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Bardakci, K.

Publication Date
1978-11-01

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LBL-8559 c.2
Preprint

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November 1978

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UCB-PTH-78/4 November 1978

FIELD THEORY FOR SOLITONS. II.

K. Bardakci

University of California
Department of Physics
and
Lawrence Berkeley Laboratory
Berkeley, CA 94720

Abstract

A method previously developed for constructing field theories of solitons is extended to non-Abelian vortex models in (2+1) dimensions and to both Abelian and non-Abelian models in (3+1) dimensions. In (2+1) dimensions, a local field theory is obtained, and in (3+1) dimensions, string theories with local interaction emerge. Various features of these models are investigated.

I. Introduction

In a previous paper,[1] referred to as (1), a method for constructing local field theories for solitons was developed and applied to the Abelian Higgs model in (2+1) dimensions and to the Georgi-Glashow model in (3+1) dimensions. The present paper is a continuation and extension of (1) in two different directions:

a) In (2+1) dimensions, a local soliton Lagrangian is constructed for a class of non-Abelian gauge theories, known to have classical soliton solutions.[2] These models have a sufficient number of Higgs scalars belonging to the adjoint representation, so that after spontaneous symmetry breaking, all vector mesons acquire finite masses. There has been some interest recently in the soliton solutions of these models, especially in connection with their transformation properties under the center Z(N) of the gauge group SU(N) and their role in charge confinement.[3] Sections 2 and 3 are devoted to the construction of the soliton Lagrangian and to the discussion of some of its features.

b) In section 4, both the Abelian Higgs model of (1) and the non-Abelian Higgs model of section 2 are extended to (3+1) dimensions, and the solitons of one lower dimension are shown to turn into closed strings. Although this result has been known for some time,[2] even in the Abelian case the string interaction is not the standard one, and in the non-Abelian case, the string possesses additional internal quantum numbers as well. We also point out the well known problems[4] of tachyonic mass and unphysical dimension associated with string theories, and offer some speculative remedies.

Recently, a series of papers[5-7] have appeared, dealing with the soliton content of various gauge theories, and we would like to compare our approach to the same problem with theirs. These papers all treat field theories on a lattice, and they use standard duality transformations[8] of statistical mechanics to transform the original action into a "dual" soliton action. Topological considerations do not seem to play a direct role, since it is difficult to do topology on a lattice. In contrast, from the beginning, we work in the continuum limit and single out fields on non-trivial topological configuration. Also, our treatment is exact at each stage, and there is no need for any approximation such as the Villain trick.[8] Our method is also able to handle models based on non-Abelian gauge groups, such as the Georgi-Glashow model discussed in reference (1), whereas in the lattice approach, only abelian models have been considered so far. However, the Abelian analogue of the Georgi-Glashow model, compact Q.E.D. on a lattice, has a structure very similar to the non-Abelian model.[9] On the other hand, to our best knowledge, the lattice version of the non-Abelian Higgs model treated in this paper has not yet been investigated.

II. Solitons in Non-Abelian Higgs Model in (2+1) Dimensions: Topology of Higgs Fields

For the ease of exposition, a model based on the gauge group SU(2) will be treated first, and the generalization to an arbitrary group will be given at the end of the section. The model is built out of an SU(2) gauge boson A^α_μ, and two Higgs scalar isotriplets ϕ^α and ψ^α, where $\alpha = 1,2,3$ and μ runs from 0 to 2. The Higgs potential is so adjusted that the vacuum expectation values of ϕ^α, ψ^α and their vector product form a non-degenerate coordinate system in the SU(2) space, and all the vector mesons acquire non-vanishing masses. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4} (G^\alpha_{\mu\nu})^2 + \frac{1}{2} (D_\mu\phi^\alpha)^2 + \frac{1}{2} (D_\mu\psi^\alpha)^2 - V(\phi,\psi), \qquad (2.1)$$

where,

$$V(\phi,\psi) = \frac{1}{4} \lambda_1^2 (\phi^2-h_1^2) + \frac{1}{4} \lambda_2^2 (\psi^2-h_2^2) + \frac{1}{2} \lambda_3^2 (\phi^\alpha\psi^\alpha-h_3)^2,$$

and

$$G^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + e\, \varepsilon^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu,$$

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha + e\, \varepsilon^{\alpha\beta\gamma} A^\beta_\mu \phi^\gamma.$$

The indices μ,ν,λ refer to space, and α,β,γ to isospin [SU(2)]. It is also convenient to express the field variables in matrix notation:

$$\phi = \frac{1}{2}\phi^{\alpha}\tau^{\alpha}, \quad A_{\mu} = \frac{1}{2}\tau^{\alpha}A^{\alpha}_{\mu},$$

$$G_{\mu\nu} = \frac{1}{2}\tau^{\alpha}\, G^{\alpha}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ie\,(A_{\mu},A_{\nu}),$$

$$D_{\mu}\phi = \partial_{\mu}\phi - ie\,(A_{\mu},\phi), \tag{2.2}$$

where τ^{α} are the Pauli matrices. In this notation, the gauge transformations that leave (2.1) invariant are given by

$$\phi \to S\phi S^{-1}, \quad \psi \to S\psi S^{-1}$$

$$A_{\mu} \to SA_{\mu}S^{-1} - \frac{i}{e}(\partial_{\mu}S)\,S^{-1}, \tag{2.3}$$

where S is a unitary two by two matrix. The standard generating functional is

$$Z(J) = \int DA \int D\phi \int D\psi \quad (\delta\Delta)$$

$$\times \; \exp\{i\int d^3x\,[\mathcal{L}(x) + J(x)\,C(x)]\}\,, \tag{2.4}$$

where J is the source coupled to some field or combination of fields denoted by C, and $(\delta\Delta)$ is a suitable gauge fixing term with its proper measure.

We are interested in the contribution to the functional integral from unusual topological configurations of the fields ϕ^{α} and ψ^{α} called kinks. The definition of a kink is as follows: The isovectors ϕ^{α} and ψ^{α}, along with their vector product, in general define a unique coordinate system in the SU(2) space. For the time being, we assume that ϕ and ψ are never parallel. At each point on a given path in space-time, one can define a unique group element of SU(2) by the parallel transport of the coordinate system established by ϕ and ψ. If the coordinate system is transported around a closed path, arriving at the starting point, the corresponding group element must be a rotation by $2\pi n$ around some axis, where n is an integer. All even n are topologically equivalent to the case n=0 and all odd n to n=1.[10] The case n=0 is trivial and n=1 corresponds to a kink. From the foregoing discussion, it is clear that kinks carry a conserved multiplicative quantum number (-1), and that kinks and antikinks are equivalent. When this analysis is extended to SU(N), it turns out that there are as many distinct kinks as the number of non-trivial elements of Z(N), the center of the group, and that kinks carry a corresponding multiplicative quantum number.[3]

If a closed path of non-trivial topology is shrunk to a point, eventually it must cross singular point(s), where a unique coordinate system cannot be defined. These points are defined to be the locations of pointlike (bare) solitons. The possibility of defining such pointlike objects is what makes a local field thoery of solitons possible.

A kink can be "straightened out" by means of a singular gauge transformation, which maps it into a trivial configuration of the fields ϕ and ψ. A simple example of such a transformation is the following:

$$S_s = \exp\left(\frac{i}{2}\,\theta\, n^\alpha \tau^\alpha\right), \tag{2.5}$$

where S_s is defined by eq. (2.3), θ is the polar angle defined by $\tan(\theta) = \frac{x_2}{x_1}$, and n is a vector of unit magnitude in SU(2) space whose direction is a smooth function of space coordinates. It is clear that the singular gauge transformation defined by eq. (2.5) maps an ordinary configuration of fields ϕ and ψ into a static kink located at the origin, and vice versa. Strictly speaking, singular gauge transformations are not gauge transformations; they carry flux and they do not leave the action invariant. Defining

$$B_\mu = \frac{1}{2}\tau^\alpha B^\alpha_\mu = -\frac{i}{e}\,(\partial_\mu S_s)\, S_s^{-1}, \tag{2.6a}$$

and,

$$F_{\mu\nu} = \frac{1}{2}\tau^\alpha F^\alpha_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - ie\,(B_\mu, B_\nu),$$

one finds, using Stokes' theorem for a closed path around the origin, the following result:

$$F^\alpha_{12} = \frac{2\pi}{e}\, n^\alpha\, \delta^2(x), \tag{2.6b}$$

$$F^\alpha_{01} = F^\alpha_{02} = 0.$$

In this special case, the kink has the time independent trajectory $x_{1,2} = 0$ for all x_0; however, it is easy to generalize eq. (2.6b) to an arbitrary number of trajectories of general form. Defining the dual vector to $F_{\mu\nu}$ by

$$\tilde{F}^{\alpha}_{\mu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda} F^{\nu\lambda,\alpha}, \tag{2.7a}$$

eq. (2.6b) can be generalized to

$$\tilde{F}^{\alpha}_{\mu}(x) = \frac{2\pi}{e} \sum_{\ell} \int d\tau_{\ell}\, n^{\alpha}_{\ell}(\tau_{\ell})\, \dot{\bar{x}}_{\mu,\ell}(\tau_{\ell})\, \delta^{3}[x-\bar{x}_{\ell}(\tau_{\ell})]. \tag{2.7b}$$

In this equation, $x^{\mu} = \bar{x}^{\mu}_{\ell}(\tau_{\ell})$ defines the trajectory of the ℓ'th kink, the τ's are internal variables parametrizing the trajectories, $\dot{\bar{x}}^{\mu}$ is the tangent to the trajectory, and n^{α}_{ℓ} is a unit vector that specifies the direction of the flux in the SU(2) space. This result is similar to eq. (2.12) of (1), except for the appearance of the isovector n^{α}_{ℓ} and the necessity of using the non-abelian definition of the field strength $F_{\mu\nu}$ in eq. (2.6a). If the vector n of eq. (2.5) is constant over space, the problem becomes Abelian, and the commutator term in the definition of $F_{\mu\nu}$ can be dropped. If n depends on position, however, the non-Abelian definition of $F_{\mu\nu}$ is needed.

Another important difference between the Abelian and non-Abelian theories is the manner in which Bianchi identities are satisfied. These identities follow from the definition (2.6a):

$$\partial_{\mu} \tilde{F}^{\mu,\alpha} + e\, \varepsilon^{\alpha\beta\gamma} B^{\mu,\beta} \tilde{F}^{\gamma}_{\mu} = 0. \tag{2.8a}$$

The Bianchi identities, together with the expression for $\tilde{F}^{\alpha}_{\mu}$ given by eq. (2.7b), yield the following equations of motion for n^{α}:

$$\dot{n}^{\alpha}(\tau) + e\, \varepsilon^{\alpha\beta\gamma} B^{\beta}_{\mu}[\bar{x}(\tau)]\, \dot{\bar{x}}^{\mu}(\tau)\, n^{\gamma}(\tau) = 0, \tag{2.8b}$$

where the dot implies differentiation with respect to τ. In the Abelian case, $n = \pm 1$ and the second term in eq. (2.8b) is absent, and the equation truly becomes an identity. In the non-Abelian case, however, (2.8b) is a non-trivial equation of motion governing the rate of precession of the isospin vector n around the external field B.

III. Lagrangian for Non-Abelian Solitons

The functional integration over the fields ϕ and ψ in eq. (2.4) includes arbitrary number of kinks with all possible trajectories. These kinks can be eliminated by means of a singular transformation similar to the one given by (2.5), and the remaining integration over ϕ and ψ is then restricted to kink free configurations. The singular transformation induces new terms in the Lagrangian, which can be computed through eq. (2.7b). Letting

$$\phi \to S_s^{-1}\,\phi S_s, \quad \psi \to S_s^{-1}\,\psi S_s,$$

$$A_\mu \to S_s^{-1} A_\mu S_s + \frac{i}{e}\, S_s^{-1}(\partial_\mu S_s) = S_s^{-1}(A_\mu - B_\mu)\; S_s, \tag{3.1a}$$

in the Lagrangian of eq. (2.1), the following transformed Lagrangian is obtained:

$$\mathcal{L}' \;=\; -\frac{1}{4}\,(G^\alpha_{\mu\nu} - F^\alpha_{\mu\nu})^2 + \frac{1}{2}\,(D_\mu\phi^\alpha)^2$$

$$+\frac{1}{2}\,(D_\mu\psi^\alpha)^2 - V(\phi,\psi), \tag{3.1b}$$

where $F^\alpha_{\mu\nu}$, defined by eq. (2.6a), is expressible as a sum over trajectories of kinks through eq. (2.7b). This new Lagrangian has to be supplemented by the constraints expressed by eq.'s (2.7b) and (2.8b), and the most convenient way of imposing these constraints is through Lagrange multipliers. The functional integral of (2.4) can then be rewritten as an integral over the kink trajectories and over fields free of kinks. In the naive form of the integral over the trajectories, however, there is a divergence due to the reparametrization invariance of eq. (2.7b) under the transformations

$$\tau_\ell \rightarrow f_\ell(\tau_\ell). \tag{3.2}$$

This is very similar to the infinity in the functional integral resulting from gauge invariance of the action in a gauge theory, and it can be cured by adding a parameter fixing term, similar to a gauge fixing term, to the Lagrangian. In the appendix of reference (1), the following was shown to be a suitable parameter fixing term:

$$\int d^3x \; (\Delta \mathcal{L}) = \sum_\ell \int d\tau_\ell \; [-\frac{1}{2} \dot{x}_\ell^2 \; (\tau_\ell)]. \tag{3.3}$$

An identical result is obtained in the continuum limit of lattice field theories.[6]

Multiplying the constraints (2.7b), (2.8b) and the constraint that n^α has unit length by Lagrange multipliers H_μ^α, s_ℓ^α and λ_ℓ respectively, and adding the sum to $\mathcal{L}' + \Delta\mathcal{L}$ yields the following action:

$$\int d^3x \ \mathcal{L}(x;\bar{x}_1, \cdots, \bar{x}_N) = \int d^3x \ \{-\frac{1}{4}(G^{\alpha}_{\mu\nu} - F^{\alpha}_{\mu\nu})^2$$

$$+ \frac{1}{2}(D_\mu \phi^\alpha)^2 + \frac{1}{2}(D_\mu \psi^\alpha)^2 - V(\phi,\psi)$$

$$+ H^{\alpha}_{\mu}(x)\ \tilde{F}^{\mu,\alpha}(x)\} + \sum_{\ell=1}^{N} \int d\tau_\ell \ \{-\frac{1}{2}(\dot{\bar{x}}_\ell)^2$$

$$+ \lambda_\ell[(n^\alpha_\ell)^2 - 1] + s^\alpha_\ell[\dot{n}^\alpha_\ell + e\ \epsilon^{\alpha\beta\gamma}\ B^\beta_\mu(\bar{x}_\ell)\ n^\alpha_\ell\ \dot{\bar{x}}^\mu_\ell]$$

$$- \frac{2\pi}{e}\ H^\alpha_\mu(\bar{x}_\ell)\ n^\alpha_\ell\ \dot{\bar{x}}^\mu_\ell\}. \qquad (3.4a)$$

In terms of this new action, the functional integral (2.4) can be written as follows:

$$Z(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int DA \int DH \int \bar{D}\phi \int \bar{D}\psi \int DB$$

$$\prod_{\ell=1}^{N} \int\int\int\int D\bar{x}_\ell Dn_\ell D\lambda_\ell Ds_\ell \ \delta(\Delta)$$

$$\exp\{ i \int d^3x[J(x)C(x) + \mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N)]\}, \qquad (3.4b)$$

where the bars over the ϕ and ψ integrations restrict the function space to fields free of kinks. This condition can be implemented either by lining up fields ϕ and ψ to form a fixed coordinate system by a suitable choice of gauge, or alternatively, expanding fields perturbatively around a constant configuration.

Our next task is to convert the summation over particle trajectories in eq. (3.4b) into a functional integral over a field. This problem is similar to the one encountered in reference (1); however, the non-Abelian internal symmetry group introduces additional complications. The first step is to suspend the integrations over A,B,H,ϕ and ψ, as well as the summation over N. Eq. (3.4a) can then be considered to be the action of N relativistic particles in <u>fixed external</u> fields B and H. The crucial idea is to pass from the action formulation of particle dynamics to the Hamiltonian formulation in terms of quantized canonical variables.(11),(12) Since there is no direct interaction between the particles, it is sufficient to consider the single particle action given by

$$I = \int d\tau \left\{ -\frac{1}{2} \dot{\bar{x}}^2(\tau) + \lambda(\tau) \left([n^\alpha(\tau)]^2 - 1\right) \right.$$

$$\left. - \frac{2\pi}{e} H^\alpha_\mu(\bar{x})\, n^\alpha\, \dot{\bar{x}}^\mu + s^\alpha(\tau)\, [\dot{n}^\alpha + e\, \epsilon^{\alpha\beta\gamma}\, B^\beta_\mu(\bar{x})\, \dot{\bar{x}}^\mu\, n^\gamma] \right\}. \qquad (3.5a)$$

The momenta conjugate to $\bar{x}^\mu$ and n^α are

$$p^\mu = \frac{\delta I}{\delta(\dot{\bar{x}}_\mu)} = -\dot{\bar{x}}^\mu - \frac{2\pi}{e} n^\alpha H^{\mu,\alpha}(\bar{x}) + e\, \epsilon^{\alpha\beta\gamma}\, s^\alpha B^{\mu,\beta}(\bar{x})\, n^\gamma,$$

$$s^\alpha = \frac{\delta I}{\delta(\dot{n}^\alpha)}. \qquad (3.5b)$$

These canonical variables are not all independent.
n can be used to eliminate n^3 in favor of the other components:

$$n^3 = [1-(n^1)^2 - (n^2)^2]^{1/2}.$$

We also note that the action is invariant under the transformation

$$s^\alpha \to s^\alpha + n^\alpha f(\tau),$$

$$\lambda \to \lambda + \frac{1}{2} f'(\tau), \qquad (3.6)$$

where f is an arbitrary function. By a suitable choice of f, one can set

$$s^3 = 0$$

and remain with the independent canonical variables $\dot{\bar{x}}^\mu$, p^μ, n^1, n^2, s^1 and s^2. The system is quantized by imposing the commutation relations(13)

$$[n^\alpha(\tau), s^\beta(\tau)] = i\, \delta^{\alpha\beta}, \; (\alpha,\beta = 1,2),$$

and

$$[\bar{x}^\mu(\tau), p^\nu(\tau)] = i\, g^{\mu\nu}, \qquad (3.7a)$$

in addition to specifying the Hamiltonian to be

$$H = p_\mu \dot{\bar{x}}^\mu + s^\alpha \dot{n}^\alpha - \text{Lagrangian}$$

$$= -\frac{1}{2}\,[p_\mu + \frac{2\pi}{e}\; H^\alpha_\mu(\bar{x})\, n^\alpha + e\, \varepsilon^{\alpha\beta\gamma}\, s^\alpha n^\beta\, B^\gamma_\mu(\bar{x})]^2. \qquad (3.7b)$$

It is easily verified that the Hamiltonian equations of motion derived from (3.7a) and (3.7b) are identical to the Lagrangian equations that follow from (3.5a), establishing consistency. It is also helpful to recognize that the operator

$$T^{\alpha} = - \epsilon^{\alpha\beta\gamma} s^{\beta} n^{\alpha} \tag{3.8a}$$

is the SU(2) angular momentum operator, with the commutation relations

$$(T^{\alpha},T^{\beta}) = i\, \epsilon^{\alpha\beta\gamma}\, T^{\gamma},$$

$$(T^{\alpha},n^{\beta}) = i\, \epsilon^{\alpha\beta\gamma}\, n^{\gamma}. \tag{3.8b}$$

The restriction of the variable n^{α} to the surface of the unit sphere is no problem, and in fact becomes irrelevant if T^{α} is expressed in terms of angular variables.

Finally, the passage to field theory is accomplished via the following set of rules:[14] The trajectory dependent part of the action of eq. (3.4a) is replaced by

$$\int d^3x \int d\hat{n} \quad \chi^*(x,n)\ (-2H)\ \chi(x,n), \tag{3.9a}$$

where χ is a complex scalar field that depends on n^{α}, and the integral is over all directions of n^{α} with equal weight (group invariant integration). The operator H is given by eq. (3.7b), with the following indentification of the canonical variables:

$$\bar{x}^\mu \to x^\mu,$$

$$p^\mu \to -\, i \frac{\delta}{\delta x_\mu},$$

$$s^\alpha \to -\, i \frac{\delta}{\delta n^\alpha}, \; T^\alpha = -\, i\, \epsilon^{\alpha\beta\gamma}\, n^\beta \frac{\delta}{\delta n^\gamma}. \tag{3.9b}$$

Putting eq.'s (3.4) and (3.9) together, the following action for solitons (kinks) is obtained:

$$Z(J) = \int DA \int DB \int DH \int \bar{D}\phi \int \bar{D}\psi \;\delta(\Delta)$$

$$\exp\{i \int d^3x\, [J(x)C(x) + \mathcal{L}_\chi(x)]\}, \tag{3.10a}$$

where,

$$\mathcal{L}_\chi(x) = -\frac{1}{4}(G^\alpha_{\mu\nu} - F^\alpha_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi^\alpha)^2 + \frac{1}{2}(D_\mu\psi^\alpha)^2$$

$$- \; V(\phi,\psi) + \frac{1}{2} H^\alpha_\mu F^\alpha_{\nu\lambda}\, \epsilon^{\mu\nu\lambda}$$

$$+ \int d\hat{n} \;|\; \partial_\mu\chi + \frac{2\pi i}{e} H^\alpha_\mu\, n^\alpha\, \chi - ie\, B^\alpha_\mu\, T^\alpha\, \chi|^2,$$

$$T^\alpha\chi = -\, i\, \epsilon^{\alpha\beta\gamma}\, n^\beta \frac{\partial\chi}{\partial n^\gamma}. \tag{3.10b}$$

Eq. (3.10) is the main result of this section. It has the unusual feature that the soliton field χ depends on the unit vector n^α, which can be viewed as a continuous isospin [SU(2)] variable that fixes the direction of the flux of the soliton in the SU(2) space through eq. (2.7b). Alternatively, the field χ can be expanded into irreducible representations of SU(2) (spherical harmonics) as a function of the angles of n, and χ can be replaced by an infinite component field labeled by eigenvalues of $(\vec{T})^2$ and T^3. Since SU(2) symmetry is (spontaneously) broken, the multiplet is not degenerate and presumably forms an infinite tower of increasing mass.

The appearance of an infinite multiplet is perhaps not so surprising, if the Abelian Higgs model treated in (1) is reconsidered. The solitons of that model are labeled by an integer n that specifies the number of flux units. In reference (1), since only the solitons with $n = \pm 1$ were considered, only a single complex field was needed. However, if solitons with all possible values of n are taken into account, it is then necessary to introduce an infinite number of fields, one for each value of n. However, in this case, one has the option of restricting n to the values ± 1, whereas in the non-Abelian model, Bianchi identities require an infinite component soliton field. This will be shown in the next section.

Some simple properties of χ follow from particle-antiparticle symmetry and from invariance under Z(2), the center of SU(2). It is natural to identify a soliton traveling forward in time with an antisoliton traveling backward in time, with the direction of the flux vector, n^α, reversed. This is expressed by the following relation:

$$\chi(x,n^\alpha) = \chi^*(x,-n^\alpha). \qquad (3.11)$$

Invariance under Z(2) means that solitons carry a conserved, multiplicative isoparity quantum number. This requires invariance under the transformation

$$\chi \to -\chi . \tag{3.12}$$

So far, our treatment has been semi-classical and problems of renormalization, operator ordering and possible (infinite) mass counter terms for the soliton field have been ignored. We hope to return to these problems in the near future. There is also a question of the existence of $|\chi|^4$ type self coupling term. No such term is present in the Lagrangian; however, it was pointed out in reference (6) that such a term may be needed in lattice theories to overcome the overcounting of intersecting trajectories. This question seems difficult to settle in a continuum theory.

IV. Discussion and Extension of the Non-Abelian Soliton Lagrangian.

The soliton Lagrangian of eq. (3.10b) possesses several invariances. One such invariance is related to the non-uniqueness of the singular transformation S_s needed to straighten out a kink. Two singular transformations that differ by a regular transformation S are topologically equivalent, which implies invariance under

$$S_s \to SS_s. \tag{4.1}$$

If S_s is transformed according to (4.1) in the redefinition of fields given by (3.1a) and in eq. (2.7b), the following set of transformations are obtained:

$$\phi \to S\phi S^{-1},\ \psi \to S\psi S^{-1},$$

$$H_\mu \to SH_\mu S^{-1},$$

$$A_\mu \to SA_\mu S^{-1} - \frac{i}{e}(\partial_\mu S)\,S^{-1},$$

$$B_\mu \to SB_\mu S^{-1} - \frac{i}{e}(\partial_\mu S)\,S^{-1},$$

$$\chi(x,n) \to \chi(x,n'),$$

where,

$$n' = S\,n\,S^{-1},\ n = \frac{1}{2}\tau^\alpha n^\alpha. \tag{4.2}$$

The invariance of the soliton Lagrangian under these transformations can also be directly established. This new invariance should not be confused with the gauge invariance of the original Lagrangian given by (2.3). The gauge transformations of eq. (2.3) can be reexpressed in terms of the fields defined by eq. (3.1a) as follows:

$$\phi \to U\phi U^{-1}, \; \psi \to U\psi U^{-1},$$

$$A_\mu \to UA_\mu U^{-1} - \frac{i}{e}(\partial_\mu U)\, U^{-1}, \qquad (4.3)$$

where

$U = S_s S S_s^{-1}$, and the fields B_μ, H_μ and χ remain unchanged.

In order to express these transformations in terms of the basic fields that appear in the Lagrangian, it is necessary to solve for S_s in terms of B_μ through eq. (2.6a). Since the resulting expression is a complicated non-linear and non-local function of B_μ, we see no point in writing it out explicitly.

Another symmetry of the soliton Lagrangian is invariance under what we shall call the Bianchi transformations:

$$H^\alpha_\mu \to H^\alpha_\mu + \partial_\mu \Lambda^\alpha + e\,\epsilon^{\alpha\beta\gamma} B^\beta_\mu \Lambda^\gamma,$$

$$\chi \to \chi \exp\{-\frac{2\pi i}{e} n^\alpha \Lambda^\alpha\}, \qquad (4.4)$$

where Λ^α is an arbitrary function of the coordinates.

Invariance under the transformations (4.2) and (4.3) ensures that the vector fields B^α_μ and A^α_μ are coupled to conserved currents, and that the equations of motion are therefore consistent, whereas invariance under (4.2) secures the consistency of the equations of motion with Bianchi identities. The equation of motion obtained by varying the Lagrangian with respect to H^α_μ is

$$\tilde{F}^\alpha_\mu = \frac{2\pi i}{e} n^\alpha [\chi^*(\partial_\mu \chi) - \chi(\partial_\mu \chi^*)$$

$$+ ie B^\beta_\mu \chi^*(T^\beta \chi) - ie B^\beta_\mu \chi(T^\beta \chi^*)]. \qquad (4.5)$$

Since the left hand side of this equation satisfies the Bianchi identity of eq. (2.8a), so must the right hand side, and consistency demands that this identity should follow from the equation of motion for χ:

$$(\partial_\mu + \frac{2\pi i}{e} H^\alpha_\mu n^\alpha - ie B^\alpha_\mu T^\alpha)^2 \chi = 0. \qquad (4.6)$$

Multiplying the equation by $\chi^* n^\alpha$ and subtracting from its complex conjugate, the desired Bianchi identity is easily established.

In arriving at this result, use is made of the commutativity of the components of n^α:

$$(n^\alpha, n^\beta) = 0. \qquad (4.7)$$

This innocent looking relation is the reason behind the infinite component soliton field. In fact, if eq. (4.7) is added to eq.'s (3.8b), a non-compact group, whose only unitary representations are infinite dimensional, is obtained. One attempt at avoiding this is to replace n^α's by finite dimensional (and necessarily non-commuting) Hermitian matrices, and let χ belong to the finite dimensional carrier space. This approach, however, runs into trouble with Bianchi identities, since the non-commuting components of n^α then introduce additional terms into the equations of motion, which violate these identities. At a more fundamental level, the only way we know of avoiding an infinite component soliton field is to replace the commutation relations between n^α and s^α by anticommutation relations. Fermi statistics then forbids the build up of arbitrarily large isospin by repeated applications of n and s. The canonical variables n and s must then be Hermitian conjugates of each other, which means that the term

$$\frac{2\pi}{e} \; H^\alpha_\mu n^\alpha \tag{4.8}$$

in the Hamiltonian (3.7) is non-Hermitian, which is disastrous. Therefore, an infinite component field theory seems inescapable.

By virtue of the gauge invariances expressed by eq.'s (4.2), (4.3) and (4.4), it is possible (and necessary) to impose gauge conditions on the vector fields A^α_μ, B^α_μ and H^α_μ. A simple choice is the Landau gauge

$$\partial_\mu A^{\mu,\alpha} \;=\; \partial_\mu B^{\mu,\alpha} \;=\; \partial_\mu H^{\mu,\alpha} \;=\; 0. \tag{4.9}$$

The transformation (4.4) has a Faddeev-Popov factor one, and (4.2) and (4.3) have the usual factors:

$$\Delta(A) \;=\; \det\{\delta^{\alpha\beta}\, \delta^4(x-y) - e\, \varepsilon^{\alpha\beta\gamma}\, A^\gamma_\mu\, \partial^{(y)}_\mu\, \Delta_F(x-y)\}, \tag{4.10}$$

and a similar expression for $\Delta(B)$.

Once the gauge conditions (4.9) are imposed, the bilinear (free) part of the Lagrangian (3.10b) becomes non-singular, and it is possible to quantize it in the traditional manner. Unfortunately, unlike in the Abelian case, it does not seem possible to eliminate any of the auxiliary fields H_μ^α and B_μ^α by explicit functional integration.

It remains to extend our results to a general group of the form SU(N). Only the case N=3 will be treated in detail, and larger values of N will be left as an exercise for the reader. In the case of SU(3), the indices α, β and γ in eq.s (2.1) and (2.2) range from 1 to 8, and the well-known λ matrices replace the Pauli matrices. Also, the antisymmetric symbol $\epsilon^{\alpha\beta\gamma}$ gets replaced by $f^{\alpha\beta\gamma}$. Instead of a single singular transformation given by (2.5), there are now four distinct singular transformations:

$$S_s^{(\pm 1)} = \exp\left[\pm \frac{i}{\sqrt{3}} \; (\lambda^\alpha n^\alpha) \; \theta \right],$$

$$S_s^{(\pm 2)} = \exp\left[\pm \frac{2i}{\sqrt{3}} \; (\lambda^\alpha n^\alpha) \; \theta \right], \tag{4.11}$$

where $\lambda^\alpha n^\alpha = S \; \lambda_8 \; S^{-1}$, and S is a three by three unitary matrix. The space spanned by the eight-vector n^α can be characterized by

$$\det\,(\kappa - n^\alpha\lambda^\alpha) = (\kappa + \frac{1}{\sqrt{3}})^2 \; (\kappa - \frac{2}{\sqrt{3}}) \; , \tag{4.12}$$

where κ is an arbitrary constant. Each singular transformation corresponds to a distinct soliton, whose flux is given by an expression similar to (2.7b):

$$F^{\alpha}_{\mu}(x) = \frac{4\pi j}{e\sqrt{3}} \sum_{\ell} \int d\tau_{\ell}\; n^{\alpha}_{\ell}(\tau_{\ell})\; \dot{\bar{x}}_{\mu,\ell}\;(\tau_{\ell})\; \delta^{3}[x - \bar{x}_{\ell}(\tau_{\ell})], \qquad (4.13a)$$

where $j = \pm 1$ or ± 2 depending on the type of soliton. Eqs. (3.5a) through (3.9b) are still valid with obvious modifications, such as

$$T^{\alpha} = -\, i\; f^{\alpha\beta\gamma}\; n^{\beta}\; \frac{\delta}{\delta n^{\gamma}} \;. \qquad (4.13b)$$

It is now necessary to introduce four scalar fields $\chi_j(x,n^{\alpha})$, $j = \pm 1, \pm 2$, with relations between them analogous to eq. (3.11):

$$\chi^{*}_{j}\,(x,n^{\alpha}) = \chi_{-j}(x,n^{\alpha}), \qquad (4.14)$$

and the Lagrangian of eq. (3.10b) is now replaced by

$$\mathcal{L}_{\chi} = -\frac{1}{4}\,(G^{\alpha}_{\mu\nu} - F^{\alpha}_{\mu\nu})^{2} + \frac{1}{2}\,(D_{\mu}\phi^{\alpha})^{2} + \frac{1}{2}\,(D_{\mu}\psi^{\alpha})^{2} \qquad (4.15)$$

$$-\; V(\phi,\psi) + \frac{1}{2}\; H^{\alpha}_{\mu}\, F^{\alpha}_{\nu\lambda}\; \varepsilon^{\mu\nu\lambda}$$

$$+\; \int d\hat{n} \sum_{j} \left|\partial_{\mu}\chi_{j} + \frac{2\pi i}{e}\; H^{\alpha}_{\mu} n^{\alpha}\, \chi_{j} - ie\; B^{\alpha}_{\mu}\; T^{\alpha}\; \chi_{j}\right|^{2},$$

where the n integration is a suitable group invariant integration over the manifold defined by (4.12). The transformation properties of χ under the Z(3) subgroup is given by

$$\chi_{j} \rightarrow \omega^{j}\; \chi_{j}, \qquad (4.16)$$

where ω is a cube root of unit.

V. Gauge Theories in (3+1) Dimensions: Strings

In this section, the results of both (1) and of section 3 will be extended to (3+1) dimensions, and the solitons of one lower dimension will become strings. The Abelian Higgs model, which will be treated first, has the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |\partial_\mu \phi + ie A_\mu \phi|^2 \tag{5.1}$$

$$- V(\phi),$$

where ϕ is complex scalar field,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \text{ and}$$

$$V(\phi) = \frac{\lambda^2}{4} (|\phi|^2 - h^2)^2 .$$

Notice that the symbols have different meanings compared to the ones used in the previous sections.

The topologically non-trivial configurations of field ϕ are discussed extensively in literature[15] and also in (1). They are most elegantly described in terms of an antisymmetric tensor current $k^{\mu\nu}$:

$$k^{\mu\nu}(x) = i\, \varepsilon^{\mu\nu\alpha\beta} (\partial_\alpha \hat{\phi}) (\partial_\beta \hat{\phi}^*), \tag{5.1}$$

where $\hat{\phi} = \phi/|\phi|$. This tensor is invariant under non-singular gauge transformations

$$\phi \to e^{-i\Lambda}\,\phi \tag{5.2}$$

$$A_\mu \to A_\mu + \Delta A_\mu , \quad \Delta A_\mu = \frac{1}{e}\,\partial_\mu \Lambda ,$$

and it is also conserved:

$$\partial_\mu k^{\mu\nu} = 0. \tag{5.3}$$

Another important property of $k^{\mu\nu}$ is that it vanishes except at the locations of topolgoical singularities, and so it can be written as a sum over surfaces traced by topological singularities, which form strings in (3+1) dimensions:

$$k^{\mu\nu}(x) = \sum_\ell 2\pi\, n_\ell \int d\tau_\ell \int d\sigma_\ell \left[\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \sigma_\ell} \right] \times \delta^4[x - \bar{x}_\ell(\sigma_\ell, \tau_\ell)], \tag{5.4}$$

where n_ℓ are integers and σ_ℓ and τ_ℓ are two internal variables that parametrize the surface traced by the kink, and the sum extends over all surfaces. The conservation eq. (5.3) is automatically satisfied for closed surfaces, and it can also easily be shown that the integral in eq. (5.4) is invariant under a general parametrization

$$\sigma' = f(\sigma,\tau) , \quad \tau' = g(\sigma,\tau). \tag{5.5}$$

From now on, to avoid end point problems, only closed strings will be considered.

The basic idea is again to remove the topological singularities by means of a singular gauge transformation. Generalizing the result obtained in (1) to (3+1) dimensions, one can show that the singular gauge transformation carries an amount of flux given by the following equation:

$$\Delta F_{\mu\nu} = \partial_\mu(\Delta A_\nu) - \partial_\nu(\Delta A_\mu) \tag{5.6}$$

$$= -\frac{2\pi}{e}\,\varepsilon_{\mu\nu\alpha\beta}\sum_\ell n_\ell \int d\tau_\ell \int d\sigma_\ell \frac{\partial \bar{x}^\alpha_\ell}{\partial \sigma_\ell}\frac{\partial \bar{x}^\beta_\ell}{\partial \tau_\ell}\,\delta^4[x - \bar{x}_\ell(\sigma_\ell,\tau_\ell)],$$

where ΔA_μ is defined by eq. (5.2). To prove this relation, it is easiest first to consider a special singular gauge transformation

$$\Lambda_s = n \tan^{-1}\left(\frac{x_3}{x_2}\right), \tag{5.7}$$

and to choose $\sigma = x_1$, $\tau = x_0$, reducing the problem to the simple case discussed in (1). Lorentz and reparametrization invariances can then be used to establish the general result. Notice that the Bianchi identity

$$\varepsilon^{\mu\nu\alpha\beta}\,\partial_\nu(\Delta F_{\alpha\beta}) = 0 \tag{5.8}$$

is automatically satisfied for closed surfaces.

After eliminating the kinks by a singular gauge transformation and after taking into account the extra terms induced in the action through eq. (5.6), the generating functional can be written as

$$Z(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int DA \int \bar{D}\phi \prod_{\ell=1}^{N} D\bar{x}_\ell \, (\delta\Delta) \tag{5.9}$$

$$\exp\left\{ i \int d^4x \, [J(x)\, C(x) + \mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N)] \right\}$$

where

$$\mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N) = |\partial_\mu\phi + ie\, A_\mu\phi|^2 - V(\phi)$$

$$- \frac{1}{4} \left[F_{\mu\nu}(x) - \frac{2\pi}{e} \, \epsilon_{\mu\nu\alpha\beta} \sum_{\ell=1}^{N} n_\ell \iint d\sigma_\ell d\tau_\ell \right.$$

$$\left. \times \frac{\partial\bar{x}^\alpha_\ell}{\partial\sigma_\ell} \frac{\partial\bar{x}^\beta_\ell}{\partial\tau_\ell} \, \delta^4[x - \bar{x}_\ell(\sigma_\ell,\tau_\ell)] \right]^2 .$$

Again, since the contribution of the kinks is explicitly taken into account, the ϕ integration is over configurations free of kinks.

It is convenient to separate this Lagrangian into several terms:

$$\mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N) = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \tag{5.9a}$$

where $\mathcal{L}$ is given by (5.1) and

$$\mathcal{L}_1 = \frac{\pi}{e} \, \epsilon_{\mu\nu\alpha\beta} \, F^{\mu\nu}(x) \sum_\ell n_\ell \tag{5.9a}$$

$$\iint d\tau_\ell \, d\sigma_\ell \, \frac{\partial\bar{x}^\alpha_\ell}{\partial\sigma_\ell} \frac{\partial\bar{x}^\beta_\ell}{\partial\tau_\ell} \, \delta^4[x - \bar{x}_\ell(\sigma_\ell,\tau_\ell)],$$

$$\mathcal{L}_2 = \frac{2\pi^2}{e^2} \sum_{\ell} \sum_{\ell' \neq \ell} n_\ell \, n_{\ell'} \iint d\tau_\ell \, d\sigma_\ell \iint d\tau_{\ell'} \, d\sigma_{\ell'} \tag{5.9b}$$

$$\left(\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \sigma_\ell} \right) \frac{\partial \bar{x}_{\ell',\mu}}{\partial \sigma_{\ell'}} \frac{\partial \bar{x}_{\ell',\nu}}{\partial \tau_{\ell'}}$$

$$\delta^4[x - \bar{x}_\ell(\sigma_\ell, \tau_\ell)] \; \delta^4[x - \bar{x}_{\ell'}(\sigma_{\ell'}, \tau_{\ell'})],$$

$$\mathcal{L}_3 = \frac{\pi^2}{e^2} \sum_\ell \Bigg[n_\ell \iint d\tau_\ell \, d\sigma_\ell \; \delta^4[x - \bar{x}_\ell(\sigma_\ell, \tau_\ell)]$$

$$\left(\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \sigma_\ell} \right) \Bigg]^2 .$$

Clearly strings interact either directly or through the exchange of the vector meson represented by the field A_μ. The string-vector meson interaction is described by $\mathcal{L}_1$ and the direct string-string interaction by $\mathcal{L}_2$. The two delta functions that appear in the expression for the direct interaction imply that this interaction takes place at the point of the intersection of two strings when they cross. Finally, the last term, $\mathcal{L}_3$, represents the self-interaction of a string and it is proportional to the area of the surface traced by it. Unfortunately, the constant of proportionality is infinite, and although this infinity is expected to be renormalized to a finite value, we are unable to carry out such a renormalization program in this paper. Instead, we provisionally replace this infinite constant by a finite one:

$$\int d^4x\, \mathcal{L}_3 \to -\frac{1}{2\pi\alpha'} \sum_\ell \iint d\sigma_\ell\, d\tau_\ell \left[\left(\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell}\ \frac{\partial \bar{x}_{\ell,\mu}}{\partial \tau_\ell}\right)^2 - \left(\frac{\partial \bar{x}_\ell}{\partial \tau_\ell}\right)^2 \left(\frac{\partial \bar{x}_\ell}{\partial \sigma_\ell}\right)^2\right]^{1/2} \tag{5.10}$$

This is the standard action for the free string model, where α' is to be identified with the slope of the particle trajectory. The standard string interaction, however, takes place by the joining and splitting of strings. In contrast, field theoretic strings interact through the exchange of a vector meson and through a direct contact interaction. Interactions of this type were already proposed by Kalb and Ramond[(16)], and by Nambu.[(17)]

In the strong coupling limit $e^2 \gg 1$, the string interaction, having coupling constant proportional to $1/e$, becomes weak, and it should be possible to treat the system as a collection of weakly interacting strings. The standard approach is then to quantize the free action given by (5.10) first, and then treat the interaction perturbatively. It is well-known that quantized free string theory suffers from several diseases,[(4)] and we make some speculative suggestions about their possible cure. The first disease is the presence of a tachyon at mass $m^2=-2\alpha'$ in the spectrum of the string. This need not be a serious problem if energy is bounded from below; it simply means that the perturbative vacuum is unstable and there will be transition to another stable vacuum, just as in the case of spontaneous symmetry breakdown. The second disease is the absence of

Lorentz invariance except in 26 dimensional space-time. This difficulty can be circumvented by using a covariant quantization scheme, which amounts to replacing eq. (5.10) by

$$\int d^4x\, \mathcal{L}_3 \rightarrow -\frac{1}{2\pi\alpha'} \sum_{\ell} \iint d\sigma_\ell\, d\tau_\ell \left(\frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell}\right)^2 . \tag{5.11}$$

This new action implies that the time coordinate $\bar{x}^0$ is quantized along with space coordinates, introducing negative metric in the theory. Nevertheless, states with negative metric can in general be eliminated by means of Ward identities that follow from reparametrization invariance.(4) For these identities to be valid, however, the following conditions must hold:

a) The lowest string state must be at $m^2 = -2\alpha'$. This is the tachyon discussed earlier.

b) The dimensions of space-time must be less than or equal to 26. Therefore, there is no necessity for the unphysical condition d = 26.

c) Any external field that interacts with strings must have the same mass and coupling as one of the string states. In the present case, the vector meson coupling given by $\mathcal{L}_1$ of eq. (5.9b) coincides with the coupling of a vector state at mass zero, and so, for consistency, the vector meson mass must be zero. This implies h = 0 in eq. (5.1) and no spontaneous symmetry breaking. We have therefore arrived at the surprising conclusion that only in the absence of spontaneous symmetry breaking in the original action given by (5.1), can one hope to have a consistent string theory. Of course, even in this case, there may be other hidden difficulties present.

Finally, we briefly describe the extension of the non-Abelian model of section 3 to (3+1) dimensions. The solitons of (2+1) dimensions again turn into strings in one higher dimension, and now the unit vector n^α defines the direction of flux in the SU(2) space at each point on the string. Eq.'s (2.7a) and (2.7b) are replaced by

$$\tilde{F}^\alpha_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\mu'\nu'} F^{\mu'\nu',\alpha} ,$$

$$\tilde{F}^\alpha_{\mu\nu} = \frac{2\pi}{e} \sum_\ell \iint d\sigma_\ell \, d\tau_\ell \, n^\alpha_\ell(\sigma_\ell,\tau_\ell) \, \delta^4[x - \bar{x}_\ell(\sigma_\ell,\tau_\ell)]$$
$$\times \left(\frac{\partial \bar{x}_{\ell,\mu}}{\partial \sigma_\ell} \frac{\partial \bar{x}_{\ell,\nu}}{\partial \tau_\ell} - \frac{\partial \bar{x}_{\ell,\mu}}{\partial \tau_\ell} \frac{\partial \bar{x}_{\ell,\nu}}{\partial \sigma_\ell} \right) , \qquad (5.12)$$

and the Bianchi identities require that

$$\frac{\partial \bar{x}^\mu}{\partial \sigma} \frac{\partial n^\alpha}{\partial \tau} - \frac{\partial \bar{x}^\mu}{\partial \tau} \frac{\partial n^\alpha}{\partial \sigma} + e \, \varepsilon^{\alpha\beta\gamma} B^\beta_\nu(\bar{x})$$

$$\left(\frac{\partial \bar{x}^\mu}{\partial \sigma} \frac{\partial \bar{x}^\nu}{\partial \tau} - \frac{\partial \bar{x}^\mu}{\partial \tau} \frac{\partial \bar{x}^\nu}{\partial \sigma} \right) n^\gamma = 0, \qquad (5.13)$$

which replaces replaces eq. (2.8b). In addition, the "gauge fixing" term given by eq. (3.3) has to be replaced by (5.11). With these modifications, the analogue of eq. (3.4a) is

$$\int d^4x\, \mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N) = \int d^4x\, \{\mathcal{L}'(x) + H^{\alpha}_{\mu\nu}(x)\, \tilde{F}^{\mu\nu,\alpha}(x)\} \qquad (5.14)$$

$$+ \sum_{\ell=1}^{N} \iint d\sigma_\ell\, d\tau_\ell \left\{ - \frac{1}{2\pi\alpha'} \left(\frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell}\right)^2 + \lambda_\ell\, [(n^\alpha_\ell)^2 - 1] \right.$$

$$- \frac{2\pi}{e}\, H^{\alpha}_{\mu\nu}(\bar{x}_\ell) \left(\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \sigma_\ell} \right) n^\alpha_\ell$$

$$+ s^\alpha_{\mu,\ell} \left[\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial n^\alpha_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial n^\alpha_\ell}{\partial \sigma_\ell} \right.$$

$$\left.\left. + e\, \epsilon^{\alpha\beta\gamma}\, B^\beta_\nu(\bar{x}_\ell)\, n^\gamma_\ell \left(\frac{\partial \bar{x}^\mu_\ell}{\partial \sigma_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \tau_\ell} - \frac{\partial \bar{x}^\mu_\ell}{\partial \tau_\ell} \frac{\partial \bar{x}^\nu_\ell}{\partial \sigma_\ell} \right) \right] \right\},$$

where $\mathcal{L}'$ is given by (3.1b). The functional integral can now be expressed in terms of the action of (5.14) by means of a formula similar to eq. (3.4b).

The next step, which will not be attempted here, is the quantization of the string variables that appear in eq. (5.14). If it is carried out successfully, this would yield a new string model.

VI. Concluding Remarks

In the preceding sections, we have extended the construction given in reference (1) to non-Abelian gauge groups and to physical space-time dimensions. The method produces local field theories for solitons in (2+1) dimensions and string theories with local interaction in (3+1) dimensions.

An interesting problem for future research is to investigate the classical solutions of these new field theories. One should then be able to recover the well-known soliton solutions in some approximation, and hopefully, some new and unexpected classical solutions may emerge. This approach may also shed some light on the confinement problem. On the other hand, severe difficulties are encountered if one tries to go beyond the classical equations of motion, and they are connected with the absence of a consistent renormalization scheme. This problem remains a serious obstacle to further progress.

Acknowledgements

I would like to thank S. Samuel for his helpful comments and criticism. I have also benefited from conversations with S. Mandelstam, N. Papanicolaou, J. Polchinski and E. Rabinovici. This work was supported by the National Science Foundation under grant number PHY77-23512 and by the High Energy Division of the U.S. Department of Energy.

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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