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Combinatorial Theory

Title

Ramsey's witnesses

Permalink

<https://escholarship.org/uc/item/2dt685nf>

Journal

Combinatorial Theory, 6(3)

ISSN

2766-1334

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Publication Date

2026-10-15

DOI

10.5070/C6.69927

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Peer reviewed

RAMSEY'S WITNESSES

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Submitted: Apr 16, 2025; Accepted: Feb 18, 2026; Published: Oct 15, 2026

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Abstract. We introduce the notion of *Ramsey partition regularity*, a generalisation of partition regularity involving infinitary configurations. We provide characterisations of this notion in terms of certain ultrafilters related to tensor products and dubbed *Ramsey's witnesses*; and we also consider their nonstandard counterparts as pairs of hypernatural numbers, called *Ramsey pairs*. These characterisations are then used to determine whether various configurations involving polynomials and exponentials are Ramsey partition regular over the natural numbers. In particular, this provides negative answers to several questions recently posed by Kra, Moreira, Richter and Robertson.

Keywords. Arithmetic Ramsey theory, partition regularity, ultrafilter, nonstandard analysis, tensor pair, Ramsey's witness

Mathematics Subject Classifications. Primary: 05D10. Secondary: 26E35, 03H15, 11U10, 05A17, 54D80

^{*}Supported by the project PRIN 2022 “Logical methods in combinatorics”, 2022BXH4R5, Italian Ministry of University and Research. We acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001. MDN is a member of the INdAM research group GNSAGA.

[†]Supported by the project PRIN 2022 “Logical methods in combinatorics”, 2022BXH4R5, Italian Ministry of University and Research.

[‡]Supported by the project PRIN 2022: “Models, sets and classifications” Prot. 2022TECZJA. We acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001.

[§]Supported by the project PRIN 2022: “Models, sets and classifications” Prot. 2022TECZJA. Supported by the project PRIN 2022 “Logical methods in combinatorics”, 2022BXH4R5, Italian Ministry of University and Research.

[¶]Supported by the project PRIN 2022 “Logical methods in combinatorics”, 2022BXH4R5, Italian Ministry of University and Research. We acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001. Supported by project 25-15571S of the Czech Science Foundation (GAČR). This work has been supported by Charles University Research Centre programme No.UNCE/24/SSH/026.

1. Introduction

Since the 1970s, applications of ultrafilters to combinatorics have been widely studied, producing a wealth of results, especially in the area of Ramsey theory; see the comprehensive monograph [HS11]. In recent years, the close connection between the space of ultrafilters $\beta\mathbb{N}$ and nonstandard extensions ${}^*\mathbb{N}$ of the natural numbers has been thoroughly investigated, providing an alternative approach that has revealed useful for formalising Ramsey properties in a simple and manageable way, and that has also made it possible to prove some new results in the area of *arithmetic Ramsey theory*.

The latter is a branch of combinatorics concerned with the “abundance” of certain arithmetic patterns in the natural numbers. A typical problem in this area is to determine whether a given configuration is ubiquitous enough that one is forced to find instances of it in some piece of every finite partition of $\mathbb{N}$.

In this area, it is customary to call partitions *colourings*. For instance, a classical theorem by I. Schur [Sch16] states that, whenever the natural numbers are coloured with finitely many colours, say $\mathbb{N} = C_1 \cup \dots \cup C_r$, then one of the C_i contains a triple of distinct elements a, b, c such that $a + b = c$. Such a triple is called a *monochromatic* solution of the equation $x + y = z$, and the existence of monochromatic solutions in all colourings is the statement that the aforementioned equation is *partition regular*.

It is natural to ask which other equations have the same property. Albeit the linear case is well-understood by a classical result of R. Rado [Rad33], the polynomial one is still very far from being settled; for instance, whether the Pythagorean equation $x^2 + y^2 = z^2$ is partition regular is still unknown, and one of the long-standing open problems in the area.

The notion of partition regularity is not limited to solutions to Diophantine equations. In general, a family $\mathcal{F}$ of subsets of a set X is *partition regular* (PR) over X if it is closed under taking supersets and, for all finite colourings $X = C_1 \cup \dots \cup C_r$ there exists $i \leq r$ such that $C_i \in \mathcal{F}$; and it is *strongly partition regular* (SPR) over X if, furthermore, for all $Y \in \mathcal{F}$ the family $\mathcal{F} \cap \mathcal{P}(Y)$ is PR over Y , i.e. if for all finite colourings $Y = C_1 \cup \dots \cup C_r$ there exists $i \leq r$ such that $C_i \in \mathcal{F}$.

If P is some property that can be satisfied by a set, with a small abuse we will say that the property P is PR if the family $\mathcal{F}_P := \{A \subseteq X \mid P \text{ is true for } A\}$ is PR. So, for example, Schur’s Theorem says that the property of containing some x, y, z with $x + y = z$ is partition regular. Another celebrated example is van der Waerden’s Theorem [va27], establishing strong partition regularity for containing arbitrarily long arithmetic progressions.

Problems in arithmetic Ramsey theory have been tackled by employing a variety of methods; as mentioned above, one that attracted significant attention in recent years is to exploit the strict connection between spaces of ultrafilters and their algebra and *nonstandard extensions* of $\mathbb{N}$. We now briefly recall what this amounts to; see e.g. [DN15, LB19, DNGL19] for a more detailed treatment.

The link between partition regularity of a family $\mathcal{F}$ of subsets of X and ultrafilters on X stems from the well-known facts that

1. $\mathcal{F}$ is PR over X if and only if there exists an ultrafilter u over X such that $u \subseteq \mathcal{F}$; such a u is called a *witness* of the partition regularity of $\mathcal{F}$; and
2. $\mathcal{F}$ is SPR over X if and only if $\mathcal{F} \neq \emptyset$ and for all $Y \in \mathcal{F}$ there is a witness $u \subseteq \mathcal{F}$ containing Y .

As for nonstandard extensions of $\mathbb{N}$, recall that these are *elementary extensions* of the natural numbers, that is, larger structures satisfying the *transfer principle* for first-order formulas.¹ For example, this means that, for every subset $A \subseteq \mathbb{N}^k$, there is an associated nonstandard extension ${}^*A \subseteq {}^*\mathbb{N}^k$ that satisfies the same first-order properties as A : e.g., if A is the set of even natural numbers, then ${}^*A$ is precisely the set of those elements of ${}^*\mathbb{N}$ of the form 2α , for a suitable $\alpha \in {}^*\mathbb{N}$. Sets of the form ${}^*A$, and more generally *internal* subsets of ${}^*\mathbb{N}$, are well-behaved in a number of senses. Crucially, they satisfy the same elementary properties as subsets of $\mathbb{N}$, e.g. every nonempty internal subset has a least element. Furthermore, they obey the *overspill* principle: whenever an internal set contains arbitrarily large natural numbers, then it must also contain an element larger than $\mathbb{N}$; so, for instance, $\mathbb{N}$ is not internal.

Elements of some nonstandard extension ${}^*\mathbb{N}$ are called *hypernatural numbers*, and they and their tuples bear a natural relation to ultrafilters on $\mathbb{N}$ and its Cartesian powers. Namely, for $(\alpha_1, \dots, \alpha_k) \in {}^*\mathbb{N}^k$, the set²

$$u(\alpha_1, \dots, \alpha_k) := \{A \subseteq \mathbb{N}^k \mid (\alpha_1, \dots, \alpha_k) \in {}^*A\}$$

belongs to $\beta\mathbb{N}^k$ —that is, it is an ultrafilter on $\mathbb{N}^k$. Conversely, every $u \in \beta\mathbb{N}^k$ is of the form $u = u(\alpha_1, \dots, \alpha_k)$ for a suitable $(\alpha_1, \dots, \alpha_k) \in {}^*\mathbb{N}^k$, provided ${}^*\mathbb{N}$ is $\mathfrak{c}^+$ -saturated.

Throughout the paper, we work in such a ${}^*\mathbb{N}$. We write $(\alpha_1, \dots, \alpha_k) \sim (\beta_1, \dots, \beta_k)$, and say that these tuples are *equivalent*, whenever $u(\alpha_1, \dots, \alpha_k) = u(\beta_1, \dots, \beta_k)$. The equivalence classes of this relation correspond, as explained above, to ultrafilters $u \in \beta\mathbb{N}^k$. For each such u , the associated $\sim$ -equivalence class called the *monad* of u , and its elements *generators* of u . It is well-known that combinatorial properties of u can be rephrased in terms of properties of its monad. For finitary configurations, that is, properties expressible by a first-order formula, this can be done by using the following fundamental property. See e.g. [DNGL19, Proposition 9.1], or [LB19, Theorem 5.4] for a more general version.

Fact 1.1. *A formula $\varphi(x_1, \dots, x_n)$ is PR, that is, for every finite colouring $\mathbb{N} = C_1 \cup \dots \cup C_r$ there is a monochromatic $(a_1, \dots, a_n)$ such that $\mathbb{N} \models \varphi(a_1, \dots, a_n)$, if and only if there are $\alpha_1 \sim \dots \sim \alpha_n$ such that ${}^*\mathbb{N} \models \varphi(\alpha_1, \dots, \alpha_n)$.*

Above, ${}^*\mathbb{N} \models \varphi(\alpha_1, \dots, \alpha_n)$ means that the formula $\varphi(\alpha_1, \dots, \alpha_n)$ holds in ${}^*\mathbb{N}$. Combining this with the ultrafilter characterisation of partition regularity we see that, for instance, Schur's Theorem is, in this language, the statement that there are $\alpha \sim \beta \sim \gamma$ in ${}^*\mathbb{N}$ such that ${}^*\mathbb{N} \models \alpha + \beta = \gamma$.

Over the years Schur's result has been strengthened in various directions. Most notably, in [Hin74] N. Hindman proved the following infinitary extension:

¹Here $\mathbb{N}$ is equipped with its full first-order structure, that is, with relation symbols for every subset of each of its Cartesian powers.

²Other notations common in the literature are $u_{(\alpha_1, \dots, \alpha_k)}$ and $\text{tp}(\alpha_1, \dots, \alpha_k)$.

Hindman's Theorem. For every finite colouring of $\mathbb{N}$ there exists an infinite increasing sequence $(x_i)_{i \in \mathbb{N}}$ such that the set of all finite sums $\text{FS}((x_i)_{i \in \mathbb{N}}) = \{x_{i_1} + \dots + x_{i_k} \mid k \in \mathbb{N}, i_1 < \dots < i_k\}$ is monochromatic.

Hindman's Theorem is the prototype of the results on the existence of infinite monochromatic configurations, which can be seen as leaps from the realm of finite mathematics to the world of infinite combinatorics.

Typically these statements are closely related, in one way or another, to Ramsey's Theorem [Ram30]. Let us recall the 2-dimensional case of the latter which, when applied to $X = \mathbb{N}$, is the case that we will study in the rest of the paper. Write $[X]^2 := \{\{x_1, x_2\} \mid x_1, x_2 \in X, x_1 \neq x_2\}$.

Ramsey's Theorem for pairs. Let X be an infinite set. For all finite partitions $[X]^2 = C_1 \cup \dots \cup C_r$ there exist $i \leq r$ and an infinite $H \subseteq X$ such that $[H]^2 \subseteq C_i$.

For convenience, when $X = \mathbb{N}$, we identify $[H]^2$ with $\{(h_1, h_2) \in H^2 \mid h_1 < h_2\}$.

A direct application of **Ramsey's Theorem for pairs** shows that the family $\mathcal{F}$ of those $A \subseteq \mathbb{N}$ such that there is an infinite H with $\{h_1 + h_2 \mid (h_1, h_2) \in [H]^2\} \subseteq A$ is SPR. The analogous statement also holds if sums are replaced by products but, notably, sums and products cannot be combined. Indeed, a theorem proved by N. Hindman in [Hin84, Theorem 2.11], another proof of which recently appeared in [HIL23, Theorem 3.1], states the following.

Pairwise Sum-Product Theorem. Let $\mathcal{F}$ be the family of those $A \subseteq \mathbb{N}$ such that

$$\exists H \subseteq \mathbb{N} \text{ } H \text{ is infinite and } \forall (x, y) \in [H]^2 \text{ } (x + y, x \cdot y \in A).$$

Then $\mathcal{F}$ is not PR.

This is (a failure of) the kind of ‘‘Ramsey partition regularity’’ statements (see Definition 3.6) that we study in this paper, namely: for which formulas $\varphi(x, y, z_1, \dots, z_n)$, given a finite colouring of $\mathbb{N}$, one can find an infinite set H such that for all $(x, y) \in [H]^2$ there exist monochromatic $z_1, \dots, z_n$ such that $\varphi(x, y, z_1, \dots, z_n)$ holds? E.g., in the theorem above, $\varphi(x, y, z, t)$ is the formula $(x \neq y) \wedge (x + y = z) \wedge (x \cdot y = t)$.

Because of the infinitary nature of these statements, Fact 1.1 does not directly apply. Nevertheless, one may still study such problems via nonstandard methods by employing the following definition, not to be confused with that of ‘‘Ramsey ultrafilters’’ (also known as ‘‘selective ultrafilters’’).

Definition 1.2. We call an ultrafilter $u \in \beta\mathbb{N}^2$ a *Ramsey's witness* (RW) if for all $A \in u$ there exists an infinite $H \subseteq \mathbb{N}$ such that $[H]^2 \subseteq A$. We also write RW for the subset of $\beta\mathbb{N}^2$ consisting of all u as above, and say that $(\alpha, \beta) \in {}^*\mathbb{N}^2$ is a *Ramsey pair*, if $u(\alpha, \beta) \in \text{RW}$. In this case, we write $(\alpha, \beta) \models \text{RW}$.

By **Ramsey's Theorem for pairs**, the family of all $A \subseteq \mathbb{N}^2$ such that there exists an infinite $H \subseteq \mathbb{N}$ with $[H]^2 \subseteq A$ is SPR. Equivalently, every such A is contained in some $u \in \text{RW}$; equivalently, ${}^*A$ contains a Ramsey pair.

Theorem 3.8 is our main technical result. It provides a ‘‘Ramsey’’ version of Fact 1.1, that is, a characterisation of Ramsey partition regularity statements in terms of properties of Ramsey pairs. From this characterisation we deduce several results in the style of the **Pairwise Sum-Product Theorem**. Let us state some of them in the language we just introduced.

Theorem A. If (α, β) is a Ramsey pair, then

1. $\alpha + \beta \approx \alpha \cdot \beta$ (Theorem 4.1),
2. $\alpha \cdot \beta \approx \alpha^\beta$ (Example 5.24),
3. $\alpha^\beta \approx \beta^\alpha$ (Example 5.23), and
4. $f(\alpha) \cdot 2^\beta \approx g(\alpha, \beta)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g \in \mathbb{Z}[x, y]$ (special case of Example 5.26).

Point 1 is a restatement of the [Pairwise Sum-Product Theorem](#), of which we provide a short nonstandard proof. It is known [[Sah18](#), Theorem 2] that every finite colouring of the naturals contains a monochromatic quadruple $a, b, a \cdot b, a^b$. Point 2 says that the Ramsey version of this does not hold, not even for $a \cdot b, a^b$, i.e., there is a finite colouring of $\mathbb{N}$ such that no increasing sequence $(a_i)_{i \in \mathbb{N}}$ is such that the set $\{a_i \cdot a_j \mid i < j\} \cup \{a_i^{a_j} \mid i < j\}$ is monochromatic. See also [[DNR24](#), [DNR26](#)] for partition regularity of exponential configurations. Points 3 and 4 correspond to similar Ramsey properties.

We also study the Ramsey partition regularity of polynomial equations. Namely, given a nonzero polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$, we investigate whether the formula $P(x_1, \dots, x_n) = 0$ is Ramsey PR. In the two-variable case, there are no Ramsey PR equations (Proposition 5.2). Our next result concerns certain 3-variable polynomials.

Theorem B (Corollary 5.10 and Proposition 5.11). Let $P, Q \in \mathbb{Z}[x]$ be nonzero polynomials with $P(0) = Q(0) = 0$ and $a, n \in \mathbb{N}$. The following are equivalent.

1. There are $\alpha \sim \beta \sim \gamma$ with (α, β) a Ramsey pair such that $a\alpha^n + P(\beta) = Q(\gamma)$;
2. $ax^n + P(y) - Q(z)$ is either $a(x + y - z)$ or $a(x - y + z)$.

In other words, up to multiplying everything by a constant, the only two equations of the form $ax^n + P(y) = Q(z)$ that are Ramsey PR in x, y are Schur's equation $x + y = z$ and the equation $x - y = -z$. An interesting property of the latter is that, contrarily to Schur's equation, it cannot be solved by tensor pairs (Remark 5.13). Theorem B rules out the Ramsey partition regularity of plenty of equations, ranging from some whose partition regularity is open, such as the Pythagorean equation $x^2 + y^2 = z^2$, to equations of the form $y - x = z^n$, for $n \geq 2$, that are known to be PR, see [[BFM96](#), Theorem C].

Similar Ramsey partition regularity problems were posed in [[KMRR25](#)], which contains questions arising from the solution [[MRR19](#)] to Erdős' $B + C$ conjecture, arguably the most important recent result in combinatorics. In particular, Theorem B answers in the negative several questions from Sections 3.3 and 3.4 of [[KMRR25](#)], in the stronger PR version, albeit in the form without shifts.

For instance, in the language of the present work, the shiftless PR version of [[KMRR25](#), Question 3.15] amounts to asking whether the equation $x^2 + y = z$ is Ramsey PR. In [[Ack25](#), Theorem 1.5] it is shown that this equation is Ramsey PR over $\mathbb{Q}$, and in fact that the same holds with x^2 replaced by any $P(x) \in \mathbb{Q}[x]$ of degree at least 2. This fails over $\mathbb{N}$ by Theorem B and, after a preprint version of this paper was circulated, E. Ackelsberg gave an explicit 5-colouring

witnessing that $x^2 + y = z$ is not Ramsey PR, see [Ack25, Section 11.1]. Analogously, Theorem B answers in the negative the PR shiftless versions of [KMRR25, Questions 3.10, 3.11, 3.12, 3.15, 3.19, 3.20], as well as the variant of [KMRR25, Question 3.16] with monomials on the left hand side but arbitrary polynomials without constant term on the right hand one.

Theorems A and B demonstrate that being Ramsey PR significantly strengthens being merely PR. One may wonder what happens to other properties classically known to be PR. In this direction, another application of our results is a proof that van der Waerden's Theorem does not admit a "Ramsey version", not even in the case of 3-term arithmetic progressions.

Theorem C (Proposition 5.14). If $\alpha \sim \beta \sim \gamma$ and (α, β) is a Ramsey pair, then α, β, γ do not form, in any order, an arithmetic progression.

The paper is structured as follows. In Section 2 we will prove several properties of Ramsey pairs and characterise them in terms of tensor pairs. In Section 3, we introduce several versions of Ramsey partition regularity and characterise them in ultrafilter and nonstandard terms. In Section 4 we study these notions for the formula $\varphi(x, y, z, t)$ that appears in the Pairwise Sum-Product Theorem, and give a very compact proof of the latter. In Section 5 we prove Theorems A, B and C, and in our final Section 6 we collect some open problems that we believe deserve further study.

2. Ramsey's witnesses

In this paper, $\mathbb{N}$ denotes the set of *positive* integers. We assume familiarity with the basics of nonstandard methods, for which we refer the reader to, e.g., [Gol98], but we briefly recall some well-known facts on tensor pairs.

The relationship between ultrafilters on $\mathbb{N}$ and on $\mathbb{N}^2$ is intricate, as the product filter $u \times v$ of two ultrafilters on $\mathbb{N}$, generated by the family $\{A \times B \mid A \in u, B \in v\}$, is not in general an ultrafilter on $\mathbb{N}^2$. Among all the ultrafilters that extend $u \times v$, one plays a special role in many applications, namely the tensor product $u \otimes v$.

Definition 2.1. Given $u, v \in \beta\mathbb{N}$, the *tensor product* $u \otimes v$ is the ultrafilter on $\mathbb{N}^2$ defined as follows: for all $A \subseteq \mathbb{N}^2$, we let

$$A \in u \otimes v \Leftrightarrow \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid (n, m) \in A\} \in v\} \in u.$$

Given $(\alpha, \beta) \in {}^*\mathbb{N}^2$, we say that (α, β) is a *tensor pair* if $u(\alpha, \beta)$ is a tensor product.

The tensor product can also be considered in arbitrary dimensions and is associative, hence it makes sense to talk of tensor triples, tensor k -uples, etc. However, in this paper we will mostly work with tensor products in dimension 2. Note that, almost by definition, if (α, β) is a tensor pair then $u(\alpha, \beta) = u(\alpha) \otimes u(\beta)$. For a detailed study of tensor products we refer to [LB19]. In more model-theoretic language, (α, β) is a tensor pair if the type of α over $\mathbb{N} \cup \{\beta\}$ is a *coheir* of its restriction to $\mathbb{N}$, that is, is finitely satisfiable in $\mathbb{N}$, which amounts to the following.

Fact 2.2. The pair (α, β) is tensor if and only if for every $A \in u(\alpha, \beta)$ there is $n \in \mathbb{N}$ such that $A \in u(n, \beta)$.

This point of view is exploited for example in [CZ22].

Combinatorial applications of tensor pairs by means of nonstandard methods are based on their characterisation given by C. Puritz in [Pur72, Theorem 3.4] (for higher dimensions, see [LB19, Theorem 4.22]³). Before recalling Puritz' Theorem, let us review some prerequisites. See e.g. [DN14, DN15, DNGL19, LB19] for further details.

Notation 2.3. Throughout the paper we treat functions $f : \mathbb{N}^k \rightarrow \mathbb{N}$ as symbols in the language of the full structure on $\mathbb{N}$; in practice this means that, if $\alpha \in {}^*\mathbb{N}^k$, its image under the nonstandard extension of f to ${}^*\mathbb{N}^k$ will be denoted by $f(\alpha)$ instead of $*f(\alpha)$. Similarly, we use the symbol f also for the induced map $\beta\mathbb{N}^k \rightarrow \beta\mathbb{N}$ (which is well-defined by Fact 2.4 below). We write $\mathbb{N}(\alpha)$ for the definable closure of α , namely

$$\mathbb{N}(\alpha) := \{f(\alpha) \mid f : \mathbb{N} \rightarrow \mathbb{N}\}.$$

Fact 2.4. Let $\alpha \in {}^*\mathbb{N}^n$, $\beta \in {}^*\mathbb{N}^m$, $f : \mathbb{N}^n \rightarrow \mathbb{N}^l$, and $g : \mathbb{N}^m \rightarrow \mathbb{N}^l$.

1. If $\alpha \sim \beta$ then $f(\alpha) \sim f(\beta)$.
2. If $f(\alpha) \sim \alpha$ then $f(\alpha) = \alpha$.
3. The following are equivalent.
 - (a) $f(\alpha) \sim g(\beta)$.
 - (b) There exists $\beta' \sim \beta$ such that $f(\alpha) = g(\beta')$.
4. If $\alpha, \beta \in {}^*\mathbb{N}$, $\alpha \sim \beta$, and $\alpha < \beta$, then $\beta - \alpha$ is infinite.

Fact 2.5 (Puritz). Let $(\alpha, \beta) \in {}^*\mathbb{N}^2$. Then (α, β) is a tensor pair if and only if for all $f : \mathbb{N} \rightarrow \mathbb{N}$ we have $f(\beta) \in \mathbb{N}$ or $\alpha < f(\beta)$.

The following remarks are well-known, and can be easily proven by using Fact 2.5.

Remark 2.6. If (α, β) is a tensor pair and $\beta \notin \mathbb{N}$, then $\beta > \alpha$.

Remark 2.7. If (α, β) is a tensor pair, and $f, g : \mathbb{N} \rightarrow \mathbb{N}$, then $(f(\alpha), g(\beta))$ is a tensor pair.

In this paper, we focus on the relationship between tensor products and Ramsey's witnesses, which is grounded on the following fact. Its proof is implicit in the usual ultrafilter proof of [Ramsey's Theorem for pairs](#), see e.g. [CK90, Theorem 3.3.7]; below, we make it explicit for the reader's convenience.

Proposition 2.8. Let $u \in \beta\mathbb{N}$ be a nonprincipal ultrafilter. Then $u \otimes u \in \text{RW}$.

Proof. For $A \subseteq \mathbb{N}^2$ and $n \in \mathbb{N}$, write $A_n := \{m \mid (n, m) \in A\}$, and let $A_u := \{n \mid A_n \in u\}$. By definition, $A \in u \otimes u \iff A_u \in u$. Fix $A \in u \otimes u$. We construct inductively an increasing sequence $(x_i)_{i \in \mathbb{N}}$ such that, for each i , we have $A_{x_i} \in u$, and if $i < j$ then $(x_i, x_j) \in A$. To construct x_{k+1} , given $x_1 < \dots < x_k$, we observe that the set $A_u \cap \bigcap_{i \leq k} A_{x_i}$ belongs to u , hence is infinite, so it contains some $x_{k+1} > x_k$. $\square$

³In the cited result, there is a missing assumption that $\alpha_1, \dots, \alpha_n$ are not standard.

Therefore, all products $u \otimes u$ with u nonprincipal are Ramsey's witnesses. Actually, we can be more precise: the set RW is the closure of the set of such products. Recall that the topology on $\beta\mathbb{N}^k$ has as basic clopen sets those of the form $\{u \in \beta\mathbb{N}^k \mid A \in u\}$, for $A \subseteq \mathbb{N}^k$.

Definition 2.9. We define the set of *tensor squares* as $\text{TS} := \{u \otimes u \in \beta\mathbb{N}^2 \mid u \in \beta\mathbb{N} \setminus \mathbb{N}\}$. We adopt similar conventions as in Definition 1.2, e.g. write $(\alpha, \beta) \models \text{TS}$ when $u(\alpha, \beta) \in \text{TS}$.

Theorem 2.10. *The topological closure $\overline{\text{TS}}$ coincides with RW .*

Proof. It is clear from Definition 1.2 that RW is a closed subset of $\beta\mathbb{N}^2$, and it contains every $u \otimes u$ by Proposition 2.8, so it clearly includes $\overline{\text{TS}}$. Conversely, let $u \in \text{RW}$, fix $X \in u$, and choose an infinite $H \subseteq \mathbb{N}$ with $[H]^2 \subseteq X$. For $h \in H$ let $H_h = \{x \in H \mid x > h\}$; the family $\{H_h \mid h \in H\}$ has the FIP, and furthermore every finite intersection of this family is infinite. Therefore, it can be extended to a nonprincipal ultrafilter v . For every $h \in H$ the fiber $([H]^2)_h = H_h$ belongs to v , and so $H = \{n \in \mathbb{N} \mid ([H]^2)_n \in v\} \in v$, i.e. $X \supseteq [H]^2 \in v \otimes v$. $\square$

We will need the following slight strengthening of Fact 2.4(2).

Proposition 2.11. *Let $f, g : \mathbb{N}^n \rightarrow \mathbb{N}^l$. If g is bounded-to-one, i.e. there is $k \in \mathbb{N}$ such that every fiber of g has size at most k , and $f(\alpha) \sim g(\alpha)$, then $f(\alpha) = g(\alpha)$.*

Proof. Without loss of generality $\alpha \notin \mathbb{N}^n$, otherwise the conclusion trivially holds.

If g is bijective, then by Fact 2.4(1) $\alpha \sim g^{-1}(f(\alpha))$, so it suffices to apply Fact 2.4(2) to $g^{-1} \circ f$. The general case can be reduced to that of a bijection as follows.

For $i \leq k$, let $C_i \subseteq \mathbb{N}^n$ be the set of those $a \in \mathbb{N}^n$ that are the i -th point of $g^{-1}(g(a))$ in the lexicographical order. Let i be such that $\alpha \in {}^*C_i$. By repeatedly splitting C_i and selecting the piece belonging to $u(\alpha)$, we now find some $C \in u(\alpha)$ such that $C \subseteq C_i$ and all of $C_i \setminus C$, $g(C)$ and $g(\mathbb{N}^n \setminus C)$ are infinite. By a routine argument, there is a bijective h that agrees with g on C , and in particular $h(\alpha) = g(\alpha)$, which reduces the problem to the case above. $\square$

Proposition 2.12. *Let $\alpha, \beta \in {}^*\mathbb{N} \setminus \mathbb{N}$. The following properties hold.*

1. $(\alpha, \beta) \models \text{RW}$ if and only if, for all $A \subseteq \mathbb{N}^2$, if $(\alpha, \beta) \in {}^*A$ then there exists a pair $(\gamma, \delta) \in {}^*A$ with $(\gamma, \delta) \models \text{TS}$.
2. If $(\alpha, \beta) \models \text{TS}$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ is such that $f(\alpha) \neq f(\beta)$ then $(f(\alpha), f(\beta)) \models \text{TS}$.
3. If $(\alpha, \beta) \models \text{TS}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ is such that $g(\beta) \neq g(\alpha)$, then for all $f : \mathbb{N} \rightarrow \mathbb{N}$ $g(\beta) > f(\alpha)$.

Proof. 1. This is just a reformulation of Theorem 2.10 in terms of generators.

2. By Remark 2.7 $(f(\alpha), f(\beta))$ is a tensor pair, and by Fact 2.4(1) $f(\alpha) \sim f(\beta)$. This, together with $f(\alpha) \neq f(\beta)$, implies $f(\alpha) \notin \mathbb{N}$, hence the conclusion.

3. By Remark 2.7 $(f(\alpha), g(\beta))$ is a tensor pair. As $g(\beta) \neq g(\alpha)$ and $\beta \sim \alpha$, we must have $g(\beta) \notin \mathbb{N}$, hence we conclude by Remark 2.6. $\square$

Proposition 2.13 (Properties of Ramsey pairs). *Let $(\alpha, \beta) \models \text{RW}$ and $f, g : \mathbb{N} \rightarrow \mathbb{N}$. The following properties hold.*

1. *If $f(\alpha) \neq f(\beta)$ then $(f(\alpha), f(\beta)) \models \text{RW}$.*
2. *If $g(\alpha) \neq g(\beta)$, then $f(\alpha) < g(\beta)$.*
3. *Either $f(\alpha) = f(\beta)$ or $\alpha < f(\beta)$.*
4. *$f(\alpha) < \beta$. In particular $\alpha < \beta$.*
5. *If $P, Q \in \mathbb{Z}[x]$, with Q of degree at least 1, then $|P(\alpha)| < |Q(\beta)|$.*
6. *If g is finite-to-one, then $f(\alpha) < g(\beta)$.*
7. *$\alpha \sim \beta$.*

Proof. 1. Fix $A \in \mathfrak{u}(f(\alpha), f(\beta))$ and let $B = (f, f)^{-1}(A) \cap \{(a, b) \mid f(a) \neq f(b)\}$. By assumption, $(\alpha, \beta) \in {}^*B$, so by Proposition 2.12(1) there exists $(\gamma, \delta) \in {}^*B$ with $(\gamma, \delta) \models \text{TS}$. As $f(\gamma) \neq f(\delta)$, the pair $(f(\gamma), f(\delta)) \in {}^*A$ satisfies $(f(\gamma), f(\delta)) \models \text{TS}$ by Proposition 2.12(2). We conclude by Proposition 2.12(1).

2. For fixed f, g , the set $\{(x, y) \mid g(x) \neq g(y) \rightarrow f(x) < g(y)\}$ belongs to every element of TS by Proposition 2.12(3). The conclusion then follows by Theorem 2.10.

3. Immediate from 2.

4. This is a special case of 2.

5. This too is a special case of 2.

6. This follows by applying 4 to the function $h(n) := \max g^{-1}([1, f(n)])$, which is well-defined because g is finite-to-one.

7. Equivalence of α and β is a closed condition and holds on TS, so we conclude by Theorem 2.10. $\square$

Theorem 2.10 allows us to give a nonstandard description of RW, Theorem 2.16 below, that makes use of internal ultrafilters, that is, elements of ${}^*(\beta\mathbb{N}^k)$.

Definition 2.14. Let $\mathfrak{u} \in {}^*(\beta\mathbb{N}^2)$. We let $\text{st}(\mathfrak{u}) := \{A \subseteq \mathbb{N}^2 \mid {}^*A \in \mathfrak{u}\} \in \beta\mathbb{N}^2$.

Remark 2.15. The notation $\text{st}(\mathfrak{u})$ is consistent with viewing ${}^*(\beta\mathbb{N}^2)$ topologically as a nonstandard extension of $\beta\mathbb{N}^2$, namely $\text{st}(\mathfrak{u}) = u$ if and only if $\mathfrak{u} \in {}^*A$ for all neighbourhoods A of u .

Below, we abuse the notation and write $\otimes$ for the nonstandard extension of $\otimes$ to ${}^*(\beta\mathbb{N}^2)$.

Theorem 2.16. $v \in \text{RW}$ if and only if there is $\mathfrak{u} \in {}^*(\beta\mathbb{N} \setminus \mathbb{N})$ such that $v = \text{st}(\mathfrak{u} \otimes \mathfrak{u})$.

Proof. This is a special case of the nonstandard characterisation of topological closure, but we spell out some details for the reader's convenience.

Left to right, for $X \subseteq \mathbb{N}^2$, define

$$\Lambda_X := \{u \in \beta\mathbb{N} \setminus \mathbb{N} \mid X \in u \otimes u\}.$$

By assumption and Theorem 2.10, for each $X \in v$ the set Λ_X is nonempty. In fact, the family $\{\Lambda_X \mid X \in v\}$ has the FIP, and it follows that there is $u \in {}^*(\beta\mathbb{N} \setminus \mathbb{N})$ such that, for every $X \in v$ we have ${}^*X \in u \otimes u$. This implies that $\text{st}(u \otimes u) = v$.

Right to left, let $u \in {}^*(\beta\mathbb{N} \setminus \mathbb{N})$. By transfer, for every internal $B \subseteq {}^*\mathbb{N}^2$ such that $B \in u \otimes u$, there is a * infinite $H \subseteq {}^*\mathbb{N}$ such that $[H]^2 \subseteq B$. In particular, this applies to those B of the form ${}^*X$ for $X \subseteq \mathbb{N}^2$. By applying downward transfer to the formula “there is a * infinite $H \subseteq {}^*\mathbb{N}$ such that $[H]^2 \subseteq {}^*X$ ”, we find an infinite $K \subseteq \mathbb{N}$ such that $[K]^2 \subseteq X$. It follows that $\text{st}(u \otimes u) \in \text{RW}$. $\square$

Proposition 2.17. *If (α, β, γ) is a tensor triple, $\beta \sim \gamma$, and $f : \mathbb{N}^2 \rightarrow \mathbb{N}$, is such that $f(\alpha, \beta) \neq f(\alpha, \gamma)$, then $(f(\alpha, \beta), f(\alpha, \gamma)) \models \text{RW}$.*

Proof. Given $A \in u(f(\alpha, \beta), f(\alpha, \gamma))$, we need to find an element of TS containing A . By associativity, the pair $(\alpha, (\beta, \gamma))$ is tensor, hence by Fact 2.2 and assumption there is $n \in \mathbb{N}$ such that $A \in u(f(n, \beta), f(n, \gamma))$ and $f(n, \beta) \neq f(n, \gamma)$. The latter, together with $(\beta, \gamma) \models \text{TS}$, gives us by Proposition 2.12(2) that $(f(n, \beta), f(n, \gamma)) \models \text{TS}$, hence the conclusion. $\square$

Fact 2.5 combined with Proposition 2.17 allows us to build explicit Ramsey pairs that are not tensor pairs.

Example 2.18. Let $u, v \in \beta\mathbb{N}$ be non-principal and let $w \in \beta\mathbb{N}^2$ be the image (or pushforward) of $u \otimes v \otimes v$ under the function $h : \mathbb{N}^3 \rightarrow \mathbb{N}^2$ defined as $h(x, y, z) = (2^x 3^y, 2^x 3^z)$. That is, for all $A \subseteq \mathbb{N}^2$

$$A \in w \Leftrightarrow \{(a, b, c) \mid (2^a 3^b, 2^a 3^c) \in A\} \in u \otimes v \otimes v.$$

To check that $w \in \text{RW}$, let (α, β, γ) be a generator of $u \otimes v \otimes v$; so (α, β, γ) is a tensor triple and $\beta \sim \gamma$. Therefore, by Proposition 2.17 we have $(2^\alpha 3^\beta, 2^\alpha 3^\gamma) \models \text{RW}$, and by Fact 2.5 applied with the 2-adic valuation we obtain $(2^\alpha 3^\beta, 2^\alpha 3^\gamma) \not\models \text{TS}$.

Remark 2.19. By Example 2.18 the inclusion $\text{TS} \subseteq \text{RW}$ is strict, hence TS is not closed.

3. Ramsey partition regularity

Throughout this section, we will consider formulas with multiple variables playing different roles; all of these are first order formulas in the full language on $\mathbb{N}$. To avoid handling a terrible amount of indices, and to increase the readability, we will use the notation $\vec{x}$ to denote a list of variables of some length $l(\vec{x})$, whose i -th variable, for $i \leq l(\vec{x})$, will be denoted by $(\vec{x})_i$; namely, $\vec{x} = ((\vec{x})_1, \dots, (\vec{x})_{l(\vec{x})})$. We explicitly allow the degenerate case $l(\vec{x}) = 0$. Similar notations will be used for lists of numbers; in this case, we will write $\vec{a} \in A$ as a shortened notation for $\vec{a} \in A^{l(\vec{a})}$.

The most studied problem in arithmetic Ramsey theory regards the partition regularity of formulas. Often, a related notion that is considered is that of partial partition regularity:

Definition 3.1. A formula $\varphi(\vec{x}, \vec{y})$ is *partially partition regular* (PPR) in the variables $\vec{x}$ if for all finite partitions $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exist $i \leq r$, $\vec{a} \in C_i$, and $\vec{b} \in \mathbb{N}$ such that $\varphi(\vec{a}, \vec{b})$ holds.

Example 3.2. As recently shown in [FKM25], the formula $x^2 + y^2 = z^2$ is PPR in the variables x, y .

Partial partition regularity can be seen as a particular case of the following general definition:

Definition 3.3. A formula $\varphi(\vec{x}_1, \dots, \vec{x}_n)$ is *block partition regular* in the tuples of variables $\vec{x}_1, \dots, \vec{x}_n$ (notation: BPR in $\vec{x}_1 \mid \dots \mid \vec{x}_n$) if for all finite partitions $\mathbb{N} = C_1 \cup \dots \cup C_r$, for every $j \leq n$ there exist $i_j \leq r$ and $\vec{a}_{i_j} \in C_{i_j}$ such that $\varphi(\vec{a}_1, \dots, \vec{a}_n)$.

PR is the case where $n = 1$, PPR in $\vec{x}_1$ is the case where $l(\vec{x}_i) = 1$ for all $i \geq 2$.

Remark 3.4. In Definition 3.3 a partition of the variables of φ is given; it is trivial to observe that this notion is preserved if the partition is refined, namely: if the variables appearing in $\vec{x}_i$ are partitioned in $\vec{y}_{i,1}, \vec{y}_{i,2}$ and φ is BPR in $\vec{x}_1 \mid \dots \mid \vec{x}_n$, then it is BPR in $\vec{x}_1 \mid \dots \mid \vec{x}_{i-1} \mid \vec{y}_{i,1} \mid \vec{y}_{i,2} \mid \vec{x}_{i+1} \mid \dots \mid \vec{x}_n$.

Remark 3.5. It is easy to see that Fact 1.1 generalises to block partition regularity as follows. A formula φ is BPR in $\vec{x}_1 \mid \dots \mid \vec{x}_n$ if and only if, for $i \leq n$, there are tuples $\vec{\alpha}_i \in {}^*\mathbb{N}$ such that, for every i , we have $(\vec{\alpha}_i)_1 \sim \dots \sim (\vec{\alpha}_i)_{l(\vec{\alpha}_i)}$.

In this paper, we are not going to study general properties of block partition regularity; in fact, as said in the introduction, we are mainly interested to use Ramsey pairs to study “infinitary statements” in combinatorics close in spirit to [Ramsey’s Theorem for pairs](#). “Infinitary statements” is a vague concept, that could be formalised in several different ways; in this paper, we define it as the following strengthening of block partition regularity:

Definition 3.6. We say that $\varphi(x, y, \vec{z}_1, \dots, \vec{z}_n)$ is *Ramsey partition regular* (Ramsey PR) in $(x, y), \vec{z}_1 \mid \dots \mid \vec{z}_n$ if for all finite colourings $\mathbb{N} = C_1 \cup \dots \cup C_r$ there are a colour $i_1 \leq r$, an infinite set $H \subseteq C_{i_1}$, and colours $i_2, \dots, i_n$ such that the following holds: for all $h_1 < h_2 \in H$ there exist $\vec{a}_1 \in C_{i_1}, \dots, \vec{a}_n \in C_{i_n}$ satisfying $\varphi(h_1, h_2, \vec{a}_1, \dots, \vec{a}_n)$.

Remark 3.7. Let us make a few comments about the above definition.

1. If $\varphi(x, y, \vec{z}_1, \dots, \vec{z}_n)$ is Ramsey PR in $(x, y), \vec{z}_1 \mid \dots \mid \vec{z}_n$, then it is in particular BPR in $x, y, \vec{z}_1 \mid \dots \mid \vec{z}_n$.
2. In Definition 3.6, $\vec{z}_1$ plays a special role: when searching for “block monochromatic witnesses” of $\varphi(x, y, \vec{z}_1, \dots, \vec{z}_n)$, we search for them with $x, y, \vec{z}_1$ monochromatic; in particular, when $n = 1$, if the formula $\varphi(x, y, \vec{z}_1)$ is Ramsey PR in $(x, y), \vec{z}_1$ then it is in particular PR, but much more is true. For example, the fact that the formula $x + y = z$ is PR is the content of Schur’s Theorem, whilst the fact that $x + y = z$ is Ramsey PR in $(x, y), z$ amounts to the fact that, if we denote $\text{PS}(H) := \{h_1 + h_2 \mid h_1 < h_2 \in H\}$, then for every finite colouring of $\mathbb{N}$ there are an infinite set H and a colour C such that $H \cup \text{PS}(H) \subseteq C$.

3. In our definition, it is allowed to have $l(\vec{z}_1) = 0$. In this case, Definition 3.6 could be slightly simplified: the request of monochromaticity on H could be dropped, as if an infinite set H with the property of Definition 3.6 exists, then a monochromatic infinite set with the same property exists by the pigeonhole principle.
4. The analogue of Remark 3.4 holds also for Ramsey partition regularity, provided one does not split (x, y) into separate blocks.

In Section 4, we will see which of the above partition regularity notions are satisfied by an important and well-studied example, namely the formula

$$\varphi(x, y, z, t) := (x + y = z) \wedge (x \cdot y = t) \wedge x \neq y.$$

As recalled in the introduction, partition regularity problems can be rephrased both in ultrafilters and in nonstandard terms. The main result of this section is the analogous characterisation of Ramsey partition regularity, which will be central for our applications.

Theorem 3.8. *Let $\varphi(x, y, \vec{z}_1, \dots, \vec{z}_n)$ be given. The following are equivalent:*

1. $\varphi(x, y, \vec{z}_1, \dots, \vec{z}_n)$ is Ramsey PR in $(x, y), \vec{z}_1 \mid \dots \mid \vec{z}_n$; that is, for all finite colourings $\mathbb{N} = C_1 \cup \dots \cup C_r$ there are a colour $i_1 \leq r$, an infinite set $H \subseteq C_{i_1}$, and colours $i_2, \dots, i_n$ such that for all $h_1 < h_2 \in H$ there exist $\vec{a}_1 \in C_{i_1}, \dots, \vec{a}_n \in C_{i_n}$ satisfying $\varphi(h_1, h_2, \vec{a}_1, \dots, \vec{a}_n)$.
2. For all finite colourings $\mathbb{N} = C_1 \cup \dots \cup C_r$ there is an infinite set H such that for all $h_1 < h_2 \in H$ there exist colours $i_1, \dots, i_n$ with $h_1, h_2 \in C_{i_1}$, and there exist $\vec{a}_1 \in C_{i_1}, \dots, \vec{a}_n \in C_{i_n}$ satisfying $\varphi(h_1, h_2, \vec{a}_1, \dots, \vec{a}_n)$.
3. There is $(\alpha, \beta) \models \text{RW}$ and there are $\vec{\gamma}_1, \dots, \vec{\gamma}_n \in {}^*\mathbb{N}$ such that
 - for all $i \leq n$, for all $h, k \leq l(\vec{z}_i)$ we have $(\vec{\gamma}_i)_h \sim (\vec{\gamma}_i)_k$;
 - if $l(\vec{z}_1) \geq 1$ then $\alpha \sim (\vec{\gamma}_1)_1$; and
 - $\varphi(\alpha, \beta, \vec{\gamma}_1, \dots, \vec{\gamma}_n)$ holds.
4. There exist $u \in \text{RW} \subseteq \beta\mathbb{N}^2$ and $u_1, \dots, u_n \in \beta\mathbb{N}$ such that
 - $\pi_1(u) = u_1$, where π_1 denotes the projection on the first coordinate; and
 - for all $A \in u, B_1 \in u_1, \dots, B_n \in u_n$ there exists an infinite set $H \subseteq B_1$ such that $[H]^2 \subseteq A$ and for all $h_1 < h_2 \in H$ there are $\vec{b}_1 \in B_1, \dots, \vec{b}_n \in B_n$ such that $\varphi(h_1, h_2, \vec{b}_1, \dots, \vec{b}_n)$ holds.

Proof. Left to right in 1 $\Leftrightarrow$ 2 is clear, as in 1 the colours $i_1, \dots, i_n$ do not depend on h_1, h_2 , while in 2 they may, and right to left is a standard application of Ramsey's Theorem for pairs.

1 $\Rightarrow$ 3 Given a finite partition P of $\mathbb{N}$, say that a tuple of natural numbers is P -monochromatic if they all belong to the same piece of P . Define

$$\begin{aligned} \Lambda_P := \left\{ (h_1, h_2, \vec{z}_1, \dots, \vec{z}_n) \mid h_1 < h_2 \wedge \varphi(h_1, h_2, \vec{z}_1, \dots, \vec{z}_n) \wedge \right. \\ \wedge (h_1, h_2, \vec{z}_1 \text{ is } P\text{-monochromatic}) \\ \wedge (\vec{z}_2 \text{ is } P\text{-monochromatic}) \\ \wedge \dots \\ \left. \wedge (\vec{z}_n \text{ is } P\text{-monochromatic}) \right\}. \end{aligned}$$

The family $\{\Lambda_P \mid P \text{ partition of } \mathbb{N}\}$, ordered by inclusion, is downward directed because $\bigcap_{i \leq \ell} \Lambda_{P_i}$ contains Λ_Q , where Q is the common refinement of the partitions $P_1, \dots, P_\ell$. By assumption, as pointed out immediately after Definition 1.2, if we denote by π the projection on the first two coordinates, each $\pi(*\Lambda_P)$ contains a Ramsey pair. It follows that $\{(\alpha, \beta) \mid (\alpha, \beta) \models \text{RW}\} \cap \bigcap_P \pi(*\Lambda_P)$ is the intersection of a family of closed sets with the FIP, hence contains a pair (α, β) , which is by construction Ramsey. Finally, by saturation and downward directedness, $\bigcap_P \pi(*\Lambda_P) = \pi(\bigcap_P (*\Lambda_P))$, and for any (α, β) in $\{(\alpha, \beta) \mid (\alpha, \beta) \models \text{RW}\} \cap \pi(\bigcap_P (*\Lambda_P))$ we can find the required $\vec{\gamma}_i$.

3 $\Rightarrow$ 4 Let $u = u(\alpha, \beta)$ and, for $i \leq n$, let $u_i = u((\vec{\gamma}_i)_1)$. As $\pi_1(u(\alpha, \beta)) = u(\alpha)$, and $\alpha \sim (\vec{\gamma}_1)_1$ by hypothesis, we have that $\pi_1(u) = u_1$. Given $A \in u, B_1 \in u_1, \dots, B_n \in u_n$, define

$$I := \{(a, b) \in A \mid a, b \in B_1 \wedge \exists \vec{z}_1 \in B_1, \dots, \exists \vec{z}_n \in B_n \varphi(a, b, \vec{z}_1, \dots, \vec{z}_n)\}.$$

Because $\varphi(\alpha, \beta, \gamma_1, \dots, \gamma_m)$ holds we have that $(\alpha, \beta) \in *I$, hence $I \in u$. As $u \in \text{RW}$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $[H]^2 \subseteq I$, which gives us the conclusion.

4 $\Rightarrow$ 1 Let the partition $\mathbb{N} = C_1 \cup \dots \cup C_r$ be given. For $j \leq n$, let i_j be such that $C_{i_j} \in u_j$. This suffices to prove the conclusion. $\square$

Theorem 3.8 reduces Ramsey partition regularity problems to existence problems for Ramsey pairs with certain combinatorial properties. We are going to exploit this idea, look at formulas of the form $\bigwedge_{i \leq n} f_i(x, y) = z_i$, and study their Ramsey partition regularity in $(x, y) \mid z_1, \dots, z_n$. By Theorem 3.8, this can be rephrased as follows.

Corollary 3.9. *Let $f_1, \dots, f_n : \mathbb{N}^2 \rightarrow \mathbb{N}$. The following are equivalent:*

1. *The formula $\bigwedge_{i \leq n} f_i(x, y) = z_i$ is Ramsey PR in $(x, y) \mid z_1, \dots, z_n$.*
2. *There is a Ramsey pair (α, β) such that $f_1(\alpha, \beta) \sim \dots \sim f_n(\alpha, \beta)$.*
3. *There is $w \in \text{RW}$ such that $f_1(w) = \dots = f_n(w)$.*

Note that, if one of the f_i is the projection on the first (or second) coordinate, then we are actually studying the Ramsey partition regularity in $(x, y), z_1, \dots, z_n$. One might wonder whether, in condition 3, RW could be strengthened to TS. This is not the case, as we will see in Remark 5.13.

4. Pairwise sums and products: a case study

In the language introduced in this paper, the [Pairwise Sum-Product Theorem](#) says that the formula $\varphi(x, y, z, t) := (x + y = z) \wedge (x \cdot y = t)$ is not Ramsey PR in $(x, y) \mid z, t$. By Corollary 3.9 this is, in disguise, a result about Ramsey pairs, namely the following.

Theorem 4.1. *Let $(\alpha, \beta) \models \text{RW}$. Then $\alpha + \beta \approx \alpha\beta$.*

We show how our techniques can be used to give a short nonstandard proof of this, and in fact prove a much stronger result. Let us begin by fixing some notation.

Definition 4.2. Let $p, m \in \mathbb{N} \setminus \{1\}$, with p prime. We denote by v_p the p -adic valuation, and we let $\ell_m(x) := \lfloor \log_m(x) \rfloor$, both regarded as functions $\mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$. We denote by $\text{smod}_p : \mathbb{N} \rightarrow \{1, \dots, p-1\}$ the function that maps x to $x/p^{v_p(x)} \pmod p$.

When p, m are clear from context, we write ℓ, v, smod for brevity.

Note that, if x is written in base p , and we count digit positions right to left⁴ and starting from 0, then $v_p(x)$ is the position of the first nonzero digit, $\text{smod}_p(x)$ is that digit, and $\ell_p(x)$ is the position of the last nonzero digit.

Lemma 4.3. *Let $n, m \in \mathbb{Z} \setminus \{0\}$ with $n + m \neq 0$ and $p > |n| + |m|$ a standard prime. If $\alpha \sim \beta$ and $v_p(\alpha) \leq v_p(\beta)$, then $v_p(n\alpha + m\beta) = v_p(\alpha)$.*

Proof. Write $v := v_p$ and $\text{smod} := \text{smod}_p$. If $v(\alpha) < v(\beta)$ we conclude by basic valuation theory and the fact that $p \nmid n$, so we may assume that $v(\alpha) = v(\beta)$. Observe that $v(n\alpha + m\beta) \geq v(\alpha)$. Moreover, as smod has finite image and $\alpha \sim \beta$, we have $\text{smod}(\alpha) = \text{smod}(\beta)$, hence the digit in position $v(\alpha)$ of $n\alpha + m\beta$ is $(n + m) \text{smod}(\alpha) \pmod p$, which is not 0 by the assumptions on p and $n + m$. $\square$

The main result of this section is the following:

Theorem 4.4. *Let $(\alpha, \beta) \models \text{RW}$ and $f, g : \mathbb{N} \rightarrow \mathbb{Z}$. Then, provided that $f(\alpha), g(\alpha) \notin \mathbb{Z}$,*

$$(f(\alpha) + f(\beta), g(\alpha) + g(\beta)) \approx (f(\alpha), g(\beta)).$$

More generally, for every $k_1, k_2, k_3, k_4 \in \mathbb{Z}$,

$$(f(\alpha) + f(\beta) + k_1, g(\alpha) + g(\beta) + k_2) \approx (f(\alpha) + k_3, g(\beta) + k_4).$$

Proof. Up to replacing f by $-f$, and similarly for g , we may assume that $f(\alpha), g(\alpha) > \mathbb{N}$. By Fact 2.4(1), by replacing f by $f + k_3 - k_1$ and g by $g + k_4 - k_2$ we may assume that $k_1 = k_3$ and $k_2 = k_4$. By applying $(\text{id} - k_1, \text{id} - k_2)$ we then reduce to the case $k_1, k_2, k_3, k_4 = 0$.

Suppose that $(f(\alpha) + f(\beta), g(\alpha) + g(\beta)) \sim (f(\alpha), g(\beta))$. Fix a prime $p > 2$ and, for brevity, let $v := v_p$, and similarly for ℓ and smod .

Observe that $v(f(\alpha)) \neq v(f(\beta))$; in fact, by assumption $f(\alpha), f(\beta) \neq 0$ and, if they had different valuation, because $p > 2$ and $f(\alpha) \sim f(\beta)$, we would have $\text{smod}(f(\alpha) + f(\beta)) \equiv 2 \text{smod}(f(\alpha)) \pmod p$, contradicting $f(\alpha) + f(\beta) \sim f(\alpha)$.

⁴So “first” means “least significant”.

Thus, since $(\alpha, \beta) \models \text{RW}$, by Proposition 2.13(2) we have $v(f(\beta)) > \mathbb{N}(\alpha)$. Similarly, $v(g(\beta)) > \mathbb{N}(\alpha)$, which implies $v(g(\alpha) + g(\beta)) = v(g(\alpha))$. Now observe that

$$\ell(f(\alpha) + f(\beta)) \geq v(f(\beta)) > \ell(g(\alpha)) + 1 > v(g(\alpha)) = v(g(\alpha) + g(\beta)).$$

As the pairs are equivalent, it follows that $\ell(f(\alpha)) > v(g(\beta))$, against $v(g(\beta)) > \mathbb{N}(\alpha)$. $\square$

Theorem 4.1 can now be deduced from Theorem 4.4 as follows.

Proof of Theorem 4.1. Suppose by contradiction this is the case, fix a sufficiently large standard prime p and write $v := v_p$ and similarly for ℓ and smod . By applying the function (v, ℓ) we get $(v(\alpha\beta), \ell(\alpha\beta)) \sim (v(\alpha + \beta), \ell(\alpha + \beta))$. We have

- $v(\alpha\beta) = v(\alpha) + v(\beta)$.
- $\ell(\alpha\beta) = \ell(\alpha) + \ell(\beta) + \varepsilon$ with $\varepsilon \in \{0, 1\}$.
- $v(\alpha + \beta) = v(\alpha)$: in fact, $v(\alpha) \leq v(\beta)$ by Proposition 2.13(2), so we may apply Lemma 4.3.
- $\ell(\alpha + \beta) = \ell(\beta) + \varepsilon'$ with $\varepsilon' \in \{0, 1\}$.

Thus $(v(\alpha) + v(\beta), \ell(\alpha) + \ell(\beta) + \varepsilon) \sim (v(\alpha), \ell(\beta) + \varepsilon')$, and since $\ell(\alpha) > \mathbb{N}$ Theorem 4.4 implies that $v(\alpha) \in \mathbb{N} \cup \{0\}$.

The ultrafilter $u(\alpha)$ determines, for each power p^k , a remainder class modulo p^k . In other words, it determines the p -adic class $a \in \mathbb{Z}_p$ of α . Therefore, since $\alpha\beta \sim \alpha + \beta$ we obtain $a^2 = 2a$ in $\mathbb{Z}_p$. Because we proved above that $v(\alpha) \in \mathbb{N} \cup \{0\}$, it follows that $a = 2$.

The pair $(\gamma, \delta) := (\alpha - 2, \beta - 2)$ is Ramsey by Proposition 2.13(1), and $\alpha\beta \sim \alpha + \beta$ yields

$$\gamma\delta + 2(\gamma + \delta) \sim \gamma + \delta. \quad (4.1)$$

As $\alpha \notin \mathbb{N}$ we have $\gamma \neq 0$, and since the class of γ in $\mathbb{Z}_p$ is 0 it follows that $v(\gamma) > \mathbb{N}$. By Lemma 4.3 and the fact that $p > 2$,

$$v(\gamma\delta) > v(\gamma) = v(\gamma + \delta) = v(2(\gamma + \delta)).$$

Therefore, $\text{smod}(\gamma\delta + 2(\gamma + \delta)) = \text{smod}(2(\gamma + \delta))$, contradicting (4.1). $\square$

Let us observe that, with the same idea as in the proof of Theorem 4.4, one proves the following.

Remark 4.5. Let $(\alpha, \beta) \models \text{RW}$. For every f, g, h, k , if there are φ, ψ such that $\varphi(g(\beta)) \neq \varphi(g(\alpha))$ and $\varphi(k(\alpha, \beta)) \leq \psi(h(\alpha, \beta))$, then $(h(\alpha, \beta), k(\alpha, \beta)) \approx (f(\alpha), g(\beta))$. Otherwise we get $\varphi(g(\beta)) > \mathbb{N}(\alpha)$, and in particular $\varphi(g(\beta)) > \psi(f(\alpha))$, against $\varphi(k(\alpha, \beta)) \leq \psi(h(\alpha, \beta))$.

Blocks	Is it BPR?	Reason
x, y, z, t	Unknown	Open problem since the 1970's
$x, y, z \mid t$	Yes	Additive Schur's Theorem
$x, y, t \mid z$	Yes	Multiplicative Schur's Theorem
$x, z, t \mid y$	Yes	Moreira's Theorem
$x, y \mid z, t$	Unknown	Never considered
$x, z \mid y, t$	Unknown	Never considered
$x, t \mid y, z$	Unknown	Never considered
$x, y \mid z \mid t$	Yes	Trivial
$x, z \mid y \mid t$	Yes	Follows from $x, z, t \mid y$
$x, t \mid y \mid z$	Yes	Follows from $x, z, t \mid y$
$z, t \mid x \mid y$	Yes	Follows from $x, z, t \mid y$
$x \mid y \mid z \mid t$	Yes	Trivial

Table 4.1: Block Partition Regularity of $\varphi(x, y, z, t)$.

Theorem 4.1 says that the formula

$$\varphi(x, y, z, t) := (x + y = z) \wedge (x \cdot y = t) \wedge x \neq y.$$

is not Ramsey PR in $(x, y) \mid z, t$. We now look at its other block and Ramsey partition regularity properties.

As known, the partition regularity of $\varphi(x, y, z, t)$ is one of the major open problems in this area of combinatorics. It has been recently solved over $\mathbb{Q}$ [BS24]; see also [Alw24] for an extension to finite arbitrarily large sums and products. On $\mathbb{N}$, the most general result we are aware of is the following theorem proven by J. Moreira in [Mor17, Corollary 1.5], which we state here using the language of block partition regularity.

Theorem 4.6 (Moreira). *$\varphi(x, y, z, t)$ is BPR in $x, z, t \mid y$.*

Moreira's Theorem, together with some basic considerations, solves most block partition regularity problems for $\varphi(x, y, z, t)$. We list the problems and their status in Table 4.1. We write “Never considered” when we are not aware of any work on the problem and, for compactness, we use the following observation.

Remark 4.7. We may omit redundant choices of blocks, arising from the symmetry in x, y of the formula $\varphi(x, y, z, t)$, or from the fact that variables inside a block can be permuted.

The picture is much richer when it comes to the different kinds of Ramsey partition regularity one might ask for $\varphi(x, y, z, t)$. First, we summarise what happens when variables in the “Ramsey part” are in a block of their own. Three such cases need some simple computations to be carried out, which we do in the following lemma.

Lemma 4.8. *The formula $\varphi(x, y, z, t)$ is Ramsey PR in $(x, t) \mid y \mid z$ and $(x, z) \mid y \mid t$, but it is not Ramsey PR in $(z, t) \mid x \mid y$.*

Blocks	Is it Ramsey PR?	Reason
$(x, y) \mid z \mid t$	Yes	Trivial
$(x, t) \mid y \mid z$	Yes	Lemma 4.8
$(x, z) \mid y \mid t$	Yes	Lemma 4.8
$(z, t) \mid x \mid y$	No	Lemma 4.8
$(x, y) \mid z, t$	No	Theorem 4.1
$(x, z) \mid y, t$	Unknown	Never considered
$(x, t) \mid y, z$	Unknown	Never considered
$(z, t) \mid x, y$	No	Follows from $(z, t) \mid x \mid y$

Table 4.2: Ramsey partition regularity of $\varphi(x, y, z, t)$, case with separate Ramsey block.

Blocks	Is it Ramsey PR?	Reason
$(x, y), z \mid t$	Yes	Use additive idempotents
$(x, y), t \mid z$	Yes	Use multiplicative idempotents
$(x, z), y \mid t$	Yes	Lemma 4.10
$(x, z), t \mid y$	Unknown	Never considered
$(x, t), y \mid z$	Yes	Lemma 4.10
$(x, t), z \mid y$	Unknown	Never considered
$(z, t), x \mid y$	No	Follows from $(z, t) \mid x \mid y$
$(x, y), z, t$	No	Follows from $(x, y) \mid z, t$
$(x, z), y, t$	Unknown	Stronger than x, y, z, t
$(x, t), y, z$	Unknown	Stronger than x, y, z, t
$(z, t), x, y$	No	Follows from $(z, t) \mid x \mid y$

Table 4.3: Ramsey partition regularity of $\varphi(x, y, z, t)$, remaining cases.

Proof. For the Ramsey partition regularity in $(x, t) \mid y \mid z$, fix $(\alpha, \beta) \models \text{RW}$ and let $(x, t) = (2^\alpha, 2^\beta)$, while $y = 2^{\beta-\alpha}$ and $z = 2^\alpha + 2^{\beta-\alpha}$.

For the Ramsey partition regularity in $(x, z) \mid y \mid t$, fix any $(x, z) = (\alpha, \beta) \models \text{RW}$ and let $y = \beta - \alpha$ and $t = \alpha \cdot (\beta - \alpha)$.

For the non Ramsey partition regularity in $(z, t) \mid x \mid y$, assume by contrast that there are $(\alpha, \beta) \models \text{RW}$ and $\gamma, \delta \in {}^*\mathbb{N}$ such that $\alpha = \gamma + \delta$ and $\beta = \gamma\delta$. Then $\alpha^2 = (\gamma + \delta)^2 = \gamma^2 + 2\gamma\delta + \delta^2 > \gamma\delta = \beta$, against Proposition 2.13(4). $\square$

Table 4.2 lists all the cases where the “Ramsey part” is in a block of its own, except those excluded by Remark 4.7 or by the following Remark 4.9.

Remark 4.9. We may omit cases containing (z, x) , (t, x) or (t, z) which, if there were a pair $(\alpha, \beta) \models \text{RW}$ witnessing Ramsey PR, would yield $\alpha + \beta < \alpha$, $\alpha \cdot \beta < \alpha$, and $\alpha \cdot \beta < \alpha + \beta$ respectively.

Finally, we are left to consider the cases where variables in the “Ramsey part” share a block with other variables. Some such cases can be easily proven by simple computations.

Lemma 4.10. *The formula $\varphi(x, y, z, t)$ is Ramsey PR in $(x, z), y \mid t$ and $(x, t), y \mid z$.*

Proof. Let (α, β, γ) be a tensor triple of infinite equivalent elements.

For the $(x, z), y \mid t$ case we let $x = \beta - \alpha, z = \gamma - \alpha, y = \gamma - \beta$ and $t = (\beta - \alpha)(\gamma - \beta)$.

For the $(x, t), y \mid z$ case we additionally require that α is divisible by all $n \in \mathbb{N}$, and we let $x = \beta/\alpha, t = \gamma/\alpha, y = \gamma/\beta$ and $z = (\beta/\alpha) + (\gamma/\beta)$. $\square$

We use Lemma 4.10 to list all remaining cases in Table 4.3, again with the omissions arising from Remarks 4.7 and 4.9.

5. Ramsey PR of equations

In this section we apply the characterisations obtained in Corollary 3.9 to study the Ramsey partition regularity of several equations. We begin with polynomial ones. The two-variable case is trivial, for the following reason.

Fact 5.1. *If $\alpha \sim \beta$ are infinite, then $P(\alpha, \beta) = 0$ if and only if $\alpha = \beta$ and $P(x, y)$ is a multiple of $(x - y)$.*

Proof. By [ALB24, Corollary 3.2] and Fact 1.1. $\square$

Proposition 5.2. *No two-variable polynomial equation is Ramsey PR.*

Proof. We use Fact 5.1. Since every Ramsey pair (α, β) satisfies $\alpha < \beta$ and $\alpha \sim \beta$ (Proposition 2.13, points 2 and 7 respectively), no such pair can satisfy the formula $P(x, y) = 0$. $\square$

We start our discussion by fixing some notations and proving some simple, yet useful, general result.

Definition 5.3. For $P(x) \in \mathbb{Z}[x]$, we let $\deg(P)$ be its degree and $c(P)$ be its leading coefficient.

When evaluating polynomials at infinite points, it is natural to consider the following notion:

Definition 5.4. Let $\alpha, \beta \in {}^*\mathbb{N}$. We say that α, β are in the same *archimedean class*, and write $\alpha \asymp \beta$, if there exists $n \in \mathbb{N}$ such that $(1/n) \leq (\alpha/\beta) \leq n$. We write $\alpha \ll \beta$ if the archimedean class of α is strictly smaller than that of β , that is, if for all $n \in \mathbb{N}$ we have $n \cdot \alpha < \beta$.

A related way to think about $\asymp$ is the following: let us recall that a hyperreal $\rho \in {}^*\mathbb{R}$ is called *finite* if there exists $n \in \mathbb{N}$ such $|\rho| \leq n$. In this case, ρ admits a *standard part*, denoted by $\text{st}(\rho)$ and defined as the unique standard real r whose distance from ρ is *infinitesimal*, i.e. smaller than $1/n$ for all $n \in \mathbb{N}$. Saying $\alpha \asymp \beta$ means saying that α/β is finite and its standard part is not 0, while saying that $\alpha \ll \beta$ means that $\text{st}(\alpha/\beta) = 0$.

By Fact 2.4(2), if $\alpha \sim \beta$ and $q \in \mathbb{Q}$, then $\alpha = q\beta$ forces $q = 1$. The first part of next lemma is a simple generalisation of this fact where $=$ is replaced by $\asymp$.

Lemma 5.5. *Let $\alpha \sim \beta$ be infinite and $P, Q \in \mathbb{Z}[x]$ be polynomials of degrees $\deg(P) = n$ and $\deg(Q) = m$. If $n, m \geq 1$, then:*

1. *if $\alpha \asymp \beta$ then $\text{st}(\alpha/\beta) = 1$;*
2. *if $P(\alpha) \asymp Q(\beta)$ then $n = m$;*
3. *if $P(\alpha) \sim Q(\beta)$ then $P = Q$.*

Proof. Fix an arbitrary finite prime p and let $\ell := \ell_p$.

1. Write α, β in base p . In the notations of our previous section, $\ell(\alpha) \sim \ell(\beta)$. Hence either $\ell(\alpha) = \ell(\beta)$ or the value $|\ell(\alpha) - \ell(\beta)|$ is infinite, and the latter is impossible as in this case α/β would either be infinite or infinitesimal, against our hypothesis. Let $\lambda = \ell(\alpha) = \ell(\beta)$. As $\alpha \sim \beta$, for all finite i the coefficients of $p^{\lambda-i}$ in the base p expansions of α and β coincide; then, by overspill, there exists an infinite τ such that for every $i \leq \tau$ the $(\lambda-i)$ -th coefficient $c_{\lambda-i}$ in the base p -expansions of α and β is the same. Let $\gamma = \sum_{i=0}^{\tau} c_{\lambda-i} p^{\lambda-i}$. Then:

- $\alpha = \gamma + \varepsilon$, with $\varepsilon \ll \gamma$;
- $\beta = \gamma + \varepsilon'$, with $\varepsilon' \ll \gamma$.

Hence $\alpha/\beta = (\gamma + \varepsilon)/(\gamma + \varepsilon')$ has standard part 1.

2. First of all, we observe that

$$\alpha^n \asymp P(\alpha) \asymp Q(\beta) \asymp \beta^m,$$

so $\alpha^n \asymp \beta^m$. Therefore, $\ell(\alpha^n)$ is at finite distance from $\ell(\beta^m)$, so there is a finite $k \in \mathbb{Z}$ such that $n\ell(\alpha) + k = m\ell(\beta)$. As $\ell(\alpha) \sim \ell(\beta)$, if this happens then the equation $nx - my = k$ is PR. By Rado's Theorem, this can happen only when $n = m$ (and $k = 0$).

3. The assumptions imply $P(\alpha) \sim Q(\alpha)$. As polynomials are bounded-to-one, by Proposition 2.11 we must have $P(\alpha) = Q(\alpha)$. So $P(\alpha) - Q(\alpha) = 0$, and since α is infinite we must have $P = Q$. $\square$

Lemma 5.5 allows some simple, yet interesting, observations. For instance, we may combine it with Rado's Theorem and van der Waerden's Theorem to obtain the following propositions.

Proposition 5.6. *Let $a, b, c \in \mathbb{N}$.*

1. *If $a = c \neq b$, then for all finite partitions $\mathbb{N} = C_1 \cup \dots \cup C_r$ and all $N \in \mathbb{N}$, there exist $i \leq r$ and $x, y, z \in C_i$ with $x > Ny$ and such that $ax + by = cz$. Moreover, the analogous result with $x > Ny$ replaced by $y > Nx$ is false.*
2. *If $a + b = c$ then for all finite partitions $\mathbb{N} = C_1 \cup \dots \cup C_r$ and all $\varepsilon > 0$ there are $i \leq r$ and $x, y, z \in C_i$ with $|(x/y) - 1|, |(x/z) - 1| < \varepsilon$ and such that $ax + by = cz$. Moreover, for all $N > 1$, there is a finite partition $\mathbb{N} = C_1 \cup \dots \cup C_r$ such that for all $i \leq r$ and all $x, y, z \in C_i$ with $x > Ny$ we have $ax + by \neq cz$. Similarly with x, y replaced, in $x > Ny$, by any two distinct elements of $\{x, y, z\}$.*

Proof. By Rado's Theorem, the equation $ax + by = cz$ is PR if and only if $a = c$, or $b = c$, or $a + b = c$. In nonstandard terms, this equation is PR if and only if there exist $\alpha \sim \beta \sim \gamma$ such that $a\alpha + b\beta = c\gamma$. Fix such α, β, γ .

1. It suffices to show that $\beta \ll \alpha$. If $\alpha \asymp \beta$ then $a\alpha + b\beta \asymp (a + b)\alpha$, so $c\gamma \asymp (a + b)\alpha$, hence by Lemma 5.5(1) we should have $a + b = c$, against the assumption. If $\alpha \ll \beta$, it similarly follows that $b = c$.

2. It suffices to show that $\text{st}(\alpha/\beta) = \text{st}(\beta/\gamma) = 1$, and by Lemma 5.5(1) it is enough to prove $\alpha \asymp \beta \asymp \gamma$. This follows easily from the equation $a\alpha + b\beta = (a + b)\gamma$ and again Lemma 5.5(1). $\square$

Proposition 5.7. *For every $N \in \mathbb{N}$, every $k > 1$, and every finite partition $\mathbb{N} = C_1 \cup \dots \cup C_r$, there exist $i \leq r$ and $x, y \in \mathbb{N}$ such that $x > Ny$ and for all $0 \leq j \leq k$ we have $x + jy \in C_i$. Moreover, the analogous result with $x > Ny$ replaced by $y > Nx$ is false.*

Proof. By the nonstandard characterisation of van der Waerden's Theorem there exists α, β such that for all $n \in \mathbb{N}$ we have $\alpha \sim \alpha + n\beta$. Arguing as in the proof of Proposition 5.6 we deduce that $\beta \ll \alpha$. From this, the conclusion follows easily. $\square$

The last technical result that we will need is the following:

Lemma 5.8. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be such that $\deg(P) = \deg(Q) = d > 1$. Assume that $c(P), c(Q) > 0$ and let $c := c(P)/c(Q)$. For all $\beta \in {}^*\mathbb{N} \setminus \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $Q(\sqrt[d]{c}\beta - n) \leq P(\beta) \leq Q(\sqrt[d]{c}\beta + n)$.*

Proof. Let $c'(P), c'(Q)$ be the coefficients of x^{d-1} in P, Q respectively. Observe that, as β is infinite, our hypotheses on P, Q give us that $Q(\sqrt[d]{c}x - n), P(x), Q(\sqrt[d]{c}x + n)$ all have the same leading monomial, hence the inequality in the conclusion holds if and only if the same inequality holds for the coefficients of x^{d-1} in $Q(\sqrt[d]{c}x - n), P(x), Q(\sqrt[d]{c}x + n)$ respectively. This coefficient is $(c'(Q) - dn \cdot c(Q))c^{\frac{d-1}{d}}$ in $Q(\sqrt[d]{c}x - n)$ and it is $(c'(Q) + dn \cdot c(Q))c^{\frac{d-1}{d}}$ in $Q(\sqrt[d]{c}x + n)$. To conclude, it suffices to find some n such that

$$(c'(Q) - dn \cdot c(Q))c^{\frac{d-1}{d}} \leq c'(P) \leq (c'(Q) + dn \cdot c(Q))c^{\frac{d-1}{d}}.$$

Because $d \cdot c(Q)$ and c are positive, it is enough to take n sufficiently large. $\square$

We are now ready to discuss the Ramsey partition regularity of polynomial equations in three variables. Our first result is very general, as it involves arbitrary functions in x .

Theorem 5.9. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be nonzero polynomials with $P(0) = Q(0) = 0$. Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ and let $\varphi(x, y, z) := (f(x) + P(y) = Q(z)) \wedge (f(x) \neq 0)$. If $\varphi(x, y, z)$ is Ramsey PR in $(x, y), z$ then $P = Q$ and $\deg(P) = 1$.*

Proof. Let $d_P = \deg(P), d_Q = \deg(Q)$. By Theorem 3.8, as $\varphi(x, y, z)$ is Ramsey PR in $(x, y), z$ there exist $(\alpha, \beta) \models \text{RW}$ and $\gamma \sim \alpha$ such that $f(\alpha) + P(\beta) = Q(\gamma)$ and $f(\alpha) \neq 0$; note that, by Proposition 2.13(4), $f(\alpha) \ll \beta$. Hence

$$\beta^{d_P} \asymp f(\alpha) + P(\beta) = Q(\gamma) \asymp \gamma^{d_Q}.$$

By Lemma 5.5(2) we deduce that $d_Q = d_P$.

Observe that the sign of $f(\alpha) + P(\beta)$ coincides with that of $c(P)$, and that of $Q(\gamma)$ coincides with that of $c(Q)$; as $f(\alpha) + P(\beta) = Q(\gamma)$, necessarily $c := c(P)/c(Q) > 0$. Without loss of generality, both $c(P)$ and $c(Q)$ are positive.

Let $d = d_Q = d_P$. Assume that $d > 1$. By Lemma 5.8 applied twice, once to the polynomials $P(x) - x$ and $Q(x)$ and once to the polynomials $P(x) + x$ and $Q(x)$, there is $n \in \mathbb{N}$ such that

$$Q(\sqrt[d]{c}\beta - n) \leq P(\beta) - \beta \leq f(\alpha) + P(\beta) \leq P(\beta) + \beta \leq Q(\sqrt[d]{c}\beta + n).$$

So $Q(\sqrt[d]{c}\beta - n) \leq Q(\gamma) \leq Q(\sqrt[d]{c}\beta + n)$, and as Q is increasing on ${}^*\mathbb{N} \setminus \mathbb{N}$, this entails that $\gamma \in [\sqrt[d]{c}\beta - n, \sqrt[d]{c}\beta + n]$. So there exists $i \in [-n, n]$ such that $\gamma = \lfloor \sqrt[d]{c}\beta \rfloor + i$, in particular, $\lfloor \sqrt[d]{c}\beta \rfloor + i \sim \beta$, hence $\lfloor \sqrt[d]{c}\beta \rfloor + i = \beta$. As β is infinite, Fact 2.4(2) gives $\sqrt[d]{c} = 1$ and $i = 0$, hence $\gamma = \beta$. But then $f(\alpha) = (Q - P)(\beta)$, which forces $Q - P$ to be constant by Proposition 2.13(6). Since $P(0) = 0 = Q(0)$, this forces $Q = P$, which gives $f(\alpha) = 0$, against the assumptions. Hence $d = 1$.

So we are in the following case:

$$f(\alpha) + c(P)\beta = c(Q)\gamma.$$

As $c(Q)\gamma \sim c(Q)\beta$ and $f(\alpha) \ll \beta$, we conclude by Lemma 5.5(1). $\square$

In the particular case where $f(\alpha)$ is a monomial in α , we can say more:

Corollary 5.10. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be nonzero polynomials with $P(0) = Q(0) = 0$. Let $a, n \in \mathbb{N}$, and let $\varphi(x, y, z) := ax^n + P(y) = Q(z)$. If $\varphi(x, y, z)$ is Ramsey PR in $(x, y), z$ then $\varphi(x, y, z)$ is an integer multiple of $x + y = z$ or of $-x + y = z$.*

Proof. Straight from Theorem 5.9, and using the same notations, we get that, by our hypotheses, there is some $c \in \mathbb{Z}$ such that $\varphi(x, y, z)$ is $ax^n + cy = cz$. Again, by Theorem 3.8, as $\varphi(x, y, z)$ is Ramsey PR in $(x, y), z$ there exist $(\alpha, \beta) \models \text{RW}$ and $\gamma \sim \alpha$ such that $a\alpha^n + c\beta = c\gamma$.

Fix a prime number p much larger than a, c (in particular, $v(a) = 0 = v(c)$), and write α, β, γ in base p . In the same notation of Section 4, as $(\alpha, \beta) \models \text{RW}$, by Proposition 2.13(1) and Proposition 2.13(4) we have only two possible cases: $v(\alpha) = v(\beta)$ or $v(\alpha) \ll v(\beta)$.

Case $v(\alpha) = v(\beta)$. We divide this case in the two subcases $n > 1$ and $n = 1$.

If $n > 1$, we observe that $v(a\alpha^n) = v(a) + nv(\alpha) = nv(\alpha)$. If $v(\alpha) \neq 0$ then $nv(\alpha) > v(\alpha) = v(\beta)$, so necessarily $v(\beta) = v(\gamma)$. The same also holds if $v(\alpha) = 0$ because $\alpha \sim \beta \sim \gamma$. Let $d(t)$ be the coefficient of $p^{nv(t)}$ in the base p expansion of t . The $nv(\gamma)$ position in the base p expansions of both sides of the equation $a\alpha^n + c\beta = c\gamma$ gives us that $a \text{ smod}(\alpha)^n + d(c\beta) \equiv d(c\gamma) \pmod{p}$. As d has finite image we have $d(c\beta) = d(c\gamma)$. This implies $a \text{ smod}(\alpha)^n \equiv 0 \pmod{p}$, which cannot happen since $a, \text{ smod}(\alpha)$ are not divided by p .

If $n = 1$, then $v(a\alpha^n) = v(\alpha) = v(\beta)$. Using the same notations as above, if $v(\beta) = v(\gamma)$ then again the equality $a\alpha + c\beta = c\gamma$ gives us $(a + c) \text{ smod}(\alpha) \equiv c \text{ smod}(\alpha) \pmod{p}$, which is impossible. Hence $v(\beta) < v(\gamma)$, so it must be $ad + cd \equiv 0 \pmod{p}$, namely $a \equiv -c \pmod{p}$. As p is much larger than a, c , this tells us that $a = -c$.

Case $v(\alpha) \ll v(\beta)$. In this case, $v(\gamma) = v(a\alpha^n + c\beta) = nv(\alpha)$ which, as $v(\gamma) \sim v(\alpha)$, forces $n = 1$. It follows that $a \text{ smod}(\alpha) \equiv c \text{ smod}(\gamma) \pmod{p}$, which forces $a \equiv c \pmod{p}$. Again, as p is much larger than a, c , this tells us that $a = c$. $\square$

The above result tells us that several well-known PR equations of the form $ax^n + P(y) = Q(z)$ are not Ramsey PR in $(x, y), z$, for example all equations of the form $x - y = Q(z)$ for $\deg(Q) > 1$ and $Q(0) = 0$ (see [BFM96] for a proof of their partition regularity). Interestingly, such equations can be Ramsey PR over $\mathbb{Q}$, as discussed in the introduction. The reason that our argument breaks over $\mathbb{Q}$ is simply due to the fact that $\mathbb{Q}$ contains numbers of arbitrarily small (negative) p -adic valuation.

As for $\pm x + y = z$ (over $\mathbb{N}$), we have the following.

Proposition 5.11. *Both $x + y = z$ and $-x + y = z$ are Ramsey PR in $(x, y), z$.*

Proof. As always, by Theorem 3.8 it suffices to find $(\alpha, \beta) \models \text{RW}$ and $\gamma \sim \alpha$ such that $\alpha + \beta = \gamma$ or $-\alpha + \beta = \gamma$. The $x + y = z$ case is well-known: it suffices to take α such that $u(\alpha)$ is additively idempotent⁵ and β such that $(\alpha, \beta) \models \text{TS}$, so that by letting $\gamma = \alpha + \beta$ we have that $\gamma \sim \alpha$.

For $-x + y = z$, pick any infinite $\eta_1 \sim \eta_2 \sim \eta_3$ with (η_1, η_2, η_3) tensor. Let $\alpha = \eta_2 - \eta_1$, $\beta = \eta_3 - \eta_1$, $\gamma = \eta_3 - \eta_2$. Then $(\alpha, \beta) \models \text{RW}$ by Proposition 2.17 and $\alpha \sim \beta \sim \gamma$ as they are all images of equivalent pairs under the same standard function. Clearly, $\beta - \alpha = \gamma$. $\square$

Corollary 5.12. *The equations $xy = z$ and $y = zx$ are Ramsey PR in $(x, y), z$.*

Proof. By Proposition 5.11 there is $(\alpha, \beta) \models \text{RW}$ with $\pm\alpha + \beta \sim \alpha$. It then suffices to consider the pair $(2^\alpha, 2^\beta)$. $\square$

The equation $-x + y = z$ is the first example we are aware of with the following property.

Remark 5.13. The formula $-x + y = z$ is Ramsey PR in $(x, y), z$, but there are no $(\alpha, \beta) \models \text{TS}$ such that $-\alpha + \beta \sim \alpha$. Namely, the equation $(-u) \oplus u = u$ has no solutions in $\beta\mathbb{Z} \setminus \mathbb{Z}$.

Proof. The fact that such (α, β) cannot exist is well-known, see e.g. [HS11, Corollary 13.19], but here is a quick argument. If (α, β) are as above, by Puritz' Theorem we have that $v_3(\beta) \in \mathbb{N}$ (otherwise $\text{smod}_3(\alpha) = \text{smod}_3(-\alpha) = 3 - \text{smod}_3(\alpha)$, which cannot happen). But then $v_3(\alpha) = v_3(\beta - \alpha)$, a contradiction because $\text{smod}_3(\beta) = \text{smod}_3(\alpha)$, hence $v_3(\beta - \alpha) > v_3(\alpha)$. $\square$

Corollary 5.10 has several more interesting consequences. One regards the non-Ramsey partition regularity of arithmetic progressions, not even in the simplest case of arithmetic progressions of length 3. In particular, there is no ‘‘Ramsey version’’ of van der Waerden’s Theorem.

Proposition 5.14. *The formula*

$$\varphi(x, y, z) := \text{‘‘}x, y, z \text{ form, in some order, an arithmetic progression of length 3’’}$$

is not Ramsey PR in $(x, y), z$.

⁵The existence of such ultrafilters is also well-known, see e.g. [HS11, Theorem 2.5].

Proof. First of all, let us observe that $\varphi(x, y, z)$ is equivalent to

$$(x + y = 2z) \vee (x + z = 2y) \vee (y + z = 2x), \quad (5.1)$$

whence by Theorem 3.8 to conclude it suffices to prove that there are no $\alpha \sim \beta \sim \gamma$ satisfying (5.1) and such that $(\alpha, \beta) \models \text{RW}$. But $\alpha + \beta = 2\gamma$ cannot happen due to Corollary 5.10 applied to $P(x) = x$ and $Q(x) = 2x$. The other two cases are dealt with similarly, by using Corollary 5.10 respectively with $P(x) = -2x$ and $Q(x) = -x$ and with $P(x) = -x$ and $Q(x) = x$. $\square$

Another application of Corollary 5.10 regards ultrafilter equations. It is known that equations of the form $au \oplus bu = cu$, where $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $u \in \beta\mathbb{N}$, can be solved only when $a = b = c$ (see [HS11, Theorem 13.18], and see [Mal00] for other results of this kind). By Corollary 5.10 we know that much more is true:

Proposition 5.15. *Let $P(x), Q(x) \in \mathbb{Z}[x]$ be nonzero polynomials with no constant term. Let $a, n \in \mathbb{N}$. The only equations of the form⁶ $au^n \oplus P(u) = Q(u)$ that have a solution in $\beta\mathbb{N}$ are the multiples of $u \oplus u = u$.*

Proof. Left to right, if the equation $au^n \oplus P(u) = Q(u)$ holds, then for $(\alpha, \beta) \models u \otimes u$ one has $a\alpha^n + P(\beta) \sim Q(\alpha)$. As $\text{TS} \subseteq \text{RW}$, Corollary 5.10 forces $\deg(P) = \deg(Q) = n = 1$ and $a = c(P) = c(Q)$. By Remark 5.13, $a = 1$. Right to left, any additively idempotent ultrafilter u solves the given equation. $\square$

Another application of Lemma 5.5 is the following generalisation of the **Pairwise Sum-Product Theorem**.

Proposition 5.16. *Let $n, m \in \mathbb{Z}$, $k, r, s \in \mathbb{N}$. Assume that $n + m \neq 0$. The following are equivalent.*

1. *There is $(\alpha, \beta) \models \text{TS}$ such that $n\alpha + m\beta \sim k\alpha^r\beta^s$.*
2. *There is $(\alpha, \beta) \models \text{RW}$ such that $n\alpha + m\beta \sim k\alpha^r\beta^s$.*
3. *$r = s = 1$, one of n, m equals 0, and the other one is positive.*

Proof. $1 \Rightarrow 2$ follows from Proposition 2.8.

For $2 \Rightarrow 3$, assume that $(\alpha, \beta) \models \text{RW}$. Fix a sufficiently large standard prime p and let v, ℓ, smod be relative to this prime. If $n\alpha + m\beta \sim k\alpha^r\beta^s$ then, by applying ℓ to both sides, and possibly replacing $\ell(\alpha)$ by $\ell(\beta)$ on the left hand side in the case where $m = 0$, we find $\varepsilon_0, \varepsilon_1 \in \mathbb{N} \cup \{0\}$ such that $\ell(\beta) + \varepsilon_0 \sim r\ell(\alpha) + s\ell(\beta) + \varepsilon_1$. By Lemma 5.5(1) we obtain $s = 1$. If $n, m \neq 0$, by Lemma 4.3 we obtain $v(n\alpha + m\beta) = v(\alpha)$, hence $v(\alpha) \sim rv(\alpha) + v(\beta)$. The same conclusion still holds if one of n, m equals 0, by an easy calculation. If $c = \text{smod}(v(\alpha))$

⁶Here u^n is the image of the ultrafilter u under the polynomial function $x \mapsto x^n$. This is distinct from $\underbrace{u \odot \dots \odot u}_{n \text{ times}}$.

this yields $c \equiv rc \pmod{p}$ or $c \equiv (r+1)c \pmod{p}$ depending on whether $v(v(\alpha)) = v(v(\beta))$ or not. As r is positive and much smaller than p , this implies $r = 1$.

We have now reduced to the case $n\alpha + m\beta \sim k\alpha\beta$. If n, m are both nonzero, an easy modification of the proof of Theorem 4.1 shows that the required (α, β) cannot be found. As $k\alpha\beta > 0$, if $n = 0$ then we must have $m > 0$, and symmetrically with the roles of n and m interchanged.

For $3 \Rightarrow 1$, we assume that $n = 0 < m$, the other case being analogous. We need to find $(\alpha, \beta) \models \text{TS}$ such that $m\beta \sim k\alpha\beta$. Let α' be such that $u(\alpha')$ is idempotent for $\odot$ and divisible by k , and let β' be such that $(\alpha', \beta') \models \text{TS}$. It then suffices to set $\alpha := m\alpha'/k$ and $\beta := m\beta'/k$. $\square$

Remark 5.17. If we allow $r, s \in \mathbb{Z}$, similar arguments as above yield that, assuming $n + m \neq 0$, if there is $(\alpha, \beta) \models \text{RW}$ with $n\alpha + m\beta \sim k\alpha^r\beta^s$ then $(r, s) \in \{(1, 1), (0, 1), (1, 0), (-1, 1)\}$. The cases $(0, 1)$ and $(1, 0)$ are equivalent and yield $n\alpha + m\beta \sim k\beta$. By Corollary 5.10 this entails $m = k = \pm n$. We do not know what happens in the case $(r, s) = (-1, 1)$, except for (trivial consequences of) Corollary 5.12.

Let us consider what happens when we relax the Ramsey partition regularity from $(x, y), z$ to $(x, y) \mid z$.

Proposition 5.18. *Let $a, b, c, k_1, k_2, k_3 \in \mathbb{N}$. The following are equivalent:*

1. $ax^{k_1} + by^{k_2} = cz^{k_3}$ is Ramsey PR in $(x, y) \mid z$;
2. $k_3 = 1$.

Proof. $1 \Rightarrow 2$ Assume, by contrast, that $k_3 > 1$. By Theorem 3.8 there exist $(\alpha, \beta) \models \text{RW}$ and $\gamma \in {}^*\mathbb{N}$ such that $a\alpha^{k_1} + b\beta^{k_2} = c\gamma^{k_3}$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as follows:

$$f(y) := \min\{z \in \mathbb{N} \mid cz^{k_3} \geq by^{k_2}\}.$$

We claim that $f(\beta) = \gamma$. In fact, trivially $f(\beta) \leq \gamma$. If $\gamma \geq f(\beta) + 1$ then $c\gamma^{k_3} \geq c(f(\beta) + 1)^{k_3}$. Write $c(f(\beta) + 1)^{k_3} = cf(\beta)^{k_3} + R(\beta)$, hence $a\alpha^{k_1} + b\beta^{k_2} \geq cf(\beta)^{k_3} + R(\beta)$. Observe that, as $k_3 \geq 2$, we have that $R(\beta)$ is a polynomial in β of degree at least 1. As $cf(\beta)^{k_3} \geq b\beta^{k_2}$ by definition of f , we deduce that $a\alpha^{k_1} \geq R(\beta)$, and this is in contrast with Proposition 2.13(6).

$2 \Rightarrow 1$ We just have to take $(\alpha, \beta) \models \text{RW}$ with $c \mid \alpha$ and let $\gamma = (a\alpha^{k_1} + b\beta^{k_2})/c$. $\square$

Example 5.19. Let us show two examples.

1. As recalled before, all equations of the form $-x + y = cz^n$ are PR, but the only ones that are Ramsey PR in $(x, y) \mid z$ are those of the form $-x + y = cz$, among which the only one that is Ramsey PR in $(x, y), z$ is $-x + y = z$.
2. Recall (Example 3.2) that the Pythagorean equation $x^2 + y^2 = z^2$ is PPR in x, y , equivalently, BPR in $x, y \mid z$. However, by Proposition 5.18 it is not Ramsey PR in $(x, y) \mid z$, hence it is also not Ramsey PR in $(x, y), z$.⁷

⁷This could have also been deduced directly from Corollary 5.10.

The following general result, close in spirit to the polynomial applications developed so far, shows that many equations involving sums of arbitrary functions are not Ramsey PR.

Proposition 5.20. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be such that $n \mapsto f(n+1) - f(n)$ is finite-to-one. Then for every $g : \mathbb{N} \rightarrow \mathbb{N}$ the equation $g(x) + f(y) = f(z)$ is not Ramsey PR in $(x, y) \mid z$.*

Proof. As always, by Theorem 3.8 it suffices to fix $(\alpha, \beta) \models \text{RW}$ and $\gamma \in {}^*\mathbb{N}$ and prove that $g(\alpha) + f(\beta) \neq f(\gamma)$. By Proposition 2.13(6) we have $f(\beta+1) - f(\beta) > g(\alpha)$, hence $f(\beta) < f(\beta) + g(\alpha) < f(\beta+1)$. It follows that $f(\beta) + g(\alpha)$ is not in the image of f . $\square$

Example 5.21. By Proposition 5.20 the equation $x + 2^y = 2^z$ is not Ramsey PR in $(x, y) \mid z$.

One may ask whether there is a version of Theorem 4.1 involving products and exponentials in place of sums and products. This, and much more, is indeed true, and can be proven by using the following observation.

Proposition 5.22. *Let α, β be infinite and $f, g : \mathbb{N}^2 \rightarrow \mathbb{N}$. If there are standard functions φ_0, φ_1, h such that*

1. $\varphi_0(f(\alpha, \beta)) = \varphi_1(\beta)$,
2. $\varphi_0(g(\alpha, \beta)) = h(\varphi_1(\beta))$, and
3. $h(\varphi_1(\beta)) \neq \varphi_1(\beta)$,

then $f(\alpha, \beta) \approx g(\alpha, \beta)$.

Proof. Otherwise $\varphi_0(f(\alpha, \beta)) \sim \varphi_0(g(\alpha, \beta))$, hence $\varphi_1(\beta) \sim h(\varphi_1(\beta))$, and this would imply $h(\varphi_1(\beta)) = \varphi_1(\beta)$. $\square$

Note that assumption 3 is true if, e.g., h has finitely many fixed points.

Let us see some example applications of Proposition 5.22 to some equations involving exponentiation. We begin with the promised “products and exponentials” version of Theorem 4.1.

Example 5.23. The formula $(x^y = z) \wedge (xy = t)$ is not Ramsey PR in $(x, y) \mid z, t$.

Proof. Let $f(x, y) = x^y$ and $g(x, y) = xy$, and suppose there is $(\alpha, \beta) \models \text{RW}$ with $f(\alpha, \beta) \sim g(\alpha, \beta)$. We apply Proposition 5.22 by showing that these two objects have different, but very close, exponential level (see e.g. [BM18, Definition 5.2]).

Let $\ell := \ell_2$, and denote by $\ell^{(k)}$ the k -fold composition of ℓ with itself. An easy computation shows that $\ell^{(3)}(f(\alpha, \beta))$ is at finite distance from $\ell^{(2)}(\beta)$, while $\ell^{(3)}(g(\alpha, \beta))$ is at finite distance from $\ell^{(3)}(\beta)$. From this observation we can conclude by applying Proposition 5.22 with $\varphi_0 := \ell^{(3)}$, $\varphi_1 := \ell^{(2)}$ and $h(x) := \ell(x) + n$ for a suitable finite $n \in \mathbb{Z}$. $\square$

Example 5.24. The formula $(x^y = z) \wedge (y^x = t)$ is not Ramsey PR in $(x, y) \mid z, t$.

Proof. This is similar to the proof of Example 5.23, with the same φ_0 and φ_1 . $\square$

In particular, there is no pair $(\alpha, \beta) \models \text{TS}$ such that $\alpha^\beta \sim \beta^\alpha$, solving a question from [LB25].

Example 5.25. The formula $(x2^y = z) \wedge (x + y = t)$ is not Ramsey PR in $(x, y) \mid z, t$.

Proof. If there were $(\alpha, \beta) \models \text{RW}$ with $\alpha 2^\beta \sim \alpha + \beta$, then $(\gamma, \delta) := (2^\alpha, 2^\beta)$ would be in RW and satisfy $\gamma^\delta \sim \gamma\delta$, against Example 5.23. $\square$

Although we deduced the previous example from Example 5.23, it is also a special case of the following.

Example 5.26. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary function, $m > 1$ a natural number, and $g(x, y) : \mathbb{N}^2 \rightarrow \mathbb{N}$ a function as follows. There is d such that, for every n , we have $g(n, y) \in \mathcal{O}(y^d) \cap \Omega(y^{\frac{1}{d}})$, that is, there are functions $h_0, h_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that for all n, y we have $h_0(n)y^{\frac{1}{d}} \leq g(n, y) \leq h_1(n)y^d$. If $(\alpha, \beta) \models \text{RW}$, then $f(\alpha) \cdot m^\beta \asymp g(\alpha, \beta)$.

Proof. This too is similar to the proof of Example 5.23, using $\varphi_0 = \ell_m^{(2)}$, $\varphi_1 = \ell_m$, and h of the form $\ell_m(x) + n$. $\square$

We conclude this section with a result on polynomial functions. We state it directly in terms of Ramsey's witnesses.

Proposition 5.27. Let $n, r \in \mathbb{Z}$, $m, s \in \mathbb{N}$, and $(\alpha, \beta) \models \text{RW}$.

1. If $n, r, n + m, r + s \neq 0$ and $n\alpha + m\beta \sim r\alpha + s\beta$, then $(n, m) = (r, s)$.
2. Let $f(x, y) = \sum_{i=0}^m f_i(x)y^i$ and $g(x, y) = \sum_{j=0}^s g_j(x)y^j$, where $f_i, g_j : \mathbb{N} \rightarrow \mathbb{Z}$ and $f_m(\alpha) \neq 0 \neq g_s(\alpha)$. If $f(\alpha, \beta) \sim g(\alpha, \beta)$, then $m = s$.
3. If $n, r > 0$ and $\alpha^n \beta^m \sim \alpha^r \beta^s$ then $(n, m) = (r, s)$.

Proof. 1. By Proposition 2.13(4) we have $n\alpha + m\beta \asymp \beta \asymp r\alpha + s\beta$. We apply Lemma 5.5(1) and conclude that $m = s$. Now let p be a finite prime much larger than $n + m + r$. Let $c := \text{smod}_p(\alpha) = \text{smod}_p(\beta)$. If $v_p(\alpha) < v_p(\beta)$, then $\text{smod}_p(n\alpha + m\beta) = nc$ and $\text{smod}_p(r\alpha + m\beta) = rc$, and it follows that $n = r$. If $v_p(\alpha) = v_p(\beta)$ then, by Lemma 4.3,

$$(n + m)c \equiv \text{smod}_p(n\alpha + m\beta) \equiv \text{smod}_p(r\alpha + m\beta) \equiv (r + m)c \pmod{p},$$

hence $n = r$.

2. Observe that $\ell_2(f(\alpha, \beta)) \sim \ell_2(g(\alpha, \beta))$. There is $\varepsilon \in \{0, 1\}$ such that $\ell_2(f(\alpha, \beta)) = m\ell_2(\beta) + \ell_2(f_m(\alpha)) + \varepsilon$, hence $\text{st}(f(\alpha, \beta)/m\ell_2(\beta)) = 1$. An analogous computation for $g(\alpha, \beta)$ shows that $\text{st}(g(\alpha, \beta)/s\ell_2(\beta)) = 1$. It follows from Lemma 5.5(1) applied to $f(\alpha, \beta)$ and $g(\alpha, \beta)$ that $\text{st}(f(\alpha, \beta)/g(\alpha, \beta)) = 1$, hence $m = s$.

3. We know from the previous point that $m = s$. Let $\pi(n)$ be the smallest prime dividing n . Observe that $\pi(\alpha^a \beta^b) = \pi(\alpha)$. Write

$$v_{\pi(\alpha)}(\alpha^a \beta^b) = av_{\pi(\alpha)}(\alpha) + bv_{\pi(\alpha)}(\beta). \quad (5.2)$$

If $\pi(\alpha) < \pi(\beta)$, then the last summand is 0. It follows that $nv_{\pi(\alpha)}(\alpha) \sim rv_{\pi(\alpha)}(\alpha)$, and since $v_{\pi(\alpha)}(\alpha) \neq 0$ we conclude by Proposition 2.11 that $n = r$.

If $\pi(\alpha) = \pi(\beta)$ we have two subcases. If $v_{\pi(\alpha)}(\alpha) < v_{\pi(\beta)}(\beta) = v_{\pi(\alpha)}(\beta)$ then this pair is in RW by Proposition 2.13(1). We then conclude by applying point 1 above to this pair, using (5.2). If instead $v_{\pi(\alpha)}(\alpha) = v_{\pi(\beta)}(\beta)$, then as $v_{\pi(\alpha)}(\beta) = v_{\pi(\beta)}(\beta)$, by (5.2) we have $(n + m)v_{\pi(\alpha)}(\alpha) = (r + m)v_{\pi(\alpha)}(\alpha)$, hence again $n = r$. $\square$

6. Open problems

The results obtained in this paper open some lines of enquiry that we believe worth investigating further.

Puritz' Theorem (Fact 2.5) gives a simple characterisation of tensor pairs in terms of standard functions. As for Ramsey pairs, in Proposition 2.13, we have shown that they have several properties that can be expressed in terms of standard functions. Most of our applications actually used some of these properties, so finding a Puritz-like characterisation of Ramsey pairs might be helpful in studying other similar combinatorial problems.

Problem 6.1. Characterise Ramsey pairs in the style of Puritz' Theorem.

Problem 6.1 and nonstandard characterisations of related Ramsey properties are the topic of a paper in preparation. Another interesting question regards polynomial equations.

Problem 6.2. Is it possible to extend Corollary 5.10 from monomials to arbitrary polynomials in x ? To arbitrary multivariate polynomials?

Our proof of Corollary 5.10 uses the fact that it is simple to write the valuation of a monomial in α in terms of the valuation of α ; the same does not hold for a generic polynomial in α . An extension of Corollary 5.10 to arbitrary polynomials in x would give an analogous extension of its corollaries, and in particular of Proposition 5.15.

As observed in Remark 5.17, some variants of the [Pairwise Sum-Product Theorem](#) remain open. The most interesting ones seem to be the following. We suspect that solving these should provide enough information to complete the classification.

Problem 6.3. Which of these formulas are Ramsey PR in $(x, y) \mid z, t$?

1. $(-x + y = z) \wedge (xy = t)$.
2. $(-x + y = z) \wedge (y = tx)$.
3. $(x + y = z) \wedge (y = tx)$.

Our final question essentially asks to deal with the “never considered” cases of Tables 4.1 to 4.3 from Section 4.

Problem 6.4. In which partitions of the variables is the formula $(x \neq y) \wedge (x + y = z) \wedge (x \cdot y = t)$ BPR? In which ones is it Ramsey PR?

Acknowledgements

We thank the anonymous referees for their careful reading of this manuscript and for many useful suggestions. We thank Ethan Ackelsberg for useful conversations around Theorem B.

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