Clifford Index of ACM Curves in $\mathbb{P}^3$

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In this paper we review the notions of gonality and Clifford index of an abstract curve. For a curve embedded in a projective space, we investigate the connection between the Clifford index of the curve and the geometrical properties of its embedding . In particular if C is a curve of degree d in $\mathbb{P}^3$, and if L is a multisecant of maximum order k, then the pencil of planes through L cuts out a g^1_{d-k} on C. If the gonality of C is equal to $d-k$ we say the gonality of C can be computed by multisecants. We discuss the question whether the gonality of every smooth ACM curve in $\mathbb{P}^3$ can be computed by multisecants, and we show the answer is yes in some special cases.

1 Gonality and Clifford index of a curve

Let C be a nonsingular projective curve over an algebraically closed field k. A linear system of degree d and (projective) dimension r will be denoted by g^r_d. The least integer d for which there exists a complete linear system g^1_d without base points is called the *gonality* of C. Thus a curve is rational if and only if its gonality is 1. Curves of genus 1 and 2 have gonality 2. For curves of genus $g \geq 2$, the curve is hyperelliptic if and only if the gonality is 2. It is well known that for curves of genus $g \geq 3$ the gonality d lies between 2 and $\left[\dfrac{g+3}{2}\right]$; there exist curves having each possible gonality in this range; and a curves of genus g of general moduli has gonality $\left[\dfrac{g+3}{2}\right]$. See [1] for references to proofs of these results.

Thus the gonality of a curve provides a stratification of the variety of moduli $\mathcal{M}_g$ of curves of genus g, with the hyperelliptic curves at one end, and the curves of general moduli at the other end.

To illustrate this principle, let us describe some different types of curves of genus g for small values of g.

For $g = 0$ there is just one curve, $\mathbb{P}^1$, having gonality 1 and a unique g^1_1.

For $g = 1$ there is a one-parameter family of non-isomorphic curves. They all have gonality 2, and each one has infinitely many g^1_2's.

For $g = 2$, the curves are hyperelliptic, each having a unique g^1_2.

For $g = 3$, there are hyperelliptic curves, with a unique g^1_2, and there are non-hyperelliptic curves, each having infinitely many g^1_3's. These are called *trigonal* (meaning a curve with gonality 3). The canonical embedding of a trigonal curve of genus 3 is a nonsingular plane quartic curve. The g^1_3's on the curve are cut out by the pencils of lines through a point on the curve.

For $g = 4$, there are again two types, hyperelliptic and trigonal. The canonical embedding of a trigonal curves of genus 4 is a complete intersection of an irreducible quadric surface Q and a cubic surface F_3 in $\mathbb{P}^3$. If Q is nonsingular (the general case), the the curve C has exactly two g^1_3's cut out by the two families of lines on Q. If Q is a cone, then C has a unique g^1_3.

For $g = 5$ there are three types of curves: the hyperelliptic curves, the trigonal curves, and the general curves. The canonical embedding of a non-hyperelliptic curve of 5 is a curve of degree 8 in $\mathbb{P}^4$. If the curve is trigonal, then it lies on a rational ruled cubic surface in $\mathbb{P}^4$, and the unique g^1_3 is cut out by the rulings of that surface. In the general case, the curve is a complete intersection of three quadric hypersurfaces in $\mathbb{P}^4$, and has infinitely many g^1_4's.

For $g = 6$ the situation becomes more complicated and more interesting. We can distinguish (at least) five different types of curves:

a) The hyperelliptic curves, having a unique g^1_2.

b) The trigonal curves, having a unique g^1_3.

c) Plane quartic curves, having a unique g^2_5. These curves have infinitely many g^1_4's, cut out by the pencils of lines through a point of the curve.

d) Double cover of an elliptic curve, having infinitely many g^1_4's, but no g^2_5.

e) Curves having only finitely many g^1_4's. The general curve has exactly five g^1_4's; some others may have fewer.

For references, see e.g. [1, Ch.V].

From these few examples it is already clear that the gonality does not tell the whole story in distinguishing different types of curves of genus g. More generally, we should take into account all possible special linear systems g^r_d that might exist on the curve. Here *special* means $r > d - g$, or equivalently $H^1(C, \mathcal{O}_C(D)) \neq 0$, where D is a divisor in the linear system.

In this connection we consider the *Brill-Noether number*

$$\rho = g - (r+1)(g - d + r) \ .$$

Then one knows, for given g, d, r that if $\rho \geq 0$, every curves of genus g has a g^r_d, and if $\rho < 0$, then a general curves of genus g does not have a g^r_d [1, Ch.V]. So to distinguish among curves of genus g, we will be interested in the existence of linear systems g^r_d for which $\rho < 0$.

Now we can introduce the *Clifford index* of a curve C of genus $g \geq 4$ (first defined by H. Martens [13]). For a particular linear system g^r_d we define its Clifford index to be $d - 2r$. Then the *Clifford index of the curve* is the minimum of Clifford indices of certain special linear series, namely Cliff C is the minimum of $d - 2r$, taken over all linear systems g^r_d with $r \geq 1$ and $1 \leq d \leq g-1$. Equivalently, one can take the minimum of $d - 2r$ over all g^r_d containing a divisor D with $h^0(\mathcal{O}_C(D)) \geq 2$ and $h^1((\mathcal{O}_C(D)) \geq 2$. (The equivalence of these two criteria is easy using the Riemann-Roch theorem, and replacing D by $K - D$ if $d > g - 1$.)

Recall that Clifford's theorem tells us that $r \leq \frac{1}{2}d$ for a special linear system g^r_d on a curve, with equality if and only if the corresponding divisor D is 0 or the canonical divisor K, or the curve is hyperelliptic. Since the inequalities in the definition rule out the possibility $D = 0$ or K, we see that the Clifford index is always ≥ 0, with equality if and only if the curve is hyperelliptic. On the other hand, the Brill-Noether theory tells us that Cliff $C \leq \left[\dfrac{d-1}{2}\right]$, and is equal to this value for a curve of general moduli.

In most cases, the Clifford index can be computed by a pencil, that is, there exists a g^1_d with Cliff $C = d - 2$. In this case Cliff $C = \operatorname{gon} C - 2$, where $\operatorname{gon} C$ denotes the gonality. This suggests the definition of the *Clifford dimension* of the curve, which is the least r for which there exists a g^r_d with Cliff $C = d - 2r$, i.e., the g^r_d computes the Clifford index. Then $r = 1$ is the normal situation. Curves with Clifford dimension > 1 are rare. The first example is the curve of genus 6 with a g^2_5 mentioned above. In this case Cliff $C = 1$ while

$\operatorname{gon} C = 4$, and so the Clifford dimension of the g^1_4 is 2.

The nonsingular plane curves of degree ≥ 5 all have Clifford dimension 2, and these are the only such. Any curve of Clifford dimension 3 must be a complete intersection of two cubic surfaces in $\mathbb{P}^3$, having degree 9 and genus 10 [11]. There exist curves of every possible Clifford dimension $r \geq 1$, and for $r \geq 3$ conjecturally only one possible degree-genus pair in $\mathbb{P}^r$ [7].

2 Curves in projective space

We now consider a nonsingular curve C embedded in a projective space $\mathbb{P}^n_k$, and we ask, how are the gonality and the Clifford index of C related to the geometry of the embedding? The prototype for this kind of question is the following well-known theorem about plane curves.

Theorem 2.1 *Let $C \subseteq \mathbb{P}^2$ be a nonsingular plane curve of degree $d \geq 2$. Then*

(a) *There is no g^1_{d-2} on C, but there are g^1_{d-1}'s on C, so the gonality is $d-1$.*

(b) *Every g^1_{d-1} on C is cut out by the pencil of lines in $\mathbb{P}^2$ through some fixed point of C.*

Proof. This result was known to M.Noether and has received a number of modern proofs [4],[10],... We will give an elementary proof to illustrate the ideas involved.

(a) Suppose D is a divisor of degree $e \leq d-2$ on C with $h^0(\mathcal{O}(D)) \geq 2$. Since D moves in a pencil, we may assume that D consists of e distinct points $P_1, \ldots, P_e$. By the Riemann-Roch theorem $h^0(K-D) \geq g-e+1$. This means that the e points $P_1, \ldots, P_e$ do not impose independent conditions on the canonical divisors K containing them. Now the canonical divisor K on C is cut out by curves of degree $d-3$ in $\mathbb{P}^2$. Any $d-2$ distinct points impose independent conditions on these curves, a contradiction. So no such D exists. On the other hand, for any $P \in C$, the lines through P cut out a g^1_{d-1}, so these do exist.

(b) Now let D be any divisor of degree $d-1$ with $h^0(\mathcal{O}(D)) \geq 2$. The argument above shows that the $d-1$ points of D impose dependent conditions on plane curves of degree $d-3$, and this can only happen if these points lie on a line in $\mathbb{P}^2$. This line L will meet C in one further point P, and then it is clear that the g^1_{d-1} is equal to the one cut out by lines through P.

This result has been generalized to irreducible plane curves C of degree D with δ nodes and cusps, when δ is not too large in relation to d [5],[6]. In those cases, the desired result would be that the gonality of the normalization $\tilde{C}$ is $d-2$, and is given by the linear systems cut out by lines through one of the double points. We cannot expect such a result to hold for arbitrary plane curves with nodes, however, as the following example shows.

Example 2.2. Let C be a smooth curve in $\mathbb{P}^3$, of degree 6 and genus 3, not lying on a quadric surface. Such a curve arises of type $(4;1^6)$ on a nonsingular cubic surface X in $\mathbb{P}^3$, for example. This is the proper transform of a plane curve of degree 4 passing through the 6 points blown up to get X. It has gonality 3 by (2.1). On the other hand, its general projection to $\mathbb{P}^2$ is a plane curve of degree 6 with 7 nodes. The pencil of lines through one of the nodes cuts out a g^1_4 on the normalization, which does not give us the correct gonality.

Passing now to curves in higher dimensional projective spaces, let us first consider a nonsingular curve C of degree d in $\mathbb{P}^3$. Let L be a multisecant of maximum order k (that is, the scheme-theoretic intersection of C and L has length k). Then the pencil of planes through L cuts out a g^1_{d-k} on C. If the gonality of C is equal to $d-k$, we say the gonality of C can be *computed by multisecants.* We can also ask the stronger question, whether every g^1_{d-k} on C arises in this way.

For a curve C of degree d in $\mathbb{P}^n$, with $n \geq 4$, the corresponding situation would be to look for a multisecant linear space L of codimension 2 in $\mathbb{P}^n$, meeting C in a scheme of length k. The hyperplanes through L will cut out a g^1_{d-k} on C, and if this gives the gonality, we say again that the gonality can be computed by multisecants.

Example 2.3. Let $C \hookrightarrow \mathbb{P}^n$ be the canonical embedding of a nonhyperelliptic curve of genus $g \geq 3$ in $\mathbb{P}^n$, with $n = g-1$. Let g^1_e be a special complete linear system without base points, and let D be any divisor of the g^1_e. Then $h^0(\mathcal{O}(D)) = 2$. Let $F = K - D$, where K is the canonical divisor. Then $h^0(\mathcal{O}(K-F)) = 2$. Since K is cut out by hyperplanes in $\mathbb{P}^n$, this means that the divisor F is contained in two distinct hyperplanes. Let them meet in the linear space L of codimension 2. Then the pencil of hyperplanes through L, after removing the fixed points F, cuts out the original linear system g^1_e on C. In particular, the gonality can be computed by multisecants.

If $g = 3$, we have a plane curve of degree 4, and recover the result of (2.1) in this case.

If $g = 4$, the curve C is the complete intersection of a quadric surface Q with a cubic surface F_3. The pencil of planes through a line of Q cuts out the other family of lines on Q, if Q is nonsingular, or the only family of lines, if Q is a cone. Thus the g^1_3's are computed by multisecants.

If $g = 5$, there are two cases. When C is trigonal, it lies on a rational ruled cubic surface S in $\mathbb{P}^4$. This surface contains conics meeting the curve in 5 points. The plane of the conic is therefore a 5-secant plane, and the hyperplane sections of S containing this conic cut out the rulings of S, which in turn cut out the unique g^1_3 on C.

If C is not trigonal, then our result tells us that the curve C has 4-secant planes, so that the pencil of hypersurface through them cut out the g^1_4's on C.

Example 2.4. Basili [3, 4.2] shows that if C is a smooth complete intersection curve in $\mathbb{P}^3$, not contained in a plane, then the gonality can be computed by multisecant lines. Furthermore every g^1_e with $e = \operatorname{gon} C$ arises in this manner.

One can also ask what are the possible orders of multisecant lines, and hence what are the possible gonalities of these complete intersection curves. Let C be the complete intersection F_a, F_b of surfaces of degrees $2 \leq a \leq b$. Nollet [15] has shown that the maximum order k of a multisecant is either $\leq a$ or $= b$. If $a = 2$, then C is a curve of bidegree (b, b) on the quadric surface, so $k = b$ and $\operatorname{gon} C = d - b$. If $a = 3$, again $k = b$ since there are lines on the cubic surface, so $\operatorname{gon} C = d - b$. If $a \geq 4$, then Ellia and Franco [8] have shown that every value of k satisfying $4 \leq k \leq a$ or $k = b$ can occur. In particular, the general complete intersection curve with $a, b \geq 4$ has at most 4-secants, and gonality $d - 4$.

Example 2.5. If C is a smooth curve of bidegree (a, b) on a nonsingular quadric surface, with $a \leq b$, then the maximum order of a multisecant is b. In this case G. Martens [11] and Ballico [2] have shown that the gonality of C is a, and thus is computed by multisecants.

Example 2.6. The case of complete intersection curves in $\mathbb{P}^3$ (2.4) has been generalized by Ellia and Franco [8] to curves arising as the zero locus of a section of a rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^3$, twisted sufficiently: $s \in H^0(\mathcal{E}(t))$ with $t >> 0$. This includes "most" subcanonical curves in $\mathbb{P}^3$, but it is hard to get specific results for small degree curves. They show the gonality of these curves can also be computed by multisecants.

Example 2.7. Farkas [9, 2.4] using the method of Mori, shows the existence of curves C on a nonsingular quartic surface X in $\mathbb{P}^3$, for certain values of degree d and genus g

satisfying complicated conditions, for which the gonality is $d-4$ and can be computed by 4-secants to the curve.

Example 2.8. Eisenbud et al [7], also studying curves on $K3$ surfaces, show the existence, for every $r \geq 3$ of a smooth curve in $\mathbb{P}^r$ having $d = 4r-3$, $g = 4r-2$, whose gonality is computed by multisecants, and having Clifford dimension r. The gonality is $2r$ and the Clifford index is $2r-3$.

Lest the reader begin to think that all these examples are evidence for supposing that the gonality of any space curve is computed by multisecants, let us give a few counterexamples.

Example 2.9. We consider rational curves C of degree d in $\mathbb{P}^3$. There exist such curves having a $(d-1)$-secant, for example, curves of bidegree $(1, d-1)$ on a quadric surface. In this case the gonality is computed by multisecants. However, for $d \geq 5$, Ellia and Franco [8] have shown that there exist smooth rational curves whose maximum order of a multisecant k can take any value $4 \leq k \leq d-1$. In particular, the general such curve has only 4-secants, and if $d \geq 6$ these do not give the correct gonality.

Example 2.10. For an example of curves of higher genus, take a plane curve of degree a, blow up 6 points not on the curve, and let C be the image curve in the nonsingular cubic surface X in $\mathbb{P}^3$. Then C has degree $d = 3a$. It has multisecants of order a and $2a$ on X. The pencil of planes through one of the latter cut out a g_a^1 on C. But the gonality is $a-1$, so the gonality is not computed by multisecants.

From all this evidence, it seems reasonable to pose the following question.

> **Question 2.11** (Peskine) If C is a smooth ACM (i.e. projectively normal) curve in $\mathbb{P}^3$, is its gonality computable by multisecants?

We will discuss this question in the following sections.

3 Behavior of gonality in a family

We consider a smooth surface X, together with a proper morphism $f : X \to T$, where T is a nonsingular curve. Then the fibers of f from a flat family of curves on X. We assume that the general fiber $f^{-1}(t) = C_t$ is irreducible and nonsingular for $t \in T$, $t \neq 0$, and that the special fiber $f^{-1}(0) = C_1 \cup C_2$, a union of two smooth irreducible curves C_1, C_2, meeting transversally at s distinct points.

Theorem 3.1 *In a flat family as above, whose general curve C_t is smooth, and whose special curve $C_0 = C_1 \cup C_2$ is a union of two smooth curves meeting transversally at s points, we have*

$$\mathrm{gon}(C_t) \geq \min\{s, \mathrm{gon}(C_1) + \mathrm{gon}(C_2)\}$$

for all sufficiently general $t \in T$.

Proof. Suppose that the general curves C_t in the family all have a g^1_d with $d \leq s$. Then we can find an open set $T' \leq T$ and an invertible sheaf $\mathcal{L}'$ on $X' = f^{-1}(T')$ inducing a g^1_d on each fiber. This invertible sheaf extends to an invertible sheaf $\mathcal{L}$ on X, but the extension is not unique, because we can replace $\mathcal{L}$ by $\mathcal{L}(mC_1)$ for any $m \in \mathbb{Z}$ and still get the same $\mathcal{L}'$ on restricting to X'.

Let us compute some intersection numbers. Since $C_1 + C_2 \sim C_t$, we have $C_1 \cdot (C_1 + C_2) = 0$, so $C_1^2 = -C_1 \cdot C_2 = -s$. Similarly $C_2^2 = -s$. Let us denote by $\mathcal{L}_0$ the restriction of $\mathcal{L}$ to C_0, and by $\mathcal{L}_1, \mathcal{L}_2$, the restrictions to C_1, C_2. Let $a = \deg \mathcal{L}_1$ and $b = \deg \mathcal{L}_2$. Then $a + b = \deg \mathcal{L} = d$. If we replace $\mathcal{L}$ by $\mathcal{L}(mC_1)$ then a becomes $a - ms$ while b becomes $b + ms$. Thus by choosing m appropriately, we may assume that $1 \leq a \leq s$ and consequently $1 - s \leq b \leq s - 1$.

Since $\mathcal{L}$ cuts out a g^1_d on the general curve C_t, we have $h^0(\mathcal{L}_t) \geq 2$ for general $t \in T$. Hence by semicontinuity, $h^0(\mathcal{L}_0) \geq 2$. We consider the exact sequence

$$0 \to \mathcal{L}_0 \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{O}_D \to 0$$

where D is the set of s points $C_1 \cap C_2$. On cohomology this gives

$$0 \to H^0(\mathcal{L}_0) \to H^0(\mathcal{L}_1) \oplus H^0(\mathcal{L}_2) \to H^0(\mathcal{O}_D)$$

we consider three cases, depending on the dimensions of $h^0(\mathcal{L}_1)$, $h^0(\mathcal{L}_2)$.

Case 1. If one of $h^0(\mathcal{L}_i)$, $i = 1, 2$ is zero, then the other must be 2. So we have a g^1_d on one of the curves, say C_1, and since the section of $\mathcal{L}_0$ giving this g^1_d is identically zero on

C_2, the divisor of the g^1_d must contain D as a fixed component. But then $d > s$, contrary to our assumptions.

CASE 2. If one of the $h^0(\mathcal{L}_i)$ is 1, say $h^0(\mathcal{L}_2) = 1$, then we can find a section of $\mathcal{L}_0$ inducing 0 on C_2 and a nonzero section of $\mathcal{L}_1$ on C_1. In this case $a \geq s$, and $b \geq 0$ since $h^0(\mathcal{L}_2) \neq 0$, so $d \geq s$ and we have $\operatorname{gon}(C_t) \geq s$.

CASE 3. If both of $h^0(\mathcal{L}_i)$ are ≥ 2, then we have a g^1_a as C_1 and a g^1_b on C_2, so $d = a + b \geq \operatorname{gon}(C_1) + \operatorname{gon}(C_2)$, as required.

Corollary 3.2 *In the statement of the theorem, if* $\operatorname{gon}(C_t) = \operatorname{gon}(C_1) + \operatorname{gon}(C_2) < s$, *then there exist morphisms* $\pi_1 : C_1 \to \mathbb{P}^1$ *and* $\pi_2 : C_2 \to \mathbb{P}^1$ *of degrees equal to the gonality, such that* π_1 *and* π_2 *agree on the* s *points* $C_1 \cap C_2$.

Proof. Indeed, if $\operatorname{gon}(C_t) < s$, then we must be in Case 3 of the proof above, and the g^1_a on C_1 and g^1_b on C_2 are induced by $H^0(\mathcal{L}_0)$, so must agree on D.

Note: A special case of this kind of argument appears in a paper of Ballico [2].

Example 3.3. We can use the theorem to give another proof of a weak form of (2.1), namely, a general curve C of degree $d \geq 2$ in $\mathbb{P}^2$ has gonality $d-1$. The pencil of lines through a point on C cuts out a g^1_{d-1}, so we always have $\operatorname{gon}(C) \leq d-1$. To prove the reverse inequality, we use induction on d.

For $d = 2$, the conic is isomorphic to $\mathbb{P}^1$, so has gonality 1.

For $d \geq 3$, consider a family of smooth curves C_t of degree d degenerating to the union of a general smooth curve C_1 of degree $d-1$ and a transversal line $C_2 = L$. Then $C_1 \cap L$ is $d-1$ points, and $\operatorname{gon}(C_1) = d-2$, $\operatorname{gon} C_2 = 1$, so by the theorem we find $\operatorname{gon} C \geq (d-2) + 1 = d-1$. Note that $s = \operatorname{gon}(C_1) + \operatorname{gon}(C_2)$ in this proof, so we do not get any additional information from the Corollary.

4 Curves on quadric and cubic surfaces

Proposition 4.1 *Let* Q *be a nonsingular quadric surface in* $\mathbb{P}^3$, *and let* C *be a general smooth curve of bidegree* (a, b) *with* $0 < a \leq b$. *Then* $\operatorname{gon}(C) = a$.

Proof. By projection onto one of the factors of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ we know that $\mathrm{gon}(C) \leq a$. For $a = 1$, the curve is rational, so has gonality 1.

For $a \geq 2$, we let C move in a family specializing to the union of a general curve C_1 of bidegree $(a-1, b-1)$ and a conic C_2 of bidegree $(1,1)$. Then $s = C_1 \cdot C_2 = a + b - 2 \geq a$. Also by induction $\mathrm{gon}(C_1) = a - 1$ and $\mathrm{gon}(C_2) = 1$. So by (3.1), $\mathrm{gon}(C) \geq a$, as required.

We recover a slightly weaker version of (2.5), since our method of proof applies only to the general curve in a family.

For curves on a cubic surface, we have seen (2.10) that not every smooth curve on a smooth cubic surface has its gonality determined by multisecants. However, we can obtain a result for sufficiently general ACM curves on a cubic surface.

Proposition 4.2 *Let C be a smooth ACM curve on a nonsingular cubic surface X in $\mathbb{P}^3$. If C is sufficiently general in its linear system on X, then C has a multisecant L such that the pencil of planes through L cuts out a pencil on C computing the gonality of C.*

Proof. First we must identify the smooth ACM curves on X. Using the postulation character γ of [14, 2.11, p. 34] we see that if C is not contained in any quadric surface, then its γ-character has $s_0 = 3$, and is positive, and connected. This means it must have one of the following four types, where $a \geq 0$:

a) $-1 \ -1 \ -1 \ 0^a \ 3$

b) $-1 \ -1 \ -1 \ 0^a \ 2 \ 1$

c) $-1 \ -1 \ -1 \ 0^a \ 1 \ 2$

d) $-1 \ -1 \ -1 \ 0^a \ 1 \ 1 \ 1$.

These can all be obtained by ascending biliaison on X from one of the following curves on a quadric surface:

a) $\gamma = -1 \ -1 \ 2$ $\qquad d = 3,\ g = 0$

b) $\gamma = -1 \ -1 \ 1 \ 1$ $\qquad d = 4,\ g = 1$

c) $\gamma = -1 - 1 \ 0 \ 2$ $\qquad d = 5,\ g = 2$

d) $\gamma = -1 \ -1 \ 0 \ 1 \ 1$ $\qquad d = 6,\ g = 4.$

Now, for suitable choice of the basis of Pic $X = \mathbb{Z}^7$, we can represent these curves by the following divisor classes on X:

a) $(1;0^6)$ gon $= 1$, line $L = G_1$ meets C in 2 points

b) $(3;1^5,0)$ gon $= 2$, line $L = F_{16}$ meets C in 2 points

c) $(4;2,1^5)$ gon $= 2$, line $L = G_1$ meets C in 3 points

d) $(6;2^6)$ gon $= 3$, has a trisecant not on X.

Thus for each of these curves the gonality is computed by a multisecant L, and in the first three cases, we can choose L to be a line lying on X. In th fourth case, we cannot find a trisecant lying on X, so we make one biliaison (i.e., replace C by $C+H$ on X, where H is the plane section) and obtain

d$'$) $(9;3^6)$.

This last curve C in case d$'$) is a complete intersection of two cubic surfaces. This is the exceptional case of Clifford dimension 3 studied by Martens [11]. He shows it has gonality 6, but Clifford index 3 given by the linear system g^3_9 giving the embedding in $\mathbb{P}^3$. Now this curve does have a trisecant L on X, say E_1, and this line computes the gonality.

To prove our result, we use the fact that every smooth ACM curve on the cubic surface X is obtained from one of a), b), c), d) by biliaison on the surface X. These curves all have gonality computed by multisecants. We prove our result then, by induction on the degree. Our induction statement is the stronger claim that if C is sufficiently general, then there is a multisecant L on X of order k, such that the gonality of C is $d-k$. We begin the induction with cases a), b), c), d$'$).

For the induction step, suppose that C on X has degree d, a multisecant L on X of order k, and gonality $d-k$. We take C' a general member of the linear system $|C+H|$, and let it specialize to $C_1 = C$ union $C_2 = H$. Then $s = C_1 \cdot C_2 = d$, and gon$(H) = 2$, since H is a plane cubic curve. Then by (3.1), gon$(C') \geq \min\{d, d-k+2\}$. Since $k \geq 2$ in all our starting cases, we conclude gon$(C') \geq d-k+2$. On the other hand, the degree of C' is $d+3$, and $C' \cdot L = k+1$. So the pencil of planes through L cuts out a g^1_{d-k+2} and we find gon$(C') = d-k+2$ as required.

In particular, for the complete intersection curves on X, we recover a weak form of Basili's result [3], since our proof works only for sufficiently general curves.

5 Conclusion

For curves on quartic surfaces, Farkas [9] has shown that the gonality of some special classes of curves is computed by 4-secants. His method does not cover all ACM curves on quartic surfaces, because he always assumes the surface contains no rational and no elliptic curves.

If we apply the methods of this paper to ACM curves lying on surfaces of degree four and higher, we obtain only an inequality for the gonality, not an exact figure, and so we are unable to answer Question 2.11 in general. If the surface contains a line, and the curve is either in the biliaison class of the line, or residual to the line, then the line becomes a multisecant of high order that computes the gonality. This case was also observed by Paoletti [16].

To make further progress on Question 2.11 will require some other technique.

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