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Abstract

Techniques are developed in order to study static magnetic screening and other nonperturbative aspects of QCD at high temperatures. In particular, a covariant derivative expansion of the one loop effective action is presented and then modified by an infinite resummation so as to provide agreement with the exactly calculable one loop effective potential. Essential to this technique is a self-consistently defined infrared cutoff which determines the prefactor in semiclassical calculations. Using this prefactor, densities of monopole and dyon plasmas are calculated, and it is found that if such plasmas do exist at finite temperature, then the solitons involved must be overlapping one another. It is also shown that no consistent perturbative or nonperturbative approximation can give rise to a linear term in the SU(2) effective potential, since such a term would not be gauge invariant. Finally, contour plots of the SU(3) A_0 effective potential are presented.

1 Introduction

It has long been predicted that QCD features a phase transition from hadronic matter to a quark-gluon plasma at sufficiently high temperatures or densities[1]. Creating such a plasma is in fact the aim of many of the heavy ion experimental programs at the AGS, SPS and RHIC. Since the quarks and gluons in such a hot plasma would be very energetic, they would also be weakly interacting due to asymptotic freedom. Consequently, extensive work has been done in developing perturbative techniques for finite temperature QCD[2, 3]. One of the most interesting results of this perturbation theory is a resummation of infrared divergent diagrams which gives rise to an A_0 Debye mass of order gT that screens static color electric fields[2]. Unfortunately, no such resummation has yet been found for the magnetic sector. Consequently, for diagrams above a certain order, infrared divergences become intractable and perturbation theory breaks down[4]. These divergences are a result of loops involving massless (n=0) Matsubara modes, so they do not occur in in QED since the photon only couples to fermions which always have Matsubara frequencies of order πT.

A constant A_0 field cannot in general be gauged away at finite temperature the way that it can at zero temperature; consequently quantum effects give rise to an effective potential for the A_0 field when $T > 0$[5]. One way that QCD could generate a magnetic screening mass would be if the A_0 effective potential were to feature an absolute minimum which was not simply a gauge transformation of $A_0 = 0$. The A_0 field could then possess a nonzero vacuum expectation value (vev), thus behaving like a Higgs field and giving a magnetic mass to the A_i^a fields through the gauge-gauge coupling terms. Unfortunately, no such minimum exists at the one loop level[5, 6]. At the two loop level, on the other hand, the presence of a negative linear term in the effective potential does produce a vev at $A_0 \sim \mathcal{O}(gT)$, thus giving rise to a magnetic mass of order g^2T[7]. This vev and magnetic mass is spurious, however, since the linear term is exactly cancelled by a term arising from the summation of the ring diagrams[8]. Beyond the order of the ring diagrams, perturbation theory breaks down due to the magnetic infrared divergences mentioned earlier. There is therefore no way that perturbation theory alone can generate a nontrivial absolute minimum in the A_0 effective potential. One of the results of this paper is to show in addition that no gauge invariant resummation or non-perturbative technique can give rise to a linear term in the effective potential, since such a term would not be gauge invariant.

In a more general context, it is well known that perturbation theory is limited in its application and by its very nature is not able to shed light on a number of very important unsolved physical problems. For this reason, non-perturbative techniques have been increasingly sought after and explored in recent years. Perhaps the most successful and well-developed of these techniques is the semiclassical method of expanding around classical solutions. In the language of the path integral formalism, the idea behind this method is that by integrating over field configurations which are small fluctuations around nontrivial classical solutions, as well as over ones which are close to the perturbative vacuum, one can better approximate the full functional integral, which should in principle be performed over *all* possible field configurations. In field theory for example, integrating around instanton solutions allows one to gain insight into quantum tunneling processes which can never be described by any finite order of perturbation theory[9, 10, 11].

Similarly, for finite temperature QCD, it has been pointed out that integrating around a plasma of magnetic monopoles could possibly provide magnetic screening as $T \to \infty$ [6, 12, 13]. Is there any evidence for the presence of such monopoles? At zero temperature, Mandelstam showed that if the ground state of QCD is a coherent superposition of monopoles, then confinement could be understood as the dual analog of superconductivity[14]. In other words, just as a condensate of electrically charged Cooper pairs will adjust to confine magnetic fields inside a superconductor, a condensate of color magnetic charges would adjust to confine color electric fields in the QCD vacuum. It has never been proven that such a condensate actually forms the ground state of QCD, however Savvidy has shown that a constant color magnetic field H has negative energy compared to the perturbative vacuum at $T = 0$[15]. Although Savvidy's configuration violates Lorentz invariance, his result suggests that the ground state of QCD does have some nontrivial magnetic structure. Based on this idea, Oleson has advocated a picture in which random distributions of magnetic vortices form a Lorentz invariant ground state featuring $< H >= 0$ but $< H^2 > \neq 0$[16]. This picture is not contradictory with one involving a monopole condensate since magnetic vorices of finite length must begin and end at monopoles, and both pictures feature strongly enhanced low frequency fluctuations[14]. At high temperatures, even though the Savvidy effect disappears[17], the presence of low frequency (infrared) magnetic instabilities could be indicating the presence of monopoles or other magnetically charged objects. In this paper, we consider only high temperature monopoles and dyons and do not specifically address condensate formation or other issues relating to confinement at $T = 0$.

In SU(N) at zero temperature the A_0^a field can always be gauged away, so if there are monopole solutions, one must be able to create them from the A_i^a fields alone. Infinite energy monopole solutions and finite energy monopole configurations which are not solutions have been found[18, 19], but no finite energy monopole solutions are known for $T = 0$. In order to find a solution which sufficiently smoothes out the $1/r$ singularities in the A_i fields at the origin, one usually introduces a scalar field in the adjoint representation, as is done for the 't Hooft-Polyakov[20, 21] or Prasad-Sommerfield[22] monopoles. At finite temperature, however, the A_0^a field cannot in general be gauged away, and it is therefore able to play the role of the Higgs field in a monopole configuration. Making this substitution, Prasad-Sommerfield monopoles become dyon solutions in pure gauge theories, possessing electric as well as magnetic charge. Although the dual charge of dyons makes them necessarily more complicated than monopoles, they are at present the only available magnetically charged classical solutions with finite energy at the tree level, so they are a logical object of study. In addition to knowing the classical mass of these dyons, it is obviously important to know how dense of a gas or plasma they might form at any given temperature.

Finding the density of a soliton plasma can be a highly nontrivial task. In order to derive an expression for the density, I will present a brief outline of the semiclassical method for a field theory at finite temperature. The first step is to find a nontrivial field configuration with energy $E_1 < \infty$ which is a local minimum of the classical action[1]. Since the solution has finite energy, it must be localized, approaching the perturbative vacuum solution (or one of them if there is more than one) as $r \to \infty$. For the sake of simplicity, we will assume that the solution is a time independent soliton. It is plausible that a configuration with two solitons which are separated by a distance much larger than their size would be a close approximation to another solution. One therefore proceeds by either proving or assuming that configurations with N well-separated identical solitons are also local minima of the action[9, 23, 24, 25]. Often it is shown or assumed in addition that the solitons are weakly interacting. If this is the case, then the relative positions of solitons in an N soliton configuration are arbitrary and must be integrated over as well, giving a factor of volume V for each soliton. Putting together these ideas, one can write down a rough approximation to the partition function of a plasma of these solitons[23]:

$$Z = 1 + \gamma T^3 V \exp(-E_1/T) + \frac{1}{2!}[\gamma T^3 V \exp(-E_1/T)]^2 + \ldots \tag{1.1}$$

where the first term corresponds to no solitons, the second to one soliton, the third to two, etc. Since the position of each identical soliton is being integrated over, a symmetry factor of $\frac{1}{N!}$ must be included for N soliton configurations. In addition, there is a dimensionless "prefactor" γ included for each soliton which can in general be some complicated function of the coupling constant g.

The density of a plasma of solitons is determined by noting that for $Z = \sum x^n/n!$,

[1]It is not enough to find a classical solution which maximizes the action in some functional direction, because integrating over all small fluctuations around such a configuration would give an infinite result. This is the problem of negative eigenmodes which we address later in the paper.

the average n is given by $< n >= x$. Thus, the density is simply:

$$< n > /V \sim \gamma T^3 \exp(-E_1/T)\ . \tag{1.2}$$

To determine γ in the one loop approximation, one must calculate functional determinants around a soliton background. This is a very formidable task since no general method is known for calculating these determinants exactly. For this reason, the value of γ is often simply estimated by heuristic arguments.

The main thrust of this paper is to develop new approximation techniques for determining prefactors γ around a large class of background configurations at finite temperature. As examples of their utility, these techniques are applied to dyon and monopole configurations in pure gauge SU(2). It is found that if plasmas of such configurations do exist, then either they are unstable, infinitely massive, or else their density is so high that they are strongly overlapping. For these types of configurations, semiclassical techniques are therefore not applicable. However, it is not ruled out that other soliton solutions may be found in the future which are not plagued by the above problems. In that event, the density of plasmas of those solitons could then be found by using the techniques developed here. For example, magnetically charged meron-antimeron solutions are known to exist at finite temperature[26], though no explicit solutions are available. Alternatively for pure gauge theories, Coleman has found topologically stable monopole solutions which have a singularity at the origin[19]. It is possible that singularity-free monopole configurations could be found which would approach the above solutions as $r \rightarrow \infty$ and would also minimize the one loop effective action.

I begin this paper by presenting the basic notation and formulas for finding the regularized one loop effective action for a pure gauge non-Abelian theory. Next, I generalize the methods of ref. [27] to finite temperature and derive a covariant derivative expansion for the effective action. The increasing dimension of successive terms of this expansion is balanced by an infrared cutoff mass which is self consistently determined so as to optimize the expansion. This infrared scale is shown to uniquely determine the semiclassical prefactor γ. Comparison of the results of the expansion to the known effective potential for a constant A_0^a field in SU(2) suggests that the lowest order form of this expansion should be reliable for slowly varying configurations in which $|A_0^a| < \mathcal{O}(T/g)$ when $g \to 0$ $(T \to \infty)$. After showing that dyons meeting the above qualifications must necessarily be overlapping, I extend the covariant derivative expansion by performing a resummation in order to find an expression which is valid for static background configurations with $|A_0^a| = \mathcal{O}(T/g)$. Since the effective potential of the A_0^a field can have periodic minima at $4n\pi T/Ng$ for pure gauge SU(N) (see Appendix B), I also examine dyon solutions and monopole configurations in which the magnitude of the A_0^a field approaches one of these minima as $r \to \infty$. I show that these monopoles are unstable and that depending on the temperature, the corresponding dyons are either infinitely massive or else overlapping. I complete the discussion of monopoles and dyons by showing that introducing fermions into the theory does not improve the situation.

2 Preliminaries

We consider a pure gauge, Euclidean, non-Abelian theory with the Lagrangian:

$$\mathcal{L} = -\tfrac{1}{4}\left(F_{\mu\nu}^a\right)^2 , \tag{2.1}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - D_\nu^{ab} A_\mu^b \tag{2.2}$$

and

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c \, . \tag{2.3}$$

Since we are interested in finite temperature, the fields have periodic temporal boundary conditions $A_\mu^a(\tau) = A_\mu^a(\tau + \beta)$, where $\beta = 1/T$[2]. The equations of motion for this Lagrangian are

$$D_\mu^{ab} F_{\mu\nu}^b = 0 \, . \tag{2.4}$$

Let $\bar{A}^a_\mu$ be solutions to the above equations which transform as normal Yang-Mills gauge fields, and let B^a_μ be quantum fluctuations around those solutions which transform in the adjoint representation. To consider one loop effects, we make the substitution[28] ($A^a_\mu = \bar{A}^a_\mu + B^a_\mu$) in the Lagrangian and expand the action up to terms quadratic in B^a_μ:

$$S(\bar{A}+B) = \bar{S} - \tfrac{1}{2} \int d^4x \, B^a_\mu W^{ab}_{\mu\nu} B^b_\nu \,, \tag{2.5}$$

where $\bar{S} = S(\bar{A})$. Note that there are no terms linear in B since $\bar{A}$ is a classical solution and hence a saddle point. We choose to work in the background gauge,

$$\bar{D}^{ab}_\mu B^b_\mu = 0 \,, \tag{2.6}$$

where $\bar{D}^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{abc} \bar{A}^c_\mu$, since it is manifestly covariant and because Pauli-Villars regularization takes a particularly simple form in this gauge[29]. By adding a gauge fixing term of $\frac{1}{2}(\bar{D}_\mu B_\mu)^2$ to the Lagrangian, we get:

$$W^{ab}_{\mu\nu} = -(\bar{D}^2)^{ab} \delta_{\mu\nu} + 2g f^{abc} \bar{F}^c_{\mu\nu} \,, \tag{2.7}$$

where $\bar{F}^a_{\mu\nu} = \partial_\mu \bar{A}^a_\nu - \bar{D}^{ab}_\nu \bar{A}^b_\mu$.

The functional integral needed to calculate the one loop effective potential is given by:

$$Z[\bar{A}] = e^{\bar{S}} \int [\mathcal{D}B^a_\mu \bar{\xi} \xi] \exp\{-\int d^4x [\tfrac{1}{2} B^a_\mu W^{ab}_{\mu\nu} B^b_\nu - \bar{\xi}^a (-\bar{D}^2)^{ab} \xi^b]\} \,. \tag{2.8}$$

Pauli-Villars regularization can be performed by introducing auxiliary fields B' and ξ' which transform like B^a_μ and ξ^a, but have mass Λ which will later be allowed to become infinite. Because all of the field fluctuations are in the adjoint representation, the mass terms $\Lambda^2 B'^2$ and $\bar{\xi}' \Lambda^2 \xi'$ are gauge invariant. Application of this procedure produces the following regulated partition function[29]:

$$Z[\bar{A}]|_{\text{reg}} = Z[\bar{A}] / Z'[\bar{A}, \Lambda^2] \,, \tag{2.9}$$

where Z' has the same form as eqn. (2.8), except that mass terms are included. Note that for convenience we have used the same Pauli-Villars mass Λ for both the B' and ξ' fields.

2.1 Zero Modes

We assume that the classical solutions depend on p parameters γ_i but that the total gauge-fixed action is independent of these parameters. There are then p remaining zero modes of the Lagrangian given by $\partial \bar{A}^a_\mu/\partial\gamma_i$ where i runs from 0 to p. Actually, any gauge transformation of one of these modes will also be a zero mode, so a more general expression for these zero modes is[23]:

$$\sqrt{N_i}\chi^{a(i)}_\mu = \partial \bar{A}^a_\mu/\partial\gamma_i + D^{ab}_\mu \theta^b_i \; , \tag{2.10}$$

where the second term is pure gauge and θ^b_i are gauge functions. By fixing the gauge, we have already removed all of the modes which do not satisfy (2.6), so in order to determine the remaining zero modes, we need to find specific functions θ^b_i such that $\bar{D}^{ab}_\mu \chi^{b(i)}_\mu = 0$. For the cases that we are studying, these zero modes can be made to be orthonormal, so we will demand:

$$\int d^4x \chi^{a(i)}_\mu \chi^{a(j)}_\mu = \delta^{ij} \; . \tag{2.11}$$

As a concrete example, consider a soliton solution which is centered around some point in space denoted by the vector $\vec{R}$. The solution then has the form $\bar{A}^a_\mu = \bar{A}^a_\mu(\tau, \vec{x} - \vec{R})$. Since the Lagrangian has no preferred points, a change in $\vec{R}$ will not change the action and therefore represents a zero mode. In this case $\gamma_i = R_i$, and due to the functional form of the solution $\partial\bar{A}^a_\mu/\partial R_i = -\partial_i \bar{A}^a_\mu$. We then choose $\theta^b_i = \bar{A}^b_i$ so that

$$\sqrt{N_i}\chi^{a(i)}_\mu = -\bar{F}^a_{\mu i} \; , \tag{2.12}$$

and the background gauge requirement (2.6) is trivially satisfied by the equations of motion (2.4). The normalization for this mode is then given by:

$$N_i = \int d^4x (\bar{F}^a_{\mu i})^2 \; , \tag{2.13}$$

where i is a label which is not summed over. If the soliton is a self-dual solution, then[30]

$$E^a_i \equiv F^a_{0i} = -\tfrac{1}{2}\, \epsilon_{ijk} F^a_{jk} \equiv B^a_i \; . \tag{2.14}$$

If, in addition, the soliton is spherically symmetric, then the normalization takes the remarkably simple form of $N_i = -\bar{S}$.

There are infinities due to the functional integration over non-gauge zero modes which can be isolated by the collective coordinate method[9, 23]. First, we expand an arbitrary field configuration as follows:

$$A(x) = \bar{A}(x, \gamma_i) + \sum_n \xi_n b_n(x) , \tag{2.15}$$

where b_n are orthonormal eigenfunctions of W with positive eigenvalues, and we have explicitly included the γ_i dependence of $\bar{A}$ in order to allow for zero mode fluctuations. To perform the functional integration, we must find the Jacobian associated with expressing the metric in terms of the eigenfunctions. For finite matrices, the Jacobian for a transformation from a vector X in one basis to a vector Y in another basis is found by calculating $\det J$, where $\delta X = J\delta Y$. If both bases under consideration are orthogonal, then J can always be diagonalized by a unitary transformation $J' = UJU^{-1}$ so that the determinant is given by $\det J = \prod_k J'_{kk}$. Calculating the length element then defines the determinant by isolating the diagonal elements of J':

$$(\delta\ell)^2 = (\delta X)^2 = (J'U\delta Y)^2 = \sum_k J'^2_{kk}(\delta Y)^2 . \tag{2.16}$$

Generalizing this technique to field theory and applying it to our problem, we have:

$$(\delta\ell)^2 = \int d^4x(\delta A(x))^2 = \sum_{i=1}^{p} N_i(\delta\gamma_i)^2 + \sum_n(\delta\xi_n)^2 , \tag{2.17}$$

so that after Gaussian integrations[23],

$$Z[\bar{A}] = (\prod_{i=1}^{p} \sqrt{N_i} \int d\gamma_i) e^{\bar{S}} \det(-\bar{D}^2)[\det{}'(W/2\pi)]^{-1/2} , \tag{2.18}$$

where $\det'(W/2\pi)$ means to take the determinant with respect to the nonzero modes of $W/2\pi$ only.

Since we are using Pauli-Villars regulators, we will also encounter the operators $W + \Lambda^2$, which do not have any zero modes. When taking determinants of these, however, it is still convenient to split the results into two factors:

$$\det[(W + \Lambda^2)/2\pi]^{-1/2} = (\frac{2\pi}{\Lambda^2})^{p/2} \det{}'[(W + \Lambda^2)/2\pi]^{-1/2} . \tag{2.19}$$

The full regulated expression, therefore becomes:

$$Z[\bar{A}]|_{\rm reg} = (\prod_{i=1}^{p} \sqrt{N_i} \int d\gamma_i)(\frac{\Lambda^2}{2\pi})^{p/2} e^{S(\bar{A})} \det(-\bar{D}^2)|_{\rm reg}[\det{}'(W)]|_{\rm reg}^{-1/2} , \tag{2.20}$$

where we use the following notation for some operator K:

$$\det(K)|_{\rm reg} = \frac{\det(K)}{\det(K+\Lambda^2)} . \tag{2.21}$$

Note that we have left out the factors of 2π in the regulated determinants of eqn. (2.20), since this expression only involves ratios of determinants, and multiplicative constants drop out. For the remainder of this paper we will drop the bars on $\bar{A}$, $\bar{D}$ and $\bar{F}$ except where they are needed for clarity, keeping in mind that we are always referring to functions of the background field and not of the full field with quantum fluctuations included.

3 Covariant Derivative Expansion

Now comes the difficult problem of evaluating the functional determinants. For a few select cases, the determinants can be evaluated exactly, but in order to find a general expression, some approximation procedure must be used. The most common method is to make a covariant derivative expansion. There have been many papers written suggesting a variety of ways to make such an expansion at zero temperature[27, 31, 32, 33], but the literature on finite temperature expansions is much more limited[34]. Each of the zero temperature methods that deals with a massless theory is forced to introduce some form of infrared cutoff mass in order to balance the dimension of new derivative terms. In most schemes, this cutoff mass remains unspecified with the argument that in a complete calculation of an observable it will drop out anyway. Alternatively, D'yakonov et al.[27] proposed a scheme in which the infrared cutoff is actually chosen in such a way that it optimizes the accuracy of any desired order of derivative expansion. To check their method, they calculated the one loop quantum correction to the action of the SU(2) instanton and obtained a result which was within

3% of the exact value calculated by 't Hooft[11]. It is this method that we have chosen to extend to finite temperatures.

In order to determine the free energy Ω of some nontrivial background configuration A, we need to calculate the ratio of the partition function of that configuration to the trivial $A = 0$ configuration:

$$\exp(-\Omega/T) = \frac{Z(A)}{Z(0)}|_{\rm reg} . \tag{3.1}$$

From eqn. (2.20), we can see that the calculation will entail finding ratios of determinants of various operators. These ratios can be evaluated by using the following expression for the difference of two logarithms:

$$\frac{\det K}{\det K_0}|_{\rm reg} = \exp\{-\int_0^\infty \frac{dt}{t} R(t) \mathrm{Tr}(e^{-tK} - e^{-tK_0})\} \tag{3.2}$$

where Tr is a functional trace over all indices and coordinates and

$$R(t) = 1 - e^{-t\Lambda^2} . \tag{3.3}$$

Note that t is formally of dimension M^{-2}. As long as both of the operators that we are interested in $(-D^2,\ W)$ are positive definite, they will have continuous spectra of eigenvalues beginning with zero, as do their vacuum operator counterparts $(-\partial^2, -\partial^2)$. One expects, therefore, that for sufficiently smooth and rapidly falling background fields, the integrand of (3.2) will be a rapidly decaying function of t[27]. This suggests the possibility of an approximation whereby the infinite upper limit of the t integration is replaced by an infrared cutoff δ. In addition to this approximation, we will make an expansion of the exponential operators in powers of covariant derivatives. After integrating with respect to t, the optimum δ for any given number of terms in this expansion can be determined by finding the extremum in the resulting expression.

The functional trace in eqn. (3.2) can be taken relative to any complete set of states, so we are free to use plane waves $\exp(ip_\alpha x_\alpha)$. These have the effect of shifting the derivatives:

$$\mathrm{Tr}e^{-tK} = \mathrm{tr}\int d^4xT\sum_n\int\frac{d^3p}{(2\pi)^3}\exp[-K(\partial_\alpha\to\partial_\alpha+ip_\alpha)t]\,\mathbf{1}\,, \tag{3.4}$$

where tr is a simple trace over spacetime and color indices. Due to the periodic temporal boundary conditions, we have replaced the normal zero temperature p_0 integral for a sum over the modes $p_0 = 2n\pi T$. Also the x_0 integral in d^4x is from 0 to $\beta = 1/T$. A **1** has been included at the end of the equation to emphasize the fact that the shifted $\exp(-Kt)$ operates on unity; so that, for example, any term in the expansion of the exponent with a ∂_α all the way to the right will vanish.

3.1 Ghosts

We now present the covariant derivative expansion of the ghost determinant. According to eqn. (3.4), we have

$$I^{gh} = \mathrm{Tr}\exp(D^2t) = \mathrm{tr}\int d^4x\,T\sum_n\int\frac{d^3p}{(2\pi)^3}e^{-p^2t}\exp[(D^2+2ip_\alpha D_\alpha)t]\mathbf{1}\,. \tag{3.5}$$

The expansion amounts to expressing

$$I^{gh} = \sum_n I^{gh}_n\,,$$

where I^{gh}_n is comprised of terms involving n covariant derivatives. I^{gh}_0 is simply given by the zeroth order term in the t expansion of eqn. (3.5), but is exactly cancelled in our calculation by the vacuum contribution seen in eqn. (3.2). Moreover, any term in the expansion with an odd number of D_α's will vanish upon p integration.

Thus the first nonzero term in the covariant derivative expansion is given by:

$$\begin{aligned} I^{gh}_2 &= \mathrm{tr}\int d^4x\,T\sum_n\int\frac{d^3p}{(2\pi)^3}e^{-p^2t}[D^2t+\frac{(2i)^2}{2!}p_\alpha p_\beta D_\alpha D_\beta t^2]\mathbf{1} \\ &= \frac{t}{(4\pi t)^{3/2}}(\mathrm{tr}\int d^4x D_0^2)T\sum_n(1-2p_0^2t)\exp(-p_0^2t)\,, \end{aligned} \tag{3.6}$$

where we have performed the momentum integral by using equations (A.1) and (A.2) in Appendix A. We would like to separate the $T=0$ and $T\neq 0$ parts of the above expression. This can be done by using equations (A.4) and (A.5) which have been derived from the Poisson summation formula (A.7):

$$I_2^{gh} = \frac{H_1}{8\pi^2 t}\sum_n \frac{n^2}{4T^2t}\exp(-\frac{n^2}{4T^2t})\,, \tag{3.7}$$

where

$$H_1 = \mathrm{tr}\int d^4x D_0^2 = -g^2N\int d^4x A_0^2\,. \tag{3.8}$$

Each term in eqn. (3.7) vanishes in the $T\to 0$ limit. This is reassuring since H_1 can be gauged away in the $T=0$ limit.

Using similar techniques, the next term in the expansion is given by:

$$I_4^{gh} \;=\; \frac{1}{48\pi^2}\sum_n \exp(-\frac{n^2}{4T^2t})[\tfrac{1}{4}F_2 + (\frac{n^2}{4T^2t})(G_2 - D_2) + 2(\frac{n^2}{4T^2t})^2 H_2]\,, \tag{3.9}$$

with the functionals F_2, D_2, G_n and H_n defined in Appendix A. Here the only term surviving when $T\to 0$ is the F_2 term[2] with $n=0$, in agreement with the result of d'Yakonov et al. [27]. This expansion can of course be continued, but for our purposes we will only need the first two terms.

To find the determinant, we must integrate over t as in eqn. (3.2). In all of our expressions, the zero temperature ($n=0$) terms are the only ones with ultraviolet divergences. For the rest however, we can immediately let $\Lambda\to\infty$ so that $R(t)=1$ and perform the remaining elementary integrals by using the variable $u=1/t$. For the case of I_2^{gh} we get:

$$\begin{aligned}\int_0^\delta \frac{dt}{t} I_2^{gh} \;&=\; \frac{H_1}{4\pi^2}\sum_{n=1}^{\infty}\exp(-\frac{n^2}{4T^2\delta})[1/\delta + \frac{4T^2}{n^2}] \\ &\simeq\; \tfrac{1}{6}T^2H_1 - \frac{T}{4\sqrt{\pi^3\delta}}H_1\end{aligned} \tag{3.10}$$

[2]Note that after using the Poisson summation formula, the sum over n is no longer a sum over Matsubara frequencies; in fact, $n=0$ terms correspond to $T=0$, while the $n\neq 0$ terms provide the temperature corrections

where the second equality is found after using the approximate expressions in Appendix A which become exact as $4T^2\delta \to \infty$. In this paper, we will only consider infrared cutoffs $T^2\delta \sim \mathcal{O}(1/g^\alpha)$ with $\alpha \geq 0$. For $\alpha > 0$, the approximations used are obviously very good at high temperatures, but surprisingly enough, even when $4T^2\delta = 1$, they are accurate to within a few percent. On the other hand, these approximations are not valid for $T = 0$, and consequently many of the following equations will not reduce correctly to their zero temperature counterparts in the limit as $T \to 0$. After using eqns. (A.16-A.19) to perform the t integration and high temperature approximations on I_4^{gh}, we arrive at the following expression for the regulated ghost determinant:

$$\ln[\frac{\det(-D^2)}{\det(-\partial^2)}]|_{\rm reg} \simeq -(\frac{T^2}{6} - \frac{T}{4\sqrt{\pi^3\delta}})H_1 - \frac{T\sqrt{\delta}}{48\sqrt{\pi^3}}[F_2 + 2(G_2 - D_2) + 6H_2]$$

$$- \frac{1}{48\pi^2}\{\tfrac{1}{4}[\gamma_E - 3.1 + \ln(\frac{\Lambda^2}{4T^2})]F_2 - G_2 + D_2 - 2H_2\} \quad (3.11)$$

3.2 Gauge Fields

For the gauge fields, we must only take the trace over the nonzero modes of $\exp(-Wt)$. If, however, we take the trace over *all* eigenfunctions of W, p of them will just give us a 1. This contribution can be subtracted out by hand, so that we get:

$$\frac{\det' W}{\det W_0}|_{\rm reg} = \exp\{-\int_0^\infty \frac{dt}{t}R(t)[\mathrm{Tr}(e^{-tW} - e^{-tW_0}) - p]\}\mathbf{1} \quad (3.12)$$

Since the trace is now over all modes, we can just take it with respect to the functions $b^a_\mu \exp(ip_\alpha x_\alpha)$, where $b^a_\mu b^b_\nu = \delta^{ab}_{\mu\nu}$. The calculations for gauge fields are similar to the ones for ghosts and one finds:

$$\ln[\frac{\det(W)}{\det(-\partial^2)}]|_{\rm reg} = 4\ln[\frac{\det(-D^2)}{\det(-\partial^2)}]|_{\rm reg} + p[\gamma_E + \ln(\Lambda^2\delta)]$$

$$+ \frac{1}{8\pi^2}[\gamma_E - 3.1 + \ln(\frac{\Lambda^2}{4T^2}) + 4T\sqrt{\pi\delta}]F_2\,. \quad (3.13)$$

Note that if $F_2 = p = 0$ (as for a constant field), then $W_{\mu\nu} = -D^2\delta_{\mu\nu}$ and the log of the gauge determinant is simply 4 times that of the ghosts, since the former involves a trace over spacetime indices.

From eqn. (2.20), we can see that the quantity that we will be interested in will be

$$\ln[\frac{\det(-D^2)}{\det(-\partial^2)}]|_{\rm reg} - \tfrac{1}{2}\ln[\frac{\det(W)}{\det(-\partial^2)}]|_{\rm reg} \,. \tag{3.14}$$

Using the expressions in (3.11) and (3.13), we can optimize the derivative expansion by differentiating (3.14) with respect to δ and finding an extremum. The resulting δ must obey the equation:

$$12H_1 + [-11F_2 + 2(G_2 - D_2) + 6H_2]\delta - 48p\sqrt{\pi^3\delta}/T = 0 \,. \tag{3.15}$$

Plugging in this δ, we get:

$$\frac{Z}{Z_0} = (\prod_{i=1}^{p}\sqrt{N_i}\int d\gamma_i)(\frac{1}{2\pi\delta})^{p/2}\exp(S_{\rm eff}) \,, \tag{3.16}$$

where

$$\begin{aligned} S_{\rm eff} &= \frac{1}{4Ng^2}F_2 - \tfrac{1}{2}p(\gamma_E - 2) + \tfrac{1}{2}(\frac{T^2}{3} - \frac{T}{\sqrt{\pi^3\delta}})H_1 \\ &+ \frac{1}{48\pi^2}\{-\frac{11}{4}[\gamma_E - 3.1 + \ln(\frac{\Lambda^2}{4T^2})]F_2 - G_2 + D_2 - 2H_2\} \,. \end{aligned} \tag{3.17}$$

3.3 Renormalization

It is worth noticing that the last term on the first line of (3.13) cancelled the Pauli-Villars ultraviolet regulator (Λ^2) in the prefactor of eqn. (2.20) and replaced it by an infrared cutoff mass ($1/\delta$) in eqn. (3.16). One might at first suspect this as being an anomalous artifact of our derivative expansion, but it is worth noting that in 't Hooft's exact one loop instanton calculation, his ultraviolet regulator in the prefactor was also replaced by an infrared scale – the size of the instanton (ρ). Moreover, renormalization can always be performed by using counterterms in the original Lagrangian which have the same symmetry as that of the background field at zero temperature. Although the gauge of the background field has been fixed, it has not been specified; consequently, the counterterms must take the form $CF_{\mu\nu}^2$, where C is some constant depending on Λ^2. It is therefore reassuring that the only Λ^2 dependence comes in the coefficient of a term multiplying F_2, so that all ultraviolet divergences can be removed by normal counterterms.

Because the counterterms in the background gauge have the same form as the original Lagrangian, one can create a renormalized Lagrangian simply by multiplying the original bare Lagrangian by the factor:

$$Z_3 = 1 - \frac{11Ng^2}{48\pi^2}\ln(\frac{Q^2}{\Lambda^2}) , \tag{3.18}$$

where g now represents the running coupling. At very high temperature, it is most convenient to choose the renormalization scale to be $Q^2 \simeq 4T^2\exp(3.1-\gamma_E) \sim 50T^2$ in order to absorb all of the one loop coefficients of F_2 into the definition of the renormalized (running) coupling. This running coupling constant is then defined in terms of the bare coupling by:

$$\frac{1}{g^2}[1 - \frac{11g^2N}{48\pi^2}[3.1-\gamma_E+\ln(\frac{4T^2}{\Lambda^2})] = \frac{1}{g_0^2} . \tag{3.19}$$

Just as at zero temperature, the running coupling can be defined in terms of an experimentally determined mass scale[35]. We will denote this scale by Λ_{QCD} even for theories other than QCD. The running coupling can then be expressed

$$\frac{g^2}{4\pi} = \frac{12\pi}{11N\ln(T^2/\Lambda_{\text{QCD}}^2)} , \tag{3.20}$$

and the renormalized effective action takes the form:

$$S_{\text{eff}} = \frac{1}{4Ng^2}F_2 + \tfrac{1}{2}p(2-\gamma_E) + \tfrac{1}{2}(\frac{T^2}{3} - \frac{T}{\sqrt{\pi^3\delta}})H_1 + \frac{1}{48\pi^2}\{D_2 - G_2 - 2H_2\} . \tag{3.21}$$

If a different renormalization scale Q^2 is chosen, the coefficient of F_2 in (3.21) will be altered by an O(1) term, and Λ_{QCD} in (3.20) will be multiplied by a calculable factor.

3.4 Constant Background A_0 Field

To test the accuracy of this covariant derivative expansion, we can plug in a constant SU(2) background field of $A_0^3 = \eta$, with all other field components vanishing. There are no non-gauge zero modes in this configuration, so $p = 0$. It can also be shown that $D_2 = F_2 = G_n = 0$ and that

$$H_2 = -(g\eta)^2 H_1 = 2(g\eta)^4 V/T\ , \tag{3.22}$$

where $V = \int d^3x$ is an infinite spatial volume. The infrared cutoffs and effective action from eqns. (3.15) and (3.21) take the simple forms:

$$\begin{aligned} \delta &= \frac{2}{(g\eta)^2} \\ S_{\rm eff} &= -\frac{V}{T}[\frac{T^2}{3}(g\eta)^2 - \frac{T}{\sqrt{2\pi^3}}(g\eta)^3 + \frac{1}{12\pi^2}(g\eta)^4]\ . \end{aligned} \tag{3.23}$$

The exact answer is well-known to be (see Appendix B):

$$S_{\rm eff} = -\frac{V}{T}V_{\rm eff}(\eta) = -\frac{V}{T}[\frac{T^2}{3}(g\eta)^2 - \frac{T}{3\pi}(g\eta)^3 + \frac{1}{12\pi^2}(g\eta)^4]_{\rm mod 2\pi T}\ . \tag{3.24}$$

where mod $2\pi T$ applies to each factor of $g\eta$ in $S_{\rm eff}$. $V_{\rm eff}$ is plotted in figure 3.1. It is apparent that no finite number of terms in the derivative expansion outlined above will be able to produce a periodic effective potential for the A_0 field. Nevertheless, if one is interested in field configurations for which $A_0 \sim \mathcal{O}(g^\alpha T)$ with $\alpha > -1$ in the $T \to \infty$ limit, then only the quadratic term in the effective potential will be important. Since the derivative expansion correctly reproduces this term (fig. 3.1), it is reasonable to use the expansion to describe the above class of configurations. It is important to note on the other hand that the derivative expansion is a bad approximation for configurations with $\alpha = -1$ even when $T \to \infty$, because in this case the cubic term and periodic nature of $V_{\rm eff}$ become important. For example, one should not use this expansion to study configurations in which the A_0 field approaches one of the minima at $2n\pi T/g$.

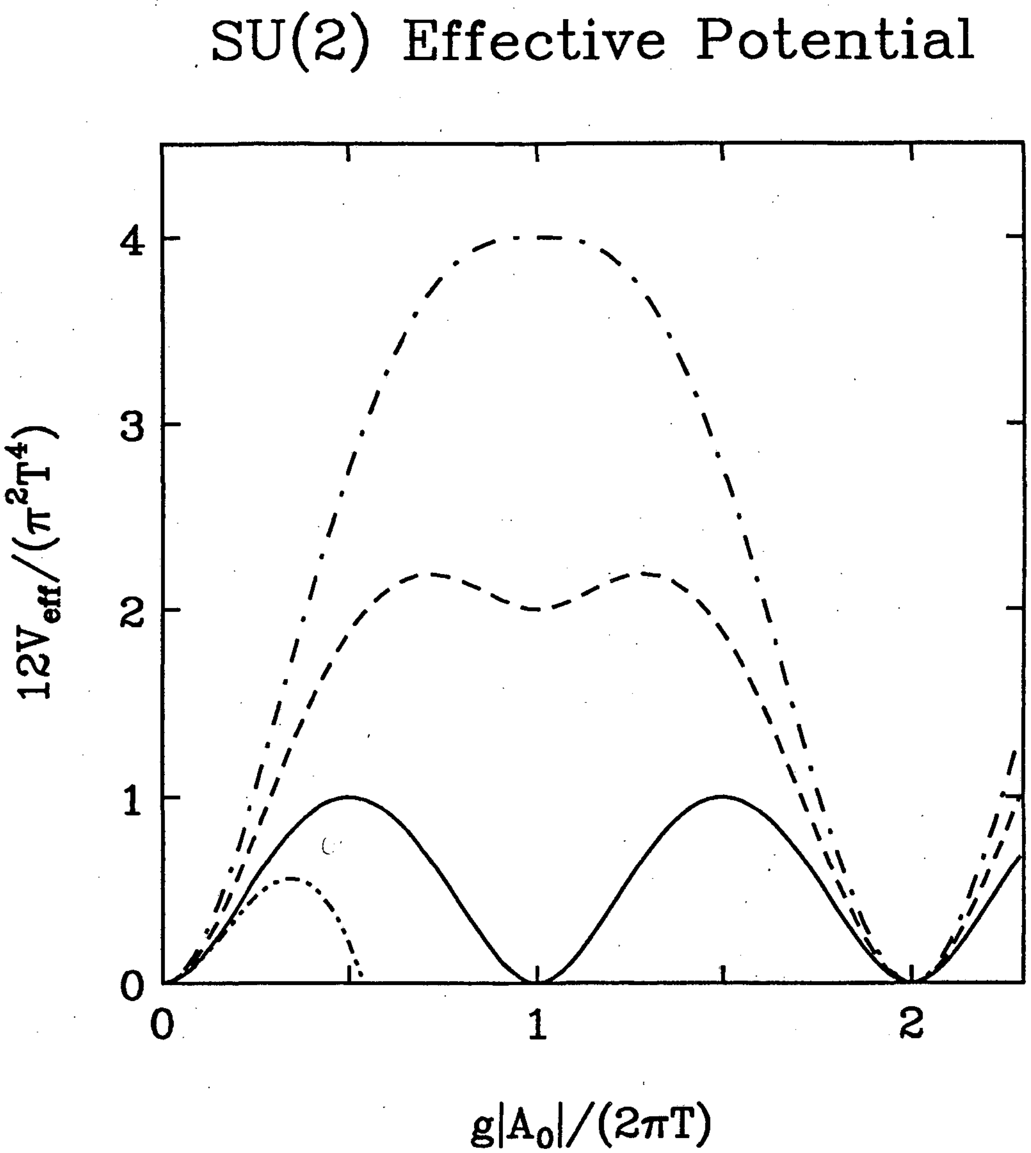

Figure 3.1 The one loop SU(2) A_0 effective potential with no fermions (solid), 1 massless fermion (dashed), and 2 fermions (dot-dashed). The dot-dot-dashed curve shows the lowest order covariant derivative expansion result of eqn. (3.23).

4 Application to Dyons

It has been suggested[13] that a plasma of magnetically charged solitons featuring $A_0 \to \mathcal{O}(gT)$ as $r \to \infty$ could possibly self-stabilize in the $T \to \infty$ limit of SU(2), even though there is no $\mathcal{O}(gT)$ minimum in the A_0 effective potential. The derivative expansion can be used to study this idea more carefully. We make the following ansatz for spherically symmetric soliton configurations:

$$A_0^a = \eta \frac{r_a}{r} f(x) \qquad A_i^a = \eta \epsilon_{iaj} \frac{r_j}{r} h(x) \;, \tag{4.1}$$

where η is the expectation value of $|A_0|$ at infinity and $x = g\eta r$. With this ansatz, the equations of motion (2.4) take the dimensionless form[36]:

$$\begin{aligned} x^2 f'' + 2xf' - 2f(1+xh)^2 &= 0 \\ x^2 h'' + 2xh' - (1+xh)(2h + xh^2 + xf^2) &= 0 \;, \end{aligned} \tag{4.2}$$

where the primes denote derivatives with respect to x. Since f, h and $x = g\eta r$ are dimensionless, any solution of the above equations will have a characteristic length scale of $\mathcal{O}(1/g\eta)$.

Note that in ansatz (4.1), the magnitude of the A_0 field approaches a nonzero constant value as $r \to \infty$. Rather than compare such configurations to the perturbative $A_0 = 0$ vacuum, it is more useful to compare them to a background with a constant $|A_0| = \eta$ field. From the form of V_{eff} in eqn. (3.24), it is apparent that such a background has infinitely more free energy (by a volume factor) than the perturbative vacuum, but it is possible that the infinite increase of entropy gained by introducing a plasma of solitons will offset the infinite background energy and allow such a plasma to self-stabilize. In other words, we would like to determine whether the free energy of a plasma of dyons in a constant background field is lower than that of the perturbative $A_0 = 0$ vacuum. To do this, we must calculate Z/Z_η, where Z is the partition function for a background dyon configuration and Z_η is that for a constant $|A_0| = \eta$ background field. All of our previous calculations have been for Z/Z_0 where Z_0 refers to the perturbative vacuum, so some of our expressions must be modified. Fortunately, $F_2 = D_2 = G_n = 0$ for both a constant field and the perturbative vacuum, so only the H_n are different. In fact all of the necessary modifications can be made by simply subtracting from each H_n the value of H_n for a constant field:

$$H_n \to H_n - 2(-g^2\eta^2)^n V/T \ . \tag{4.3}$$

Much of our discussion will center around the Prasad-Sommerfield-Julia-Zee (PSJZ) dyon, which is a magnetic and electrically charged self-dual solution of the classical Euclidean SU(2) Lagrangian for any value of η. It is defined by:

$$f(x) = \pm(\coth(x) - 1/x) \qquad h(x) = \pm(\operatorname{csch}(x) - 1/x) \ , \tag{4.4}$$

where the $\pm$ reflects the fact that both dyons and antidyons are solutions to the equations of motion, each having a tree level action given by $\bar{S} = -4\pi\eta/gT$ [30].

In addition to three translational zero modes which were treated previously as an example, these dyons each have a global gauge zero mode which is not eliminated by the background gauge requirement[37]. To find the prefactor associated with this zero mode, it is best to consider the monopole in the string gauge. In this gauge, $A_0^a = \delta^{a3}\eta f$, and the A_i^a field has a Dirac string singularity along the $-\hat{z}$ axis. The string gauge form of the solution can be obtained from the spherically symmetric form by making a gauge transformation with the following gauge function[19]:

$$U(\theta, \phi) = \exp(i\sigma_3\phi/2)\exp(i\sigma_2\theta/2)\exp(-i\sigma_3\phi/2) \ , \tag{4.5}$$

where σ_a are the Pauli matrices.

Consider the following global gauge transformation:

$$A_\mu \to A'_\mu = G A_\mu G^{-1} \ , \tag{4.6}$$

where $A_\mu = \sigma_a A_\mu^a$ and G is given by:

$$G = \exp(i\sigma_3\omega g\eta/2) \ . \tag{4.7}$$

Treating ω as an infinitessimal collective coordinate, we find:

$$\frac{\partial A'_\mu}{\partial\omega} = -\tfrac{1}{2}\, ig\eta[A_\mu, \sigma_3] = \sigma_a D_\mu^{ab}(\delta_{b3}\eta) \tag{4.8}$$

By making a careful choice of the gauge function θ_a from eqn. (2.10), we get the following zero mode:

$$\chi_\mu^a = \frac{\partial A_\mu^{a\prime}}{\partial \omega} + D_\mu^{ab}[\eta \delta_{b3}(f-1)] = F_{\mu 0}^a \;, \tag{4.9}$$

which satisfies the background gauge requirement (2.6) through the equations of motion (2.4). Like the translational modes, the normalization of this mode is $N_0 = -\bar{S}$, and the partition function involves an integral over the collective coordinate ω. However, unlike the translational modes, ω has a finite range of $0 < \omega < 4\pi/g\eta$, as can be easily seen by examining the form of G in eqn. (4.7). The entire prefactor for the dyon can now be expressed in terms of the infrared cutoff δ:

$$\xi = \frac{Z}{Z_\eta} = \frac{16\pi\eta V}{g^3 T^2 \delta^2} \exp(S_{\text{eff}}) \;, \tag{4.10}$$

where V is the volume of space and S_{eff} is defined by eqn. (3.24) with the replacement (4.3).

As we mentioned in the Introduction, if we assume that identical dyons are noninteracting, then we can approximate the one loop functional integral around two well separated dyons by $\xi^2/2$. The factor of $1/2$ is included in order to avoid double counting when the positions of the identical dyons are switched. Similarly, for a solution with N identical dyons, there will be a symmetry factor of $1/N!$. A full one loop calculation of the partition function should incorporate quadratic fluctuations around every single saddle point of the original Lagrangian which has the same boundary conditions at infinity. If we demand that $A_0^3 \to \eta$ as $r \to \infty$, then the saddle points include any number of dyons and antidyons, as well as a constant background field with no dyons:

$$Z/Z_0 = Z_\eta/Z_0 \exp(2\xi) = \exp(-V_{\text{eff}}(\eta)V/T + 2\xi) \;, \tag{4.11}$$

where the factor of 2 reflects the sum over both dyon and antidyon saddle points, and Z_0 is the partition function of the perturbative vacuum. Using $V_{\text{eff}}(\eta)$ from (3.24) and dropping all but the quadratic term, we get the following expression for the free energy density of a dyon plasma compared to that of the perturbative vacuum:

$$\Omega = -(T/V)\ln Z = \tfrac{1}{3}\,(g\eta T)^2 - \frac{32\pi\eta}{g^3 T\delta^2}\exp(S_{\text{eff}})\,. \tag{4.12}$$

The trick of self-stabilization as $T \to \infty$ is to see if a minimum of Ω can be found for some nonzero value of η.

For the moment, let us assume that as $T \to \infty$ ($g \to 0$) one loop corrections are parametrically smaller than the tree level action (i.e. we assume that infrared divergences do not destroy this property). We can therefore replace S_{eff} in (4.12) by $\bar{S} = -4\pi\eta/gT$. Because of the exponential dependence of the second term, we can see that the only hope of finding a nontrivial minimum would be for $\eta \sim \mathcal{O}(g^\alpha T)$ with $\alpha \geq 1$. Furthermore, the prefactor of the second term could be of no higher order in g than $g^{2+2\alpha}$ since that would be the order of the first term. From the discussion in the Introduction, we can see that the density of the plasma would be $\sim \mathcal{O}(g^{2+2\alpha}T^3)$, while from the discussion after eqn. (4.2), we know that the size of a dyon is $\sim \mathcal{O}(1/g^{1+\alpha}T)$. In other words, for $\alpha \geq 1$ the dyons would have to be strongly overlapping. Furthermore, since the difference in length scales is a parametric one, the overlapping would get infinitely worse as $g \to 0$.

Is this really a problem? If the plasma was comprised only of identical dyons with no antidyons, then overlapping might not be a problem since topologically stable, overlapping dyon solutions which are classically noninteracting have already been found[25]. On the other hand, a dyon and an antidyon can annihilate, so the approximation that we have been using that they are noninteracting would be a very bad one for a strongly overlapping plasma of dyons and antidyons. If an overlapping neutral plasma did in fact exist, it would have to be strongly interacting and consequently very difficult to describe using semi-classical methods. Furthermore, as Gross et al.[6] pointed out when making a similar argument about a plasma of Wu-Yang monopoles, such a plasma, with typical field strengths on the order of gT, would be difficult to distinguish from normal fluctuations around the perturbative vacuum. Perhaps the only clue to its existence might be the enhancement of low frequency fluctuations[14]. In order to avoid the problem of annihilation, it has been suggested that some mechanism could be found which would stabilize large domains of dyons and antidyons[38]. Even with such a mechanism, the fact that each dyon has zero field strength at the origin would still make a parametrically overlapping plasma domain locally very difficult to distinguish from the perturbative vacuum.

It is interesting to see what value of α would be necessary to make a plasma of dyons nonoverlapping in the $g \to 0$ limit. Suppose that infrared divergences in one loop terms miraculously caused them to be of the same order as tree level terms and were able to render $S_{\text{eff}} \sim \mathcal{O}(1)$, even when $\eta \sim \mathcal{O}(T)$. The prefactor of the second term in eqn. (4.12) would then have to be at most $\mathcal{O}(g^2)$ in order to create a nontrivial minimum. In such a scenario, the typical separation would be $\sim 1/g^{2/3}T$ while the size of a dyon would be $\sim 1/gT$. Again, the plasma would be parametrically overlapping in the $g \to 0$ limit. Using similar reasoning, it can be shown that the only hope of creating a self-stabilized, nonoverlapping plasma of dyons would be for $\alpha \leq -1$, which is exactly the range of α for which the covariant derivative expansion becomes unreliable. We can therefore conclude that no weakly interacting, nonoverlapping plasma of Prasad-Sommerfield dyons with $\alpha > -1$ will be able to self-stabilize in the $g \to 0$ limit.

I would like to make a couple of remarks before continuing. It has been suggested that by using a Coleman-Weinberg type mechanism[39] to minimize the effective action rather than the classical action, one may be able to to find monopole solutions with $A_0 \to \mathcal{O}(gT)$[13]. The idea would be that after combining the one loop effective potential with the tree Lagrangian, solutions could be found for which A_i drops off like $1/x$ at large distances, but A_0 only approaches η like $\exp(-Cx)$. Such a solution would not have a long range electric field and would consequently be a magnetic monopole rather than a dyon. As we shall show later however, in order to find such a monopole, it is necessary that the A_0 field approaches a local minimum of the effective potential as $r \to \infty$ (see section 7). Unfortunately, no evidence has been found for such a minimum[8], except for the periodic minima at $2n\pi T/g$ mentioned earlier. It is still possible that a plasma of Wu-Yang-type monopoles as suggested in [6] or a strongly interacting plasma of dyons could provide a magnetic screening mass of $\sim \mathcal{O}(g^2T)$ as $g \to 0$, but if so, it is not clear that semiclassical methods would be useful in describing these effects. On the other hand, it would be interesting to see whether the situation changes at all for dyons with $\eta \sim \mathcal{O}(T/g)$. To do so, we must perform some infinite resummations which will improve our covariant derivative expansion.

5 Improved Expansion

The periodicity of V_{eff} in eqn. (3.24) is simply a consequence of invariance under temporal gauge transformations. To see this, we first note that due to unitarity and the temporal boundary conditions at finite temperature, the most general gauge transformation for pure gauge SU(2) (see appendix C) is given by:

$$U(\tau, \vec{x}) = \exp\{i\sigma_a[\theta^a(\tau, \vec{x}) + n_a \pi T \tau]\}\,, \tag{5.1}$$

where θ^a is periodic in τ, n_a are integers, and σ_a are the Pauli matrices. Since the gauge of the background field is never specified in the background field formalism, any effective potential for the A_0 field must be gauge independent. The most general gauge invariant expressions involving A_0 but not A_i are integral powers of the Polyakov line

$$\operatorname{tr}\exp[ign\sigma_a \int_0^\beta A_0^a d\tau]\,.$$

Thus the most general possible gauge invariant expression for the pure gauge SU(2) A_0 effective potential is:

$$V_{\text{eff}}(A_0) = \sum_{n=0}^{\infty} a_n \cos(gn\sigma_a \int_0^\beta A_0^a d\tau)\,. \tag{5.2}$$

Because the above effective potential is a general expression which should hold for any field configuration, our knowledge of the exact answer for a constant A_0 field uniquely determines the coefficients a_n in the one loop approximation. For SU(2), we have:

$$a_0 = \frac{2\pi^2 T^4}{45} \qquad a_{n>0} = -\frac{4T^4}{\pi^2 n^4}\,, \tag{5.3}$$

which leads to the correct expression (3.24) for $A_0^3 = \eta$.

Since the form of eqn. (5.2) is only a result of gauge invariance, V_{eff} for all consistent higher order calculations must also take that form, though the coefficients will of course be modified. As a consequence, V_{eff} can never feature a linear term at the origin (i.e. $A_0 = 0$ must always be an extremum of the potential). This is significant, since if one makes a two loop calculation of the effective potential, a linear term does appear which seems to create a minimum of $\mathcal{O}(gT)$ in the effective potential[7, 12]. From the above arguments, however, we know that such a linear term is spurious and must vanish in a consistent $\mathcal{O}(g^n)$ calculation (which does not always coincide with a loop expansion). It is therefore not surprising that summing the infrared divergent Debye ring diagrams (with more than two loops) in SU(2) gives rise to a linear term which exactly cancels the one found at the two loop order[8].

Where is V_{eff} hidden in our covariant derivative expansion? The main problem with our expansion is that we are expanding a gauge invariant effective action in terms of functionals like H_n and G_n which are gauge dependent. Nevertheless, if we had had the patience and fortitude to calculate all terms in the expansion out to infinite order, making no approximations and letting $\delta \to \infty$, we would have arrived at an exact and gauge invariant expression for the effective action. In particular for SU(2), all of the terms H_n would have summed up to form the effective potential of eqns. (5.2) and (5.3). We can therefore improve our approximation of S_{eff} by including the known form of V_{eff} and dropping all H_n terms. By construction, our effective action will then exactly reproduce $V_{\text{eff}}(\eta)$ from eqn.(3.24).

After having resummed the H_n terms, the only remaining gauge dependent terms are G_n. The main problem with these terms is that they do not reflect the equivalence between configurations with A_0 near the different minima at $2n\pi T/g$. We can solve this problem by introducing new functionals G'_n which do reflect that equivalence:

$$G'_{n+1} = -2T^2 \mathrm{tr} \int d^4x [D_i, [D_i, \cos(nD_0/T)]] \, . \tag{5.4}$$

In particular, for static SU(2) fields with $g|A_0| << T$, we get:

$$G_2 = G'_2 = -\frac{2g^2}{T} \int d^3x \partial_i^2 A_0^2 \tag{5.5}$$

Thus to the order that we are working, if we replace G_2 by G'_2, we not only reproduce the correct behavior for static fields with small magnitudes, we also introduce the periodicity necessary to describe configurations with $|A_0|$ near each of the minima at $2n\pi T$. If we wanted to take the derivative expansion to the next order, we would get some terms involving G_3. We could then replace G_2 and G_3 by their primed counterparts, choosing coefficients such that the behavior of static fields with $g|A_0| << T$ was not altered. In addition, new gauge dependent terms involving more derivatives of D_0^2 could be replaced by terms having the same small $|A_0|$ behavior, but which reflect the equivalence of the A_0 minima. In this way, a modified covariant derivative expansion for static SU(2) fields can be continued to higher orders with the gauge equivalence of the A_0 minima manifest at each step.

Looking back at eqn. (3.15), we can see that after resumming the H_n terms, the new δ is given by:

$$\sqrt{\delta} = \frac{48\sqrt{\pi^3}p}{T(-11F_2 + 2G'_2 - 2D_2)} \tag{5.6}$$

Notice that $\delta = 0$ for configurations without zero modes. This just means that for these configurations, we would need to keep more terms in the derivative expansion to get a reliable value for δ. However, since we are primarily interested in calculating prefactors for configurations with zero modes, the above definition of δ is sufficient provided that it turns out that $\delta > 0$. Assuming this, we can write down a partially resummed, renormalized effective action for pure gauge SU(2):

$$S_{\text{eff}} = \frac{1}{4Ng^2}F_2 - \tfrac{1}{2}p(\gamma_E - 2) + \frac{1}{48\pi^2}(D_2 - G'_2) - \frac{1}{T}\int d^3x V_{\text{eff}}(A_0) \, , \tag{5.7}$$

where V_{eff} is given by eqns. (5.2) and (5.3). Equations (5.6) and (5.7) along with eqn. (3.16) are the main results of this paper.

6 More Dyons

We would now like to apply our improved formalism to the case of a PSJZ dyon for which $|A_0| \to \eta = 2\pi T/g$ as $r \to \infty$. Since A_0 approaches one of the absolute minima of the effective potential at infinity, a plasma of these dyons would not have to "self-stabilize" its entropy against an infinite background energy, as was the case of the dyons considered previously. Since the PSJZ dyon is self dual,

$$F_2 = 4D_2 = 8g^2\bar{S} = -64\pi^2 \tag{6.1}$$

The integral for G_2' is convergent and can be found to be:

$$G_2' = -16\pi^2 \,. \tag{6.2}$$

Since the dyon has four zero modes, the infrared cutoff can be found from (5.6) to be:

$$\delta = \frac{1}{\pi}\left(\frac{3}{11T}\right)^2 . \tag{6.3}$$

Note that $\delta \sim \mathcal{O}(1/(g\eta)^2)$ just as it was for a constant A_0 field. Keeping more terms in the derivative expansion will not affect the order of δ, though it will affect the size of the $\mathcal{O}(1)$ coefficient. Looking at eqn. (4.10), we can see that the entire plasma prefactor is determined, and we only need to evaluate S_{eff} in order to determine the density of the plasma.

Here is where we run into problems. We might at first think that we can simply replace S_{eff} by $\bar{S}$ in the exponent of (4.10) because the one loop corrections $S_{\text{eff}}^{(1)}$ are down by $\mathcal{O}(g^2)$. However, the fact is that for an isolated dyon, $S_{\text{eff}}^{(1)}$ diverges like a distance at infinity since A_0 only approaches the minimum at $2\pi T/g$ like $1/x$. We can see this by cutting off the integral over V_{eff} at some large radius R:

$$\frac{4\pi}{T}\int_0^R r^2 dr V_{\text{eff}}(A_0) \simeq 8\pi^2(2\pi TR) \,. \tag{6.4}$$

For a neutral plasma, we could argue that the highest electric multipole moment at infinity would be a dipole and so this divergence would not really occur. Let us assume that this is the case and try to find some sensible procedure for estimating R in the $g \rightarrow 0$ limit. The simplest guess would be that $2\pi TR \sim \mathcal{O}(g^{-\alpha})$. For any positive α, R would be parametrically larger than the typical size of a dyon $\sim \mathcal{O}(1/(2\pi T))$. On the other hand, as long as $\alpha < 2$, $\bar{S}$ will dominate S_{eff} and the density of the plasma can be found from (4.10) to be $\mathcal{O}(g^{-4}\exp(-8\pi^2/g^2)T^3)$. In the $g \rightarrow 0$ limit, one would expect R to be of the same order in g as the typical separation between dyons, but we can see that due to the exponential dependence of the density on $1/g^2$, this cannot be achieved in the $g \rightarrow 0$ limit. In fact, trying to find an equivalence between R and the typical dyon separation will drive $R \rightarrow \infty$ in the $g \rightarrow 0$ limit. Thus, due to one loop effects, PSJZ dyons with $\eta = 2\pi T/g$ will become infinitely heavy and decouple from the theory as $T \rightarrow \infty$.

On the other hand, we should not dismiss these dyons so easily for finite temperatures, in particular when $T \rightarrow \Lambda_{QCD}$. For a neutral plasma at finite T, it might be that a scale could be found for R which would be in qualitative agreement with the typical dyon separation which we will hereafter call R_s. In other words, we would like to find an R for which:

$$R \simeq \tfrac{1}{2}\, R_s = \tfrac{1}{2}\,(\frac{3}{4\pi V}\frac{Z}{Z_\eta})^{-1/3}\,, \tag{6.5}$$

where R_s depends on R through S_{eff} and Z/Z_η is given by eqns. (4.10) and (4.9). It turns out that this equation only starts having solutions for $g > 4$. Obviously at this point, we have left the regime of weak coupling, so the one loop approximation becomes dubious at best. In addition, it can be shown that the R's which solve (6.5) are typically between $1/(4\pi T)$ and $1/(2\pi T)$ which is the same scale as the size of the dyon, so dyons and antidyons would again begin to overlap.

7 Monopoles

One way that we could dispose of the troublesome divergence of $\int d^3x V_{\text{eff}}$ would be if we could find a way to make A_0 approach $2\pi T/g$ faster than $1/x$. In the 't Hooft-

Polyakov monopole, the Higgs field approaches its vacuum expectation value like $\exp(-Mx)$, which is just a consequence of it going to a quadratic minimum. If we use the Coleman-Weinberg mechanism[39] to find configurations which minimize S_{eff} rather than classical solutions which minimize $\bar{S}$, we should be able to achieve the desired behavior for $A_0 \to 2\pi T/g$ as $r \to \infty$ since there is a quadratic minimum in the effective potential there. To really use the Coleman-Weinberg mechanism with a clear conscience, we should include all orders of the derivative expansion in our expression for S_{eff} before we minimize, and we should verify that the configurations that we are interested in have no negative eigenmodes associated with them. Nevertheless, we shall proceed in the most naive manner, keeping only the effective potential and not worrying about negative eigenmodes for the time being.

For $r \to \infty$, the extrema of S_{eff} can be found by solving the following equations:

$$D_\mu^{ab} F_{\mu\nu}^b - \delta_{\nu 0} \frac{\partial V_{\text{eff}}}{\partial A_0^a} = 0\,. \tag{7.1}$$

These equations are greatly simplified by using the ansatz (4.1) along with the definition:

$$f(x) = 1 + \frac{F(x)}{x} \qquad h(x) = \frac{H(x) - 1}{x}\,. \tag{7.2}$$

Equations (7.1) then become[30]:

$$x^2 H'' \;=\; H(H^2 - 1 + (x+F)^2) \tag{7.3}$$

$$x^2 F'' \;=\; 2(x+F)H^2 + \frac{1}{3\pi^2} F(F + \tfrac{1}{2}x)(F+x)\,, \tag{7.4}$$

where the primes denote derivatives with respect to the variable $x = g\eta r$, and we have assumed that $\eta = 2\pi T/g$. For a monopole configuration, the A_i fields should drop off like $-1/x$ far from the origin. From the definitions of (7.2), then, we expect H and F to be small as $x \to \infty$. In this limit, the equations of motion become $H'' = H$ and $F'' = F/6\pi^2$, so that:

$$H \to C_1 \exp(-x) \qquad F \to C_2 \exp\left(-\frac{x}{\sqrt{6\pi^2}}\right)\,. \tag{7.5}$$

If we try to find a monopole for which A_0 asymptotically approaches a value which is not a minimum, then we find an equation like $F'' = Cx$, which does not feature solutions which vanish as $x \to \infty$. We can conclude that only monopole configurations in which the A_0 field approaches a minimum of the effective potential have any chance of minimizing the effective action.

7.1 Negative Modes

Unfortunately, in deriving S_{eff} for the monopole, we have implicitly integrated over negative eigenmodes. To see this, let us look a little more closely at what it means to integrate around a configuration which minimizes the effective action rather than the classical action. Suppose we have a monopole configuration $\bar{A}^a_\mu$ defined by eqns. (4.1), (7.2) and (7.5). Since $\bar{A}^a_\mu$ is not a classical solution, when we make the replacement $A^a_\mu = \bar{A}^a_\mu + B^a_\mu$ there will be terms linear in B^a_μ. Nevertheless, by adding an appropriate current term $J^a_\mu A^a_\mu$ to the original Lagrangian, the linear terms can be exactly cancelled and the monopole configuration becomes a solution to the modified equations of motion:

$$D^{ab}_\mu F^b_{\mu\nu} = J^a_\mu \,. \tag{7.6}$$

It is now possible to perform gaussian functional integrals over the terms which are quadratic in B^a_μ as long as none of the operators involved have negative eigenmodes (i.e. the configuration is stable). If, on the other hand, there are negative eigenmodes, then some of our "gaussian" functional integrals would actually be integrals of the type $\int \exp(+\alpha x^2)dx$ which diverge and render the one loop approximation useless. In the absense of negative modes, the current J is set equal to zero at the one loop level if the original configuration turns out to be an extremum of the effective action[39]. In a sense, we have gone about things a bit backwards by first finding a configuration which sets $J = 0$. We must now go back and check whether or not the configuration was classically stable to begin with.

Far from the center of the monopole, exponentially falling functions are unimportant, so we can approximate the configuration by using (4.1) and (7.2) with $H = F = 0$. We can then find an explicit expression for the operator inside the ghost determinant:

$$-D^2 = [-i\partial_0 - 2\pi T\,(I\cdot\hat{r})]^2 - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}[J^2 - (I\cdot\hat{r})^2]\,, \tag{7.7}$$

where I, L and J are isospin, orbital, and total angular momentum operators given by:

$$\begin{aligned} (I_c)^{ab} &= -i\epsilon^{abc} \\ L_i &= -i\epsilon_{ijk}r_j\partial_k \\ J &= I + L\,. \end{aligned} \tag{7.8}$$

We are interested to see whether this operator has any negative eigenvalues. For static configurations, we can use temporal eigenfunctions of $\exp(i2n\pi T\tau)$ and see that the first term of $-D^2$ is positive semi-definite by making the replacement $-i\partial_0 \rightarrow 2n\pi T$. In addition, we can see that the last term is positive definite by noting that $(L\cdot\hat{r}) = 0$ and replacing $(I\cdot\hat{r})$ by $(J\cdot\hat{r})$. Furthermore, the radial derivative terms are positive definite since

$$-\partial_i^2 = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{L^2}{r^2} \tag{7.9}$$

is positive definite even when $L^2 = 0$. Therefore the whole ghost operator is positive definite.

What about the gauge operator? To begin examining W, we first note that far from the monopole, there is no electric field and consequently $F^a_{0i} = 0$. From eqn. (2.7), this implies that $W_{0i} = W_{i0} = 0$. The gauge determinant can then be separated into two determinants:

$$\det(W_{\mu\nu}) = \det(-D^2)\det(W_{ij})\,, \tag{7.10}$$

where we have already shown that the first is positive definite. Dropping the spatial indices on W_{ij}, we can use techniques similar to those used for the ghosts to write:

$$W = (2\pi T)^2(n-(I\cdot\hat{r}))^2 - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\{K^2 + S^2 - 2S\cdot[K - \hat{r}(I\cdot\hat{r})] - (I\cdot\hat{r})^2\} \tag{7.11}$$

where S and K are spin and total angular momenta defined by:

$$\begin{aligned} (S_k)^{ij} &= -i\epsilon_{ijk} \\ K &= I + S + L\,. \end{aligned} \tag{7.12}$$

The only nonzero commutator among the operators of (7.11) is between $S\cdot K$ and $S\cdot\hat{r}$. Even with this difficuly, however, we can still make W block diagonal by quantizing with respect to S^2, K^2, $m = (I\cdot\hat{r})$, $s = (S\cdot\hat{r})$ and $l = K_z$.

The dangerous modes of this operator are when $K < 2$ and $n = m = -s = \pm 1$. For the $K = 0$ modes the operator reduces to:

$$W = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\,. \tag{7.13}$$

In ref. [19], Coleman presented an elegant way to show that operators which take the above form far away from the origin always have negative eigenvalues due to their attractive centrifugal potential. Consider the following radial function:

$$\begin{aligned} \psi &= \frac{1}{r}(\sqrt{r} - \sqrt{R})\exp(-r/a)\,, \qquad r \geq R \\ &= 0\,, \qquad r < R\,, \end{aligned} \tag{7.14}$$

where R and a are positive numbers. The expectation value of W from eqn. (7.13) for this function is:

$$\begin{aligned} < W > &= \int_0^\infty r^2 dr \psi(W)\psi \\ &= \int_0^\infty dr[r^2(d\psi/dr)^2 - \psi^2] \\ &= -\frac{3}{8}\ln(a/R) + \ldots \end{aligned} \tag{7.15}$$

where the triple dots denote terms that have a finite limit as $a \to \infty$. For any fixed R, this expression becomes negative for sufficiently large a. To get a negative expectation value for some function, there must be eigenfunctions with negative eigenvalues, since any function can be formed from linear combinations of eigenfunctions. Furthermore, since the proof works for arbitrarily large R, no behavior of the fields near the center of the monopole where F and H are nonzero can save W from having negative eigenvalues.

Another way to see that monopole configurations like the one suggested above would not be stable is to see that, unlike the normal 't Hooft-Polyakov monopole with a Higg's, these monopoles are not protected by topology at infinity. As $r \to \infty$, the A_0 field approaches a constant value of $\eta = 2\pi T/g$ which is simply a temporal gauge transformation of $A_0 = 0$. If $A_0 \to 0$, it doesn't matter whether it looks like a hedgehog or is in a uniform color direction, and consequently topology is lost. There is nothing to stop a configuration which has $A_0 = 0$ at both $r = 0$ and $r = \infty$ from reducing A_0 to 0 at intermediate values of r in order to minimize its action. This will be a general problem with any monopoles in pure gauge Yang-Mills theories: finite energy monopole configurations which minimize the effective action will feature the A_0^a fields approaching minima of the effective potential as $r \to \infty$. These minima, however, will be gauge equivalent to $A_0^a = 0$, so the monopole configuration will not be stable.

7.2 Generalization to SU(3) with fermions

To better illustrate these points, I will consider SU(3). From Appendix B, we know that we only need consider field configurations in which $gA_0/(2\pi T) = \nu\lambda_3 + \sqrt{3}\rho\lambda_8$, where λ_a are the Gell-Mann matrices. Again using Appendix B, we have plotted the effective potential as a function of ρ for $\nu = 0$ in fig. 7.1. The only minima of the potential in this direction occur at the points $\rho = 2n/3$ which are just gauge transformations of $\rho = 0$ (see Appendix C). Now let us look in the λ_3 direction by setting $\rho = 0$ and plotting V_{eff} as a function of ν (fig. 7.1). The absolute minima are again gauge transformations of $\nu = 0$, but in addition there appear to be local minima at $\nu = 2n + 1$. By making a contour plot with both ν and ρ (fig. 7.2), however, we can see that the apparent "local minimum" at $\nu = 1$ is actually just the side of a crater which falls to an absolute minimum at $(\nu, \rho) = (1, 1/3)$. The A_0 matrix at this minimum has the same eigenvalues as the minimum at $\nu = 0$ and $\rho = -2/3$, so we know that it is also a gauge transformation of $A_0 = 0$.

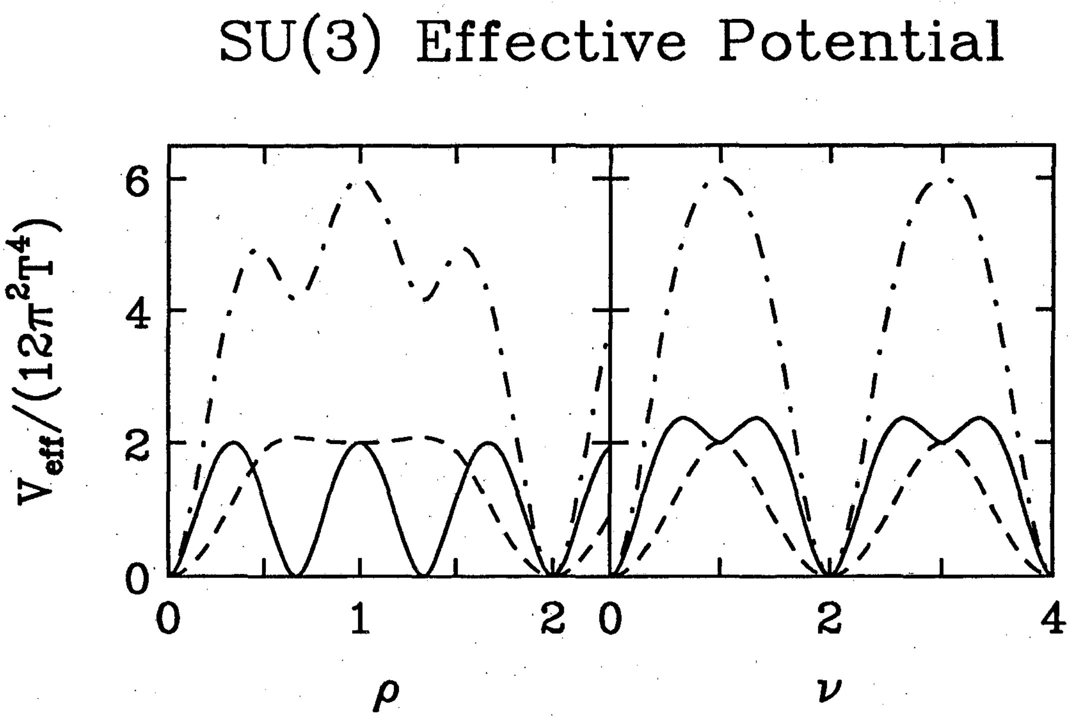

Figure 7.1: The one loop SU(3) effective potential for a constant A_0 field with no fermions (solid), 2 fermions (dot-dashed), and fermions only (dashed). The left frame is for $gA_0/(2\pi T) = \sqrt{3}\rho\lambda_8$, while the right frame is for $gA_0/(2\pi T) = \nu\lambda_3$.

On the other hand, true local minima of the effective potential can be created by introducing fermions into the theory and thereby breaking the center symmetry of the gauge group (see Appendix C). For example, if one massless fermion is introduced into SU(2), the absolute minimum at $g|A_0| = 2\pi T$ is transformed into a local minimum (fig. 3.1) [40]. Since there is no longer an allowed gauge transformation which takes this minimum to the $A_0 = 0$ configuration, one might be tempted to believe that a stable monopole configuration would exist with the $|A_0| \to 2\pi T/g$ as $r \to \infty$. Unfortunately, the presence of fermions induces no change in the gauge operator W, so there are still negative eigenmodes and the monopole is still unstable. It is interesting to note that if a minimum of the effective potential with fermions had occured at any point other than one which was an absolute minimum of the pure gauge theory, then it would have been possible to create a stable monopole configuration which minimized the effective action.

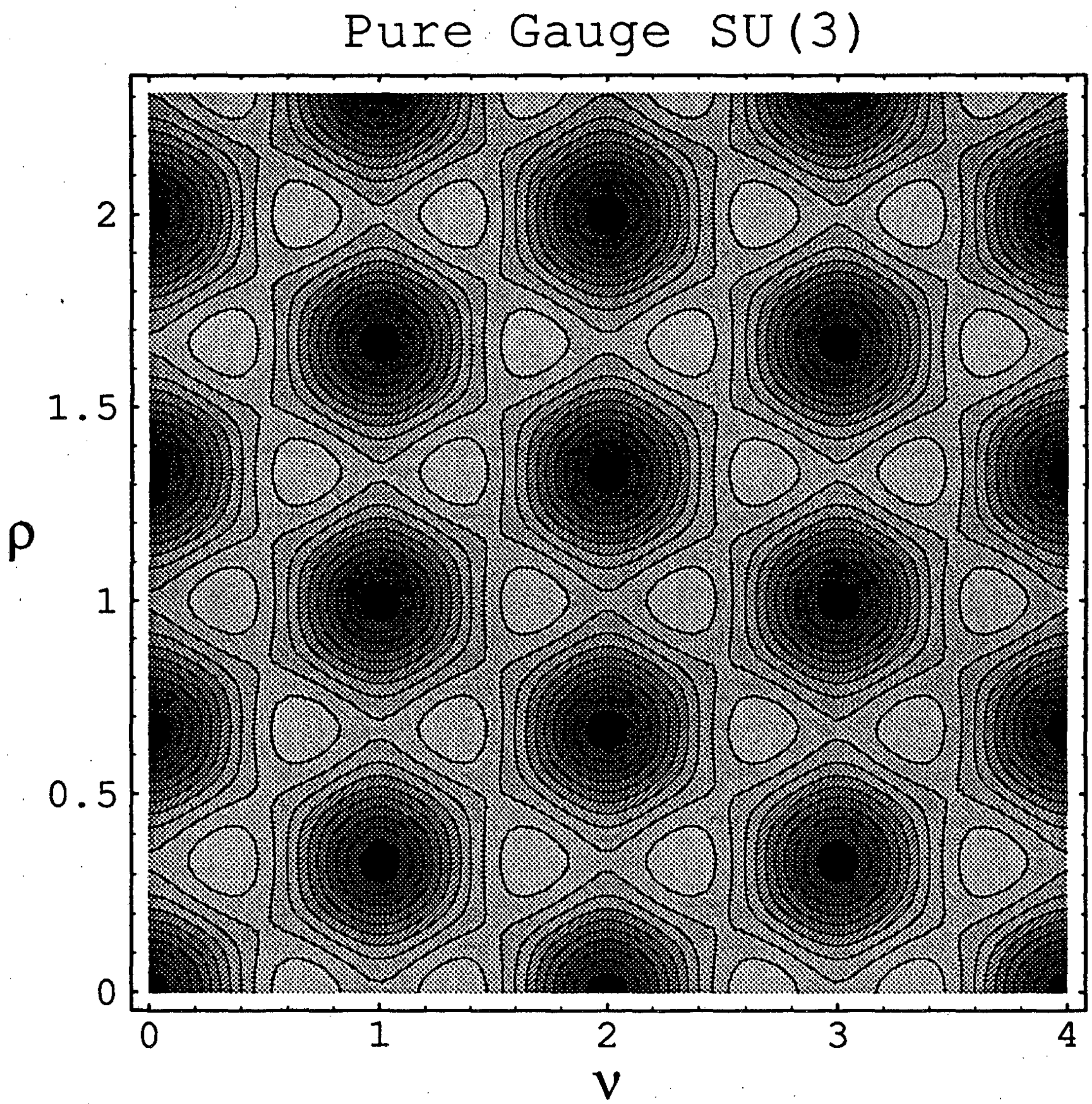

Figure 7.2: A contour plot of the one loop pure gauge SU(3) effective potential for a constant field given by $gA_0/(2\pi T) = \nu\lambda_3 + \sqrt{3}\rho\lambda_8$.

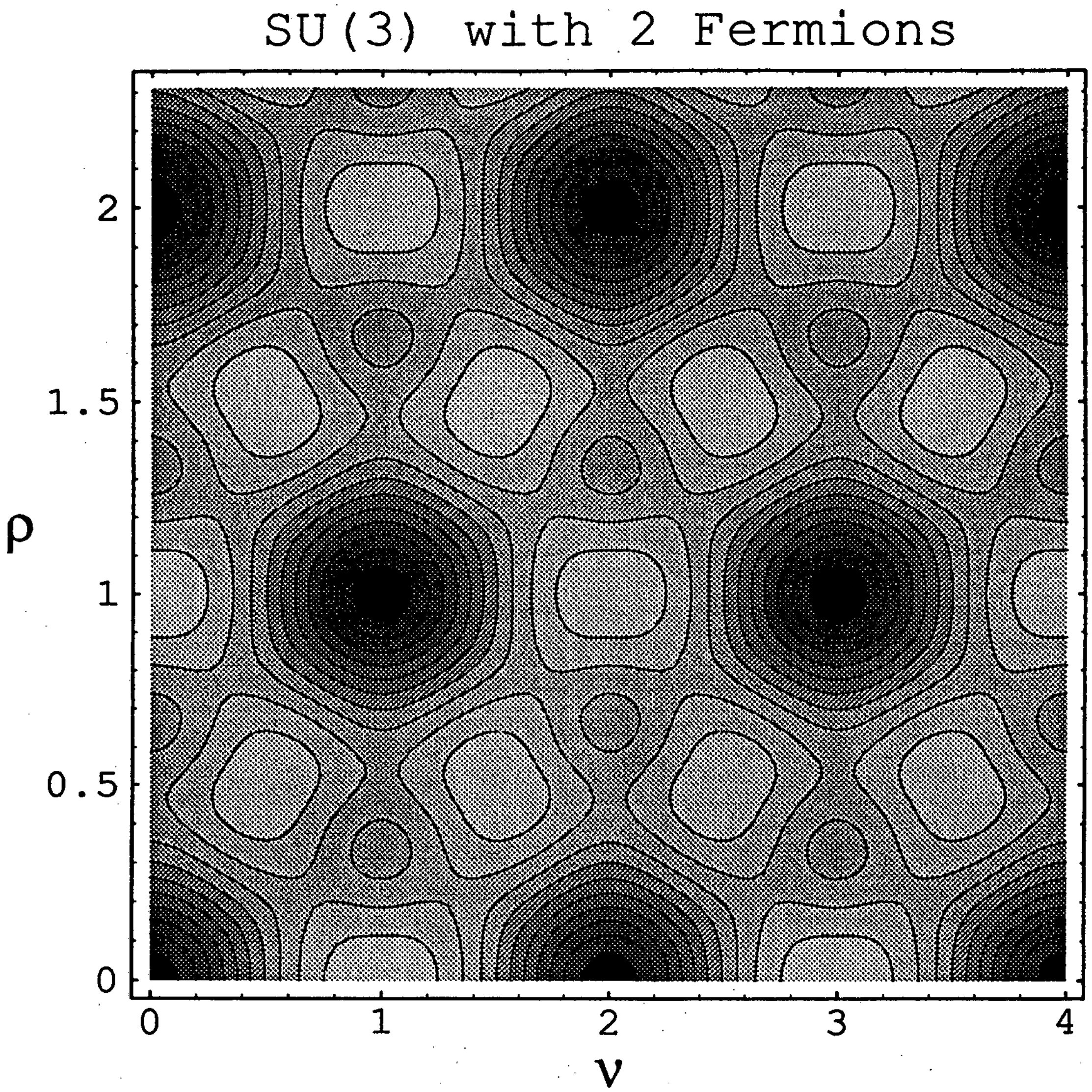

Figure 7.3: The SU(3) effective potential as in figure 7.2, but with 2 massless fermions.

Now we will examine the effect that fermions have on the SU(3) effective potential. Figure 7.1 shows this potential as a function of ρ for $\nu = 0$. It is not immediately obvious by looking at the figure that the local minima with fermions will be positioned at exactly $2n\pi T/3$. Nevertheless, this is the case since the absolute minima of the gauge part of V_{eff} precisely line up with maxima of the fermionic part (Appendix B).

Similarly, each of the local minima of (ν, ρ) shown in fig. 7.3 corresponds exactly to an absolute minimum of the pure gauge theory. By the same reasoning used for SU(2) then, any monopole configuration with A_0 approaching one of these local minima is still unstable.

8 Conclusion

We have developed a covariant derivative expansion of the one loop SU(N) effective action at finite temperature. The main use of this expansion is that it self-consistently produces an infrared cutoff mass which can be used to determine the density of a plasma of solitons in the semiclassical approximation. We have used our technique to evaluate suggestions in the literature[12, 13] that magnetically charged solutions to pure gauge SU(N) could self-stabilize at finite temperature, providing a nonperturbative mechanism for screening static magnetic fluctuations. We have found that classical dyon solutions have infinite energy at the one loop level unless they form an overlapping plasma, in which case they may be difficult to differentiate from thermal fluctuations. In addition, we have found finite energy monopole configurations in SU(2) and SU(3) which minimize the effective action but which are unstable. Therefore, at least these two types of semi-classical magnetic configurations do not solve the magnetic screening problem in hot QCD. Nevertheless, if stable, localized, finite energy solutions to pure gauge SU(N) at $T \neq 0$ are found in the future, then the methods developed here should be useful for estimating their density at high temperatures.

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A Integrals, Sums and Functionals

In this appendix, we present some of the tools that were used in deriving expressions for the effective action. In order to derive eqns. (3.6), (3.7), (3.9) and (3.13), it is necessary to use the following integrals and sums:

$$\int \frac{d^3p}{(2\pi)^3} \exp(-p_i^2 t) = \frac{1}{(4\pi t)^{3/2}} \tag{A.1}$$

$$\int \frac{d^3p}{(2\pi)^3} \exp(-p_i^2 t) p_j p_k = \frac{1}{2t(4\pi t)^{3/2}} \delta_{jk} \tag{A.2}$$

$$\int \frac{d^3p}{(2\pi)^3} \exp(-p_i^2 t) p_j p_k p_l p_m = \frac{1}{(2t)^2(4\pi t)^{3/2}} (\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) \tag{A.3}$$

$$T \sum_{n=-\infty}^{\infty} \exp(-p_0^2 t) = \frac{1}{2\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} \exp(-\frac{n^2}{4T^2 t}) \tag{A.4}$$

$$T \sum_n p_0^2 t \exp(-p_0^2 t) = \frac{1}{2\sqrt{\pi t}} \sum_n (\tfrac{1}{2} - \frac{n^2}{4T^2 t}) \exp(-\frac{n^2}{4T^2 t}) \tag{A.5}$$

$$T \sum_n p_0^4 t^2 \exp(-p_0^2 t) = \frac{1}{2\sqrt{\pi t}} \sum_n [\frac{3}{4} - 3\frac{n^2}{4T^2 t} + (\frac{n^2}{4T^2 t})^2] \exp(-\frac{n^2}{4T^2 t}) . \tag{A.6}$$

The above sums were obtained by using the Poisson summation formula:

$$\sqrt{\beta} \sum_{n=-\infty}^{\infty} F(n\beta) = \sqrt{\alpha} \sum_{n=-\infty}^{\infty} f(n\alpha) , \tag{A.7}$$

where $\alpha\beta = 2\pi$ and $F(x)$ and $f(p)$ are Fourier transforms of each other. The sums on the left sides of (A.4-A.6) are over Matsubara frequencies, while those on the right side are over $T = 0$ ($n = 0$) and $T \neq 0$ ($n \neq 0$) pieces. The latter can be seen by noticing that in the limit as $T \to 0$, only the $n = 0$ terms survive.

In addition, we define the following functionals for notational convenience:

$$H_n = \mathrm{tr} \int d^4x D_0^{2n} \tag{A.8}$$

$$\begin{aligned}
H_1 &= -g^2N\int d^4x A_0^2 && \text{(A.9)}\\
F_2 &= \mathrm{tr}\int d^4x[D_\mu, D_\nu]^2 = -g^2N\int d^4x(F^a_{\mu\nu})^2 && \text{(A.10)}\\
D_2 &= \mathrm{tr}\int d^4x[D_j, D_0]^2 = -g^2N\int d^4x(F^a_{j0})^2 && \text{(A.11)}\\
G_{n+1} &= \mathrm{tr}\int d^4x[D_j,[D_j, D_0^{2n}]] && \text{(A.12)}\\
G'_{n+1} &= -2T^2\mathrm{tr}\int d^4x[D_i,[D_i,\cos(nD_0/T)]]\ , && \text{(A.13)}
\end{aligned}$$

where the last definition was introduced in eqn. (5.4) while developing the improved expansion. For a constant A_0^a field, all of the above functionals vanish except

$$H_n = \mathrm{tr}\int d^4x(gA_0)^{2n}\ , \tag{A.14}$$

where we use the matrix notation $A_0 = f^{abc}A_0^c$. For SU(2) with $|A_0| = \eta$, we get the simple form:

$$H_n = 2(-g^2\eta^2)^n V/T\ . \tag{A.15}$$

The following high temperature approximations were used in deriving eqns. (3.10), (3.11) and (3.13):

$$\sum_{n=1}^{\infty} n^{2p}\exp(-\epsilon n^2) \simeq \tfrac{1}{2}\sqrt{\pi/\epsilon}\frac{(2p-1)!!}{(2\epsilon)^p} - \tfrac{1}{2}\delta_{p0} \tag{A.16}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-\epsilon n^2) \simeq \tfrac{1}{2}\epsilon - \sqrt{\pi\epsilon} + \tfrac{1}{6}\pi^2 \tag{A.17}$$

$$\sum_{n=1}^{\infty}\mathrm{Ei}(-\epsilon n^2) \simeq -\sqrt{\pi/\epsilon} - \tfrac{1}{2}\ln\epsilon + 1.55\ . \tag{A.18}$$

The above sums become exact in the limit as $\epsilon \to 0$ and are even good to within a few percent when $\epsilon = 1$. We also used the following integral to regulate the ultraviolet divergences in (3.11) and (3.13):

$$\int_0^\delta \frac{dt}{t}(1 - e^{-\Lambda^2 t}) = \gamma_E + \ln(\delta\Lambda^2)\ , \tag{A.19}$$

where $\gamma_E \simeq 0.577$ is Euler's constant.

B Effective Potentials for $T \neq 0$ SU(N)

The A_0 fields in SU(N) can always be expressed in terms of a traceless Hermitian $N \times N$ matrix by defining $A_0 = \frac{1}{2} A_0^a \lambda_a$ where $\frac{1}{2} \lambda_a$ are the $N^2 - 1$ fundamental generators of SU(N). Any matrix of this form can be diagonalized by a unitary transformation. However, making such a unitary transformation is equivalent to making a time-independent gauge transformation on A_0. Since the effective potential must be invariant under all gauge transformations, it can only depend on the eigenvalues of A_0, so it is sufficient to study configurations in which A_0 is diagonal. In Appendix D of ref.[6], Gross et al. evaluate functional determinants for constant fields which are diagonal in color. We use their results to write down a general expression for the effective potential of any traceless, diagonal A_0 matrix. Let

$$gA_0 = 2\pi T q \tag{B.1}$$

where q is a diagonal, real and traceless matrix whose elements are given by

$$(q)_{jk} = q^j \delta_{jk} \,. \tag{B.2}$$

The effective potential for this field configuration is given by[6]:

$$V_{\text{eff}} = \frac{2T^4}{\pi^2} \sum_{n=1}^{\infty} \sum_{j=1}^{N} \{2N_f(-1)^n \frac{\cos(n\pi q^j)}{n^4} - \sum_{k=1}^{N} \frac{\cos(n\pi(q^j - q^k))}{n^4}\} + \frac{\pi^2 T^4}{45} \,. \tag{B.3}$$

The sums over n can be done by using the following relations:

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{n^4} = \frac{\pi^4}{90} - \frac{\pi^4}{48}[x]_+^2([x]_+ - 2)^2 \tag{B.4}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi x)}{n^4} = -\frac{7\pi^4}{720} + \frac{\pi^4}{48}(2[x]_-^2 - [x]_-^4) \,, \tag{B.5}$$

where $[x]_+ = [x(\text{mod}2)]$ and $[x]_- = [(x+1)\text{mod}2] - 1$.

For SU(2), there is only one possible form of traceless diagonal matrix: $q = \nu\sigma_3$. The effective potential then takes the form:

$$V_{\text{eff}} = \frac{\pi^2 T^4}{12}\{2N_f(2[\nu]_-^2 - [\nu]_-^4) + [2\nu]_+^2([2\nu]_+ - 2)^2\} - \frac{\pi^2 T^4}{15}(1 + \frac{7N_f}{6}) \,. \tag{B.6}$$

Dropping the constant term at the end, V_{eff} for SU(2) is plotted in fig. 3.1. For SU(3), there are two diagonal generators, so an arbitrary diagonal SU(3) matrix can be expressed by $q = \nu\lambda_3 + \sqrt{3}\rho\lambda_8$. The effective potential then takes the form:

$$\begin{aligned} V_{\text{eff}} &= \frac{\pi^2 T^4}{12}\{N_f(2[\nu+\rho]_-^2 - [\nu+\rho]_-^4 + 2[\nu-\rho]_-^2 - [\nu-\rho]_-^4 + 2[2\rho]_-^2 - [2\rho]_-^4) \\ &+ [2\nu]_+^2([2\nu]_+ - 2)^2 + [\nu+3\rho]_+^2([\nu+3\rho]_+ - 2)^2 + [\nu-3\rho]_+^2([\nu-3\rho]_+ - 2)^2\} \\ &- \frac{\pi^2 T^4}{45}(8 + \frac{21N_f}{4}) . \end{aligned} \tag{B.7}$$

V_{eff} for SU(3) is plotted in figs. 7.1 - 7.3.

There are more allowed gauge transformations in pure gauge SU(N) than there are in SU(N) with fermions (see Appendix C). For this reason, some of the degenerate absolute minima of the pure gauge effective potential are no longer absolute minima when fermions are included in the theory. Nevertheless, we show here that these points remain stationary points of the complete effective potential with fermions. A general diagonal SU(N) matrix can always be written as a linear combination of matrices having at least one zero on the diagonal and the matrix λ_{N^2-1} given by:

$$\lambda_{N^2-1} = \text{diag}(1, 1, ..., 1-N)\nu . \tag{B.8}$$

Only this last matrix will feature the minima we seek (see Appendix C), so we only need to consider its effective potential:

$$\begin{aligned} V_{\text{eff}} &= \frac{2T^4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\{2N_f(-1)^n[(N-1)\cos(n\pi\nu) + \cos(n(N-1)\pi\nu)] \\ &- 2(N-1)\cos(nN\pi\nu) - (N-1)^2\} . \end{aligned} \tag{B.9}$$

By simple differentiation, it is easy to verify that the minima at $\nu = 2m/N$ of the pure gauge part correspond exactly to maxima of the fermionic part. Consequently, for any value of N_f, the full effective potential will always have stationary points at $\nu = 2m/N$.

C Allowed Gauge Transformations

Since there are periodic temporal boundary conditions for the fields at finite temperature, the only allowed gauge transformations are those which preserve the boundary conditions. We would like to determine the most general form of these allowed gauge transformations. A general unitary transformation can always be written as an exponential:

$$U = \exp[i\lambda_a \theta^a(\tau, \vec{x})] , \tag{C.1}$$

where $\frac{1}{2}\lambda_a$ are the generators of the group. Let us now perform a gauge transformation on $A_0 = \frac{1}{2}\lambda_a A_0^a$:

$$A_0 \rightarrow A_0' = U A_0 U^{-1} - \frac{i}{g}[\partial_0 U]U^{-1} . \tag{C.2}$$

As usual, the first term simply rotates A_0^a in color space, while the second term changes its magnitude. Just looking at the second term, we can see that the magnitude of $A_0'(\tau)$ will only be the same as that of $A_0'(\tau+\beta)$ if $\theta^a(\tau, \vec{x})$ takes the form:

$$\theta^a(\tau, \vec{x}) = f_1^a(\tau, \vec{x}) + f_2^a(\vec{x})\tau , \tag{C.3}$$

where $f_1^a(\tau+\beta, \vec{x}) = f_1^a(\tau, \vec{x})$. Now if we examine the first term of eqn. (C.2), we can see that periodicity for a pure gauge theory also implies:

$$U(\tau+\beta, \vec{x}) = \exp(i\alpha)U(\tau, \vec{x}) , \tag{C.4}$$

where α is a global scalar phase (multiplied by the unit matrix). Taking the determinant of each side, and knowing that $\det(U) = 1$, we can see that for SU(N), the only allowed values of α are when $\det[\exp(i\alpha)] = 1$ or $\alpha = 2n\pi/N$, where n is an integer. Since f_1^a is periodic, only f_2^a will be able to generate a nonzero α. Furthermore, since only discrete values of α are allowed, this implies that there can be no $\vec{x}$ dependence for f_2, since such a dependence would be continuous rather than discrete.

For SU(2),

$$U = \cos(|\theta^a|) + i\lambda_a \hat{\theta}^a \sin(|\theta^a|) , \tag{C.5}$$

where $\hat{\theta}^a = \theta^a/|\theta^a|$, and it is easy to see that only $f_2^a = n_a \pi T$ with integer n_a will satisfy eqn. (C.4). For SU(N) with $N > 2$, it is always possible to choose a fundamental representation in which all but one of the generators have at least one zero eigenvalue (for example the Gell-Mann matrices for SU(3)). The f_2^a terms corresponding to each of the generators with a zero eigenvalue must be of the form $f_2^a = 2n_a \pi T$. The remaining generator λ_{N^2-1} is given in its unnormalized form by (B.8) in Appendix B. It can be verified that $f_2^{N^2-1} = 2n\pi T/N$ gives rise to allowed gauge transformations.

The situation changes a bit if there are fermions in the theory. Since fermions transform like $\psi \to U\psi$, there are no factors of U^{-1} to cancel global phases. Thus in order for fermion temporal boundary conditions to remain unaffected by gauge transformations, only transformations satisfying eqn. (C.4) with $\alpha = 0$ are permissible. In other words, fermions break the center symmetry which is present in pure gauge theories. Therefore, the most general form of f_2^a for SU(N) with fermions is $f_2^a = 2n_a \pi T$.

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