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A parallel hierarchical blocked adaptive cross approximation algorithm

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Abstract
This article presents a low-rank decomposition algorithm based on subsampling of matrix entries. The proposed algo- rithm first computes rank-revealing decompositions of submatrices with a blocked adaptive cross approximation (BACA) algorithm, and then applies a hierarchical merge operation via truncated singular value decompositions (H-BACA). The proposed algorithm significantly improves the convergence of the baseline ACA algorithm and achieves reduced com- putational complexity compared to the traditional decompositions such as rank-revealing QR. Numerical results demonstrate the efficiency, accuracy, and parallel scalability of the proposed algorithm.

Keywords
Adaptive cross approximation, singular value decomposition, rank-revealing decomposition, parallelization, multilevel algorithms

1. Introduction

Rank-revealing decomposition algorithms are important numerical linear algebra tools for compressing high- dimensional data, accelerating solution of integral and par- tial differential equations, constructing efficient machine learning algorithms, analyzing numerical algorithms, and so on, as matrices arising from many science and engineer- ing applications oftentimes exhibit numerical rank- deficiency. Despite the favorable $O(nr)$ memory footprint of such decompositions with n and r, respectively, denoting the matrix dimension (assuming a square matrix) and the numerical rank, the computational cost can be expensive. Existing rank-revealing decompositions such as truncated singular value decomposition (SVD), column-pivoted QR (QRCP), CUR decomposition, interpolative decomposition (ID), and rank-revealing LU typically require at least $O(n^2r)$ operations (Cheng et al., 2005; Gu and Eisenstat, 1996; Mahoney and Drineas, 2009; Voronin and Martins- son, 2017). This complexity can be reduced to
$O(n^2 \log r + nr^2)$ by structured random matrix projection-

cross approximation (ACA) (Bebendorf, 2000; Bebendorf and Grzhibovskis, 2006; Zhao et al., 2005) algorithms can achieve $O(nr^2)$ complexity. However, the robustness of these algorithms relies heavily on matrix properties that are not always present in practice.

2. When the matrix can be rapidly applied to arbitrary vectors, algorithms such as randomized SVD, QR, and UTV (T lower or upper triangular) (Feng et al., 2019; Liberty et al., 2007; Martinsson et al., 2019; Xiao et al., 2017) can be utilized to achieve quasi- linear complexity.
3. Finally, given a matrix with missing entries, the low- rank decomposition can be constructed via matrix completion algorithms (Balzano et al., 2010; Candès and Recht, 2009) in quasi-linear time assuming inco- herence properties of the matrices (i.e. projection of natural basis vectors onto the space spanned by sin- gular vectors of the matrix should not be very sparse).

based algorithms (Liberty et al., 2007; Voronin and Mar- tinsson, 2017). In addition, faster algorithms are available in the following three scenarios.

1. When each element entry can be computed in $O(1)$ CPU time with prior knowledge (i.e. smoothness, sparsity, or leverage scores) about the matrix, faster algorithms such as randomized CUR and adaptive

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This work concerns the development of a practical algo- rithm, in application scenario 1, that improves the robust- ness of ACA algorithms while maintaining reduced complexity for broad classes of matrices.

The partially pivoted ACA algorithm, closely related to LU with rook pivoting (Foster, 1997; Neal and Poole, 1992; Poole and Neal, 1992, 1991), constructs an LU-type decomposition upon accessing one row and column per iteration. For matrices resulting from asymptotically smooth kernels, ðÞACA is a rank-revealing and optimal- complexity algorithm that converges in $O\ r$ iterations (Bebendorf, 2000). Despite its favorable computational complexity, it is well-known that the ACA algorithm suf- fers from deteriorated convergence and/or premature ter- mination for non-smooth, sparse, and/or coherent matrices (Heldring et al., 2014). Hybrid methods or improved con- vergence criteria (e.g. hybrid ACA-CUR, averaging, statis- tical norm estimation) have been proposed to partially alleviate the problem (Grasedyck and Hackbusch, 2005; Heldring et al., 2015). The main difficulty of leveraging ACA as robust algebraic tools for general low-rank matrices results from ACA's partial pivot-search strategy to attain low complexity. In addition to the above-mentioned remedies, another possibility to improve ACA's robustness is to search for pivots in a wider range of rows/ columns without sacrificing too much computational effi- ciency. Here we consider two different strategies:

1. Instead of searching one row/column per iteration as in ACA, it is possible to search a block of rows/ columns to find multiple pivots together.
2. Instead of applying ACA directly on the entire matrix, it is possible to start with compressing sub- matrices via ACA and then merge the results as one low-rank product.

In extreme cases (e.g. when block size equals matrix dimension or submatrix dimension equals one), these stra- tegies lead to quadratic computational costs. Therefore, it is valuable to address the question: for what matrix kernels and under what block/submatrix sizes will these strategies retain low complexity.

For the first strategy, this work proposes a blocked ACA algorithm (BACA) that extracts a block row/column per iteration to significantly improve convergence of the base- line ACA algorithms. The blocked version also enjoys higher flop performance as it involves mainly BLAS-3 operations. Compared to the aforementioned remedies, the proposed algorithm provides a unified framework to bal-

H-BACA algorithm is at most $O\left(\sqrt{n_b}\, n r^2\right)$ assuming the block size in BACA is less than the rank. In other words, the proposed H-BACA algorithm is a general numerical linear algebra tool as an alternative to ACA, SVD, QR, and so on. In addition, the overall algorithm can be parallelized using distributed-memory linear algebra packages such as ScaLAPACK (Blackford et al., 1997) which avoids the dif- ficulty of efficient parallelization of plain ACA algorithms. Numerical results illustrate good accuracy, efficiency, and parallel performance. In addition, the proposed algorithm can be used as a general low-rank compression tool for constructing hierarchical matrices (Rebrova et al., 2018).

2. Notation

Throughout this article, we adopt the Matlab notation of matrices and vectors. Submatrices of a matrix A are denoted $A(I, J)$, $A(:, J)$, or $A(I, :)$ where I and J are index sets. Similarly, subvectors of a column vector u are denoted $u(I)$. An index set I permuted by J reads $I(J)$. Transpose, inverse, pseudo-inverse of A are A^t, A^{-1}, $A^{\dagger}$. $\|A\|$ and $\|u\|$ denote Frobenius norm and 2-norm. Note that u refers to a $n \times 1$ column vector. Vertical and horizontal

ance robustness and efficiency. Upon increasing the block size (i.e. the number of rows/columns per iteration), the algorithm gradually changes from ACA to ID. For the sec- ond strategy, the proposed algorithm further subdivides the matrix into n_b submatrices compressed via BACA, followed by a hierarchical merge algorithm leveraging low-rank arithmetic (Grasedyck and Hackbusch, 2003; Hackbusch et al., 2002). The overall cost of this

concatenations of A, B are $[A; B]$ and $[A, B]$. Element-wise multiplication of A and B is $A \circ B$. All matrices are real-valued unless other- wise stated. It is assumed for $A \in \mathbb{R}^{m\times n}$, $m \sim O(n)$, but the proposed algorithms also apply to complex-valued and tall- skinny/short-fat matrices. We denote truncated SVD as $[U, \Sigma, V, r] = SVD(A, \epsilon)$ with $U \in \mathbb{R}^{m\times r}$, $V^t \in \mathbb{R}^{n\times r}$ col- umn orthogonal, $\Sigma \in \mathbb{R}^{r\times r}$ diagonal, and r being ϵ-rank defined by $r = \min\{k \in \mathbb{N} : \Sigma_{k+1,k+1} < \epsilon \Sigma_{1,1}\}$. We denote QRCP as $[Q, T, J] = QR(A, r)$ or $[Q, T, J] = QR(A, \epsilon)$ with $Q \in \mathbb{R}^{m\times r}$ column orthogonal, $T \in \mathbb{R}^{r\times n}$ upper triangular, J being column pivots, and ϵ and r being the prescribed accuracy and rank, respectively. QR without column- pivoting is simply written as $[Q, T] = QR(A)$. Cholesky decomposition without pivoting is written as $T = Chol(A)$ with T upper triangular. $\log n$ means logarithm of n to the base 2.

3. Algorithm description

3.1. Adaptive cross approximation

Before describing the proposed algorithm, we first briefly summarize the baseline ACA algorithm (Zhao et al., 2005). Consider a matrix $A \in \mathbb{R}^{m\times n}$ of ϵ-rank r, the ACA algorithm approximates A by a sequence of rank-1 outer products as

$$\widehat{A} \approx UV = \sum_{k=1}^{r} u_k v_k^t \tag{1}$$

Define the residual matrix E_k as

$$E_0 = A; \quad E_k = E_{k-1} - u_k v_k^t. \tag{2}$$

At each iteration k, the algorithm selects column u_k (pivot j_k from remaining columns) and row v_k^t (pivot i_k from remaining rows) from the residual matrix E_{k-1}

Algorithm 1. Adaptive cross approximation algorithm (ACA).

input : Matrix $A \in \mathbb{R}^{m\times n}$, relative tolerance ϵ
output: Low-rank approximation of $A \approx UV$ with rank r

$U = 0, V = 0, \mu = 0, r_0 = 0, j_1$ is a random column index;
for $k = 1$ **to** $\min\{m, n\}$ **do**
3 $u_k = E_{k-1}(:, j_k) = A(:, j_k) - UV(:, j_k)$;
4 $i_k = \arg\max_i |u_k(i)|$;
5 $u_k \leftarrow u_k / u_k(i_k)$;
6 $v_k^t = E_{k-1}(i_k, :) = A(i_k, :) - U(i_k, :)V$;
7 $j_{k+1} = \arg\max_j |v_k(j)|$;
8 $\nu^2 = \|u_k\|_2^2 \, \|v_k\|_2^2$;
9 $\mu^2 \leftarrow \mu^2 + \nu^2 + 2(u_k^t U)(V v_k)$;
10 $U \leftarrow [U, u_k], V \leftarrow [V; v_k^t], r_k = r_{k-1} + 1$;
11 Terminate if $\nu < \epsilon\mu$.

corresponding to an element denoted by $E_{k-1}(i_k, j_k)$ with sufficiently large magnitude. Note that u_k and v_k are $m \times 1$ and $n \times 1$ vectors. The partially pivoted ACA algorithm (ACA for short), selecting j_k ; i_k by only looking at previously selected rows and columns, is described as Algorithm 1. Specifically, each iteration k selects pivot i_k used in the current iteration and pivot j_{k+1} for the next iteration (via lines 4 and 7) as

$$i_k = \arg\max_{i \neq i_1, \ldots, i_{k-1}} |E_{k-1}(:, j_k)| \quad (3)$$

$$j_{k+1} = \arg\max_{j \neq j_1, \ldots, j_k} |E_{k-1}(i_k, :)| \quad (4)$$

and j_1 is a random initial column index. Note that $i_k \neq i_1, \ldots, i_{k-1}$ and $j_k \neq j_1, \ldots, j_{k-1}$ are enforced. The iteration is terminated when $\nu < \epsilon\mu$

$$\nu = \|u_k v_k^t\|_F \approx \|E_k\|_F, \quad \mu = \|UV\|_F \approx \|A\|_F \quad (5)$$

and ϵ is the prescribed tolerance. Note that each iteration requires only $O(nr_k)$ flop operations with r_k denoting currently revealed numerical rank. The overall complexity of partially pivoted ACA scales as $O(nr^2)$ when the algorithm converges in $O(r)$ iterations. Despite the favorable complexity, the convergence of ACA for general rank-

been developed but they do not generalize to a broad range of applications.

3.2. Blocked adaptive cross approximation

Instead of selecting only one column and row from the residual matrix in each ACA iteration, we can select a fixed-size block of columns and rows per iteration to improve the convergence and accuracy of ACA. In addition, many BLAS-1 and BLAS-2 operations of ACA become BLAS-3 operations and hence higher flop performance can be achieved.

Specifically, the proposed BACA algorithm factorizes A

$$A \approx UV = \sum_{k=1}^{n_d} U_k V_k$$

where $U_k \in \mathbb{R}^{m\times d_k}$ and $V_k \in \mathbb{R}^{d_k\times n}$. In principle, the algorithm selects a block of d rows and columns via cross approximations in the residual matrix and then $d_k \leq d$ ones via rank-revealing algorithms to form a low-rank update at iteration k. The total number of iterations is approximately $n_d \approx r/d$ if $d_k \approx d$. Instead of selecting row/column pivots via lines 4 and 7 of Algorithm 1, the proposed algorithm selects row and column index sets I_k and J_k by performing QRCP on d columns (more precisely their transpose) and rows of the residual matrices. This proposed strategy is described in Algorithm 2.

Each BACA iteration is composed of three steps.

- Find block row I_k and block column J_{k+1} by QRCP. Starting with a random column index set J_1, the block row I_k and the next iteration's block column J_{k+1} are selected by (lines 4 and 7)

$$[Q_k^c, T_k^c, I_k] = QR(E_{k-1}(:, J_k)^t, d) \quad (7)$$

$$[Q_{k+1}^r, T_{k+1}^r, J_{k+1}] = QR(E_{k-1}(I_k, :), d) \quad (8)$$

deficient matrices is unsatisfactory. For many rank-deficient matrices arising from the numerical solution of PDEs, signal processing and data science, ACA oftentimes either requires $O(n)$ iterations or exhibits premature termination. First, as ACA does not search the full residual matrices for the largest element, it cannot avoid selection of smaller pivots for general rank-deficient matrices and may require $O(n)$ iterations. Second, the approximation $||u_k v_k^t||_F$ in (5) often causes the premature termination with the selection of smaller pivots. Remedies such as averaged stopping criteria (Zhou et al., 2017), stochastic error estimation (Heldring et al., 2015), ACA (Grasedyck and Hackbusch, 2005), and hybrid ACA (Grasedyck and Hackbusch, 2005) have

Here the algorithm first selects d skeleton rows from the submatrix $E_{k-1}(:, J_k)$ (i.e. d columns from its transpose) and then selects d skeleton columns from the submatrix $E_{k-1}(I_k, :)$ by leveraging the LAPACK implementation of QRCP as it provides a simple way of greedily selecting well- conditioned columns by examining column norms in the R factor at each iteration. Note that many other subset selection algorithms exist in both the machine learning and numerical linear algebra communities (e.g. strong rank-revealing QR (Gu and Eisenstat, 1996), spectrum-revealing QR (Feng et al., 2019), and column subset selection problems (Boutsi- dis et al., 2009)), which ideally pick d matrix columns with maximum volumes. Note that I_k excludes rows selected in previous iterations. To efficiently enforce such condition, the QRCP is performed on the submatrix of $E_{k-1}(:, J_k)^t$ excluding previously selected rows rather than directly on $E_{k-1}(:, J_k)^t$. Similarly, J_k excludes columns selected in pre- vious iterations. See Figure 1(a) for an illustration of the procedure. I_k and J_{k+1} are selected by QRCP on the column

Algorithm 2. Blocked adaptive cross approximation (BACA) algorithm.

input : Matrix $A \in \mathbb{R}^{m\times n}$, block size d, relative tolerance
output: Low-rank approximation of $A \approx UV$ with rank r

1 $U = 0$, $V = 0$, $r_0 = 0$, $\mu = 0$, $\bar{J}_1$ is a random index set of cardinality d;
2 **for** $k = 1$ **to** $\min\{m, n\}$ **do**
3 $C_k = E_{k-1}(:, J_k) = A(:, J_k) - UV(:, J_k)$;
4 $[Q_k^c, T_k^c, I_k] = QR(C_k^t, d)$, I_k denotes selected skeleton rows;
5 $R_k = E_{k-1}(I_k, :) = A(I_k, :) - U(I_k, :)V$;
6 $[Q_k^r, T_k^r, \bar{J}_{k+1}] = QR(R_k, d)$, $\bar{J}_{k+1}$ denotes selected skeleton columns;
7 $W_k = E_{k-1}(I_k, J_k) = A(I_k, J_k) - U(I_k, :)V(:, J_k)$;
8 $[U_k, V_k, d_k, \bar{J}] = LRID(C_k, W_k, R_k,)$;
9 $I_k \leftarrow I_k(\bar{J}[1 : d_k])$, $J_k \leftarrow J_k(\bar{J})$;
10 $r_k = r_{k-1} + d_k$;
11 $\nu = LRnorm(U_k, V_k)$;
12 $\mu \leftarrow LRnormUp(U, V, \mu, U_k, V_k, \nu)$;
13 $U \leftarrow [U, U_k]$, $V \leftarrow [V; V_k]$;
14 Terminate if $\nu < \mu$.

15 **Function** $LRID\ (C, W, R,)$
input : $C = A(:, J)$, $R = A(I, :)$, $W = A(I, J)$ with I, J of same cardinality
output: $A \approx UV$ with $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{r\times n}$
16 $[Q, T, \bar{J}, r] = QR(W,)$;
17 $U = C(:, \bar{J})$;
18 $V = T^{-1}Q^tR$;
19 **return** $U, V, r, \bar{J}$

20 **Function** $LRnorm\ (U, V)$
input : $A = UV$
output: $\|A\|_F$
21 $T_1 = Chol(U^tU)$;
22 $T_2 = Chol(VV^t)$;
23 **return** $\|T_1T_2^t\|_F$;

24 **Function** $LRnormUp\ (U, V, \nu, \bar{U}, \bar{V}, \bar{\nu})$
input : $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{r\times n}$, $\bar{U} \in \mathbb{R}^{m\times \bar{r}}$, $\bar{V} \in \mathbb{R}^{\bar{r}\times n}$, $\nu = \|UV\|_F$, $\bar{\nu} = \|\bar{U}\bar{V}\|_F$

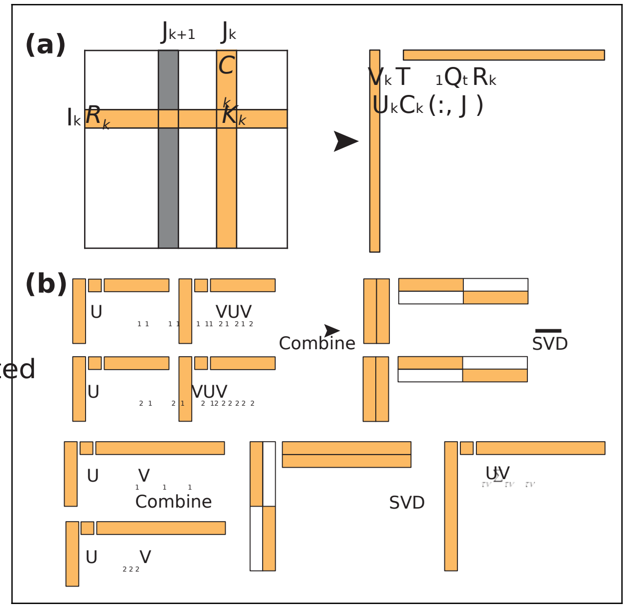

Figure 1. (a) Selection of I_k/J_k and form the low-rank update U_kV_k. (b) Low-rank merge operation.

$E_{k-1} \approx C_kW_k^{\dagger}R_k = U_kV_k$ (Voronin and Martinsson, 2017) by (9) and (10). Note that the pseudo inverse is computed via rank-revealing QR (also see the LRID algo- rithm at line 8). The rank-revealing algorithm is needed as the $d \times d$ block W_k can be further compressed with rank d_k. Particularly, for matrices where the ACA algorithm tends to fail, the corresponding $d \times d$ matrices W_k in BACA are often rank-deficient. In this case, BACA becomes more robust than ACA as the effective d_k pivots can still be used to generate d columns J_{k+1} for the next iteration (as long as $d_k > 0$). Consequently, the effective rank increase is $d_k \leq d$ and the pivot pair I_k; J_k is updated in (11) by the column pivots $\bar{J}$ of QRCP in (9).

$$[Q, T, \bar{J}] = QR(W_k, \epsilon) \text{ with } Q \in \mathbb{R}^{d\times d_k} \tag{9}$$

$$U_k = C_k(:, \bar{J}), \quad V_k = T^{-1}Q^tR_k \tag{10}$$

$$I_k = I_k(\bar{J}[1 : d_k]), \quad J_k = J_k(\bar{J}) \tag{11}$$

output: $\overline{\mathbf{1}}[\ \ ][\ \ ;\ ]1$ F F

$s = \nu^2 + \nu^{-2} + 2\nu$ $V(i, j)$ with

r

$i=1$ $j=1$

$\tilde{V} = (V V^t) \circ (U^t \overline{U});$

26 **return** $\sqrt{s}$

- Compute n ¼ jj$U_k V_k$jj F and update ¼ jjUV F jj

Assuming constant block size d, the norm of the low-rank update can be computed in $O(nd^2)$ operations (line 11) via

T_U ¼ Chol$(U^t_k U_k)$; T_V ¼ Chol$(V^t_k V_k)$ (12)

n ¼ jj$T_U T^t_V$ jj (13)

and transpose of the row marked in yellow, respectively. The column marked in grey is used to select I_{k+1} in the next iteration. For illustration purpose, index sets in Figure 1(a)

$_k \; _F V k$

Once n is computed, the norm of UV can be updated efficiently in $O(nr_k d_k)$ operations (line 12) as

$$\sum^{r_k -} {}_1 \sum^{d_k}$$

consist of contiguous indices.

- Form the factors of the low-rank product

$U_k V_k$. Let $C_k =$

E $_{k-1}$ can be

E_{k-1}ð:; J_kÞ, R_k ¼ E_{k-1}ðI_k; :Þ, and W_k ¼ E_{k-1}ðI_k; J_kÞ,

2^2 ^{2} þ n^2 þ

approximated by an ID-type decomposition

i¼1 j¼1

*V ði; jÞ

$$\tilde{V} = (VV^t)o(U^tU_k) \qquad (14)$$

where r_k represents the column dimension of U_k at iteration k. Note that the matrix multiplications in (12) and (14) involving V and V (and similarly for those involving U) can be performed as $T_U T^t_k$, V; V_k V_k^t to further improve the computational efficiency. Then the algorithm updates U, V as U; U_k, V; V_k and tests the stopping criterion n < e. Note that n; with larger d provides better approximations to the exact stop criterion compared to those in (5) hence can significantly reduce the chance of premature termination.

We would like to highlight the difference between the proposed BACA algorithm and existing ACA algorithms. First, as BACA selects a block of rows and columns per iteration as opposed to a single row and column in the base- line ACA algorithm, the convergence behavior and flop per- formance can be significantly improved. In the existing ACA algorithms, convergence can also be improved by leveraging averaged stopping criteria (Zhou et al., 2017) or searching a single pivot in a broader range of rows and columns (e.g. fully pivoted ACA). However, they still find one row or column at a time in each iteration and hence suffer from poor flop performance. Moreover, they cannot utilize strong rank-revealing algorithms to select skeleton rows and columns with better volume (determinant in mod- ulus) qualities. Second, BACA also has important connec- tions to the hybrid ACA algorithm (Grasedyck and Hackbusch, 2005). The hybrid ACA algorithm assumes prior knowledge about the skeleton rows and columns to leverage interpolation algorithms (e.g. ID and CUR) on a skeleton submatrix and use ACA to refine the skeletons. In contrast, BACA uses cross approximations with QRCP to select skeleton rows and columns and uses interpolation algorithms (LRID at line 8) to form the low-rank update in each iteration. In other words, hybrid ACA can be treated as embedding ACA into interpolation algorithms while BACA can be thought of as embedding interpolation algorithms into ACA iterations. In addition, BACA is purely algebraic and requires no prior knowledge of the row/column skeletons or geometrical information about the rows/columns.

It is worth mentioning that the choice of d affects the trade-off between efficiency and robustness of the BACA algorithm. When $d < r$, the algorithm requires $O(nr^2)$ operations assuming convergence in $O(r/d)$ iterations as each iteration requires $O(nr_kd)$ operations. For example, BACA (Algorithm 2) precisely reduces to ACA (Algo- rithm 1) when $d = 1$. In what follows we refer to the base- line ACA algorithm as BACA with $d = 1$. On the other hand, BACA converges in a constant number of iterations when $d \approx r$. In the extreme case, BACA reduces to QRCP- based ID when $d = \min(m; n)$ (note that the LRID algo- rithm at line 8 remains the only nontrivial operation). In this case, the algorithm requires $O(n^2r)$ operations but enjoys the provable convergence of QRCP. Detailed com- plexity analysis of the BACA algorithm will be provided in Section 4.

The BACA algorithm oftentimes exhibits overestimated ranks compared to those revealed by truncated SVD. Therefore, an SVD re-compression step of U and V may be needed via first computing a QR of U and V as $[Q_U; T_U] = QR(U)$, $[Q_V; T_V] = QR(V)$, and then a trun- cated SVD of $T_U T_V^t$ (Heldring et al., 2015). The result can

be viewed as an approximate truncated SVD of A and we
assume this is the output of the BACA algorithm in the rest of this article.

3.3. Parallel hierarchical low-rank merge

The distributed-memory implementations of the proposed BACA algorithm and the baseline ACA algorithm can pose performance challenges as straightforward parallelization of all operations in Algorithm 2 and 1 involves many col- lective communications. To see this, assuming the U and V factors in Algorithm 1 follow 1-D block row and column data layouts, then every operation from line 3 to line 9 requires one or more collective communications. Instead, one can assign one process to perform BACA/ACA on submatrices without any communication and then leverage parallel low-rank arithmetic to merge the results into one single low-rank product. To elucidate the proposed algo- rithm, we first describe the hierarchical low-rank merge algorithm, then outline its parallel implementation.

Given a matrix A 2 $R^{m \times n}$ with m � n, the algorithm first creates L-level binary trees for index vectors ½1; m] and ½1; n] with index set I_{t} and J_{n} for nodes t and n at each level, upon recursively dividing each index set into I_{t_i}/J_{n_j} of approximately equal sizes, i 1; 2, j 1; 2. Here, t_i and n_j are children of t and n, respectively. The leaf and root levels are denoted 0 and L, respectively. This process generates n_b leaf-level submatrices of similar sizes. For simplicity, it is assumed n_b 4^L. We denote submatrices associated with t; n as A_{tn} A I_{t}; J_{n} and their truncated SVD as U_{tn}; S_{tn}; V_{tn}; r_{tn} *SVD* A_{tn}; e . Here r_{tn} is the e-rank of A_{tn}. As submatrices A_{tn} have significantly smaller dimensions than A (e.g. when n_b O n^2 as an extreme case), both BACA and ACA algorithms become more robust to attain the truncated SVD. Following com- pression of n_b submatrices A_{tn} by BACA or ACA at step l 0, there are multiple approaches to combine them into one low-rank product including randomized algorithms via applying A to random matrices, and deterministic algo- rithms via recursively pair-wise re-compressing the blocks using low-rank arithmetic. Here we choose the determinis- tic algorithm for simplicity of rank estimation and parallelization. Here, we deploy truncated SVD as the re-compression tool but other tools such as ID, QR, UTV can also be applied. Figure 1(b) illustrates one re-compression operation for transforming SVDs of $A_{\mathrm{t}_i\mathrm{n}_j}$; i ¼ 1; 2; j ¼ 1; 2 into that of A_{tn}. The operation first horizontally compresses SVDs of $A_{\mathrm{t}_i\mathrm{n}_j}$; i ¼ 1; 2; j ¼ 1; 2 at step l − and then vertically compresses the results, that is, SVDs of $A_{\mathrm{t}_i\mathrm{n}}$; i 1; 2 at step l, l 1; ::; L. Specifically, the horizontal compression step is composed of one

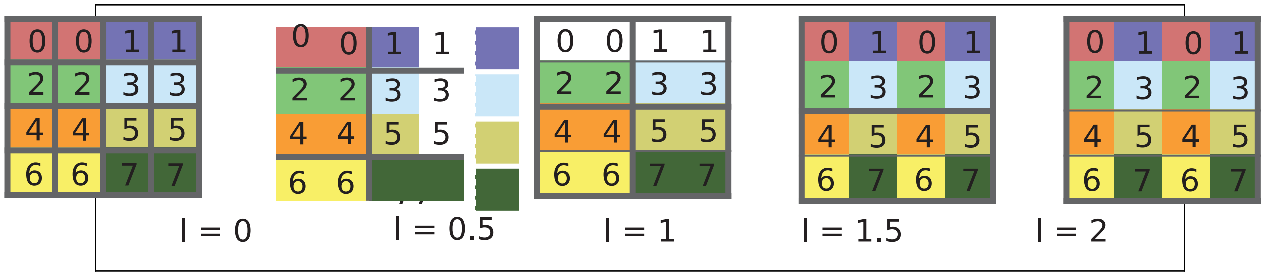

Figure 2. Parallel hierarchical merge with eight processes. Blocks surrounded by solid lines represent $A_{\tau\nu}$ after compression at each step l. Blocks surrounded by dashed lines represent ScaLAPACK blocks.

concatenation operation in (15) and one compression oper- ation in (16):

$$\bar U_{\tau_i\nu} = [U_{\tau_i\nu_1}\Sigma_{\tau_i\nu_1}; U_{\tau_i\nu_2}\Sigma_{\tau_i\nu_2}]; \quad \bar V_{\tau_i\nu} = diag(V_{\tau_i\nu_1}; V_{\tau_i\nu_2}) \quad (15)$$

$$[U_{\tau_i\nu}, \Sigma_{\tau_i\nu}, V_{\tau_i\nu}, r_{\tau_i\nu}] \leftarrow SVD(\bar U_{\tau_i\nu}, \epsilon); \quad V_{\tau_i\nu} \leftarrow \bar V_{\tau_i\nu} V_{\tau_i\nu} \quad (16)$$

with $i = 1, 2$. Let $\bar U_{\tau_i\nu} \bar V_{\tau_i\nu}$ and $U_{\tau_i\nu}\Sigma_{\tau_i\nu}V_{\tau_i\nu}$ denote the submatrix before and after the SVD truncation, respectively. Similarly, the vertical compression step can be performed via horizontal merge of $A^t_{\tau_i\nu}$; $i = 1, 2$. Let s_l represent the maximum rank $r_{\tau\nu}$ among all blocks at steps $l = 0, 1, \ldots, L$. Note that the algorithm returns an approx- imate truncated SVD after L steps. As an example, the hierarchical merge algorithm with the level count of the hierarchical merge $L = 2$ and $n_b = 16$ is illustrated in Figure 2. At step $l = 0$, the algorithm compresses all n_b submatrices with BACA; at step $l = 0.5, 1.5$, the algorithm merges every horizontal pair of blocks; similarly at level $l = 1, 2$, the algorithm merges every vertical pair of blocks. Note that blocks surrounded by solid lines represent results after compression at each step l.

The above-described hierarchical algorithm with BACA for leaf-level compressions is dubbed H-BACA (Algorithm 3). In the following, a distributed-memory imple- mentation of the H-BACA algorithm is described. Without loss of generality, it is assumed that $m = n2^i$ and $p = 2^j$. The proposed parallel implementation first creates two $\log p$-level binary trees with p denoting the total number of MPI processes. One process performs BACA compression of one or two leaf-level submatrices and low-rank merge operations from the bottom up until it reaches a submatrix

Algorithm 3. Hierarchical low-rank merge algorithm with BACA (H-BACA).

input : Matrix $A \in R^{m\times n}$, number of leaf-level subblocks n_b, block size d of leaf-level BACA, relative tolerance ϵ

output: Truncated SVD of $A \approx U\Sigma V$ with rank r

Create L-level trees on index vectors $[1, m]$ and $[1, n]$ with index set I_τ and J_ν for nodes τ and ν at each level, $L = \sqrt{\log n_b}$, the leaf and root levels are denoted 0 and L, respectively;

for $l = 0$ **to** L **do**

  foreach $A_{\tau\nu} = A(I_\tau, J_\nu)$ at level l **do**

    if *leaf-level* **then**

      $[U_{\tau\nu}, \Sigma_{\tau\nu}, V_{\tau\nu}, r_{\tau\nu}] = BACA(A_{\tau\nu}, d, \epsilon)$;

    else

      Let τ_1, τ_2 and ν_1, ν_2 denote children of τ and ν;

      for $i = 1$ **to** 2 **do**

        $\bar U_{\tau_i\nu} = [U_{\tau_i\nu_1}\Sigma_{\tau_i\nu_1}, U_{\tau_i\nu_2}\Sigma_{\tau_i\nu_2}]$;

        $\bar V_{\tau_i\nu} = diag(V_{\tau_i\nu_1}, V_{\tau_i\nu_2})$;

        $[U_{\tau_i\nu}, \Sigma_{\tau_i\nu}, V_{\tau_i\nu}, r_{\tau_i\nu}] \leftarrow SVD(\bar U_{\tau_i\nu}, \epsilon)$;

        $V_{\tau_i\nu} \leftarrow \bar V_{\tau_i\nu} V_{\tau_i\nu}$;

      $\bar U_{\tau\nu} = diag(U_{\tau_1\nu}, U_{\tau_2\nu})$;

      $\bar V_{\tau\nu} = [\Sigma_{\tau_1\nu}V_{\tau_1\nu}; \Sigma_{\tau_2\nu}V_{\tau_2\nu}]$;

$[U_{\tau\nu}, \Sigma_{\tau\nu}, V_{\tau\nu}, r_{\tau\nu}] \leftarrow SVD(V_{\tau\nu},);$ $U_{\tau\nu} \leftarrow \bar{U}_{\tau\nu} U_{\tau\nu};$

return $U = U_{\tau\nu}, V = V_{\tau\nu}, \Sigma = \Sigma_{\tau\nu}, r = r_{\tau\nu};$

shared by more than one process. Then, all such blocks are handled by PBLAS and ScaLAPACK with BLACS process grids that aggregate those in corresponding submatrices. Con- sider the example in Figure 2 with process count p ¼ 8. The workload of each process is labeled with its process rank and highlighted with one color. The dashed lines represent the ScaLAPACK blocks. First, BACA compressions and merge operations at l ¼ 0; 0:5 are handled locally by one process without any communication. Next, merge operations at l ¼ 1; 1:5; 2 are handled by BLACS grids of 2 × 1, 2 × 2, and 4 × 2, respectively. For illustration purposes, we select the ScaLAPACK block size in Figure 2 as n_0 × n_0 where n_0 is the dimension of the finest-level submatrices in the hierarchical merge algorithm and n ¼ $\sqrt{n_b}\, n_0$. In this case, the only required data redistribution is from step l ¼ 1 to l ¼ 1:5. However, the ScaLAPACK block size may be set to much smaller numbers in practice, requiring data redistribution at each row/column re-compression step. Similarly, the require- ment of m ¼ n¼ 2^i and p¼ 2^j is not needed in practice.

4. Cost analysis

In this section, the costs for computation and communica- tion of the proposed BACA and H-BACA algorithms are analyzed.

4.1. Computational cost

First, the costs for BACA can be summarized as follows. Assuming BACA converges in $O(r/d)$ iterations, each iteration performs entry evaluation from the residual matrices, QRCP for pivot selection, LRID for forming the and estimation of matrix norms. The entry evaluation computes $O(nd)$ entries each requiring $O(r_k)$ operations; QRCP on block rows requires $O(nd^2)$ operations; the LRID algorithm requires $O(ndd_k + d_k d^2)$ operations; norm estimation requires $O(nr_k d_k)$ operations. Summing up these costs, the overall cost for the BACA algorithm is

$$c_{BACA} = \sum_{k=1}^{O(r/d)} O(nd^2 + nr_k d + d_k d^2) \leq O(nd^2 + rd^2 + nrd)\,O(r/d) = O(nr^2) \tag{17}$$

Here we assume the block size $d \ll r$. Note that when $d = r$ (e.g. $d = O(n)$), it follows that the worst-case complexity is $c_{BACA} = O(n^2 r)$ by bypassing the pivot selection step that causes the nd^2 term. In practice, one would always avoid the case of $d = r$.

Next, the computational costs of the H-BACA algorithm are analyzed. The costs are analyzed for two cases of distributions of the maximum ranks s_l at each level, that is, $s_l = r$ (ranks stay constant during the merge) and $s_l = 2^{l-L} r = \sqrt{n_b}^{-1} 2^l r$ (rank increases by a factor of 2 per level), $l = 0, 1, \ldots, L$. The constant rank case is often valid for matrices with their numerical ranks independent of matrix dimensions (e.g. random low-rank matrices, matrices representing well-separated interactions from low-frequency and static wave equations and certain quantum chemistry matrices); the increasing-rank case holds true for matrices whose ranks depend polynomially (with order no bigger than 1) on the matrix dimensions (e.g. those arising from high-frequency wave equations, matrices representing equations, and certain classes of kernel methods on high dimensional data sets). Note that these rank distributions often follow from a proper hierarchical partitioning tree and may not be valid using an arbitrary partitioning tree. From the aforementioned analysis of BACA, the computational costs for the leaf-level compression $c_b = c_{BACA}\frac{n_b}{n}$ are:

Table 1. Flop counts and communication costs for the leaf-level compression and hierarchical merge operations in Algorithm 3 for two classes of low-rank matrices.[a]

	Constant rank $s_l \asymp r$	Increasing rank $s_l \asymp r/\sqrt{n_b} \times 2^l$
BACA $d \asymp s_0$	$O(nr^2/\sqrt{n})$	$O(nr^2/(\sqrt{n}\,n_b))$
Merge compute	$O(nr^2\sqrt{n_b})$	$O(nr^2)$
Merge communicate	$[O(r\log^2 p), O(nr\log^2 p/\sqrt{p})]$	$[O(r\log p), O(nr\log p/\sqrt{p})]$

[a] n and r denote matrix dimension and rank. d denotes the block size in BACA. p and n_b denote number of processes and leaf-level submatrices. s_l denotes maximum ranks among all level-l submatrices.

summarized in Table 1. Note that the costs of the BACA algorithm can also be extracted from Table 1 upon setting $n_b = 1$. Not surprisingly, the hierarchical merge algorithm induces a computational overhead of at most $\sqrt{n_b}$ when ranks stay constant; the leaf-level compression can have a $1/\sqrt{n}$ reduction factor for the increasing rank case and $\sqrt{n_b}$ overhead for the constant rank case.

For completeness, the comparison between the proposed BACA, H-BACA algorithms (assuming $d \ll r_0$) and existing ACA algorithms are given in Table 2. In contrast to existing ACA algorithms that select one pivot at a time, BACA and H-BACA select d and $n_b d$ pivots simultaneously. As such, H-BACA is the most robust algorithm among all listed here. Not surprisingly, H-BACA can induce a computational overhead of $\sqrt{n_b}$.

4.2. Communication cost

As the leaf-level BACA compression requires no communication, only the communication costs for the hierarchical merge operations are analyzed here. Since the merge operations may introduce an $O(\sqrt{n_b})$ computational overhead, to create more parallelization of the process count $p \leq n_b$. Consider the one level l merge operation, that is, transformation of the SVDs of $A_{\tau_i \nu_j}$; $i = 1, 2$; $j = 1, 2$ into that of $A_{\tau\nu}$ shown in Figure 1(b). Let $p_l = 4^l$ denote the number of processes sharing one level l block $A_{\tau\nu}$, $l = 0, \ldots, L$. The horizontal compression step in (15, 16) requires redistribution from the process grids of size p

$$c_b = O(\sqrt{n_0}\, s^2 n_b); \quad \text{if } d \le s_0 \tag{18}$$

which represent the complexity with ACA when $n_b = 1$.

Let $n_l = 2^{l-1} n_b = n_{\tau;n}$ denote the size of submatrices $A_{\tau;n}$ at level l. The computational costs c_m of hierarchical merge operations can be estimated as

$$c_m = \sum_{l=1}^{L} O(4^{L-l} n_l s_l^2) \tag{19}$$

Accounting for the two cases of rank distributions, the computational costs for the leaf-level BACA and hierarch- ical merge operations of the H-BACA algorithm are

sharing $A_{\tau_i n_j}; i = 1, 2; j = 1, 2$ to the process grids of size $2p_{l-1}$ sharing $A_{\tau_i n}; i = 1, 2$.

After redistribution, each process grid involves PDGESVD function in ScaLAPACK (see (16)) to compute the new rank after the combination in (15), and a PDGEMM function in PBLAS to multiply factors $V_{\tau_i n}$ with $\bar{V}_{\tau_i n}$ (see 16). Similarly, the vertical compression step requires redistribution from the process grids of size $2p_{l-1}$ sharing $A_{\tau_i n}; i = 1, 2$ to the process grids of size p_l sharing $A_{\tau n}$, and calling PDGESVD and PDGEMM func- tions in the new grids. Let the pair [#messages, volume] denote the communication cost including the number of messages and the number of words transferred along the critical path. Then the communication costs for each

Table 2. Comparisons between proposed BACA, H-BACA algorithms and existing ACA algorithms.[a]

Algorithm	ACA/ACA+	Hybrid-ACA	BACA	H-BACA
Pivot count per iteration	1	1	d	$n_b d$
Cost (constant rank)	$O(nr^2)$	$O(nr^2)$	$O(nr^2)$	$O(\sqrt{n_b}\, nr^2)$
Cost (increasing rank)	$O(nr^3)$	$O(nr^2)$	$O(nr^3)$	$O(nr^2)$
Preselection of submatrices	No	No	No	Yes

[a] Note that the algorithms show increasing robustness from left to right.

(BLACS) grid redistribution, PDGEMM and PDGESVD during the hierarchical merge are $[O(1), O(n_l s_l/\sqrt{p_l})]$, $[O(s_l \log p_l), O(n_l s_l/\sqrt{p_l})]$, and $[O(s_l \log p_l), O(n_l s_l \log p_l/\sqrt{p_l})]$, respectively (Blackford et al., 1997). Recall that $n_l = 2^l n/\sqrt{p}$ and s_l denote the size and rank of submatrices at level l and note that $n_l \geq s_l$. Therefore the communication cost v_m of the hierarchical merge (and H-BACA) can be estimated as

$$v_m = \left[\sum_{l=1}^{L} O(s_l \log p_l),\ O\left(\sum_{l=1}^{L} \frac{n_l s_l \log p_l}{\sqrt{p_l}}\right)\right] = \left[O\left(\sum_{l=1}^{L} l s_l\right),\ O\left(\sum_{l=1}^{L} \frac{l n_l s_l}{\sqrt{p}}\right)\right] \quad (20)$$

Consider the two cases of rank distributions, that is, $s_l = r$ and $s_l = 2^{l-L} r$, the overall communication costs of H-BACA are $v_m = [O(r \log^2 p), O(nr \log^2 p/\sqrt{p})]$ and $v_m = [O(r \log p), O(nr \log p/\sqrt{p})]$, respectively (see Table

5. Numerical results

This section presents several numerical results to demon- strate the accuracy and efficiency of the proposed H-BACA algorithm. The matrices in all numerical examples are gen- erated from the following kernels:

1. Gaussian kernel: $A_{i,j} = \exp\left(\frac{-\|x_i - x_j\|^2}{2h^2}\right)$, $i, j = 1, \ldots, 2n$. Here h is the Gaussian width, and $x_i \in \mathbb{R}^{8\times1}$ and $\mathbb{R}^{784\times1}$ are feature vectors in one subset of the SUSY and MNIST Data Sets from the UCI Machine Learn- ing Repository (Dheeru and Karra Taniskidou, 2017), respectively. Note that the Gaussian kernel permits low-rank compression as shown in (Bach, 2013; Musco and Musco, 2017; Wang et al., 2018).

4. Frontal3D kernel: A is a dense frontal matrix that arises from the multifrontal sparse elimination for the finite-difference frequency-domain solution of the homogeneous-coefficient Helmholtz equation inside a unit cube.
5. Polynomial kernel: $A_{i,j} = (x_i^t x_j + h)^2$. Here $x_i, x_j \in \mathbb{R}^{50\times1}$ are points from a randomly generated data set, and h is a regularization parameter.
6. Product-of-random kernel: $A = UV$ with $U \in \mathbb{R}^{n\times r}$ and $V \in \mathbb{R}^{r\times n}$ being random matrices with i.i.d. entries.

Note that the EFIE2D, EFIE3D, and Frontal3D kernels result in complex-valued matrices. Throughout this section, we refer to ACA as a special case of BACA when $d = 1$. In all examples except for the product-of-random kernel, the algorithm is applied to the off-diagonal submatrix $A_{12} = A(1:n, 1+n:2n)$ assuming rows/columns of A have been properly permuted (e.g. by a KD-tree partitioning scheme). Note that the permutation may yield a hierarchical matrix representation of A, but in this article we only use the permutation to define the partition trees for H-BACA com- pression of one off-diagonal subblock of A with H-BACA. All experiments are performed on the Cori Haswell machine at NERSC, which is a Cray XC40 system and consists of 2388 dual-socket nodes with Intel Xeon E5-2698v3 proces- sors running 16 cores per socket. The nodes are configured with 128 GB of DDR4 memory at 2133 MHz.

5.1. Convergence

2. EFIE2D kernel: $A_{ij} = H_0^{(2)}(k|x_i - x_j|)$ resulting from the Nyström discretization of the electric field integral equation (EFIE) for electromagnetic scattering from 2-D curves. Here $H_0^{(2)}$ is the second kind Hankel function of order 0, k is the free-space wave- number, $x_i, x_j \in \mathbb{R}^{2\times1}$ are discretization points (15 points per wavelength) of two 2-D parallel strips of length 1 and distance 1.
3. EFIE3D kernel: A is obtained by the Galerkin method for EFIE to analyze electromagnetic scatter- ing from 3-D surfaces.

First, the convergence of the proposed BACA algorithm is investigated using several matrices: Gaussian-SUSY matrices with $n = 5000$, $h = 1.0, 0.2$, an EFIE3D matrix for a unit sphere with $n = 21788$ and approximately 20 points per wavelength, and a Frontal3D matrix with $n = 1250$ and 10 points per wavelength. The corresponding ε-ranks are $r = 4683, 1723, 1488, 718$ for $\epsilon = 10^{-6}$. The residual histories versus revealed ranks r_k, at each iteration k of BACA with $1 \le d \le 256$ are plotted in Figure 3. The residual error is defined as $\|UV - U_kV_k\|_F / \|UV\|_F$ from (13). As a reference, the singular value spectra $\Sigma(k, k)$, $k = 1, \ldots$ computed from $[U, \Sigma, V, r] = SVD(A, \epsilon)$ are also plotted.

For the Gaussian-SUSY matrices, the baseline ACA algorithm ($d = 1$) behaves poorly with smaller h due to the exponential decay of the Gaussian kernel. As a result, the matrix becomes increasingly sparse and coherent for small h particularly for high dimensional data sets. In fact, ACA

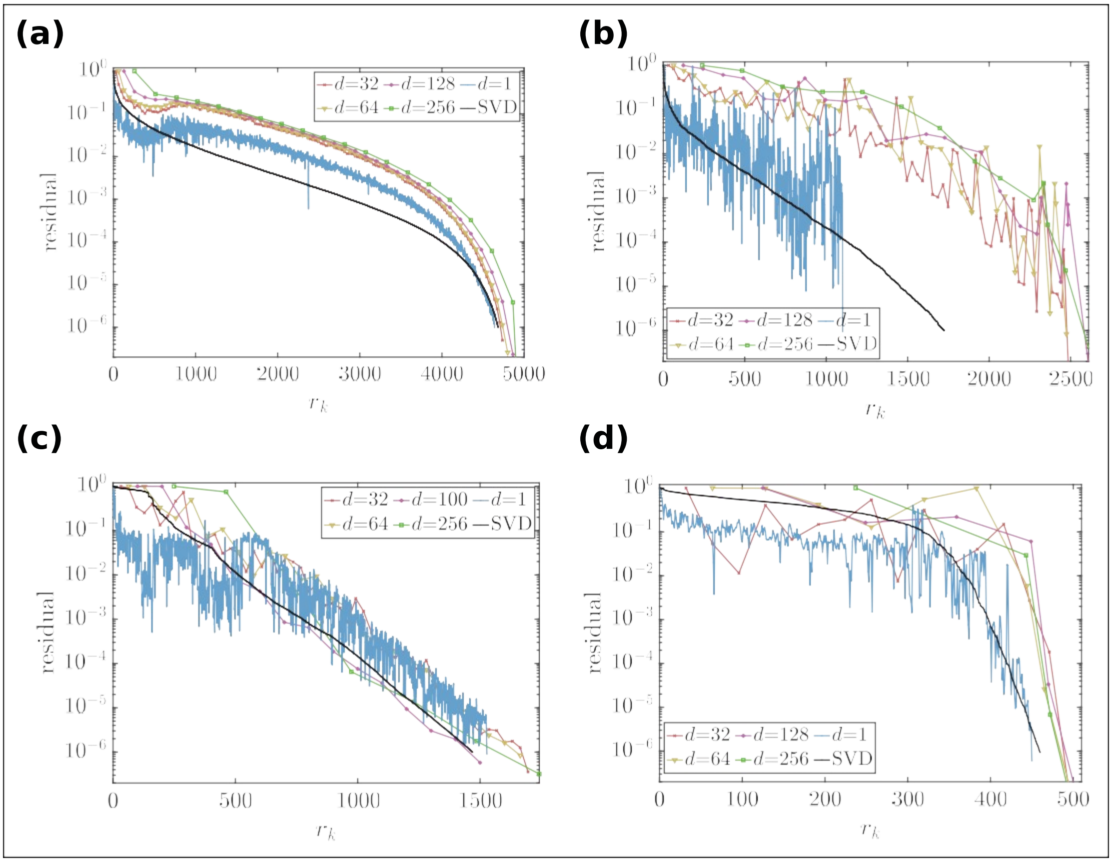

Figure 3. Convergence history of BACA for the (a) Gaussian-SUSY kernel with *h* ¼ 1:0, *n* ¼ 5000, e¼ 10^{-6}, *r* ¼ 4683, (b) Gaussian- SUSY kernel with *h* ¼ 0:2, *n* ¼ 5000, e¼ 10^{-6}, *r* ¼ 1723, (c) EFIE3D kernel for a unit sphere with *n* ¼ 21; 788, e¼ 10^{-6}, *r* ¼ 1488, and (d) Frontal3D kernel with *n* ¼ 1250, e¼ 10^{-6}, *r* ¼ 718.

constantly selects smaller pivots and the residual exhibits wild oscillations particularly for smaller *h* (e.g. when *h* ¼ 0:2 in Figure 3(b)). Similarly, the analytical and numerical Green's functions respectively for the EFIE3D (Figure 3(c)) and Frontal3D (Figure 3(d)) matrices are not asymptotically smooth for ACA to converge rapidly. For all examples in Figure 3, significant portions of the residual curves lie below the singular value spectra which causes premature iteration termination for certain given residual errors. In stark contrast, the proposed BACA algorithm (*d* ¼ 32; 64; 100; 128; 256) shows increasingly smooth residual histories residing above the singular value spectra as the block size *d* increases. Although BACA may over- estimate the matrix ranks particularly for larger *d*, the SVD re-compression step mentioned in Section 3.2 can effec- tively reduce the ranks.

5.2. Accuracy

Next, the accuracy of the H-BACA algorithm is demon- strated using the following matrices: two Gaussian-SUSY matrices with *n* ¼ 5000, *h* ¼ 1:0; 0:2, one EFIE3D matrix for a unit sphere with *n* 1707 and approximately 20 points per wavelength, and a Frontal3D matrix with *n* ¼ 1250 and 10 points per wavelength. The relative

Frobenius-norm error jj$A$ − UV jj$_F$ =jj$A$jj$_F$ is computed for changing number of leaf-level submatrices n_b and block size d. When h 1:0 for the Gaussian-SUSY matrix (Figure 4(a)), the H-BACA algorithms achieve desired accuracies (e ¼ 10^{-2}; 10^{-6}; 10^{-10}) using the baseline ACA (d 1), and BACA (d 32) when n_b 1 and the hier- archical merge operation only causes slight error increases as n_b increases. However when h 0:2 for the Gaussian- SUSY matrix (Figure 4(b)), all data points for H-BACA with d 1 fail due to the wildly oscillating residual his- tories. In contrast, H-BACA with d 32 achieves signifi- cantly better accuracies for most data points particularly as n_b increases. For the EFIE3D (Figure 4(c)) and Frontal3D (Figure 4(d)) matrices, H-BACA with d 32 achieves comparable accuracies as H-BACA with d 1 for most data points. Note that d 32 is significantly better than d 1 when the prescribed residual error is large (e 10^{-2}). This agrees with the residual histories in Figure 3(c) and (d) as they lie below the singular value spectra when iteration count k is small.

5.3. Efficiency

This subsection provides six examples to verify the computational complexity estimates in Table 1. H-BACA

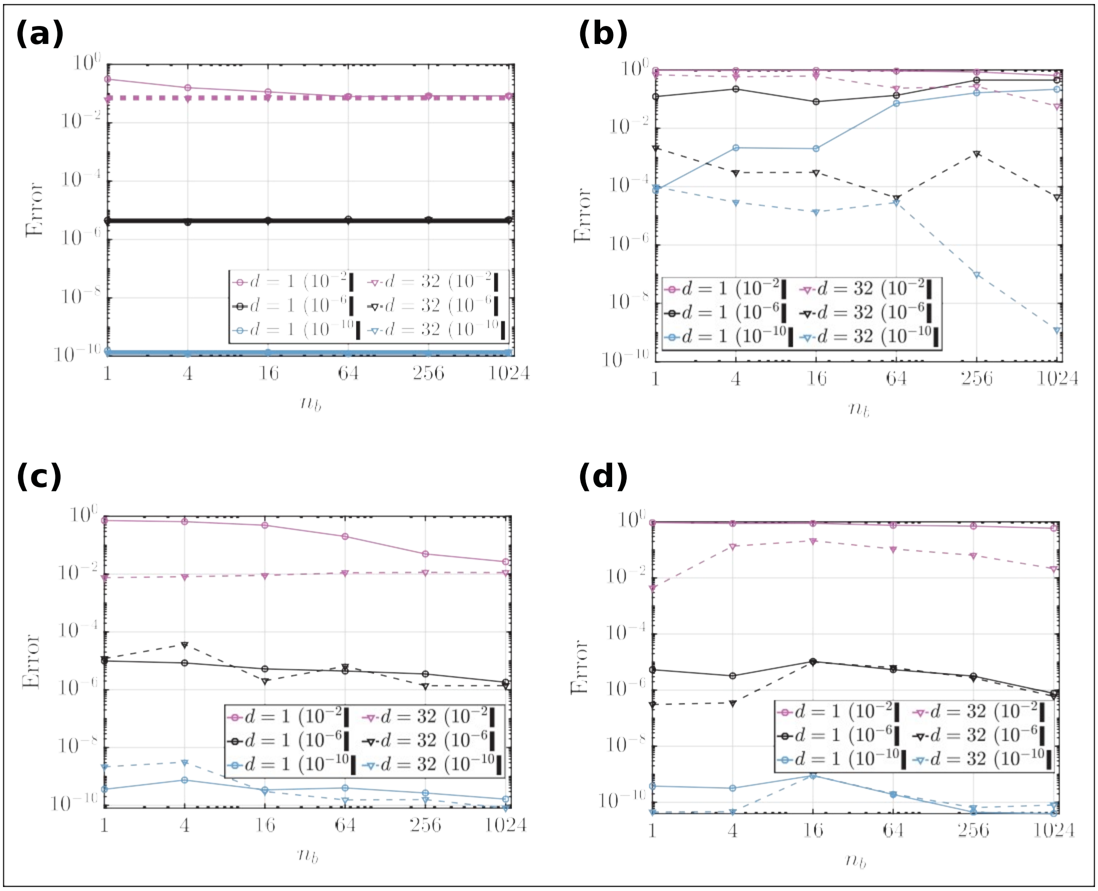

Figure 4. Measured error of H-BACA with $\epsilon = 10^{-2}, 10^{-6}, 10^{-10}$ for the (a) Gaussian-SUSY kernel with $h = 1.0$, $n = 5000$, (b) Gaussian-SUSY kernel with $h = 0.2$, $n = 5000$, (c) EFIE3D kernel for a unit sphere with $n = 1707$, and (d) Frontal3D kernel with $n = 1250$.

with leaf-level ACA ($d = 1$) and BACA ($d = 8, 16, 32, 64, 128$) is tested for the following matrices: one Gaussian-SUSY matrix with $n = 50,000$, $h = 1.0$, $\epsilon = 10^{-2}$, one Gaussian-MNIST matrix with $n = 5000$, $h = 3.0$, $\epsilon = 10^{-2}$, one EFIE3D matrix for a unit sphere with $n = 26268$, $\epsilon = 10^{-6}$ and 20 points per wavelength, one Frontal3D matrix with $n = 1250$, $\epsilon = 10^{-6}$ and 10 points per wavelength, one polynomial matrix with $n = 10,000$, $h = 0.2$, $\epsilon = 10^{-4}$, and one product-of-random matrix with $n = 2500$, $\epsilon = 10^{-4}$. The corresponding ϵ-ranks are 298, 137, 1488, 788, 450, and 1000, respectively. It can be vali- dated that the hierarchical merge operation attains increas- ing ranks for the Gaussian, EFIE3D and Frontal3D matrices, and relatively constant ranks for the polynomial, and product-of-random matrices. All examples use one pro- cess except that the Gaussian-SUSY example uses 16 pro- cesses. The CPU times are measured and plotted in Figure 5.

Table 1 predicts that H-BACA exhibits increasing (with a factor of $\sqrt{n_b}$) and constant time when s_l stays constant and increases, respectively. Note that the rank assumption $s_l \lesssim r$ leading to the $O(\sqrt{n_b})$ computational overhead may not be fully observed for practical values of n_b and n. Given one matrix, s_l may stay approximately constant for a limited number of subdivision levels l. For example, s_l stay constant for bottom levels of EFIE3D and Frontal3D matrices, and top levels of Polynomial and product-of-random matrices. This agrees with the observed scalings (w.r.t n_b) in Figure 5(c) to (f). As a reference, the $O(\sqrt{n_b})$ curves are plotted and only small ranges of n_b exhibit the $O(\sqrt{n_b})$ overhead. For the Gaussian matrices, we even observe nonincreasing CPU time w.r.t. n_b when n_b is not too big (see Figure 5(a) and (b)).

The effects of varying block size d also deserve further discussions. First, larger block size d can significantly improve the robustness of H-BACA for the Gaussian matrices. For example, H-BACA does not achieve desired accuracies due to premature termination for all data points on the $d = 1$ curve in Figure 5(a) and $d = 1$ and $d = 8$ curves in Figure 5(b). In contrast, H-BACA with larger d attains desired accuracies. Second, larger block size d results in reduced CPU time for the Polynomial and Frontal3D matrices due to better BLAS performance (see Figure 5(d) and (e)). For the other tested matrices, no significant performance differences have been observed by changing block size d. However, for matrices with ranks $s_0 \lesssim d$, larger d and n_b can introduce significant overheads.

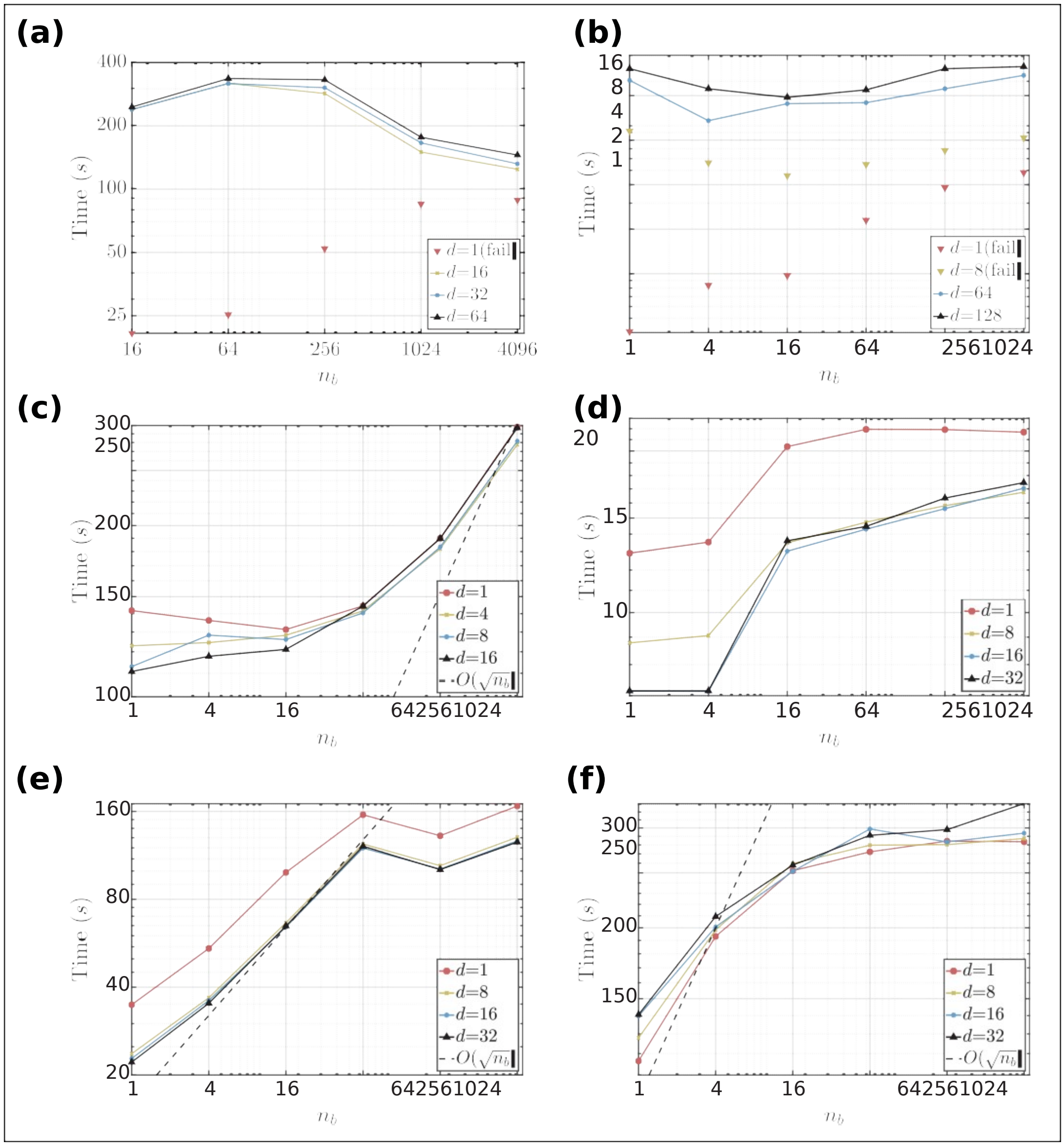

Figure 5. Computation time of H-BACA with varying n_b and d for the (a) Gaussian-SUSY kernel with h ¼ 1:0, n ¼ 50; 000, e¼ 10^{-2}, r ¼ 298, (b) Gaussian-MNIST kernel with h ¼ 3:0, n ¼ 5000, e¼ 10^{-2}, r ¼ 137, (c) EFIE3D kernel for a unit sphere with n ¼ 26; 268, e ¼ 10^{-6}, r ¼ 1488, (d) Frontal3D kernel with n ¼ 1250, e ¼ 10^{-6}, r ¼ 788, (e) polynomial kernel with h ¼ 0:2, n ¼ 10; 000, e ¼ 10^{-4}, r ¼ 450, and (f) product-of-random kernel with n ¼ 2500, r ¼ 1000. Note that the data points where the algorithm fails are shown as triangular markers without lines.

5.4. Parallel performance

Finally, the parallel performance of the H-BACA algorithm is demonstrated via strong scaling studies with the EFIE2D, EFIE3D, product-of-random and Gaussian matrices with process counts p ¼ 8; ... ; 1024. For the EFIE2D matrices, n ¼ 160; 000 and the wavenumbers are chosen such that the e-ranks with e ¼ 10^{-4} are 937 and 107, respectively. For the EFIE3D matrices for a unit square, n ¼ 21; 788 and the wavenumbers are chosen such that the e-ranks with e ¼ 10^{-6} are 1007 and 598, respectively. For the product- of-random matrices, n ¼ 10; 000 and the inner dimension of the product is set to r ¼ 2000 and 800, respectively. For the Gaussian matrices with a randomly generated data set of

dimension 50 and *n* ¼ 10; 000, we choose *h* ¼ 1:0 and *h* 1:6 such that the e-ranks with e 10^{-3} are 2106 and 191, respectively. In all examples, the block size and num- ber of leaf-level subblocks in H-BACA are chosen as *d* ¼ 8 n_b *p*
64 64. As the reference, we compare to a straightforward parallel implementation of the baseline ACA algorithm which essentially parallelize every operation in ACA with collective MPI communications.

For all examples, the parallel ACA algorithm stops scal- ing when *p* is sufficiently large (see Figure 6). In contrast, the proposed parallel H-BACA algorithm scales up to *p* ¼ 1024. In most examples, H-BACA achieves better

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p p ¼and . The ScaLAPACK block size x is set to

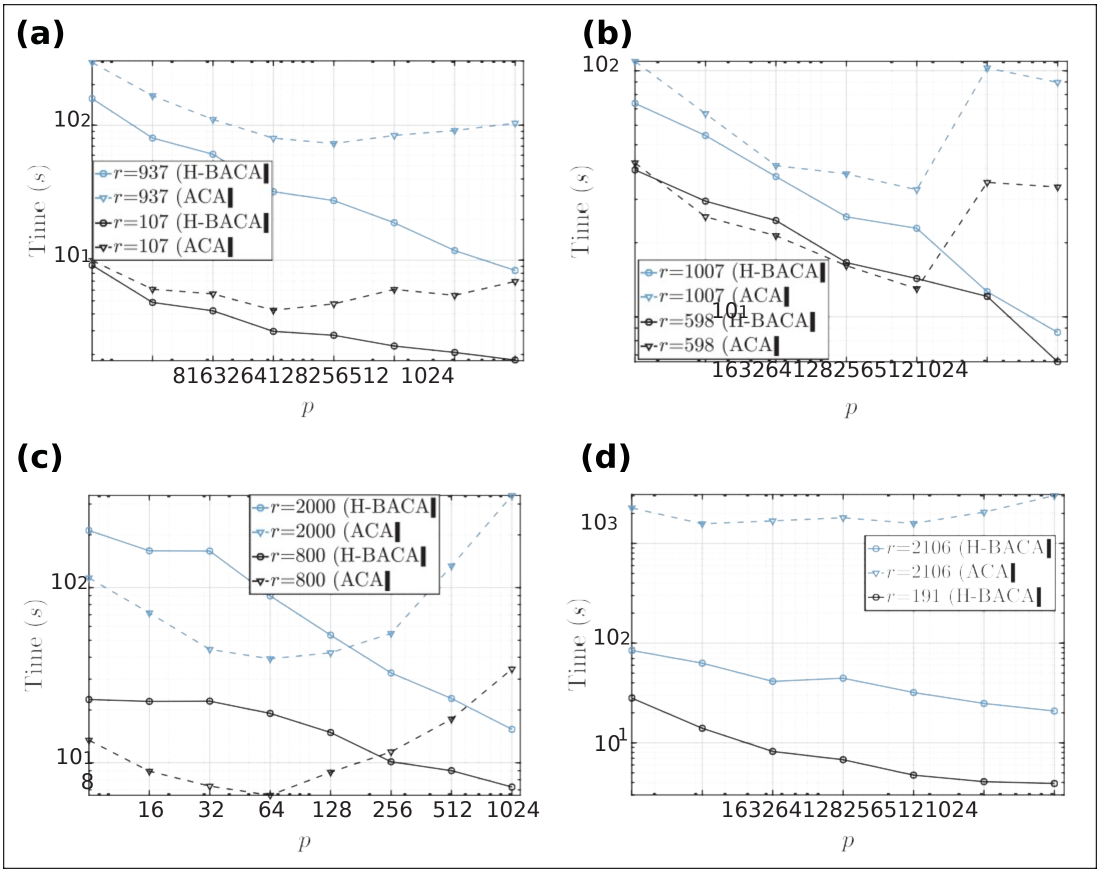

Figure 6. Computation time of H-BACA with varying process counts for the (a) EFIE2D kernel with n ¼ 160; 000, e¼ 10^{-4}, r ¼ 107 and 937, (b) EFIE3D kernel for a unit square with n ¼ 21; 788, e¼ 10^{-6}, r ¼ 598 and 1007, (c) product-of-random kernel with n ¼ 10; 000, r ¼ 800 and 2000, and (d) Gaussian kernel for a randomly generated data set with h ¼ 1:0; 1:6, e¼ 10^{-3}, r ¼ 2106 and 191. Note that for the Gaussian matrix with r ¼ 191, ACA fails to provide accurate results and is not plotted.

parallel efficiencies with larger ranks due to better process utilization during the hierarchical merge operation. We also note that ACA outperforms H-BACA for the Product-of-random matrices with small process count p (and n_b). This is partially attributed to the $O(\sqrt{n_b})$ overhead observed in Figure 5(f).

Overall, the parallel H-BACA algorithm can achieve reasonably good parallel performances for rank-deficient matrices with modest to large numerical ranks. Not surprisingly, the parallel runtime is dominated by that of ScaLA- PACK computation and possible redistributions between each re-compression step as analyzed in Section 4. Also note that the leaf-level BACA compression is embarrassingly parallel for all test cases.

6. Conclusion

This article presents a parallel and algebraic ACA-type matrix decomposition algorithm given that any matrix entry can be evaluated in $O(1)$ time. Two proposed strate- gies, BACA and H-BACA, are leveraged to improve the robustness and parallel efficiency of the (baseline) ACA algorithm for general rank-deficient matrices.

First, the BACA algorithm searches for blocks of row/ column pivots via QRCP on the column/row submatrices at each iteration. The blocking nature of BACA provides a closer estimation of the true residual error and reduces the chance of selecting smaller pivots when compared to ACA. Therefore, BACA exhibits a much smoother and more reli- able convergence history. Moreover, blocked operations also benefit from higher flop performance compared to non- blocked ones. For a rank-deficient matrix with dimension n and e-rank r, the computational cost of BACA is $O(nr^2)$ assuming the block size constant and iteration count $O(r)$.

Second, the H-BACA algorithm divides the matrix into n_b similar-sized submatrices each compressed with BACA and then hierarchically merges the results using low-rank arithmetic. Depending on the rank behaviors of subma-trices during the merge, the H-BACA may have a compu- tational overhead of $O(\sqrt{n_b})$ yielding the overall computational cost at most $O(nr^2\sqrt{n_b})$. The H-BACA algorithm can be parallelized with distributed-memory machines by assigning each process to one submatrix and leveraging PBLAS and ScaLAPACK for the hierarchical merge operation. Such parallelization strategy yields a much more favorable communication cost when compared to the straightforward parallelization of ACA/BACA with collective MPI routines. Not surprisingly, good parallel performance can be achieved for matrices with modest to large numerical ranks which increases process utilization for each merge operation.

In contrast to the baseline ACA algorithm, the proposed algorithms exhibit improved robustness and favorable par- allel performance with low computational overheads for broad ranges of matrices arising from many science and engineering applications.

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