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A PERTURBATION THEOREM FOR SENSITIVITY ANALYSIS OF SVD BASED ALGORITHMS

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ABSTRACT

We present a perturbation theorem on perturbations in the SVD truncated matrices and SVD truncated pseudo inverses. The theorem can be easily applied for sensitivity analysis of any SVD based algorithm that can be formulated in terms of SVD truncated matrices or/and SVD truncated pseudoinverses. The theorem is applied to an SVD based polynomial method and an SVD based direct matrix pencil method for estimating parameters of complex exponential signals in noise. With the theorem, it is simple to show that TLS-ESPRIT, Pro-ESPRIT and the state space method are equivalent to the direct matrix pencil method to the first order approximation.

1. INTRODUCTION

Singular value decomposition (SVD) has been used extensively in signal processing and especially for estimating parameters of superimposed exponential signals in noise [1-8]. Various kinds of SVD based algorithms have been proposed and tested by numerical simulations. Recently, there is a strong interest among several researchers in perturbation analysis of SVD based algorithms [9-19] since SVD plays a major role as a noise filter in all SVD based algorithms. But many analyses have heavily relied on the perturbations of singular values and singular vectors [14-19]. Those approaches have led to complicated expressions which are difficult to understand except for simple cases (typically, single exponential case). However, we have observed that many SVD based algorithms can be formulated in terms of SVD truncated matrices or/and SVD truncated pseudoinverses [9-13]. For those algorithms, we do not have to rely on perturbation theory of singular values and singular vectors. Instead, we can base our analysis directly on the perturbations of the SVD truncated matrices and the SVD truncated pseudoinverses. As will be shown by the theorem in Section 2, the first order perturbations in the SVD truncated matrices or the SVD truncated pseudoinverses can be simply expressed in terms of the perturbations in the original data matrices.

It is important to note that for the case where two or more singular vectors are very close, the perturbations in the corresponding singular vectors can be very high [21], but the perturbations in the SVD truncated matrices or the SVD truncated pseudoinverses are virtually not affected, which can be seen from the theorem in the next section.

In Section 3 and 4, we apply the theorem for the perturbation analysis of an SVD based polynomial method and an SVD based direct matrix pencil method. The two methods are used for estimating parameters of complex exponential signals in noise. In contrast to the analyses in [14-19], our analysis is straightforward and the resulting perturbation expressions are simple and general enough for further study. In Section 5, we formulate TLS-ESPRIT [7], Pro-ESPRIT [8] and the state space method [22] in terms of the SVD truncations so that they are easily shown with the theorem to be equivalent to the direct matrix pencil method to the first order approximation.

2. A PERTURBATION THEOREM

Define an $N_1 \times N_2$ matrix as

$$Y = X + \delta Y \qquad (2.1)$$

where X is a rank-M matrix, and δY is a small (in norm) perturbation matrix. We write the SVD of Y as

$$Y = \Sigma_{i=1,min}\ \sigma_i\ \underline{u}_i\ \underline{v}_i^H \qquad (2.2)$$

where σ_i, i=1,2,...,min, are singular values in descending order; $\underline{u}_i$, i=1,..., min, are the corresponding left singular vectors; and $\underline{v}_i$, i=1,...,min, are the corresponding right singular vectors. min is the smaller number of N_1 and N_2. The superscript "H" denotes conjugate transpose. It is clear that if $\delta Y = 0$ then $\sigma_i = 0$ for i > M. Now we write the SVD truncated matrix of Y as

$$Y_T = \Sigma_{i=1,M}\ \sigma_i\ \underline{u}_i\ \underline{v}_i^H$$
$$= U\ \Sigma\ V^H \qquad (2.3)$$

where $U = [\underline{u}_1, \ldots, \underline{u}_M]$, $V = [\underline{v}_1, \ldots, \underline{v}_M]$, and $\Sigma = diag[\sigma_1, \ldots, \sigma_M]$. The SVD truncated pseudoinverse of Y is denoted by

$$Y_T^+ = \Sigma_{i=1,M}\ 1/\sigma_i\ \underline{v}_i\ \underline{u}_i^H$$
$$= V\ \Sigma^{-1}\ U^H \qquad (2.4)$$

where the superscript "+" denotes pseudoinverse. If $\delta Y = 0$, then $Y_T = X$ and $Y_T^+ = X^+$. But if δY is not equal to zero, we write

$$Y_T = X + \delta Y_T \qquad (2.5)$$
$$Y_T^+ = X^+ + \delta Y_T^+ \qquad (2.6)$$

where δY_T and δY_T^+ are called the perturbations in the truncated matrix and in the truncated pseudoinverse respectively. Now we are ready to present the following:

<u>Theorem</u>: To the first order approximation,

$$\underline{u}_0^H\ \delta Y_T = \underline{u}_0^H\ \delta Y \qquad (2.7a)$$
$$\delta Y_T\ \underline{v}_0 = \delta Y\ \underline{v}_0 \qquad (2.7b)$$
$$\underline{v}_0^H\ \delta Y_T^+\ \underline{u}_0 = -\underline{v}_0^H\ X^+\ \delta Y\ X^+\ \underline{u}_0 \qquad (2.8)$$

where $\underline{u}_0$ is any vector from R(X), and $\underline{v}_0$ is any vector from $R(X^H)$. R( ) denotes the column span (i.e., range) of the corresponding matrix.

The proof is omitted here. (2.7a) and (2.7b) imply that the SVD truncations do not affect the first order perturbations.

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3. PERTURBATION ANALYSIS OF AN SVD BASED POLYNOMIAL METHOD

Assume a data sequence is given by

$$y(k) = \sum_{i=1,M} a_i \; z_i^{\,k} + n(k) \quad (3.1)$$

where $k=0,1,\dots,N-1$, z_i's and a_i's are unknown signal poles and unknown amplitudes. $n(k)$ is the noise. If z_i's are known, a_i's can be easily estimated by minimizing the quadratic function:

$$J = \sum_{k=0,N-1} |y(k) - \sum_{i=1,M} a_i \; z_i^{\,k}|^2 \quad (3.2)$$

To estimate z_i's, Kumaresan and Tufts [1] proposed the following algorithms (assuming $|z_i| \le 1$ for $i=1,\dots M$):

1) Define the data matrix:

$$Y' = [\underline{y}_L \;\; \underline{y}_{L-1} \;\; \dots \;\; \underline{y}_0 \,]$$

$$= \begin{bmatrix} y(L) & y(L-1) & \dots & y(0) \\ y(L+1) & y(L) & \dots & y(1) \\ & & \dots & \\ y(N-1) & y(N-2) & \dots & y(N-L-1) \end{bmatrix} \quad (3.3)$$

where $M \le L \le N-M$. The parameter L can be adjusted to minimize the noise sensitivity.

2) Find the backward minimum-norm polynomial coefficients by

$$\underline{b} = -\, Y_T^{\,+} \;\; \underline{y}_0 \quad (3.4)$$

where

$$\underline{b} = [b_0 \; , \; b_1 \; , \; \dots, \; b_{L-1} \,]^T \quad (3.4a)$$

$$Y = [\underline{y}_L \;\; \underline{y}_{L-1} \;\; \dots \;\; \underline{y}_1 \,] \quad (3.4b)$$

$Y_T^{\,+}$ is the SVD rank-M truncated pseudo-inverse of Y.

3) Estimate the signal poles by the M roots, with magnitudes less than or equal to one, of the (backward) polynomial:

$$P_B(z) = 1 + \sum_{j=1,L} b_{L-j} \;\; z^j \quad (3.5)$$

If $n(k) = 0$ for $k=0,1,\dots,N-1$, Kumaresan [1] showed that the M signal poles are M roots of $P_B(z)$ and the L-M extraneous roots of $P_B(z)$ are outside the unit circle in the complex plane.

To evaluate the first order perturbations in the estimated signal poles due to the noise $n(k)$, we proceed as follows. Since $P_B(z_i) = 0$, the perturbation in z_i (i.e., δz_i) is related to the perturbations in b_j's (i.e., δb_j's) according to (by differentiating (3.5)):

$$\sum_{j=1,L} \; \delta b_{L-j} \;\; z_i^{\,j} \; + \; \sum_{j=1,L} \; b_{L-j} \;\; j \; z_i^{\,j-1} \;\; \delta z_i \; = 0 \quad (3.6)$$

This can be written as

$$\delta z_i = N(z_i)/D(z_i) \quad (3.7)$$

where

$$N(z_i) = -\, \underline{z}_i^{\,T} \;\; \delta\underline{b} \quad (3.8)$$

$$\underline{z}_i = [z_i^{\,L} \; , \; \dots, \; z_i^{\,1} \,]^T \quad (3.9)$$

$$D(z_i) = \sum_{j=1,L} \; b_{L-j} \;\; j \; z_i^{\,j-1} \quad (3.10)$$

In (3.6)-(3.10), only δz_i and $\delta\underline{b}$ are noise perturbed. Differentiating (3.4), we can write

$$\delta\underline{b} = -\, \delta Y_T^{\,+} \;\; \underline{y}_0 \; - \; Y^+ \;\; \delta\underline{y}_0 \quad (3.11)$$

Substituting (3.11) into (3.8) yields

$$N(z_i) = \underline{z}_i^{\,T} \;\; \delta Y_T^{\,+} \;\; \underline{y}_0 \; + \; \underline{z}_i^{\,T} \;\; Y^+ \;\; \delta\underline{y}_0 \quad (3.12)$$

Now we note that the conjugate of $\underline{z}_i$ belongs to $R(Y^H)$ and $\underline{y}_0$ belongs to $R(Y)$. Then applying (2.8) of the theorem to (3.12) leads to

$$N(z_i)$$

$$= -\, \underline{z}_i^{\,T} \;\; Y^+ \;\; \delta Y_T \;\; Y^+ \;\; \underline{y}_0 \; + \; \underline{z}_i^{\,T} \;\; Y^+ \;\; \delta\underline{y}_0$$

$$= \underline{z}_i^{\,T} \;\; Y^+ \;\; \delta Y' \;\; \underline{b}' \quad (3.13)$$

where

$$\underline{b}' = [1, \; \underline{b}^T \,]^T \quad (3.14)$$

$\delta Y'$ is defined by (3.3) with $y(k)$ replaced by $n(k)$. $N(z_i)$ can be written more explicitly in terms of noise as follows.

$$N(z_i) = \underline{z}_i^{\,T} \;\; Y^+ \;\; B \; \underline{n} \quad (3.15)$$

where

$$\underline{n} = [n(0), \; n(1), \; \dots, \; n(N-1)]^T \quad (3.16)$$

$$B = \begin{bmatrix} b_L & b_{L-1} & \dots & b_1 & 1 & & & \\ & b_L & b_{L-1} & \dots & b_1 & 1 & & \\ & & & & \dots & & & \\ & & b_L & b_{L-1} & & \dots & b_1 & 1 \end{bmatrix} \quad (3.17)$$

For any given signal, (3.7) and (3.15) can be used to evaluate the first order perturbations. Comparing to the results in [14-19], (3.7) and (3.15) are not only very simple but also more general. Detailed study of (3.7) and (3.15) is available in [11,13].

For the simple case where $y(k) = a_1 \; z_1^{\,k} + n(k)$, $|z_i| = 1$ and $n(k)$ is white with the variance σ^2, it is straight forward to show from (3.7) and (3.15) that

$$\mathrm{Var}(\delta z_i) = 1/\mathrm{SNR} \begin{cases} \dfrac{2(2L+1)}{3(N-L)^2 \; L(L+1)} & \text{for } L \le N/2 \\[2ex] \dfrac{2(-(N-L)^2 + 3L^2 + 3L+1)}{3(N-L)L^2 \; (L+1)^2} & \text{for } L \ge N/2 \end{cases} \quad (3.18)$$

where $\mathrm{SNR} = |a_1|^2 / 2\sigma^2$.

4. PERTURBATION ANALYSIS OF AN SVD BASED DIRECT MATRIX PENCIL METHOD

Given the data of (3.1), the direct matrix pencil method [10,12] estimates signal poles by the M generalized eigenvalues of the SVD truncated data matrix pencil:

$$Y_1 \; - zY_2$$

$$\approx Y_{1T} \; - zY_{2T}$$

$$= U_1 \;\; \Sigma_1 \;\; V_1^{\,H} \; - z \; U_2 \;\; \Sigma_2 \;\; V_2^{\,H} \quad (4.1)$$

where

$$Y_1 = [\underline{y}_L \;\; \underline{y}_{L-1} \;\; \dots \;\; \underline{y}_1 \,] \quad (4.2)$$

$$Y_2 = [\underline{y}_{L-1} \;\; \underline{y}_{L-2} \;\; \dots \;\; \underline{y}_0 \,] \quad (4.3)$$

Y_{1T} and Y_{2T} are rank-M SVD truncations of Y_1 and Y_2 respectively. The M generalized eigenvalues of (4.1) are the M eigenvalues of $Y_{2T}^{\,+} \, Y_{1T}$ or $Y_{1T} \, Y_{2T}^{\,+}$. The parameter L satisfies $M \le L \le N-M$ and can be used to minimize the noise effect. In noiseless case, the M signal poles are the exact generalized eigenvalues of $Y_1 \; - zY_2$, i.e., $Y_1 \; - zY_2$ decreases its rank by one if and only if z is equal to the exact signal poles z_i , $i=1,\dots,M$.

In noisy case, there exist a noisy z_i , a corresponding noisy $\underline{p}_i$ in $R(Y_{2T})$ and a corresponding noisy $\underline{q}_i$ in $R(Y_{2T}^{\,H})$ such that

$$\underline{p}_i^{\,H} \;\; (Y_{1T} \; - z_i \; Y_{2T} \,) = 0 \quad (4.4)$$

$$(Y_{1T} \; - z_i \; Y_{2T} \,) \; \underline{q}_i \;\; = 0 \quad (4.5)$$

The first order perturbation in the estimated z_i can be easily derived from the above two equations. One can verify that

$$\delta z_i = \frac{\underline{p}_i^{\,H} \; \delta Y_{1T} \; \underline{q}_i \; - z_i \; \underline{p}_i^{\,H} \; \delta Y_{2T} \; \underline{q}_i}{\underline{p}_i^{\,H} \; Y_2 \; \underline{q}_i} \quad (4.6)$$

Note that in (4.6), all quantities except for δz_i , δY_{1T} and δY_{2T} are noiseless quantities. It can be shown [13,25] that $p_i^H Y_2 q_i = a_i$. Applying (2.7a) and (2.7b) of the theorem to (4.6) yuelds

$$\delta z_i = 1/a_i \ (p_i^H \delta Y_1 q_i - z_i \ p_i^H \delta Y_2 q_i) \quad (4.7)$$

where δY_1 and δY_2 are defined by (4.2) and (4.3) with y(k) replaced by n(k). Explicitly in terms of the noise vector $\mathbf{n}$, δz_i can be rewritten as

$$\delta z_i = 1/a_i \ p_i^H \ Q_i \ \mathbf{n} \quad (4.8)$$

where

$$Q_i = \begin{bmatrix} 0 & q_{i,L} & q_{i,L-1} & \dots & q_{i,1} & \\ . & & & \dots & & \\ 0 & & q_{i,L} & q_{i,L-1} & \dots & q_{i,1} \end{bmatrix} - z_i \begin{bmatrix} q_{i,L} & q_{i,L-1} & \dots & q_{i,1} & & 0 \\ & & & \dots & & . \\ & q_{i,L} & q_{i,L-1} & \dots & q_{i,1} & 0 \end{bmatrix} \quad (4.9)$$

$q_{i,j}$ is the jth element of q_i.

For the simple case defined in the previous section, it can be shown that

$$\mathrm{Var}(\delta z_i) = 1/SNR \begin{cases} \dfrac{1}{(N-L)^2 L} & \text{for } L \le N/2 \\ \dfrac{1}{(N-L)L^2} & \text{for } L \ge N/2 \end{cases} \quad (4.10)$$

It is simple to verify that

$$\mathrm{Var}(\delta z_i)_{polynomial} \le \mathrm{Var}(\delta z_i)_{matrix\ pencil} \quad (4.11)$$

5. PERTURBATION ANALYSIS OF OTHER MATRIX PENCIL ALGORITHMS

In this section, we show that Pro-ESPRIT [8], TLS-ESPRIT [7] and the state space method [22] have the same first order perturbations as the direct matrix pencil method [10,12] as discussed in the previous section. Note that the covariance filtering incorporated in Pro-ESPRIT and TLS-ESPRIT is not considered.

<u>Pro-ESPRIT:</u>

This algorithm can be described based on (4.1). Multiplying (4.1) by U_2^H from the left and by V_2 from the right, one obtains the equivalent MxM pencil:

$$U_2^H U_1 \Sigma_1 V_1^H V_2 - z \Sigma_2 \quad (5.1)$$

Zoltowski [8] suggests that $U_2^H U_1$ and $V_2^H V_1$ be replaced by their best unitary approximations since in noiseless case they are unitary. In other words, he replaces (5.1) by the "cleaned" pencil:

$$Q_U \Sigma_1 Q_V^H - z\Sigma_2 \quad (5.2)$$

where

$$Q_U = (U_2^H U_1)_{unitary} \quad (5.3)$$
$$Q_V = (V_2^H V_1)_{unitary} \quad (5.4)$$

The unitary operator in (5.3) works as follows. If $U_2^H U_1$ has the SVD $U_Q \Sigma_Q V_Q^H$, then $Q_U = U_Q V_Q^H$. Q_V is similarly obtained.

To carry out the first order perturbation analysis, we present a matrix pencil which is equivalent to (5.2). Since $[U_1, U_2]$ and $[V_1, V_2]$ each span the same M-dimensional column space in the noise case, one may compute the joint rank-M SVD truncations:

$$[U_1, U_2]_T = [U_{1T}, U_{2T}] = U_U \ \Sigma_U \ [V_{U1}^H, V_{U2}^H] \quad (5.5)$$
$$[V_1, V_2]_T = [V_{1T}, V_{2T}] = U_V \ \Sigma_V \ [V_{V1}^H, V_{V2}^H] \quad (5.6)$$

Then (4.1) can be replaced by the "cleaned" pencil:

$$U_{1T} \ \Sigma_1 \ V_{1T}^H - z \ U_{2T} \ \Sigma_2 \ V_{2T}^H \quad (5.7)$$

which is equivalent to the MxM pencil

$$V_{U1}^H \ \Sigma_1 \ V_{V1} - z \ V_{U2}^H \ \Sigma_2 \ V_{V2} \quad (5.8)$$

We can show [23] that (5.8) and (5.2) are equivalent. (Also (5.8) can be shown to be equivalent to the TLS-Pro-ESPRIT [8], i.e., Pro-ESPRIT is equivalent to TLS-Pro-ESPRIT.)

Following the same approach which leads to (4.6), one can verify that the first order perturbations in the generalized eigenvalues obtained from (5.7) are given by (4.6) with its numerator equal to

$$p_i^H \delta(U_{1T} \Sigma_1 V_{1T}^H) q_i - z_i \ p_i^H \delta(U_{2T} \Sigma_2 V_{2T}^H) q_i \quad (5.9)$$

Applying (2.7a) and (2.7b), one can verify that

$$p_i^H \delta U_{1T} q_i = p_i^H \delta U_1 q_i \quad (5.10)$$
$$p_i^H \delta U_{2T} q_i = p_i^H \delta U_2 q_i \quad (5.11)$$
$$p_i^H \delta V_{1T}^H q_i = p_i^H \delta V_1^H q_i \quad (5.12)$$
$$p_i^H \delta V_{2T}^H q_i = p_i^H \delta V_2^H q_i \quad (5.13)$$

Substituting (5.10)-(5.13) into (5.9) yields that (5.9) is equal to

$$p_i^H \delta(U_1 \Sigma_1 V_1^H) q_i - z_i \ p_i^H \delta(U_2 \Sigma_2 V_2^H) q_i = p_i^H \delta Y_{1T} q_i - z_i \ p_i^H \delta Y_{2T} q_i = p_i^H \delta Y_1 q_i - z_i \ p_i^H \delta Y_2 q_i \quad (5.14)$$

Now it is proved that Pro-ESPRIT (i.e., (5.2), (5.7) or (5.8)) is equivalent to the direct matrix pencil method to the first order approximation.

<u>TLS-ESPRIT:</u>

This algorithm consists of two steps [24] of joint SVD truncations. The first step is to compute the joint SVD of $[Y_1, Y_2]$ as follows

$$[Y_1, Y_2]_T = U_{Y3} \ \Sigma_{Y3} \ [V_{1Y3}^H, V_{2Y3}^H] \quad (5.15)$$

The second step is to compute the joint SVD of $[V_{1Y3}, V_{2Y3}]$ as follows:

$$[V_{1Y3}, V_{2Y3}]_T = U_{VY3} \ \Sigma_{VY3} \ [V_{1V3}^H, V_{2V3}^H] \quad (5.16)$$

Then using (5.15) and (5.16), we can write

$$Y_1 - z Y_2 \approx U_{Y3} \Sigma_{Y3} [V_{1V3} - z V_{2V3}] \Sigma_{VY3} U_{VY3}^H \quad (5.17)$$

which is equivalent to the MxM pencil

$$V_{1V3} - z \ V_{2V3} \quad (5.18)$$

This pencil can be shown [24] to be equivalent to the pencil used in TLS-ESPRIT.

With the above formulation of TLS-ESPRIT, one can follow the approach used for the direct matrix pencil method and Pro-ESPRIT to show that TLS-ESPRIT yields the same first order perturbations given by (4.6).

<u>The State Space Method:</u>

This method computes the truncations of Y_1 and Y_2 as follows. Let Y' have the SVD truncation

$$Y'_T = U \ \Sigma \ V^H \quad (5.19)$$

Then one defines that V_1 be V with its last row deleted and V_2 be V with its first row deleted. Hence,

$$Y_1 - zY_2 \approx U\Sigma V_1^H - z \ U\Sigma V_2^H$$

$$= U\Sigma(V_1^H - z V_2^H) \quad (5.20)$$

This is equivalent to the pencil

$$V_1 - z V_2 \quad (5.21)$$

which is used in the state space method [22]. Now it is a simple matter to show that the space space method is equivalent to all the above matrix pencil algorihtms to the first order approximation.

CONCLUSION

We have presented a perturbation theorem of SVD truncated matrices and SVD truncated pseudoinverses. The theorem indicates that SVD truncations do not affect the first order perturbations. For any method which can be expressed in terms of SVD truncations, the theorem can be directly applied for perturbation analysis without using complicated perturbations of singular values and singular vectors. The theorem has been applied for perturbation analysis of an SVD based polynomial method (i.e., SVD Prony method) and an SVD based direct matrix pencil method. The application of the theorem to Pro-ESPRIT, TLS-ESPRIT and the state space method has shown that all those algorithms are equivalent to the direct matrix pencil method to the first order approximation. We note finally that formulating algorithms directly in terms of SVD truncations is vital for the application of the theorem.

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