provide an overview of the chapters in the dissertation.

1.1 Culturally and mathematically responsive teaching

Mathematically responsive teaching places students' mathematical sensemaking and emerging ideas at the center of the teaching and learning process. In contrast with most U.S. mathematics instruction, mathematically responsive teaching creates

roles for students as authors, critics, and revisers of mathematical knowledge (see Franke et al., 2007, for a review). It thus creates a better alignment between the roles available in a particular classroom (e.g., from “Nyah is a central participant in this classroom because she gets the right answer quickly” to “Nyah is a central participant in this classroom because she critiques and revises mathematical arguments”) and disciplinary roles beyond the classroom (“Nyah is an accomplished mathematician because she critiques and revises mathematical arguments”).

Culturally responsive teaching (Ladson-Billings, 1995; Varelas, Martin & Kane, 2012) supports nondominant students to excel academically while developing robust racial, academic and disciplinary identities. Research has shown that school in general, and mathematics classrooms in particular, are imbued with racialized narratives that position nondominant students as intellectually and academically inferior (Nasir, Snyder, Shah & Ross, 2012). Therefore, supporting these students requires making new aspects or combinations of identities available. To form an initial theory of culturally responsive teaching, researchers have studied the practices of excellent teachers of nondominant students, many of whom are themselves teachers of color. Ladson-Billings (1995) identified three components of culturally relevant teaching: supporting high achievement, developing cultural competence (helping nondominant students learn to navigate the dominant culture while sustaining their community practices), and learning to critique unjust social systems.

Studies of experienced teachers of color have identified culturally specific practices that can be leveraged for culturally relevant teaching. Many of the best known examples of culturally relevant teaching are about excellent teachers of Black students, many of whom are Black teachers. For example, Black teachers’ use of Black cultural language repertoires and speech modes supports Black students’ access to learning opportunities and development of cultural competence (e.g., Johnson et al., 2013), and teachers’ recognition of and resistance to racism supports Black students’ development of cultural critique (e.g., Ladson-Billings, 1995; Beauboeuf-Lafontant, 1999). This does not imply that a simple racial match between teacher and students is either sufficient or necessary to ensure culturally responsive teaching. Indeed, there have been efforts for the last several decades to identify particular teaching practices that can leverage particular funds of knowledge of various nondominant communities in the classroom (e.g., Lee, 1993; González, Moll & Amanti, 2013). Underlying these various efforts is a principle that culturally relevant teaching practices should affirm the unique cultural heritage (supporting racial identity development), brilliance and potential of nondominant students (supporting academic identity development) while drawing on common cultural referents to engage students in the work of the discipline (supporting disciplinary identity development).

There is a potential for synergy between culturally responsive and mathematically responsive teaching. However, the synthesis into a coherent model of *culturally and mathematically responsive teaching* is decidedly nontrivial. By centering student

thinking and inviting students into roles as authors and critics of disciplinary ideas, mathematically responsive teaching potentially values a much broader range of ways of knowing and participating than traditional mathematics instruction. This has great potential to make new combinations of racial, academic and disciplinary identities available to nondominant students, whose ways of knowing and participating have historically been marginalized in mathematics classrooms. However, efforts at mathematically responsive teaching often fall short of addressing issues of power and racialization that culturally responsive approaches consider central factors in limiting or expanding learning opportunities for students (Gutiérrez, 2002; Martin, 2009a). With a few notable exceptions (González, Andrade, Civil & Moll, 2001; Hudicourt-Barnes, 2003; Bang, Warren, Rosebery & Medin, 2012), little has been written about practices that leverage nondominant students' culturally specific funds of knowledge and focus on an expansive view of mathematical or scientific sensemaking.

In the next section, I will introduce the construct of *framing* as a way of conceptualizing the new roles and types of mathematical activity that mathematically responsive teaching seeks to make available to students and teachers. I will then revisit the question of how power and positioning affect which students gain access to these new roles and activities.

1.2 Framing

A powerful way to understand the differences between typical U.S. classrooms and mathematically responsive classrooms is through shifts in *framing*. A frame defines the meaning of a situation for participants interacting within it; it allows individuals to respond to the question: "What is it that is going on here?" (Goffman, 1974). Frames define the activity within the scene and the roles available to various participants. Frames are socially negotiated between participants in an activity, and power operates both in the negotiation of frames and in the positioning of participants in particular roles within a frame (Hand, Penuel & Gutiérrez, 2012).

In my dissertation, a central question will be to identify the frame at play. Here I briefly describe two different frames that are fairly well documented in the literature. The first frame, *knowledge transmission*, is a synthesis¹ of the framing most typical of U.S. classrooms (Hiebert et al., 2003). In this frame, the primary activity of the mathematics classroom is the transmission of a fixed body of mathematical knowledge to students. The teacher bears primary responsibility for clear exposition of this body of knowledge as well as organization and pacing of the class to make the knowledge accessible to students. Students are responsible for demonstrating that they can reproduce parts of the mathematical knowledge the

¹ What I call a *knowledge transmission frame* is not dissimilar from Hand et al.'s *doing school frame*. However, the term *doing school* emphasizes a type of student participation whereas I would like to emphasize the underlying model of learning and its sharp contrast with a constructivist perspective.

teacher has presented, for example by reproducing definitions of vocabulary words and using standard procedures that have been modeled for them (Lappan & Phillips, 2009). Table 1.1 summarizes the main aspects of a knowledge transmission frame.

Knowledge Transmission Frame	
Mathematics Content	Mathematics content as experienced by students in this frame tends to be organized around students' development of particular skills and procedures in a fixed order. It provides few opportunities for students to develop experience and fluency with higher-level mathematical practices such as reasoning and problem solving.
Typical discourse patterns	Typical discourse patterns in this frame include: teacher exposition, chains of Initiation-Response-Evaluation (IRE) sequences, ² and student requests for the teacher to slow down or repeat something that was confusing.
Teacher and Student Roles	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are typically made available only to the teacher (and textbook authors, in absentia).

Table 1.1. Knowledge transmission frame for learning.

The second frame, *productive disciplinary engagement*, represents a synthesis of the framing operating in classrooms that are considered exemplars of mathematically responsive teaching (e.g., Lampert, 1990; Ball et al., 2008). The term productive disciplinary engagement was coined by Engle & Conant (2002) to describe learning environments in which students are encouraged to take on intellectual problems, given authority to address such problems, made accountable to others and to disciplinary norms, and provided sufficient resources to participate in these ways.

The use of the term productive disciplinary engagement in connection with framing was introduced by Hand et al. (2012). The authors used the term both to indicate the rich content learning opportunities afforded by centering students' ideas in the classroom, and how centering the ideas of nondominant students in particular can help organize classroom power relationships in equitable ways. This dissertation is true to the spirit of these authors' work, but provides a somewhat more detailed operationalization of the meaning of this frame.

In a productive disciplinary engagement frame, the primary activity of the mathematics classroom is joint consideration of important disciplinary ideas. Students and the teacher bear responsibility for making sense of the problem at hand, sharing their emerging thinking, and refining their initial ideas according to

² IRE sequences (Mehan, 1979) are one of the most common discourse forms in U.S. classrooms. The teacher Initiates the sequence with a known-answer question, the student Responds, typically with only a few words, and the teacher Evaluates the response as correct or incorrect.

the norms of the discipline of mathematics. The teacher bears additional responsibilities for posing rich problems, establishing and reinforcing both social and mathematical norms, and connecting student thinking with important disciplinary ideas.

	Productive Disciplinary Engagement Frame with explicit attention to issues of power
Mathematics Content	<p>Students begin from their prior knowledge and often discuss multiple approaches to solving a problem. Because students have opportunities to connect, critique, and revise ideas, they gradually refine their prior knowledge into powerful and general disciplinary understandings.</p> <p>Power relationships are negotiated in ways that position the multiple mathematical knowledge bases of nondominant students and communities as central to the learning process.</p>
Typical Discourse Patterns	<p>The discourse pattern consistent with this frame can be quite varied. There must be space for students to share their thinking in some depth. There are many possibilities for discourse structures that support the gradual refinement of student thinking in connection with important disciplinary ideas.</p> <p>Power relationships are negotiated in ways that welcome nondominant speech patterns and forms of expression and support the development of cultural competence.</p>
Teacher and Student Roles	<p>In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are made available to students and the teacher.</p> <p>Power relationships are negotiated in ways that position and support nondominant students to act as authors and critics of powerful ideas, thus expanding their learning opportunities.</p>

Table 1.2. Productive disciplinary frame for learning with attention to power.

Establishing a productive disciplinary engagement frame makes new roles available to students that afford powerful learning opportunities. These new roles and learning opportunities should be made available to all students. However, issues of power and positioning too often restrict the access of nondominant students to roles of mathematical competence and authority. Therefore, particular attention is needed to the extent to which these new roles and opportunities are made available to nondominant students.

Table 1.2 represents a model of *culturally and mathematically responsive teaching* that will be used throughout the dissertation. In summary, culturally and mathematically responsive teaching involves the creation of a productive disciplinary engagement frame with sufficient attention to issues of power and positioning to create inclusive power relationships.

The frames given in Table 1.1 and Table 1.2 represent extremes. Not all mathematics learning environments will fall neatly into one or the other of these frames. We can expect that many learning environments will include some elements of both frames and/or elements of other frames. The case studies presented later in the dissertation will help illuminate these details.

1.3 A continuity approach to teacher learning

Most research portrays the gulf between typical U.S. teaching practices and mathematically responsive practices as dramatic. Beyond shifting a few talk moves, teachers are being asked to develop a new set of resources, goals, and orientations (Schoenfeld, 2010) that place student sensemaking at the center of the teaching and learning process. However, this dissertation takes the position that describing a gulf can only go so far toward supporting teachers to cross it. A much deeper theoretical understanding is needed of the points of continuity between typical teaching practices and culturally and mathematically responsive teaching (e.g., Russ et al., 2016). In other words, what existing teaching practices can experienced teachers most easily adapt and repurpose to make a classroom significantly more culturally and mathematically responsive?

A continuity approach to teacher learning invites us to look for teacher knowledge bases that have been previously overlooked as learning resources. Specifically, this dissertation will look at the knowledge bases of experienced Black teachers for culturally and mathematically responsive teaching. A growing body of literature documents the strengths of teachers of color for culturally responsive teaching. For example, there is a growing literature on culturally relevant classroom management. However, there is insufficient literature about the culturally specific resources that nondominant teachers bring to teaching mathematics content (Birky, Chazan & Morris, 2013; Frank, Khalil, Scates & Odoms, 2018) or to mathematically responsive teaching. Indeed, some authors have suggested that Black mathematics teachers are categorically opposed to mathematics reform (Delpit, 1995). I agree with many of Delpit's points about the operation of race and power within mathematics reform efforts. However, I disagree that these issues preclude the potential for mathematics reform to become a tool of liberation for nondominant communities.

This dissertation focuses on the insights of four experienced Black mathematics teachers in a high poverty urban school serving predominantly Latinx and Black students. I will argue that these teachers had a strong prior knowledge base for culturally responsive teaching at the beginning of the study. These teachers, with

others in their department learning community, set out to learn about mathematically responsive teaching, and were largely successful. Yet, it proved both empirically difficult and unproductive to separate their work on mathematically responsive teaching from the social and cultural context at the school, and these teachers' sense of mission to empower nondominant students through their teaching. For this reason, my analysis focuses on these teachers' learning about culturally and mathematically responsive teaching.

Some of the analyses presented, particularly in Chapter 4, foreground issues of race and power in the particular local context of the school site. For example, one case study in Chapter 4 shows a Black male teacher positioning two Black male students as authors of mathematical ideas through affirmation of Blackness and hip hop identity and inviting one of them to present a novel idea to the class. This is just one example of a combination of identity moves that a teacher can use to position nondominant students as authors of mathematical ideas. The analysis will unpack the local particularities with an eye toward general principles that can be applied in a variety of contexts.

Other analyses, particularly in Chapter 5, foreground the creation of new mathematical roles for students. In this second set of cases, issues of race and power are backgrounded for the bulk of the analysis. All students in the classroom are nondominant students, and the analytical focus is on the teachers' mathematical positioning of these students as authors and critics of ideas consistent with a productive disciplinary engagement frame. Nevertheless, I provide some indicators that the new mathematical roles are made broadly available to all students in the class and in particular to those who are potentially vulnerable to being marginalized. Toward the end of the chapter, I provide some contextual data from teacher interviews about the learning histories of the two main student presenters, a Black boy who could have been positioned as having "anger management issues" and a Latina girl who could have been positioned as "not knowing English." Both students were frequent presenters in this classroom, and their teacher views the presentations as an important opportunity to help them and their peers see their mathematical competence, in order to set them up for success both within and beyond her classroom.

1.4 Dissertation outline

Chapter 2 reviews literature on mathematically and culturally responsive teaching. The review of mathematically responsive teaching includes classroom discourse literature on IRE sequences and their alternatives, as well as the importance of a strengths-based or continuity view of teachers' prior knowledge and practices. The review of literature on culturally responsive teaching discusses the potential of mathematically responsive teaching to open more powerful learning opportunities to nondominant students, the way race and power operate to constrain learning opportunities available within classrooms to nondominant students, the methodology of studying excellent teachers of nondominant students to better

understand culturally responsive teaching, and the potential for this methodology to be applied to the study of mathematically responsive teaching.

Chapter 3, “Methods,” includes a description of the site selection process and research site, data collection methods, and a brief overview of data analysis methods. The description of the research site includes the social context of the school site, demographic information about the students and teachers, and enough description of the work of the mathematics department to contextualize teacher learning. The description of data collection methods includes how focal teachers were selected for classroom observations and a description of observation procedures, frequency, and data sources obtained. Data analysis methods vary significantly from chapter to chapter; they are briefly summarized in Chapter 3 and elaborated on more fully in Chapters 4–6.

Chapter 4, “Supporting nondominant students’ co-development of racial and mathematical identities: Insights from Black mathematics teachers,” presents two classroom case studies in which experienced Black teachers who are fairly early in their engagement with mathematics reform leverage specific Black cultural practices to create mathematically open and responsive discourse spaces in their classrooms. These Black cultural practices both support student engagement and enrich the mathematics being discussed. In the first case study, “Write up why while you whippin and nae naein,” an Algebra 1 teacher listens to a Black male student’s novel solution method and affirms its mathematical value while also affirming the student’s Black identity and hip hop affiliation. The teacher then supports this student to present his novel idea to the class and other students to restate the important ideas in their own words, which further positions this student as an author of mathematical ideas, an important element of a productive disciplinary engagement frame. In the second case study, “I don’t want to work more if they’re almost paying me the same,” a sixth-grade teacher supports his Latinx and Black students to go beyond discussing the answers to a homework problem and instead draw on a range of inside- and outside-of-school knowledge to debate which of two jobs they would prefer to have. The debate format supports students to engage in critiquing and refining mathematical ideas and to apply these ideas in an authentic context, two important areas of consistency with a productive disciplinary engagement frame. This results in both high engagement and a much richer conversation about important mathematical ideas than would be afforded by a narrow interpretation of the homework problem.

Chapter 5, “Surprising functions of IRE sequences in mathematically responsive classrooms,” discusses a discourse move that most teachers have in their toolkit, but that few teacher educators would think of as a resource for creating a productive disciplinary engagement frame: the Initiation-Response-Evaluation (IRE) sequence. Although the majority of literature about the IRE discourse form describes it as one of the main tools for constructing a knowledge transmission frame, in my dissertation data it was common to see teachers use IRE sequences for a much wider variety of purposes and to support various epistemological frames. The

chapter presents two case studies of classroom discourse that include many elements of a productive disciplinary engagement frame, but nevertheless make substantial use of IRE sequences.

Chapter 6, “Developmental trajectory of one culturally responsive teacher toward increasing mathematical responsiveness,” shows one teacher’s efforts to adopt a new teaching practice, student presentations, and to sustain and adapt this new practice over the course of the school year. In doing so, he makes significant progress toward a productive disciplinary engagement frame. Quantitative and qualitative methods are used to characterize the teacher’s baseline teaching practices prior to attempting the new presentation activity, his first attempt at the presentation activity, and the ways he sustained and adapted the activity throughout the rest of the school year. The quantitative analysis is based on a coding scheme, Talk Quality Quantified (TQQ), inspired by Reinholz & Shah’s EQUIP (2018), that captures shifts in student talk at the micro level (talk turn grain size). This quantitative analysis is supplemented with qualitative analysis of each aspect of framing: the mathematics content, meso-level discourse structure and function, and student and teacher roles. This analysis shows that the student presentation activity led to immediate changes in the micro-level discourse structures in the classroom: students took much longer talk turns and described their mathematical processes as well as answers. Despite this immediate increase in student “air time,” the mathematical content of what students said in the first presentation episode was not very rich. Students repeated a standard procedure for all problems and made significant conceptual errors that were not addressed. The framing remained largely consistent with a knowledge transmission frame. Nevertheless, as the teacher sustained and adapted the student presentation activity throughout the year, he continued the desirable features of the initial activity while substantially enriching the mathematics content and supporting students to take on a much broader role in authoring and to a limited extent critiquing mathematical ideas. In this way, his teaching practice progressed significantly toward a productive disciplinary engagement frame.

Chapter 7, “Discussion,” describes the substantive and methodological contributions of this dissertation. The main findings include the model and exemplars of culturally and mathematically responsive teaching and the description of several implications of a continuity perspective on teacher learning about classroom discourse. The chapter then notes implications for professional development. Finally, it suggests productive avenues for future research, particularly on teacher knowledge bases for culturally and mathematically responsive teaching and the development of these knowledge bases.

2 Literature Review

International studies show mathematics education in the United States provides students with limited opportunities to engage in mathematical sensemaking, compared to other countries such as Japan (Hiebert et al., 2003), which results in the lower performance of U.S. students on international assessments such as PISA compared to almost all other OECD countries (Darling-Hammond, 2010). These findings point to a need to better understand and support the development of teacher capacity for mathematically responsive teaching for all students.

The need for culturally and mathematically responsive teaching is especially urgent in high poverty urban schools, where public divestment is all too common. The distribution of learning opportunities within the U.S. is much more inequitable than in many other countries. Specifically, high poverty urban schools serving a majority of nondominant students are less likely to have access to qualified and experienced teachers, less likely to be organized in ways that support the long-term professional growth of teachers (Little & Bartlett, 2010), and more likely to emphasize disconnected vocabulary, facts and procedures, with little space for students to generate their own mathematical strategies or critique or refine each other's ideas (Means & Knapp, 1991; Ladson-Billings, 1997; Lubienski, 2002). Even when nondominant students do gain access to powerful mathematics learning environments, they are often marginalized *within* these learning environments in ways that narrow their learning opportunities (Hand et al., 2012; Nasir et al., 2012).

These findings represent a call to action to better understand and support teacher learning about culturally and mathematically responsive teaching. This dissertation takes up a piece of this dual call to action. It focuses on teaching and learning within classrooms, and also attends to the ways that teachers learn in collaboration with each other and ways that issues of power and inequality in society as a whole push into schools and classrooms.

This chapter has two overlapping functions. The first function, which is somewhat atypical for a literature review, is to develop and make clear the conceptual underpinnings of the primary analytical construct used in the dissertation: a productive disciplinary frame for learning with explicit attention to issues of power and equity. The second function, which is more typical of a literature review, is to outline the development of the major strands of research that the dissertation builds on and to which it aims to contribute.

Section 2.1 serves mainly the first purpose. This section defines the concept of framing used in the dissertation. Framing includes attention to: (1) approaches to mathematics content, (2) typical discourse moves, and (3) student and teacher roles. The section then uses framing to provide an overview of a productive disciplinary engagement frame with explicit attention to issues of power and equity,

a central construct of the dissertation that is also used as the organizing principle of the remainder of the chapter.

Sections 2.2 through 2.4 form the bulk of the chapter and follow a more typical literature review format, using the construct of culturally and mathematically responsive teaching as an organizing principle. Section 2.2 discusses research on mathematically responsive teaching. Section 2.3 discusses research on culturally responsive teaching. Section 2.4 discusses research that synthesizes both perspectives, that is, research on culturally and mathematically responsive teaching.

Within Sections 2.2 through 2.4, subsections are organized based on the three aspects of framing introduced in Section 2.1. That is, Section 2.2.1 discusses approaches to mathematics content in mathematically responsive teaching, Section 2.2.2 discusses approaches to classroom discourse in mathematically responsive teaching, and Section 2.2.3 discusses approaches to teacher and student roles in mathematically responsive teaching. Sections 2.3 and 2.4 discuss culturally relevant teaching and culturally and mathematically responsive teaching, respectively, with similarly organized subsections.

The model of culturally and mathematically responsive teaching outlined in Sections 2.1 through 2.4 represents a goal for teacher learning. There remains a need to discuss processes by which teachers can make progress toward this goal. Accordingly, Section 2.5 summarizes important principles from the literature on teacher learning that inform the approach taken in the dissertation.

2.1 Framing

A powerful way to understand the differences between typical U.S. classrooms and culturally and mathematically responsive classrooms is through shifts in *framing*. A frame defines the meaning of a situation for participants interacting within it; it allows individuals to respond to the question: “What is it that is going on here?” (Goffman, 1974). Frames define the activity within the scene and the roles available to various participants. They may be most visible in moments of breach or negotiation. They are often less visible, though still present, at other times.

The most typical frame for mathematics instruction in the United States is a *knowledge transmission (KT) frame*. In this frame, mathematics content tends to be narrowly conceptualized in terms of a series of definitions and procedures that students should learn and practice, typically in a particular order (Hiebert et al., 2003). Discourse patterns in U.S. mathematics classrooms typically include exposition and demonstration by the teacher, followed by student practice of many similar, short exercises (Hiebert et al., 2003) and teacher questions that elicit student responses a few words in length and test students’ knowledge of facts and procedures (Mehan, 1979, 1985). The prescribed student role is somewhat narrow and opportunities to author, critique, and revise mathematical ideas are the purview of the teacher alone.

Research on mathematically responsive teaching can be synthesized into a substantially different frame for learning, a *productive disciplinary engagement (PDE) frame*. The term *productive disciplinary engagement* was coined by Engle and Conant (2002), and refers to the following set of guiding principles:

- 1) **Problematizing:** Students are encouraged to take on intellectual problems;
- 2) **Authority:** Students are given authority in addressing such problems;
- 3) **Accountability:** Students' intellectual work is made accountable to others and to disciplinary norms;
- 4) **Resources:** Students are provided with sufficient resources to do all of the above.

In mathematics classrooms, a PDE frame implies posing problems with multiple possible strategies (problematizing), making students' mathematical thinking central to the learning process (authority), examining student strategies for the underlying knowledge about important mathematical ideas and orchestrating discussions that allow students to connect and refine their ideas (accountability), and providing adequate resources for this work.

Table 2.1 and Table 2.2 summarize important contrasts between the knowledge transmission and productive disciplinary engagement frames, each of which will receive analytical and empirical attention in the dissertation.

	Knowledge Transmission Frame
Mathematics Content	Mathematics content as experienced by students in this frame tends to be organized around students' development of particular skills and procedures in a fixed order. It provides few opportunities for students to develop experience and fluency with higher-level mathematical practices such as reasoning and problem solving.
Typical discourse patterns	Typical discourse patterns in this frame include: teacher exposition, chains of IRE sequences, and student requests for the teacher to slow down or repeat something that was confusing.
Teacher and Student Roles	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are typically available only to the teacher (and textbook authors, in absentia).

Table 2.1. Summary of the knowledge transmission frame.

Frames require a high degree of coordination among participants in order to function smoothly. For example, if students are accustomed to a "demonstrate and practice" mode of learning, it may not be easy for a teacher to delegate roles of mathematical authorship and critique to students. Students may even protest that such a teacher is not teaching. On the other hand, if a teacher calls on students for brief answers but students have substantial experience explaining their mathematical reasoning, students may make bids to take on greater mathematical ownership than the teacher has asked for, for example by volunteering explanations.

Either way, framing is an emergent characteristic of group activity. All participants have some influence and elements of one or more frames could be in play at a time.

	Productive Disciplinary Engagement frame with explicit attention to issues of power
Mathematics Content	<p>Students begin from their prior knowledge and often discuss multiple approaches to solving a problem. Because students have opportunities to connect, critique, and revise ideas, they gradually refine their prior knowledge into powerful and general disciplinary understandings.</p> <p>Power relationships are negotiated in ways that position the multiple mathematical knowledge bases of nondominant students and communities as central to the learning process.</p>
Typical Discourse Patterns	<p>The discourse patterns consistent with this frame can be quite varied. There must be space for students to share their thinking in some depth. There are many possibilities for discourse structures that support the gradual refinement of student thinking in connection with important disciplinary ideas.</p> <p>Power relationships are negotiated in ways that welcome nondominant speech patterns and forms of expression and support the development of cultural competence.</p>
Teacher and Student Roles	<p>In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are made available to students and the teacher.</p> <p>Power relationships are negotiated in ways that position and support nondominant students to act as authors and critics of powerful ideas, thus expanding their learning opportunities.</p>

Table 2.2. A frame for mathematically and culturally responsive teaching.

Hand et al. (2012) argue that power operates both in the negotiation of which frame is in play and in the positioning of participants in particular roles within a frame. The authors note that in addition to systemic inequalities that affect nondominant students' *access to* rigorous learning environments, these students are often marginalized *within* classrooms. It is possible, even likely, that a classroom learning environment could position some students as authors and critics of mathematical ideas, but that nondominant students could still be shut out of these roles and the corresponding learning opportunities because they are marginalized within the classroom learning community. Explicit attention is needed to renegotiate

dominant power relationships in order to position nondominant students as central participants in classroom activity. Hand and colleagues use the construct of framing to analyze both epistemological innovations within classrooms and policy campaigns regarding educational inequity. In both cases, the authors use the construct of framing to afford explicit attention to how issues of power support or constrain the learning opportunities available to nondominant students. In my work, I focus within classrooms, but do use the construct of framing to support attention to both the mathematical responsiveness and cultural responsiveness of classrooms. More specifically, I use framing to highlight epistemological differences between typical U.S. mathematics instruction (KT frame) and mathematically responsive instruction (PDE frame) and also to afford explicit attention to ways that related issues of power can play out. Table 2.2 summarizes the frame for culturally and mathematically responsive teaching used for this dissertation.

The ideas behind Table 2.2 are not new. I draw on the terminology and contributions of Engle & Conant (2002) and Hand et al. (2012), but the ideas behind these terms are not the creation of any small handful of authors. They incorporate some of the major ideas from decades of research on mathematics learning and on issues of race, culture and power in schooling. Here, I do not attempt a comprehensive review. However, I do my best to acknowledge the long intellectual history of the central constructs of the dissertation and to present the main findings of each strand of literature in sufficient detail to convince readers that the parallels I draw are true to the spirit of each strand.

The remainder of the chapter uses the model presented in Table 2.2 as an organizational principle. Section 2.2 elaborates on the various aspects of a productive disciplinary engagement frame. Section 2.3 summarizes the literature on culturally relevant teaching with particular attention to how issues of race and power can influence each aspect of a productive disciplinary engagement frame: mathematics content, discourse patterns, and teacher and student roles. Section 2.4 reviews in more detail the existing literature that synthesizes culturally and mathematically responsive approaches. Section 2.5 briefly summarizes a few essential ideas from the teacher learning literature: that teachers, like students, learn by building on their prior knowledge, and that teacher learning occurs within communities of practice.

2.2 Mathematically Responsive Teaching

This section discusses literature on mathematically responsive teaching, with subsections organized based on the three aspects of framing introduced in Section 2.1: Section 2.2.1 discusses approaches to mathematics content, Section 2.2.2 discusses approaches to classroom discourse moves at a fine grain size, and Section 2.2.3 discusses approaches to teacher and student roles.

2.2.1 Coherent and connected mathematics content

International research shows that most mathematics instruction in the United States reflects a relatively disconnected view of content in terms of isolated facts and

procedures. A noteworthy example is the TIMSS video study, which compared a random sample of eighth-grade mathematics classrooms in the United States, Germany, and Japan (Stigler & Hiebert 1999; Hiebert et al., 2003). The TIMSS study found that the variation within countries was much less than the variation between countries, and that eighth graders in the United States tended to experience mathematics instruction that was organized around students' development of particular skills and procedures, often in a conventional order, and provided few opportunities for students to develop experience and fluency with higher-level mathematical practices such as reasoning and problem solving. This approach to mathematics content is consistent with a knowledge transmission frame.

A number of efforts in the U.S. have sought to better understand and implement a dramatically different view of mathematics content. According to this body of research, a knowledge transmission frame severely limits opportunities for student learning. This is not because there is anything wrong per se with activities such as teacher exposition or student practice of procedures; each has its place in the mathematics classroom. However, students also deserve opportunities to participate in the processes of doing mathematics including advancing conjectures, authoring innovative strategies, critiquing the ideas of others. Knowledge transmission, as a primary frame for learning, severely limits these opportunities (Schoenfeld, 1988; Hiebert et al., 2003).

The approach to mathematics content that informs a productive disciplinary engagement frame is consistent both with recent standards documents (NCTM, 2000; National Governors Association, 2010) and with the large research literature pointing to the importance of building on students' mathematical thinking in the classroom (NRC, 2000). Historically, the development of this research literature informed the development of standards documents and curricula, and vice versa.

Beginning in the 1970s, a growing research literature on mathematical thinking and problem solving helped us understand the types of mathematical activities that students should engage in in order to develop rich understandings. For example, a traditional geometry approach of teaching proof and construction as separate collections of procedures left students unable to draw on their knowledge of proof to solve problems. This was true even when the construction problem they were asked to carry out was very similar to a proof problem they had just solved. Effective problem solving required more than a solid mathematical knowledge base; students also needed problem-solving heuristics, metacognitive control strategies to monitor their process, and productive beliefs about mathematics (Schoenfeld, 1985). In 1980, the National Council of Teachers of Mathematics [NCTM] (1980) issued a call that "problem solving must be the focus of school mathematics" (p.1). In hindsight, much of the early implementation of problem solving curricula was still consistent with a knowledge transmission frame. For example, numerical exercises were turned into "real-world" word problems, but the applications were superficial: Students were still given many similar, short exercises intended help them practice one skill many times (Schoenfeld, 1992).

Subsequent research brought increasing clarity about how students could make strong connections between procedures, concepts and contexts, and become flexible and resourceful thinkers and problem solvers. This vision of mathematics content is consistent with a productive disciplinary engagement frame. In 1989, the NCTM released (1989) the *Curriculum and evaluation standards for school mathematics (Standards)*, and several efforts to develop “reform” mathematics curricula were launched on the basis of these curricula. Reform curricula were far from perfect and were the topic of heated political debate, often referred to as the “math wars” (Schoenfeld, 2004). However, by the early 2000s, initial studies of the impact of these curricula agreed: students who studied reform curricula outperformed their peers who studied traditional curricula on tests of conceptual understanding and problem-solving, while still performing equally well on measures of basic skills (Schoenfeld, 2002).

	Knowledge transmission frame	Productive disciplinary engagement frame
Mathematics Content	Mathematics content as experienced by students in this frame tends to be organized around students’ development of particular skills and procedures in a fixed order. It provides few opportunities for students to develop experience and fluency with higher-level mathematical practices such as reasoning and problem solving.	Students begin from their prior knowledge and often discuss multiple approaches to solving a problem. Because students have opportunities to connect, critique, and revise ideas, they gradually refine their prior knowledge into powerful and general disciplinary understandings.

Table 2.3. Mathematics content in KT and PDE frames.

The core ideas of the *Standards* were elaborated substantially in the *Principles and Standards for School Mathematics* (NCTM, 2000), and also form the foundation of the *Common Core Standards in Mathematics* (National Governors Association, 2010). Curricula in the United States continue to change often. Indeed, there is currently a patchwork of available curricula that vary in quality and in alignment with the vision of mathematics content outlined here. Furthermore, additional work is needed to engage various stakeholders in the development and adoption of curricula and standards. However, the shift from mathematics content consistent with a knowledge transmission frame toward mathematics content consistent with a productive disciplinary engagement frame is not merely a fad. Research on the effectiveness of standards-based curricula, as well as a large body of literature on

student learning, show that shifting mathematics content in this direction can have powerful impact on student learning.

An important epistemological distinction between the KT and PDE frames lies in the status accorded to students' emerging understandings. In a knowledge transmission frame, teachers typically minimize "air time" accorded to incorrect or incomplete student work, out of concern that it will be confusing or distracting to others. In contrast, a productive disciplinary engagement frame attends more carefully to students' underlying knowledge. The seed of a powerful and generative idea may often be buried in an incorrect solution. Therefore, students' mathematical thinking is actively elicited in the classroom and students are invited to try to share multiple approaches to solving a problem. Although some of students' initial ideas may be incorrect, incomplete, or mathematically less sophisticated, they are still viewed as resources for learning within a larger mathematical conversation. Students are provided with opportunities to connect, critique, and revise ideas, allowing them to gradually refine their prior knowledge into powerful and general disciplinary understandings (Schoenfeld and the Teaching for Robust Understanding Framework, 2016; Franke et al., 2007). To provide students with these opportunities, the structure of classroom discourse as well as the roles taken on by the teacher and students in classroom activity both shift substantially.

2.2.2 Classroom discourse in a PDE frame

Since at least the 1980s, studies of classroom discourse have drawn connections between the talk moves made by teachers and students and the types of mathematical activity that students are able to engage in. I will give an overview of this literature here, and also indicate where I will elaborate more in subsequent chapters.

Mehan (1979) analyzed one very common discourse patterns the Initiation-Response-Evaluation (IRE) sequence in which the teacher Initiates with a known-answer question, the student briefly Responds, and the teacher Evaluates the student's answer. This discourse pattern is very common in U.S. classrooms and has very limited affordances for students to explain their thinking. It is typically associated with a knowledge transmission frame for teaching.

Following Mehan's work, a number of researchers studied the classroom discourse patterns in exemplary classrooms that did reflect a productive disciplinary engagement frame. These analyses led to a number of alternatives to IRE sequences, including: (1) a teacher move called revoicing in which the teacher restates or elaborates on a student's mathematical idea and provides the student an opportunity to affirm or reject this restatement (O'Connor & Michaels, 1993, 1996); (2) Initiate-Describe-Elaborate sequences which provide opportunities for students to present various approaches with the teacher elaborating on the mathematical connections (Nathan, Eilam & Kim, 2007); and more elaborate structures such as

accountable argumentation (Horn, 2008) in which students take on various roles in critiquing or defending mathematical ideas according to the norms of the discipline.

A necessary, but not sufficient, aspect of classroom discourse in a productive disciplinary engagement frame is to create space for students to explain their ideas. For example, students may be invited to regularly “share out” their ideas, giving multiple sentence explanations and demonstrations either at the front of the classroom or from their seats. It is clearly necessary to elicit student thinking in some detail order to build on it. To see why this is not sufficient, consider a classroom in which students are invited to use multiple approaches to solving a task and then a class discussion is organized in which students present these strategies one after another. If these discussions are organized as “show and tell,” with little attention to making students accountable to each other or the norms of the discipline, connecting student methods or unpacking underlying conceptual understanding, they can still reinforce a knowledge transmission frame. See Stein et al. (2008) for an important critique of, and alternative to, “show and tell” presentations. It is essential that student thinking is not only elicited, but also refined and connected to important disciplinary ideas.

	Knowledge transmission frame	Mathematically Responsive Pedagogy (PDE frame)
Typical discourse patterns	Typical discourse patterns consistent with this frame include: teacher exposition, chains of IRE sequences, and student requests for the teacher to slow down or repeat something that was confusing.	The discourse patterns consistent with this frame can be quite varied. There must be space for students to share their thinking in some depth. There are many possibilities for discourse structures that support the gradual refinement of student thinking in connection with important disciplinary ideas.

Table 2.4. Classroom discourse in KT and PDE frames.

Table 2.4 summarizes the ideas in this section. It is important to note that the teacher plays an important role in this process, both through setting up classroom norms that support students to do some of the work of critiquing and refining ideas, and by introducing powerful ideas and connections into the discussion. The question of teacher and student roles is taken up in more detail in the next sections.

2.2.3 Teacher and student roles in a PDE frame

Establishing a productive disciplinary engagement frame for mathematics learning is not simply a matter of adopting a reform curriculum nor of incorporating a few new talk moves into one’s classroom practice. Various authors, reviewed by Franke, Kazemi and Battey (2007), argue that a more fundamental shift is needed in how classroom activity is organized for learning. Specifically, in a PDE frame, it is essential that students be provided with more opportunities to take active roles in the production and refinement of knowledge than is typical in U.S. classrooms.

Students as well as the teacher need opportunities to take on roles as authors, critics, revisers and refiners of mathematical ideas.

Lampert (1990, 2001) and Ball (Ball, Lewis & Thames, 2008; Horn, 2008) provide early exemplars of mathematics teaching in which students and the teacher share responsibility for both authoring and critiquing mathematical ideas. Lampert (1990, 2001) designs nonstandard arithmetic problems that afford multiple strategies, where the purpose is not to solve the problem but rather to grapple with key concepts through the comparison and refinement of these strategies. In one example (Lampert, 1990), students find powers of single digit numbers and make and defend conjectures about the patterns in the last digits of these numbers. To sustain this discussion, they explicitly disambiguate comments like “square the five two times” (25×2 or 25^2 ?), leading to important conversations about the meaning of exponentiation and its relationship to other operations. In Ball’s classroom, students engage in a system of accountable argumentation (Horn, 2008) in which students advance, critique and refine conjectures according to disciplinary norms. Within this system, students and the teacher use a variety of talk moves to position students as either defending or critiquing ideas. This positioning allows students to not only share their reasoning but also allows the class to refine ideas over time according to disciplinary norms in mathematics.

Yackel & Cobb (1996) introduce the term *sociomathematical norms* to discuss this issue. The authors use the following examples to distinguish between social and sociomathematical norms:

Normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom are sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm.... The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. (p. 461)

They provide empirical examples of a class of elementary students developing sociomathematical norms while engaged in a series of tasks that afford multiple solution strategies. For example, students and the teacher negotiated a sociomathematical norm of *mathematical difference*. Initially students provided solution strategies that were arithmetically similar (no regrouping), but once one student shared a novel method for regrouping it sparked a discussion in which several other students shared other new regrouping strategies. The classroom learning community also negotiated sociomathematical norms for sufficient justification throughout the year.

	Knowledge transmission frame	Mathematically Responsive Pedagogy
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		(PDE frame)
Roles	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are typically made available only to the teacher (and textbook authors, in absentia).	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are made available to students and the teacher.

Table 2.5. Student and teacher roles in KT and PDE frames.

In each of these examples, students have opportunities to learn about the process of reasoning mathematically. These learning opportunities were both created by, and in turn supported, students' positioning as authors and critics of mathematical ideas within a productive disciplinary engagement frame. There is now a sufficient range of documented examples to extract general principles. Contemporary frameworks tend to describe these principles in terms of supporting students' development of particular types of agency and identity in mathematics classrooms.

In the Teaching for Robust Understanding Framework [TRU] (Schoenfeld and the Teaching for Robust Understanding Project, 2016), teacher and student roles are addressed through the Agency, Ownership, and Accountability dimension. The TRU framework provides a synthesis of research on powerful classroom learning environments, separated into equivalence classes (as described in detail in Schoenfeld, 2013, 2014, 2018; Schoenfeld et al., 2016). The focus of TRU is at the same grain size as my focus on student and teacher roles: it outlines general principles that can be synthesized across exemplars of mathematically responsive teaching in the literature, while still remaining specific enough to be actionable by teacher learning communities. One practical TRU tool for members of teacher learning communities is the *TRU observation guide* (Schoenfeld and the Teaching for Robust Understanding Project, 2018), which provides the following set of observable indicators of Agency, Ownership, Ownership:

- Each student takes ownership of the learning process in planning, monitoring, and reflecting on individual and/or collective work;
- Each student asks questions and makes suggestions that support analyzing, evaluating, applying and synthesizing mathematical ideas;
- Each student builds on the contributions of others and help others see or make connections;
- Each student holds classmates and themselves accountable for justifying their positions, through the use of evidence and/or elaborating on their reasoning.

These indicators elaborate on teacher and student roles in a mathematically responsive classroom, consistent with a productive disciplinary engagement frame. Note that the TRU framework does not prescribe particular teaching methods. Consistent with a continuity perspective, it outlines goals at a medium grain size that teacher learning communities can work toward in a variety of ways in order to

create learning environments that are increasingly mathematically responsive, as well as some milestones for progress.

Table 2.5 summarizes the literature reviewed in this section.

2.3 Culturally Responsive Teaching

Enacting the various aspects of a productive disciplinary engagement frame is not a politically neutral endeavor. For example, positioning students in more authoritative roles in the classroom requires careful attention to how issues of power, race, gender, and class, among others, influence existing authority structures and how these relationships can be reimagined. In other words, efforts at mathematically responsive teaching are just as vulnerable to the processes of inclusion and marginalization in society as any other sphere of human endeavor. This section reviews research on issues of power, inequity and justice in education with a focus on how these issues affect each aspect of a productive disciplinary engagement frame. Section 2.3.1 addresses how issues of power impact mathematics content. Section 2.3.2 addresses how issues of power impact discourse practices. Section 2.3.3 addresses how issues of power impact teacher and student roles.

2.3.1 Equity and Mathematics Content

In this section, I sketch some of the primary ways that issues of equity in mathematics education have been researched, in relationship to mathematics content. Section 2.3.1.1 summarizes how equity has been attended to within the standards movement, primarily through a focus on access and achievement for various demographic groups. Section 2.3.1.2 summarizes a second line of research, which defines equity in terms of agency and identity (Gutiérrez, 2002, 2013). One claim of this line of research with respect to mathematics content is that the funds of knowledge of nondominant communities are often marginalized both in curricula and in classrooms. A second, related claim is that school content, including mathematics content, should empower students to work for a more just society.

This section contextualizes my dissertation with respect to broader issues in the field. Sections 2.3.2 and 2.3.3 show how these issues can “push into classrooms” in ways that are more directly relevant for my analytical and empirical focus.

2.3.1.1 Power and Access

In 1989, anticipating the release of the *Standards*, the National Research Council [NRC] published the report *Everybody Counts* (NRC, 1989) about the state of mathematics education in the United States. Issues of equity were a major theme of the report. *Everybody Counts* problematized the organization of mathematics education around the purpose of preparing a select few students for calculus and for careers in science and engineering. The report argued that it was unnecessary and unacceptable for the discipline to function as a filter:

More than any other subject, mathematics filters students out of programs leading to scientific and professional careers. From high school through

graduate school, the half-life of students in the mathematics pipeline is about one year; on average, we lose half the students from mathematics each year, although various requirements hold some students in class temporarily for an extra term or a year. Mathematics is the worst curricular villain in driving students to failure in school. When mathematics acts as a filter, it not only filters students out of careers, but frequently out of school itself. (p. 7)

The report argued that mathematics education should become a pump, rather than a filter, that could help all students develop the knowledge based needed for mathematical literacy, college coursework in various fields including STEM fields, and informed citizenship. In addition to “mathematics for all,” the report included subsections on issues of access for various subgroups, including racial minorities, women, and disabled persons. A central argument was that the reorganization of mathematics curricula, combined with explicit attention to issues of access for particular groups, would result in excellent and equitable mathematics education for all students.

A great deal of literature has framed issues of equity in terms of access and achievement, specifically through a focus on racial and SES achievement gaps, for example between Black and White or Latinx and White students. Various perspectives have been given for the reasons for these gaps, including deficit perspectives and interpretations in terms of structural inequality. However, I align with various authors, notably Rochelle Gutiérrez (2008), who have argued that a focus on achievement gaps is problematic. Gutiérrez argues that even well-intentioned studies that seek to identify causes of racial achievement gaps for the purpose of reducing these gaps are problematic for several reasons, including: (1) the focus on between-group variance over within-group variance hides the reality that achievement distributions for Latinx, African American and European American students are highly overlapping; (2) the constant comparison of Black and Latinx student populations to White and Asian student populations implies that nondominant groups are not worth studying in their own right, and (3) the narrow definitions of learning and equity reflected in existing achievement measures lead to a focus on technical solutions rather than either attention to the larger patterns of social inequality that shape the structure of educational systems or attention to the spaces for agency for nondominant students to excel despite unjust conditions. In the late 1990s and early 2000s, large scale studies of the “mathematics for all” approach showed promising results (reviewed in Schoenfeld, 2002). Students who studied reform curricula did equally well on tests of basic skills as their peers who studied traditional curricula, and they did substantially better on tests of conceptual understanding and problem solving. Various studies also reported that the use of reform curricula correlated with substantial narrowing of achievement gaps based on race and SES (e.g., Briars & Resnick, 2000). In hindsight and in light of the critique by Gutiérrez, this finding can be re-examined with an effort to avoid an overemphasis on between-group comparison. It can be said instead that reform assessments set a new standard for excellence and that both Black and White students rose to the occasion. For example, the Briars and Resnick study shows that fewer than 20% of students from any background at weak implementation schools

met the conceptual understanding or problem solving standards. However, 30–40% of Black students at strong implementation schools achieved these standards, and the achievement of White students improved as well. This and other similar studies show that the “education for democracy” and “education for all” rhetoric of reform documents and associated improvements in curriculum led to substantial increases in opportunities in low-income and nondominant students’ opportunities to learn powerful mathematics.

Nevertheless, it is important to note that although achievement gaps have both increased and decreased over the last several decades, our educational system as a whole continues to perpetuate racial inequalities. Systematic divestment from the educational opportunities of nondominant students continues, as do entrenched patterns of inequality in society as a whole. Mathematics education is not immune from broader social forces. Specifically, despite the long history of attention to issues of access, mathematics reform efforts have had an inconsistent or missing focus on the way issues of power play out in larger social contexts of education. This has led to some hard lessons for the field. For example, curricula that could have benefitted all students were shelved because of how they played out politically (e.g., Oakes, 1997). Furthermore, some of the gains after *Brown vs. Board* have been reversed in the last few decades. The NRC’s monograph, *Adding it Up* (2001), reports that that “Although the gap between black students and white students had narrowed through the 1980s, it widened between 1990 and 1999, especially among students of the best-educated parents.” Racial and economic segregation of urban schools has *increased* in the last few decades, coupled with dramatic disparities between majority-White and majority-nondominant schools in terms of per pupil funding, class sizes, and the ability to recruit and retain qualified teachers (Darling-Hammond 2010).

2.3.1.2 Power and Agency

During the same time period as the “mathematics for all” approach, parallel efforts focused on the specific needs and strengths of nondominant students. One of the most important efforts was a study by Ladson-Billings of excellent teachers of African American children, leading to the development of her framework for culturally relevant pedagogy (1994). This framework includes three main elements. First, students must experience academic success. Culturally relevant pedagogy is not primarily about students’ feelings; students must be supported to develop strong content knowledge, specifically, “all students need literacy, numeracy, technological, social, and political skills in order to be active participants in a democracy” (Ladson-Billings, 1995, p. 160). Second, students must develop cultural competence, meaning that students retain cultural integrity – something that requires explicit effort against the grain of a school system that has historically been used for cultural subjugation. Third, students must develop “a critical consciousness through which they challenge the status quo of the current social order” (p. 160). That is, education should not only be about the advancement of individuals despite an unjust society but a means toward building a more just society as well. In general, culturally relevant pedagogy represents a commitment

to collective empowerment of nondominant communities and not only the individual empowerment of a few students in isolation.

Another major strand of research that attended explicitly to the connections between knowledge and power is research on the funds of knowledge of nondominant communities (Moll, Amanti, Neff & González, 1992), and how these funds of knowledge can be leveraged to inform and transform learning in various disciplines including language arts (Lee, 1993), science (González et al., 1995; Rosebery, Ogonowski, DiSchino & Warren, 2010; Bang & Medin, 2010; Bang et al., 2012) and mathematics (González et al., 2001). These studies paid close attention to the particular social contexts of education. They addressed how dominant power relationships constrain *whose* knowledge is seen as valuable. The marginalization of nondominant communities' funds of knowledge is not accidental, but the product of historical and current racial projects (Martin, 2013). As one example, consider the many Black teachers who lost their jobs after *Brown vs. Board* because it was socially and institutionally unacceptable for them to teach white children in newly integrated schools. Although school desegregation certainly marked an important step forward, the knowledge of Black teachers was marginalized in the reorganization of schooling. A second example is the systemic use of boarding schools to eradicate Indigenous cultures and educational processes in the United States.

	How power operates in society and in classrooms	Culturally responsive pedagogy (sociopolitical perspective)
Disciplinary Content	Issues of power often operate to marginalize nondominant funds of knowledge in society and in the classroom.	Explicit attention to power is needed in order to integrate nondominant perspectives on the important ideas of the discipline and to empower students to use disciplinary knowledge to work for a more just society.

Table 2.6. How issues of power influence the framing of disciplinary content.

Funds of knowledge studies helped reforge more just power relationships at the local level and showed important ways that disciplinary knowledge can be reimagined in partnership with nondominant communities. For example, the studies broadened the definition of expertise by either having teachers conduct knowledge-seeking home visits within nondominant communities, partnering with community elders in the design of curriculum, and/or inviting community experts such as parents or elders into the classroom. These new arrangements often had lasting impact beyond the particular classroom learning activity, such as restoring trust between schools and communities and/or creating an impetus for community

experts to continue their formal education toward advanced degrees. Although localized and still fairly vulnerable to disruption by the operation of power in society more broadly, these efforts represent a substantial step toward reimagining educational and social relationships in a more just way.

Table 2.6 summarizes the literature reviewed in this section. The next section will address the synthesis of a productive disciplinary engagement frame with what we know about the importance of explicit attention to issues of power, all with respect to the mathematics content.

2.3.2 How issues of power influence “acceptable” discourse moves

There is a large literature about culturally specific discourse practices of nondominant communities, how these practices are marginalized in schooling, and how culturally relevant pedagogies can better recognize and leverage these practices. Here, I mention only a few general, research-based principles that guide my approach.

First, it is important to avoid essentializing the practices of nondominant communities. Individuals develop repertoires of discourse practices through participating in various communities, and we cannot expect a single categorization of group membership to define any student’s learning style (Gutiérrez & Rogoff, 2003).

Second, while it is valuable to recognize the repertoires of practice that students bring into a learning space, part of learning is to grow and extend these repertoires. Furthermore, as both schools and the societies they prepare students for are increasingly interconnected, it is less and less common for individuals to participate in environments where everyone shares the same cultural background. What is needed is a balance so that each student experiences some “mirrors” in which their own traditions are valued and reflected and some “windows” into other traditions and cultural practices (Gutiérrez, 2012).

	How power operates in society and in classrooms	Culturally responsive pedagogy (sociopolitical perspective)
Typical discourse forms	Issues of power often operate to marginalize nondominant discourse forms.	Explicit effort is needed to recognize and welcome nondominant discourse patterns, translate between discourse modes as needed (an aspect of cultural competence) and recognize and overcome biases and blind spots (an aspect of cultural critique).

Table 2.7. How issues of power influence the framing of classroom discourse.

Third, it is essential to recognize the operation of power. Nondominant discourse practices are often marginalized. For example, a teacher who tries to leverage Spanish or African American English in the classroom may face pressure from administration to teach only Academic English. Political battles over English-only

education continue in many states. Constant work is needed by all parties to recognize and overcome biases as well as to recognize blind spots. Research on funds of knowledge in nondominant communities suggests effective methodology for engaging parents, elders, and other community experts as partners in the co-construction of knowledge about community realities (Bang & Medin, 2010; Bang et al., 2012).

Table 2.7 summarizes the findings of this section.

2.3.3 How issues of power influence student positioning

Perhaps the most promising area for joint attention to issues of mathematics content and issues of power in society is in research on student positioning. Recent research has offered a much more detailed conceptualization than previously of how issues of race, culture and power play out *within* classrooms and affect identity development and opportunities for content learning for nondominant students. Rochelle Gutiérrez (2013) terms this trend a *sociopolitical turn* in mathematics education, defining the term as follows:

Educators who take a sociopolitical stance recognize that mathematics education is identity work. Learners are always positioning themselves with respect to the doing of mathematics, their peers, their sense of themselves and their communities, and their futures. So, when an astute teacher sees that a student is not doing what is expected in the mathematics classroom, the teacher can recognize that the reason for this unexpected activity is connected to identity-in-the-making: resisting narratives that position the student as inferior, unworthy, abnormal, or on the margins of the local (e.g., classroom) culture. (p. 53)

A central aspect of culturally relevant teaching is expanding classroom learning opportunities such that there is a closer connection between important disciplinary ideas and practices and nondominant students' existing knowledge bases. Individual teachers have gone to great lengths to do this, often substantially reorganizing existing curricula to push their students beyond what others think they can do (Ladson-Billings, 1994). To the extent that research on mathematically responsive teaching expands the range of meanings of mathematical competence, it has great potential to assist culturally relevant teachers in their efforts to expand *who* is positioned as competent. For example, Birky et al. (2013) document the efforts of one Black mathematics teacher to reorganize their Algebra curriculum to teach systems of equations earlier, because the teacher believes that students can better learn about the mechanics of solving equations after they had some experience with using the relationships between variables to situations that are relevant to them.

	How power operates in society and in	Culturally responsive pedagogy (sociopolitical
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	classrooms	perspective)
Roles	Issues of power often operate to narrow the opportunities for nondominant students to participate centrally in classroom activity, which in turn narrows their learning opportunities.	Explicit attention to power is needed order to position and support nondominant students central participants in classroom activity, and thus expand their learning opportunities.

Table 2.8 How issues of power influence student and teacher roles.

However, attempts to position students in greater roles of authority require explicit and sustained attention to how of power and inequality often affect *who* can acceptably be positioned as an authority. One way this can manifest is through *status problems* that limit which students are considered competent and thus which students are provided opportunities to do serious intellectual work in the classroom (Cohen & Lotan 1995, 1997). Perhaps the best-known endeavor to combine mathematically responsive teaching with explicit attention to status at the classroom and institutional levels is Railside High School (Boaler and Staples, 2014). Teachers at Railside High supported high levels of student achievement and enrollment in advanced courses by nondominant students. Furthermore, mathematical practices such as justification were common and served both mathematical and equity-related functions. Issues of status and relational equity received consistent, joint attention from Railside teachers. In the absence of such attention, dominant narratives about how power and authority should be distributed along racial, gender, and class lines, are likely to persist and even be reinforced in mathematics classrooms.

In particular, the negotiation of student and teacher roles does not occur in isolation from dominant ideologies that create few subject positions for young people of color to be seen as good at school or good at math. *Racial storylines*, or stereotypes, are ubiquitous in schools and constrain the learning identities available to students (Nasir et al., 2012). In the discipline of mathematics, there is a strong overlap between discourse about race and discourse about mathematics ability in U.S. society: both discourses are hierarchical, treat race (or math ability) as innate, and link race (or math ability) with general intelligence (Shah, 2013). Interview studies with African American students (Nasir & Shah, 2011) and students of various backgrounds in urban schools (DosAlmas, 2012) find that racial storylines specific to mathematics ability, such as “Asians are good at math,” are salient for students. In particular, students note the limited subject positions available at their school for smart African American and/or Latinx students and limited opportunities for African American and/or Latinx students to participate in advanced math classes. Too often, nondominant students are seen by those with status and authority as either thugs, unintellectual beings or an endangered species (Nasir et al. 2012; McGee & Martin 2011). Nasir and colleagues call this being “sutured into” an identity, meaning that although there is always some space for student and teacher

agency in contesting these negative identities, it goes against the weight of current institutional and social arrangements to do so.

2.4 Culturally and Mathematically Responsive Teaching

This section describes attempts to synthesize culturally and mathematically responsive perspectives on teaching and learning. Like the previous sections, it begins with a focus on mathematics content in Section 2.4.1. Attention to issues of power in conceptualizing mathematics content is not new, and I briefly review some of the most fruitful efforts. However, as these efforts themselves show, an integrated model of culturally and mathematically responsive teaching needs to go well beyond the creation or adoption of new curricula. Careful attention is needed to how to draw out and build on nondominant students' multiple knowledge bases in interaction. Readers might expect that research on culturally and responsive classroom discourse would provide the necessary insights. Unfortunately, as I will describe in Section 2.4.2, most literature on culturally responsive discourse in mathematics classrooms conceptualizes the mathematics content in ways that reinforce a knowledge transmission frame. This conceptualization severely limits the potential synergy between culturally and mathematically responsive approaches. Fortunately, there is an emerging literature on student positioning in mathematics classrooms that combines careful attention to issues of power with a sufficiently rich conceptualization of mathematics content to begin to explore these synergies. Central research questions of this emerging literature are: "Which students are positioned as authors and critics of mathematical ideas? How and why?" This strand of research is reviewed in Section 2.4.3 and is the primary intellectual home of this dissertation.

2.4.1 Culturally and mathematically responsive mathematics content

Mathematics proved to be a particularly challenging discipline for connecting the funds of knowledge of nondominant communities for disciplinary learning in school. A study of community funds of knowledge in mathematics (González et al., 2001) readily found funds of knowledge that *researchers* saw as relevant to learning mathematics, for example tailors' embodied calculations of various geometric quantities when cutting and sewing angles and arcs. However, the pathway to leveraging this knowledge for students' school learning remained unclear, partly because it appeared necessary to teach students a great deal about the specific application contexts (e.g. a month learning sewing before being able to leverage this knowledge for a related geometry problem) and partly because school mathematics continued to be so narrowly defined.

Teachers and researchers who do successfully cue students' outside-of-school knowledge often find that it is a nontrivial task to make explicit connections to mathematics standards. For example, Lane and Silver (1995) discuss a "Busy Bus Company Problem," which was designed to assess middle school students' problem-solving knowledge in a culturally relevant context. In the Busy Bus Company

Problem, students are told that a one-way bus ticket costs \$1.00 and a weekly pass costs \$9.00. They are also given a description of Yvonne's weekly bus use, which they can interpret to mean that she needs to ride the bus 8 times during the week. They are asked, "Should Yvonne buy a weekly bus pass?" Teachers and researchers who assigned this problem expected students to respond that Yvonne should not purchase the bus pass because the one way fares cost \$8.00, which was cheaper than the weekly pass. Teachers who administered this task were surprised by the answer:

In particular, teachers reported that many students indicated that Yvonne should purchase the weekly pass rather than paying the daily fare, which the teachers believed to be the more economical choice. Curious about this unexpected answer to what the teachers believed to be a rather straightforward question – a multi-step arithmetic story problem involving multiplication of whole numbers – they decided to discuss the problem in class and ask students to explain their thinking. The ensuing discussion with students provided an interesting illustration of students applying outside-of-school knowledge and problem-solving strategies to a mathematics problem. Many students argued that purchasing the weekly bus pass was a much better decision because the pass could allow many members of a family to use it (e.g., after work and in the evenings), and it could also be used by a family member on weekends. Students' reasoning about this problem – situated in the context of urban living and the cost-effective use of public transportation – demonstrated to the teachers that there was more than one 'correct' answer to this problem...

Lane and Silver go on to explain that students who used this outside-of-school knowledge still scored poorly on the test question, because they did not show their relevant mathematical work (for example, adding up the cost of Yvonne's rides). Teachers interpreted this as a need and opportunity to hold a class discussion about the criteria for "a complete, high quality explanation that was based both on solid mathematics and on sensible reasoning." While I view this as a reasonable response and a significant move by teachers to value and build on students' outside-of-school reasoning, an alternate interpretation would be that students were given a real-world problem and responded with situated explanations that would be appropriate to the situation. For example, they may have given explanations that would be sufficient to justify to a family member that it was worth purchasing the weekly bus pass. Such a family member might find the student explanation convincing and the teacher's criteria unreasonable or even annoying.

A different approach to integrating mathematical and culturally responsive perspectives on content is *mathematics for social justice* (Moses & Cobb, 2001; Gutstein, 2006). This approach frames mathematical literacy, or "reading and writing the world with mathematics," as a civil rights issue. It asks: What mathematical knowledge do nondominant students need to thrive within, and ultimately transform, an unjust world? Like funds of knowledge approaches, mathematics for social justice seeks to leverage the knowledge and also the organizing capability of nondominant communities. It emphasizes cultural critique

(Ladson-Billings, 1994) as one of several essential components of mathematically and culturally responsive pedagogy, and one that can easily be minimized in other approaches.

In this dissertation, I have less data about cultural critique than about other aspects of culturally relevant pedagogy. For example, teachers were not observed to teach or collaborate around social-justice oriented mathematics projects. Nevertheless, teachers did adapt the curricula they were assigned to teach in ways that they felt would better support student success in life. For example, see Mr. B’s explanation in Chapter 4 about how he believes a classroom discussion comparing salaries will support students to survive, thrive, and advocate for themselves when confronted with injustice.

	Culturally and Mathematically Responsive Pedagogy
Mathematics Content	<p>Students begin from their prior knowledge and often discuss multiple approaches to solving a problem. Because students have opportunities to connect, critique, and revise ideas, they gradually refine their prior knowledge into powerful and general disciplinary understandings.</p> <p>Power relationships are negotiated in ways that position the multiple mathematical knowledge bases of nondominant students and communities as central to the learning process.</p>

Table 2.9. Content in a culturally and mathematically responsive frame.

2.4.2 Culturally and Mathematically Responsive Classroom Discourse

There is a dearth of literature that addresses mathematics classroom discourse with sufficient attention both to students’ mathematical thinking and to issues of power. Moschkovich (1999, 2007) is a notable exception. Moschkovich (1999) laments that research on supporting English Language Learners in mathematical discussions tends to focus on vocabulary and comprehension, with little guidance about how to attend to and build on the mathematical content of what these students have to say. She argues that an excessive focus on scaffolding language development often ignores students’ need for *conceptual* development, and argues that teachers instead should listen for the mathematical content of English Learners’ ideas and support them to build on these ideas. Moschkovich (2007) also makes explicit connections between psychological and sociolinguistic research on bilingualism to refute deficit positionings of students who language switch and code switch during mathematical discussions. Briefly, students’ fluid transitions between English and Spanish should not be interpreted as indicating a deficit in either their linguistic or mathematical knowledge, but can be understood in terms of the student’s history of participation in various communities, and their efficient and

adaptive use of a wide variety of linguistic resources. To give a few examples, students may language switch or code switch in order to indicate warmth and familiarity, may import technical words from English because they have been denied formal schooling in Spanish or because the English word is more informal and the Spanish word feels overly cold and formal, or because they will eventually be expected to express their solution using English technical vocabulary. In short, students' language choices reflect student agency and flexibility in drawing on their repertoires of practice (Gutiérrez & Rogoff, 2003), not deficiencies in their knowledge.

	Culturally and Mathematically Responsive Pedagogy
Typical discourse patterns	<p>The discourse patterns consistent with this frame can be quite varied. There must be space for students to share their thinking in some depth. There are many possibilities for discourse structures that support the gradual refinement of student thinking in connection with important disciplinary ideas.</p> <p>Power relationships are negotiated in ways that position and support nondominant students to act as authors and critics of powerful ideas, thus expanding their learning opportunities.</p>

Table 2.10. Discourse in a culturally and mathematically responsive frame.

Table 2.10 summarizes the findings of this section. In Chapter 4, I will take a fairly new approach to these issues. In that chapter, I will show two case studies of Black teachers leveraging Black discourse practices to position nondominant students as authors and critics of mathematical ideas. The analysis will primarily combine literature on power, race, and positioning students as authors and critics of mathematical ideas reviewed in Section 2.2, but will note the specific cultural discourse practices leveraged by teachers to do this work, and where possible the roots of these discourse practices in Black community activities.

2.4.3 Roles in a Culturally and Mathematically Responsive Classroom

A promising area of synthesis between mathematical and culturally responsive perspectives is the study of how *mathematical competence* is co-constructed by teachers and students within classroom learning environments. Because this literature is comparatively small, but especially important to the conceptual framework of my dissertation, I will provide somewhat more detail about the individual studies reviewed in this section.

An important paper in this vein is by Hand (2010) about the co-construction of opposition in a low-track mathematics classroom. Hand argues that traits such as “opposition” should not be conceptualized as traits of individual students but rather as co-constructed as different participants interpret and respond to each other’s actions. Her analyses show the trajectory of one low-track mathematics class over the course of one school year as the environment becomes increasingly

oppositional. Toward the beginning of the school year, the teacher presents mathematical tasks in a way that invites student sensemaking, does significant work to ensure students have access to the conversation (e.g., making sure everyone has their books open to the correct page and are ready to start), students share their mathematical ideas and respond to redirection of their behavior, and the teacher takes up students' ideas in a way that both positions them as competent participants in the classroom and moves the mathematical discussion forward. In the middle of the school year, the mathematical activity has somewhat narrower affordances for student sensemaking and there is somewhat more oppositional behavior, but it is not interpreted by the teacher or students in a way that blocks access to the mathematical conversation. For example, one student draws a big line through his worksheet in marker, which is marked by various parties as an oppositional move, but the student then re-engages with the mathematics and fills out other problems on the worksheet. Late in the school year, however, things have degenerated substantially. The mathematical opportunities being offered to students are quite narrow – students are expected to give very short answers to known-answer questions as the teacher directs a mathematical procedure – and students engage in more oppositional behavior, which is interpreted by all parties in ways that block their access to continue participating in the mathematics.

A paper by Gresalfi, Martin, Hand and Greeno (2009) shows how mathematical competence can also be co-constructed in mathematics classrooms. The paper included a case study of a sixth-grade classroom in which students were positioned as authors and critics of mathematical ideas and took agency for generating, comparing and critiquing their own representations to model different scenarios (in this case, bus trips). Within this learning environment, the teacher attends to both mathematical and social positioning in making identities of competence (and the learning opportunities that come with them) broadly available for students. For example, the teacher helps two students at a potentially vulnerable moment in which one student, Callie, comes to the teacher with a critique of another student's work. The teacher calls over the other student, James, indicates that she thinks he is capable of justifying his own work, and asks questions to help him see that his chosen representation does not yet clearly convey the needed information. James says he'll revise his work and says "darn you, Callie" and the teacher makes a positioning move to indicate that she sees both James and Callie's contributions as productive. James then revises his work. In presenting the Callie and James example, Gresalfi and colleagues showed one way that a teacher could jointly support mathematical openness and broad opportunities for student competence. They do not claim that these two will always go together. In fact, the authors also present a contrasting example from an eighth-grade classroom that operates smoothly in a demonstrate-and-practice mode – what I call a knowledge transmission frame. In this classroom, opportunities to participate are mathematically narrow, but access to participate in these ways and be seen as competent were broadly available to all students. In summary, Gresalfi and colleagues present one example of how to analyze classroom interactions and positioning with a focus on both the mathematical framing and negotiation of

student status, and how these mathematical and social positionings interact to broaden or narrow the learning opportunities available.

Hand (2010) and Gresalfi and colleagues (2009) consider issues of status and positioning, but do not explicitly discuss race, gender, or other ways that power may impact students' positioning. Other authors take up these issues more explicitly (Langer-Osuna, 2011; Esmonde & Langer-Osuna, 2013). Langer-Osuna (2011) shows how leadership roles in a project-based algebra classroom can be gendered. Displays of authority by a female group leader, Briana, were positioned as inappropriate by others, leading her to be seen as bossy. Displays of authority by a male group leader, Kofi, were positioned as desirable and helpful by others, leading him to be seen as smart. This paper shows that although students in the project-based algebra class were positioned as authors and critics of mathematical ideas in ways that afforded their learning of powerful mathematics, issues of power and gender restricted *which* students accessed and benefitted from these learning opportunities. The paper also provides some methodological tools for analyzing mathematical and social positionings together in interaction.

Esmonde and Langer-Osuna (2013) take up issues of gender and race explicitly in their analysis of one small group's negotiation of the meaning of mathematical competence. The small group consists of two Black, female, tenth-grade students and a White, male, ninth-grade student. The students move somewhat fluidly between a figured world of school mathematics in which the boy had more authority and a figured world of social relationships and romance in which the girls had more authority. The conditions that widened or narrowed learning opportunities were complex. For example, the boy made a bid to ask the girls guiding questions, indirectly positioning them as less knowledgeable and himself in a position to help them think through the problem, but they resisted this positioning and little learning occurred. Later, the boy more explicitly told one girl how to solve a problem, positioning her as less knowledgeable, but she responded by critiquing a step of his arithmetic, which increased both her mathematical and social authority and created a learning opportunity for all three students.

Culturally and Mathematically Responsive Pedagogy	
Roles	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are made available to students and the teacher. Power relationships are negotiated in ways that position and support nondominant students to act as authors and critics of powerful ideas, thus expanding their learning opportunities.

Table 2.11. Roles in a culturally and mathematically responsive frame.

Table 2.11 distills core principles from the literature reviewed here, as they relate to my dissertation. Taken together, these papers show that analyses of *which* students are positioned as authors and critics of mathematical ideas can help us better understand the mechanisms of mathematical and social positioning as they affect learning opportunities.

More research is needed in this area. This is one of the major threads I take up in the dissertation. In Chapter 4, I will analyze two case studies in which two Black teachers leverage Black cultural practices to position nondominant students as authors and critics of mathematical ideas.

2.5 Teacher learning

The preceding sections outline a conceptualization of mathematically and culturally relevant teaching. This section summarizes theoretical and empirical work on teacher learning that can address the question: “How can teachers develop mathematically and culturally relevant pedagogy over the long term?”

Across the United States, there is far less support for teachers’ professional learning than is considered necessary and common in other countries (Hiebert et al., 2003; Darling-Hammond, 2010; Little & Bartlett, 2010). In educational policy circles in the United States, teacher quality is usually measured in terms of teachers’ academic preparation and subject matter credentials (Little & Bartlett, 2010). Insufficient attention is given to teachers’ enthusiasm and preparation for teaching nondominant students (Ladson-Billings, 1994). Furthermore, teacher learning takes time. Countries that value excellent teaching support teachers’ professional growth over at least the first decade of their careers, for example by ensuring lower teaching loads, enough stability in curriculum to allow teachers to build their knowledge of student thinking from year to year, and opportunities for sustained professional collaboration. In contrast, most teachers in the United States experience induction programs lasting a year or two or no induction program at all (Darling-Hammond, 2010; Little & Bartlett, 2010), a patchwork of curricula (NRC, 2000), and an “egg crate model” (Lortie, 1975) in which teachers are usually isolated in their own classrooms with little opportunity to collaborate. Even when opportunities for professional collaboration do exist, and are considered successful by teachers and administrators, this collaboration is rarely focused on what counts mathematically (Horn, Garner, Kane and Brasel, 2017) or sociopolitically (Gutiérrez, 2012).

In this context, I consider it essential to outline a few principles of teacher learning that inform my study.

First, I take a *continuity perspective* on learning in general and on teacher learning in particular. The term continuity perspective has a long history in the learning sciences, and refers broadly to a conceptualization of learning as a dynamic process of building on prior knowledge. Research on learning in mathematics (Schoenfeld, Smith & Arcavi, 1993; Pratt & Noss, 2002), physics (diSessa, 1993; Smith, diSessa & Roschelle, 1994), and other areas (Hammer & Elby, 2002; Philip, 2011) demonstrates that students build on prior knowledge in fairly complex ways. Snapshots of “novice” and “expert” knowledge are insufficient to conceptualize learning, because learning is not a straightforward process of “adding” or “replacing” knowledge but often involves extending and adapting prior knowledge to be more coherent, connected, and robust. The general principle that students

learn by building on prior knowledge is applicable to teacher learning as well (NRC, 2000).

A continuity perspective on teacher learning holds that teachers also learn by adapting and building on what they already know (Russ et al., 2016). More specifically, teaching practice is part of a complex constellation of teachers' resources, goals and orientations; as a result, significant change can be expected to take many years and cannot be expected to follow a straightforward or linear trajectory (Schoenfeld, 2010). On the other hand, teachers' everyday knowledge offers a substantial repertoire of resources that teachers can use and refine as they develop more specialized teaching practices. For example, although facilitating classroom discussions certainly involves some specialized knowledge and routines, it could not be carried out without the underlying foundation of teachers' knowledge and routines for carrying out everyday conversations (Russ et al., 2016).

A continuity perspective can explain why research and professional development approaches that assume teachers can or will make a gestalt shift from a knowledge transmission to productive disciplinary engagement frame have proven difficult. For example, consider Cohen's (1990) famous description of "Mrs. Oublier," a teacher who adopts many of the materials, activities, and even the language of mathematics reform documents while in many ways continuing to enact a knowledge transmission frame in her classroom:

Mrs. O seemed to treat new mathematical topics as though they were a part of traditional school mathematics. She used the new materials, but used them as though mathematics contained only right and wrong answers. She has revised the curriculum to help students understand math, but she conducts the class in ways that discourage exploration of students' understanding. (p. 312)

From a continuity perspective, a shift from a knowledge transmission to productive disciplinary engagement frame is not a simple matter of adopting a new curriculum. Teachers need time to make sense of the underlying values of a productive disciplinary engagement frame, develop the knowledge bases to understand new content goals and how student ideas may develop towards these goals, and develop and/or adapt new talk moves and teaching routines. This type of generative change in teaching practice typically takes years (Franke, Carpenter, Levi & Fennema, 2001; Visnovska & Cobb, 2009), although even novice teachers can take important steps along this trajectory (Levin, Hammer & Coffey, 2009). In the context of a long-term learning trajectory, Mrs. Oublier's adoption of a new curriculum and routines, though still largely consistent with a knowledge transmission frame for teaching, takes on new significance. Cohen's study, like most similar work, does not address the potential long-term effects of these changes, in particular whether the changes affect the classroom activity system enough to create new opportunities for teacher learning about students' mathematical understandings. I will take up these questions in Chapter 6, where I analyze a teacher's adoption of a new activity structure, student presentations, in ways that are initially consistent with a

knowledge transmission frame but then shift to become more consistent with a productive disciplinary engagement frame.

In the area of classroom discourse, the bulk of prior research has focused on either documenting inadequacies of typical U.S. classrooms (Mehan, 1979; Hiebert et al., 2003), studies of a few exemplary teachers that can inform a richer conceptualization of expert teaching practice (e.g., Horn, 2008, Ladson-Billings, 1994), and more recently studies of teacher learning programs that seek to support the development of culturally and/or mathematically responsive teaching by novice teachers (Rosebury & Puttick, 1998; van Es & Sherin, 2002; Philip, 2011; Turner et al., 2012) and experienced teachers (Horn & Little, 2010; Takeuchi & Esmonde, 2011; van Es & Sherin, 2008; Battey & Franke 2013).

Although many of the professional development efforts studied do create affordances for teachers to shift their new orientations, routines and practices over the long term, teachers' prior knowledge and practices still tend to be conceptualized as negative ideas to be replaced. There is little serious attention to how the existing knowledge and practices of experienced teachers can be adapted, repurposed, or extended, as they develop more culturally and mathematically responsive teaching practices. Much more research is needed about how the knowledge bases that teachers already possess can be leveraged for learning about culturally and mathematically responsive teaching. Chapter 5 takes up this issue with a particular focus on IRE sequences. IRE sequences are ubiquitous in U.S. classrooms and we can expect that many teachers would be familiar with this practice. In typical U.S. classrooms, IRE sequences reinforce a knowledge transmission frame, but I argue that in mathematically responsive classrooms they can serve a much broader range of functions. This reconceptualization of IRE sequences in particular, and teachers' prior knowledge and practices in general, can support a much richer understanding of teacher learning about culturally and mathematically responsive teaching.

A second, related point concerns the funds of knowledge of nondominant teachers in particular. Just as the funds of knowledge for content learning of nondominant communities have been marginalized, so have their funds of knowledge for teaching. However, a substantial literature now documents teaching practices of nondominant teachers and communities (e.g., Rogoff, Paradise, Mejía Arauz, Correa-Chávez & Angelillo, 2003) that can be powerful for all learners and particularly for nondominant learners. A majority of this literature addresses funds of knowledge in nondominant communities for organizing instruction and creating a positive learning environment. A much smaller, but emerging literature addresses specific mathematics disciplinary content, primarily through investigating the funds of knowledge of nondominant mathematics teachers (Birky, Chazan & Morris, 2013; Frank, Khalil, Scates & Odoms, 2018).

Third, teacher learning is not an individual activity, though it may be treated as such in most U.S. policy documents (Little & Bartlett, 2010). Learning occurs in

communities of practice (Lave & Wenger, 1991; Horn, 2005; Bannister, 2018). Conceptualizing teacher learning as situated in communities of practice has important implications for both research (Horn, 2005; Bannister, 2018) and policy (Little & Bartlett, 2010). At a research level, individual conceptions of teacher learning perpetuate an “egg crate model” of teaching expertise and miss many of the mechanisms by which change is possible. At a policy level, differences in teachers’ academic credentials is one indicator of unequal teaching quality between schools and districts (Darling-Hammond, 2010), but attempting to recruit and retain teachers with more academic credentials is not an adequate approach to solving this problem (Little & Bartlett, 2010). If we conceptualize the development of expertise in teaching as at least a decade-long process, then we will at best be able to hire teachers who have great potential. If we want to convert that potential into excellence, we need much more robust professional development programs. Developing such programs would also improve teacher retention, especially in districts that are hardest hit by unequal school funding and where teachers are least likely to be treated either by administrators or by the popular media as professionals.

2.5.1 Implications for the study

Taken together, these principles of teacher learning have significant implications for research conceptualization and methodology. First, any teacher learning trajectories that can be documented over the course of one school year will necessarily be partial. One cannot expect to find a “phase shift” in which teachers who operate largely within a knowledge transmission frame and/or deficit frame suddenly transform into a productive disciplinary engagement and/or culturally responsive frame. Elements of multiple frames will likely overlap and compete in the classroom, and change takes place on the time scale of years or decades. However, we do expect to document significant portions of a learning trajectory over the course of a school year; Chapter 6 gives one example.

Second, in a continuity perspective, documenting learning trajectories is a high-end goal that often occurs toward the end of a long research program. This is because, in order to document learning at a fine enough grain size, a fairly deep understanding of individual knowledge elements is needed. For example, the detailed learning trajectory given by diSessa (2014) for students’ conceptual change in coming to understand a central aspect of thermal equilibration was based on substantial prior work that mapped several of the important knowledge elements (diSessa, 1993) that were invoked and recombined in the learning trajectory. Although I do not attempt nearly the same level of detail, these papers serve as an important caution about the relevant general principle that mapping learning trajectories depends on a sufficiently specific conceptualization of the “states” of the thing to be learned to be able to describe points of continuity and adaptation between the initial and final states. In this dissertation, the learning trajectory documented in Chapter 6 builds on significant groundwork laid in Chapters 4 and 5.

3 Methods

3.1 Study participants

3.1.1 Site Selection and Description

The school site for this research is Adams School (a pseudonym), a neighborhood public school in a high-poverty urban district in Northern California. Adams School is located in an historically Black, working-class neighborhood, although recently the neighborhood demographics are shifting to include many immigrant families from Mexico and Central America. In 2016–17, the school served a majority of Hispanic or Latinx students (80–85%), a large minority of Black or African American students (10–15%), and a small minority of Asian and Pacific Islander students (1–5%). There were very few students from other backgrounds (1% or less from each of these groups: American Indian or Alaska Native, White, Two or More Races, Other). Almost all students (90–95%) at the school are considered socioeconomically disadvantaged. (California Department of Education, 2017).

The site selection process had some parallels with the site selection process described by Ladson-Billings (1994), although with notably less formality and parent input. Before selecting my school site, I spent two years observing and student teaching in several nearby schools, and used this as an opportunity to learn more about excellent teaching and learning in a couple of neighboring, urban school districts. I spent time in more than 10 nearby schools, seeking out experienced and successful teachers of nondominant students. I visited a wide variety of schools, including large comprehensive high schools, small public and charter high schools, and neighborhood middle schools. At each school, I either contacted particular teachers recommended by colleagues or reached out to the mathematics department chairs to recommend teachers. Most teachers responded positively and invited me to visit their classrooms. I visited the classrooms of approximately 15 teachers for at least one day and also carried out 6 observation or student teaching placements lasting a month or more. During this process, a colleague mentioned the teacher learning community (TLC) at Adams school. Mathematics teachers at Adams had previously contacted our research group to request a professional development partnership. This was an early signal of the department's potential for strong collaboration. I soon began a semester of student teaching at Adams and became a participant-observer in the teacher-research partnership.

3.1.2 Focal Teachers

After a year as a student-teacher and participant-observer at Adams, my university colleagues who had been facilitating professional development completed their own studies and ended their partnership with the school. I contacted the mathematics teachers about continuing the partnership for my own dissertation research, noting that I did not yet have the teaching background needed to facilitate professional

development, but would be very interested to be a participant-observer and conduct a classroom video study.

All seven teachers in the mathematics department, plus the full-time mathematics teaching coach, agreed to participate in my research study. Of the seven teachers, four had more than one year of teaching experience, and all four were selected as focal teachers: Ms. A (sixth grade), Mr. B (sixth grade), Mr. X (high school Algebra 1) and Mr. Y (high school Algebra 1).

Three of the focal teachers self-identified during interviews as Black and/or African American. One of the focal teachers self-identified as mixed race in the beginning-of-year interview, and also mentioned that others racialized him as a Black man. He had strong ties to Black culture and in more casual conversations typically referred to himself as a Black man both when talking about his own cultural affiliations and when talking about his experiences of racial discrimination. I will therefore refer to all four teachers as Black teachers. Non-focal participating teachers included one mixed-race teacher, one Asian teacher, one White teacher, and the teaching coach, who identified as White. Although teachers' race and ethnicity were not selection factors, issues of racial identity emerged as a research focus because of the perspectives and strengths of the teachers in the study.

3.1.3 Teacher Learning Community

This section describes the local context in which teacher learning was situated. At Adams school, there were strong supports for collaboration in the mathematics department and collective approaches to problems of practice.

3.1.3.1 Mathematics Department

The mathematics department includes two sixth-grade math and science teachers, two full-time middle school math teachers, three full-time high school math teachers, and a full-time math coach. There are several department-wide classroom teaching commitments and goals that reflect a collective vision of high quality teaching. Department meetings occur about once per month and focus on collective support and accountability to work toward the department's goals, which are:

- 1) 1 year of student growth on the **SMI**³ exam;
- 2) Teaching 2 **FALs**⁴ per semester;
- 3) Having each student present in front of the class once per marking period;

³ The Scholastic Mathematics Index (SMI) is an adaptive, computerized test of procedural mathematics knowledge that can take from 20 minutes to 2 hours. It is designed to be most accurate in measuring growth, but in practice is often used to assign a single number to students corresponding to their grade level in math. The test is similar to the Scholastic Reading Index (SRI) in English Language Arts.

⁴ Formative Assessment Lessons (FALs) created by the Shell Center, available at <http://map.mathshell.org/lessons.php>.

- 4) At least 80% of middle school students and 70% of high school students pass their classes.

Goals 2 and 3 are of particular significance to the study. Goal 2 represents a common commitment to use a curricular resource with especially rich content and pedagogical affordances (Herman et al., 2014; Research for Action, 2015). Goal 3 represents a common commitment to explore a pedagogical strategy that included substantial “air time” for students to share their mathematical thinking. Thus, individual and collective work on these goals supported the teachers to try and refine ambitious practices for what I call mathematically responsive teaching, well beyond what is typical in U.S. mathematics classrooms.

The department has several other commitments and goals, which are less relevant to this dissertation, but are summarized briefly here. Goals 1 and 4 represent a collective commitment to student achievement as evaluated by quantitative measures. This type of goal is seen as a political necessity at many schools. It is unusual that standardized test and grade data give sufficiently specific formative data to support teacher learning. However, these teachers made a good faith effort to get the most out of these data as one part of their commitment to access and achievement for their students. In addition to these formal goals, the Adams math department had several years previously advocated for and received grant money for individual laptop computers to support their blended learning model. In this model, students in all math classrooms worked on individual laptop computers for a portion of class time, typically on computer skills programs such as Khan Academy or Practutor. Several teachers in the department saw student success on this computer skills work as directly tied to student success on high stakes tests, especially since the introduction of the computerized SBAC⁵ tests two years ago and the computerized SMI test this year. Although the computer skills programs appeared effective as a way of tailoring practice problems to student needs, they tended to reinforce a knowledge transmission frame for learning. Teachers also had a shared commitment to using other resources to support problem solving. For example, several teachers had a “Problem solving Wednesday” routine that used more open-ended tasks, often MARS⁶ tasks, to support students’ problem solving. Some of these resources were consistent with a productive disciplinary engagement frame in that they supported multiple solution methods and student sensemaking about meaning and application of important mathematical ideas. In short, although the resources available to teachers conveyed inconsistent messages about the frame at play, teachers nevertheless combined these resources in an effort to provide their students with a “balanced diet” of procedures, concepts and problem-solving.

⁵ The Smarter Balanced Assessment Consortium (SBAC) is one of two testing companies contracted to assess student achievement relative to the Common Core State Standards and currently the main standardized test used in California.

⁶ Rich contextual tasks created by the Shell Center, similar to but shorter than the Formative Assessment Lessons and available at <http://map.mathshell.org/tasks.php>.

To support these goals and commitments, the department organized several teacher learning opportunities. All teachers except one received coaching which included cycles of planning, lesson-observation and debriefs with the coach. These cycles happened weekly for two of the three beginning teachers and approximately every two weeks for the four experienced (focal) teachers. The third beginning teacher received similar, weekly coaching from an outside coach. The four experienced (focal) teachers also participated in weekly co-planning meetings, which were primarily teacher-led. The coach attended one of the weekly co-planning meetings and I attended both meetings; we both offered suggestions when invited to do so during these meetings. Finally, all teachers in the department conducted occasional observations of each other's teaching (roughly 3–4 colleague observations during the year). In addition to department-organized learning opportunities, the district organized monthly mathematics professional development which the beginning teachers usually attended and the experienced teachers sometimes attended.

3.1.3.2 Co-Planning Teams

The focal teachers were paired by grade level in two co-planning teams that met weekly: one consisting of two sixth-grade teachers and another consisting of the two high school Algebra 1 teachers. Like the department meetings, teacher team meetings included some significant opportunities for learning about mathematically responsive teaching, but were not wholly dedicated to this purpose. There was a balanced focus that included some attention to the needs of the moment, such as lesson planning for the next day, and support toward longer-term goals, including the use of student presentations. Teacher buy-in for the learning spaces was strong in part because of the flexibility to meet teachers' needs. On the other hand, important learning opportunities were sometimes missed. For example, after a lesson where students unexpectedly studied with an "alphabetizing" procedure for ordering decimals (described in more detail in Chapter 6), I offered to bring a few resources to the next sixth-grade co-planning meeting for thinking about students conceptual knowledge relevant to ordering decimals. Mr. B agreed that this would be very valuable, but the conversation was postponed and eventually cancelled because lesson planning took precedence. Nevertheless, as explained in the next section, some longer-term goals such as student presentations were given enough attention during this time to support teachers' sustained use and adaptation of new practices.

3.1.3.3 Supports for Teacher Learning – Example of Student Presentation Practice

To briefly illustrate the supports for teacher learning created by the department structures, I will consider the example of Goal #3, student presentations. This goal is particularly salient to the teacher learning trajectory for Mr. B documented in Chapter 6, and as such provides important context for the analysis in that chapter.

One of the six monthly department meetings toward the beginning of the year was spent discussing what "student presentations" meant. This was important because teachers could potentially have very different goals for using student presentations,

such as to have students develop public speaking skills or confidence with mathematical vocabulary. However, the coach wanted teachers to see that student presentations could be used as part of a mathematical conversation. During this session, the coach showed a video from her own classroom when she had been a full-time teacher and had had multiple students present on the same problem, followed by a comparison and contrast of the methods used. During this meeting, the department also began developing a simple rubric rating student presentations on a 1–4 scale, which emphasized that presentations should be an opportunity to show conceptual understanding and academic language as well as correct answers and procedures. The rubric was finalized by the sixth-grade planning team in one of their weekly meetings, and will be discussed in more detail in Chapter 6.

These department-wide activities set the stage for teachers' initial efforts at student presentations. As will be discussed in detail in Chapter 6, the initial use of student presentations was not necessarily aligned with a productive disciplinary engagement frame. For example, Mr. B had an initial goal of having every student come to the front of the class to share their ideas, and he achieved this goal the first time he tried the routine. However, creating space for students to explain their thinking at length created new opportunities for teacher learning. Mr. B did not stop with his original practice, but instead built on the opportunities he saw to get students to express their own sensemaking and then to build on the ideas they expressed as part of an increasingly rich mathematical conversation.

As teachers experimented with the new practice, their continued learning was supported by their interactions with their coach and with their co-planning teams. For example, teachers typically prepared to teach significant tasks, especially Formative Assessment Lessons, by doing the math together with the coach, co-planning team, or both, and trying to anticipate how students would approach the main ideas. This type of anticipation of student thinking is a major technique for supporting mathematical discussions (Stein et al., 2008).

3.2 Data Collection

Classroom observations serve as the primary data source for the dissertation. Focal class periods were selected in consultation with each teacher. The two sixth-grade teachers each taught two math classes; I videotaped the morning class of one teacher and the afternoon class of the other teacher. The two experienced high school teachers each taught 2–3 sections of Algebra 1; they selected a section that they preferred for me to videotape. Student and parent permissions were obtained for each focal class period. I visited the four focal classes about once per week at a time that was scheduled in advance. Although I did not visit the classes unannounced, teachers sometimes forgot that we had a visit scheduled. When this occurred, they still allowed me to videotape unless they were giving an exam. It did not appear that teachers changed their lesson plans because of my visits.

Roughly every other class visit was a data collection visit and every other visit was a relationship building visit. During relationship building visits, I did not bring any research equipment and did not collect any formal data, but instead assisted the teacher and students however the teacher directed. This typically involved assisting individual students or groups during work time, but sometimes included other duties such as making photocopies or pulling out a group of students who were struggling to have a focus session in another room. During data collection visits, I video- and audio-recorded the class period, took field notes, photographed or collected and scanned instructional materials and student work when possible, and conducted audio-recorded debrief sessions with teachers after class when possible.

Secondary data sources include video recordings of monthly department meetings, audio recordings of the weekly meetings of both grade-level teams, video-recorded pre-interviews with all teachers, and video-recorded post-interviews with focal teachers and the coach. The purpose of the pre-interview was to get to know the teachers and their goals for instruction. They provided useful context for the study; however, the data obtained was not specific enough to be used directly. The purpose of the post-interview was to view video clips with the teachers and get their reactions to the emerging findings; again, the data obtained were not used directly.

At the end of the study, I conducted end-of-study interviews with Ms. A, Mr. B, and Mr. Y to share the draft results. Mr. X was invited to participate in an end-of-study interview, but was unable to participate due to poor health. End-of-study interviews with teachers took about an hour each and included a verbal conversation about the findings, time to read a few pages of the draft manuscript and an invitation to review the full manuscript if they wished to do so. In addition to teachers, students Enoch and Frank from the primary case in Mr. X's classroom also participated together in an end-of-study interview. The student interview was shorter, less formal, and included a verbal conversation and video watching. Only a few sentences of the draft manuscript were read by students. The end-of-study interviews were not audio- or video-recorded, but I took notes during the

interviews. No teachers requested changes to the descriptions of their classroom teaching; however, they did add contextual detail about students and suggested a few wording changes. For example, a teacher asked to be identified as Black rather than African American. A teacher also asked me to adapt the language in my model of culturally and mathematically responsive teaching to say “Power relationships are negotiated” instead of “Dominant power relationships are renegotiated.” While small, I believe these wording changes reflect a respectful conversation with the focal teachers about the sociopolitical stance taken by the study and their own stances. Students did not request any changes, but did add detail to the description of the dance moves in their episode. I made all of the suggested changes.

Data source	# Observations 2014–2015			Ch4	Ch5	Ch6
	Total observations	With Student Presentations	With FAL			
Department Meetings	6 (Monthly)					+
6 th grade Collaboration Audio	16 (Weekly)					+
Ms. A (6 th grade) Classroom Observation	23	20	15		***	
Mr. B (6 th grade) Classroom Observation	16	5	10	***	***	***
9 th grade Collaboration Audio	11 (Most weeks)			+		
Mr. X (9 th grade) Classroom Observation	15	3	6	***		
Mr. Y (9 th grade) Classroom Observation	15	About 3–5	8			
Pre Interviews	8					
End-of-Year Interviews	5					
End-of-Study Interviews	1+			+	+	+

Table 3.1. Data Sources.

Table 3.1 shows the primary data sources (***) and secondary data sources (+) used in each chapter. Classroom observations of three focal teachers serve as the primary data source, with teacher debriefs, collaboration sessions and post interviews serving as secondary sources that help uncover the causes of and orientations toward the observed practices or shifts in practice.

3.3 Data Analysis

Data analysis methods differ substantially from chapter to chapter and are primarily described within each data chapter. An overview is provided here.

There was a methodological tension between documenting classroom practice, which is carried out by one teacher in isolation from teacher colleagues, and documenting collective learning by teacher learning communities, which is most visible in teacher collaborative meetings. For this dissertation, I initially planned to use video of teacher collaborative meetings as my primary data source, in order to capture collective learning. Classroom data, along with teacher pre- and post-interviews, were intended to be a secondary focus for the purpose of triangulating results and documenting that teachers actually “walked the walk” as well as “talking the talk.” However, I soon found that significantly more work was needed to conceptualize classroom discourse in a way consistent with a continuity approach to teacher learning. This conceptual work required careful, qualitative analyses of classroom data, of the kind carried out in Chapters 4 and 5. A next step was to demonstrate that the conceptualization thus developed actually supports research on teacher learning trajectories; Chapter 6 serves as a proof of concept. Again, classroom video was central to Chapter 6 because I aimed to document small shifts in classroom practice that opened up space for frame shifts. This is not meant to imply that Mr. B accomplished the learning shown in Chapter 6 by himself. He was part of a robust teacher learning community and I document some general learning affordances of this community in relation to the documented shifts in his classroom practice. Future research using this same dataset could provide a more thorough analysis of the particular learning opportunities in teacher collaborative meetings that may have supported Mr. B to make the documented shifts in his classroom.

3.3.1 Summary of chapter-specific data analyses

Chapter 4 uses a case study method to conceptualize and then document how culturally specific practices can be leveraged for culturally and mathematically responsive instruction. It presents a case study analysis of two classroom episodes from the dissertation dataset. The cases differ in grade level, mathematical content, classroom activity structures, and the particular cultural practices used. However, in both cases, teachers leverage Black cultural practices to position nondominant students in roles consistent with a productive disciplinary engagement frame, specifically as authors and/or critics of mathematical ideas. In both cases, the qualitative analysis addresses the contextualized meaning of the Black cultural practices used, and how these practices and others support the unfolding classroom activity. Specifically, I analyze:

How did the teacher support student engagement?

How did the teacher invite students’ outside-the-box sensemaking, that is, sensemaking outside the knowledge transmission paradigm?

How did the teacher position students as authors and elicit their mathematical thinking?

How and to what extent did the teacher support the take-up, critique and revision of students’ mathematical ideas?

In Case 1, “write up why while you whippin and nae naein,” Mr. X supports a Black male student to present a novel and unexpected strategy at the front of the classroom. He uses a variety of culturally specific practices to affirm the student’s Black identity, hip hop affiliation, and competence to explain mathematical ideas. In

Case 2, “I don’t want to work more... if they’re almost paying me the same,” Mr. B positions his Latinx and Black students as defenders and critics of important mathematical ideas by turning a homework review into a debate about a choice between two jobs. The teacher makes several culturally specific moves to blur the boundary between school mathematics and students’ outside-of-school knowledge. This positions the outside-of-school knowledge of nondominant students as a central resource for the discussion while supporting students to dig into important mathematical ideas related to proportionality and mathematical modeling.

Chapter 5 also uses a case study method to conceptualize and then document classroom discourse practices, specifically IRE sequences, in a way that is consistent with a continuity perspective on teacher learning. Specifically, I argue that IRE sequences, a very common discourse move in U.S. classrooms that is often indicative of a knowledge transmission frame, can be adapted and repurposed to be more consistent with a productive disciplinary engagement frame. A strict operational definition of an IRE sequence was used, consistent with Mehan (1979), and cases were selected that included the use of IRE sequences within a mathematically responsive classroom episode. The primary case is an episode from Ms. A’s classroom that includes two student presentations and eight IRE sequences. After demonstrating that the episode as a whole was mathematically responsive and that IRE sequences made up a significant fraction of the talk, the analysis unpacks the function of the various IRE sequences within the episode.

Chapter 6 builds on the conceptual work done in Chapters 4 and 5 to analyze substantially more data. The chapter analyzes Mr. B’s learning trajectory over the course of the whole school year as he implements and adapts a new teaching practice: student presentations. The primary data used for this chapter include classroom observations of four baseline episodes before the first observed use of the student presentation activity, and four episodes in which the student presentation activity was observed. A combination of quantitative and qualitative analyses are used. Quantitative codes adapted from the EQUIP framework (Reinholtz & Shah, 2018) are used to code all public student talk turns in the eight episodes. This coding reveals substantial shifts in both the length and nature of student talk from the baseline episodes to the first presentation episode and subtler shifts between the student presentation episodes. Qualitative analysis is also used for selected episodes to analyze the various aspects of framing: mathematics content, discourse moves, and teacher and student roles. Qualitative analysis consisted in selecting relevant segments, transcribing, and determining functions of teacher and student contributions. The two analyses together reveal several ways that Mr. B shifted the student presentation activity to become increasingly consistent with a productive disciplinary engagement frame as he sustained and adapted this ambitious teaching practice over the course of the school year.

4 Sustaining and developing nondominant students' joint racial and mathematical identities: Insights from Black mathematics teachers

4.1 Introduction

Two Black, male students are having a seated “dance off” in math class. They are not off task. On the contrary, they are celebrating the fact that both of them have correctly solved a difficult mathematics problem that had stumped most of the class. The teacher, far from reprimanding the dancing behavior, gives one student an affectionate shoulder bump and tells him to “write up why, while you whippin and nae naein.” Both students continue grooving while bending over their papers to write explanations of their mathematical strategies. One of the students has used a fairly standard, algebraic strategy. The other student has invented a novel, numerical strategy that the teacher had not considered. Before the end of the class period, the teacher calls up the student with the novel strategy to present his strategy to the class, and the teacher and several other students make connections between this unexpected strategy and important ideas in the discipline of mathematics.

This vignette, which is expanded as a focal case for analysis later in the chapter, is unfortunately rare in United States mathematics classrooms in that the classroom episode it describes supports students' nondominant racial identities, academic identities, and mathematics disciplinary identities all at once. It is rare for students of any background to be invited to author and share novel mathematical ideas, which invites a productive combination of academic and mathematical identities. It is rare for nondominant students' culturally specific emotional displays to be valued in the classroom. In this case, the emotional displays in question celebrate students' cultural affiliation and mathematical success simultaneously, which invites a productive combination of racial and mathematical identities.

This dissertation aims to contribute to knowledge about culturally and mathematically responsive teaching, specifically how classroom discourse structures can position and support nondominant students to take up roles as authors and critics of mathematical ideas within a productive disciplinary engagement frame. All students should have the opportunity to participate in this type of mathematics instruction. However, particular attention is paid to the participation opportunities of nondominant students since it is all too common for these students to be shut out from such opportunities.

This chapter attends explicitly to the racialized positioning of nondominant students and the cultural and mathematical knowledge that teachers can bring to bear to position these students as authors and critics of mathematical ideas. The focal teachers in this study had common goals which, in my analysis, supported their learning about mathematically responsive teaching, including: the use of rich mathematics tasks, asking open-ended questions, and supporting students to present their mathematical ideas at the front of the classroom. The teachers did not state an explicit, common goal of supporting students' racial identity development. However, observations of their teaching practice revealed that many of their efforts at mathematically responsive teaching were deeply intertwined with cultural knowledge and practices for culturally responsive teaching. Specifically, in the process of creating new mathematical opportunities for nondominant students, teachers also did a great deal of work to affirm students' cultural identities and position students as academically successful.

Research has shown that, in most U.S. mathematics classrooms, only a narrow spectrum of positionings are made available to students. For nondominant students, this spectrum is even narrower. Many mathematics classrooms operate in a *knowledge transmission frame*, in which students are positioned as passive recipients of knowledge. A few students who are quick and accurate in demonstrating their mastery of this knowledge tend to be positioned as mathematically successful and gain a greater share of learning opportunities, while students who share incorrect, partial, or outside-of-school knowledge tend to be positioned as unsuccessful in ways that limit their access to future learning opportunities (Cohen & Lotan, 1995). For nondominant students, racial stereotypes further constrain how students are likely to be positioned. Research has shown that dominant racial narratives continue to be pervasive in schools and constrain the academic identities and learning opportunities available to nondominant students (e.g., Nasir et al., 2012). Furthermore, although the domain of mathematics may appear to be an objective and apolitical discipline that has nothing to do with race, recent research has shown a significant convergence between racial narratives and narratives about mathematical ability. For example, the mathematical narrative that mathematics ability is innate and connected to a person's general intelligence can converge with a dominant racial narrative that nondominant individuals have low intelligence⁷ to create a racial-mathematical narrative that nondominant students are particularly incapable of success in mathematics (Nasir & Shah, 2011).

This research highlights the need to broaden and shift the spectrum of mathematical positionings available to all students, and particularly to nondominant students.

⁷ Some scholars argue that even the development of a "scientific" and measurable concept of intelligence, notably through IQ testing, is historically inseparable from the racialized early functions of IQ tests, which included ranking and sorting army recruits and immigrants at Ellis Island based on perceived mental capabilities (Oakes et al., 1997).

Mathematics reform, at its best, aims to challenge the mathematical parts of these converging narratives, for example by promoting a *growth mindset* (Dweck, 2006) as a counter to the narrative that mathematical ability is fixed. Yet, mathematics reform can also background issues of power in ways that deflect attention from the way these narratives get racialized (Gutiérrez, 2002, 2013; DiME, 2007).

This chapter attends explicitly to both (1) the racial, academic, and disciplinary positionings that are made available to nondominant students in mathematics classrooms, and (2) to the specific opportunities thus created for students to learn powerful mathematics. The chapter foregrounds the teaching practices, and when possible the knowledge and goals underlying these practices, of four experienced Black mathematics teachers in one urban school. These teachers are part of a learning community focused on mathematically responsive teaching. They also bring particular insights about race and power through their experiences as Black people and Black teachers. The classroom practices of these teachers draw consciously or unconsciously on tools from both culturally responsive teaching and mathematically responsive teaching to make new combinations of racial and mathematical identities available to students. The analyses will show how these teachers leverage Black cultural practices not only to support student engagement but also to enrich the mathematics beyond traditional curricula.

Both case studies in the chapter take place in the classrooms of experienced Black teachers. In both cases, the teachers leverage specific Black cultural practices to expand the racial, academic, and mathematical identities available to students. By doing so, they support students' mathematical engagement, willingness to contribute, debate and refine important mathematical ideas, and likelihood of applying their mathematical learning to improve their lives outside of the classroom. This assertion is not intended to imply that only Black teachers can do this type of identity work. Instead, this dissertation follows in the tradition of those who study excellent teachers of nondominant students, many of whom are themselves nondominant teachers, in an effort to find powerful and general insights about culturally relevant teaching that can be adapted and applied by teachers of all backgrounds (e.g., Ladson-Billings, 2009).

Furthermore, I argue that in positioning nondominant students as authors and critics of mathematical ideas, teachers are both supporting the central engagement of these students in classroom activity and simultaneously enriching the mathematical activity that students are engaging *in*. In short, the culturally specific positioning moves used by these teachers have the effect of creating many elements of a productive disciplinary engagement frame in their classrooms.

4.1.1 Chapter Overview

Section 4.2 describes the literature foundation of the conceptual tools used for analysis. In this section, I provide a literature-based argument about (1) dominant narratives about mathematics and how mathematically responsive teaching seeks to make new mathematical identities available to students, (2) dominant narratives

about race and schooling and how culturally responsive teaching seeks to make new racial and academic identities available to students, (3) points of convergence between mathematical and racial narratives and their impact on students. This section is fairly narrowly focused on the roles made available to students in classrooms and the role of mathematical and racial narratives in positioning students. However, it also marks places where these theoretical constructs are aligned with much larger bodies of literature about identity, teacher beliefs, and learning theory.

Section 4.3 describes the methods used in the chapter. This section is fairly brief because data collection and general analysis methods have previously been discussed in Chapter 3. The section describes the case study method used for this chapter including data selection and analysis methods.

Section 4.4 presents a case study analysis of two classroom episodes from the dissertation dataset. Case 1, "Write up why while you whippin and nae naein," takes place in a ninth-grade Algebra 1 class. In this episode, the teacher tries something new and mathematically responsive in his teaching by having a student present a novel and unexpected strategy at the front of the classroom. To support the new mathematical activity and position this student (a Black male student) as an author of mathematical ideas, the teacher uses a variety of culturally specific practices that affirm the students' Black identity, competence to explain mathematical ideas, and hip hop affiliation. Case 2, "I don't want to work more... if they're almost paying me the same," takes place during a homework review session in a sixth-grade mathematics class. The teacher quickly establishes the correct answers to several homework problems, and then tries something new and mathematically responsive in his teaching: having students debate an extension question with multiple possible answers that are connected to real-world value judgments and their implications. To support students in taking up new roles as defenders and critics of mathematical ideas, the teacher makes several culturally specific moves to blur the boundary between school mathematics and students' outside-of-school knowledge.

Section 4.5 presents a summary and concluding remarks about the implications of these findings for research and professional development. A primary implication is that teachers' culturally specific knowledge about how to engage nondominant students forms an important, but overlooked, part of their knowledge base for mathematics reform. Consider that both case study teachers were fairly early in their engagement with mathematics reform, but achieved significant success in leveraging Black cultural practices to create mathematically open and responsive discourse spaces in their classrooms. Now consider that teachers of color are an important audience of mathematics reform professional development, but the field of mathematics education that researches, designs and implements this professional development continues to be a White space (Martin, 2009a). It is important for researchers and teacher educators to understand how nondominant cultural practices that teachers are already familiar with can be leveraged for teachers' learning about mathematics reform. This understanding can help center the

experiences and contributions of nondominant teachers in an arena where their voices are often unheard or silenced.

4.2 Background

This dissertation asks the question, “How can we help students, especially nondominant students, take on roles as authors, critics and revisers of mathematical ideas?” It is taken as a given that sustained participation in these roles leads to both deep mathematics learning and students’ development of robust mathematical identities. The focus of this literature-based argument is on the ways that learning environments make the desired participant roles available or unavailable to students in general and to nondominant students in particular.

The literature reviewed comes from a wide variety of perspectives, including cognitive and situated learning theories and developmental and poststructuralist perspectives on the way language and power can operate to shut nondominant students out of particular roles. Throughout the chapter, the term *identity* is used in an active and moment-to-moment way to refer to the roles and forms of participation that students make bids to take on and/or are “animated” into by others (e.g., Nasir & Cooks, 2009). The terms “narratives” and “storylines” are used interchangeably to describe socially negotiated ideas that get mobilized to expand or constrain the identities available to students (e.g., Nasir et al., 2012). Some examples of mathematical storylines are “Only geniuses can do mathematics” or “Everyday knowledge is a valuable resource for learning mathematics.” Some examples of racial storylines are “Black people are unintelligent and uneducated” or “Black people have a long, proud history of fighting for educational access under unjust conditions.” Although these narratives exist broadly in society, the analytical focus here will be local in scope, with attention to how narratives are invoked in particular classroom moments to shut down or create positive conditions for learning.

Section 4.2.1 explores mathematical narratives common in U.S. classrooms and counternarratives that seek to make new mathematical identities, corresponding to the roles consistent with a productive disciplinary engagement frame, available to students. Section 4.2.2 explores dominant racial narratives, their pervasiveness in U.S. classrooms as a central obstacle to the participation and learning of nondominant students, and counternarratives that seek to make new racial and academic identities, as well as the learning opportunities that come from central participation in classroom activity, available to nondominant students. Section 4.2.3 explores the points of convergence of mathematical and racial narratives, which may initially appear to have little relationship to each other, but which, it will be argued, overlap significantly and mutually reinforce each other.

The empirical case studies used in the chapter return to these points of convergence between racial and mathematical narratives, and show how teachers’ positioning of students and mathematics counters multiple narratives and makes new combinations of identities available to students in order to position nondominant

students as central participants in classroom activity that largely aligns with a productive disciplinary engagement frame for learning.

4.2.1 Mathematical narratives

There is a large literature on different teacher orientations toward what it means to “do mathematics” and how these teacher orientations affect the types of mathematical participation teachers make available to students (e.g., Lampert, 1990; Schoenfeld, 2010). Much of this literature can be framed in terms of “mathematical narratives.” The discussion here begins with a more detailed presentation of mathematical narratives from two authors who explicitly study intersections of racial and mathematical narratives. Shah (2013) uses discourse analysis to identify three main narratives about mathematics:

- **Hierarchy:** Some people are better at math than others; people could be ordered by math ability;
- **Innateness:** Mathematical ability comes from a born or natural talent;
- **Intelligence:** Mathematical ability is related to a person’s overall intelligence.

DosAlmas (2012) conducted interviews with high school students about what it means to be good at mathematics. Students said that they thought there were two kinds of kids who were successful in math class: the “smart kids” who could get it quickly without effort and the “hard-working kids” who studied hard and eventually got it. Many students saw the “smart kids” category as inaccessible or distant from themselves; they perceived that ordinary people could never be part of this category. Many students with high grades in math courses, who admitted to working hard and learning a lot to achieve those grades, denied being good at mathematics. Furthermore, students who did identify as good at mathematics often went to great discursive lengths to deny that they had worked hard to get to where they were. From this work we can add two narratives:

- **Genius:** only a few people are good at math; ordinary people will never be good at math;
- **Effortless Achievement:** people who are very good at math excel without effort.

The mathematical narratives referenced by Shah and DosAlmas connect to a much larger literature on teacher beliefs about mathematical knowledge, teaching and learning and how these beliefs affect the types of mathematical participation teachers make available to students. Here only two additional narratives that arise from a synthesis of this literature will be briefly referenced. Literature about typical U.S. mathematics classrooms makes it clear that a high value is placed on speed and correctness when positioning students as competent or not. This is a much narrower set of criteria for competence than is typical of the discipline of mathematics as a whole, in which thoughtfulness, thoroughness, innovation and elegance are often valued over speed and a valid process of conjecture and proof is often considered more important than correct answers per se. The following two

narratives can therefore be identified, particular to school mathematics, which are used to position students as more or less competent:

- **Speed:** Solving problems quickly indicates mathematical competence and solving them slowly indicates lack of competence (e.g., Schoenfeld, 1988; Seeley, 2009).
- **Correctness:** Correct answers indicate mathematical competence and wrong answers indicate lack of competence. There is little space for partial or incorrect answers to be valued as part of a problem solving process. (Lappan & Phillips, 2009).

The dominant narratives mentioned here, though pervasive, can be contested. Perhaps the best-known example is *growth mindset* work (Dweck, 2006), which provides an explicit counternarrative to the Innateness narrative: “Mathematical ability is developed through effort.” When students experience struggle while grappling with challenging mathematical ideas, this counternarrative can be used to position this struggle as a sign of progress, rather than as a sign of failure. A second counternarrative is *multiple abilities* (Cohen & Lotan, 1995), which counters the Hierarchy and Genius narratives by arguing that, rather than one-dimensional and restricted to a few “best” people, mathematical ability is multidimensional and everyone has different mathematical strengths to contribute to a learning community as well as something to learn from their peers. As a third example, many mathematically responsive classrooms place a high value on principled revision and thoughtful refinement of student ideas. To support this practice, teachers counter the Speed, Correctness, and Effortless Achievement narratives and support students’ sustained and sometimes slow engagement with the work of revision and refinement of ideas.

4.2.2 Racial narratives

Racial narratives are ubiquitous in all areas of society, including schools, and constrain the identities available to nondominant students (Nasir et al., 2012). They can function in many domains, from aesthetics (“Light skin is beautiful”) to intelligence (“Asians are smart”). Importantly, even narratives that could appear positive on the surface function to position racial groups relative to each other in ways that are harmful for all groups. In the examples given, “Light skin is beautiful” implicates the idea that dark skin is ugly, and “Asians are smart” implicates the idea that other groups, for example Black and Latinx individuals, are less intelligent. The cumulative effect of these narratives, according to Nasir and colleagues, is that small actions get racialized and mobilized to forcibly position or “suture” nondominant kids into these negative identities. These forced positionings can always be contested by the student or others, but they can carry enough power and institutional weight to make contesting them difficult (Oakes et al., 1997; Horn, 2007; Langer-Osuna, 2011; Nasir & de Royston, 2013; Esmonde & Langer-Osuna, 2013).

A comprehensive overview of all racial narratives will not be attempted here. However, there are some racial narratives that are especially salient in schools.

Shah (2013) names three such, which closely parallel mathematical narratives:

- **Hierarchy:** Some races are seen as better than others; people could be ordered by race;
- **Innateness:** Race is something you are born with;
- **Intelligence:** White people are more intelligent than Black and Brown people.

Too often, nondominant students are positioned in school spaces as unintelligent, deviant, or lazy. But powerful counternarratives also exist in schools. One example, which will be called “Valuing Diversity,” holds that multiple perspectives are valuable, even necessary, for learning to take place. A second example, “Education as a Civil Right,” is that nondominant peoples have fought for centuries to gain access to educational opportunities under unjust conditions. The first example backgrounds issues of power while the second foregrounds these issues. Both counternarratives can be useful in different contexts to position nondominant students in ways that support their central participation in learning activities.

4.2.3 Racial-mathematical narratives

Racial and mathematical narratives can combine in harmful ways. Shah’s description of the convergence of both kinds of narratives around issues of hierarchy, innateness, and intelligence begins to suggest these combinations. A few specifics are considered below.

For example, the Speed and Correctness mathematical narratives can create pressure for students of all races to try to get correct answers quickly and fear looking stupid by being too slow or getting wrong answers. But for students of color, stereotype threat (Steele & Aronson, 1995) can compound this problem, creating a risk not only of individually looking bad but also being identified with a negative “stupid” or “lazy” narrative about their whole race.

In her interview study, DosAlmas (2012) finds that racial and mathematical narratives can overlap to make it very difficult for any behavior by Black students to be positioned as academically successful. Specifically, the students she interviews invoke the Effortless Achievement narrative when asked to describe success in mathematics: successful mathematics students just “get it” without working too hard. However, the interviewees also articulated that anytime a Black student was not working hard, racial narratives would quickly be invoked to position that student as lazy. That is, the same behavior of trying to “get it” without working hard led to very different student positionings depending on the student’s race: a positive mathematical identity of effortless achievement if the student was White and a negative mathematical identity of being lazy if the student was Black. In short, the combination of these racial and mathematical narratives created very narrow opportunities for Black students to be perceived as good at mathematics.

The existence of dominant narratives that are both racial and mathematical suggests a need for racial-mathematical counternarratives as well. Thus far, there is limited

literature about counternarratives that explicitly combine racial and mathematical elements. Racial counternarratives seek to explicitly uplift the brilliance and history of particular nondominant groups, without denigrating other groups. Narratives of this type have little overt connection to mathematics, but are nevertheless useful in supporting the participation of nondominant students in classroom activity, including in mathematics classrooms. Mathematical counternarratives have the potential to open up space for participation by more students and for valuing more ways of knowing and learning mathematics. Such efforts have the potential to create more opportunities for nondominant students. However, it is unlikely that related racial narratives can be fully countered, even locally, without explicit attention to issues of power in building counternarratives. Examples like DosAlmas' work show that when issues of power are ignored, they continue to negatively impact the learning opportunities of nondominant students. This chapter provides some initial examples of what racial-mathematical identity work can look like.

The stage has now been set for the empirical work of the chapter, which foregrounds the insights of experienced Black mathematics teachers who are learning together about mathematics reform. This analysis looks at moments where teachers seem to be using new moves from mathematics reform, along with their prior knowledge about culturally relevant teaching, to support students' co-construction of racial and mathematical identities. Some of this identity work appears to be a result of conscious effort on the part of the teacher; in other cases it appears unconscious or it is difficult to tell. Nevertheless, it will be argued that both conscious and unconscious teacher moves directly counter dominant racial and mathematical narratives in ways that create new spaces for students' racial and mathematical identity development.

4.3 Methods

This chapter uses a case study methodology (Yin, 1994). Two focal episodes of classroom video were selected that seemed especially powerful for (a) students' co-development of racial and mathematical identity and (b) students' opportunities to learn rich mathematics.

Selected episodes were transcribed and moments were noted where teachers seemed to be (a) supporting students' racial and/or mathematical identity development or (b) supporting students to engage as authors or critics of important ideas within a mathematically rich conversation. Both episodes are presented chronologically according to the following themes:

How did the teacher support student engagement?

How did the teacher invite students' outside-the-box sensemaking, that is, sensemaking outside the knowledge transmission paradigm?

How did the teacher position students as authors and elicit their mathematical thinking?

How and to what extent did the teacher support the take-up, critique and revision of students' mathematical ideas?

These themes are elaborated in sufficient depth to demonstrate that both episodes are culturally responsive, in the sense that they support students' co-development of racial, academic and disciplinary identities, and mathematically responsive, in the sense that student thinking plays a central role in the unfolding mathematical discussions and students are positioned as authors and/or critics of important mathematical ideas. Furthermore, the analysis of particular teacher moves sheds light on how the cultural and mathematical dimensions of classroom activity support each other.

Although classroom video of the focal episodes was the primary data source for the chapter, additional classroom video, teacher interviews, and teacher collaboration sessions were used for triangulation regarding teachers' intentions and the typicality of teacher practice on the given days. Focal teachers were also invited to comment on a draft of the dissertation in end-of-year interviews. Mr. B completed an end-of-year interview, with particular attention to discussing Case 2 in this chapter; his feedback is summarized in Section 4.4.3.6. Mr. X was unable to participate in an end-of-year interview due to poor health.

4.4 Analysis and Results

In teacher interviews, all four Black mathematics teachers expressed a desire to be a role model for students and a strong ethic of caring about student success both in school and in life. Three of the four mentioned their own Black identity in connection with wanting to be a role model. In the interviews, teachers expressed various combinations of affiliation and non-affiliation with specific aspects of Black cultural activities and communities, such as hip-hop, the Black church, athletics, and political activism. Several expressed that it was important for students to have role models of a variety of different ways to be Black and successful.

In their classroom practice, teachers were observed using a variety of strategies to support students' racial, mathematical, and academic identity development. To give a few examples, all four teachers used Black cultural speech modes, including African-American English (AAE) and "speeches" (Johnson, 2013) about the importance of classroom participation, to support students' racial and academic identity development. To support mathematical identity development, all four teachers expected all students to give mathematical presentations at the front of their classroom and supported students to feel successful about their presentations. Two of the four teachers, who had more experience with reform mathematics pedagogy, engaged in assigning competence based on the validity of students' thinking process and made an extra effort to assign competence to low status students. The other two teachers, who used more traditional pedagogies, typically assigned competence for correct answers.

Many prior studies of the cultural knowledge bases of Black teachers study Black teachers of Black students. However, the majority of students at Adams school were Latinx. This raises several important questions, for example, how effective are the teaching practices mentioned in engaging Latinx and other nondominant, non-Black

students? To what extent is there overlap between the Black speech modes and cultural referents used and urban speech modes and cultural referents shared by non-Black students at Adams? All teachers in the study appeared effective in engaging their Latinx students and I noticed no cultural “misfires” in their efforts to do so. Throughout this chapter, I will make an initial effort to sort out the intersections between various racial and cultural identities and affiliations where possible and relevant. However, these issues are quite complex. Significant additional research is needed in this area, and it is likely that additional data sources, such as student interviews, will eventually be needed. The chapter also points to opportunities for future work.

4.4.1 Overview of cases

The remainder of this chapter analyzes two classroom cases. During case selection, I looked at classroom video from the two teachers who had more teacher-centered pedagogy for moments when the mathematical discourse space opened up in a way that centered on student ideas and gave significant mathematical authority to students, that is, moments when teachers tried something new that was consistent with elements of a productive disciplinary frame. In each case, the mathematical openness of the discourse space is somewhat atypical for the teacher, but it is built on a foundation of student trust and comfort with the teacher that has been established throughout the year. Both teachers use Black cultural practices to establish these trusting relationships with students. These cultural practices appear consistently throughout the year and through these mathematically unusual lessons.

The cases differ in grade level, mathematics content, which mathematically responsive teaching practice was tried by the teacher, and which Black cultural practices were used. Furthermore, Case 1 focuses on a learning interaction that began between a Black teacher and two Black students, which then was brought to the whole class in a way that created learning opportunities for all students (most of whom were Latinx). Case 2 takes place entirely in a whole-class activity format, again with mostly Latinx and some Black students.

I will argue that in both cases, the mathematical and cultural responsiveness of the discourse are closely intertwined. That is, the same Black cultural practices that have been present throughout the year of traditional teaching become an instrument driving force in sustaining mathematically responsive discourse.

Both case study presentations are organized similarly, with analysis of the following issues:

- 1) Classroom context and summary of the mathematics discussed;
- 2) How students were supported to engage in classroom activity;
- 3) How outside-the-box sensemaking was encouraged;
- 4) How students were positioned as authors of mathematical ideas;
- 5) How student-authored ideas were taken up by peers and the teacher;
- 6) Connections to important disciplinary ideas and opportunities (or lack thereof) for critique and revision.

Case 1 is from a high school Algebra 1 classroom with primarily ninth graders. The lesson involves a student presentation of a novel strategy for solving a challenging problem, which is taken up in some detail by the class. Before the presentation, the teacher supports engagement and outside-the-box sensemaking by listening to and praising a novel student idea. The language the teacher uses to praise the student's mathematical success also affirms the student's Black identity, sensemaking, and hip hop affiliation. To support the student in taking up the new mathematical role of presenter, the teacher used a new frame, which I call a *giving a talk* frame. There was significant peer uptake of the presenter's ideas, but limited opportunities for student refinement ideas because only correct answers were presented.

Case 2 is from a sixth-grade classroom. The lesson begins as a homework review of a few problems where students had used decimal multiplication to calculate the weekly salaries of different jobs. Correct answers for the homework are established within the first five minutes, but the teacher poses an additional, mathematically rich question to students and supports them to discuss this question for an additional 25 minutes. The teacher animates the problem in a way that invited students to blend inside- and outside-of-school sensemaking and language repertoires. He uses what I call a *debate frame* to support students to critique and refine each other's ideas over an extended period of time.

4.4.2 Case 1: "Write up why, while you whippin and nae naein"

This case study demonstrates multiple opportunities for juxtaposition of racial and mathematical identities: first offered by the teacher, then taken up by one student with the teacher's affirmation, and finally taken up by two students together as the teacher walks away. It is culturally responsive because Black cultural practices are used to celebrate mathematical success. This is in contrast to how dominant narratives in society can position nondominant cultural practices for expressing various emotions, including excitement, reluctance and frustration, as disruptive or even dangerous. The case is mathematically responsive because a novel student idea is given "air time" in both an individual conversation with the teacher and a student presentation of the novel strategy to the whole class.

The episode took place in Mr. X's ninth-grade Algebra 1 classroom. Students were beginning a unit on solving linear equations. The task used is shown in Figure 4.1. This task uses the scenario of a tug of war to cue student intuitions about equality that are relevant to solving equations, especially the transitivity of equality and substitution.

Tug of War

Use the information below to figure out who will win the third round of tug of war.

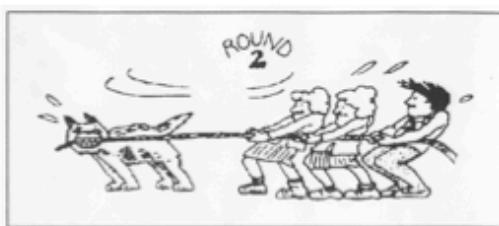
Round 1:

On one side there are four acrobats, each of equal strength. On the other side are five neighborhood grandmas, each of equal strength. The result is dead even — a perfect tie.



Round 2:

On one side is Ivan the dog. On the other side are two grandmas and one acrobat. Again, it is an equal pull — a tie.



Round 3:

Three grandmas and Ivan are pulling against four acrobats. Which team wins? Be ready to justify your answer!



Figure 4.1. Task used in Case 1.

4.4.2.1 Supporting engagement and welcoming outside-the-box sensemaking

During his check-ins with individual students, Mr. X found that students had used a variety of strategies to approach the problem. His check-in with Enoch, who used a numerical strategy, is shown first. In a debrief conversation, Mr. X said that Enoch's strategy "surprised the heck out of me" because he had expected students to use variables and substitution, but that the novel method "worked, and was awesome."

Looking at Round 1, Enoch chose a numerical value of 25 for each acrobat and 20 for each grandma such that both sides added up to 100. He explained his strategy as follows.

Enoch: I was like, five out of five is a hundred. Four out of four is a hundred. And then I was like, so then right here, so one grandma is 20 because 20, 40, 60, 80, then 100. And then one acrobat is 25.

Mr. X: So you gave them values.

Enoch: Yeah.

Mr. X: That's what's up.

Here, Mr. X uses the AAE expression “That’s what’s up” to celebrate student success in mathematics. As argued below, it appears to be taken by Enoch as an affirmation of both his Black identity and mathematical identity. Mr. X also used the expressions “That’s what’s up” and “That’s whassup” with non-Black students. They seemed to be positively received.

Enoch goes on to explain how his method applied to Problems 2 and 3, implicitly using the transitivity of equality to assign a numerical value of 65 to the dog in Round 2 and comparing numerical totals to find who wins Round 3. Figure 4.2 shows a reproduction of his work.

After listening carefully to Enoch’s novel explanation, Mr. X praises his correct answer, again saying “that’s whassup.” He gives substantial positive feedback, “that’ll work!” which Enoch celebrates with a couple of dance moves:

Mr. X: Ah, that'll work, that'll work! That'll work. So

(Enoch does a whip⁸ with his right hand)

write up why while you whippin and nae naein,⁹

(Enoch shakes both hands)¹⁰

Mr. X: write up why.

(Mr. X walks away. Enoch bends over his paper and starts writing).

Notice that Enoch is using the whip and nae nae dance moves, which are associated with both Black and urban culture, to celebrate mathematical success. This is another juxtaposition of racial and mathematical identity, this time initiated by the student. Mr. X affirms both that Enoch is justified in celebrating his mathematical success and that he is welcome to use Black cultural modes of celebration in this classroom. First, he reacts with affection to the dance moves, which affirms that

⁸ The whip is a dance move invented by Famous 2 Most in 2014 and widely viewed in Silentó’s 2015 video “Watch Me.” According to Enoch, “you squat down a little bit and you bring one foot up, then the opposite arm comes out while your leg drops.” Enoch did only the arm movement in the video, while remaining seated.

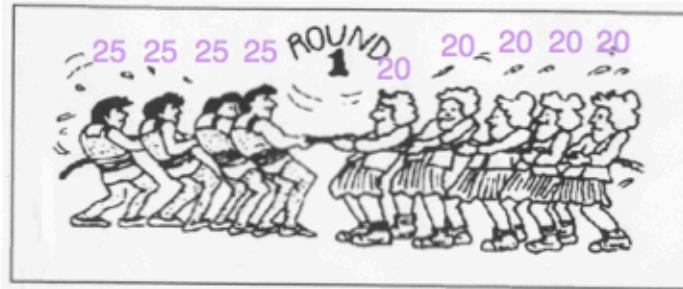
⁹ The nae nae is a dance move invented by We Are Toonz and widely viewed in Silentó’s “Watch Me.” According to Frank, the move involves putting one hand in the air and a “snake-like movement” of the hips.

¹⁰ In the end-of-study interview, Enoch explained that this was not intended to be a specific dance move. It was a “set up” to continue dancing.

Enoch is welcome to “be himself” as a Black and urban person in this classroom. Next, but within the same breath, Mr. X encourages Enoch to write up a mathematical justification. This positions Enoch as a mathematically competent person whose ideas are valued in this classroom. Furthermore, Mr. X’s instruction to “write up why” communicates to Enoch that his mathematical work does not end with a correct answer. He has a further obligation (and further learning opportunity) to provide a written explanation of his novel strategy.

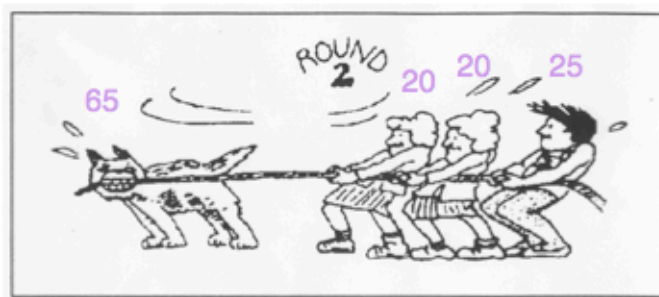
Round 1:

On one side there are four acrobats, each of equal strength. On the other side are five neighborhood grandmas, each of equal strength. The result is dead even — a perfect tie.



Round 2:

On one side is Ivan the dog. On the other side are two grandmas and one acrobat. Again, it is an equal pull – a tie.



Round 3:

Three grandmas and Ivan are pulling against four acrobats. Which team wins? Be ready to justify your answer!

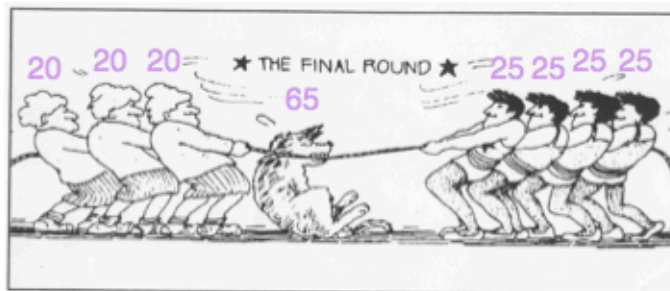


Figure 4.2 Enoch's strategy

In summary, during his individual interaction with Enoch, Mr. X affirms various aspects of Enoch's identity: Blackness, hip hop affiliation, smartness, math success, and continued effort. Affirmation of a student's cultural and mathematical identities together can support the participation and learning of students from any background. But in this particular context, these moves are especially significant because of their contrast to dominant racial narratives that too often create obstacles for Blackness and mathematical competence to be seen as complementary.

Mr. X next checks in with another student, Frank, who has used a more standard algebraic method to solve the problem.

Mr. X (to Frank): What's up?

Frank: Wouldn't, uh, wouldn't the grandmas and Ivan win? Because you get $5G + A = 4A$. And then when you get to Round 1, four acrobats and five grandmas win.

Mr. X: Exactly, because

Frank: So you subtract A.

Mr. X: I agree with you, because five grandmas equals four acrobats, so we got five grandmas plus an acrobat. So in other words, the three grandmas plus the dog should beat the acrobats. Good call, good call.

Here, Frank's explanation is quite abbreviated, but it is pretty clear that he has used the information from Round 1 ($5G = 4A$) to substitute into the "equation" $5G + A = 4A$ that is written on the board. Mr. X praises this strategy and emphasizes the concept of equality used for substitution. In this exchange, Mr. X affirmed Frank's mathematical identity using a formal speech mode common to mathematics classrooms ("I agree with you because"). There is a more subtle continued use of AAE in expressions like "we got." But the real affirmation of students' racial identity comes after Mr. X finishes talking.

As Mr. X walks away, Frank and Enoch celebrate with a seated "dance off," going back and forth with each youth dancing three times before they both bend over their papers to continue writing. This moment shows a significant level of student buy-in to the juxtaposition of racial and mathematical identities initiated by Mr. X. Frank and Enoch are friends, and both showed through their eye gaze and posture that they were attending to each other's individual check-in with Mr. X. The "dance off" celebrates each other's mathematical success, using elements of Black and hip hop culture to celebrate engagement in school and in mathematics. This counters dominant narratives that position hip hop and school engagement as in tension with each other. Furthermore, after about 30 seconds of dancing, both Frank and Enoch simultaneously bend over their desks and resume writing, presumably working on the explanations that Mr. X has asked for. This shows that they have accepted Mr. X's message that they are not finished just because they have correct answers; their positioning as mathematically competent students carries an expectation of continued effort and learning as they write up their explanations. This message from Mr. X counters the Effortless Achievement narrative because, instead of positioning mathematically competent students as those who finish quickly, he links success with an expectation of continued effort to write an explanation, which has its own

learning affordances. In this way, Mr. X offers, and students take up, a positioning different from the position made available by the dominant Effortless Achievement narrative.

Furthermore, the dance off is quite revealing of what it means to do mathematics in Mr. X's classroom. It clearly does not disrupt the class: Frank and Enoch remain in their seats, few other students even glance in their direction, and within about 30 seconds Frank and Enoch have both bent studiously over their desks and resumed writing. Looking at other classroom videos from Mr. X's class without sound and in fast forward reveals that Enoch moves much more than most other students, and often carries a silent beat with his hands and/or feet as he works. In another classroom this could have led to a racialized positioning of Enoch as a disruptive Black boy; in Mr. X's classroom there is little attention to his beats and much more attention paid to the quality of his ideas and the mathematical practices he engages in to develop and share these ideas.

Mr. X uses several moves to support new or nascent positionings for Enoch. First, Mr. X uses AAE, "That's whassup" to affirm Enoch's Black identity and mathematical success together. Mr. X's follow-up to Enoch's correct answer is also inconsistent with a knowledge transmission frame, in which correct answers are an end in themselves. Mr. X praises Enoch's answer and in the next breath encourages him to continue learning by continuing to work on his ideas by writing up an explanation. This sends the message that Enoch has made a valuable mathematical contribution, but his task does not end with finding the correct answer. Effort is still expected as he explains his answer. In short, Enoch is being positioned as a Black person who is both intelligent and hard working, leading to his success and continued learning opportunities in mathematics.

4.4.2.2 Positioning a student as an author: student presentation

After a period of individual work and check-ins, Mr. X calls Enoch up to the front of the class to present his strategy:

Mr. X (to Enoch): Would you do me a favor, sir? Would you explain what you did?

(Mr. X erases board space) Um, you can work it out right here. Here you go. (Hands Enoch a marker) Here you go.

Enoch: Do I have to draw it?

Mr. X: You, well, well, begin explaining to them how you determined the values. You don't have to draw the whole thing, no.

Enoch: (Body facing board) Ah, Round 1 was four acrobats and ah

Mr. X: They over there.

Enoch: (Turns body out toward the class, speaks a little louder) so I, I gave each of them numbers, like the grandma was a five so I was like, five out of five is a hundred, and the acrobats I said four out of four is a hundred, so um, each one, each grandma would get 20, because, ah, 20 goes into 100 five times, so each grandma gets 20, and 4 goes into 100 25 times, so each acrobat would have 25.

By giving “air time” to a student with a surprising idea, Mr. X is positioning Enoch as an author of novel and valuable mathematical ideas. This is a significant break with most U.S. classrooms, in which students are expected to reproduce standard methods. Hence, it is a significant indication of the mathematical responsiveness of the moment.

It is unusual for students to give presentations in Mr. X’s class – only 3 out of 15 recorded class days included student presentations and Enoch’s is the only observed presentation of a novel, student-generated idea. Nevertheless, by the time Enoch finishes the first sentence of his presentation, he appears quite comfortable: he faces the class with a confident posture, speaks loudly and clearly, and presents an organized sequence of ideas. Mr. X supports Enoch in several ways. Specifically, he frames Enoch’s presentation as a “favor” to him, clears off board space, and answers Enoch’s question about how to frame the presentation. All of these moves reinforce the idea that Enoch’s presentation will be valuable to the class and/or help establish concrete expectations for how Enoch should proceed. In short, Mr. X is positioning Enoch as an author of mathematical ideas, an important student role consistent with a PDE frame.

Of particular note is Mr. X’s use of the phrase “they over there.” It is typical in U.S. classrooms, and in Mr. X’s classroom, for public student speech about mathematics to be directed toward their teacher. It is therefore a highly significant shift that Mr. X asks Enoch to speak to his peers instead. Nevertheless, this shift is accomplished quickly and gracefully. Enoch has already begun his presentation, and during a very brief pause Mr. X speaks the direction “they over there.” After another very brief pause and without dropping his train of thought, Enoch re-orientes his body toward his peers and continues speaking. The intervention is so brief and fluent that it appears to be some type of shorthand or reminder to Enoch of a shared class value, something like “we face toward people when we are presenting.” If one only watched this one video episode, one would have guessed that student presentations were common in Mr. X’s class, but this is not the case. Then where does this shared understanding come from?

It is plausible that Mr. X is referencing a general frame of a *presentation* that is broadly shared among people in the U.S. Presentations happen in a variety of professional settings and the actors involved include a presenter, who should speak in a clear, audible and organized way, and the audience, who should listen attentively and applaud at the end. It is no small thing to import an outside-of-school frame into a classroom setting. Nevertheless, Mr. X layers the two frames seamlessly through his use of the phrase “they over there” and overlapping speech rhythm characteristic of AAE. In other words, Mr. X uses culturally specific shared knowledge of AAE to make a general and powerful shift in the framing of classroom activity.

4.4.2.3 Publicly unpacking Enoch's ideas

After Enoch explains his reasoning about Round 1 of the tug of war, Mr. X pauses to make sure everyone has understood. He asks several other students to explain parts of Enoch's strategy in their own words, and also adds some of his own comments about how Enoch's work connects to important mathematical ideas including equality.

Mr. X: Let me stop. Do you, does everybody understand what he said by looking at the first picture? He's giving a value to each grandma and to each acrobat. Can somebody else explain to me why each acrobat has a value of 25 according to Enoch and why each grandma has a value of 20? (Visible hands from Frank, Gabrielle, Humberto Iglesias, Jimena.) Brother Iglesias.

Humberto Iglesias: Uh, he said that four, well for the acrobats it's four, because each one gets 25 because there's four, and four goes into a hundred 25 times

Mr. X: (Sneezes) Correct. (Sneezes).

Humberto: (So it would be 25 for the acrobats)

Student: (Bless you.)

Mr. X: Correct. Thank you.

Humberto: For the grandmas, if you give a value to each grandma of 20, because 5 goes into 20 a hundred times, I mean 5 goes into a hundred 20 times.

Student: Mhmm.

Mr. X: If y'all was just paying attention, these two gentlemen just explained to you how percentages work. Mk? We have a whole. We have one and one. How do we know they're a whole and one and one, because they tied. No one won the tug of war. But if I divide one by four, each one of those acrobats is 25 percent of the whole. If I divide one by five, each one of those grandmas represents 20 percent of the whole. Does that piece make sense?

Mr. X: Okay, so that's why it's a value of 25 and 20. (To Enoch:) Keep going, brother, I'll shut up.

Although Mr. X takes a long talk turn here and gives his own explanation of mathematical ideas, his speech reinforces the sense of Enoch's authorship of a valuable mathematical idea. His final talk turn in the episode, "Keep going brother, I'll shut up," reinforces a sense that Enoch is independently able to explain his thinking.

Throughout the episode, Mr. X uses language like "brother" and "sir" to refer to students in a culturally affirming way. This aspect of the episode is typical of Mr. X's classroom practice throughout the year. Calling another man "brother" in Black culture is an affirmation of shared humanity in the face of oppression. Mr. X uses the term "brother" with all his male students regardless of their ethnicity (though all are nondominant students) and sometimes uses "sister" to refer to his female students. Calling another person "sir" in many parts of the United States, signifies a level of formality, respect and authority. The term is especially common in the South, but broadly recognizable across the country. By using this language with students, Mr. X is using some of his power as a teacher to position students as

having status and authority in society. Mr. X uses the term “sir” with male students regardless of their ethnicity and uses several other linguistic forms to indicate a similar level of respect and deference toward female students.

4.4.2.4 Limited opportunities to connect, refine and critique ideas

So far, we have considered the substantial cultural and mathematical work that Mr. X did in order to position students as authors of mathematical ideas and to ratify their culturally specific affiliations and performances as legitimate, in his classroom. But a productive disciplinary engagement frame implies more than student authorship of ideas. Students should also be engaged in connecting their strategies to big ideas in the discipline of mathematics and refining their ideas through critique and revision.

It is worth considering how Enoch’s numerical method relates to Mr. X’s algebraic goals for the lesson. The connection is nontrivial; I will mark ways that Mr. X’s take-up of Enoch’s ideas is sensible and powerful as well as places that it could be further enriched. Perhaps because Enoch’s method is so novel to him, Mr. X does not make any explicit connections between it and algebraic notation. Nevertheless, Enoch has used important mathematical ideas including the transitivity of equality. Mr. X emphasizes the concept of equality when he says, “because they tied. No one won the tug of war.”

However, Enoch’s language around equality is conflated with the idea of percentages here. Recall that in his private explanation to Mr. X, Enoch used the language “five out of five is a hundred” and “four out of four is a hundred.” Mr. X interpreted these statements as an implicit use of the concept of percentages and later expounded on the idea that each side of the equation would be equal to “a whole” because of the tie. This interpretation of percentages is somewhat problematic because it emphasizes a fixed total value (“one and one” or “one hundred”) for the tug of war, when in fact only one round of tug of war has a total value of 100 in Enoch’s scheme. Round 2 has a total value of 65, and cannot easily be interpreted using this percentage scheme. Mathematically, it might be more effective to mention that Enoch could have chosen to set the total value of the first round to any number, and that his choice of 100 was a nice number to work with especially since students were familiar with dividing up 100 from their previous work with percentages. Mr. X may have perceived a tension between “running with” his best understanding of Enoch’s percentages method or going back to more familiar, standard methods that would avoid the potentially messy conceptual issue of percentages. This is a classic conundrum in student-centered teaching since the richness of student ideas is almost always accompanied by some messiness to sort through together. Mr. X’s choice to validate a novel student method and iron out the details later is a very powerful one.

To further enrich the mathematics, Mr. X could have asked both Enoch and Frank to present their work, and have the class make connections between the two methods. Using Frank’s method would allow a continued focus on the algebraic goals of the

lesson, while having the numerical details of Enoch's method might help scaffold more of the class into understanding the most important, but abstract, parts of Frank's method. For example, when walking through Frank's method and reaching a substitution step, Mr. X could refer back to Enoch's numerical values for the substitution to help make it clear to students that the symbols being substituted have the same numerical value, and where that equality comes from.

It is also worth discussing one element of framing the nature of mathematics, common in U.S. classrooms, that persisted in Mr. X's classroom despite the dramatic shift toward student authorship: a focus on correctness. Mr. X shared in a debrief conversation that Enoch's correctness played a significant role in his willingness to call Enoch up to the front, saying that Enoch's method "worked, and was awesome." He went on to explain that when he taught the same lesson in another period (not observed), a student wanted to go up to the board and share a novel method, but Mr. X said no because the student's method was incorrect. Students had already seen the correct, standard method to solve the problem and he did not want to confuse them with an incorrect method. Mr. X stated explicitly that if the student's method had been correct or offered before the standard method was explained, he would have been happy to give the student air time.

These statements indicate that Mr. X believes that completeness and correctness is a good measure of the value of students' mathematical ideas. An alternative belief would be that students' emerging work, including incomplete and incorrect solutions, can be extremely valuable for deeply exploring important mathematical ideas. The second set of beliefs is more consistent with a productive disciplinary engagement frame because it creates much more space for students to refine their ideas according to the principles of the discipline and to take on roles as critics and revisers of mathematical ideas. If partial and incorrect ideas are not made public, this severely limits student opportunities to engage in critique and revision and see these mathematical practices modeled. Indeed, students were not seen publicly critiquing or revising ideas in this case study, and it was uncommon to see students take on these roles in classroom practice in general.

However, there is a possible tension here. At this point in the term, a lot of the sense of success that students are feeling, and the counternarratives flourishing around this success, builds on acknowledgement of student correctness. It is not immediately obvious if students would feel the same sense of success in response to teacher praise focused on justified reasoning of emergent or partial ideas that they demonstrated in response to teacher praise focused on correctness. It is outside of the scope of this case study to fully resolve this tension. The central argument here is not whether the teacher is implementing all aspects of a productive disciplinary engagement frame. The teacher has made important steps toward a productive disciplinary engagement frame, particularly with respect to positioning students as authors of mathematical ideas. These steps required significant cultural and mathematical work to counter dominant narratives.

4.4.2.5 Case 1 Summary

This case study showed Mr. X leveraging Black cultural practices both to support student engagement and to enrich the mathematical content and practices with which students engaged. Table 4.1 summarizes the messages sent to students about the nature of mathematics and their roles as learners, most of which depart from what is typical in U.S. classrooms.

	Typical U.S. classroom	Mr. X's classroom / this episode
Mathematical narratives	Only geniuses generate mathematical knowledge.	The role of students is to creatively use what they know to solve new and challenging problems. Students' novel ideas will get "air time" in the class and can help their peers learn about important mathematical ideas.
	The role of the teacher and text is to clearly expound standard strategies. The role of students is to practice and internalize these strategies.	
	Speed; Effortless achievement.	Quick correct answers should be celebrated, but they are only part of the learning process. Good ideas should be built into careful explanations.
	Correctness is essential; the class should spend a majority of time considering correct answers.	
Racial narratives	Black cultural practices and affiliation with hip hop culture are racialized and portrayed as antithetical to school success.	Black cultural practices and affiliation with hip hop culture are valid ways to celebrate school success.
Racial-mathematical narratives	"Asians are good at math" and implication that Blacks are not good at math.	Affirmation/celebration of Blackness and mathematical success together.

Table 4.1. Summary of mathematical and racial narratives in Case 1.

These positioning moves and counternarratives are intimately connected with the central question of this dissertation, "How can we help students, especially nondominant students, take on roles as authors, critics, and revisers of student ideas?"

Mr. X positioned students as authors of mathematical ideas both publicly and privately. Privately, Mr. X listened to Enoch and Frank explain two different strategies for solving the problem, affirming both their mathematical success and Black identity. Publicly, Mr. X asked Enoch to present a novel strategy to the class and positioned this strategy as something that everyone could learn from. Furthermore, Mr. X asked students to rephrase and build on Enoch's ideas, which

supported their sensemaking about a novel strategy while reinforcing Enoch's position as an author.

Although Mr. X's praise and encouragement of students served many powerful roles, his emphasis on correctness in the public classroom space limited the opportunities for students to publicly revise and critique mathematical ideas. We do not see any students taking on a role of critic or reviser in this episode. Nevertheless, it is significant that upon seeing many students were stuck, Mr. X called on a student to present a novel strategy rather than going over the problem himself. This is an important step toward students seeing each other as resources for resolving challenges, rather than having the teacher be the only critic and reviser of ideas.

4.4.3 Case 2: "I don't want to work more... if they're almost paying me the same"

This episode takes place in Mr. B's sixth-grade classroom in mid-October. The class is nearing the end of their first unit on decimal arithmetic and will soon begin a unit on proportions. The full 45-minute lesson is spent discussing the previous night's homework, which many students have not completed. A majority of class time (25.5 minutes) is spent discussing an extension question posed by the teacher. The extension question, "Who would rather work where Rosa does?" is not part of the homework assignment itself but is grounded in the real-world context of the homework problems. Over the course of this discussion, the teacher and students make flexible use of integer and decimal arithmetic and number sense, and they preview many key ideas about comparing proportions. Interwoven with these rich and connected mathematical ideas, they discuss social ideas including working to help one's family, the health and social costs of working too many hours, the benefits of overtime pay, responsible money management, credit cards, bankruptcy, and the extreme salary differences between owners, managers and employees.

Students work on the following homework problems:¹¹ Rosa earned \$8.25 per hour and worked 30 hours per week. Leah earned \$9.50 per hour and worked 25 hours per week. In Problem 1, students calculate Rosa's weekly salary, which is \$247.50. In Problem 2, students do another decimal multiplication calculation that is not related to the Rosa vs. Leah debate that follows. In Problem 3 students are asked who earns more each week, Rosa or Leah. This requires them to calculate Leah's weekly salary, which is \$237.50 and conclude that Rosa earns slightly more. Most students subtract and conclude that Rosa earns exactly \$10 more per week.

Beyond the assigned homework problems, which only take about 5 minutes to review, the class spends an additional 25.5 minutes discussing an extension problem posed by Mr. B: "Who would rather work where Rosa does?" As argued below, this extension discussion was both mathematically and culturally responsive

¹¹ This information was extracted from the video recording. No photographs of the homework problem were collected.

to students' ideas. Furthermore, the mathematical and cultural dimensions of the conversation were thoroughly integrated and mutually sustaining. The teacher clearly cared that his students learn both the mathematical and social content of the lesson in order to prosper in their lives. He continually grounded the discussion in the real-world context through the use of cultural levers such as Black cultural modes of speech, personal knowledge about the economic realities of low-income jobs, and individual and cultural values around work, family and fairness. These cultural levers helped sustain and animate the mathematical discussion.

Table 4.2 offers a glimpse of the richness of the discussion. It provides a brief summary of mathematical and social ideas generated by the teacher and students over the course of the extension discussion. The ideas presented in the table are not direct quotations, but an attempt to summarize and clarify their statements while staying as close to the original language as possible. No evaluation is made yet as to the correctness of these ideas, but the claims made by participants are simply restated in order to show the rich variety of sensemaking elicited. Whether and how these ideas are connected to important mathematics is discussed later.

Mathematical Ideas	Social Ideas
<p>Subtraction can be used to find the difference in weekly earnings between two jobs.</p> <p>Comparing weekly earnings alone is not enough to tell you which job is better if the hours worked per week are different.</p> <p>Subtraction can be used to find the difference in weekly hours for similar weekly earnings.</p> <p>The hourly salary (unit rate) is a good point of comparison between jobs.</p> <p>A large difference in weekly hours is more important than a small difference in weekly earnings.</p> <p>It is useful to compare how many hours would need to be worked in order to get the same salary at both jobs. In this case, Leah could earn almost the salary as Rosa while working four hours less per week.</p> <p>A small difference in weekly earnings, such as ten dollars per week, does add up to something significant over the</p>	<p>A salary of \$8.25/hour probably indicates a job in fast food, retail and/or part time work.</p> <p>Earning more money can help your family buy meals.</p> <p>Working too many hours has a cost in health and family time.</p> <p>A standard work week is 40 hours, but many adults work overtime in order to earn time-and-a-half or double pay.</p> <p>Over a long-term period, such as one year, you could probably get a raise.</p> <p>If you earned about 10 dollars more per week you could help make things easier for your family.</p> <p>Careful management of your money is as important as earnings.</p> <p>Saving a little money every month can add up to something larger, such as a Christmas vacation.</p> <p>If you know you are not careful with money, you should work more hours.</p>

<p>course of a year.</p> <p>A yearly difference in earnings can be calculated by either multiplying the weekly difference in earnings times the number of weeks per year or the monthly difference in earnings times the number of months per year.</p> <p>A difference in weekly earnings of fifty cents per week is small enough to be insignificant, even when multiplied by 52 to scale up to a difference in yearly earnings.</p> <p>It is useful to compare what the salaries would be if both employees worked the same length work week. In this case, if both employees worked a 30-hour work week, Leah would have a much higher salary.</p> <p>A number in the thousands of dollars is not a realistic hourly salary.</p>	<p>Bankruptcy can be caused by buying items that you know you can't afford, especially with credit cards that let you spend money you don't have.</p> <p>Taxes get taken out of your paycheck right away.</p> <p>Your friend might call you on your break and make you feel bad about still being stuck at work, but it will feel good when you earn more money.</p> <p>Managers earn much more per hour than the employees they supervise.</p> <p>If you are going to work in fast food or retail, you should try to become a manager in order to earn more money.</p>
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Table 4.2. Summary of mathematical and social ideas discussed in the episode.

4.4.3.1 Supporting Engagement

At the beginning of the whole-class discussion, before doing any mathematics, Mr. B talks for about three minutes about the importance of the problem context. He makes a case for why the content is relevant to him in his own daily life as an adult and why it will be important to students' lives:

Mr. B: "Stuff like this is something I do on a regular basis, because hey, if I do a summer job, and I work this many hours, and I'm sposed to get paid, this, you better believe I'm already calculating, here's what I'm expecting to see on my check. And when I don't see it, then I call down to payroll.

"Because if you cain't, if you cannot calculate what you supposed to expect, this is how people take advantage of you. Because you don't know what you are supposed to be getting. If I'm at the register, you better believe, I'm already calculating in my head. If I spent this much, and I know what taxes are, here's what I'm expecting to pay. I look at my receipt; I want to be charged for exactly what I paid for.

"You don't just take people's word. That's why we go over this stuff. This ain't boring stuff; this is life skills that you supposed to have. You should know how to calculate what you are expected to be paid. It's like a grade. Now, if you know for a hundred percent fact that you did the work a hundred percent correctly, you are supposed to expect a good grade. When you don't get it, you know something is wrong."

This speech from Mr. B strongly parallels the idea of teacher speeches in the Black teacher literature. Johnson and colleagues (2013) describe the use of teacher speeches by a well-respected Black algebra teacher, pseudonym Floyd Lee. Roughly once per observed class day, Floyd Lee would stop the class to address problematic student behavior or comments through an extended speech. The mode of presentation of these speeches was culturally specific in the use of Black speech modes and shared cultural values to engage students. The content of the presentations included an important piece of social critique: positioning low-track students as intelligent and on a good path in life, in contrast to how the school system had positioned them. The impact of these speeches was to re-engage students in the behaviors they needed to succeed in school in a culturally relevant way. In terms of identity, Floyd Lee's speeches supported a juxtaposition of nondominant racial identities and positive academic identities.

The form of Mr. B's "speech" here is very similar to Floyd Lee's speeches. Like Floyd Lee, Mr. B changes the rhythm and tone of his talk, employs elements of AAE, shows a level of passion about what he is saying, and expects and receives students' avid attention. Like Floyd Lee, Mr. B demonstrates a sense of caring about students' lives outside of the mathematics classroom, and frames his statements in terms of what will benefit them in life. The content of Mr. B's speech positions students' outside-of-school funds of knowledge as valuable to classroom activity, which is different from the prevailing norms in most mathematics classrooms.

Like Floyd Lee, Mr. B uses speeches to engage students in classroom activity. However, the mathematical nature of this classroom activity differs sharply between the two teachers. Floyd Lee primarily gave speeches about classroom management issues. He explicitly acknowledged that some math content was arbitrary, and instead emphasized the importance of a high stakes algebra exam to students' lives. ("Our goal is to whup the test.") His speeches supported the validity of the high stakes test and the superficial and procedural content it covered, while also affirming his students' ability to do well on the test no matter what others might think of them. In contrast, Mr. B's speech is about the importance of particular mathematical content. He does not treat the mathematics content as arbitrary or disconnected from students' lives but instead makes a direct case that students will need to use this type of mathematical reasoning in real life to avoid being taken advantage of. The following section describes how Mr. B invites students to bring both inside- and outside-of-school sensemaking to bear on this rich mathematics problem.

4.4.3.2 Inviting outside-the-box sensemaking

Immediately following his speech about the importance of this type of mathematics, Mr. B leads the class into the first problem. He introduces the problem in a way that invites students to bring their outside-of-school sensemaking to bear on solving the mathematics problem.

Mr. B: *So if she works, as it clearly says, 30 hours per week, at 8 dollars 25 cents. First of all, what kind of job she got? Granted, it's not on here, but what kind of job you think she got?*

Students (overlapping): *janitor, burger king, cleaning*

Mr. B: *Huh? Say it again?*

S: *Burger king*

Mr. B: *Probably some fast food.*

Student laughter.

Mr. B: *I mean it's, this is probably part time work!*

This section has several affordances. First, it establishes to students that they can bring their life experiences to bear on this problem. The teacher is guiding the class in a certain direction but welcomes a variety of responses. Although the class has not yet gotten into serious mathematical work, these initial teacher signals mark a break from the authoritative role of a teacher in traditional math classrooms and are much more consistent with the teacher's role in reform math classrooms.

Second, the teacher's use of AAE signals that students can use language repertoires from their everyday life, rather than feeling pressure to formulate their ideas in Academic English before speaking. When supported to do so, multilingual students can use multiple language repertoires to communicate mathematical and affective nuance, develop Academic English while developing deep conceptual understanding, sustain their own cultural traditions and/or engage in sociopolitical critique (Moschkovich, 2007; Lee, 2006).

Third, the teacher's comments cue some number sense and contextual meaning for the numbers in the problem. Specifically, it establishes that \$8.25 per hour is a relatively small hourly salary (in fact, it is significantly below minimum wage at the time, which was around \$12 per hour).

All three of these affordances are part of a frame that Mr. B is establishing to support students' sensemaking. He has implicitly rejected some elements of a knowledge transmission frame, specifically, that only school knowledge is valued, students should only speak in formal Academic English, and there is only one right answer. In their place, he has incorporated some elements of a new epistemological frame in which the line between school and outside-of-school knowledge is blurred, a variety of speech registers are welcomed, and a variety of responses will be valued. We will soon see how this frame supports students to take up roles of defending and critiquing stances in a debate using both school mathematics and outside-of-school knowledge to justify their reasoning, while still respecting the mathematical integrity of the problem.

4.4.3.3 *Establishing correct answers and procedures*

The subsequent five minutes of instruction are not central to this analysis, and therefore are only summarized briefly here. They consist of a teacher-directed discussion of the correct procedures and answers for Problems 1–3. This portion of

the class discussion moves at a brisk pace. Mr. B calls on several student volunteers individually and asks some questions to the whole class that receive choral responses. The student contributions included single words or numbers such as “multiplication” and “two hundred forty seven and fifty cents” as well as explanations about one sentence long about how or why they did a particular calculation.

These five minutes of instruction are consistent with a knowledge transmission frame, which students are clearly familiar with, but which does not appear to disrupt or conflict with a productive disciplinary engagement framing of the subsequent discussion. The apparent purpose of this segment is to get students on the same page about what they were supposed to have done on the homework, which establishes common ground for the more open-ended discussion that follows.

4.4.3.4 Positioning students as authors through a debate

At this point, Mr. B poses his extension question, “Who would rather work where Rosa does?” and creates a sense of drama by giving a long wait time as students raise their hands eagerly. He first calls for a vote by show of hands. Most students vote that they would rather work where Rosa does, but two or three high-status students in the class vote that they would rather have Leah’s job. A few students say “what?” or “huh?” but Mr. B does not allow explanations yet, instead creating a sense of suspense.

Next, Mr. B asks for the students who voted for Rosa’s job to defend their position:

Mr. B: *Those of you who picked Rosa, this one (points to board), the top. Why did you pick Rosa?*

He calls on multiple students in brisk succession to answer this question and accepts their answers without evaluation. The student responses (teacher talk omitted) are as follows.

S1: *I picked Rosa because she earns more.*

S2: *Um, because it's 10 dollars for your family to have, for meals and like (inaudible)*

S3: *Much better pay! And then like money!*

From a perspective of school mathematics, all three of these student responses are equivalent. All three students have focused on the difference in weekly salary between Rosa and Leah, noting that Rosa earns more. Student 1 offers a straightforward statement of this rationale, consistent with a knowledge transmission frame. But when Mr. B calls on other students, they begin to add a little more mathematical detail and a lot more contextual and affective detail. Student 2 assigns a number to the difference in salaries: Rosa earns 10 dollars more. He also adds that 10 dollars would be a valuable contribution to one’s family well-being, for example for meals. The third student uses the most dramatic voice (most variety in volume and tone to emphasize his point) implying that the choice between earning more money or less money is an obvious one.

Next, Mr. B offers students another chance to vote for Leah’s job.

Mr. B: *Okay, watch. Okay. How many would rather work here? (Points to Leah's job on the board)*

This time, four or five students raise their hands eagerly. Mr. B again uses a long wait time as students gradually raise their hands and try to get his attention. This supports access by allowing students time to think and supports engagement by creating a feeling of suspense. He then calls on several students to explain their reasoning. The first student he calls is Katia, who draws attention to the fact that Leah works significantly fewer hours per week than Rosa and then minimizes the difference in salaries in light of this difference in work hours.

T: Okay, uh, Katia, why do you say, rather work here?

Katia: Uh, because, it's almost like um, I don't want to go there five more hours?

Mr. B: Hey, I

(Two students raise hands eagerly)

Katia: It's almost um

Student: Ooohhh!

Katia: They're almost paying me the same as (Rosa).

(Now there are 4 eager hands).

Mr. B: Can you say that louder? Why, why did you choose...

Katia: Because I don't want to work more five hours and if they're almost paying me the same as Rosa.

As Katia speaks, her peers are clearly listening attentively, and several make public shows of being convinced by her argument, for example by saying "ooohhh!" and raising their hands. This a high level of engagement from other students, likely because Katia's ideas cue a new way of thinking about the problem and this generates excitement.

Note how Katia's contribution connects to important mathematical ideas, particularly about the comparison of ratios, which will be the content focus of this class in the upcoming unit. In this problem, there are four numbers to compare (Rosa's salary, Rosa's hours, Leah's salary, and Leah's hours), which can be thought of as two ratios (Rosa's salary divided by hours worked and Leah's salary divided by hours worked). Before Katia's contribution, students were only considering two of these four numbers: Rosa's salary and Leah's salary. Katia's contribution explicitly mentions all four numbers, and compares the difference in salaries to the difference in hours worked. She does not mention ratios explicitly and we have no evidence that she has cued any formal school knowledge about fractions. But she implies that 10 dollars per week is a small number compared to 5 hours per week, even though in an abstract context 10 is a larger number than 5. She must be drawing on some additional knowledge to make this comparison. She could be drawing from two separate life experiences that 5 hours is a long time and 10 dollars is relatively little. She could be imagining working five hours to earn 10 dollars, and deciding that this would not be worth it. Or she possibly could be qualitatively comparing 5 hours to the total hour amounts in the problem, 25 and 30 (quantitatively, this would be a 20% increase in hours) and 10 dollars to the total dollar amounts in the problem (quantitatively, this would be a 4% increase in pay). Any or all of these intuitions

can be productively leveraged and connected to important ideas in formal mathematics. We will see how the discussion unfolds.

Rather than commenting directly on Katia's contribution, Mr. B gives chances for several other students to explain why they would prefer Leah's job. Here are their replies in sequence.

S4: Leah because if I worked 30 hours like Rosa I would get paid more than Rosa.

This statement compares the pay for an equal number of hours. It is more sophisticated than several of the previously presented strategies in that it introduces hypothetical information not included in the problem – Leah's hypothetical pay if worked 30 hours – in order to make a fair comparison. Although formal language about proportions has not yet been introduced, this student's argument is potentially generative as a strategy for comparing proportions in the future.

S5: I put Leah because you work less hours and make more money per hour. And you could work less five hours and you get 10 dollars less.

Student 5 mentions multiple points of comparison between the two jobs: total hours worked, hourly salary (unit rate), and difference in salaries. He is the first to mention unit rate as a point of comparison, which is a very general and powerful strategy for comparing proportions. Additionally, he seems to be making an implicit argument that working for five hours less is a large difference whereas getting 10 dollars less is not a very big difference. This echoes Katia's claim and requires some number sense.

S7: I put Leah because she only works 25 hours and Rosa works 30 hours. And um Leah gets paid more. But in total Rosa gets paid more but she has to work 5 extra hours for 10 dollars more.

This student contribution is very similar to Student 5's contribution except that Student 7 first mentions the numbers of hours worked by each girl and then the difference in hours worked and difference in price. He does not use unit rate language "money per hour" but does make a verbal distinction between hourly salary "Leah gets paid more" and weekly salary "in total Rosa gets paid more."

Next, Mr. B moves to summarize the discussion so far. He appears very pleased with how his students have individually and collectively used mathematics to make progress on a complex problem.

Mr. B: Now that is a, what you just done, okay, is what we adults have to decide on a regular basis. Cause I love that, do I want to work five more hours

Student: for just ten

Mr. B: I gotta work five more hours for just ten dollars extra. Now, how much. Now, we think about how much sense does that make when you start considering fatigue, you know, myself, married, kids. You start thinking about, is the 10 dollars worth it?

(At least 3 students): No.

Mr. B: I get almost the same for five hours less. So, now, I'm not saying that is a bad choice.

Student: It is though.

Mr. B: I am not saying that is a bad choice. But what you just done, see, that's the kind of thinking I want to hear when we're doing stuff, because this is more practical. This is applying what you just went through to a real-world application. Which one makes more sense. Now. You know, who wouldn't want the 10 bucks but then again, I gotta work five more hours. When if I just did one hour over here, I already made up the 10 dollars. And still work four hours less.

Here, Mr. B returns to the themes of his initial speech, emphasizing the relevance of the mathematics students have just done to real-world decisions. He also summarizes the main student idea about comparing differences, saying, "You know, who wouldn't want the 10 bucks but then again, I gotta work five more hours."

There are two levels of frame negotiation going on here. First, Mr. B frames the problem as grounded in real-world decision making with space for subjective judgment. He aligns himself fairly clearly with one side of the debate, and restates the justification that working "five more hours for just ten dollars extra" may not be worth it. Additionally, he builds on the dramatic moment with some personal and affective details about fatigue and wanting to spend time with his wife and family. Nevertheless, he avoids making an explicit evaluation that typically characterizes school problems. He frames the problem as requiring subjective decision making by saying it is "what we adults have to decide on a regular basis." A few sentences later, he says "I'm not saying that is a bad choice." A student contests this framing, saying "it is though," presumably implying that working so many hours for so little pay is clearly a bad choice. But Mr. B holds firm "I am not saying that is a bad choice." Although not explicitly stated, it is implied people might have their own reasons for making particular decisions even if someone else might disagree with them. Consistent with this framing, Mr. B makes his personal position clear but refrains from judging or evaluating those who might make a different choice.

Secondly, Mr. B indicates that the type of real-world reasoning students have just done is welcome in math class in general. He says, "see, that's the kind of thinking I want to hear when we're doing stuff, because this is more practical." This statement furthers the blurring of the line between the activities of the mathematics classroom and real-life decision making. The statement provides one indication that Mr. B will continue to welcome student sensemaking, real-world experiences, number sense and intuitions, as the class moves forward. In practice, as the year continues, Mr. B's classroom has some days that sustain this openness to students' real-world experiences and some days that are more narrowly focused on an abstract mathematical procedure without clear connection to students' lived experience. In other words, like most learning, the joint progress that the teacher and students made in this lesson toward a new frame is not indelible, but part of a long term and contextualized process. Nevertheless, it is powerful that after giving students a

glimpse of a new frame, Mr. B presents the new type of reasoning as something that he wants them to continue using inside as well as outside of math class.

4.4.3.5 Sustained opportunities to connect, refine and critique ideas

At this point, Mr. B appears ready to wrap up the discussion of homework and move on to the day's lesson plan.¹² However, students continued to suggest reasons for one side or the other of the debate. Deferring to his highly engaged students and their creative ideas, Mr. B spun out the homework conversation for the rest of the class period.

The first student question is grounded in the real-world context of the problem.

Student: Couldn't you get another job and make more money?

Mr. B: I mean. Uh, oh, yeah, you can do overtime, true. But then overtime means you workin more hours. That just depends.

Student: But you're getting double.

Mr. B: But you're getting double, yeah, that's true. And that's why a whole lotta people do overtime. Now, uh, in the real world, that's why, you know, uh, instead of a 40-hour work week which is normal, but most do more than that because overtime pay, like it says here is usually time-and-a-half or double....But the only downside is, and I think Katia touched on it, is the balance of your health, because that's what it eventually boils down to, and you got family and kids, well you workin all the time, you don't have time for that other stuff. Now, are you more worried, you more interested in making money? Or having, you know, health and spend it with family. Okay, so you wait. That's what it boils down to.

This excerpt is noteworthy for several reasons. A student has asked a question that opens a new line of thinking, which the teacher pursues at some length. Here, Mr. B does not connect the students' comment back to the problem but instead connects it outward toward the life lessons he is trying to impart. Furthermore, he continues to describe money as only one factor that adults need to consider when making decisions.

The next student comment re-opens the mathematical debate about whether Rosa's or Leah's job is better. Student 8 argues that Rosa's job is better because the 10 dollars per week extra that she earns will add up over time.

S8: *If you do like for a whole month and there's 4 weeks in a month you get an extra 40 dollars.*

Mr. B takes up this student idea as worth exploring and spends six minutes extending it with the class. He introduces the idea of multiplying to find the pay difference between Rosa's and Leah's jobs over the course of a year. He initially has students multiply 40×52 . Then he realizes his mistake and it becomes clear that

¹² In fact, I was only present as an observer because Mr. B had said he planned to teach the first day of a Formative Assessment Lesson. After class, he apologized for the change in plans but said that he could not let the low homework completion slide by.

some students have realized it too. He gives a correction that it should be either 10×52 (dollars per week times weeks per year) or 40×12 (dollars per month times months per year). He has students do both calculations. This part of the episode is significant for several reasons. A student has taken up the role of defendant in the debate, and has introduced a new line of reasoning accompanied by formal school mathematics. Of course, whole number multiplication is much easier formal mathematics than this class of sixth graders usually engages with, but it is still striking that the student spontaneously mentions it because it shows the student is taking an “authorship” role with initiative over the use of the tools of school mathematics to make inroads into novel and complex problems. Additionally, Mr. B makes a mistake in his unit analysis, but is comfortable working through this mistake in front of the class. This reinforces the framing that revision based on sound logic is more important in mathematics than quick, correct answers.

The conversation continues for 13 more minutes, until the end of the class period. Students continue to add both mathematical and practical ideas to the conversation and Mr. B takes them all seriously. One student argues that it is not fair to calculate the pay difference for a whole year because “during the whole year, you could probably get a raise.” Several students ask brief, clarifying questions about topics ranging from the meaning of numbers previously discussed to when and how taxes get taken out of a paycheck, which Mr. B answers briefly. A student comment about “going bankrupt” leads to a longer speech from Mr. B about responsible management of money and the danger of overusing credit cards and spending money you don’t have.

Some mathematical ideas that were not discussed in detail during the initial debate get revisited later on. Recall that in the initial debate, students mentioned comparing the two jobs by using the unit rate (hourly salary) or salaries for an equal number of hours. These are both very important strategies for comparing ratios that Mr. B did not initially take up. However, these ideas are revisited by both students and the teacher. Student 5 revisits the unit rate as the class consideration of Student 8’s idea is wrapping up:

S5 – If you wanted extra hours could you get the same 9 dollars and 25 cents.
Mr. B: True. You work one more hour and it’s all even. I mean, well, 50 cents [*sic*] times, uh, whatever, uh, okay 50 cents times 52. Now we talking do I really want to work 4 more hours or whatever. But you’re right though, there’s a lot you could do.

Remember that Student 8 has just argued that 10 extra dollars per week becomes significant over time. Student 5 responds by proposing an easier way to earn extra money: work extra hours at the higher salary job. To make this argument, Student 5 re-introduces hourly salary as a point of comparison.

Later in the conversation, Mr. B also takes up both the idea of unit rate and the idea of comparing salaries for the same number of hours worked.

Mr. B: You forget I make more per hour than you. So you gotta work more hours to make what I make. So the reason why you makin 10 dollars more is

cause you working 5 hours more. You working 30 hours, I'm working 25. So you working more to make what I make. So I'm happy. Let's think if I work 5 more hours we don't even have this conversation. If I choose to work more hours, we don't even have this conversation. Cause now I making much more than you, cause we working the same amount of hours.

In this excerpt, Mr. B mentions both the salary “per hour” and “working the same amount of hours” as points of comparison, while continuing to animate the debate through his use of personal pronouns “I” and “you” and appeal to students’ sense of competition through language like “So I’m happy” and “we don’t even have this conversation.”

These excerpts demonstrate the affordances of the sustained conversation about the salary problem for students’ content learning and identity development. Students have the opportunity to strengthen their understanding about important mathematical ideas including decimals, proportions, and unit rate. The conversation also supports students to develop strong disciplinary and racial/cultural identities because they have sustained opportunities to propose and critique mathematical ideas and strategies while working on a problem that is relevant to their lives. This rich conversation is supported by Mr. B’s use of a debate frame to indicate that students should not only justify their reasoning but also be accountable to responding to the reasoning of their peers. Students have made bids for roles of authorship over mathematical ideas, and these bids have been ratified when both their peers and teacher respond to the details of their thinking. Furthermore, outside-of-school knowledge and language repertoires have been welcomed in the debate, adding both emotional flavor and intuitive grounding for mathematical sensemaking.

4.4.3.6 Case 2 Summary

In this case, Mr. B used culturally specific practices to establish relationships between himself and the students and between the students and the mathematics. Specifically, he used AAE and “speeches” to indicate the value he placed on students’ outside-of-school experiences and to connect classwork to a sense of deep caring about students’ success in life. He referenced shared cultural values, such as hard work, family, and fairness, as a point of connection with students. But he also respectfully acknowledged the validity of different choices in life and different approaches to money in particular, such as managing money carefully or working more hours and spending more.

Mr. B’s culturally relevant discourse practices are not new, either in his own practice or in the research literature. Mr. B used AAE and “speeches” throughout the year, regardless of the mathematics content. Similar practices have been documented in the literature as practices that many Black teachers use that are helpful for building relationships with nondominant students. However, a majority of this literature focuses on culturally relevant classroom management and relationship building with students, not specific content. Content may be referred to

in a vague way that is driven by outside pressures – recall “Floyd Lee’s” speeches about whupping a high stakes test (Johnson et al., 2013).

Floyd Lee primarily gave speeches about classroom management issues. He explicitly acknowledged that some math content was arbitrary, and instead emphasized the importance of a high stakes algebra exam to students’ lives. (“Our goal is to whup the test.”) His speeches supported the validity of the high stakes test and the superficial and procedural content it covered, while also affirming his students’ ability to do well on the test no matter what others might think of them. In contrast, Mr. B’s speech is about the importance of particular mathematical content. He does not treat the mathematics content as arbitrary or disconnected from students’ lives but instead makes a direct case that students will need to use this type of mathematical reasoning in real life to avoid being taken advantage of.

This case study, in contrast, highlighted how similar, culturally specific practices were used to frame content learning in a way that supported *mathematically* responsive instruction, specifically making it easier to both elicit and build on student thinking about key mathematical ideas. The literature shows that “speeches” can be used to support student engagement with procedural mathematics dictated by standardized tests. Mr. B uses them very differently: to cue particular knowledge resources, such as intuitions about the magnitude of numbers, that help involve students in sensemaking within a complex modeling task. His use of the frame *adults making tough decisions* sends a social message to students that it is okay to be wrong or to disagree with each other. But it also sends an epistemological message that doing mathematics involves critiquing and justifying ideas. His use of a debate frame has the social affordance of generating a feeling of suspense and supporting student engagement in the classroom activity. But it also has a mathematical affordance of giving students sustained experience in generating mathematical justifications and critiques of each other’s ideas.

Is race a central part of this case study? As in the previous case, the teacher leverages cultural practices particular to Black culture. Unlike the previous case, however, students don’t visibly take up or respond to these ideas in obviously racialized ways. However, it is important to note that the topic at hand – economic decisions related to low-income jobs – could easily have become a point of classed or racialized “othering” of students. It would be all too easy for a mathematics teacher to bring up the need for math-savviness in the real world in ways that would intentionally or unintentionally position people with low-income jobs as ignorant. For students whose parents have less formal education and/or low-income jobs, this might create a feeling of being forced to choose between aligning with their parents or aligning with mathematics, a major obstacle to their development of a robust mathematical identity. Instead, Mr. B positions his job as a teacher – a middle-class job – in a way that can parallel what low-income families may experience. He emphasizes being paid by the hour, potentially getting underpaid, and needing to call down to payroll. He thus resists making a class distinction

between himself and his students by emphasizing aspects of working that are more universal.

Although there is no explicit mention of race in this episode in connection with fast food or retail jobs, it is important to note that there is a strong stereotype in society that jobs in these industries are generally filled by Black and Latinx individuals. If a White teacher were to position people with low-wage jobs as ignorant, even unintentionally, this would reinforce a racial stereotype that would close down opportunities for students. Even if Mr. B, as a Black teacher of predominantly Latinx students, had created a distance between himself and students' families through his positioning, this could have reinforced a racial stereotype for his Latinx students. Instead, the combination of Mr. B's implicit positioning as an educated Black person and his explicit moves to align his own positioning with that of students contest this racial stereotype. Furthermore, there is no reason to expect that only Mr. B's Black students would profit from this discussion. The way he positions himself counters racial stereotypes in a way that can potentially benefit all of his students.

In a wrap-up interview with Mr. B, I asked him directly whether he agreed with my claim that his life experience as an African American mattered to being able to teach the lesson in Case 2. He replied yes, and mentioned several ways that it mattered. He mentioned that there are good people in the world, but there are also many people who try to take advantage of you: You could be charged extra at the cash register if you are not calculating in your head how much you're supposed to pay. You could be paid a lower wage than you deserve because people assume you don't know the difference.

During the interview, which took place in his classroom, Mr. B pointed to his whiteboard at problems from an April 2018 lesson on unit rate, which included problems about converting between hourly and daily salary. He explained that he wants students to understand clearly that a small difference in hourly salary can make a big difference in their final paychecks. Furthermore, he explained that he wants students to be able to protect themselves against "the stereotype that hey, Blacks and Mexicans don't pay attention when it comes to detail. The hell we don't!" He continued that although "it's not us versus them" with respect to issues of race, students may be in a situation where a colleague of another race is being paid more than they are for the same work, and he wants students to be able to recognize that situation. Mr. B referred to his own work experience, which included three jobs in the food industry and one as a custodian. He mentioned taking classes at 9 pm to finish his own education. Based on this experience, he wants students to persevere through difficult or unjust situations if they have to in order to achieve high goals in life. Furthermore, Mr. B took the stance that he, and other teachers, need to be invested in the success of nondominant students rather than afraid of this success:

"Whether I'm Black, White, or whatever, I need to teach you because I ain't afraid of you getting more education. It ain't gonna be no sweat off me if you surpass me. I'm just gonna say hell yeah, you did it." (Field notes from Mr. B final interview, April 2018).

I interpret these comments to mean that, in short, Mr. B recognizes injustice in the world because of his personal experience, and makes a direct connection between students developing a deep understanding of the mathematics in the focal lesson and being able to navigate an unjust world with open eyes, dignity and honesty.

The only revision requested by Mr. B was that I remove the language “\$8.25/hour is a low-wage job.” He argued that while this statement was true, it undermined the main message he wanted to convey to students which is that a job in fast food is still honest work. He wanted students to know that even though they may get turned down from a job because of their race, there is a second, third, and hundredth job out there. At the end of the day, they only need to have one job to be able to provide for their family. He wanted them to know that they can always choose to keep going and they never have to choose to endanger their lives by “slinging rocks”¹³ (Field notes from Mr. B final interview, April 2018). In response to Mr. B’s concern, I removed all nine instances of the term “low-wage job” in the chapter and clarified whether the reference should be to the economic realities of low-income jobs or to work in particular industries such as food service.

4.4.4 Summary

The two case studies presented showed remarkable parallels despite significant differences in grade levels, mathematical content, classroom activity structures, and teachers’ cultural affiliations. Case 1 involved ninth graders, algebra content, seat work and student presentations, and a teacher who affiliated with hip hop and Black activist traditions. Case 2 involved sixth graders, decimal arithmetic and proportional reasoning, a teacher-led whole-class discussion, and a teacher who affiliated with the Black church. In both cases, teachers leveraged Black linguistic practices as well as common cultural referents – dance moves in Case 1, work experiences and values in Case 2 – to support students’ engagement in mathematically responsive discourse.

The intention here is not to focus on the particular cultural practices used; these cases provide only a few examples. However, the parallel functions of these cultural practices are highly significant. Engaging in mathematically responsive discourse can involve significant risk-taking by students. Instead of simply stating answers – some of which can be inferred from the way the teacher poses questions – students are being asked to justify their thinking at length, and to interpret and respond to the ideas of others. These risks are often multiplied for nondominant students, for example because of stereotype threat, pressure to formulate answers in Academic English before speaking, and/or the positioning of these students’ outside-of-school knowledge as invalid or even antithetical to mathematics classroom learning.

Both case studied here showed teachers deploying a variety of Black cultural practices that contribute to reframing the conversation and lowering these risks. In

¹³ Dealing crack cocaine.

Case 1, the teacher affirmed mathematical success in language that also affirmed Blackness (“That’s what’s up”), the value of explanation in addition to answers (“Write up why”) and hip hop affiliation (“while you whippin and nae naein”). This type of cultural and mathematical affirmation appeared to be typical of his teaching practice throughout the year. However, in the case study, the teacher tried something new and mathematically responsive in his teaching: having a student present his own sensemaking to the class. To support the student in taking up this new mathematical role, the teacher drew on the frame – not commonly used in mathematics classrooms but presumably familiar to students in other settings – of giving a talk. He encouraged the student to address his peers as required by the giving a talk frame using the same type of culturally affirming language he had used previously (“they over there”).

In Case 2, the teacher also used African American English blended with Academic English and both his Black and Latinx students appeared comfortable sharing ideas in a variety of formal and informal registers. He also gave speeches indicating a deep level of caring about student success in life, and connecting that sense of caring to strict expectations that they work hard in his class. These supports for student engagement appeared typical of his teaching practice throughout the year. In the case study, this teacher also tried something new and mathematically responsive in his teaching: having students debate an extension problem to the homework set. To support students in taking up new roles as defenders and critics of mathematical ideas, this teacher also drew on a frame familiar to students from outside of the mathematics classroom – in this case, a debate frame. Furthermore, he used a second frame of adults making tough decisions to blur the boundary between school mathematics and outside-of-school knowledge. Using this frame, he indicated to students that the particular mathematics content of the debate would be useful to them in life, that their outside-of-school knowledge would be valued in the debate, and that they would not be shamed for getting an incorrect answer because *making tough decisions* requires making subjective judgments and respecting a variety of answers. He used familiar teaching practices, specifically AAE and “speeches,” to draw students into these new frames and support their sustained participation in mathematically new roles.

It is worth noting here how much, and in important ways, Mr. X and Mr. B’s actions align with the goals of reform mathematics. In Case 1, student ownership is taken seriously in ways that enrich the mathematical learning opportunities for the whole class. Competence is assigned for a novel idea, one student speaks publicly at length and other students respond, and the teacher connects the student’s mathematical thinking to larger mathematical ideas. In Case 2, the “real-world application” used is not artificial. Previous literature, in particular the “bus problem” described by Lane & Silver (1995), has pointed to the potential affordances and also to the difficulty of effectively leveraging students’ outside-of-school funds of knowledge to explore real-world modeling tasks. Case 2 shows an example in which similar funds of knowledge that are effectively leveraged the classroom. The extensions of the

mathematical discussion deal with real, and meaningful, mathematics and its applications.

4.5 Discussion

This section addresses three themes that connect the results of this chapter to the literature:

- i. The power of classroom moments to shape identity;
- ii. General principles for leveraging culturally specific practices;
- iii. Centrality of issues of race and power to the analysis.

Within each subsection, the significance of these findings for the field are discussed; responses are also given to potential challenges to the power and generality of these findings.

4.5.1 The power of classroom moments to shape identity

The studies cited in the literature background of this chapter are mostly student interview studies with some classroom data. Interviews focus on identities that have thickened over time that students now carry with them across spaces. This study is part of a slightly different tradition oriented toward local and contextualized identity work in classroom observation data (e.g., Esmonde & Langer-Osuna, 2013; Langer-Osuna, 2011). It is important to look at classroom video because:

- Students are unlikely to be able to remember and reliably report particular moments of significance at the level of detail needed by theorists or practitioners;
- Teacher interviews are very useful for triangulation, but teachers often self-report practices that are more different from prevailing norms than what video of their classroom shows.

Nevertheless, it is worth discussing the potential problems that could arise from working with classroom video, and how this work addresses them.

First, classroom moments are chosen based on the researcher's post facto judgment of theoretical significance. How does one know these moments are powerful for students? To address this issue, moments were selected where teachers' identity moves are actively taken up by students, in ways that contrast with the prevailing norms in society as a whole. That is, not only do teacher moves *offer* positions to students that contrast with dominant narratives (e.g., Mr. X affirming Enoch's Black racial identity and mathematics disciplinary identity at the same time), but students take up and act on these identity bids by teachers (e.g., Enoch dancing and then writing a mathematical explanation in response to Mr. X's encouragement).

Second, how can we know these local and contextualized moments have any power to stay with students in the long term and/or into other spaces? This is a challenging question to answer based on microanalysis of one learning moment. However, in this work, I make note of ways that the particular video episodes are continuous with and/or break from typical classroom practice over the course of a year's observation in each teacher's classroom. In both case studies, patterns of

student engagement are seen that break with prevailing and detrimental norms but are sustained throughout the remainder of the year. Future work could use additional data sources, such as video over longer periods of time, participant observation across learning spaces, and/or student interviews.

4.5.2 General principles for leveraging culturally specific practices

This chapter highlights a few particular, culturally specific teaching practices of Black teachers for supporting students' identity development. As such, it is important to avoid simplistic arguments about a cultural match between Black teachers and Black students.

This potential problem is avoided by keeping the analytical focus on the general issues of racial and mathematical identity, as they play out in locally contextualized ways. In this way, the analysis presented goes well beyond a description of "some good things Black teachers do to connect with Black students." Although some particular common cultural referents between Black teachers and Latinx and Black students are highlighted, the most important thread of analysis relates to the identities made available to students. The particular cultural moves used can be conceptualized as part of a large toolkit to be developed for supporting the racial and mathematical identity development of nondominant students.

This chapter unpacked several specific practices of Black teachers with Latinx and Black students. It is not claimed that non-Black teachers could not use these practices, nor that the practices could not be used or adapted to support the learning of other groups of students. However, it is essential to treat students' cultural practices with respect and sincerity. Gutiérrez and Rogoff (2003) make this point very clearly, citing the example of Lee's research on the Black literary practice of signifying. Signifying is a culturally specific example of important literary structures, and as such can be very valuable for engaging Black students in literary analysis (Lee, 1993). However, Rogoff and Gutiérrez warn strongly against an approach to culture that assumes all Black students are familiar with signifying just because they are Black! Before trying to leverage this culturally specific practice in the classroom, the teacher and students need sufficiently deep knowledge of (and respect for) the particulars of this practice. The racial identifications of the teacher and students are much less important than their shared knowledge of relevant cultural repertoires of practice and the positioning of these practices as resources for powerful content learning (Gutiérrez & Rogoff, 2003).

Similarly, in one of the case studies, hip hop dance moves become a shared cultural referent for a Black teacher and two Black students to celebrate mathematical success. Although hip hop has deep roots in Black culture, one can easily imagine non-Black teachers and/or students who genuinely value hip hop drawing on these same practices. One can also easily imagine a Black teacher who does not value hip hop and does not welcome dancing in the classroom. This teacher might use a very different set of common cultural referents to connect with students, but still apply

the principles of this chapter to support the co-development of students' racial and mathematical identities.

4.5.3 Centrality of issues of race and power to the analysis

To borrow from Ladson-Billings (1995), a potential critique is, "But that's just good teaching!" In other words, when described at a general level, the teaching strategies employed by both teachers appear to fit with broad classes of teacher moves that are already known to be necessary to support all students in mathematically responsive discourse: building relationships with students, lowering the level of risk-taking needed to speak up, and affirming students for their process rather than for answers. Indeed, in the remainder of this dissertation, there are many moments when issues of race are backgrounded to focus on the mathematical content and practices. Why foreground these issues here?

Foregrounding issues of race and culture was essential to the analysis of these case studies because the moves that teachers used to position students as authors and critics of mathematical ideas were culturally and contextually specific.

In Case 1, Mr. X invoked an outside-of-school frame of giving a talk to position Enoch as an author and support him to share his novel mathematical strategy with the class. In addition to general classroom strategies such as inviting Enoch to the front of the room and clearing space for his work on the whiteboard, the teacher used culturally specific language, "they over there," to communicate a shift in the classroom's mathematical norms to Enoch: instead of speaking for the purpose of demonstrating his knowledge to Mr. X, which was the primary purpose of most student talk in Mr. X's classroom, Enoch should address his classmates with the purpose of explaining an important and novel mathematical idea to them. Although the student presentation activity was atypical of Mr. X's classroom practice, it was typical for Enoch and his friend Frank to be positioned as mathematically knowledgeable participants *and* have their Black cultural identity affirmed in this classroom. Mr. X used culturally specific moves to align these racial and mathematical positionings, for example using AAE to praise the students and affirming their use of hip hop dance moves to celebrate mathematical success.

In Case 2, Mr. B invoked the outside-of-school frames of a debate and adults making tough decisions to position students as critics and connectors of mathematical ideas and supported them to use a variety of mathematical and social concepts to make inroads into a complex, real-world mathematics problem. He used Black cultural practices such as giving a "speech" and shared his personal experience as a working adult and parent. In doing so, Mr. B both conveyed to students that he cared about their lives and also opened up the discussion to include outside-of-school knowledge that is not typically valued in a mathematics classroom. As students voiced their knowledge and questions about related social issues like working extra to support one's family, working overtime, and credit cards, they were also supported to engage in estimation, mathematical modeling, and the flexible use of procedures.

These are only two examples of a much more general phenomenon. Students have multiple knowledge bases for learning mathematics, including prior school knowledge and outside-of-school knowledge. Mathematically responsive teaching seeks to center students' ideas, but these ideas are culturally and contextually specific. Some types of knowledge, for example knowledge of hip hop moves or the health impact of low income jobs, can easily be positioned as "non-mathematical" or even "disruptive." It should not surprise us that excellent teachers of nondominant students find ways to elevate and integrate this knowledge with powerful disciplinary ideas and practices. This integration can be as simple as affirming the use of a hip hop move to celebrate mathematical correctness or as complex as teaching a lesson about comparing proportions through a discussion about the advantages and disadvantages of working many more hours for just a little more money.

This chapter begins with the premise that, when the experiences of nondominant students and teachers are centered in education research, issues of race and power that impact nondominant groups become impossible to ignore. Many of the obstacles to the engagement and learning of nondominant students can be located in either the racist structural organization of schooling or the operation of racialized narratives within classrooms to restrict the identities and learning opportunities available to these students. It is the premise of this chapter that the general principles of mathematically responsive teaching can benefit all students regardless of race, but the particularities of implementing these principles in local contexts are racialized. Without explicit attention to issues of race, power, and status, it is hypothesized, nondominant students would be unlikely to derive equal benefit from these beneficial principles.

However, the findings of this chapter go well beyond identifying barriers to the success of nondominant students. Instead, the findings are quite hopeful and positive. By foregrounding the practices of excellent teachers of nondominant students – who in this case are themselves Black teachers – two different examples are given of how common cultural referents can be leveraged to position and support nondominant students as authors and critics of powerful mathematical ideas while simultaneously affirming their racial and cultural identities. These examples point to how insights from excellent teachers of nondominant students can deepen our understanding of culturally relevant teaching, mathematically responsive teaching, and intersections of the two.

5 Surprising Functions of IRE sequences in responsive classrooms

5.1 Introduction

To this point, this dissertation has focused on classroom discourse at a fairly coarse grain size: at the frames operating in a classroom and the consequences of these frames for student learning and identity development. In particular, we have discussed a *knowledge transmission* frame, which positions students as passive recipients of knowledge, and a *productive disciplinary engagement* frame, which positions students as authors and critics of powerful disciplinary ideas. We now turn our attention to a finer grain size and ask: how do teacher moves support particular frames?

Most of the existing discourse literature examines two ends of a one-dimensional spectrum. The classrooms studied contain either very narrow or very broad opportunities for students to make their thinking public, that is, they tended to strongly align with either a *knowledge transmission* frame or *productive disciplinary engagement* frame, with little in between. At a smaller grain size, the individual teacher moves in these studies are characterized as either squashing student creativity and reasoning or fundamentally empowering students as mathematical thinkers. Now that these studies have achieved their purpose and the field has clear exemplars of both very closed and very open discourse, a growing literature investigates how to support teachers to learn the teacher moves and corresponding orientations toward student thinking associated with a *productive disciplinary engagement* frame, such as “open-ended questioning,” revoicing, or inviting students to present their work at the front of the class (Franke, et al., 2001; Visnovska & Cobb, 2009). These efforts, as noted in my literature review, have proven challenging.

I will argue here that we need a much richer understanding of how teacher moves support particular frames. It is improbable that any discourse move is always good or always bad across all contexts. Furthermore, teachers will not progress from a knowledge transmission frame to a productive disciplinary engagement frame overnight, and it is likely their practice will support a variety of “in-between” and/or hybrid frames. Similarly, it is unrealistic (and I believe misguided) to assume that teachers will discard their entire current repertoire of current teacher moves and acquire a new set of moves consistent with a new frame they wish to enact. We should ask: Do teacher moves that have been researched in the context of a particular frame always align with and support that frame? For example, if a teacher invites a student to the front of the class to present his/her work, is that a sure indication of a productive disciplinary engagement frame? Is the presence of chains of IRE sequences in a classroom a sure indication of a knowledge

transmission frame? Or might the impact of these teacher moves on framing be sensitive to context?

In my dissertation data, teacher moves somewhere in the middle of the discourse spectrum were common. Teachers integrated a few specific reform-motivated, high-leverage discourse moves into their existing teaching practice, such as inviting students to present their work at the front of the class (discussed in Chapter 6). Other teacher moves, associated in the literature with a knowledge transmission frame, persisted, but seemed to be serving new purposes. Within this complex, but rich learning context, my analysis will focus on how teacher moves thought to have very narrow affordances for building on student thinking function in a classroom culture that has very broad affordances for the same. More broadly, my analysis of re-purposed and hybrid teacher moves contributes to the literature on discourse and learning by enriching our understanding of how moment-to-moment discourse moves support particular frames.

5.2 Background

Classroom discourse has been studied at very different grain sizes, from micro-analyses of a few talk turns to accounts of teachers' shifting discourse moves over a year or more. To describe these different grain sizes, I adopt the language of Nathan and colleagues (2007). These authors refer to classroom discourse at three "interdependent but partially decomposable levels": (a) the *micro level*, referring to the "moment-to-moment flow of information" between participants, (b) the *meso level*, referring to "the nature and purposes of classroom scaffolding," and (c) the *macro level*, referring to "global patterns of interaction that occur across an entire discourse." In my work, I use the term micro level to refer to turn-by-turn analysis and consideration of individual teacher moves. I use the term meso level to refer to the norms and participant roles made available during a particular class activity such as a whole-class discussion or a period of small group work at the time. I use the term macro level more rarely. While I agree with Nathan and colleagues that it is essential to understand how social narratives and structures can push into classrooms, I find the term macro level too broad to be analytically useful. In my work I tend to take on particular slices of the macro level in more detail. For example, the previous chapter analyzed the effect of macro-level racial narratives and counternarratives, combined with teachers' micro-level positioning moves, to afford particular participation roles to students.

This chapter is primarily about the interaction between teacher moves at the micro level and framing at the meso level. To situate my own analysis of IRE sequences in responsive classrooms, I will briefly review the literature on micro-meso interactions in general and IRE sequences in particular.

5.2.1 IRE sequences

One of the first micro-level discourse moves to be studied in detail is the Initiation-Response-Evaluation (IRE) sequence, in which the teacher Initiates the sequence with a known-answer question, a student Responds with no more than a few words,

and the teacher Evaluates the response as correct or incorrect (Mehan, 1979; Cazden & Beck, 2003).

According to Mehan's original analysis (1979), IRE sequences are a form of discourse that is quite common in classrooms but would seem very strange in everyday interactions. First the teacher asks "known information questions" to which he or she already knows the answer, as opposed to "information seeking questions" typical of everyday discourse, in which the questioner is genuinely curious as to the answer. According to Mehan, "The respondent in a 'known information question' is placed in the position of trying to match the questioner's predetermined knowledge, or at least fall within the previously established parameters." The teacher's follow-up is also different from everyday discourse. In an IRE sequence, the teacher *evaluates* student replies, certifying correct answers and either rejecting incorrect answers or repeating the known-answer question for another student to answer. In other words, the IRE sequence discourse form includes both a prompt for students to say something specific and an evaluation certifying whether their response is correct. Mehan ascribes this discourse form one specific function: to "test the knowledge of the respondent." Additionally, although Mehan does not explicitly specify the length of a student Response turn, the example IRE sequences he provides tend to have students speaking a few words. The brevity of the expected responses significantly limits the complexity of the knowledge that can be "tested."

At the meso level, IRE sequences are often chained together in topically related sequences, meaning the teacher asks a sequence of IRE questions to one or more students about the same problem or a series of related problems (Mehan, 1979). A large majority of classroom discourse literature views IRE sequences and topically related chains¹⁴ as problematic, both because they narrow the space of mathematics being worked on in the classroom and because they limit student voice to statements of right or wrong answers to known-answer questions (Franke et al., 2007, provide an excellent review).

Very few authors argue that triadic dialogue *can* serve responsive roles in the classroom. Perhaps the most convincing such counterargument is advanced by Wells (1993). Wells gives examples of triadic dialogue being used to elicit student interpretations and proposed follow-up actions about an experiment in an inquiry-based science class. However, Wells defines triadic dialogue more broadly than other authors in two ways. First, he uses the term IRF (Initiation-Response-Follow-up) instead of IRE, indicating that the last turn of teacher talk need not be an evaluation of the student talk. Secondly, although he does not note this distinction in the paper, his examples of IRF sequences include much longer talk turns by

¹⁴ To avoid the potentially ambiguous wording "topically related sequences of IRE sequences," I will use the term "IRE sequence" for a single triad and "topically related chains (of IRE sequences)" for a number of triads in direct succession.

students than those provided by Mehan.¹⁵ Most of the responsiveness to student thinking that Wells describes comes either from these longer explanations by students or the more flexible Follow-up turn by the teacher. Thus, the paper does not satisfyingly demonstrate that IRE sequences, by a strict interpretation, can serve responsive functions. Nevertheless, it makes an important contribution because it is one of the few papers to explore a close variant of IRE sequences and argue that this slightly expanded IRE form can serve a variety of responsive functions.

At the macro level, international studies such as the TIMSS video study (Hiebert et al., 2003) lament the prevalence of IRE sequences in U.S. mathematics classrooms relative to other countries where the mathematical focus is richer and participant roles are available for students to act as generators and critics of mathematical ideas as well as producers of answers.

Cast in terms of framing, this literature portrays chains of IRE sequences as a strong indication that what I call a knowledge transmission frame is operating in the classroom. In this frame, learning is imagined to proceed as follows: the teacher presents certain ideas, tests students' recall of those ideas (using IRE sequences) and immediately certifies their answers (using the same IRE sequences). In this frame, knowledge both originates with and is certified by the teacher. The roles of authorship, critique and evaluation of mathematical ideas are made available to the teacher and the (absent) textbook authors. There is apparently little space for students to take on roles as authors, critics and revisers of important mathematical ideas.

5.2.2 Alternatives to IRE sequences

At the other, more open, end of the discourse spectrum, many micro-level alternatives to IRE sequences have been documented such as *IDE sequences*¹⁶ (Nathan et al., 2007) and revoicing¹⁷ (O'Connor & Michaels, 1993, 1996). There is

¹⁵ For example, two student Responses from the Wells paper were "No. you have to-- if I filled my bottle half and to make that a fair test she would fill her bottle half" (23 words) and "I learned that the time may not only be in seconds, you may . see it as a minute, a second and a second is made up of . quite a few fast counts." (32 words). The longest student Response analyzed by Mehan (1979) was only six words long, and one- or two-word student Responses were more typical. In this chapter I will be quite strict and only code IRE sequences when student Response turns are five words or less. A full operational definition of IRE sequences for this chapter can be found in the Methods section.

¹⁶ Three turns of dialogue: (1) open Initiation consisting of an information-seeking question, (2) student Demonstration of a proposed solution often involving multiple sentences and/or a novel student-generated representation, and (3) Elaboration of the preceding demonstration such as an addition, modification or query.

¹⁷ A single turn of speech positioning the speaker (typically a teacher) as making a warranted inference about a previous speaker's idea. An example would be "So Jose, you said 34 because you saw a vertical pattern in the table, is that right?"

also a rich literature on meso-level discourse structures that have broad affordances for building on student thinking (Yackel & Cobb, 1996; Horn, 2008; Stein et al., 2008).

Perhaps the best-studied mathematically responsive micro-level discourse move is revoicing. O'Connor and Michaels (1993, 1996) show how revoicing creates participant roles for students as authors and evaluators of mathematical knowledge. They argue that these roles are drastically different from the roles made available by typical U.S. teaching practice, including IRE sequences. In particular, they create a role for a student as the author of a mathematical idea, position the teacher as someone who is considering the student's idea in order to understand it, build on it, and make it public, and give the student authority to affirm or reject the teacher's interpretation of his/her idea.

Cast in terms of framing, IRE alternatives are typically associated with a productive disciplinary engagement frame (Engle & Conant, 2002; Hand et al., 2013). More specifically, they create space for the teacher and students to propose, critique, and build on ideas according to the standards of the discipline of mathematics. In this frame, students view each other as participants in knowledge generation and freely take on roles as authors, critics and revisers of important mathematical ideas.

5.2.3 Multiple functions of teacher moves

The literature on IRE alternatives is important because it shows that student-centered meso-level discourse is possible, provides exemplars of this discourse, and analyzes how a few specific teacher moves function within this responsive context. However, these exemplars of exceptional practice do not yet offer a model of teacher learning. As I argued in my introduction, it is dangerous to extrapolate a model from two extremes, because the "easy" models like "just get rid of the bad teaching practices and do the good ones instead" are quite damaging to teachers who are trying to balance many dimensions of practice. Instead, it is essential to try to map out points of *continuity* between typical teaching practices and culturally and mathematically responsive teaching (e.g., Russ et al., 2016). In other words, what existing teaching practices can experienced teachers most easily adapt and repurpose to make a classroom significantly more culturally and mathematically responsive?

This chapter will contribute to our understanding of these issues by analyzing teaching episodes that are significantly more mathematically responsive than is typical for the U.S., yet use many teacher moves that are familiar to U.S. teachers. The chapter aims to advance our understanding of how familiar teacher moves can be repurposed and adapted to serve new and responsive functions.

Very recent research on the multiple forms and functions of revoicing (Herbel-Eisenmann et al., 2009; diSessa et al., 2015) begins to explore a powerful and generative idea: that even well-studied micro-level teacher moves can take variant forms and serve multiple functions across different meso-level activities. These

studies investigate the functions of revoicing in meso-level contexts as varied as classroom teaching and clinical interviewing. Across these contexts, revoicing served some similar functions in drawing out student ideas. However, there was also some variation in function depending on context. For example, in clinical interviewing, the framing of the activity makes clear that student ideas are of interest to the researcher and hence less work is required to make clear to students that their ideas are valued. Nevertheless, revoicing is helpful in drawing out the details of student thinking while leaving the student a chance to agree or disagree with the ideas being attributed to him/her.

It seems timely to revisit IRE sequences in light of this newer literature. Could IRE sequences, like revoicing, take on different functions in different contexts? In particular, do IRE sequences, which are ubiquitous in U.S. classrooms, always reinforce a knowledge transmission frame? Or, in the right meso-level context, could IRE sequences be one of many teacher discourse moves that together support a productive disciplinary engagement frame?

5.2.4 Situating the chapter

The first goal of this chapter is to serve as an existence proof, answering the following research question:

RQ1: Is it possible for chains of IRE sequences to exist in class discussions where a knowledge transmission frame is not operating?

I intend to answer this question in a strong affirmative. I will demonstrate that such IRE sequences can not only exist, but in fact can carry a significant fraction of the mathematical flow of a conversation while actively disrupting a knowledge transmission frame. To do this, I will analyze two cases from different teachers' classrooms. In both cases, I will first show that students take on mathematical authorship roles that are clearly inconsistent with a knowledge transmission frame and more consistent with a productive disciplinary engagement frame. I will then show that chains of IRE sequences are integral to the conversation in that they both represent a significant fraction of the discourse and are part of the mathematical flow of ideas.

The above existence proof, while easily established in my data analysis, contradicts a common assumption in the literature that there is an easy one-to-one correspondence between micro-level teacher moves and meso-level framing. This shows that a more complex model of the relationships between micro-level moves and meso-level framing is required. The second goal of this chapter, therefore, is to use the case analyses to shed some light on what a more complex picture could be. Specifically, I aim to glean a deeper understanding of the function of chains of IRE sequences in these cases, in relationship to frame negotiation. This leads to a second research question.

RQ2: How can IRE sequences operate together with other teacher moves to position students in roles consistent with a productive disciplinary engagement frame?

To briefly preview the findings, both cases included a combination of IRE sequences and other teacher moves. In both cases, the teachers began by giving the student significant “air time” to explain his/her work on a problem. After this extended student explanation, the teachers both used chains of IRE sequences to “step through” the student solution. I will argue that the function of these IRE chains was to re-phrase and build on the student’s mathematical ideas while reinforcing the student’s position as an author. In short, chains of IRE sequences served some of the canonical functions of revoicing in these classrooms: reviewing or rephrasing student ideas while reinforcing students’ positioning as authors of mathematical ideas.

5.3 Methods

Chapter 3 covered data collection methods for the dissertation as well as an overview of analysis methods. In this section, I briefly explain a few remaining details of data selection and analysis procedures particular to this chapter that have not already been discussed.

5.3.1 Data Selection

For this chapter, the main data source is video recordings of classroom observations. Before selecting focal episodes, I first made an initial pass through some of the beginning- and end-of-year data, looking for shifts in teacher moves. In the process, one episode stood out to me because of the presence of IRE chains *between* two student presentations, where the literature suggested IRE sequences would be inconsistent with the meso-level framing of the episode. I flagged this episode (which would eventually become Case 1) for more detailed analysis. I then made a more systematic exploration of data episodes likely to be mathematically responsive: episodes with student presentations and extended whole-class discussions across all four teachers. IRE sequences and near-IRE sequences appeared quite common.

In selecting episodes for analysis, I wished to choose episodes that were (a) clearly mathematical responsive and (b) clearly contained IRE sequences. In reviewing the IRE literature, different authors defined IRE sequences slightly differently. As noted earlier, it was particularly difficult when authors made arguments about the benefits of triadic dialogue but defined triadic dialogue more broadly than other authors.

My next step, therefore, was to choose a fairly strict operational definition of an IRE sequence, to ensure that my argument about the responsive functions of IRE sequences would not be undermined by doubt as to whether these are *really* IRE sequences. I chose the following:

Teacher Initiation: A question with a single, known correct answer.

Student Response: A very brief (less than five-word) attempt to answer the question.

Teacher Evaluation: Either an explicit or implicit¹⁸ evaluation that leaves little doubt about whether the teacher considers the response correct. With this definition, I made a more careful pass through classroom episodes across teachers that I thought were most likely to be mathematically responsive, such as class days when student presentations or extended whole-class discussions had occurred, to determine whether there was a strong presence of IRE sequences according to my strict definition. I quickly identified two episodes for deeper analysis.

5.3.2 Analysis Methods

This chapter will use qualitative video analysis as the primary methodology, including both micro- and meso-level analyses.

Micro-level analyses will focus on the moment-to-moment flow of information, primarily through talk but also through students' publicly displayed written work when applicable. The analytic unit will typically be a few talk turns in length. This analysis will identify the form and proximal function of talk turns and short sequences of talk turns, such as IRE sequences and more open teacher and student speech.

I analyze the functions of IRE sequences in these contexts, which are, as I will demonstrate, quite different from the classic IRE functions documented in the literature.

Meso-level analyses will also be used. The analytic unit will typically be a 5–20 minute classroom episode. These analyses will focus on the functions of particular talk moves within the context of the development of mathematical ideas and student authority across the episode.

5.4 Analysis and Results

The chapter consists of two case studies. The first case, analyzed in depth, is a roughly 10-minute episode of classroom practice that was responsive to students' mathematical thinking in its meso-level structure, and included several chains of IRE sequences. The second case, presented in much less detail, is a much briefer episode of classroom instruction in which a student explains a mathematical strategy and then IRE sequences are used to elaborate and clarify that idea (similar to a function that would be more commonly carried out by revoicing, as mentioned in the literature).

¹⁸ Some of the IRE literature suggests that the Evaluation turn should be an explicit evaluation, such as "that's correct." However, there is substantial precedent for including implicit evaluations. For example, Mehan (1979) shows multiple examples of chains of IRE sequences. Negative evaluations were sometimes unspoken by the teacher. However, if the first student called did not provide the correct response, the teacher continued questioning (perhaps calling another student or making the question easier) until a correct response was obtained.

5.4.1 Detailed case study

In this section, I will provide a detailed analysis of a ten-minute whole-class discussion that included both student presentations and IRE sequences. I will begin with a brief overview of events to orient the reader. My analysis will then make three claims, sequentially.

First, I will argue that, at the meso level, the discussion was responsive to student thinking. In particular, student thinking was elicited and responded to in some depth and students took on roles as authors, critics and revisers of mathematical ideas.

Second, I will demonstrate that IRE sequences made up a significant fraction of the discourse in the episode, about one quarter of all talk turns.

Third, I will examine the function of all eight IRE sequences in this episode. I will explore the various functions served by IRE sequences in this responsive classroom. Recall that most literature on IRE sequences asserts that they strongly constrain student thinking and limit the opportunities for students to express their own ideas. My analysis will show that, to the contrary, IRE sequences were integral to the development of mathematical ideas in a way that was closely connected to the mathematical content of preceding and subsequent extended talk turns by students. In my findings, IRE sequences do often “step through” a sequence of related ideas, but they do so to much more varied effect than suggested in the literature. In particular, I will demonstrate that IRE sequences do not necessarily limit students’ authority to author and revise mathematical ideas and in fact can reinforce this authority.

5.4.1.1 Overview of events in the episode

Students in Ms. A’s sixth-grade class have been matching frequency graphs with statistical summaries (mean, median, mode and range) of the same data. The task is quite challenging for students and extends over several days. Most days include “share-outs” where students come to the document camera to present a pair of cards, one graph and one table of summary statistics, that they think represent the same dataset and therefore should be matched together. See Section 5.4.1.2 for an overview of the mathematical ideas discussed.

The case study episode is a classroom video from this lesson that is roughly ten minutes in length. The episode as a whole exemplified a number of classroom practices that were very responsive to student ideas. Students came to the front of the class to share their thinking, took talk turns that were multiple sentences in length to explain their mathematical process, asked other students questions, corrected each other, expressed being stuck and subsequently expressed being able to make progress again.

An overview of the sequence of events in the ten minutes is presented in Table 5.1. There were two student presentations by Barak and Celia. Each presentation began

with the presenter explaining what he or she did. In both cases, this student explanation was followed by a brief teacher commentary, then a space for student questions and/or commentary, and finally a teacher-led whole-class discussion.

Presentation 1 (Barak)	Presentation 2 (Celia)
Presenter explains what he did	Presenter explains what she did
Teacher commentary	Teacher commentary
Space for student questions	Student commentary
Teacher-facilitated class discussion	Teacher-led summary and connections

Table 5.1 Overview of Case 1.

The classroom discourse varied widely during the episode. There were both long and short talk turns by both teacher and students, times during which the scope of discussion was quite narrowly focused by teacher talk and times when it was quite open to directions set by students. Some of these details will be examined closely in the following analysis. For now, an overview can be seen through two rough categorizations of discourse, indicated by the colors in the table:

- red indicates a discourse dominated by student talk;
- teal indicates discourse dominated by teacher talk;
- purple indicates a complicated combination of student and teacher talk.

5.4.1.2 Claim 1: At the meso-level, the discussion was responsive to student thinking

Here I will argue that, at the meso-level, the episode did not follow a knowledge transmission frame and instead was responsive to student thinking. My working definition of responsiveness is that students' mathematical ideas are elicited and built on in some detail.

In the first presentation, Barak presented a standard strategy for finding the median of a list of numbers: ordering the numbers from least to greatest and crossing out the end numbers until only one is left. This was a familiar way of finding the median that students had used before. Barak got somewhat stuck, but was supported to stay at the front of the room and eventually produced the representation of his method shown in Figure 5.1.

Specifically, Ms. A takes up multiple student strategies and representations (both student-generated and given) in some detail in her summary comments. She walks through Barak's written work in a sequential order that parallels Celia's explanation, and concludes with an explicit statement that the two strategies are the same. Here is what Ms. A says:¹⁹

Ms. A: So, you know what, it makes sense right, because how many ones did he write here? Let's just look at what he wrote. Cause look. How many ones did he write here? Five, right? And then he wrote five and he crossed it out. Here Ms. A draws attention to the five ones written by Barak in his list of numbers (see Figure 5.1).

Ms. A: But then he wrote, how many sixes were in my bar graph? One, so he crossed off a six. How many fives were there? One, two, three, he crossed off four, four fives right?

In this statement, Ms. A explicitly references both the bar graph and list of numbers representations in explaining that Barak crossed off one six and four fives from his list.

Ms. A: And then what's in the middle? 2. So crossing it out works the same way as listing it out, right?

Finally, Ms. A concludes that two is in the middle of Barak's list. Having shown the parallels between the two strategies, she summarizes with a statement of their equivalence, "crossing it out works the same way as listing it out."

In summary, this ten-minute episode of classroom instruction showed multiple indicators that were inconsistent with the knowledge transmission model of teaching and instead indicated the episode was open and responsive to student thinking. Students did much more than reproduce a standard approach to finding the median. They had the chance to consider multiple representations of data. Students were able to share their mathematical thinking in some detail, giving multiple-sentence explanations of both standard and novel strategies. Barak's overview of a standard strategy did not stand unquestioned; Celia asked a follow-up question that prompted him and the class to elaborate on the details of the method, and then she presented a second, novel strategy. Finally, the teacher both facilitated students' sharing of their ideas and responded to student thinking in detail by systematically connecting two student strategies in a way that also showed connections between two important representations of data.

5.4.1.3 Claim 2: Chains of IRE sequences were a significant fraction of the discourse

So far I have demonstrated that the flow of mathematical ideas in this classroom was both open and responsive to novel ideas originated by students.

Now I will look at the discourse at the micro level, to show what talk moves contributed to the meso-level responsiveness noted here.

¹⁹ Again I have presented the main flow of mathematical ideas attributed to its main author, the teacher. Short student comments have been removed for now. They will be analyzed at a later time.

Figure 5.3 shows a color-coded transcript of the full episode, with mathematical teacher talk turns colored in teal, mathematical student talk turns colored in red, and non-mathematical talk turns (off task or classroom management) in grey. This figure represents the wide variety of micro-level talk turns used in the episode. Both teacher and student talk turns varied widely in length, from a single word to several sentences long. At times, students responded to each other without the teacher's spoken intervention; there were sections of student-student talk up to 10 turns long.

I was surprised by the appearance and frequency of IRE sequences. There were eight IRE sequences comprising 11/46 (24%) of all teacher talk turns in the episode.²⁰ These IRE sequences occurred after the open space for other students to question or comment on each presentation.

The next section will give a detailed analysis of each of the three groups of IRE sequences. I will show that the first topically related chain of three IRE sequences "steps through" a standard method for finding the median. The subsequent singleton IRE sequence corrects a student's misuse of the vocabulary word "mode" when the correct word would have been "median." The final topically related chain of four IRE sequences connects Barak and Celia's presentations.

²⁰ Readers may expect that since each IRE sequence includes two teacher talk turns, eight IRE sequences would equal 16 teacher talk turns. However, when IRE sequences are chained together one after another it is common for the Evaluation of one IRE and the Initiation of the next to occur within a single teacher talk turn.

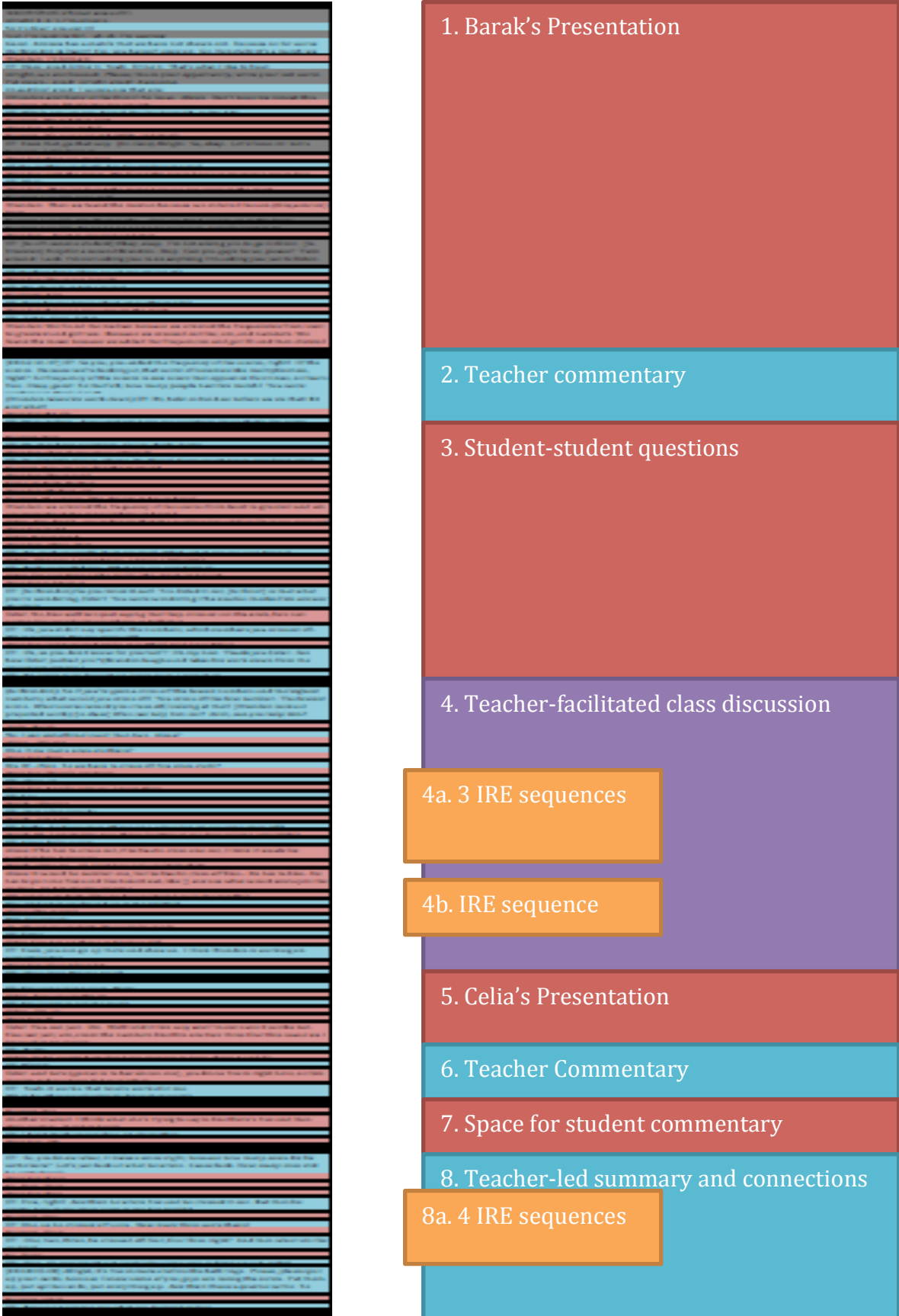


Figure 5.3. Annotated full episode transcript for Case 1.

5.4.1.4 Claim 3: Functions of IRE sequences

Here I will look in more detail at each topically related chain of IRE sequences and examine its function within the episode. I will show that the functions are inconsistent with a *knowledge transmission frame* and consistent with a mathematically responsive *PDE frame*.

5.4.1.4.1 Setting the mathematical stage

In this section, I will briefly summarize segments 1–3 of the case study episode, which occurred prior to any IRE sequences. Although not central to my analysis, these segments set the mathematical stage for the discussion that follows.

1. Barak's Presentation

2. Teacher Commentary

3. Student questions

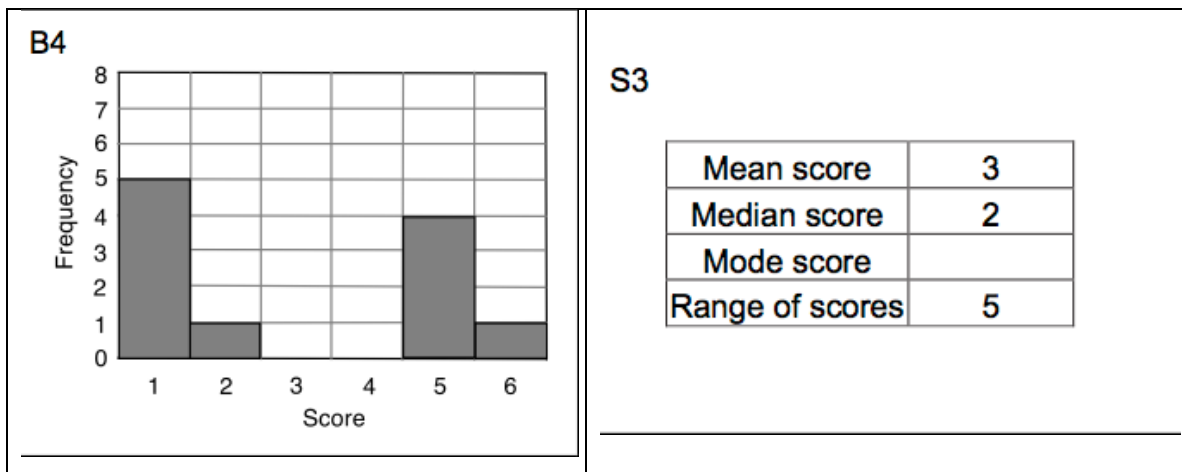


Figure 5.4. Card math presented by Barak.

Toward the end of the sixth day of work on this FAL, Barak presents a share-out of the card match shown in Figure 5.4.

Prior to the IRE sequences, Barak presents on a card match, explaining briefly how he got the mean, median, mode and range for a particular graph. When he explains how he got the median, he gives a standard, abstract explanation of lining up the numbers and crossing them off. He doesn't show any work for this part of the problem. A second student, Celia, asks Barak a question about the details of what numbers he crossed off, and it becomes apparent that he guessed and did not do this calculation.

5.4.1.4.2 First chain of IRE sequences

It soon becomes apparent that Barak is stuck. He can give an abstract explanation of how to find the median of a list of numbers, but does not know how to start when presented with a frequency graph.

4a. 3 IRE sequences

At this point, Ms. A steps in and spends about one minute walking the class through one way to calculate the median. During this segment, Ms. A uses three IRE sequences in rapid succession:

Ms. A: (to class) Who can help him out? If you're gonna cross off the lowest numbers and the highest numbers, what would you cross off? You cross off the low number. The lowest score. What scores would you cross off, looking at that? (to Dulce): Dulce, can you help him? Yeah. What's the lowest score?

Dulce: Two?

Ms. A: No, I see something lower than two. Estela?

Estela: It's one.

Ms. A: One. How many ones are there?

Barak: Five.

Ms. A: Five.

The three IRE sequences can be seen as stepping through the beginning of a process of how to find the median in this case, concretizing Barak's statement that you have to list out the numbers.

Notice that Barak is silent for the first part of this exchange, but answers the third IRE elicitation himself. Immediately following the previous excerpt, Barak shows signs of being unstuck. He continues the line of thinking begun by Ms. A and describes the remaining numbers in the frequency graph. Below are the next few talk turns:

Ms. A: Five. So we have to cross off five ones, right?

Barak: There's one twos.

Ms. A: Okay, so

Barak: And four threes, I mean fives.

The conversation continues in this vein for several more turns, with Barak continuing to describe his work while Ms. A tries to show approval to him while continuing her train of thought with the rest of the class. Because of interruptions from Barak as well as longer talk turns from other students, there are no further complete IRE sequences in this chain.

This first chain of IRE sequences serves a noteworthy function. It began with the framing of "helping Barak out" of being stuck and ended when Barak was unstuck. Specifically, Barak has come to see how to "read" the graph to get a list, from which he can cross off numbers according to his understanding of the algorithm for finding median. Although Ms. A did step through a sequence of steps to demonstrate the beginning of a strategy for finding the median, she did not carry this sequence through to its logical conclusion of presenting a complete strategy for finding the median. Instead, she presented just enough to get Barak unstuck, then deferred to him and other students to carry the conversation forward again. Far from squashing student-directed talk, it seems this IRE sequence actually fostered such talk by giving Barak just enough to get unstuck.

5.4.1.4.3 Second set of IRE sequences, analysis

The next IRE sequence is a singleton IRE that does not occur as part of a topically related chain, so it is less central to my analysis. For the sake of completeness, I will analyze it briefly.

4b. IRE sequence

Ms. A: You said what would end up in the middle?

Alexa: The mode?

Ms. A: Mm. (negative).

Off camera student: It's not the mode.

Ms. A: Celia.

Celia: Can I go up and (show it)?

Ms. A: Yeah, you can go up and show us. I think Barak is working on something too.

The first three sentences of this excerpt show an IRE sequence that occurred in response to a student using an incorrect vocabulary word (mode instead of median). Ms. A likely intended this IRE sequence to publicly repeat and correct the incorrect word.

It is fairly unusual for IRE sequences or chains to terminate without a correct answer being established. In fact, Mehan (1979) claims that the evaluation of a correct answer has an “obligatory co-occurrence relationship” with the student response. In this context, this IRE chain can be seen as interrupted rather than completed. In this case, Ms. A calls on Celia and may expect her to provide the correct answer of “median.” But Celia instead makes a bid to go up and present an alternate solution. Ms. A takes advantage of this opportunity and allows Celia space to present her work, ending her IRE questioning even though the correct vocabulary word “median” has not been provided. This “interruptability” of Ms. A’s IRE agenda suggests that Ms. A values student presentations more than establishing correct vocabulary, at least in this case – quite inconsistent with a knowledge transmission frame for learning.

5.4.1.4.4 Third set of IRE sequences, analysis

In this section, I analyze the final chain of four IRE sequences in Case 1. This IRE chain occurs after Barak’s and Celia’s presentations and highlights connections between the two presentations.

8a. 4 IRE sequences

An astute reader may notice that the transcript excerpt presented here is the same as the excerpt presented for Claim 1 as evidence that Ms. A was responding to student ideas in detail, with student comments added back in. One of my most surprising findings, relative to the existing literature, was that the discourse form used for the action of responding to student ideas was IRE sequences.

Ms. A walks through Barak’s written work in a sequential order that parallels Celia’s explanation, and concludes with an explicit statement that the two strategies are the same. Here is what Ms. A says:

Ms. A: So, you know what, it makes sense right, because how many ones did he write here? Let's just look at what he wrote. Cause look. How many ones did he write here?

Barak: Four.

Ss: Five. Four.

Barak: Five.

Ms. A: Five, right? And then he wrote five and he crossed it out.

Here Ms. A draws attention to the five ones written by Barak in his list of numbers (see Figure 4.1).

But then he wrote, how many sixes were in my bar graph?

Student: one.

Ms. A: One, so he crossed off a six. How many fives were there?

Student: Four.

Ms. A: One, two, three, he crossed off four, four fives right?

In this statement, Ms. A explicitly references both the bar graph and list of numbers representations in explaining that Barak crossed off one six and four fives from his list.

And then what's in the middle?

Ss: Two!

Ms. A: Two. So crossing it out is the same as listing it out, right?

Finally, Ms. A concludes that two is in the middle of Barak's list. Having shown the parallels between the two strategies, she summarizes with a statement of their equivalence, "crossing it out works the same way as listing it out."

Like the last chain of IRE sequences, this chain does function to "walk through some steps" of a procedure. In this case, the procedure is attributed to Barak. There is a trade-off here: Ms. A could have chosen to have Barak present his strategy, which would have increased his mathematical authority in some ways. Instead, she presents it herself while attributing it to him, which validates his ideas while keeping her in control of the flow of mathematical ideas. By doing so, Ms. A is able to present Barak's strategy in a way that implicitly and explicitly highlights connections to Celia's strategy. Ultimately, this somewhat supports both Barak's and Celia's mathematical authority by building on their ideas while also making explicit the connections between two mathematically important representations.

5.4.1.5 Issues of power, access and identity

So far, this case study has focused on the mathematical functions of particular discourse moves during one classroom episode. However, issues of power, access, and identity were also salient to Ms. A when she orchestrated student presentations. In March 2018, I conducted an end-of-study interview with Ms. A in which I gave her an opportunity to read and comment on a near-final draft of the above case study as well as the model of culturally and mathematically responsive teaching used in the dissertation. I told her that my analysis of her case study largely backgrounded issues of race, culture and power but briefly explained the examples in Chapter 4 that foregrounded these issues. I then asked Ms. A to share a little more with me about Barak and Celia's history as learners, and whether she was doing any work

“behind the scenes” of what I had written about to create a culturally relevant classroom for them and for her other students. Here is how she replied. Language in the following two paragraphs that is attributed to Ms. A, but not quoted directly, is based on field notes from our conversation, which was not audio-recorded.

Ms. A described Celia as an emerging bilingual student who had been in a newcomer program the previous year and was in her first year of mainstream coursework. It seemed to her that Celia was trying to find her voice:

Celia seemed to have “momentum to me, wanting to talk about the math, really pushing her English to talk about the math, asking questions, and even though it might not come out totally perfect” it was okay. “To keep that momentum going for her, I was like I want her to do more presentations.”

Furthermore, having Celia present was both supported by and helped reinforce the supportive social norms in the classroom as a whole.

“The way that class was set up, people were really supportive. She had a certain status in the class. People really respected her.”

Ms. A’s goal in having Celia present was therefore:

“Using her reputation to know that she could express herself and know that she was mathematically sound. So that was one reason that I wanted to give her some kind of platform. And I feel like it paid off, you know.”

During my classroom observations, Celia was the most frequent student to present. After the sixth grade, Celia continued her trajectory of academic success and leadership at Adams school. In the seventh grade, she used her mathematical and cultural knowledge and presentation skills to win a schoolwide business competition by inventing a product, creating a start-up, and producing, marketing and selling her product at the school and beyond.

Ms. A described Barak as an intellectual, Black young man who “always had his nose in a book.” Nevertheless, Barak was sometimes positioned by others in ways that could have restricted his ability to participate centrally in classroom activities. Ms. A said that before he entered her class she was “warned about” his reputation for anger management issues. However, her first impression of him was more positive. She saw a young man with a lot of ideas in his head that he didn’t always know how to express, but who always tried and participated 100%. Having him present at the front of the class was an opportunity for him to gain the confidence to express himself, to help his peers see him as competent, and to “create his own image of how he sees himself, not just having to get in fights and stuff like that.” Barak never got into any fights at Adams school. He was among the most frequent (top quintile) of student presenters in Ms. A’s class. Barak and his mother both later expressed to Ms. A that before this class, he had not seen himself as a math person or been successful in math class; her class was the first time he experienced success in mathematics.

Although Barak and Celia presented frequently in Ms. A’s class, classroom presentations were not the purview of a select few students, but were broadly available to all students in the class. I observed 26/34 (84%) of students present

during 16 video-recorded class days with presentations (Sayavedra & Seashore, 2017). Ms. A reported that all students presented during the year and my observations are consistent with this claim. In the case of group presentations, students were only counted as presenters if they both stood at the front of the classroom and spoke at least one sentence about the mathematics. This means that at least 84% of students, and likely the whole class, made important public mathematical contributions at some point during the school year.

5.4.2 Second mini case – IRE chain following a more extended student answer

This second case study also shows a chain of IRE sequences that “steps through” an idea. I present it to enrich the previous analysis. The novel quality of Case 2 is that the idea being “stepped through” is a student-generated strategy and the student largely maintains ownership over this idea. Specifically, the teacher consistently attributes the idea to the student and also defers to the student at several points to state or confirm what the next step should be. Ultimately in this case, IRE sequences take on a function very similar to the canonical function of revoicing in the literature: clarifying or elaborating student ideas while positioning students as authors who can ratify or dispute the teacher’s statements about their ideas (O’Connor & Michaels, 1993).

The case study took place within a March lesson that included a roughly 35-minute review of an especially difficult homework assignment that students had been assigned the previous night. The homework assignment was three problems from a previous year’s district benchmark exam, including two multiple-choice problems and a more extensive performance task. The review included time to talk about the problems in small groups, two student presentations on the two multiple-choice problems, and teacher-led whole-class discussion of all three problems. This excerpt is from the whole-class discussion of the second problem, shown in Figure 5.5.

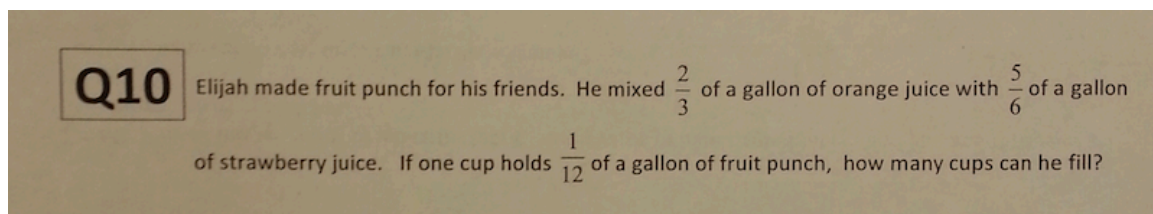


Figure 5.5. Second homework problem discussed in Mr. B-NF7.

Discussion of this problem took about 8 minutes. First, one student went to the board to present his strategy for solving the problem by converting both $\frac{2}{3}$ and $\frac{5}{6}$ to a common denominator of 12 and adding the fractions, then recognized that $\frac{18}{12}$ gallons was equivalent to 18 cups. Mr. B praised this answer, then asked if anyone did it a different way. A second student described his response, which he said was “not exactly different,” but that he had expressed the solution as a mixed number of gallons. Felicia was the third student to speak. Although she did not

present her strategy at the board, Mr. B asked her to explain it step by step and wrote the steps on the board. The exchange between Felicia and Mr. B is shown in Table 4.3.

Line	Student Talk (Speaker is Felicia unless indicated)	Teacher Talk	Coding
1	What I did was I... multiplied those by, two thirds and five sixths times two, and I added the answers and I multiplied nine sixths times two and I got eighteen twelfths. And I		
2		Whoa whoa whoa, what did you do? Because...	
3	(slower) Okay, so I did two thirds and five sixths times two, both of them	(writes $\frac{2}{3} \times$)	
4		so two thirds times (.) two (pause in writing conveys doubt about what written symbols Felicia is describing when she says "times two").	
5	Two over two	(Mr. B writes $\frac{2}{2}$)	
6		And that gave you (writes =)	IRE 1
7	Four sixths.		
8		(writes $\frac{4}{6}$) Okay. And then you multiplied five sixths (Moves pen several inches right and writes $\frac{5}{6}$) times two over two (writes $\times \frac{2}{2}$) and that gave you?	IRE 2
9	Ten twelfths.	(writes = $\frac{10}{12}$)	
10		Now what?	
11	(So four) sixths times two.		
12		And you multiplied this times two again?	
13	Yeah.		

14		Oh okay. She used four. Because he did four over four (writes $\frac{4}{4}$ on an empty part of the whiteboard). She just did two over two twice. Because what's two times two?	Teacher Initiation
15	Choral: Four.		Student Response
16		She did it twice, he did it once. And that gave her eight twelfths (writes $\frac{8}{12}$), and when you combine the two you get eighteen twelfths (writes $=\frac{18}{12}$ and circles it).	(no Evaluation)

Table 5.2. Transcript excerpt for Case 2.

In this excerpt, the student takes on the role of explaining a strategy while the teacher takes on the role of writing the student's strategy on the board. The student, Felicia, begins by explaining her full strategy verbally. The teacher, perhaps recognizing that Felicia is speaking too quickly for the class to follow, prompts her to start over and begins writing on the board.

The overall exchange here is clearly responsive to student thinking. From the first line of this exchange, there are signals that Felicia "owns," the strategy she is presenting. She is offered the floor to present a different way of doing it, signaling that the class's attention should not only be on the correctness of her answer but the details of her process. Accordingly, in line 1, she begins the exchange with an extended talk turn describing her process. Mr. B slows down the conversation and the dialogue shifts to long teacher talk turns and much shorter comments by Felicia. But Felicia is still setting the direction to a large extent. This can be seen in line 10, "now what?" in which Mr. B marks that Felicia is the only one who can say what she did next.

Two IRE sequences were identified in the episode. The first, beginning on line 5, is as follows:

Teacher Initiation: And that gave you (writes =)

Student Response: Four sixths.

Teacher Evaluation: (writes $\frac{4}{6}$) Okay.

The second IRE sequence, beginning on line 7, was

Initiation: And then you multiplied five sixths times two over two and that gave you?

Student Response: Ten twelfths.

Teacher Evaluation: (writes = $\frac{10}{12}$)

In both cases, the teacher initiation is a known-answer question, asking for the result of a fraction multiplication problem the student has indicated was part of her solution strategy. The student response is a single number, which is the correct answer. Both teacher follow-ups were considered an evaluation because this teacher typically only writes established correct answers in the flow of his boardwork; incorrect answers or answers under debate are written elsewhere on the board or not written. Also, it is not typical for him to let incorrect arithmetic steps pass without comment. Because of these established norms in the class, the teacher action of writing down the student's answer to a known-answer question without comment communicated a positive evaluation.

The function of this exchange was again to “step through” part of a strategy. However, this time, the strategy was voiced twice by the student and the teacher's utterances were restatements of her words, ratified by her providing the answers to arithmetic problems. These IRE sequences could have been replaced by revoicing with a similar function. The student remains in a role of ownership of and mathematical guidance over the strategy, evidenced by the teacher's question “Now what?” immediately after the IREs.

5.4.3 Summary: Surprisingly Responsive Functions of IRE chains

I have now presented two case studies showing the presence and surprising functions of IRE sequences in responsive class discussions. First, I argued that the main case study episode was responsive to students' thinking. I demonstrated that students' thinking was elicited and responded to in some depth. Specifically, students presented both standard and novel strategies for finding the median and the teacher compared features of their strategies across two important representations.

IRE sequences, strictly interpreted, were both frequent and integral to the exchange. IRE sequences made up about one quarter of all teacher talk turns in the episode. Furthermore, these IRE sequences were central, not peripheral, to the main mathematical flow of ideas. If the IRE sequences were removed (and not replaced by something else serving a similar function), it is unlikely that either Barak's revision of his standard strategy or the detailed comparison between Barak's and Celia's strategies would have happened. These were some of the most mathematically responsive parts of the case study.

Finally, I analyzed the functions of both topically related chains of IRE sequences that appeared in my data as well as one IRE chain from a second case study. Similarly to the IRE literature, I found that IRE sequences do prompt students to say something the teacher already knows, and chains of IREs do step through a sequence of ideas in a way directed by the teacher.

Nevertheless, the impact of these IRE chains on the meso-level discourse is more complicated than portrayed by the literature. “Stepping through” a sequence does not constrain students to mathematical passivity as is often assumed in the literature. Instead, in the first chain of IRE sequences in Case 1, Ms. A “steps through” the beginning of a student strategy to get Barak unstuck. Barak is then able to resume control of and authority over the solution strategy. In the second chain of IRE sequences in Case 1, “stepping through” student solutions allowed the teacher to draw a detailed parallel between two student strategies. This deepened, rather than lessening, student ownership of the strategies. In Case 2, the teacher “stepped through” a student’s idea in order to elaborate that idea but with the student maintaining authority for the mathematical direction of the conversation. This is a typical function of revoicing that was filled by an IRE sequence in this case.

5.5 Discussion

I have proposed that IRE sequences are a common routine that may have very different learning affordances in different contexts. In retrospect, this proposal should not seem too surprising. Let’s consider the example of another common discourse format in teaching: teacher exposition. One can easily call up a mental image of a teacher who lectures over students’ heads in a way that does not provide students with adequate opportunity to make sense of the content and is ultimately disempowering to students. Yet few teachers or researchers would claim *a priori* that lecture always shuts down student thinking. We can imagine lectures that command listener attention or inspire creativity. Some teachers use mini-lectures following a period of group work in order to respond in detail to the most interesting student ideas that emerged during the group work. In these cases, the teacher exposition would be one part of a broader activity system that was responsive to student ideas. Lecture alone is not enough to produce these results, yet it can form a critical piece of the puzzle.

My results have significant implications for both the theory of classroom discourse and the practice of supporting teacher learning. A main theoretical implication is that connections between the micro and meso levels of discourse are more complicated than previously recognized in the literature. There is a need to revisit the literature about how micro-level discourse moves create student roles in conversation. The literature says that revoicing positions students as authors (productive disciplinary engagement frame) while IRE sequences reinforce the idea that “the student proposes, the teacher disposes” (knowledge transmission frame). In contrast, my analysis shows IRE sequences supporting a PDE frame. I suggest that as a field we need to take a step back from our assumptions about the effects of various micro-level discourse structures. Instead, we should first look for good student talk, such as extended explanations and students critiquing each other’s ideas. We should be open to look for student talk anywhere, and, wherever we find it, only then look for what causes and supports this talk. We can expect that there will be a broad set of tools and teacher moves that can support good student talk in different situations.

A main practical implication of my research is that telling teachers “IREs are bad” is unlikely to be successful. Beyond making teachers defensive, this approach does not allow for gradual growth in teaching practices or making new meaning by tweaking existing practices. Instead, I recommend professional development collaborations with and between teachers that maintain focus toward good student talk over an extended period of time. For example, teachers and other collaborators can ask, “How can we get students to explain their ideas in detail?” or “Okay, students are talking, how can we get them to ask each other questions?” In order to reach these goals for student talk, teachers can be supported to gradually build broad toolkits of discourse moves. In this context, the goal is not to eliminate or replace IRE sequences but to use these or any other teacher moves appropriate to the context that can promote the desired student talk. Reflecting on practice (alone or with others) and how to tweak moves to create more opportunities for student talk can be a powerful way for teachers to expand these toolkits.

6 Developmental trajectory of one culturally responsive teacher toward increasing mathematical responsiveness

6.1 Introduction

This chapter aims for one of the most ambitious goals of a continuity approach to learning: tracing an individual's pathway to major intellectual accomplishments, noting specific resources that were used and adapted along the way. In this case, the individual in question is an experienced mathematics teacher, Mr. B. The "major accomplishment" here is a shift toward increasingly mathematically responsive teaching practices, that is, from a *knowledge transmission frame* toward a *productive disciplinary engagement frame*.

This chapter documents Mr. B's shifts in teaching practice over the course of one year. At the beginning of the year, his practice reflects a frame close to a knowledge transmission frame and, by the end of the year, his practice shifts to have some aspects (though not all) consistent with a productive disciplinary engagement frame. Specifically, Mr. B takes up and adapts a new, high-leverage practice: inviting students to the front of the room to present their thinking to the class. These student presentations open the discourse in his classroom to create significantly more space for students to take on roles as authors and critics of mathematical ideas, with the mathematics under consideration growing correspondingly richer, important elements consistent with a productive disciplinary engagement frame.

A continuity approach to learning implies that this is not an all-or-nothing change. Mr. B's adoption of a new, high-leverage practice is an important part of the story, but we do not expect his prior practices to disappear or be categorically replaced by this new practice. Snapshots of Mr. B's teaching before and after the change are important to document progress, but far from the whole story. What matters is a series of smaller and specific (not necessarily indelible) shifts as Mr. B adapts both his repertoire of prior practices to serve new functions, and integrates and adapts the new practice as well.

I will show that the student presentation practice is high-leverage for teacher learning in two ways. In the short term (the first time the practice is implemented), it creates substantially more "air time" for student talk than was previously present. At first, the presentations are an isolated activity separate from the other work of the class: there is more of a focus on presentation skills than the math content of the presentations, the things students say in this "air time" are not especially rich

mathematically, and students do not yet take up positions of either mathematical authorship or critique.

Nevertheless, as Mr. B sustains his new practice throughout the year, each instance of the activity creates new learning affordances for both teacher and students. By the end of the year, the classroom presentation activity is both mathematically richer and more integrated with Mr. B's other teaching practices. Students take up positions of authorship and share novel strategies and connections as part of a broader conversation about important mathematical ideas. To a more limited but still significant extent, Mr. B integrates other discourse practices such as questioning students, restating and generalizing ideas with the student presentation activity. This integration of teaching practices enriches the mathematical conversation and increases the opportunities for students to take on roles as authors and, to some extent, critics of mathematical ideas, consistent with a productive disciplinary engagement frame.

6.2 Background

This dissertation characterizes elements of classroom discourse as they fit with either a knowledge transmission frame or a productive disciplinary engagement frame. Throughout the dissertation, the focus is on the opportunities for students to take on roles as authors and critics of mathematical ideas, and how this enriches mathematics learning opportunities. Chapter 5 argued that certain teacher discourse moves, such as IRE sequences, can be purposed to reinforce elements of either a knowledge transmission or productive disciplinary engagement frame, depending on the meso-level context and, in particular, the positioning of students and their mathematical ideas.

This chapter takes a teacher learning approach to the same issues. A partial teacher learning trajectory of one experienced teacher is shown over the course of one school year, as the teacher's practice shifts from close to a knowledge transmission frame toward a productive disciplinary engagement frame, and in particular as he adapts several of his teaching practices to position students' mathematical ideas as an increasingly central part of the learning activity of the class. A few preliminary comments about the nature of teacher learning are in order.

The teacher in question, Mr. B, adopts a new and potentially high-leverage teaching practice early in the school year: having students present their thinking at the front of the class. Adopting this high-leverage practice is not the end goal, nor is it disconnected from the teacher's prior practices and orientations. The concern here is not just with the teacher adopting an isolated new practice, but making a more fundamental shift from a knowledge transmission frame toward a productive disciplinary engagement frame. The new student presentation practice does not accomplish this shift all at once. In fact, one of the main analyses in this chapter examines an instance where the new student presentation practice was implemented, but the framing remained primarily knowledge transmission. Nevertheless, the new student presentation practice is valuable because it creates

new circumstances in the classroom – students saying a lot more about their mathematical processes and reasoning in addition to answers – which themselves create new learning opportunities for the teacher. Over time, as the teacher gradually responds to these new learning opportunities and creates yet a different set of classroom circumstances, the discourse moves substantially toward a productive disciplinary engagement frame.

It should be noted that, although this chapter uses the term “teacher learning,” all of the classroom discourse analyzed is considered to be a joint accomplishment between the teacher and the students. Teacher talk and student talk can be helpful to separate analytically, but in reality are deeply interconnected. Relatedly, skepticism might be warranted about analyzing “teacher moves” in a way that is divorced from how students take up and respond to these moves. For example, a lot of professional development encourages teachers to ask open-ended questions (teacher focus). However, it is at least as important whether students give open-ended answers (student focus). In this chapter, student talk is analyzed first because it gives the clearest indication of how students are engaging with the activity and which of the potential learning affordances are actually being taken up by students. Afterward, the teacher moves that supported or limited the observed student learning behaviors are also analyzed. Thus, while the shifting discourse structure of the classroom is conceptualized as a joint accomplishment between teacher and students, a result of the analysis is to show changes in teaching behaviors that help produce these shifts with these students. For simplicity, these changes in teaching behaviors can be described as “teacher learning,” provided the contextual nature of this learning and of the eventual accomplishment is understood.

I will argue that Mr. B makes substantial progress from a knowledge transmission frame toward a productive disciplinary engagement frame. The working definitions of these two frames, described in Chapters 1 and 2, are reproduced here for the reader’s reference:

	Knowledge Transmission Frame	Productive Disciplinary Engagement frame with explicit attention to issues of power
Mathematics Content	Mathematics content as experienced by students in this frame tends to be organized around students’ development of particular skills and procedures in a fixed order. It provides few opportunities for students to develop	Students begin from their prior knowledge and often discuss multiple approaches to solving a problem. Because students have opportunities to connect, critique, and revise ideas, they gradually refine their prior knowledge into powerful and general disciplinary understandings. Power relationships are negotiated in

	experience and fluency with higher-level mathematical practices such as reasoning and problem solving.	ways that position the multiple mathematical knowledge bases of nondominant students and communities as central to the learning process.
Typical discourse patterns	Typical discourse patterns in this frame include: teacher exposition, chains of IRE sequences, and student requests for the teacher to slow down or repeat something that was confusing.	The discourse patterns consistent with this frame can be quite varied. There must be space for students to share their thinking in some depth. There are many possibilities for discourse structures that support the gradual refinement of student thinking in connection with important disciplinary ideas. Power relationships are negotiated in ways that welcome nondominant speech patterns and forms of expression and support the development of cultural competence.
Teacher and Student Roles	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are typically available only to the teacher (and textbook authors, in absentia).	In this frame, the mathematical roles of authoring, building on, connecting, critiquing and revising ideas are made available to students and the teacher. Power relationships are negotiated in ways that position and support nondominant students to act as authors and critics of powerful ideas, thus expanding their learning opportunities.

Table 6.1. Summary of the KT and PDE frames.

How do student presentations align with these two frames for learning? This is a complex question. Chapter 5 argued that micro-level discourse structures could serve multiple functions depending on the meso-level context in which they are implemented. Specifically, IRE sequences could either reinforce or limit the authorship roles available to students depending on how they functioned in meso-level contexts. This chapter argues that student presentations are similarly complex. Student presentations do provide an extended opportunity for student talk that is atypical of a knowledge transmission frame. Nevertheless, one can imagine presentations being used for a summative purpose, to test the presenter's knowledge about a standard procedure. Or, a student could be called on to give a "model" presentation to the class because the teacher has checked that the student has correctly completed a problem. In either case, the student would have only partial ownership of the ideas being presented, and the authority to critique these ideas would rest entirely with the teacher. In short, the purpose of presentations may be consistent with many aspects of a knowledge transmission frame. One can

also imagine various functions of student presentations consistent with a productive disciplinary engagement frame, such as: presentations that raise important questions for the class to consider; presentations that suggest multiple strategies for solving a problem, which are later connected together; or presentations justifying positions in a debate. Finally, one can imagine some functions of student presentations that have clear elements of both a knowledge transmission frame and a productive disciplinary engagement frame, and hence land somewhere in between. For example, a student who has developed familiarity with a certain mathematical procedure could present about this procedure in order to help other students who are struggling. This would position the presenting student in a teacher-like role, but the other students as passive recipients of knowledge, and would not necessarily enrich the mathematics being discussed.

In short, how any practice aligns with a frame is a function of its use. The present chapter takes up the question empirically in the case of student presentations. The shifting functions of student presentations form a significant part of the shift in Mr. B's teaching practice toward a productive disciplinary engagement frame.

The analysis in this chapter characterizes three stages of Mr. B's learning trajectory: his baseline teaching practices, initial implementation of the presentation activity structure, and continued use and adaptation of the activity structure throughout the year. In each stage, the roles, discourse structures, and mathematical content of the episodes are characterized and their fit with a knowledge transmission or productive disciplinary engagement frame is analyzed.

6.2.1 Anticipation of claims

First, I will characterize Mr. B's baseline teaching practices. The roles, discourse structures, and mathematics in the baseline episodes are consistent with a knowledge transmission frame. Specifically, during these episodes, students are expected to take on a role of demonstrating their knowledge and the teacher evaluates them as correct or incorrect. The primary discourse structure is IRE sequences plus some open-ended questions to those students who had already given correct answers. The mathematics is narrowly focused on a few procedures.

Second, I will characterize the first episode in which Mr. B tried the new presentation activity. The discourse structures in this new activity show significant alignment with a productive disciplinary engagement frame: students are invited to take much longer talk turns and describe their process for solving problems. However, the roles and mathematics remain very close to a knowledge transmission frame. All of the students use the same procedure to solve the problems, and their presentations focus on a description of this procedure. As a result, the presentations do not really position students as authors of their own mathematical ideas because the presentations function primarily to demonstrate students' knowledge of a single procedure that they had previously been expected to internalize. Additionally, there is no talk by either the teacher or the class about the mathematical ideas in the presentations. Therefore, neither the teacher nor the class

takes on a role of connecting, critiquing or revising mathematical ideas. This is true even when the presenting students make arithmetic mistakes that strongly violate common sense in the problem context. The mathematics of the lesson focuses narrowly on a single procedure, to the exclusion of considering contextual and conceptual ideas central to the task.

Third, I will show how Mr. B's subsequent use and adaptation of the classroom presentation activity moved significantly toward a productive disciplinary engagement frame. By the end of the year, students are positioned as authors who present a variety of different ideas and their own accompanying sensemaking to the class. The teacher sometimes takes on a role of connecting and critiquing student authored ideas, and there are a few brief instances where students take on a connecting or critiquing role as well. The discourse structure is still primarily long student talk turns (>80% of student talk time), but includes some additional types of talk such as teacher questioning of students and teacher summary of student ideas. The mathematical tasks used are rich, with affordances for multiple representations and connections. Some of these affordances are taken up by the teacher and students, while other affordances are missed or scaffolded away as the task is enacted. Overall however, the mathematical ideas being discussed are far richer than in either the baseline or first presentation episodes.

6.2.2 Chapter Overview

In Section 6.3, Methods, the methods used for this chapter are reviewed. This review includes an introduction to a quantitative, talk-turn-level coding scheme that is used throughout the chapter to capture some aspects of the shifts in discourse structure in Mr. B's class. The coding scheme, called Talk Quality Quantified (TQQ), is inspired by the EQUIP coding scheme developed by Reinholz & Shah (2018). The TQQ coding scheme gives a very useful summary of what the student talk looked like in each episode. However, it does not adequately capture either the roles that students took up or the richness of the mathematical content being considered. Therefore, the quantitative analyses are supplemented with qualitative analyses of selected episodes that highlight these issues. Thus, Section 3 discusses in some detail the origins of the EQUIP coding scheme, the differences between this project and the original intended use of EQUIP and how these differences led to modifications of EQUIP to create TQQ, what TQQ does and does not tell us about the presence of a knowledge transmission or productive disciplinary engagement frame, and how qualitative analyses are used to supplement these quantitative analyses.

In Section 6.4, Results, three stages of Mr. B's learning trajectory are characterized: (6.4.1) his baseline teaching at the beginning of the year, which is a near fit to a knowledge transmission frame; (6.4.2) his first use of the new student presentation activity, which leads to a large shift in discourse structure, but still aligns primarily with a knowledge transmission frame in terms of the roles and mathematics; and (6.4.3) his continued use and adaptation of the presentation activity throughout the

year, which shifts significantly toward a productive disciplinary engagement frame in terms of the discourse structures, roles and mathematics.

Section 6.5, Summary, summarizes the important findings of the chapter.

6.3 Methods

The data analysis includes three passes through the data. Each pass focuses on a smaller subset of the data, allowing for the richest analyses of a few brief excerpts to be situated within a summary of trends across the whole dataset. The first pass involves data selection and content logging. The second pass involves quantitative analysis using TQQ. Because TQQ was not used for the earlier dissertation chapters, some time is spent explaining the coding scheme. The third pass involves qualitative analysis of selected focal episodes. Table 6.2 summarizes the kinds of evidence provided by each of the three analysis passes, as they relate to demonstrating that the classroom has either a knowledge transmission frame, productive disciplinary engagement frame, or elements of both.

	Knowledge transmission frame	Productive disciplinary engagement frame
Evidence expected in the first analysis pass (content logging and selecting whole-class episodes)	Content log gives a general sense that the mathematics is skills-oriented, that only one approach is considered per problem, that the teacher guides this strategy, and that the discourse structures do not allow much space for student talk.	Content log gives a general sense that the mathematics is rich, that multiple strategies or approaches are considered per problem, that discourse structures create significant space for students to share their sensemaking.
Evidence expected in the second analysis pass (TQQ coding)	<p>Most teacher elicitations have a single expected answer (WHAT code).</p> <p>Most student talk is coded as very short (<5 word code) and answer focused (WHAT code).</p> <p>Most student talk is immediately evaluated by the teacher (evaluation codes prevalent).</p>	<p>A significant fraction of teacher elicitations are coded as WHY elicitations.</p> <p>A significant fraction of student talk coded as long (21+ words) and contains justification (WHY code).</p> <p>Most student talk is not immediately evaluated by the teacher (few or no evaluation codes).</p>

	The primary function of medium chunks is to establish a single correct answer and/or procedure.	The primary functions of medium chunks are to enrich, refine, and connect ideas. Many of these ideas originate with students.
Evidence expected in the third analysis pass (qualitative analysis)	<p>Mathematical ideas spoken by students closely reproduce what the teacher has recently modeled.</p> <p>Meso-level discourse structures position students as passive recipients and reproducers of knowledge of knowledge (e.g., the teacher uses chains of IRE sequences to lead guided practice).</p> <p>The teacher is the only one to critique student talk.</p>	<p>Long talk turns contain mathematical ideas beyond what the teacher has recently modeled.</p> <p>Students are positioned as authors of important mathematical ideas, e.g., through attribution of their ideas. (“What Luciano did was compare numerators.”)</p> <p>Student ideas are critiqued, connected, and refined by students and the teacher.</p>

Table 6.2. Summary of expected evidence from the three analysis passes.

The first pass, content logging, is expected to give an overview of all elements of framing, but with limited precision and detail. The second pass, TQQ coding, offers wide coverage of all whole-class episodes, with a fairly narrow set of indicators about the frame in play. Specifically, EQUIP includes codes about the micro-level discourse structures, such as length of student talk turns, and rough categories of quality of teacher and student talk that allow a distinction between talk focused on answers (WHAT), process and procedures (HOW), or justification and connection (WHY). Since EQUIP codes are applied at the micro level (talk turn grain size), they provide less insight about either the functions these discourse structures, meso-level function of these talk turns or the richness of the mathematics content. In creating TQQ, the coding scheme was extended to provide somewhat more information about function at the micro and meso levels. Nevertheless, supplementary qualitative analysis is needed to understand the meso-level discourse structure and function, as well as the roles and mathematics. Therefore, the third pass of data analysis elaborates and expands on meso-level issues: the mathematical richness of the task as enacted in the classroom, the extent to which students are positioned, and take up, roles as authors and critics of mathematical ideas, and the extent to which students’ mathematical ideas are taken up by others.

Throughout the results section, results from TQQ coding and qualitative analyses at the meso-level are triangulated to give a more complete picture of which elements of a knowledge transmission or productive disciplinary engagement frame are in play.

6.3.1 First Pass: Data Selection

The primary data source for this chapter is video of eight whole-class episodes from Mr. B's classroom.

Mr. B was selected out of four focal teachers (Ms. A., Mr. B., Mr. X., and Mr. Y) because he showed the clearest case of a teacher integrating a new teaching practice. Mr. B was in his 19th year of teaching when starting the study, but reported that he had never had students give presentations before this year. Ms. A and Mr. X had both used student presentations before the study year, and continued to use familiar routines during the year. Mr. Y had some elements of a productive disciplinary engagement frame already in place at the beginning of the year. He experimented with various new teaching strategies including student presentations, but showed a less clear trajectory in the sense of sustaining and adapting a single new practice.

The analysis focused on whole-class episodes because they give a window into how norms are operating in classroom activity in general. During whole-class episodes, there is one primary mathematical conversation going on, rather than many conversations around the room. Although some student thinking is still outside of the public view, there is at least a nominal single thread to the mathematical activity, so the mathematical learning affordances for students are easier to track analytically than in individual and small group activity structures. Regarding roles, teachers often use whole-class time to influence norms and roles for the class as a whole, which can carry over into small group work. Regarding classroom discourse structures, it is most challenging for students to get "air time" during whole-class time and to be positioned as authors of mathematical ideas during this time. Thus, whole-class activity is easiest to analyze, most directly influenced by the teacher (because the teacher is part of the conversation for the whole time rather than splitting attention between many groups), and still sets a high bar for what it means to position students as authors and critics of mathematical ideas.

In the first phase of data selection, I watched or skimmed once through all 19 hours of classroom video from Mr. B's classroom, and created a content log of what happened each day. In this analysis pass, the start and end times of each whole-class episode were also recorded. Typically, it was straightforward to distinguish between whole-class and other activities (e.g., small group or individual work). Occasionally, whole-class episodes were punctuated by short segments of individual work. For example, a period of teacher exposition might be broken up by a problem for students to try individually and then go over as a class. In these cases, if students spent more than about two minutes in any activity structure, this was considered a new episode. For example, 10 minutes of teacher exposition, followed by three

minutes of individual work, followed by 10 minutes of whole-class discussion would be split into three separate episodes, with both whole-class episodes flagged for future analysis. However, 10 minutes of teacher exposition, followed by one minute of individual work, followed by 10 minutes of whole-class discussion would be logged as a single, 21-minute, whole-class episode.

Most observed class days contained only one whole-class episode, which are referenced by the name of the observation day as a whole (e.g., NF1). A few observed days contained multiple whole-class activities, which are referenced by letter (e.g., NF2a, NF2b, and NF2c). Through this process, 18 whole-class episodes were identified.

6.3.2 Second Pass: TQQ coding

In the second pass, all whole-class episodes were coded using TQQ, modified from part of Reinholz & Shah's (2018) EQUIP coding scheme.

EQUIP, which stands for Equity Quantified In Participation, gives a measure of the length and quality of talk turns by individual students. In the original scheme, the intention of the coding scheme was to support teachers who wished to do the difficult work of addressing status issues that “push into” classrooms without explicit, sustained and skillful teacher attention to counter them. EQUIP collects data about the length and quality of student talk, as well as the complexity of questions posed by the teacher to different students, and allows compilation of this information by any demographic marker that the teacher thinks may be related to processes of marginalization in that school's social context (e.g., race, gender, English Learner or IEP designation). In this way, the coding scheme provides teachers working for equity with concrete, quantitative information about the learning opportunities they are providing to students from different demographic groups, in order to facilitate a process of continual learning and improvement about addressing status issues. EQUIP can be used to create statements such as “The teacher asked more high-level questions of English Learners now than she did three months ago, and students took longer talk turns and provided higher-quality responses in answer to her questions.”

This study has a somewhat different purpose. Notably, TQQ codes are not separated by demographic groups. Instead, selected codes from this scheme are used and modified to give an overview of the nature of the mathematical discourse between the teacher and the class as a whole. Even without using the feature of breaking down codes by demographic groups, the types of student talk and learning opportunities measured by EQUIP are reasonable micro-level indicators of the difference between a knowledge transmission frame and a productive disciplinary engagement frame. EQUIP is also efficient to use, since codes can be applied directly to video data without transcribing the video, requiring only 2–5 times real-time. For these reasons, EQUIP was a promising starting place for a coding scheme that could practically be applied broadly across all the data and give an initial indication of

when students had opportunities to speak at length about mathematics and/or offer high-quality talk (e.g., explanation and justification vs. answer-focused talk).

The main modification of EQUIP in creating TQQ was to add some notes and chunking that describe the meso-level activity structures used in each episode, such as “teacher asks students for their answers first, then asks only students with correct answers to explain their process and/or reasoning.” This section gives an overview of the EQUIP scheme, small modifications made to a few of the micro-level codes, and the chunking process.

6.3.2.1 Organization of Codes

In TQQ, discourse is assumed to be in a T-S-T format: the teacher elicits student talk, the students respond, and the teacher provides an evaluation and/or feedback of the student response. This means that the coding scheme may not apply to discourse structures where students respond to other students directly, without teacher talk turns in between. However, direct student-student talk was rare in the dataset. Even when students replied to each other’s ideas, there was typically a teacher talk turn between the student talk turns that facilitated the student-student conversation. This also means that the coding scheme did not attend in detail to long teacher talk turns (e.g., teacher exposition). The focus was on talk sequences that included at least some student talk. This focus is appropriate to the goal of revealing the ways that students were invited to participate in the mathematical conversation.

Accordingly, talk turns are divided into three columns: (1) teacher Elicitations, (2) student Responses, and (3) teacher Evaluation/Feedback turns. All student talk is placed in the Response column. This means that any student-student talk turns that do exist are captured as multiple Response turns in a row, which is likely to de-emphasize and somewhat misrepresent any student-student discussions if they do occur. However, this allows the assumption that student-student talk is rare to be tested. Long teacher talk turns, notably teacher exposition, does not fit in any of the three columns, and instead is placed across all three columns. Thus, long teacher talk turns are also de-emphasized by the organization of the codes. Teacher exposition turns are fairly common in the dataset. However, de-emphasizing them is appropriate given our focus is on the joint accomplishment of student learning opportunities and student talk. The coded talk includes all student talk turns as well as those teacher talk turns that directly elicit or give feedback on student talk.

6.3.2.2 Talk Type Code

After organizing the talk turns, both teacher and student talk is further coded based on talk type. Talk type codes distinguish answer-focused talk from descriptive or explanatory talk. Talk type was one of the codes from EQUIP that was kept intact in TQQ and eventually became central to the analysis in this chapter.

As in EQUIP, teacher Elicitations and student Responses are both assigned one of four talk type codes:

- WHY: justification or explanation;
- HOW: description of a method without justification or explanation;
- WHAT: talk focused on answers;
- OTHER.

For a full elaboration of inclusion and exclusion criteria for these codes, see the codebook in Appendix B.

In the EQUIP scheme, the talk type codes are considered to be of hierarchically increasing quality. The presence of WHY talk is the strongest indication of high quality mathematical learning opportunities for students, followed by HOW talk. Answer-focused (WHAT) and OTHER talk are less mathematically valuable in and of themselves, and are considered of lower value in this scheme.

If a student uses multiple kinds of talk within a talk turn, for example a description (HOW) mixed with justification (WHY), only the highest-level talk is coded. For example, a description (HOW) mixed with justification (WHY) would be assigned a WHY code only.

What would be an ideal proportion of talk types? This is a complex question. It is clear from the literature that the presence of student justification or explanation (WHY talk) is a strong indication of high quality mathematics learning for students. For example, Lampert (1990) gives an early example of student dialogue that is almost entirely focused on justification, rather than answers. If EQUIP codes were applied to the student dialogue in that paper, almost all student talk would be coded as WHY. However, Lampert's classroom episode was selected to serve as an exemplar of WHY talk, and it is not yet clear whether this very high proportion of WHY talk can or should be sustained throughout a school year.

However, the literature is mixed about the value of other talk types. It is clear that a classroom consisting of only answer-focused (WHAT) talk has limited learning affordances for students. But there is some evidence that a mixture of WHAT, HOW and WHY talk can support more students to access the desired WHY talk. For example, Nathan and Kim (2009) describe a process of teacher regulation of questioning level, in which a substantial fraction of lower-level questions was used to scaffold broad participation in higher-level questions requiring justification and reasoning. Nathan and Kim report 25% Choice and 47% Product Elicitations (both would be coded as WHAT in TQQ), 10% Process (HOW) Elicitations, and 18% Metaprocess (WHY) elicitations in a four-day, mathematically responsive and inclusive lesson. The teacher regulated the difficulty level of his/her questioning based on student responses. When students responded incorrectly, the teacher asked a lower-level question, and when students responded correctly, the teacher asked a higher-level question. The end result was broad participation in

consideration of Metaprocess (WHY) questions, but these questions were only about 20% of teacher elicitations.

In short, in a knowledge transmission frame, one would expect primarily WHAT talk from students and WHAT elicitations from teachers. In a productive disciplinary engagement frame, one would expect WHY talk to be central to the mathematical activity. I am not aware of any literature reporting a lower fraction of WHY talk in a responsive discussion than the Nathan and Kim study, so one would expect at least 20% WHY talk, and possibly much higher, in a productive disciplinary engagement frame. Along a trajectory toward a productive disciplinary engagement frame, one would expect to see an increasing fraction of WHY talk and at least some decrease in WHAT talk. One could possibly see an increase in HOW talk, and/or repurposing of WHAT talk to scaffold and support the new WHY talk.

6.3.2.3 Student Response Length

In a knowledge transmission frame, student talk turns are typically very short (<5 words). As the teaching moves toward a productive disciplinary engagement frame, one would expect student talk turns to increase substantially in length.

In the EQUIP scheme, student responses are additionally coded as one of the following response lengths: <5 words, 5–20 words, or 21+ words. Student talk was not necessarily transcribed to get an accurate word count. It was not too difficult to determine in real time whether the student talk was only a few words (<5 words), about a sentence (5–20 words), or two or more sentences (21+).

In TQQ, because of my focus on student presentations, one new code was added for student talk turns over 40 words long. This allows an additional distinction between student talk that is two or three sentences long (21–40 words) or many sentences (41+ words). In practice, almost all student talk coded 41+ words occurred when the student was standing at the front of the class giving a presentation. Some student presentations were 21–40 words, but a majority of student presentations were 41+ words long.

6.3.2.4 Teacher Evaluations

In EQUIP, Teacher Feedback turns are coded as Evaluative or Non-Evaluative. Only explicit evaluations are coded as Evaluative.

In my data, the teacher made heavy use of implicit evaluations, such as writing correct answers on the board or asking another student to “help” someone with an incorrect answer. Therefore, a code was added for implicit evaluations.

For example, “That’s correct” or an affirmative repetition of the student’s answer would be coded as evaluations in EQUIP and explicit evaluations in TQQ. Implicit evaluations in TQQ were based on the conversational norms of the classroom. For example, some teachers only write correct answers on the board or only “move on” from a question once it has been answered correctly. In these cases, the acts of

writing a student’s answer on the board and/or “moving on” would be coded as implicit evaluations. Because these codes are more subtle, the inclusion criteria used were also tracked.

The non-evaluative code was used if no explicit or implicit evaluation was present.

6.3.2.5 Chunking

In addition to EQUIP coding which focuses at the micro level, the meso-level discourse structure was attended to by chunking the transcript at three grain sizes.

Instead of “participation sequences” used by EQUIP, all talk turns were coded and the data was then chunked by small chunks. Small chunks correspond to single IRE sequences (or variations that still fit the elicitation-response-feedback format). These small chunks may include more than three talk turns if the teacher chooses to re-elicite a version of the same question, for example if the teacher is dissatisfied with the initial response to the question. This follows Mehan’s (1979) definition that “the reflexive structures that tie interactional sequences together are wide ranging, and not limited to adjacently occurring utterances.” In particular, the obligatory co-occurrence relationship between parts of an IRE sequence can be satisfied over three or more turns of talk. See Table 6.3 for an example where seven talk turns are needed to find a fully satisfactory answer to a single teacher Elicitation. This example would be considered a single IRE sequence by Mehan. In this coding scheme it is considered one small chunk.

Turn	Elicitation	Response	Evaluation/Feedback
BNF7-1-72	WHAT		
BNF7-1-73		WHAT <5	
BNF7-1-74	Re-elicite for additional details		
BNF7-1-75		WHAT <5	
BNF7-1-76	Re-elicite for additional details		
BNF7-1-77		WHAT <5	
BNF7-1-78			Explicit eval

Table 6.3. Example of a small chunk longer than than three talk turns.

Medium chunks correspond to a generalization of *topically related chains* of IRE sequences. They are made up of one or more small chunks on a single topic. For example, if a class discusses two alternate strategies for solving a problem, exploration of each strategy might correspond to one medium chunk.

Small and medium chunks were coded by the type of student talk facilitated (WHAT, HOW, or WHY). For example, if a teacher described a solution method in detail (teacher HOW talk), but only asked students for the answers to calculations (student WHAT talk), the chunks would be coded based on the primary type of student talk

invited and enacted (WHAT). This coding of chunks serves as one indicator of how framing and corresponding student roles are constructed between the micro and meso levels. For example, in a productive disciplinary engagement frame, one would expect to see students taking on a role as justifiers and critics of mathematical ideas. One would expect to see a significant fraction of WHY codes, both at the micro and meso levels. On the other hand, lack of alignment between types of talk at either the micro or meso level could mark very interesting moments for closer study. For example, misalignment between teacher and student talk types within a small chunk could mark a negotiation of the type of talk students should be sharing. Misalignment between talk types at the micro and meso levels could indicate that micro-level talk moves are being used for atypical purposes.

The largest chunks concerned the mathematics problem under consideration. These chunks were time stamped but not coded in any further way.

6.3.3 Third Pass: Qualitative analysis of selected focal episodes

Initial TQQ analyses of all whole-class episodes were also used to select focal episodes for more detailed qualitative analysis. For this chapter, the focus is on student presentations in Mr. B's class, as well as "baseline" data without presentations for contrast.

I define student presentations as any time students are physically positioned at the front of the classroom and speak about their mathematical thinking. A student explaining their thinking from their seat would not be considered a presentation. A student writing their homework solution on the board and then returning to their seat without speaking about the mathematics would also not be considered a presentation. It turns out that TQQ codes are a fairly reliable identifier of student presentations. All but one student talk turn coded as 41+ words was a presentation. A majority of student presentations were coded as talk turns of 41+ words in TQQ, though a significant minority were coded as 21–40 words.

These student presentations occurred on five out of 16 observed class days. Four of these five episodes were adaptations of the same student presentation activity structure. In this activity structure, which occurred after students had worked for some time in small groups on a challenging problem, all small groups were called to the front of the classroom to present their work. The teacher gave groups a few minutes to prepare what they would say, then called up one group after another with fairly strict time limits (typically two or three minutes) for the group to present.

There was one additional whole-class episode that included student presentations. It differed from the presentation structure of the other four episodes in a number of ways (e.g., only a few students participated in presentations rather than all groups). A detailed analysis of that episode would show it to be consistent with the main line of analysis presented here, but following through on all the interactions would

complexify the story in unnecessary ways. This episode has therefore been excluded.

To analyze Mr. B's baseline practice, all four of the recorded whole-class episodes were selected that occurred before the first instance of the student presentation activity. These four episodes also have similar TQQ code profiles and similar meso-level discourse structures to each other.

Once focal episodes were selected, qualitative analysis focused on indications of a knowledge transmission or productive disciplinary engagement frame not covered by TQQ. Since TQQ primarily provides data about the micro-level discourse structures, supplementary qualitative analysis focused on the meso-level functions of these discourse structures as well as the mathematical content and roles.

Analysis of the meso-level discourse functions includes the sequence and routines that inform the micro-level talk turns. For example, if TQQ analysis revealed that student talk in an episode was 50% WHY and 50% WHAT, the meso-level qualitative analysis would give more detail about the ordering of this talk. Did the WHAT talk appear to scaffold and provide broader access to a central WHY question? Or, conversely, did the teacher initially invite students to engage in WHY talk but then scaffold away the challenge, perhaps accidentally?

Analysis of the mathematics content began with an examination of the mathematical task for affordances for sensemaking. However, the written task materials were of less importance than the enactment of the task in the classroom and how the task's potential affordances were taken up. Some indicators of mathematical content consistent with a productive disciplinary engagement frame include: consideration of multiple strategies; connections between multiple strategies and/or multiple representations; and the extent to which procedures are unpacked and adapted using relevant conceptual and strategic reasoning.

Analysis of teacher and student roles paid special attention to indications of student positioning as authors, critics, and connectors of mathematical ideas. Some indicators of teacher and student roles consistent with a productive disciplinary engagement frame include: pronouns indicating student ownership of ideas (e.g., "his strategy was..."), take-up of student ideas by peers and/or the teacher, and meta comments and connections between ideas made by students (e.g., "I did basically the same thing except...").

6.4 Analysis and Results

A new teaching practice that Mr. B worked on throughout the year was having students come to the front of the classroom to present their ideas. Mr. B reported in an interview that, in his 19 years of teaching, he had never had students give presentations prior to this year. However, during a math department goal-setting conversation at the beginning of the year, he agreed to the department goal that "All students will present their work in front of the class at least once per [6-week]

marking period.” This section analyzes how Mr. B implemented and adapted the new practice over the course of the school year and the learning affordances for students.

Section 6.4.1 analyzes Mr. B’s baseline teaching practices prior to trying the student presentation activity. TQQ is used to give an overview of the micro-level student discourse in the baseline episodes, supplemented with qualitative analysis of the mathematics content, meso-level discourse structures and functions, and teacher and student roles in the episodes. This analysis demonstrates that the baseline episodes were a near match to a knowledge transmission frame for teaching.

Section 6.4.2 analyzes the first student presentation episode. In this episode, TQQ codes show a very different pattern of micro-level student discourse, specifically that students take much longer talk turns and describe their procedures to the class (HOW code) instead of only their answers (WHAT code). Nevertheless, qualitative analysis reveals that these changes are somewhat superficial. The mathematics content, meso-level discourse functions, and teacher and student roles in this episode remain largely consistent with a knowledge transmission frame.

Section 6.4.3 analyzes how Mr. B’s practice shifts as he sustains and adapts the student presentation activity throughout the rest of the school year. This analysis supports the view that Mr. B adapted the practice to create significantly more opportunities for student authorship and critique, which also led to a corresponding increase in the mathematical richness of the activity. There were two observed shifts in discourse structures that supported this change:

- *Increase in WHY presentations; and*
- *Meso-level discourse structures that built on students’ ideas, such as teacher summaries of presenters’ ideas and teacher follow-up questions to presenting students.*

Through both of these shifts, Mr. B significantly expanded the mathematical learning affordances of the presentation activity for students. Students were also increasingly positioned as authors and, to some extent, critics of mathematical ideas. These changes indicate significant progress toward a productive disciplinary engagement frame.

6.4.1 Baseline Teaching Episodes

In this section, Mr. B’s baseline whole-class teaching practice at the beginning of the year is characterized. All four observed whole-class activities from August to mid-October are considered, prior to the first observation with student presentation (NF1 in August, F1D2 in September, and two activities during NF2 in October). This section argues that all four baseline episodes are fairly similar, and the micro-level discourse structures, mathematics content, meso-level discourse structure and function, and teacher and student roles all align fairly closely with a knowledge transmission frame.

6.4.1.1 Quantitative Analysis: Baseline Episodes

Figure 6.1 shows TQQ length codes for the baseline episodes. All four episodes had similar code profiles. A majority of student talk in all four episodes was spoken in talk turns that were only a few words long (<5 word code: 56–82%). Students gave some answers that were short phrases or single sentences (5–20 word code: 18–44%). It was very rare for students to speak more than a sentence at a time (21–40 and 41+ codes combined: 0–14%).

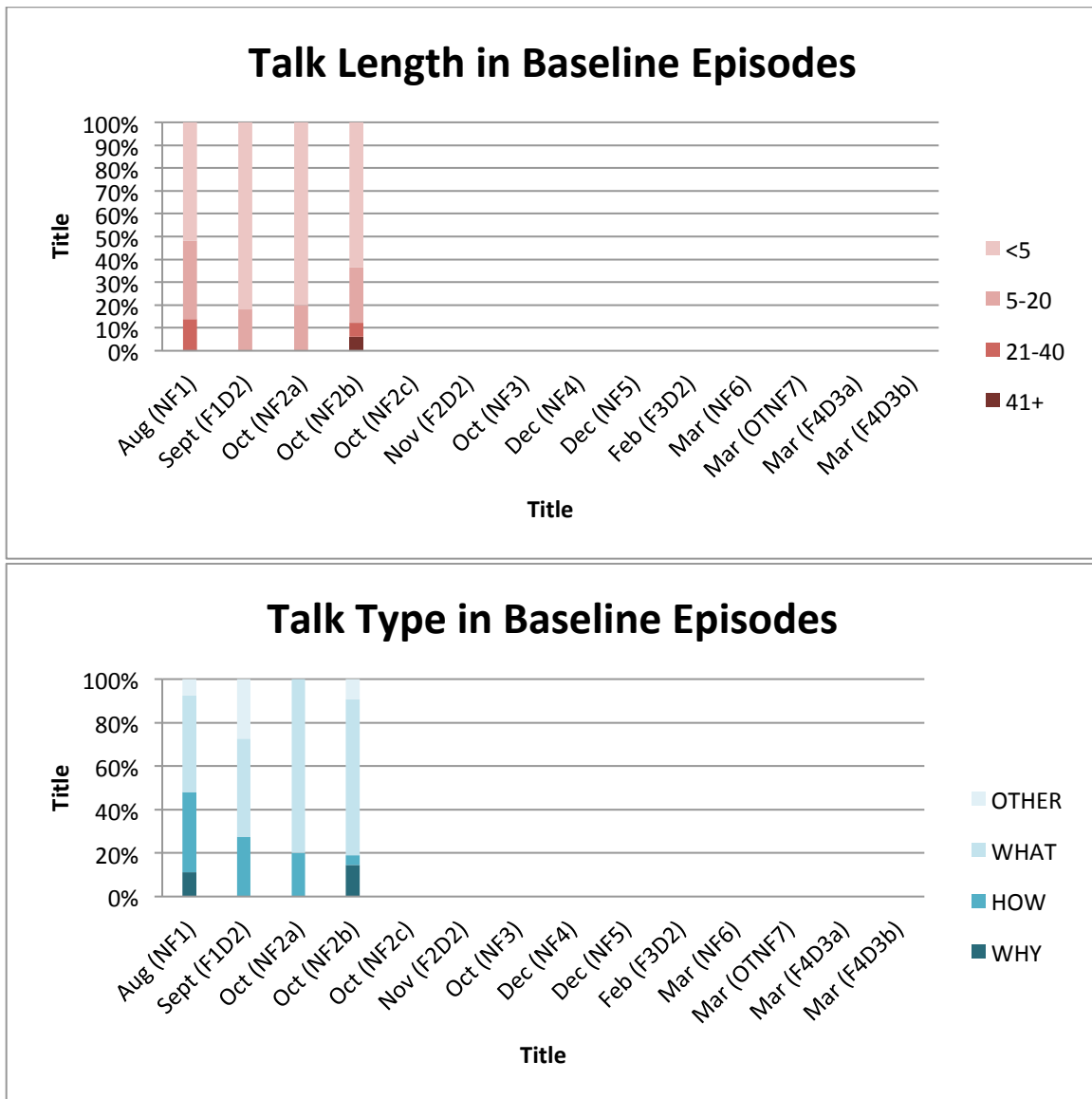


Figure 6.1. TQQ codes for baseline episodes.

Talk type codes show that a plurality of student talk was answer focused (44–80% WHAT code), with only a small fraction of student justification and reasoning (0–14% WHY code). The fraction of descriptive talk varied more from episode to episode (5–37% HOW code).

Episode NF1 has the highest frequency of both high quality student talk type codes and long student talk turns, and may appear promising based on the TQQ analysis alone. However, this episode was quite short, so a few longer student talk turns caused these high frequency percentages. It turns out that a single talk turn was both the only student talk turn coded WHY and the only student talk turn coded 21–40 words. Qualitative analysis in the next section reveals that this talk turn is not a good indication of a productive disciplinary engagement frame. Additionally, qualitative analysis of the mathematics content of each lesson reveals that episode NF1 was by far the easiest lesson: a warm-up problem from one of the first days of school. As the mathematics became more challenging, the teacher took on a greater role in demonstrating procedures and the frequency of high-quality student talk decreased.

6.4.1.2 Qualitative Analysis: Baseline Episodes

Of the four baseline episodes, three are short reviews of warm-up problems (4–6 minutes each) and the fourth was a 25-minute review of homework problems. This section presents an analysis of the mathematics content, meso-level discourse structures and functions, and teacher and student roles across the four episodes.

6.4.1.2.1 Mathematics Content

The mathematics content of the warm-up problems (episodes NF1, F2D2, and NF2a) is relatively straightforward. The first is a whole-number arithmetic word problem, “Rosa picked 6 dozen roses on Saturday. How many roses did she pick? How many flower bunches could she make if each must have 9?” The second warm-up problem is a single order-of-operations problem. The third warm-up problem consists of four two-digit whole-number multiplication problems. In each case, the mathematical activity of “going over” the warm-up problem as a class consists mostly of establishing the correct answers. Students are called on for answers and, if they are correct, they are invited to explain a little more. When students struggle or give incorrect answers, the teacher explains how to do the problems. In general, students seem comfortable, the correct process and answer are stated, and the teacher moves on fairly quickly to other mathematical activities.

The mathematics content of the homework review (episode NF2b) is about comparing decimals. Students have previously struggled with a few homework problems, such as:

Which of the following statements are true? Circle all that apply.

- A. $0.042 > 0.21$
- B. $.33 > .198$
- C. $0.56 < 0.475$
- D. $0.686 < 0.88$
- E. $0.74 > 0.597$

The homework problems are difficult because they require students to consistently apply an understanding of place value in decimals to make inferences about the relative magnitude of numbers. Furthermore, the problems are set up so that

students' intuition about the relative magnitude of whole numbers will lead them astray unless they carefully consider the place value of each decimal. For example, in problem B, students' knowledge of whole numbers and perhaps a specific intuition that "longer numbers are bigger" might lead them to draw the incorrect conclusion that $0.33 < 0.198$ because $33 < 198$.

This student confusion indicates a potential opportunity for students to grapple deeply with the implications of place value in decimals. There are many possible strategies to approach these problems, each of which would help students make some important connections:

- Adding trailing zeros to create equivalent decimals that are all in thousandths, for example turning 0.21 into 0.210.
- Helping students see that the tenths place is ten times as important as the hundredths place, and so on, for example, by:
 - Cutting off or rounding all decimals after the tenths place and using this information to make an initial comparison of their values, then going on to the hundredths and smaller place values as needed;
 - Placing the numbers on a number line divided into tenths, for example showing that 3 in the tenths place in 0.33 would be used to place the number between 0.3 and 0.4 and the 3 in the hundredths place is used to place the number about a third of the way past the 0.3 marker. Subdivisions for hundredths could be added occasionally if necessary.
- Explicitly discussing student overgeneralizations such as "longer numbers are bigger" and asking students to explain why these statements are true for whole numbers but not true for decimals.

Naturally, these strategies have different affordances. Some, such as adding trailing zeros, can lead students fairly quickly to a sense of success in solving the given homework problems, whereas others, such as the number line method or an explicit discussion of misconceptions, may lead students to experience a period of confusion as they grapple with some of the important underlying concepts. On any given day, a teacher would not teach all of these methods, but instead choose one or more methods that fit with his other goals. Nevertheless, in a productive disciplinary engagement frame, one would expect that students at some point would have time to grapple with and make their own sense of these important ideas, and make connections between several of the methods and representations listed above. In a knowledge transmission frame, one would expect the teacher to prioritize teaching a single method that would lead students to fairly easily answer the problem at hand while avoiding misconceptions.

To address student confusion, Mr. B chooses to model a strategy he calls "alphabetizing" for comparing decimals by comparing digits from left to right. He reminded students that if they were going to compare words and try to alphabetize them, they would look at each letter from left to right. Table 6.4 shows the transcript of how Mr. B guides the class through part A of the first problem.

Line	Transcript		
	Teacher Elicitation	Student Response	Teacher Feedback
B-NF2b-44	Same thing works here. Comparing left and right. We gonna start with the first digit.		
B-NF2b-45	What's this first digit? (Points to .042 on the board)		
B-NF2b-46		Choral: ZERO	
B-NF2b-47			(underlines 0)
B-NF2b-48	Go behind this decimal (points to 2 in 0.21). What's the first digit?		
B-NF2b-49		Choral: TWO	
B-NF2b-50			(underlines 2)
B-NF2b-51	Now is this greater (points to underlined 0 in .042)... greater (writes >), less (writes <) or equal (writes =) to 2?		
B-NF2b-52		Choral: Greater. Less.	
B-NF2b-53	Huh?		
B-NF2b-54		Choral: Less. Greater.	
B-NF2b-55			Now. It's less. But it says this (writes >).
B-NF2b-56	So is that correct?		
B-NF2b-57		Choral: NO.	
B-NF2b-58			(No response)
B-NF2b-59	So can it be A?		
B-NF2b-60		Choral: NO.	
B-NF2b-61			(No response)
B-NF2b-62	Because this (points to .042) is decimal zero. This is (points to 0.21), even though I got a zero, but I see a decimal. If I want I could add that. (Writes a leading 0 in front of .042 to make it 0.042). Make it the same! Go to the next one, that's zero, that's two. So A can't be involved.		

Table 6.4. Transcript excerpt from baseline episode NF2b.

The alphabetizing strategy modeled by Mr. B here is mathematically accurate and straightforward for students to apply. However, it has limited affordances for students to connect their prior knowledge of place value with whole numbers to the new context of place value with decimals. Possible misconceptions are noted, but the teacher’s response to these misconceptions is to make minimal additions to show how a standard process can be correctly applied. For example, in line B-NF2b-62, the teacher notes that students may read the 0 before the decimal point in 0.21 as the “first digit,” but then goes on to suggest that they can avoid this problem by making sure both numbers have a 0 before the decimal point. He does not mention the difference in place value between the various digits discussed. This mathematical focus on a single procedure and on minimizing distractors from the procedure rather than making explicit connections to prior knowledge is consistent with a knowledge transmission frame.

6.4.1.2.2 Meso-level Discourse Structure and Function

This section builds on the micro-level discourse analysis using TQQ to demonstrate how individual talk turns were linked together within meso-level discourse structures, and analyzes the functions of these meso-level structures. TQQ analysis reveals that in all four baseline episodes, a majority of student talk turns were short (<5 word length code), and answer focused (WHAT type). This is consistent with a discourse pattern of IRE sequences. IRE sequences appear in TQQ as follows:

- Teacher Elicitation coded WHAT;
- Student Response coded WHAT and <5 words;
- Teacher Feedback coded as an implicit or explicit evaluation.

Much of the classroom discourse does indeed show this pattern. The transcript shown in Table 6.4 is a clear example, not only of the teacher’s mathematical strategy, but also of a chain of IRE sequences. TQQ coding of the same excerpt is shown in Table 6.5. All of the teacher Elicitations in the excerpt were coded as WHAT, all student responses were coded as WHAT and <5 words, and all teacher feedback turns were coded as either explicit or implicit evaluations.²¹

Line	Transcript			Notes on function	
	Teacher Elicitation	Student Response	Teacher Feedback	Function of small chunks	Function of medium chunks
B-NF2b-44	Teacher exposition – a few sentences.				Establish whether statement A
B-NF2b-45	WHAT			Name first digit behind	
B-NF2b-46		WHAT <5			

²¹ In lines 58 and 61, the transcript shows no teacher talk or action, but the TQQ code Implicit Evaluation is still used because the TQQ codebook allows for classroom norms to be used to assign codes. In this classroom, it is typical for the teacher to correct all wrong answers, and it was empirically checked for this episode that he did not allow any incorrect answers to pass without comment. Therefore, if he allows an answer to pass without comment, it is fairly clear to students that the answer was correct, and an Implicit Evaluation code is appropriate.

B-NF2b-47			Implicit eval	decimal in .042.	is correct while walking students through a method for comparing decimals.
B-NF2b-48	WHAT			Name first digit behind the decimal in 0.21.	
B-NF2b-49		WHAT <5			
B-NF2b-50			Implicit eval		
B-NF2b-51	WHAT			Is 0 greater than, less than, or equal to 2?	
B-NF2b-52		WHAT <5			
B-NF2b-53	Re-elicite for correctness				
B-NF2b-54		WHAT <5			
B-NF2b-55			Explicit eval		
B-NF2b-56	WHAT			Establish contradiction in statement A.	
B-NF2b-57		WHAT <5			
B-NF2b-58			Implicit eval		
B-NF2b-59	WHAT			Establish that statement A is false.	
B-NF2b-60		WHAT <5			
B-NF2b-61			Implicit eval		
B-NF2b-62	Teacher recap of method noting possible difficulties – not coded.			Teacher recap of procedure.	

Table 6.5. TQQ excerpt from baseline episode NF2b.

The meso-level discourse structure shown in Table 6.5 is a topically related chain of IRE sequences. It has the classic function of guiding students step by step through a procedure that the teacher wishes them to use. This excerpt is entirely consistent with a knowledge transmission frame in its meso-level structure and function.

In addition to student WHAT talk, the micro-level TQQ coding also shows a small percentage of student HOW and WHY talk and a small percentage of longer student talk turns. These codes typically correspond with each other in these episodes: student talk turns longer than five words are typically also coded as HOW or WHY talk. At the meso level, these longer HOW and WHY talk turns often occur when Mr. B asks two questions about the same mathematical step. First, Mr. B uses an IRE sequence to establish the correct answer. Then, Mr. B invites the same student or another student to elaborate on their answer, typically asking a HOW or WHY question and receiving a somewhat longer (5–20 word) HOW or WHY response.

For example, in episode NF1, students are solving the problem “Rosa picked 6 dozen roses. How many roses did she pick?” Mr. B first uses IRE sequences to establish that the word “dozen” meant twelve. He then asks the WHY question, “How do you know?” and another student raises his hand and replies “So if there’s a dozen eggs up in an egg carton and it says dozen on the top, that’s twelve, a dozen eggs.” This

student response was coded WHY and 21–40 words long, making it one of the highest-level and longest responses across the baseline episodes. In many ways, this student response is delightful: it shows a real life example that other students can easily use to connect the academic vocabulary to their lived experiences, and the student’s tone infuses a level of enthusiasm into the conversation. Nevertheless, this student comment is somewhat peripheral to the central mathematical activity: if the student had given a completely different explanation, for example referring to a dozen cupcakes or a dozen bagels instead of a dozen eggs, the subsequent mathematical activity would be substantially unchanged.

Mr. B also repeatedly asks students for alternate strategies but rarely, if ever, receives a response, and he does not insist on a response or make the exploration of multiple strategies central to the mathematics. Again, the teaching strategy has potential to move the class toward a productive disciplinary engagement frame, but the joint activity of the teacher and class remains focused on establishing a single, correct procedure.

For ease of future reference, Table 6.6 summarizes the meso-level discourse practices in the baseline episodes.

Baseline teaching practices
Teacher exposition
IRE sequences to demonstrate and practice procedures
Teacher asks some HOW and WHY follow-up questions after establishing answers
Teacher sometimes asks for alternate strategies with little response from students

Table 6.6. Discourse structures in baseline episodes.

In summary, the majority of meso-level discourse structures and functions in the baseline episodes are consistent with a knowledge transmission frame. The teacher makes some clear efforts to break out of this routine, such as asking for alternate strategies and asking students follow-up HOW and WHY questions. However, these open discourse practices were ancillary to the primary mathematical activity of establishing correct answers and clearly demonstrating procedures.

6.4.1.2.3 Student and Teacher Roles

In all four episodes, the teacher takes on a role as arbiter of mathematical knowledge. The teacher directs the flow of the mathematical conversation. Students are engaged, even eager to participate. They give choral responses on cue and many students raise their hands asking to be called on. The teacher is attentive to students’ needs, for example, slowing his pace, repeating himself, or offering encouragement when students give wrong answers.

However, students’ positioning with respect to mathematical knowledge was passive. The teacher was the only one to make strategic decisions about which procedure to use. For example, in episode NF2a, it is clearly implied that all students should use the partial-product method to solve all four multiplication problems in the warm up. When one student presents a different, correct method –

a shortcut he has been taught in previous classes that the teacher views as procedural – he is told, “We don’t do that anymore.”

To give another example, in episode NF2b, the teacher guides the class through several problems using the same method. He breaks the problems into very small pieces and guides students through each problem step by step, receiving choral responses for each step. After the second or third problem, a few advanced students try to anticipate the teacher’s method. They shout out sentences that anticipate the teacher’s next few sentences. The teacher largely ignores these comments from students and continues to guide the class through the problems step by step. When students are incorrect, the teacher is attentive to their needs but minimizes the “air time” given to consider wrong answers. Each time he notices a student with an incorrect answer, he draws attention to it in a non-judgmental tone, repeats an explanation of the correct answer until students say, “Oh! I get it,” then moves on.

In short, the teacher and student roles in these episodes are entirely consistent with a knowledge transmission frame. The teacher is positioned as the sole author of mathematical ideas because he is the only one to make strategic choices. Students are expected to apply the procedures demonstrated by the teacher, and show their knowledge for the teacher’s evaluation. The teacher is positioned as the sole critic of mathematical ideas.

The next section discusses the new practices that Mr. B develops when he starts using the new student presentation activity. In the new activity, students take longer talk turns, but their talk is not initially very mathematically rich. The third section of analysis, discusses how Mr. B integrates and adapts his baseline and new practices to create new arrangements where students have significant “air time” and make major contributions within a mathematically rich discussion.

6.4.2 New Activity Structure: Student Presentations

The new teaching practice that Mr. B tries, and sustains throughout the year, is to have students present their work at the front of the classroom. This means that Mr. B organizes a new activity structure, which occurs temporally after students have had some time to work in small groups on a particular problem. In the new activity, all groups are called up to present their work at the front of the class. The teacher first gives all groups a reminder that they will be presenting, and a few minutes to review their written work and practice what they will say. Then he calls one group at a time to the front of the classroom, sets a timer for the group (typically for two or three minutes), and tells them when to start and stop their presentation. Depending on the mathematical task, groups are either responsible for sharing all work they have done on the problem or have some freedom to choose which part they present.

The first observed student presentation activity during the school year is a fairly brief episode, about eight minutes in length.²² During this time, two groups present their work. A total of five students present a total of six problems from posters they have previously prepared. Students give step-by-step descriptions of the procedure they used to solve each problem. All six problems use the same procedure. There is no further discussion during the observation of the mathematical ideas presented by students.

6.4.2.1 Quantitative Analysis: First Presentation Episode

Figure 6.2 shows student talk length codes for the four baseline episodes (NF1, F1D2, NF2a and NF2b) and first student presentation episode (NF2c). In the baseline episodes, more than half of student talk turns are just a few words long (<5 words), with some student talk turns that are a phrase or sentence long (5–20 words). Student talk more than a sentence long (21–40 or 41+ words) is very rare. In contrast, in the presentation episode, all six student talk turns were multiple sentences long (41+ words). This shows that the student presentation activity created dramatically more “air time” for students to talk than the discourse structures in the baseline episodes.

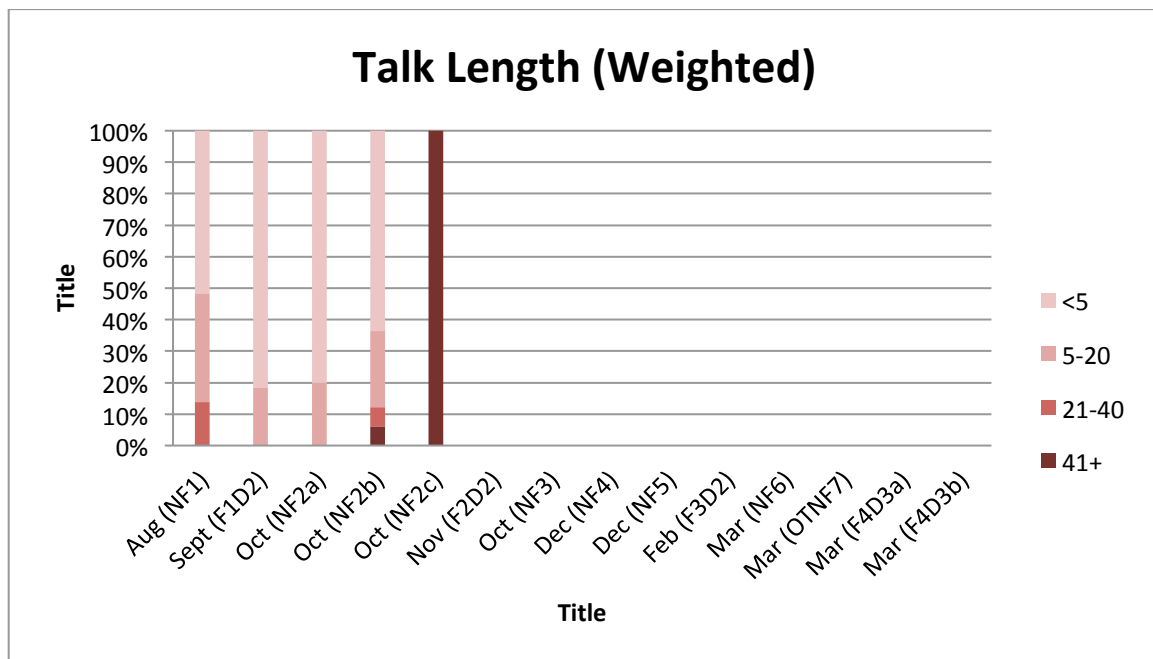


Figure 6.2. Student talk length in baseline data and presentation episode 1.

²² The teacher reported that this was only part of a longer activity begun in a previous day’s class. He had asked students to work in groups on a problem, prepare posters, and then asked all groups to present their work. During the observed lesson, he invited the last three groups to present who had not had time to do so previously. Two of these presentations were observed and the last one was saved for another day. I believe the two observed groups were typical.

Extended student talk turns are necessary, but not sufficient, for a productive disciplinary engagement frame. One would also like to see shifts in the types of things students have the opportunities to talk about: students taking responsibility for “guiding” the direction of mathematical problem solving, authoring novel ideas, and building on and/or critiquing the reasoning of others instead of looking to the teacher to evaluate answers. The TQQ coding scheme partially captures these kinds of shifts through the talk type code (OTHER, WHAT, HOW, WHY). This code is applied to all teacher and student talk turns, but student talk type is particularly significant in determining the frame in play. Do students mainly talk about answers (WHAT), describe processes and methods (HOW), or justify and critique the reasons behind these processes (WHY)?

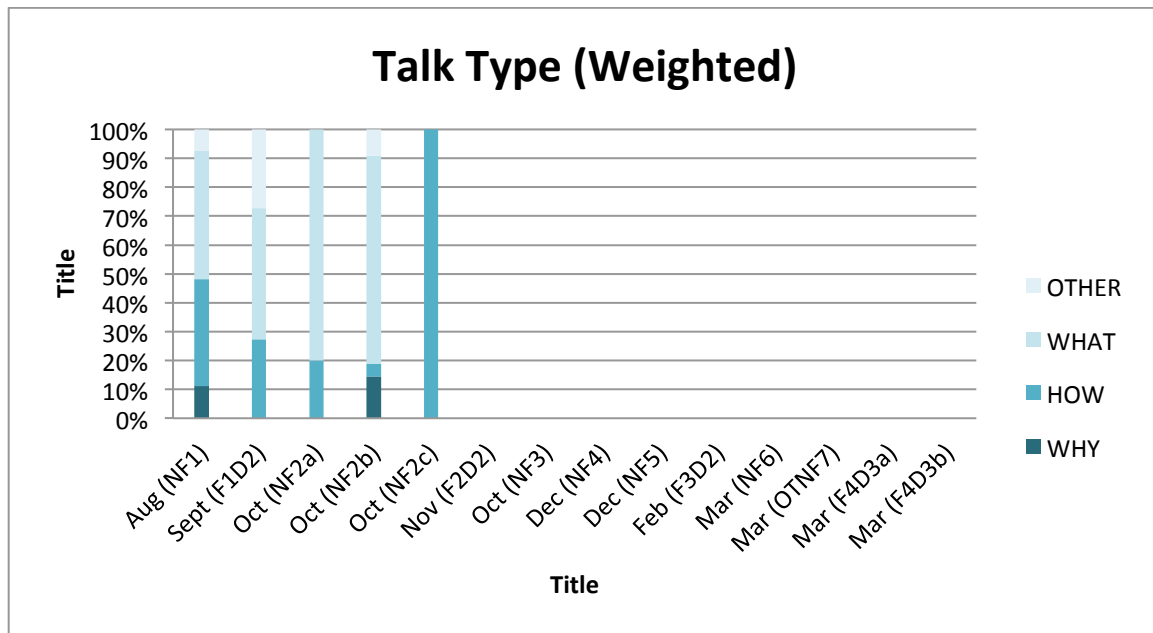


Figure 6.3. Student talk type in baseline data and presentation episode 1.

Figure 6.3 shows the student talk type codes for the four baseline episodes and the first presentation episode. In the baseline episodes, a majority of student talk is about answers (WHAT), with less than half of the talk about processes (HOW) or justification (WHY). In contrast, all six student talk turns during the presentation episode are focused on describing procedures (100% HOW).

6.4.2.2 Qualitative Analysis: First Presentation Episode

This section unpacks the first presentation episode (NF2c) and analyzes the meso-level discourse structure and function, mathematics content, and roles taken on by the teacher and students.

6.4.2.2.1 Meso-level Discourse Structure and Function

The function of the new presentation routine is for students to demonstrate what they have done to solve the problem. The discourse structures are consistent with this function. First, the teacher tells a particular group to come to the front, sets a timer, and tells them when to start. Then, each student takes a turn explaining a

problem. All student talk turns are 41+ words and coded HOW in TQQ. Finally, the teacher tells students when their time is up, typically within a few seconds after the last student in the group finishes presenting. Neither the teacher nor the students follow up on the ideas in the presentations; there are neither follow-up questions nor references to the presentations in subsequent talk by others. The teacher likely provides written feedback to students, but there is no record of this feedback in the video recordings, supplemental photographs or debrief conversations with the teacher.

Table 6.7 summarizes the discourse structures in the new activity structure.

Presentation Episode 1	
Who presents?	All groups / all students in the group.
Framing the presentations	Little framing talk; teacher only tells groups when to start and stop the presentations.
Presenter talk length and type (TQQ codes)	Presenters take long talk turns (41+ word code).
	Students describe their procedures for solving problems (HOW code).
Teacher talk framing and/or building on students	Presentations are self-contained and no one follows up on the ideas presented.

Table 6.7. Discourse structures in presentation episode 1.

6.4.2.2.2 Mathematics Content

Prior to presenting, students have completed posters about a contextualized decimal arithmetic problem. The problem involves a person making fruit salad and purchasing various fruits at various prices: 1.5 lb of strawberries for \$2.89/lb, etc. Students are asked to find the cost for each fruit and the total cost of the fruit salad.

It is expected in this class that students use a partial-product method²³ for multiplication. This method helps students keep track of all parts of a multi-digit

²³ For whole numbers, the partial-product method is as follows:

- 1) Write both whole numbers as a sum of one-digit numbers with different place values, for example, $15 = 10+5$ and $289 = 200+80+9$.
- 2) Use multiplication facts to find all partial products, for example,

$$10 \times 200 = 2000$$

$$10 \times 80 = 800$$

$$10 \times 9 = 90$$

$$5 \times 200 = 1000$$

$$5 \times 80 = 400$$

$$5 \times 9 = 45$$
- 3) Add the partial products to get the full product, for example, $2000+800+90+1000+400+45=4335$.

For decimal multiplication, students can treat the decimals as though they are whole numbers, and then add a decimal point to their answer in the appropriate place.

multiplication problem. This method has some affordances for sensemaking about place value for whole numbers. However, these affordances were not taken up and there is little public sensemaking about place value in the observed episode.

During the episode, five students present six problems, and all use the partial-product method. Table 6.8 shows a typical presentation about the purchase of 1.75 lb of kiwis for \$1.75/lb.

Reproduction of displayed work	Student talk:
<p style="text-align: center;">Kiwis</p> $1.75 = 100^A + 70^B + 5^C$ $1.75 = 100^F + 70^E + 5^D$ $A \times D = 100 \times 5 = 500$ $A \times E = 100 \times 70 = 7000$ $A \times F = 100 \times 100 = 10000$ $B \times D = 70 \times 5 = 350$ $B \times E = 70 \times 70 = 4900$ $B \times F = 70 \times 100 = 7000$ $C \times D = 5 \times 5 = 25$ $C \times E = 5 \times 70 = 350$ $C \times F = 5 \times 100 = 500$ <div style="border: 1px solid black; padding: 2px; width: fit-content; margin: 0 auto;">30.625</div>	<p>So what I did is I multiplied one and seventy-five times one and seventy-five and then I put, and then I did the column and then I put the letters on them, A, B, and C And then I put each of the top letters 3 times and multiplied by the bottom letters And then I added all of the sum up and I got thirty and six hundred twenty-five.</p>

Table 6.8. Typical student presentation from presentation episode 1.

In this presentation, the student implements the partial-product method in a clear and organized way. However, she does not unpack the procedure in relationship to the underlying concept of place value. Her description barely mentions that the numbers are decimals and reflects a limited understanding of place value. For example, she makes several mistakes related to place value. First, she reads numbers 1.75 as “one and seventy-five” and 30.625 as “thirty and six hundred twenty-five.” This could mean that she has been taught to read decimals with an explicit mention of place value (e.g., “one and seventy five hundredths”) but leaves off the place value names because she is unsure of what to say. It could also mean she is simply reading 1.75 as two whole numbers separated by a dot, without assigning place value meaning to the numbers after the decimal point.

The presenter also does not connect her procedure to the problem context. She gives an answer that is ten times the correct answer to the problem, presumably because she places the decimal point incorrectly after doing the whole-number partial-product algorithm. Furthermore, she makes no reference to the context of the problem and the meaning of \$1.75/lb and 1.75 lb. If she did, she might notice that \$30 is far too much to pay for less than 2 lb of something that costs less than \$2/lb.

This presentation is typical of the episode as a whole in that the partial-product method is used, there is significant attention to the procedure and little discussion of place value, place value errors go unremarked upon, and there is no mention of the problem context except occasionally and insubstantially as a label (e.g., “For blueberries, I did...”). The focus on a single procedure without unpacking the connections to important concepts and the problem context is consistent with a knowledge transmission frame for learning.

6.4.2.2.3 Student and Teacher Roles

In some ways, students take on very new and more authoritative roles in the new activity structure. They are invited to a physical position at the front of the classroom that is usually reserved for the teacher. Students also take long talk turns explaining procedures, taking on a teacher role of exposition.

However, students’ positioning with respect to the mathematics content is relatively unchanged, and remains consistent with a knowledge transmission frame. Students are implicitly or explicitly told which procedure to apply to solve the problem, and their role in the presentations is to demonstrate their knowledge of this procedure. Students do not make strategic decisions about what procedure to use, unpack relevant concepts such as place value, or make connections to the problem context. Therefore, they do not take up a position of mathematical authorship. Furthermore, they are not positioned by others as authors, as evidenced by the lack of reference to student ideas by others.

Neither the teacher nor the students take a role of publicly connecting or critiquing student ideas. This is a subtle, but significant difference in *teacher* role compared to a knowledge transmission frame, in which the teacher would immediately evaluate and critique student ideas. In this episode, the teacher seems to be abdicating from the critiquing role, at least publicly. This is a potentially powerful shift and a difficult one for many teachers, because teacher critique otherwise cuts off the possibility of student critique. However, in this case, students do not take up the critiquing role and the lack of teacher or student critique results in a missed opportunity to address student mistakes with place value and the illogic of student answers relative to the problem context.

6.4.2.3 Summary: First Presentation Episode

The analysis so far demonstrates that the initial student presentation practice is different from the teacher’s baseline practices. Specifically, the new practice allows

“air time” for long talk turns by students, and students talked less about answers and more about descriptions of their procedures. Students stepped into an unfamiliar role of creating posters, standing at the front of the room, taking long talk turns and giving an organized description of their work. In this sense, the first presentation episode is an important step away from a knowledge transmission frame and toward a productive disciplinary engagement frame by both teacher and students.

However, the mathematics of the presentations and the positioning of students with respect to that mathematics remained consistent with a knowledge transmission frame. The presentations have no basis to feel as though they are part of a larger mathematical conversation. The most important mathematics in the problem (considering issues of place value in decimal arithmetic, and making connections between the arithmetic and the problem context) is backgrounded during the presentation activity. Students make mistakes that are not publicly addressed within the session (though they are likely privately addressed in teacher feedback to students) and some of the most important mathematical ideas in the problem are barely mentioned during the activity.

Thus, although the new presentation routine creates new opportunities for student discourse, the missed opportunities to unpack student ideas and errors lessen the potential quality of the mathematics. Ultimately, the power of a productive disciplinary engagement frame lies not only in expanding “air time” for student talk but primarily in enriching the quality of the mathematics that students talk about. After this episode alone, it is not yet clear whether the teacher will capitalize on these new affordances. Much depends on Mr. B’s future actions as he sustains and adapts the practice during the rest of the year. Will future presentations be mathematically valuable, neutral exercises in “presentation skills,” or even harmful to students’ mathematics learning? As Mr. B sustains and becomes more familiar with the new routine, analysis shows that he also adapts the routine in ways that preserve the valuable opportunities for student talk while substantially enriching the mathematics. In doing so, he makes substantial progress toward a productive disciplinary engagement frame.

6.4.3 Shifts that enrich the student presentation activity structure

The first presentation episode shows Mr. B’s initial student presentation practice, in which students had new opportunities to speak publicly about mathematics, but were not yet part of a rich mathematical conversation. Mr. B continues and adapts the new practice throughout the rest of the year. Four total episodes of the student presentation activity were observed. This section analyzes the remaining three episodes and describes several shifts that Mr. B made to substantially enrich the student presentation practice, making significant progress toward a productive disciplinary engagement frame.

The first important shift is supporting student presenters to go beyond descriptive (HOW) talk and include more of their own justification and sensemaking (WHY)

talk. This shift occurs between the first and second episodes of student presentations. The second, third, and fourth episodes all contain substantial fractions of WHY talk from student presenters.

The second important shift is building on the ideas presented by students. In the first presentation episode, we see students present one after another without any public discussion of their ideas. In the remaining three episodes, the mathematical ideas presented by students are increasingly taken up by others, through teacher questioning of presenters and/or by incorporating presenters' ideas into his subsequent exposition. A few presenters were even questioned by other students about their work.

For each shift, TQQ codes are first used to give a high-level overview of how this shift affected micro-level discourse moves across the four presentation episodes. Qualitative analysis is then used to show how these micro-level shifts were accompanied by shifts in meso-level discourse structure and function, mathematical content, and student and teacher roles.

6.4.3.1 Shift 1: WHY Presentations

One of the most significant shifts toward sensemaking occurs within the student presentations themselves. Starting in the second presentation episode (NF5), many students start to go beyond describing their procedures (HOW code) and also justify their reasoning (WHY code). This section describes the shift in student behavior as well as the teacher moves that support the shift.

6.4.3.1.1 Appearance of WHY presentations

Figure 6.4 shows TQQ codes for student talk type across the four presentation episodes. Starting in the second presentation, a significant fraction of presenters' talk is coded as WHY, although there is still a significant fraction of descriptive (HOW) talk present. Not all student presentations are coded as WHY; there are some HOW presentations in every episode. The relative fraction of HOW and WHY presentations varies significantly based on how the task was enacted by students in their small groups, specifically the extent to which students tried various methods for solving the problems or mostly used a single procedure. In particular, the fourth presentation episode had a lower fraction of WHY talk than the other two. This is discussed further in a subsequent section, Mathematics of presentation episode 4. Nevertheless, there was a significant presence of WHY presentations in all episodes after the first.

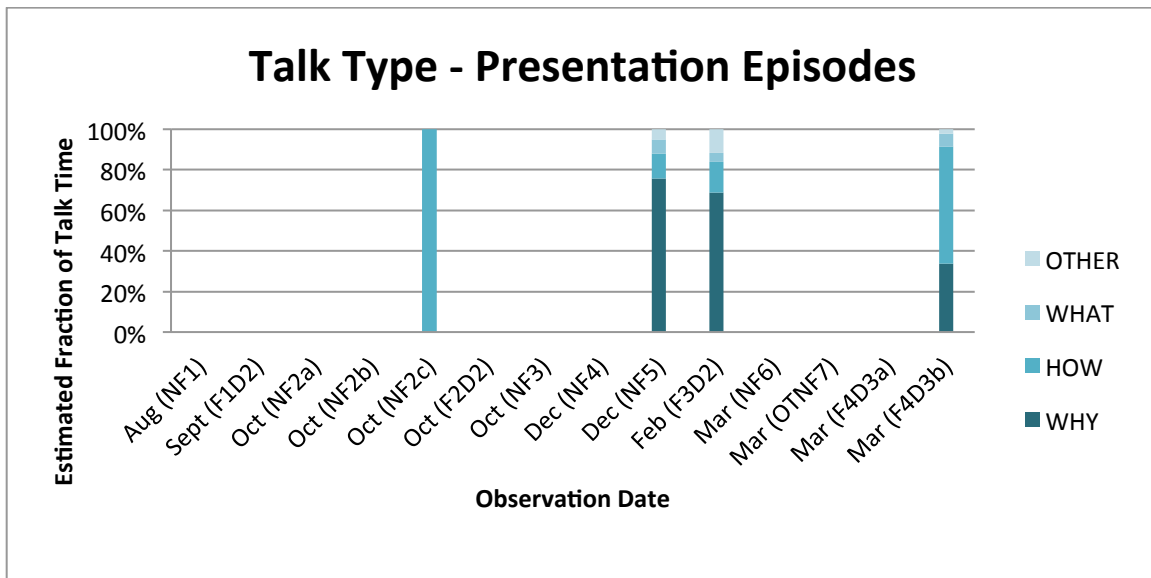


Figure 6.4. TQQ codes for student talk length during presentation episodes.

In the subsequent qualitative analysis, an example of a WHY presentation is unpacked in order to show how it is different from the purely descriptive (HOW) presentations given in presentation episode 1.

6.4.3.1.2 Significance of the WHY presentation code

In this section, I analyze an example of a student presentation that was coded as WHY. I will argue that the shift to WHY presentation codes is accompanied by an increase in richness of the mathematics content and an increase in student positioning as authors of mathematical ideas. In other words, the shift to WHY presentations moves the whole presentation activity significantly toward a productive disciplinary engagement frame.

The example analyzed is from the second presentation episode, NF5. This episode occurs during a unit on proportions. Students work on a rich contextual task involving comparing various recipes for chocolate milk.²⁴ Figure 6.5 shows a partial problem statement for part of the task that is the focus of this student presentation.

²⁴ Some affordances of this task for sensemaking include: (1) It provides easy access into the central mathematical question, “Whose hot chocolate is more chocolaty?” because students have had personal experiences with mixtures and every student can form an opinion as to which mixture would taste more “chocolaty” and provide some type of justification. (2) There are many possible approaches to this task, for example comparing cocoa only, milk only, ratio of cocoa to milk or cocoa to total liquid. To compare ratios, students could find equivalent fractions with a common denominator, with a common numerator, use an operation that preserves order such as multiplying both expressions by the product of the denominators (“cross multiplication”), or divide and compare decimals, and (3) each of these potential student methods has some general power for comparing proportions, and therefore

Hot Chocolate

Dia loves to drink hot chocolate in the morning. Her dad makes the hot chocolate the same way each day. He uses 3 ounces of cocoa and 5 ounces of milk in her special cup.

3. Dia's older brother, Dylan, likes his hot chocolate made with 2 ounces of cocoa and 3 ounces of milk. Whose hot chocolate is more chocolaty?

Explain how you know.

Figure 6.5. Partial Problem Statement for Episode NF5.

The focus here is on one presentation, by Luciano, who has an unconventional but sophisticated method for solving this problem. Luciano expresses the chocolateness of Dia's hot chocolate with the ratio $\frac{3}{5}$. He expresses the chocolateness of Dylan's hot chocolate with the ratio $\frac{2}{3}$. Luciano then finds equivalent forms of both fractions that share a common *numerator*: $\frac{6}{10}$ for Dia's hot chocolate at $\frac{6}{9}$ for Dylan's hot chocolate. He then argues that Dylan's hot chocolate is more chocolaty because $\frac{6}{9}$ is greater than $\frac{6}{10}$. Table 6.9 shows a transcript of Luciano's talk.

Likely student boardwork: ²⁵
$\frac{3}{5} \cdot \frac{2}{2} = \frac{6}{10} \qquad \frac{2}{3} \cdot \frac{3}{3} = \frac{6}{9}$
Student talk:
<p>So, for number 3, "Dia's older brother, Dylan, likes his hot chocolate made with 2 ounces of cocoa and 3 ounces of milk. Whose hot chocolate is more chocolaty?" I picked Dylan because this is, this is Dyah, and this is, <u>Deeah</u>²⁶ and this is Dylan. You multiply this by 2 gives you six tenths. Multiply this by 3 that gives you six ninths. And um that's there because fractions, the smaller number is bigger and six ninths (inaudible) six tenths.</p>

comparing various students methods helps students build connections between central mathematical ideas.

²⁵ The whiteboard was not recorded during Luciano's presentation because a group member did not give permission to be videotaped. Table 6.8 therefore shows a plausible reconstruction of what Luciano may have written on the board, for ease of interpretation of his speech.

²⁶ The nonstandard spellings of "Dia" here are phonetic transcriptions of Luciano's shifting pronunciation of the name "Dia," which is peripheral to his mathematical argument but is helpful to understand his repetitive phrasing. The underline indicates emphasis.

Table 6.9. Luciano’s talk and reproduction of his likely board work.

Luciano’s presentation is coded as length 41+ words. It is coded as type WHY because of his reference to the problem context (“this is Dia”, “this is Dylan”) and justification (e.g., “because fractions, the smaller number is bigger.”). This presentation was selected because it is especially clear that Luciano is reasoning this through on his own since comparing numerators is a relatively unconventional way to solve this type of problem. However, students who solved problems using more conventional methods but still explained their justification and/or connections to the problem context in their own words were also assigned WHY codes.

Notice that Luciano’s presentation includes a lot of descriptive (HOW) talk, such as “multiply this by two gives you six tenths.” This talk alone would be coded as HOW. However, in EQUIP, student talk turns are only assigned a single talk type code. If more than one code is possible, only the highest-level code is used. This aspect of EQUIP is also used in TQQ and is sensible for this analysis because a few references to the problem context and/or justification can set the tone that the student’s entire presentation is about sensemaking. Most presentations coded as WHY and 41+ words also included some elements of description. As argued earlier, having students describe their procedures is valuable in its own right, but is much more valuable when combined with other reasoning about what those procedures mean and when and why they are applicable.

This example shows how a shift toward WHY presentations instead of HOW presentations moves Mr. B’s classroom significantly toward a productive disciplinary engagement frame. The mathematics content of the task as enacted by students is quite connected, partly because Mr. B selects a task that affords multiple strategies, but also partly because students are supported to develop and share various strategies. Furthermore, students are acting as authors of mathematical ideas by taking ownership over the strategic choices needed to find various methods to solve the problem.

6.4.3.1.3 Teacher actions that support WHY presentations

How did Mr. B support students to shift from HOW to WHY presentations? A full answer to this question is not accessible from video data of whole-class discussions, because one can expect that a lot of this work was done as Mr. B supported small groups to solve the problems. Even without a detailed analysis of the small group time, however, one can make a few claims.

In presentation episode 1 (NF2c), all groups used the partial-product procedure to solve the problems. This procedure was not part of the written statement of the task, and one can easily imagine another group of students working on the same problem without using the partial-product method. This strongly implies that Mr. B encouraged students to use the partial-product method for solving these problems. He may have done so explicitly, in his framing of the small group task for students,

or implicitly, by emphasizing the partial-product method in other multiplication problems so strongly that students understood it was expected without an explicit statement that they should use the method for this task. Either way, the presentations are significant evidence that the task as enacted by students was focused on practicing this one procedure, rather than making sense of a contextualized situation and place value. No observation was made of any explicit framing by Mr. B that students should focus their presentation on any particular aspect of their mathematical work.

In presentation episode 2, however, Mr. B did do some explicit framing for students about what presentations should be like. Specifically, he shared a rubric with students that he had co-developed with the sixth-grade co-planning team. Students are asked to score each other based on this rubric. Table 6.10 shows the presentation rubric as posted and described during presentation episode 2 (NF5).

	<p>Mr. B's Description (from field notes)</p> <p>Students will score each other based on the rubric. You don't get a 3 or a 4 if you don't hear academic vocabulary. Conceptual understanding where we get the idea you really know. Do you give us a drawing that helps us? Listen for good explanations.</p>
--	--

Table 6.10. Presentation rubric for presentation episode 2.

Through the rubric, Mr. B makes it clear that students are expected to demonstrate “conceptual understanding” in order to score above a 2. He gives a few examples of what students could do to demonstrate “conceptual understanding”: using academic vocabulary, giving a helpful drawing or a good explanation.

In addition to the rubric, Mr. B chose a problem for presentation episode 2 that afforded many strategies, and did not “scaffold away” these options when students were working in small groups to make sense of the task. This is evident without an analysis of the small group work because the students who presented solved this problem in a variety of ways. Considering Problem 3 alone, Luciano was the only student to compare equivalent fractions with a common numerator. Other students used other methods, some more correct than others and some more sophisticated than others. Some students did not take into account both the quantity of hot chocolate and the quantity of cocoa. Several presenters formally compared the ratios $\frac{3}{5}$ and $\frac{2}{3}$. One of these presenters, Marco, found a common denominator and argued that $\frac{9}{15}$ was less than $\frac{10}{15}$. Another student, presenting about another problem, converted two equivalent fractions into the same decimal as a way of proving that the fractions were equivalent.

During our debrief conversation about presentation episode 2, Mr. B expressed satisfaction with the variety of strategies the class had come up with. He said he did wish more students had done a formal comparison of proportions. He cited this as the reason that he chose to briefly summarize two of the methods for comparing proportions during the last minute of the observed class. This wrap-up discussion, and Mr. B’s continued growth in building on student ideas as the year progressed, are described in the next section.

6.4.3.2 Shift 2: Building on presenters’ ideas

The previous section described a shift in student talk, from HOW presentations that were purely descriptive to WHY presentations that included justification and sensemaking. This shift creates significant new learning opportunities for the class listening to the presentations because they are listening to something mathematically rich and potentially different from their own strategies. However, it is not yet clear whether the class had a chance to process and understand the presenters’ ideas. It is equally important to analyze what happens after the presentations. Do presenters’ ideas ever get taken up by others? In other words, do presentations begin to function as part of a larger mathematical conversation, consistent with a productive disciplinary engagement frame? This section explores another shift in Mr. B’s practice that resulted in increased take-up of presenters’ ideas.

As in the previous section, this section begins with a high-level overview of the changes supported by quantitative analysis using TQQ codes. Qualitative analysis is then used to unpack some of the shifts that occur in more detail.

Figure 6.6 shows TQQ codes for student talk length across the four presentation episodes. Recall that a distinguishing feature of the initial student presentation activity is the prevalence of very long student talk turns (41+ word code). In subsequent episodes, this aspect of the practice was sustained fairly well. Notice that in NF5 and F3D2 a substantial fraction of presentations are 21–40 words long, possibly due to time constraints in trying to have all students present by the end of

the period. By the final episode, there are still a few presentations coded as 21–40 words long, but most are coded as 41+ words. Regardless, in all four episodes, students give presentations that are at least two sentences in length, which is longer than almost all student talk turns in the baseline episodes.

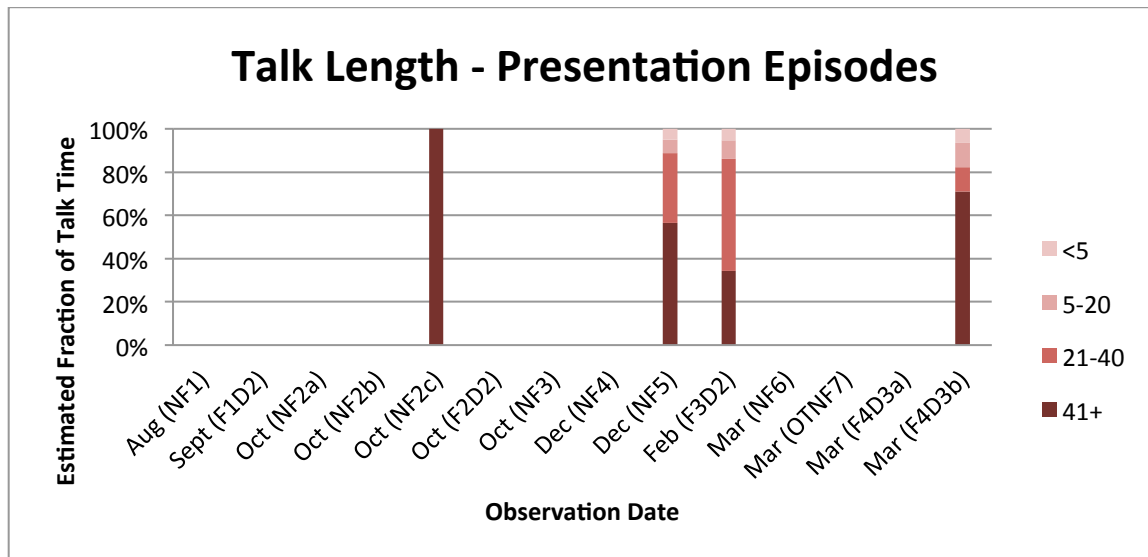


Figure 6.6. TQQ codes for student talk length during presentation episodes.

Notice a more subtle, but highly significant change in the talk length graphs: the small but increasing fraction of student talk turns coded as <5 words long or 5–20 words long. By F4D3b, a significant minority of student talk time (approx. 19%) was spent on short talk turns under 20 words in length. These shorter talk turns do not indicate that some presenters spoke for less than 20 words; this never occurred. Instead, these indicate additional student math talk beyond the presentations. The nature of this math talk is not clear from the TQQ coding. Does the teacher ask a series of less rich IRE sequences before the student presentations? Does the teacher question students during their presentations in a way that builds on their ideas? Are students discussing each other’s ideas? Any of these discourse structures, which are quite different at the meso level, could lead to these micro-level TQQ codes.

Because of the emphasis here on the value of long talk turns for a productive disciplinary engagement frame, readers may initially expect that the presence of shorter talk turns is a negative trend. However, in the next section, it is argued that these shorter talk turns are part of extremely valuable routines at the meso level that build on the presenters’ ideas. Specifically, by the end of the year, the teacher developed new meso-level routines that included asking follow-up questions of some presenters and leading a closing discussion of some of the most important mathematical ideas. These routines are essential steps toward a productive disciplinary engagement frame because they position student presenters as authors of important mathematical ideas and enrich the mathematics by building on the student ideas presented.

In short, all other things being equal, longer student talk turns are more valuable than shorter talk turns at the micro level. However, all other things are not always equal. Having students present one after another without any take-up of the ideas is not mathematically rich. The meso-level analysis proves essential to understand whether and how presenters' ideas are built on by the teacher and/or class.

The remainder of this section describes two different practices that Mr. B used to build on presenters' ideas. The first practice, which he began in the second presentation episode (NF5), was to summarize important ideas from the student presentations after all presentations had been completed. The second practice, which Mr. B began in the last presentation episode (F4D3b), was to ask questions of the presenters immediately after their presentations, before they returned to their seats. Both practices would have been appropriate in all four presentation episodes, but were notably absent from the first presentation episodes. Once begun, both practices contributed significantly to moving the presentation activity toward a productive disciplinary engagement frame.

6.4.3.2.1 Presentation Episode 2: brief teacher summary of different strategies

The first way that Mr. B began to build on presenters' ideas was to summarize important student ideas at the end of the activity, after all presentations had been completed. This practice started in presentation episode 2.

In presentation episode 2, after all student presentations were complete, Mr. B gave a roughly one-minute summary of two important student strategies. Although brief, Mr. B's summary talk does considerable work both to highlight important mathematical ideas and to position students as authors of these ideas. Specifically, Mr. B talked about Problem 3. Recall from the previous section that Problem 3 required students to compare the "chocolatyness" of two drinks, one of which was $\frac{3}{5}$ cocoa and the other was $\frac{2}{3}$ cocoa. Students shared a wide variety of strategies for solving this problem, many of which did not involve a formal comparison of proportions. In his summary, Mr. B highlighted two sophisticated student strategies for comparing fractions, each of which had been presented by only one student: Marco's strategy of using a common denominator and Luciano's strategy of using a common numerator.

Table 6.11 shows Mr. B's talk and board inscriptions for his summary of student strategies at the end of episode 2. The transcript excerpt includes all of Mr. B's math talk after student presentations had been completed. Students interject a few comments in the middle of Mr. B's summary, which he largely ignores. For clarity, student comments have been removed here to give a sense of the flow of Mr. B's ideas. Student comments are added back in in Table 6.12 when student positioning and ownership over the mathematical ideas are analyzed.

In describing both Marco's and Luciano's methods, Mr. B emphasizes which fractions they are comparing by writing these fractions on the board. In both cases,

he does *not* describe the steps of students' procedures for finding these equivalent fractions. Instead, he emphasizes the contextual interpretation of these two comparison methods: comparing denominators is "using the same amount" (of milk) while comparing numerators is "the same amount of cocoa." Mr. B takes more time to talk through the meaning of a common numerator, likely because this method is less familiar to students. In this case, he points to the denominators 9 and 10 and explains that 9 is less water, or milk, than 10, with the same amount of cocoa. All of these explanations occur at a brisk pace, and there isn't sufficient evidence to determine what percentage of the class has followed Mr. B's explanations. Nevertheless, there are significant learning affordances to giving students an opportunity to hear these valuable methods explained twice (once by the presenter and once by Mr. B) as well as to having Mr. B make an explicit contrast between the meaning of comparing numerators versus comparing denominators, a tricky idea for students.

<p>Re-creation of board work</p> $\frac{2}{3} \qquad \frac{10}{15}$ $\frac{3}{5} \qquad \frac{9}{15}$	<p>Marco used a[n approach] similar to this (circles something on board – mostly off camera but likely Luciano's work), but he compared denominators. Uh, when he used two thirds (writes $\frac{2}{3}$) and three fifths (writes $\frac{3}{5}$ below). And if you, (interrupted) Whatever. And you saw this, Marco saw that if you use the same amount (Mr. B has written $\frac{10}{15}$ and $\frac{9}{15}$, not sure when).</p>
$\frac{2}{3} \qquad \frac{10}{15} \qquad \frac{6}{9}$ $\frac{3}{5} \qquad \frac{9}{15} \qquad \frac{6}{10}$	<p>And then Luciano compared numerators instead. (Writes $\frac{6}{9}$ and $\frac{6}{10}$). You can still compare fractions that way. This is denominators (circles $\frac{10}{15}$ and $\frac{9}{15}$) because that's what Marco did, and Luciano, (circles $\frac{6}{10}$ and $\frac{6}{9}$), here's another thing you could do, using the numerator, because that's less water (point to the 9 in $\frac{6}{9}$) or less milk. Than this (points to 10 in $\frac{6}{10}$). They have the same amount of cocoa.</p>

Table 6.11. Mr. B's summary of two student strategies in episode 2.

Next, consider the extent to which Mr. B's summary positions students as authors of mathematical ideas. First, Mr. B consistently cites both Marco and Luciano as authors of their methods. These citations are important to students. In fact, at least one student is so concerned that Luciano get credit for his work that he interrupts Mr. B three times while Mr. B is explaining Marco's solution, to try to credit Luciano (lines 37, 39 and 41). Mr. B largely ignores these interruptions and the student stops interrupting once Mr. B starts to explain Luciano's work as well (and appropriately credits Luciano). Table 6.12 shows the full transcript of Mr. B's

summary plus student interruptions. The interrupting student(s) are off camera and it cannot be determined whether it is the same student each time or various students.

	Mr. B talk	Student talk
BNF5-36	Marco, used a similar to this (circles something on board – mostly off camera but likely Luciano’s work), but he compared denominators. Uh, when he used two thirds (writes $2/3$) and three fifths (writes $3/5$ below)	
BNF5-37		That’s Luciano’s!
BNF5-38	And if you	
BNF5-39		I thought that was Luciano.
BNF5-40	Whatever. And you saw this, Marco saw that if you use the same amount (Mr. B has written $10/15$ and $9/15$, not sure when).	
BNF5-41		That’s Luciano’s!
BNF5-42	And then Luciano compared numerators instead. (Writes $6/10$ and $6/9$). You can still compare fractions that way. This is denominators (circles $10/15$ and $9/16$) because that’s what Marco did, and Luciano, (circles $6/10$ and $6/9$), here’s another thing you could do, using the numerator, because that’s less water (point to the 9 in $6/9$) or less milk	
BNF5-43		Milk!
BNF5-44	than this (points to 10 in $6/10$). They have the same amount of cocoa.	

Table 6.12. Transcript of summary talk at the end of presentation episode 2.

The last student interruption is the word “Milk!” in line BNF5-43. This comment immediately follows a teacher hesitation over whether the denominators in Luciano’s fractions represent an amount of water or an amount of milk. In a sense, it doesn’t matter to Mr. B whether water or milk is being used to dilute the cocoa – either way, a larger amount of diluting liquid results in a less chocolaty beverage. However, the student comment demonstrates that this student both knows that the cocoa is made with milk and cares enough about this detail to interrupt the teacher. This is a stark contrast with presentation episode 1, in which the problem context was ignored by all students, to the point where one student did not notice that her answer would lead to someone being overcharged by a factor of 10 for their purchase of kiwis.

In summary, the second presentation episode supports a much greater variety of student contextualized sensemaking and use of multiple strategies than the first presentation episode (shift 1). As a result, the mathematics content affords more connections between procedures, concepts and contexts, an important indicator of progress toward a productive disciplinary engagement frame. After eliciting this rich variety of student ideas, the teacher takes a small but significant step toward building on these student ideas (shift 2). He does this by giving a brief, one-minute summary of two of the most important student ideas, crediting students and positioning them as authors. Furthermore, students take on a limited critiquing role within Mr. B's summary, for example when students interrupt the teacher's summary to assign credit to a different student for an idea or to clarify a detail of the problem context. The next section discusses how Mr. B expands his practices for building on student ideas by the fourth presentation episode.

6.4.3.2.2 Presentation Episode 4: Questioning presenters and discussing student difficulties

By the fourth and final presentation episode of the year, Mr. B has significantly expanded his repertoire of practices for building on presenters' ideas. He expands his practice from episode 2 of summarizing presenters' ideas. In presentation episode 4, he leads a six-minute summary and discussion of several important points from the presentations. Mr. B also begins a new practice in presentation episode 4 of asking extension questions to presenting groups immediately after their presentations, while the group is still at the front of the classroom.

This section shows how shift 2 unfolds in presentation episode 4. For ease of reading, the excerpts from episode 4 are presented in chronological order. First, I explain the mathematics being discussed in the episode. Second, I show how Mr. B asked extension questions of the presenting groups. Third, I show how Mr. B led a summary and discussion at the end of the presentation activity structure.

6.4.3.2.2.1 Mathematics of presentation episode 4

Episode 4 takes place on the third day of work on an extended task connecting equivalent arithmetic expressions with area models representing those expressions. The activity of making these connections is intended to strengthen student understanding of the laws of arithmetic, such as the distributive property. The task was a Formative Assessment Lesson created by the Shell Center for Mathematics Education (<http://map.mathshell.org/lessons.php?unit=6220&collection=8>).

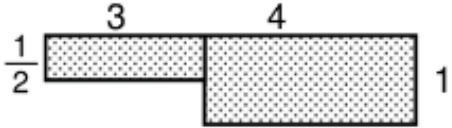
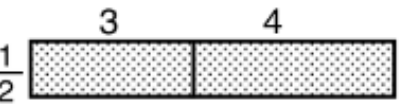
<p>A3</p> 	<p>E10</p> $4 + 3 \times \frac{1}{2}$
<p>Blank A8</p> 	<p>E14</p> $\frac{3}{2} + 4$
	<p>E6</p> $\frac{3}{2} + \frac{4}{2}$
	<p>E11</p> $\frac{1}{2}(3+4)$
<p>E13</p> $\frac{3+4}{2}$	

Figure 6.7. Partial task solutions for presentation episode 4.

Most of the expressions and areas involve small whole numbers, but two sets of equivalent cards include fractions. These are particularly difficult for students. Figure 6.7 shows these two sets of equivalent cards. Card A3 shows an area diagram that matches both expressions E10 and E14. Card A8 is blank in the student materials. Students are expected to recognize that expressions E6, E11 and E13 are equivalent to each other but do not match any of the area models pictured in their materials. They are expected to then match these with the blank card A8 and draw their own area model. The area model pictured in the solutions is one very sensible option, but students often draw other area models with the same area; it is left to the teacher's discretion how to respond to these possible student solutions.

It is important to note that the mathematics of this task, as enacted by the teacher and students, did not take advantage of all affordances in the written task. Specifically, students primarily used a single strategy to make most card matches: calculating the area of each diagram and the result of each arithmetic expression and comparing these two answers to check for equality. This focus on matching answers led to several missed opportunities for deeper connections between the two representations that could potentially strengthen students' conceptual understanding. For example, in diagram A3, a potentially fruitful mathematical

action would be to extend a horizontal line through the middle of the 4×1 rectangle, in order to create area diagram A8 within diagram A3. This new line could be used to justify many connections and card matches, for example, that E14 belongs with A3 while E6 belongs with A8. I did not observe the students or teacher engage in these types of mathematical connections during the task.

The creation of a fairly rich discussion space that still misses some important learning affordances is very typical of teachers teaching Formative Assessment Lessons (FALs) for the first time (Seashore, 2015). The FALs often use unfamiliar mathematical representations and/or require students to make connections that are outside the content of most curricula. It takes time for teachers to develop new pedagogical content knowledge about the multiple ways students can engage with these new representations and the connections between these student strategies and important mathematics. A typical teacher trajectory is to teach a specific Formative Assessment Lesson at least twice, with significant time and support to unpack the mathematical affordances of the task and make sense of unexpected student responses, before having a full grasp of the potential of the task.

The missed mathematical opportunities in this lesson are associated with the lower fraction of WHY talk in presentation episode 4 noted in the quantitative analysis section. This doesn't mean that Mr. B reversed his learning trajectory. Like many aspects of teaching practice, one can expect that he has a fairly large repertoire of possible practices and any given observation only captures a small slice of these. One can't expect his learning to be linear. In fact, continued experimentation with rich tasks, even if this occasionally leads to missed opportunities for mathematical connections, is an important indication that Mr. B is continuing to spend effort finding tasks and expanding his pedagogical content knowledge in ways that support his continuation of shift 1, increasing WHY talk.

In particular, there is no evidence that the lessening of WHY talk in presentation episode 4 was *caused* by the shift 2, teacher questioning of presenters. In the next section, an example is shown of teacher questioning and how the questions push students toward higher-level talk types, for example by asking one student presenter to elaborate on a brief WHY talk turn he had already given during his presentation, and encouraging two other groups of students to move from HOW to WHY by asking them to consider the rationale for a common mistake.

However, it is important to note that the shift toward increased WHY talk is contextualized by the mathematics and the teacher's pedagogical content knowledge in a particular mathematical domain. Having a rich mathematics task is a prerequisite for WHY talk, but the level of talk also likely depends on the teacher's and students' *familiarity* with the mathematics at hand. Connected mathematics depends on students having enough familiarity with the representations and procedures used to gain access to the central mathematical ideas, and the teacher having enough familiarity with likely student strategies that he can predict, elicit, and connect these student strategies with central ideas of the discipline.

6.4.3.2.2 Example of teacher questioning

Over the course of the presentation episode 4, 11 groups came to present at the front of the class. The teacher asked extension questions to 4 of the 11 groups and a 5th group was asked a question by another student. Below is a brief overview of all questions asked by Mr. B, followed by a more in-depth analysis of one example.

Here are the questions that Mr. B asked groups:

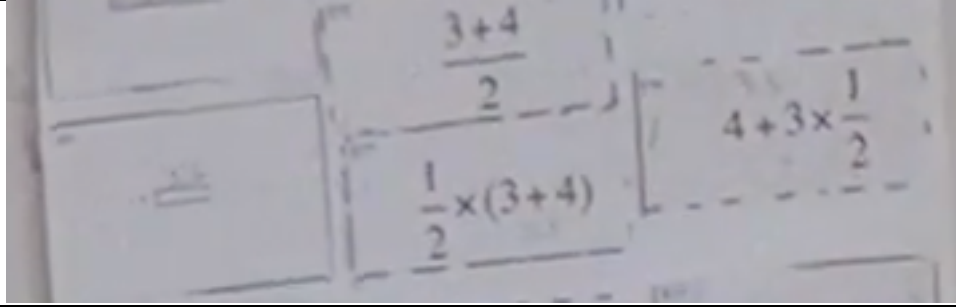
- 1) "Say which problem you are doing."
- 2) "How do you know it's 25 and not 57?"
- 3) "Okay, how do you know the answer is 25 and not 57?"
- 4) "What does your drawing say?" / "How did you label your sides?"

Question 1 is asked of the first group and is a simple re-elicitation to remind the group that their audience needs to know which question they are presenting on before they launch into a description of their arithmetic steps. Questions 2 and 3 are the same question asked of two different groups. This question has to do with the arithmetic problem $4 \times 4 + 3 \times 3$, for which a common mistake is to do the operations from left to right: 4×4 , then $16 + 3$, then $19 \times 3 = 57$. The first group that was asked this question did not adequately explain where the 57 came from, creating a lot of excitement in the class as other students asked permission to present next in order to have a chance to answer the same question.

Here, for the first time, Mr. B positions students in a critiquing role by asking them to engage with a hypothetical wrong answer. The first group asked this question didn't really engage with the critique activity, instead simply restating their answer. This was accepted by Mr. B as correct but not seen as deserving extra credit for being able to fully answer the question. This created a lot of excitement for the other students to want to engage in critique. This could be a scaffolded step toward students critiquing each other's ideas directly. It certainly marks a break from a knowledge transmission frame that Mr. B is introducing "wrong" answers into the conversation. In the baseline episodes, Mr. B avoided extended talk about wrong answers, perhaps to avoid confusing students, but now he is actively promoting the mathematical value of considering wrong answers and explicitly talking about common mistakes.

Question 4 is a little more complex, because it is actually a series of questions related to a specific part of the preceding presentation by one student, Steven. Table 6.13 shows the transcript of Steven's presentation and Mr. B's follow-up questions. Like the other presenters in this episode, Steven presents one area model card of his and all the expression cards that he thought matched that area model. Steven chose area model A8, which was the blank card that students had to draw themselves. Steven was one of the few students to present on one of the area models with fractions.

Steven's poster: card A8 at left, E13 on top, E11 bottom center, and E10 at right



Steven's presentation

[Card A8]

I did A8. I was lazy, so I did 1 times 3.5 for the area.

[Card E13]

And did 3 plus 4 which is 7, times a half, which got me 3.5.

[Card E12]

And 3 plus 4 which is 7, again, divided by 2, which is 3.5.

[Card E10 – sic]

And this is the same thing as this one and this one so I just got 3.5 as well.

Teacher Questioning

Mr. B talk	Steven's talk
Did you create a drawing for that one?	
	Yeah, this one.
What does your drawing say?	
	The area.
How did you label your sides?	
	I did 1 for the height and 3.5 for the length.
Ohh.	
	The base. Yeah.
Okay.	

Table 6.13. Transcript of Steven's presentation and Mr. B's follow-up questions.

Mr. B's questions ask Steven for more detail about his area diagram. One purpose of these questions is likely for Mr. B's own information since Steven's area diagram was both unusual and physically very small. A second purpose may have been to help Steven articulate his idea in a way the class could understand, especially because creating a diagram with fractions was a challenging part of a challenging problem and Steven was one of the only students to present on it. Mr. B's questioning leads Steven to publicly clarify what his diagram looks like, though he does not elaborate more on the strategic reasoning he used to create this diagram.

Steven describes his choice of dimensions as “lazy,” probably because he thought that using 1 as a side of a rectangle led to the very easy multiplication problem $1 \times 3.5 = 3.5$. Presumably, Steven already had the number 3.5 based on his arithmetic work on the other cards and knew this card needed to match, so he picked the easiest pair of number he could think of that he knew would have a product of 3.5: 1 and 3.5. This “lazy” reasoning is arguably a strategic choice that is very much in line with the lesson’s focus on properties of arithmetic: one important property of arithmetic is that 1 is the multiplicative identity. However, Mr. B does not further pursue the matter after asking Steven to state the dimensions that he chose.

Notice that Steven makes a mistake on card E10, but Mr. B does not comment on it. Steven says that card E10 is “the same thing,” even though in card E10 the number $\frac{1}{2}$ is only being multiplied by the 3 and not by the 4. This likely reflects a common mistake by Steven. Most likely, Steven did the operations on card E10 from left to right: first adding $3+4$ to get 7, then dividing $7/2$ to get an answer of 3.5, just as in the other problems. A correct solution for E10 would be to multiply 3 times $\frac{1}{2}$ to get 1.5 and then to add 4 to get 5.5. Most likely, Mr. B did not see this mistake. Student posters were difficult to read from the back of the room where Mr. B and I were standing, and Steven did not name the cards he used. I also missed the mistake until I analyzed the episode from videotape and I was able to look carefully at a zoomed-in version of the cards Steven was presenting.²⁷

Finally, one group was asked a question by another student, but unfortunately the question was inaudible on the videotape. Another student in the audience can be heard answering the student’s question by saying, “He said use 2.” These are promising signs that the class is on the verge of taking on ownership for critiquing and building on each other’s work as well as presenting.

Even though this episode was the first time Mr. B asked extension questions to presenters, his questions cover several of the most important mathematical ideas in the task. Question 1, though simple, orients the presenter to the need to consider the needs of his audience instead of just stating procedures. This conveys to all students that the purpose of the presentations is to deepen the understanding of the class, rather than just to summarize what has been done – a purpose much more consistent with a productive disciplinary engagement frame than presentation episode 1. Questions 2 and 3 constitute an extension beyond the given task that invite students’ reflection about common procedural errors. These questions are inconsistent with a knowledge transmission frame because the teacher is proposing an incorrect idea for the sake of learning. However, from a productive disciplinary engagement perspective, questions of this type are a valuable way to build students’

²⁷ This type of logistical issue may be common as teachers try out a student presentation routine. Other teachers in the department who were more experienced with student presentations often used a document camera to allow students to enlarge their work and make it visible to the class. However, this option was not available to Mr. B because his projector was not working at the time.

metacognitive reflection and strengthen their understanding. Question 4, really a question series, asks for a student to elaborate on the details of his unusual area model. This question is also consistent with a productive disciplinary engagement frame because the teacher's response to an unusual and novel student idea is to invite the student to explain the idea in more detail. These questions are not numerous; four questions in half an hour does not yet constitute a discussion. Nevertheless, they represent a promising beginning. These types of questions, reminding students to consider their audience's needs, prompting reflection on common mistakes, and asking students to elaborate surprising ideas or challenging problems, all align with a productive disciplinary engagement frame and have corresponding affordances to deepen student sensemaking.

6.4.3.2.2.3 Discussion of student difficulties

In addition to the new practice of questioning students, Mr. B continues and extends his previous trend of giving some concluding comments after all student presentations are complete. Similarly to episode 2, he references students' presentations in his concluding remarks. Unlike episode 2, his concluding comments in episode 4 are significantly longer (six minutes instead of one minute), covered two of the main problems that students had struggled with as well as some more general points, and include space for students to speak as well.

The first topic discussed is the area model A3, which had been challenging for students because of the side length of $\frac{1}{2}$. Mr. B calls on several students to share their answers for the problem and invites one student to share his reasoning process as well. Second, Mr. B gives feedback to one particular group about their good presentation skills and briefly shows how to do a second problem that the same group had struggled with. Third, Mr. B uses a few IRE sequences to help students attach the formal vocabulary "distributive property" to the main idea of the lesson.

Mr. B's concluding remarks in presentation episode 4 stand in stark contrast to presentation episode 1, when student mistakes went unremarked upon. It is enfranchising for students to have their ideas taken up by the teacher and to have a chance to discuss and work through their difficulties with support from their peers. This concluding discussion is another indicator that the purpose of presentations is no longer "to demonstrate the correct execution of standard methods" (a purpose which would be consistent with a knowledge transmission frame), but rather to raise important mathematical ideas for discussion (a purpose much more consistent with a productive disciplinary engagement frame).

6.4.3.3 Summary: Enriching the new practice

Over the course of the four presentation episodes, Mr. B makes two major shifts that enrich the presentation activity. Table 6.14 summarizes how Mr. B's presentation practices shift over the course of the four episodes.

Adaptation of “all groups present” practice				
Classroom practice	Episode			
	NF2c	NF5	F3D2	F4D3
Who presents?	All groups / all students in the group	All groups / all students in the group present	All groups / one student per group who was least focused during small group time	All groups / all students in the group
Framing the presentations	Little framing talk; teacher only tells groups when to start and stop	Presentation rubric prompts explanation	Presentation rubric posted in the classroom, and at least implicitly made relevant	Presentation rubric posted in the classroom, and at least implicitly made relevant
Presenter talk length and type (TQQ codes)	Presenters take long talk turns (41+ word code)	Presenters take medium to long talk turns (21+ words)	Presenters take medium to long talk turns (21+ words)	Presenters take medium to long talk turns (21+ words)
	HOW presentations	HOW and WHY presentations	HOW and WHY presentations	HOW and WHY presentations
Teacher talk framing and/or building on students’ ideas	Presentations are self-contained and no one follows up on the ideas presented.	Teacher briefly summarizes presenters’ strategies		Teacher explains big ideas missed in presentations
				Teacher asks extension questions to presenters

Table 6.14. Adaptations in the student presentation practice over time.

Shift 1 begins in presentation episode 2. Mr. B supports students explaining more about their sensemaking and conceptual justification (WHY talk) in addition to describing their procedures (HOW talk). This shift can be seen in Table 6.14 in the row “TQQ codes.” Mr. B supports this shift through the use of problems that support multiple strategies and through a presentation rubric that values conceptual understanding (“Framing the presentations” row).

Shift 2 also begins in the second presentation episode and is significantly more developed in the fourth presentation episode. Mr. B builds on presenters’ ideas by summarizing student ideas for problems that most students had struggled with. Starting in the fourth presentation episode, he also builds on presenters’ ideas by asking them follow-up questions. This shift is summarized in the row “Teacher talk framing and/or building on students’ ideas” in Table 6.14.

6.5 Summary

This chapter shows a partial learning trajectory of one experienced teacher, Mr. B, toward a productive disciplinary engagement frame. At the beginning of the school year, Mr. B is an experienced and effective teacher. The baseline teaching episodes show strong student engagement and correct mathematics. However, they are fairly closely aligned with a knowledge transmission frame: most problems have a single, correct solution procedure and students are expected to practice that procedure and demonstrate their knowledge to the teacher. During whole-class activities, students rarely speak more than a few words at a time. A few students are invited to speak a sentence or so about their process or reasoning, but only after the teacher has checked that their answer is correct. Even these slightly longer speech turns are a near fit to a knowledge transmission frame: students are not positioned as authors or critics of mathematical ideas because the teacher has pre-evaluated their ideas and the ideas they share fit well with a pre-established flow of the lesson, guided by the teacher.

Early in the year, Mr. B tries a new activity structure: bringing closure to a period of small group work by having all groups present their work at the front of the classroom. Mr. B sustains and adapts this new activity structure throughout the year, leading to four observed student presentation episodes.

Student presentations are portrayed in the literature as largely consistent with a productive disciplinary engagement frame (Stein et al., 2008; Nathan et al., 2007). However, the data presented in this chapter are more complex. The findings in this chapter are consistent with a model of classroom discourse that that similar micro-level talk turns can support different frames depending on how the meso-level structure and function of the talk positions students with respect to mathematics.

Presentation episode 1 in Mr. B's classroom is a case in point. Micro-level analysis, through TQQ codes, reveals an increase in long and very long student talk turns. This indicates that students had more "air time" to share ideas. Nevertheless, meso-level analysis finds the mathematical content of student presentations to be somewhat off the mark. Students present largely correct renditions of a particular method for multiplication that they have practiced in class. However, they fail to connect their work to either the concept of place value in decimals or to the word problem context. They occasionally report nonsensical results, such as paying \$30 for berries that should cost \$3, but these results are not publicly remarked upon. Students are not positioned as authors of mathematical ideas because they are all using the same procedure, and no one (including the teacher) critiques the ideas that students present. Thus, despite the significant shift in micro-level talk turns from the baseline episode to the first presentation episode, the framing remains largely the same: the mathematics content and student and teacher roles remain consistent with a knowledge transmission frame.

Nevertheless, Mr. B's initial implementation of the student presentation activity creates new affordances for *his own* learning. As he sustains the practice throughout the year, he also adapts it and, jointly with students, moves the activity much closer to a productive disciplinary engagement frame. He accomplishes this change through two main shifts. First, beginning in presentation episode 2, he supports students to share more of their reasoning, leading to a large fraction of explanatory (WHY) talk during the presentations. Mr. B supports this shift by assigning a task that afforded multiple strategies and using a presentation rubric that explicitly prompts students to explain their thinking.

Perhaps the most significant shift between episodes 1 and 2 is a shift in Mr. B's goals for using student presentations. His initial goal for student presentations was to help students learn public speaking skills. As a researcher, I could easily have dismissed this goal (in fact, I came close to doing so). Public speaking skills are certainly important for students to learn, but initially seemed tangential to my research focus of helping teachers elicit and build on students' mathematical ideas. Nevertheless, Mr. B and I were quickly able to agree that it would be valuable to try having students present. This convergence of our different goals allowed Mr. B to quickly create a very new activity in his classroom: having students speak at length about whatever was easiest for them to think about. It was this new activity that plausibly allowed for the rest of the teacher learning to take place, which Mr. B and I both valued highly by the end of the year. Mr. B's shift in goals is one example of a well-documented phenomenon: major shifts in teaching practices are accompanied by parallel shifts in teacher goals, resources, and orientations (Schoenfeld, 2010).

For a productive disciplinary engagement frame, it is necessary but not sufficient to elicit student thinking; student ideas also need to be refined and connected to important disciplinary ideas. Mr. B's second shift takes a productive step in this direction. He builds on student ideas in two main ways. First, in episodes 2 and 4, he summarizes important points from the student presentations, in a way that either connects student ideas to each other (in episode 2) or draws out new student comments to address areas of confusion from other students (in episode 4). In both episodes, his summary talk positions presenters' ideas as important contributions to a larger mathematical conversation. Second, in episode 4, Mr. B asks follow-up questions to presenters after they have finished presenting but while they are still at the front of the class. He does not yet ask very many questions, but the types of questions he does ask support student metacognition and explanation, further positioning students as authors and, to a limited extent, critics of mathematical ideas.

Each time Mr. B tries something new related to student presentations, it creates new affordances for his own learning. In episode 1, he and his students have a positive first experience with students taking long, public talk turns. This in itself is a major shift in the norms of the classroom. Without this experience, it could be hard to imagine students giving presentations about novel mathematics. Once students have presented *at all*, Mr. B has a new opportunity to reflect about what they should

present *about*. A teacher might or might not see this opportunity or choose to take advantage of it, but Mr. B does so. In the second presentation episode, he uses a task that affords multiple strategies and a rubric that explicitly values explanation to support students to explain their own ideas. Students rise to the occasion splendidly, which in turn creates a new opportunity for Mr. B to critique or connect these student ideas. Again, another teacher might or might not take advantage of this opportunity, but Mr. B seizes it. He immediately recognizes the value of a novel student strategy – comparing fractions by finding a common numerator – and gives a one-minute summary connecting this idea to a more standard strategy of comparing fractions by finding a common denominator. After additional reflection time, he goes even further to build on student ideas. By the fourth presentation episode, Mr. B asks a few extension questions to student presenters and leads a somewhat longer wrap-up discussion. Although his practice cannot yet be said to be fully aligned with a productive disciplinary engagement frame – there are still important missed opportunities in episode 4, both to highlight key mathematical ideas and to position students as authors and/or critics of mathematical ideas – substantial progress has been made toward a productive disciplinary engagement frame.

The partial learning trajectory presented in this chapter is consistent with the claims about classroom discourse presented in earlier chapters: the same micro-level discourse structures can support different frames depending on their function within a meso-level structure, and in particular how they position students with respect to the mathematics. In this case, the micro-level structure of a student giving a relatively isolated presentation (a long student talk turn that is not a response to a specific teacher question) supported either a knowledge transmission frame or something much closer to a productive disciplinary engagement frame. Furthermore, these data show how this perspective on discourse is helpful and necessary to understand one powerful example of teacher growth.

A more practical, yet critically important, message that emerges from these data concerns the pace of teacher learning. Major changes in practice take time. In particular, the learning described in this chapter took place over a full school year, and was only part of a hypothetical trajectory that might easily take 3–5 school years. First, the teacher needed space to try a very new practice, which was different but not initially more mathematically valuable than his prior practices. Then, the teacher sustained and adapted the new practice over the course of a full school year. Most of the learning value for students came from the adaptations of the new practice. By the end of the school year, the teacher had learned to elicit a range of student ideas and sensemaking, and was just beginning to try ways of building on the student ideas he elicited. He could easily spend another several years honing his practices of questioning, summarizing, and facilitating discussions of presenters' ideas. It is essential to give teachers time to do this type of long-term learning.

7 Discussion

This chapter summarizes the main findings of the dissertation and discusses their implications for professional development and future research.

7.1 Findings

This section summarizes two primary contributions of the dissertation. The first finding concerns the interrelationship between mathematically and culturally responsive teaching. In short, mathematically responsive teaching is a sociopolitical act in that positioning nondominant students as mathematical authorities and building on their funds of knowledge often requires explicit attention to negotiating power relationships in equitable ways. The second finding concerns points of continuity between the knowledge transmission and productive disciplinary engagement frames, and how this continuity can be leveraged to understand and support teacher learning.

7.1.1 Finding 1: Mathematically responsive teaching is a sociopolitical act

The first major contribution of this dissertation was to conceptualize, and describe in some detail, a model for integrating culturally and mathematically responsive pedagogy. Specifically, I pursued an analytical focus on how nondominant students can be positioned as authors and critics of mathematical ideas within a productive disciplinary engagement frame. Throughout the analysis, I found that teachers' sociopolitical work to position nondominant students in authoritative roles and build on their funds of knowledge was an integral part of their efforts at mathematically responsive teaching.

Chapter 4 presented two classroom case studies in which experienced Black teachers leveraged specific Black cultural practices to both support student engagement and enrich the mathematics being discussed. In both case studies, nondominant students (a) were invited to blur the line between school mathematics and outside-of-school funds of knowledge, (b) shared novel mathematical ideas, and (c) were positioned as authors and/or critics of mathematical ideas by their teachers. In the first case study, "Write up why while you whippin and nae naein," Mr. X listened to Enoch's novel solution method, invited him to present his method at the front of the class, supported another student to explain his own understanding of Enoch's strategy, and highlighted some important mathematical ideas related to the strategy. At the same time, Mr. X used elements of AAE and knowledge of hip hop to affirm that Enoch was welcome to "be himself" as a young Black man in the classroom, in particular by using hip hop dance moves to celebrate mathematical success. In the second case study, "I don't want to work more if they're almost paying me the same," Mr. B supports his Latinx and Black students to draw on a range of inside- and outside-of-school knowledge to debate which of two

jobs they would prefer to have. The debate format supports students to engage in critiquing and refining mathematical ideas and to apply these ideas in an authentic context, two important areas of consistency with a productive disciplinary engagement frame. Students are highly engaged and have a much richer conversation about mathematical ideas related to proportionality than would have been afforded by a narrow interpretation of the homework problem. Mr. B also frames the conversation in a way that communicates respect for honest work, even if the pay is low, and a deep sense of caring about students' success in life. Students demonstrate their buy-in to this framing through comments and questions about the social context of the mathematics problem, including comments about wanting to support their families and questions about overtime pay, bankruptcy and credit card debt.

Chapter 4 showed that positioning nondominant students as authors and critics of mathematical ideas is a sociopolitical act, and the line between “mathematical” and “nonmathematical” positioning is often blurry. Specifically, it showed that some teacher moves that appear at face value to be “nonmathematical” positioning could function as strong supports for students to take new mathematical roles. In Case 1, Mr. X's affirmation of students' hip hop affiliation was an important part of positioning them as central participants in the class, which in turn was an essential foundation for teacher and students to feel comfortable exploring a novel student idea through a novel activity structure. In Case 2, Mr. B drew on his knowledge of and passion for a real life context: making strategic choices about work hours and salary. He also used social moves that positioned him in solidarity with community members. This cultural knowledge, in addition to his relevant mathematical knowledge, was essential to orchestrating the culturally and mathematically responsive discussion shown in the chapter.

7.1.2 Finding 2: Continuity between KT and PDE frames

The second major contribution of this dissertation was to conceptualize, and describe in some detail, several points of continuity between the knowledge transmission and productive disciplinary engagement frames. In particular, I analyzed how similar discourse moves could be adapted and repurposed to be increasingly aligned with a productive disciplinary engagement frame. Two examples of such discourse moves, analyzed in Chapters 5 and 6 respectively, were IRE sequences and student presentations.

Chapter 5 concerned IRE sequences, the presence of which is portrayed in the literature as a strong indication that a knowledge transmission frame is operating. The cases presented in Chapter 5 complexified this picture substantially. The primary case was a conversation in a sixth-grade math classroom about two student presenters' strategies for calculating the median of a dataset. The first presenter gave a high-level description of a standard strategy, but got stuck when pushed by another student to flesh out the details. The teacher used IRE sequences as a temporary scaffold to help the presenter “step through” the initial steps of the strategy, after which he continued on his own. The second presenter described a

nonstandard strategy that applied the concept of the “middle number” directly to a frequency plot, without the need to list out the numbers. The teacher then used IRE sequences to review a few important aspects of the second presenter’s strategy while highlighting connections to the first presenter’s strategy. Both chains of IRE sequences positioned the student presenters as authors of mathematical ideas while supporting the continued development of their thinking and making that thinking more transparent to the class.

The findings in Chapter 5 complexified the prior literature finding that IREs are strongly indicative of a knowledge transmission frame. The analysis refutes the idea that all instances of “closed-ended” questions, short student responses or immediate teacher evaluations undermine student authorship. It remains true that “closed-ended questions” and short student responses, alone, provide insufficient opportunities for students to share their ideas. We can expect that as teachers move toward a productive disciplinary engagement frame, they will ask some “open-ended” questions that invite longer explanations from students. Similarly, the teacher evaluation turn of an IRE sequence can cut off opportunities for student critique. This suggests the evaluation turn should often be delayed or eliminated as teachers move toward a productive disciplinary engagement frame. However, it matters what is being evaluated. Both cases in Chapter 5 showed examples where evaluations of smaller pieces of information on the way to a big idea were not overly disruptive of a larger scale sense of student ownership over that big idea. Indeed, these short IRE chains appeared to be helpful access moves.

In short, IRE sequences, like any discourse move, can be adapted to a variety of purposes. Consistent with a continuity perspective on teacher learning, we would not expect teachers who use IRE sequences to experience a “gestalt shift” and suddenly begin asking only open-ended questions. Instead, teachers may gradually adapt and repurpose IRE sequences and combine them with other discourse moves to be increasingly consistent with a productive disciplinary engagement frame.

Chapter 6 showed just such a learning trajectory, in which a different discourse form – student presentations – was adopted by Mr. B and gradually adapted and integrated with his other teaching practices in ways that were increasingly consistent with a productive disciplinary engagement frame. The first presentation episode was aligned with a knowledge transmission frame. Several students presented the same mathematical procedure, with limited discussion of the problem context or concept of place value. This was consistent with prior findings that “show and tell” presentations do not have the same mathematical affordances as presentations that build on the details of students’ mathematical ideas (Stein et al., 2008). Nevertheless, the activity had significant affordances for both student and teacher learning because it allowed all classroom actors to gain comfort with a new routine that included substantially more student “air time” than previously. That is, students gained comfort speaking at the front of the class while the teacher had a chance to explore and reflect on the learning affordances of a new routine. Subsequent presentation episodes had strong points of alignment with a productive

disciplinary engagement frame. The chapter documents several specific shifts in Mr. B's teaching practice that supported this alignment, including: the use of tasks that afforded multiple strategies, the use of a presentation rubric that encouraged students to explain their thinking, teacher follow-up questions to presenters, and teacher summary talk highlighting important mathematical ideas presented.

Taken together, these chapters showed that it is necessary, but not sufficient, to create activity structures where students get to talk – there is no substitute for students sharing their own ideas. However, it is also critically important what students talk *about* and how students' mathematical ideas are positioned and taken into follow-up discussion. The first presentation episode analyzed in Chapter 6 showed that students can be the main speaker while the mathematical focus remains on teachers' ideas. If student speakers get "air time" but only share correct answers or only repeat standard strategies, this still has strong areas of alignment with a knowledge transmission frame. Simply reporting the volume of teacher and student talk without discussing how the contributors are positioned and where mathematical authority seems to lie cannot adequately capture to what degree a knowledge transmission, productive disciplinary engagement, or hybrid or other frame is in play. A more important question is: "Do student *ideas* get air time?" or, in a similar vein, "To what extent are students positioned as authors of mathematical ideas?" Some indicators that students are being positioned as authors include: multiple strategies, representations, connections, and partial and incorrect work contributing to a larger conversation. Various discourse structures can be used to explore and build on students' ideas. Examples given in the chapters included teacher and student follow-up questions to presenters, teacher summary talk or brief chains of IRE sequences elaborating on or connecting student ideas, and teacher-led discussions of issues raised by students during the presentations.

7.2 Implications for professional development

A continuity perspective holds that teachers trying new practices will gradually integrate them with existing practices. Some new practices can open up classroom activity in ways that allow even experienced teachers to see very different things from students. For example, student presentations were an ambitious practice that was new for Mr. B and Mr. Y, and enacted in new ways by Mr. X that led to more student explanation than previously. These adaptations of the student presentation practice were accompanied by substantial shifts in activity-level goals, and expanded learning opportunities for students. Specifically, Mr. B initially used student presentations in ways that were consistent with a goal of having students learn public speaking skills, with limited affordances for further conversation about important mathematical ideas. Similarly, prior to the presentation episode in Chapter 4 (although not analyzed in detail) Mr. X initially used student presentations in place of a quiz, in ways that were consistent with a goal of summatively assessing students' knowledge of a particular procedure and also had limited affordances for developing mathematical ideas. For both teachers, presentations developed into a space for further consideration and, to some extent, conversation about important mathematical ideas. When practices like student

presentations are sustained and developed over time, as in the case of Mr. B's learning trajectory in Chapter 6, these practices can lead to substantial shifts in overall classroom framing toward a productive disciplinary engagement frame.

The learning trajectory in Chapter 6 is analyzed at a fine grain size in order to demonstrate what a continuity analysis of teacher discourse change can look like. Here I look at a larger grain size at the learning that occurred over the course of the school year within the department teacher learning community at Adams. I attempt to collect the discourse moves used by various teachers to achieve similar goals. What follows is a rough sketch of a few general milestones in mathematically responsive teaching, that may help guide professional development at other sites. The sketch is higher-inference than the detailed analyses in the data chapters and is intended to be suggestive, not prescriptive.

A first goal that a teacher learning community can work on is how to get students to publicly speak multiple sentences about the mathematics. All focal teachers in my study accomplished this goal. A primary method used was by orchestrating various opportunities for students to present their mathematical thinking at the front of the class. These included:

- Asking students to come to the front of the class after a period of group work to present their final conclusions (Mr. B, Mr. X, Mr. Y);
- Asking students to come to the front of the class in the middle of a period of group work to “share out” their emerging ideas (Ms. A);
- Asking students to come to the front of the class with little prior preparation, to work out a familiar type of problem at the board and describe their process for solving the problem (Mr. X).

It is possible to enact any of these types of student presentations in a way that is largely consistent with a knowledge transmission frame for learning. Nevertheless, each provides opportunities for students to speak at length, which potentially creates an opportunity for future learning.

Presentations are not the only way that teachers in my study tried to get students to speak at length. Teachers also asked questions to prompt student explanation. Many of these questions were related to IRE sequences, but with expanded opportunities for student talk and/or ownership. This suggests that if teachers typically use IRE sequences in their classrooms, there are various ways that IRE sequences can be modified to create expanded opportunities for students to explain their mathematical thinking. At least one teacher in my study tried each of the following when going over work:

- The teacher first asked students for a brief statement of their answer, then asked a follow-up question that required a more extended response, such as “How did you get 5.2?” or “How would the answer change if the problem was...? Why?”
- The teacher attributed the mathematical knowledge being discussed to students.

- The teacher delayed the evaluation, instead asking students to share a few possible answers before evaluating. It is worth noting here that, in general, efforts to delay evaluations may require attention to implicit evaluations – for example, writing multiple answers on the board instead of only correct answers.

Each of these discourse moves creates significantly more opportunities for students to explain their thinking than IRE sequences alone.

Once students have had opportunities to talk at length, a possible second goal for a teacher learning community is to interpret the student thinking thus shared, note productive beginnings and common misunderstandings, and respond. Teachers in my study accomplished this in a variety of ways, including:

- After several student presentations on the same problem, several teachers briefly summarized the different student strategies used and drew connections between them.
- During or after a student presentation in which the student shows an incorrect or incomplete understanding, several teachers either provided or asked other students to provide missing information or a new way of looking at the problem.

A classroom in which student thinking is being elicited and responded to already has many aspects of a productive disciplinary engagement frame in place. But one teacher in my study, Ms. A., took things a step further. Where other teachers used student presentations as a closing activity for any given task, Ms. A often used presentations formatively, as a way to respond to students' emerging ideas while they still had time to revise and build on these strategies. These strategies are described in more detail by Sayavedra and Seashore (2017) and suggest a possible third goal for a teacher learning community: give students an opportunity for revision after student ideas have been responded to. To accomplish this goal, Ms. A did all of the following over the course of the year:

- She gave presenters who had been “helped” by others an opportunity to publicly revise their work or simply restate the “help” in the presenters' own words.
- She had students “turn and talk” to a neighbor for a minute or two about new ideas that came out in the discussion.
- She had students return to small group work after a “share-out.” This framed the presentation activity as a formative activity that raised ideas for further consideration, rather than a summative activity to bring closure to a task.
- She had students do a new problem using the strategy presented by a classmate.

Each of these practices has the potential to position students as revisers of mathematical ideas, creating corresponding opportunities for student learning.

7.3 Next steps for research

A continuity perspective suggests that we should not expect a “gestalt” change in which teachers first develop knowledge about new discourse moves and then suddenly apply this knowledge to enact ambitious teaching practices. Instead, we expect new knowledge bases and ambitious teaching practices to co-develop in complex ways. The dissertation illustrated some of the specific ways that teachers’ cultural and sociopolitical knowledge, as well as their knowledge of students’ mathematical thinking, contributes to culturally and mathematically responsive teaching. This section suggests some next steps for future research on these teacher knowledge bases and their development.

7.3.1 Cultural and Sociopolitical knowledge

This dissertation documents in some detail how mathematically responsive teaching in one particular local context is a sociopolitical act. In particular, I document a few specific ways that four experienced Black teachers leverage aspects of their personal experience as Black individuals to position nondominant students as authors and critics of mathematical ideas. I now suggest a few avenues of follow-up research.

First, as marked in the dissertation, greater attention to intersectional identities is needed. For example, in Chapter 4, Case 2, it was difficult to tease apart the operation of race and social class. To give another example, issues of gender were backgrounded in the dissertation, but can often cause status issues in mathematics classrooms in ways that overlap with the issues of race that were foregrounded. Relatedly, further research is needed on tools for building solidarity between students from different backgrounds. For example, in Chapter 4, Case 1, an affiliation to hip hop was invoked in a way that supported the racial and mathematical identity development of two Black male students. Might not some students from other cultural backgrounds at the school also affiliate with hip hop and also derive similar benefit from this type of positioning move? How was the discussion of salaries in Chapter 4, Case 2, received by students and did the discussion contribute to a sense of common experience between students of different cultural backgrounds? In general, can points of common reference be identified and invoked in ways that promote a sense of shared humanity and commitment to justice among students from different cultural backgrounds while also honoring the unique contributions, histories, and current needs of different nondominant communities?

Second, the dissertation argued that teachers’ sociopolitical knowledge was essential for their development of mathematically responsive teaching. Future research could investigate the converse: Can mathematically responsive teaching support teacher learning about students’ cultural practices and funds of knowledge? The focal teachers in the dissertation had a strong knowledge base for culturally responsive teaching, but still expressed a desire to learn more. Many teachers in urban schools lack this knowledge base, and are inadequately prepared to teach

nondominant students (Ladson-Billings, 2000). In Chapter 6, I documented an example where a teacher routine that gave more “air time” to student ideas also created more opportunities for teacher learning about building on students’ mathematical thinking. Future research could investigate whether expanding the opportunities for students to explain their mathematical ideas and draw on outside-of-school knowledge could create similarly rich opportunities for teachers’ sociopolitical learning. More specifically, can giving “air time” to student ideas help teachers develop cultural and sociopolitical knowledge about their students’ cultural heritage and lived experience of injustice and resilience?

Finally, future research could investigate nondominant teachers’ experience of teacher learning spaces associated with mathematically responsive teaching. Learning about mathematically responsive teaching often requires substantial risk-taking from any teacher. In particular, Smith (1996) articulates how traditional mathematics teaching provides a particular foundation for teachers’ sense of self-efficacy: Teachers feel effective when they give clear and accurate explanations and maintain an orderly and quiet classroom.

A strong sense of efficacy supports teachers’ efforts to face difficult challenges and persist in the face of adversity. Current reforms that de-emphasize telling and focus on enabling students’ mathematical activity undermine the basis of this efficacy. For the current reforms to generate deep and lasting changes, teachers must find new foundations for building durable efficacy beliefs that are consistent with reform-based teaching practices. (p. 1)

Although not explicitly addressed by Smith, teachers may also be implicitly or explicitly evaluated based on traditional criteria of clear explanations and quiet classrooms, which ties their job security to these criteria. Smith makes concrete suggestions for mathematically responsive teaching practices that can serve as new “moorings” for teacher efficacy: choosing problems that engage students in important mathematics, predicting student reasoning, generating and directing discourse, and judicious telling that supports students’ continued reasoning and ownership. However, such substantial shifts in individual and collective goals and efficacy judgments take time.

Risk-taking in the service of learning can be positive. However, I do not know of any research that investigates how nondominant teachers experience these risks. Scholars have argued that mathematics education is a White institutional space (Martin, 2013) and, as a consequence, much existing professional development is created, facilitated, and evaluated in ways that can marginalize the experiences of nondominant teachers. In this context, it seems important to investigate whether nondominant teachers may be positioned in ways that create obstacles to safe and productive risk-taking in the service of learning. During my time at Adams, some focal teachers expressed being exhausted by their racialized experiences both inside and outside of school, in ways that made it more difficult to take on extra risks and/or give the attention they wished to ambitious teaching practices. One teacher, despite his buy-in to the department’s goals, expressed that he saw the district’s

new (reform) curriculum as a deliberate attempt to dismantle his traditional instruction, which he felt had been effective at helping nondominant students pass his gatekeeper course.

I wonder: How might mathematics reform initiatives interact with the operation of race and power to impact nondominant teachers' sense of self-efficacy and how they are positioned by others? How can professional learning spaces be organized in ways that build on all funds of knowledge that teachers bring to culturally and mathematically responsive teaching, with particular attention to nondominant teachers' funds of knowledge?

7.3.2 Knowledge of students' mathematical thinking

Although some of the activity structures analyzed, such as student presentations, can appear independent of the content domain, the effectiveness of these activity structures was contingent on teachers' knowledge of student thinking in particular mathematical domains. Specifically, teachers needed to be able to recognize and interpret students' useful mathematical ideas, and have a flexible enough understanding of the target disciplinary knowledge to make connections with students' emerging ideas.

Teachers connected student ideas with relative ease when the teacher and students both had a rich base of relevant content knowledge. The primary cases of the dissertation provide two examples of this. In Chapter 4, Case 2, Mr. B orchestrated a rich discussion of proportional reasoning in the context of a debate about salaries. In this example, students had some contextualized knowledge about salaries and a repertoire of ways of making sense of arithmetic. The teacher had the cultural knowledge to keep the conversation grounded in the real world as well as sufficient knowledge of student thinking about modeling, ratios and proportions to run with students' novel ideas. The combination encouraged students and the teacher to stretch out the debate and sustain the learning opportunities it supported. In Chapter 6, presentation episode 2, Mr. B orchestrated a series of presentations that contrasted several important strategies for comparing fractions in the context of a contextual problem about the chocolateness of milk. In this example, student prior knowledge included some contextualized knowledge about chocolaty beverages and a repertoire of ways to make sense of rational numbers. The teacher had the necessary knowledge to quickly recognize an uncommon but sophisticated strategy, comparing the equivalent fractions with a common numerator, and connect it to both the problem context and a more typical strategy of comparing equivalent fractions with a common denominator. These two examples, together, provide an elaboration of a mechanism for how funds of outside-of-school knowledge can be leveraged for powerful disciplinary learning.

However, it was relatively difficult to create a rich mathematical conversation when students and/or the teacher have limited experience with the problem context. An example can be found in Chapter 6, presentation episode 4. In that episode, Mr. B

continued to use several of the ambitious teaching practices he had developed throughout the year for orchestrating a rich conversation. However, the task at hand, a Formative Assessment lesson on the properties of arithmetic (available at <http://map.mathshell.org/lessons.php?unit=6220>), relied on an area model representation for arithmetic expressions that was relatively new to both the students and the teacher. Students ended up using the area models primarily as indications of calculation to do, rather than *models* of the calculation. Because of this, they ended up using one similar procedure for all problems: do an arithmetic computation for each area model and each expression and compare the answers. They ended up doing similar arithmetic computations over and over again, missing opportunities to use the area models to discuss *why* various arithmetic expressions were or were not equivalent. Mr. B asked extension questions to students that reflected a strong pedagogical content knowledge of the arithmetic procedures, for example asking students to explain the reasons behind common misconceptions, but he did not invite students to use the area model representation in richer ways.

These examples show how ambitious classroom practice relies on teacher knowledge of students' mathematical thinking, but an inverse relationship also exists. Classroom activities that elicit student thinking also serve to build teachers' knowledge bases for the next time. For example, consider Case 1 in Chapter 5, which showed how Ms. A orchestrated a mathematically rich discussion of a formative assessment lesson on mean, median and mode. Two years prior to this dissertation study, when Ms. A taught this FAL for the first time, she scaffolded it down to a single procedure, much as Mr. B did with the laws of arithmetic FAL. Specifically, when students struggled with the open-ended task, she instructed them on a specific method for extracting a list of numbers from each frequency plot, then for calculating the mean, median, mode and range of the list of numbers (Sayavedra & Seashore, 2017). Two years later, during the dissertation study, she approached the task with much more nuanced attention to how the frequency plot representation could support student sensemaking about the mean, median, mode and range of different datasets. Specifically, the case study in Chapter 5 shows two student methods for finding the median, which use the frequency plot quite differently, and Ms. A connecting the methods and representations in a way that highlight the central concept that the median divides a dataset in half.

My analyses of the interrelationship between teachers' knowledge bases and their enactment of ambitious teaching practices highlights an essential aspect of teacher learning. It is important to emphasize that teachers are unlikely to develop new knowledge bases in isolation and then employ them in ambitious teaching practices. A continuity perspective on learning implies a more complicated co-evolution. This study sketched an initial understanding of some learning trajectories. Future research could flesh out these trajectories, map alternative trajectories, and eventually look for common threads and general principles between learning trajectories. This future work will likely also reveal ways that traditional discourse practices, like IRE sequences, are adapted and repurposed to meet new goals associated with culturally and mathematically responsive teaching.

Future research could also extend the ideas presented here about individual teacher learning to the level of a teacher learning community. A continuity perspective implies that teachers should not wait to teach ambitious lessons until they have fully developed the desired new knowledge bases – and that teacher learning is much more likely to be facilitated if teachers are part of a learning community as opposed to acting alone. This points to a limitation of relying on specialized content knowledge for teaching at the individual teacher level.

Finally, this dissertation has begun the enterprise of mapping teacher knowledge bases for culturally and mathematically responsive teaching from a continuity perspective. The impact of this contribution will be magnified as mechanisms are developed for teacher learning communities to generate, share, and apply this knowledge.

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A Transcript for Chapter 5, Case 1

[00:39:59.05] (Timer goes off.) Alright 3, 2, 1, I'm at zero.
So it's four, you got it?
Just, I'm looking for... uh oh, I'm waiting.
Good. Anyone has a match that we have not shown yet. Because so far we've done a lot of the matches. (Several hands) (to Barak & his partner) Yes, you haven't gone up. Go. Hopefully it's a match we haven't seen. It's a match, so.
Barak: I'll bring it.
Ms. A: Okay, good, bring it. Yeah. Bring it. That's what I like to hear. Alright, we are focused. Please, this is your opportunity, write your last word. Put down... good! alright, good! Awesome.
(inaudible) good. I wanna see that one. (Barak and his partner at the front.) So Jesus. Alexis. Don't have me repeat this. You are looking up here. Thank you.
Student: Yes! That's the one we got
Ms. A: Which one are you doing? You're doing S4, or B4, S3?
Student: We did that one!
Barak: No you didn't.
Student: It's cause we did (B7?) and that's...
Ms. A: Yeah that, go that way. (to class) Alright. So, okay. Let's focus in! Let's focus in. Let's focus in.
Barak: First we started
Ms. A (to another student): You found that one too?
Barak: with the range. We found the range because six minus one is five.
Ms. A: Okay.
Barak: Then we found the mode because one appears the most.
Student near Ms. A: (you go?)
Barak: Then we found the median because we ordered the um (frequencies) from
Student near Ms. A, simultaneously: ... that we don't wanna go in the front
Student near Ms. A: No no no no no no no nooooo. I don't want to go
Barak: ... least to greatest and then

Ms. A: (to off camera student) Okay, okay. I'm not asking you to go in front. (to Barak) Stop for a second Barak. Stop. Can you guys focus, please? Turn around. Look. I'm not asking you to do anything. I'm asking you just to listen right now. That's all I'm asking you to do.

Ms. A (to Barak): Okay, so where are we at?

Barak: The mean (now?).

Ms. A: We already did the mode?

Students: Yes!

Ms. A: How do you know which one is the mode?

Barak: Because one appears the most.

Ms. A: Got it, okay. Got it.

Barak: We found the median because we ordered the frequencies from least to greatest and got two. Because we crossed out the, um, end numbers. We found the mean because we added the frequencies and got 33 and then divided by 11 and got um 3.

[00:42:41.17] Ms. A: So you, you added the frequency of the scores, right? Of the scores. Because we're looking at, that word of becomes like multiplication, right? So frequency of the scores is one score that appeared five times, so that's five. Okay, good! So that's B, how many people had this match? You were working on this? Good!

(Barak takes his work down) Ms. A: B4, hold on hold on before we do that! B4 and what?

Barak: B4, S3.

Ms. A: Okay, before... does anybody have any questions about that? I'm sorry.

Student: Yes!

Ms. A: Oh, they have questions. Go up. Back. Sorry.

Barak: (Yes there are questions?)

Ms. A: (to class) So please, please, (to Alexis) Alexis put it down, put down your pen.

Student: How do you find the median?

Barak: The mean?

Same student: Median.

Barak: Median, um,

Student off camera: (We already did it on here).

Barak: we ordered the frequency of the scores from least to greatest and um we crossed out the end numbers and got 2.

Celia: You didn't... you didn't do that the number (inaudible) (that you crossed)

Barak: huh?

Celia: Never mind.
Barak: Okay. then.
Ms. A: No, push yourself! Push yourself. What, what are you wondering?
Celia: (Did you, I don't know. I know a little bit.)
Ms. A: Push yourself Celia. What are you wondering?
Celia: I'm wondering if he really, like, did it, or he just
Barak: I did do it.
Ms. A: (to Barak) So you listed it out? You listed it out, (to Celia) is that what you're wondering, Celia? You were wondering if he double checked his answer (laughs)
Celia: No, like well he's just saying that they crossed out the ends, he's not saying like specific the numbers and all that
Ms. A: Oh, you didn't say specific the numbers, which numbers you crossed off. What numbers did you cross off?
Barak: (embarrassed smile, inaudible) told (inaudible)
Ms. A: Oh, so you don't know for yourself? Oh my God. Thank you Celia. See how Celia pushed you? (Barak laughs and takes his work down from the document camera.)
Ms. A: No, leave it up, leave it up, leave it up. Leave it up.
(to Barak): So If you're gonna cross off the lowest numbers and the highest numbers, what would you cross off? You cross off the low number. The lowest score. What scores would you cross off, looking at that? (Barak looks at projected work) (to class) Who can help him out? Dulce, can you help him? Yeah. What's the lowest score?
Dulce: Two?
Ms. A: No, I see something lower than two. Estela?
Estela: It's one.
Ms. A: One. How many ones are there?
Barak: Five.
Ms. A: Five. So we have to cross off five ones, right?
Barak: There's one twos.
Ms. A: Okay, so
Barak: And four threes, I mean fives.
Ms. A: So=
Barak: =There's
Ms. A: then what would=
Barak: =one six.
Ms. A: Right. So if we cross off one one, what else do we gotta cross off?

Barak: Oh, I get it! One, two, three (continues at a low volume, inaudible)
Ms. A: Estela, help us out.
Estela: If he has to cross out, if he has to cross one out, I think it would be number five, because=
Barak: =One six... oh, I get how you could do that!
Estela: It would be number one, but he has to cross off like... He has to like... he has to put one fives and like take it out, like () and see what would end up in the middle. And that's the <mode.>
Off camera student: (The mode would end up in the middle).
You said what would end up in the middle?
Estela: The mode?
Mm. (negative).
An off camera student: (It's not) the mode
Ms. A: Celia.
Celia: Can I go up there and (show it)?
Ms. A: Yeah, you can go up there and show us. I think Barak is working on something too.
Barak: Okay I found it.
Ms. A: Okay, how did you do it?
Ms. A: She said scoot it over. More.
Celia: So I can see the gr
Ms. A: She wants to see the graph.
Celia: Um, so.
Barak: ??
Celia: You can just. Um. Well I did it this way and I'm not sure it works but. You can just, um, count the numbers like this one two three four five cause we I know that it's eleven.
Ms. A: Right.
Celia: And I counted um five here gestures to bars above 6 and 5)
Ms. A: Mhmm.
Celia: and here (gestures to bar above one), you know five is right here, so this (points to bar above 2) bar is left so ...
Ms. A: Yeah, it works, that totally works for me. What do other people think? Does that work?
Student: Yes.
Another student: I think what she's trying to say is like there's five and then she's saying... (hard to hear).

Ms. A: I don't think anyone has, oh, that's that.

Barak: Oh.

Ms. A: So, you know what, it makes sense right, because how many ones did he write here? Let's just look at what he wrote. Cause look. How many ones did he write here?

Barak: Four

Ss: Five. Four.

Barak: Five.

Ms. A: Five, right? And then he wrote five and he crossed it out. But then he wrote, how many sixes were in my bar graph?

Student: One.

Ms. A: One, so he crossed off a six. How many fives were there?

Student: Four.

Ms. A: One, two, three, he crossed off four, four fives right? And then what's in the middle?

Ss: Two!

Ms. A: Two. So crossing it out works the same way as listing it out, right?

[00:48:03.08] Alright, it's five minutes before the bell rings. Please, please put up your cards, because I know some of you guys are losing the cards. Put them up, put up the cards, put everything up. And then I have a journal write. So you've got one minute to clean up and then you have a journal write.

Student: why?

Ms. A: Because I want to see what you learned today!

B TQQ Codebook

B.1 Purpose

The purposes of the coding scheme are to (1) show snapshots of elements of classroom discourse consistent with either a knowledge transmission or productive disciplinary engagement frame, or something in between, (2) show shifts in classroom discourse on the time scale of a year, and (3) use the codes to identify shorter episodes for more detailed qualitative analysis that can enrich the insights gained about the teacher learning trajectory in question. In order to meet goal (2), it is important to be able to apply the codes to a year's worth of classroom data (say 5–15 hours of video data) in a reasonable amount of time.

First, I plan to code all whole-class discussions for one teacher for the whole school year. Each coded classroom observation will represent one snapshot of the teacher's practice. I hope that summary graphs of a few codes will show the following types of shifts in teaching practice over the course of the year:

- Decreasing frequency of very short student talk turns (<5 words) which are consistent with a knowledge transmission frame. Increasing frequency of long student talk turns (e.g. two sentences or more) and even student presentations (many sentences), which are consistent with a productive disciplinary engagement frame.
- Decreasing frequency of student talk about answers, which is consistent with a knowledge transmission frame. Increasing frequency of student talk describing their methods and justifying their reasoning, which are consistent with a productive disciplinary engagement frame.
- Decreasing frequency of evaluative talk from the teacher, which is consistent with a knowledge transmission frame.

After presenting these summary graphs, I can also use the code profiles of each classroom observation to select 2–3 typical episodes from the beginning, (middle?), and end of the year for more detailed qualitative analysis of the frame(s) in play.

B.2 Summary

The coding process includes:

- 1) Dividing talk into three columns;
- 2) Chunking the code into small, medium, and large chunks and recording brief descriptive notes about what is happening at the small and medium grain sizes (typically by pausing the tape after each small chunk to type a few words);
- 3) Coding talk turns;
- 4) Coding small and medium chunks by student talk type facilitated.

For experienced coders, the process should be feasible in a single pass through the recording with regular, short pauses. It should typically take about 2x real time.

B.3 Column Format

The coding scheme is intended for teacher-led discussions with a T-S-T format: the teacher elicits student talk, the students respond, and the teacher provides an evaluation and/or feedback of the student response. The coding scheme is not designed to analyze other discourse structures, such as student-student talk or teacher exposition.

First, talk turns are divided into three columns:

- (1) teacher Elicitations,
- (2) student Responses, and
- (3) teacher Feedback turns (may or may not be evaluative).

Table B.1 shows an example of this organization scheme with three lines of transcript.

Transcript			
Line	Teacher Elicitation	Student Response	Teacher Feedback
XNF1-1-4	Mr. X: Raul, how did we come up with this? Do you know?		
XNF1-1-5		Raul: The difference between the squares is 4.	
XNF1-1-6			Mr. X: Right.

Table B.1

In the real-time coding scheme, transcription is often not necessary. In this case, all talk turns are coded, but not necessarily transcribed.

Any talk that does not fit a T-S-T format will not fit well in these categories. If the episode is not primarily of a T-S-T format, the coding scheme is not applicable. If the talk is primarily T-S-T but has a few talk turns of another type of talk, the following rough conventions may be used:

- All student talk should be placed in the Response column, even if it is not in response to an elicitation. This talk WILL be included in summary statistics for the coding scheme. However, if more than 5–10% of student talk is of this type, this is an indication that the coding scheme is not capturing an essential aspect of the classroom discourse; consider using another coding scheme.
- Any teacher talk that is not an Elicitation or Feedback, notably longer teacher talk turns, should be placed across all three columns. This talk is currently NOT included in summary statistics.

B.4 Chunking

Transcripts are chunked at three grain sizes: small, medium, and large.

A new small chunk begins with every new elicitation (not counting re-elicitations).

Small chunks correspond to single IRE sequences (or variations that still fit the elicitation-response-feedback format). These small chunks may include more than three talk turns if the teacher is dissatisfied with the initial response to the question and chooses to re-elicite a version of the same elicitation. This follows Mehan's (1979) definition that an IRE sequence could be more than three turns but that there are obligatory co-occurrence relationships between the teacher Initiation, correct Reply, and positive Evaluation of that reply. See Table B.2 for an example where seven talk turns are needed to find a fully satisfactory answer to a single teacher Elicitation. This example would be considered a single IRE sequence by Mehan. In my coding scheme it is considered one small chunk.

Turn	Teacher Elicitation	Student Response	Teacher Feedback	Small Chunks
BNF7-1-72	Elicitation			Teacher asks students to summarize the given information in the problem. Students initially only provide partial information, but after a few teacher re-elicitations, students complete their summary.
BNF7-1-73		WHAT <5		
BNF7-1-74	Re-elicitation for additional details			
BNF7-1-75		WHAT <5		
BNF7-1-76	Re-elicitation for additional details			
BNF7-1-77		WHAT <5		
BNF7-1-78			Explicit Eval	

Table B.2

B.4.1 Medium chunks

Medium chunks corresponded to a generalization of *topically related chains* of IRE sequences. They are made up of one or more small chunks on a single topic. For example, if a class discusses two alternate strategies for solving a problem, exploration of each strategy might correspond to one medium chunk. Medium chunks are often 30 seconds to two minutes in length. Table B.3 shows one medium chunk.

Turn	Teacher Elicitation	Student Response	Teacher Feedback	Small Chunks	Medium Chunks
BNF7-1-12	Elicitation			Review vocabulary related to previously presented student strategy	Recap important points of preceding student presentation.
BNF7-1-13		Coded talk			
BNF7-1-14	Re-elicitation				
BNF7-1-15		Coded talk			
BNF7-1-16			Coded talk		
BNF7-1-17	Uncoded teacher talk				
BNF7-1-18	Elicitation			Summarize/generalize first half of previous student strategy	Note: Pretty straight IRE sequences.
BNF7-1-19		Coded talk			
BNF7-1-20			Coded talk		
BNF7-1-21	Uncoded teacher talk				
BNF7-1-22	Elicitation			Summarize method of to second half of previous student strategy	
BNF7-1-23		Coded talk			
BNF7-1-24			Coded talk		
BNF7-1-25		Coded talk			
BNF7-1-26	Re-elicitation				
BNF7-1-27		Coded talk			

Table B.2

B.4.2 Large chunks

Large chunks were by the mathematics problem under consideration. Large chunks were time stamped but not otherwise coded. Table B.4 shows one large chunk.

Line	Teacher Elicitation	Student Response	Teacher Feedback	Small Chunks	Medium Chunks
[10:09] Problem 9					
BNF7-1-1	Codes			Notes	
BNF7-1-2		Codes			
BNF7-1-3	Codes			Notes	
BNF7-1-4		Codes			
BNF7-1-5	Codes			Notes	
BNF7-1-6		Codes			
BNF7-1-7	Codes				
BNF7-1-8		Codes			
BNF7-1-9			Codes		
BNF7-1-10	Codes			Notes	
BNF7-1-11		Codes			
BNF7-1-12	Codes			Notes	
BNF7-1-13		Codes			
BNF7-1-14	Codes				
BNF7-1-15		Codes			
BNF7-1-16			Codes		
BNF7-1-17	Uncoded talk				
BNF7-1-18	Codes			Notes	
BNF7-1-19		Codes			
BNF7-1-20			Codes		
BNF7-1-21	Uncoded talk				
BNF7-1-22	Codes			Notes	
BNF7-1-23		Codes			
BNF7-1-24			Codes		
BNF7-1-25		Codes			
BNF7-1-26	Codes				
BNF7-1-27		Codes			
BNF7-1-28		Codes		Notes	
BNF7-1-29	Codes			Notes	
BNF7-1-30		Codes			
BNF7-1-31			Codes		
BNF7-1-32	Codes			Notes	
BNF7-1-33		Codes			
BNF7-1-34			Codes		
[Timestamp] Problem 10					

Table B.3

B.5 Coding

Talk type codes are applied to all three columns (Elicitation, Response, and Feedback). Each column also receives one additional code that is column-specific.

B.5.1 Talk Type

Talk type codes are adapted from Reinholz & Shah (2018). An excerpt from their codebook is provided below, followed by a few of my own adaptation notes.

B.5.1.1 Hierarchical Definitions of Why/How/What/Other (Reinholz & Shah 2018)

- Why: Student explains/justifies the mathematics behind an answer, procedure, or concept.
 - Keyword: because.
- How: Student reports on the steps taken to solve a problem or give a list of procedures.
 - Definition of a procedure: A sequential/algorithmic list of discrete steps that a student follows (like a recipe).
 - Keywords: “how/what I did it was”; “first I did this, then I did this, then I did this, etc.”
- What: Student reads out part of a problem statement, recalls a fact, or gives a numerical/verbal answer to a problem (with no further justification/elaboration).
- Other: Student asks a question or does not say a mathematical object, idea, statement, process, or reasoning/explanation.
 - Student says “yes” or “no” indicating that they agree or disagree with another student.
 - “I agree.”
 - “I don’t know.”
 - “Hmm,” “um” or some other filler words.

B.5.1.2 Additional notes

Sometimes there is insufficient information within a single turn of talk to code the type. In this case, contextual cues may be used as follows:

- For Elicitation turns, it is permissible to assign a type of NONE if there is no information given.
 - For example, if the teacher pauses an exposition, sees a student hand raised, and calls on the student without posing a question, this would be counted as an Elicitation with type NONE.
- Talk is assumed to be aligned to the same type as the elicitation unless otherwise indicated.
 - E.g. “Correct.” as an Evaluation is coded the same type as the elicitation.

B.5.2 Teacher elicitation

Teacher elicitation talk is coded as either a new elicitation or re-elicitation in order to facilitate chunking at the small grain size.

New elicitations mean a new question has been asked. They are simply coded 'elicitation' and mark the beginning of a new small chunk.

Re-elicitations indicate that the initial elicitation has not yet been satisfactorily answered and is still 'on the table.' They do not mark the beginning of a new small chunk. Types of re-elicitation may be significant to code for in future work. For this dissertation, instead of formal codes, I write a few extra words of text about the re-elicitation type. For example:

- (bare) Re-elicite (same question repeated with at most trivial rephrasing);
- Re-elicite for volume (e.g. "Could you say that again louder?");
- Implicit re-elicite (e.g. teacher calls on another student who attempts to answer the prior question);
- Re-elicite for additional details
- Hint question.
 - o This is the trickiest case. A new (typically easier) question is asked without the previous question being resolved yet. A new question is only considered a hint question if the teacher later goes back to the original question. In this case, the hint question is not coded as its own small chunk.

B.5.3 Student Response length

Student Response length refers to the number of words in a single continuous utterance. Length of talk *is not* aggregated over multiple turns of talk by the same student.

EACH student talk turn is assigned one of the following codes based on the length of talk:

- 1-4 words;
- 5-20 words;
- 21-40 words;
- 41+ words.

Typically, it is not necessary to transcribe student talk to obtain an accurate enough word count to code the talk. It is not too difficult to determine in real time whether the student talk was only a few words (<5 words), about a sentence (5–20 words), about two or three sentences (21–40 words), or many sentences (>40 words).

The codes for Student Response Length are borrowed from the EQUIP scheme, with the addition of the 41+ word category. Talk turns in the 41+ word category in my data are typically student presentations at the front of the class. This category was added because the distinction between a student speaking about two sentences or many sentences is potentially significant for questions of framing and student positioning as authors of mathematical ideas.

Below are a few notes from the EQUIP codebook which are also applicable for my purposes:

- If a transcript does not capture the specifics of all of a student's speech, because some words were indecipherable, the utterance is coded according to the best estimate of how many words were indecipherable (i.e. the indecipherable words still add to the length of talk).
- If a student's speech is interrupted by side talk between a teacher and another student, count both the number of words before and after the side talk, because the side talk is not part of the whole class discussion and is just ignored.
- When students say a number, we code it as a single word.
 - ◆ Even if in the transcript the number is written out (e.g., "fifty seven" gets counted as 1 word, not 2 words)

B.5.4 Evaluative teacher talk

Teacher Evaluation/Feedback turns were additionally coded as either:

- Evaluative,
- Implicit Evaluation, or
- Non-Evaluative.

The code Evaluative is used for explicit evaluations only. For example, “That’s correct” or an affirmative repetition of the student’s answer would be coded as explicit evaluations.

Implicit evaluations should be coded based on the conversational norms of the classroom. The rationale for each implicit evaluation should be noted.

- There may be norms in place throughout the entire episode that lead to the rationale for implicit evaluations. For example:
 - o If the teacher only writes correct answers on the board during the episode, or
 - o If the teacher only “moves on” from a question once it has been answered correctly.

In these cases, there should be an empirical check of whether the norm is applied consistently throughout the episode, noted at the bottom of the code profile. E.g. “Implicit evaluation inclusion criteria: *The teacher writes the answer on the board (only correct answers were written on the board this episode).” Additionally, all instances where that inclusion criterion is used should be noted (with a * in the code profile in the above example).

- There may also be judgment calls based on a more local interpretation of the meso-level meaning. For example, if the teacher asks four different students what they got for problem three, without commenting on their answers, a judgment call is needed. One interpretation could be that the teacher is “fishing” for the correct answer, and implicitly evaluates all but the last answer as incorrect. A second interpretation would be that the teacher is simply “getting a feel” for what different students answered, without any implied evaluation. Either way, the interpretation should be noted and justified either within or immediately after the code profile for the episode.
- To assign an implicit evaluation code, it should be judged that it is reasonably clear to all or most students in the class that an evaluation has taken place. The determination of whether or not to assign this code should not be based on the coder’s outside math knowledge (e.g. knowing that an answer is correct and therefore suspecting that the teacher does not intend to imply that it is incorrect)

The non-evaluative code should be used by default if clear rationale for an explicit or implicit code can be identified.

B.5.5 Chunk type

This part of the coding scheme is not used in the dissertation but may be useful for future work.

Talk type codes (OTHER/WHAT/HOW/WHY) could be applied recursively, not only to individual talk turns but also to small and medium chunks after the fact.

Talk type codes for chunks would be assigned based on the type of student talk supported or validated by the chunk as a whole. This would usually, but not always, match the highest frequency talk type code among all student talk turns in the chunk.

- For example, if a teacher described a solution method in detail (teacher HOW/Process talk), but only asked students for the answers to calculations (student WHAT/Product talk), the chunks would be coded based on the primary type of student talk invited and enacted (HOW/Process).

Applying the same codes recursively to talk turns, small chunks, and medium chunks, can potentially provide a window into how framing and corresponding student talk roles are constructed between the micro and meso levels. Recall that in a knowledge transmission frame we would expect to see primarily WHAT/Product codes for student Response turns and in a productive disciplinary engagement frame, we would expect to see a significant frequency of WHY/Metaprocess codes for student Response turns. But consider the following possible hybrid scenario:

- The teacher is doing a great deal of scaffolding work to support students to do some WHY/Metaprocess talk. The teacher asks WHY/Metaprocess questions, but students initially answer with talk of another type. The teacher uses re-elicitations to push for additional details without scaffolding away the challenge of her questions. On the second or third try, students typically do succeed in providing WHY/Metaprocess answers.

In this case, the raw percentage of WHY/Metaprocess student Responses might be fairly low. However, the percentage of small chunks coded as WHY/Metaprocess would be much higher. Furthermore, looking at the discrepancy between the turn by turn codes frequencies and small chunk code frequencies would reveal the scaffolding work being done by the teacher.