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M-MATRICES SATISFY NEWTON'S INEQUALITIES

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ABSTRACT. Newton's inequalities $c_n^2 \geq c_{n-1}c_{n+1}$ are shown to hold for the normalized coefficients c_n of the characteristic polynomial of any M- or inverse M-matrix. They are derived by establishing first an auxiliary set of inequalities also valid for both of these classes. They are also used to derive some new necessary conditions on the eigenvalues of nonnegative matrices.

1. INTRODUCTION

The goal of the paper is to prove a conjecture made in [4] about a set of inequalities satisfied by (the elementary symmetric functions of) the eigenvalues of any M- or inverse M-matrix.

Let $\langle n \rangle$ denote the collection of all increasing sequences with elements from the set $\{1, 2, \ldots, n\}$, let $\#\alpha$ denote the size of the sequence α, and let α' denote the complementary or 'dual' sequence whose elements are all the integers from $\{1, 2, \ldots, n\}$ not in α. Given a matrix $A \in \mathbb{C}^{n \times n}$, the notation $A(\alpha)$ $(A[\alpha])$ will be used for the principal submatrix (minor) of A whose rows and columns are indexed by α. By convention, $A[\emptyset] := 1$.

A matrix A is called a P-matrix if $A[\alpha] > 0$ for all $\alpha \in \langle n \rangle$. A is called a (nonsingular) M-matrix if it is a P-matrix and its off-diagonal entries are nonpositive. If in this definition the positivity of all principal minors is relaxed to nonnegativity, one obtains the class of all M-matrices, including the singular ones. The class of inverse M-matrices consists of matrices whose inverses are M-matrices. The M-matrices are an important class arising in many contexts (see, for example, [2, Chapter 6]).

Given a matrix A, let $c_j(A)$ denote the normalized coefficients of its characteristic polynomial:

$$c_j(A) := \sum_{\#\alpha=j} A[\alpha] \Big/ \binom{n}{j}, \qquad j = 0, \ldots, n.$$

The inequalities

$$(1) \qquad c_j^2(A) \geq c_{j-1}(A)c_{j+1}(A), \qquad j = 1, \ldots, n-1,$$

are known for real diagonal matrices, i.e., simply for sequences of real numbers (see [13] and references therein), as was first proved by Newton. Since the numbers c_j are invariant under similarity, Newton's inequalities (1) also hold for all diagonalizable matrices with real spectrum, and therefore also for the closure of this set, viz. for *all* matrices with real spectrum.

It was conjectured in [4] that Newton's inequalities are also satisfied by M- and inverse M-matrices (and by matrices similar to those). The next section contains proofs of several results on M-matrices and symmetric functions culminating in the proof of this fact.

2. Proof of Newton's inequalities

Let us begin by establishing a set of auxiliary inequalities first. Given an $n \times n$ matrix A and nonnegative integers m_1, m_2, k, define functions $S_{m_1,m_2,k}$ as follows:

$$(2) \qquad S_{m_1,m_2,k}(A) := \sum_{\substack{\alpha \in \langle n \rangle, \#\alpha = m_1, \\ \beta \in \langle n \rangle, \#\beta = m_2, \#\alpha \cap \beta = k}} A[\alpha]A[\beta].$$

Theorem 1. *For any M- or inverse M-matrix A of order n and nonnegative integers $m < n$, $k < m$,*

$$(3) \qquad S_{m,m,k}(A)/S_{m,m,k}(I_n) \geq S_{m+1,m-1,k}(A)/S_{m+1,m-1,k}(I_n),$$

where I_n denotes the identity matrix of order n.

Proof. By induction.

Case 1 (induction base). If $k = 0$, $n = 2m$, then (3) is a special case of Theorem 1.3 from [6]. Indeed, since $n = 2m$, the functions $S_{m.m,0}$ and $S_{m+1,m-1,0}$ are immanants, $\lambda := (m, m)$ and $\mu := (m+1, m-1)$ are partitions of n, and μ majorizes λ. Then the normalized immanant corresponding to μ does not exceed the one corresponding to λ (beware a typo in [6], where the sign is reversed). If an M-matrix A is nonsingular, then $A^{-1}[\alpha] = A[\alpha']/\det A$ (see, e.g., [3, Section 1.4]). Hence $S_{m,m,0}(A^{-1}) = S_{m,m,0}(A)/(\det A)^2$, $S_{m+1,m-1,0}(A^{-1}) = S_{m+1,m-1,0}(A)/(\det A)^2$, so the inequality (3) holds for the matrix A^{-1} as well.

Now assume (3) holds for all M- and inverse M-matrices of order smaller than n.

Case 2 (induction step of the first kind). Suppose $2m - k < n$ and A is an M- or inverse M-matrix. Then both normalized functions $S_{m,m,k}(A)/S_{m,m,k}(I_n)$ and $S_{m+1,m-1,k}(A)/S_{m+1,m-1,k}(I_n)$ can be obtained by first averaging the terms $A[\alpha]A[\beta]$ over submatrices of order $n-1$ and then taking the average of the obtained n quantities:

$$\frac{S_{m,m,k}(A)}{S_{m,m,k}(I_n)} = \frac{1}{n} \sum_{\alpha \in \langle n \rangle, \#\alpha = n-1} \frac{S_{m,m,k}(A(\alpha))}{S_{m,m,k}(I_{n-1})},$$

$$\frac{S_{m+1,m-1,k}(A)}{S_{m+1,m-1,k}(I_n)} = \frac{1}{n} \sum_{\alpha \in \langle n \rangle, \#\alpha = n-1} \frac{S_{m+1,m-1,k}(A(\alpha))}{S_{m+1,m-1,k}(I_{n-1})}.$$

But principal submatrices of M- (inverse M-) matrices are again M- (inverse M-) matrices ([5, p.113, p.119]). Therefore the inductive assumption holds for all submatrices $A(\alpha)$, $\#\alpha = n - 1$. This implies (3) for the matrix A itself.

Case 3 (induction step of the second kind). Let $2m - k = n$ and $k > 0$. First assume A is a *nonsingular* M- or inverse M-matrix. Switch to the dual case: Each

$A[\alpha]A[\beta]$ in the right-hand side of (2) equals $A^{-1}[\alpha']A^{-1}[\beta']/(\det A)^2$, the index sets α' and β' do not intersect, and $\#\alpha' + \#\beta' = 2(n - m) < n$. Hence

$$S_{m,m,k}(A) = \frac{S_{n-m,n-m,0}(A^{-1})}{(\det A)^2}, \qquad S_{m+1,m-1,k}(A) = \frac{S_{n-m+1,n-m-1,0}(A^{-1})}{(\det A)^2}$$

and the functions $S_{n-m,n-m,0}(A^{-1})$, $S_{n-m+1,n-m-1,0}(A^{-1})$ are as in Case 2 above. Thus (3) holds for the matrix A^{-1} and hence for the matrix A. So, the induction step of this kind is now proved for nonsingular M-matrices and their inverses. But the set of all M- matrices is the closure of the set of nonsingular M-matrices (see, e.g., [5, p.119]), which justifies the induction step for singular M-matrices as well.

With all possible cases considered, the theorem is proved. $\qquad\square$

Now, the theorem can be used to replace Newton's inequalities by a stronger (but simpler) set of quadratic inequalities in the variables $A[\alpha]$.

Lemma 2. *Let* $m \in \{1, \dots, n\}$ *be fixed and let* $t(m)$ *be the column vector*

$$t(m) := (t_\alpha)_{\alpha \in \langle n \rangle, \#\alpha = m}.$$

Let Ψ_m *denote the Hermitian form*

$$(4) \quad t(m) \mapsto t(m)^* \Psi_m t(m) := \sum_{j=0}^{m} (m(n-m) - (m+1)(n-m+1)\frac{m-j}{m-j+1}) \sum_{\substack{\#\alpha = \#\beta = m \\ \#\alpha \cap \beta = j}} \overline{t_\alpha} t_\beta.$$

If Ψ_m *is nonnegative definite, then the* m*th Newton's inequality* (1) *holds.*

Proof. Expanding both sides of the mth Newton's inequality yields

$$c_m^2(A) = \sum_{j=0}^{m} S_{m,m,j}(A)/\binom{n}{m}^2,$$

$$c_{m-1}(A)c_{m+1}(A) = \sum_{j=0}^{m-1} S_{m+1,m-1,j}(A)/\binom{n}{m+1}\binom{n}{m-1}.$$

So, the mth Newton's inequality is equivalent to

$$(5) \qquad m(n-m)\sum_{j=0}^{m} S_{m,m,j}(A) \geq (m+1)(n-m+1)\sum_{j=0}^{m-1} S_{m+1,m-1,j}(A).$$

On the other hand, straightforward counting gives

$$S_{m,m,j}(I_n) = \binom{n}{j}\binom{n-j}{m-j}\binom{n-m}{m-j},$$

$$S_{m+1,m-1,j}(I_n) = \binom{n}{j}\binom{n-j}{m-j-1}\binom{n-m+1}{m-j+1}.$$

Hence the inequalities (3) are equivalent to

$$(m-j)S_{m,m,j}(A) \geq (m-j+1)S_{m+1,m-1,j}(A).$$

Thus, upon replacing each $S_{m+1,m-1,j}$ in the right-hand side of inequality (5) by $\frac{(m-j)}{(m-j+1)}S_{m,m,j}$, one obtains a set of inequalities stronger than Newton's. Precisely, these stronger inequalities assert that

$$\sum_{j=0}^{m}(m(n-m) - (m+1)(n-m+1)\frac{m-j}{m-j+1})S_{m.m.j} \geq 0,$$

or, recalling the definitions of $S_{m,m,j}$ and of Ψ_m,

$$a(m)^*\Psi_m a(m) \geq 0 \qquad \text{where} \quad a(m) := (A[\alpha])_{\alpha \in \langle n \rangle, \#\alpha = m}.$$

So, if Ψ_m is nonnegative definite, then the mth Newton's inequality is satisfied. $\square$

Thus, it remains to prove the following.

Lemma 3. *With the notation of Lemma 2, $t(m)^*\Psi_m t(m) \geq 0$ for all $t(m)$ and all $m = 1, \ldots, n-1$.*

Proof. Consider first the Hermitian form

$$\Phi_m : t(m) \mapsto t(m)^*\Phi_m t(m) := \sum_{j=0}^{m} j \sum_{\substack{\#\alpha = \#\beta = m \\ \#\alpha \cap \beta = j}} \overline{t_\alpha} t_\beta.$$

The representation matrix

$$(\#\alpha \cap \beta)_{\alpha, \beta}$$

of this Hermitian form is the Gramian, with respect to the standard inner product, for the system of vectors $(v_\alpha)_\alpha$ where

$$v_\alpha(i) := \begin{cases} 1 & \text{if} \quad i \in \alpha, \\ 0 & \text{otherwise}, \end{cases}$$

hence is nonnegative definite. Moreover, the vector e of all ones (of appropriate length) is an eigenvector of Φ_m. Now consider a form

$$\widetilde{\Phi}_m : t(m) \mapsto t(m)^*\widetilde{\Phi}_m t(m) := \sum_{j=0}^{m} (m - j + 1) \sum_{\substack{\#\alpha = \#\beta = m \\ \#\alpha \cap \beta = j}} \overline{t_\alpha} t_\beta.$$

Its representation matrix is obtained by subtracting Φ_m from a positive multiple of the Hermitian rank-one matrix ee^* (precisely $(m+1)ee^*$). Therefore all eigenvalues of $\widetilde{\Phi}_m$ are nonpositive except for the one corresponding to the eigenvector e, which is strictly positive. Therefore, by [1], the Hadamard inverse $\widetilde{\Psi}_m$ of the matrix $\widetilde{\Phi}_m$, i.e., the matrix

$$\left(\frac{1}{m - \#\alpha \cap \beta + 1} \right)_{\alpha, \beta}$$

is nonnegative definite. Finally, Ψ_m is obtained from $(m+1)(n-m+1)\widetilde{\Psi}_m$ by subtracting the rank-one matrix ee^*, this time multiplied by $(n+1)$. The eigenvalue of Ψ_m corresponding to e is equal to zero, since

$$
\begin{aligned}
e^*\Psi_m e &= m(n-m)\sum_{j=0}^{m} S_{m,m,j}(I_n) - (m+1)(n-m+1)\sum_{j=0}^{m} \frac{m-j}{m-j+1} S_{m,m,j}(I_n) \\
&= m(n-m)\sum_{j=0}^{m} S_{m,m,j}(I_n) - (m+1)(n-m+1)\sum_{j=0}^{m-1} S_{m+1,m-1,j}(I_n) = 0.
\end{aligned}
$$

All the other eigenvalues of Ψ_m are nonnegative, so Ψ_m is nonnegative definite. $\square$

This lemma finishes the proof of Newton's inequalities.

Theorem 4. *Let A be similar to an M- or inverse M-matrix. Then the normalized coefficients of its characteristic polynomial satisfy Newton's inequalities (1).*

Also note that a by-product of Lemma 3 is a binomial identity:

Corollary 5. $\sum_{j=0}^{m}(m(n-m)-(m+1)(n-m+1)\frac{m-j}{m-j+1})\binom{m}{j}\binom{n-m}{m-j}=0.$

3. Newton's inequalities and the inverse eigenvalue problem for nonnegative matrices

As possible applications of Theorem 4 one can envision eigenvalue localization for M- and inverse M-matrices as well as inverse eigenvalue problems. In the rest of the paper the focus will be on the latter problem for nonnegative matrices.

The nonnegative inverse eigenvalue problem (NIEP) is that of determining necessary and sufficient conditions in order that a given n-tuple be the spectrum of an entrywise nonnegative $n \times n$ matrix. For details and history of the problem, see [2], [8], [12], and references therein.

Two known necessary conditions that an n-tuple $\Lambda:=(\lambda_1,\ldots,\lambda_n)$ be realizable as the spectrum of a nonnegative matrix are formulated in terms of its moments

$$s_k(\Lambda):=\sum_{j=1}^{n}\lambda_j^k,$$

viz.

(6) $$s_k \geq 0, \qquad \text{all} \quad k,$$
(7) $$s_k^m \leq n^{m-1}s_{km}, \quad \text{all} \quad k,m.$$

The condition (6) follows simply from the fact that $\operatorname{tr}(A^k)$ is the kth moment of the eigenvalue sequence of A, while the condition (7) is due to Loewy and London [10] and, independently, Johnson [7].

Newton's inequalities proven above result in a third set of conditions necessary for realizability of a given n-tuple. Precisely, if $\Lambda = (\lambda_1,\ldots,\lambda_n)$ is the spectrum of a nonnegative matrix A and $\lambda_1 = \max|\Lambda|$ is its spectral radius, then the set $(0,\lambda_1-\lambda_2,\ldots,\lambda_1-\lambda_n)$ is the spectrum of an M-matrix $\lambda_1 I - A$ and should therefore satisfy Newton's inequalities (1).

Newton's inequalities are independent of (6) and (7). First of all, it is clear that (1) and (6) are independent: for example, the triple $(1,-1,-1)$ does not satisfy (6) but its shifted counterpart $(0,2,2)$ satisfies (1), while the triple $(\sqrt{2},i,-i)$ satisfies (6) but the corresponding shifted triple $(0,\sqrt{2}-i,\sqrt{2}+i)$ does not satisfy (1).

Moreover, neither the two conditions (6) and (1) *together* imply (7) nor the two conditions (6) and (7) *together* imply (1).

Indeed, the conditions (6) and (1) can be satisfied while the conditions (7) may fail. To show this, consider the 10-tuple $\Lambda:=(3,1,1,1,1,1,-2,-2,-2,-2)$. Its first and third moment are equal to zero, while the rest are positive. Now, consider its perturbed version $\Lambda_t:=(3+t_1,1+t_2,1,1,1,1,-2+t_3,-2,-2,-2)$, where the t's are real and

$$t_1+t_2+t_3>0,$$
$$(3+t_1)^3+(1+t_2)^3+(-2+t_3)^3=20,$$

which is always possible according to the Linearization Lemma [11, p.163], since the system

$$t_1 + t_2 + t_3 > 0,$$
$$9t_1 + t_2 + 4t_3 = 0$$

is solvable arbitrarily close to the point $(0, 0, 0)$. The first moment of Λ_t is thus positive, while the third is still zero. All the other moments remain positive if (t_1, t_2, t_3) is sufficiently small. So, (6) is satisfied. The Newton conditions (1) are satisfied as well, since Λ_t is real. But the condition (7) with $k = 1$, $m = 3$ fails.

To construct an example where (6) and (7) are satisfied but (1) fails, consider the sequence of zeros of the polynomial $p(x) = x^6 - 6x^5 + 14x^4 - 20x^3$. It does not satisfy (1): This polynomial is obtained by cutting the expansion of $(x-1)^6$, whose coefficients satisfy (1) with strict equalities, and then decreasing slightly the value (originally 15) of one coefficient. Then the second Newton's inequality fails. The nonzero roots of p are approximately 3.6702 and $1.1649 \pm 2.0229i$. By shifting back by the largest absolute value ≈ 3.6702, one obtains the 6-tuple $\Lambda := (a, a, a, 0, b, \overline{b})$ with $a \approx 3.6702$, $b \approx 2.5054 + 2.0229i$. It is not hard, though a bit tedious, to check that Λ satisfies (7). Since $s_1(\Lambda) > 0$, this also implies that all moments of Λ are positive. This shows that (1) cannot be derived from (6) and (7).

In the case that the first moment of an n-tuple is zero, Laffey and Meehan [9] established another necessary condition, viz.,

$$(n-1)s_4 \geq s_2^2.$$

It is also not implied by (6), (7) and (1). An example is provided by the 5-tuple $(3, 3, -2, -2, -2)$.

Note, however, that the condition (7) with $k = 1$, $m = 2$ is exactly equivalent to the Newton inequality (1) for $j = 1$.

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