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Higher Symplectic Geometry

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Christopher Lee Rogers

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To Rosie with love

ABSTRACT OF THE DISSERTATION

Higher Symplectic Geometry

by

Christopher Lee Rogers

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2011
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In higher symplectic geometry, we consider generalizations of symplectic manifolds called n-plectic manifolds. We say a manifold is n-plectic if it is equipped with a closed, nondegenerate form of degree $(n+1)$. We show that certain higher algebraic and geometric structures naturally arise on these manifolds. These structures can be understood as the categorified or homotopy analogues of important structures studied in symplectic geometry and geometric quantization. Our results imply that higher symplectic geometry is closely related to several areas of current interest including string theory, loop groups, and generalized geometry.

We begin by showing that, just as a symplectic manifold gives a Poisson algebra of functions, any n-plectic manifold gives a Lie n-algebra containing certain differential forms which we call Hamiltonian. Lie n-algebras are examples of strongly homotopy Lie algebras. They consist of an n-term chain complex equipped with a collection of skew-symmetric multi-brackets that satisfy a generalized Jacobi identity.

We then develop the machinery necessary to geometrically quantize n-plectic manifolds. In particular, just as a prequantized symplectic manifold is equipped with a principal U(1)-bundle with connection, we show that a prequantized 2-plectic manifold is equipped with a U(1)-gerbe with 2-connection. A gerbe is a categorified sheaf, or stack, which generalizes the notion of a principal bundle. Furthermore, over any 2-plectic manifold there is a vector bundle equipped with extra structure called a Courant algebroid. This bundle is the 2-plectic analogue of the Atiyah algebroid over a prequantized symplectic manifold. Its space of global sections also forms a Lie 2-algebra. We use this Lie 2-algebra to prequantize the Lie 2-algebra of Hamiltonian forms.

Finally, we introduce the 2-plectic analogue of the Bohr-Sommerfeld variety associated to a real polarization, and use this to geometrically quantize 2-plectic man-

ifolds. For symplectic manifolds, the output from quantization is a Hilbert space of quantum states. Similarly, quantizing a 2-plectic manifold gives a category of quantum states. We consider a particular example in which the objects of this category can be identified with representations of the Lie group SU(2).

Contents

Chapter 1

Introduction

Higher symplectic geometry is a generalization of symplectic geometry which begins with considering manifolds equipped with a closed nondegenerate form of higher degree. This thesis explains how such a differential form gives rise to algebraic and geometric structures which act as the higher analogues of important structures found in symplectic geometry and geometric quantization. Indeed, a recurring theme in this work is the idea that basic results in symplectic geometry are specific instances of more general theorems which hold for a much larger class of structures.

In particular, we focus on manifolds equipped with a closed nondegenerate 3-form. We call such manifolds '2-plectic'. In this case, we see that higher symplectic geometry is intimately related to string theory. We use ideas from higher category theory and homotopical algebra to develop a geometric quantization procedure for 2-plectic manifolds. In doing so, we encounter structures known to play important roles in other string-inspired areas of current interest. These include the theory of L_∞-algebras, loop groups, gerbes, and generalized geometry. Our results shine new light on these structures, and suggest new relationships among the above fields. We invite the reader who has some familiarity with these ideas to skip ahead and browse Table 1.1. There we list examples of such structures and the roles they play in the quantization of 2-plectic manifolds.

We wish to provide in this introductory chapter a gentle overview of the basic ideas behind higher symplectic geometry, and describe, with some detail, the main results of this thesis. We begin with a brief survey of symplectic geometry and geometric quantization which emphasizes the role played by classical and quantum mechanics. Higher symplectic geometry is then introduced as a consequence of combining two known approaches to studying classical field theory: multisymplectic geometry and higher gauge theory. We conclude by providing a chapter-by-chapter summary of our main results.

Symplectic geometry and geometric quantization

Symplectic geometry is the study of manifolds equipped with a closed nondegenerate 2-form. nondegeneracy, in this context, means that the 2-form gives an isomorphism between the space of tangent vectors and the space of 1-forms by contraction or "lowering indices". Such a 2-form produces a variety of interesting algebraic and geometric structures. Symplectic manifolds appear in many branches of mathematics and these structures often provide useful characterizations of important phenomena. In particular, symplectic manifolds play a crucial role in classical mechanics and representation theory.

The origins of symplectic geometry, in fact, lie in classical mechanics. In classical mechanics, one studies the physics of a system of point-like particles. For many systems of interest, the state of the system at any time is uniquely determined by specifying the position and momentum of each particle. This state can be interpreted as a point in a manifold called the 'phase space' of the system. The time evolution of the system is therefore represented by a smooth path in this manifold, which is a solution to an ordinary differential equation called 'Hamilton's equation'. Physical observables of the system are smooth functions on the manifold. Measurement of an observable corresponds to evaluating the function at a particular a point of phase space. Remarkably, the structures needed to guarantee a solution to Hamilton's equation, and also to describe how measurements change in time, are provided by equipping the manifold with a symplectic 2-form.

For example, the nondegeneracy of the symplectic 2-form guarantees that Hamilton's equations have, at least for some interval of time, a solution. More interestingly, the symplectic structure makes the space of functions on the manifold into a special kind of Lie algebra called a Poisson algebra. The fact that the symplectic 2-form is closed implies that the corresponding bracket satisfies the Jacobi identity. This Lie bracket is used to compute the time evolution of observables.

There are many systems of interest, however, which must be studied by using quantum mechanics, instead of classical mechanics. In these cases, classical mechanics can be understood as a very rough approximation to the true physical behavior of the system. In their attempts to understand such quantum systems, physicists developed a process called 'quantization' in which one first considers a system classically, and then replaces these structures with their quantum analogues. Roughly speaking, in quantum mechanics the states of the system no longer correspond to points on a manifold, but rather to vectors in a Hilbert space. Observables no longer correspond to functions on a manifold, but rather to linear operators on the Hilbert space. The time evolution of a system is given by a solution to a partial differential equation called ‘Schrödinger’s equation’, rather than Hamilton’s equation. The time evolution of observables is now determined by the commutator bracket of operators, rather than the Poisson bracket of functions.

Hence, within the context of symplectic geometry, the physicists’ findings suggests that quantization is a procedure which involves assigning to a symplectic manifold a Hilbert space, and to the Poisson algebra a representation as linear operators on this space. This is, in fact, the first step of a rigorous procedure called ‘geometric quantization’ developed by Kirillov [34], Kostant [37], and Souriau [64] (KKS) in the 1960’s. It is based on the following facts: If a symplectic 2-form satisfies a certain integrality condition, then it must be the curvature of a principal U(1)-bundle equipped with a connection living over the manifold. Such a symplectic manifold is called ‘prequantizable’. Certain global sections of the associated Hermitian line bundle form a Hilbert space whose inner product is given by the symplectic structure. The connection on the bundle then determines a faithful representation of the Poisson algebra as operators on this prequantum Hilbert space.

However, in practice, this Hilbert space is “too large”. The second step in the KKS procedure involves choosing an additional structure on the manifold called a ‘polarization’. Roughly speaking, a polarization on a symplectic manifold is a special kind of integrable distribution [63, 70]. The size of the Hilbert space is reduced by considering only those sections that are covariantly constant in the directions given by vectors contained in the distribution. This smaller space is called the ‘quantum Hilbert space’, or ‘space of quantum states’.

Geometric quantization may appear, at first sight, to be a rather mysterious procedure with limited applicability. Not every symplectic manifold is prequantizable, and not every prequantized symplectic manifold admits a polarization. Even when such structures do exist, there are several non-canonical choices to be made. Furthermore, the presence of certain topological obstructions often implies that additional fine-tuning is required. Regardless, the KKS procedure is very powerful and has led to a large number of important results, for example, in the representation theory of Lie groups. Here, one typically studies the symmetry group of a geometric object by first understanding the algebraic representation theory of the group. Kirillov and Kostant’s original motivation for developing geometric quantization was, in some sense, the converse: to construct the representations of groups as geometric objects. Indeed, the central tenet of Kirillov’s orbit method [34] is, roughly, that an irreducible representation of a Lie group corresponds to a particular symplectic manifold equipped with an action of the group. The representation itself is recovered as the quantum Hilbert space obtained from geometric quantization.

Higher degree, higher dimension, and higher structure

After digesting all of this, the curious reader might ask a simple question: What is so special about 2-forms? After all, many manifolds admit interesting closed forms of higher degrees, and some of these, such as volume forms, are "nondegenerate". It is also reasonable to ask how much, if any, of the above story involving symplectic geometry and quantization carries over to manifolds equipped with such forms. The main goal of this thesis is to address these questions.

At its most basic level, higher symplectic geometry involves studying manifolds equipped with a closed, nondegenerate form of higher degree. We call such a manifold 'n-plectic' if the form has degree $(n+1)$, so that a 1-plectic manifold is a symplectic manifold. Here, nondegeneracy means that the n-plectic form injectively maps the space of tangent vectors into the space of n-forms, again by contraction. In contrast with the symplectic case, this injection is not necessarily an isomorphism. Many examples of n-plectic manifolds appear "in nature". These include orientable manifolds, exterior powers of cotangent bundles, and compact simple Lie groups.

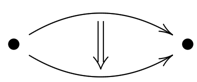
Usually, n-plectic manifolds go by the name of multisymplectic manifolds [16]. Just as symplectic geometry has its origins in the classical mechanics of particles, multisymplectic geometry was initially developed to study higher-dimensional classical field theories. Let us briefly explain what this means. As previously mentioned, the time evolution of a point-like particle is described by a path which depends on one variable: time. So, the 'world-line' of a zero-dimensional object is determined by a map from a one-dimensional manifold. A physicist might call classical mechanics a $(0+1)$-dimensional field theory. However, describing the behavior of a higher-dimensional object, such as a string, requires more variables. The amplitude of a vibrating string depends on both time and the position along the string. Hence, the time evolution of the one-dimensional string is described by a map from a 2-dimensional manifold or 'world-sheet'. In this way, string theory is a $(1+1)$-dimensional field theory. In general, the physics of a $(n-1)$-dimensional object, or 'brane', is described by a n-dimensional field theory.

The basic ideas in multisymplectic geometry can be found in Weyl's 1935 work on the calculus of variations [68]. It was further developed in the 1970's mainly by the Polish school of mathematical physics. The work of Kijowski [32], Tulczyjew [33], and others [51] showed that, just as symplectic manifolds can by used as phase spaces for $(0+1)$-dimensional field theories, multisymplectic manifolds can be used as 'multiphase' spaces for higher-dimensional field theories. Specifically, the multiphase space used to describe the physics of an $(n-1)$-dimensional object is an n-plectic manifold. A solution to a partial differential equation called the de Donder-Weyl equation corresponds to a particular n-dimensional submanifold of this space. The data encoded by these submanifolds include the value of the field as well as the value of its 'multi-momentum' at each point in space and time. The multi-momentum is a quantity that is related to the time and spatial derivatives of the field, in a manner similar to the relationship between the velocity of a point particle and its momentum. This formalism has several attractive mathematical features, but it still needs further development before it can replace more common frameworks used by physicists to study field theories.

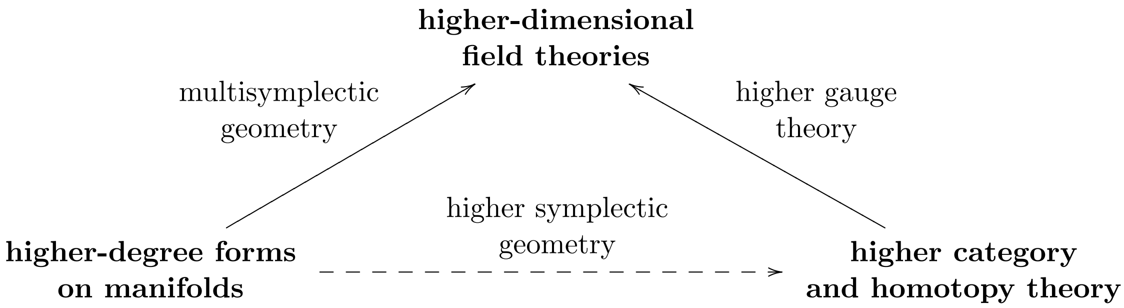
The work of Baez and Schreiber [7], Freed [20], Schreiber [60], Sati, Schreiber, and Stasheff [56] suggests that structures found in classical mechanics can be generalized by using higher category and homotopy theory and then applied to the study of higher-dimensional field theories. So far this viewpoint has been most fruitful in studying the string and brane-theoretic generalizations of gauge theory. Although the details are quite technical, the basic philosophy behind higher gauge theory is very simple. While a classical particle has a position nicely modelled by an element of a set, namely a point in space:

the position of a classical string is better modelled by a morphism in a category, namely an unparametrized path in space:

Similarly, the time evolution of a particle can be thought of as a morphism, while the time evolution of a string can be thought of as a 2-morphism, or 2-cell:



So, both higher degree forms on manifolds and higher structures can be used to study higher-dimensional field theories. Motivated by this idea, we suspect that the higher analogues of well-known structures on symplectic manifolds should naturally arise on n-plectic manifolds. The work presented in this thesis confirms this hunch, and we understand higher symplectic geometry as the formalism which completes the following diagram:



Overview of main results

We now describe the main results in this thesis. In Table 1.1, we list some particular examples to keep in mind while reading this section.

We first present some basic facts about n-plectic manifolds in Chapter 2. We say an $(n+1)$-form ω on M is n-plectic if ω is closed and nondegenerate. By nondegenerate, we mean the contraction

$$\begin{aligned} TM &\to \Lambda^n T^*M \\ v &\mapsto \omega(v,-) \end{aligned} \tag{1.1}$$

is injective. For the most part, we follow Cantrijn, Ibort, and de León's work on multisymplectic manifolds [16]. In particular, we use their generalizations of the familiar notions of Lagrangian submanifolds and real polarizations found in symplectic geometry.

Next, in Chapter 3, we extend the algebraic structures found in symplectic geometry to the n-plectic setting. Given an n-plectic manifold (M,ω), we show that the n-plectic structure naturally induces a skew-symmetric bracket on a particular subspace of $(n-1)$-forms, which we call Hamiltonian. An $(n-1)$- form α is Hamiltonian if there exists a vector field v such that

$$d\alpha = -\omega(v,-).$$

The vector field v is called the Hamiltonian vector field associated to α. In the 1-plectic/symplectic case, we see that every 0-form is Hamiltonian, and our bracket reduces to the Poisson bracket of functions. However, for higher values of n, the bracket only satisfies the Jacobi identity up to an exact form. This leads us to the notion of a Lie n-algebra. Lie n-algebras (equivalently, n-term L_∞-algebras [39]) are higher analogs of

	symplectic geometry	**2-plectic geometry**
degree of differential form	2	3
examples	cotangent bundles	exterior square of cotangent bundles
	coadjoint orbits of Lie groups	compact simple Lie groups
classical field theory		
physical objects	particles	strings
observables	Lie algebra of functions	Lie 2-algebra of Hamiltonian 1-forms
measurement	$x =$ point in phase space $x \mapsto f(x)$	$\gamma =$ path in multiphase space $\gamma \mapsto \int_\gamma \alpha$
prequantization		
prequantum structure	principal U(1)-bundle with connection **or** Hermitian line bundle with connection	U(1)-gerbe with 2-connection **or** 2-line stack with 2-connection
local data for prequantum structure	Deligne 1-cocycle: transition functions, 1-forms	Deligne 2-cocycle: transition functions, 1-forms, 2-forms
infinitesimal symmetries of prequantum structure	Atiyah algebroid (Lie algebroid)	Courant algebroid (Lie 2-algebroid)
quantization		
example	$\mathbb{R}^2 \setminus \{0\}$, $\omega = d\theta$	$\mathbb{R}^3 \setminus \{0\}$, $\omega = dB$
polarization	concentric circles	concentric spheres
Bohr-Sommerfeld condition	$\int_{S^1} \theta \in 2\pi i\mathbb{Z}$	$\int_{S^2} B \in 2\pi i\mathbb{Z}$
quantum states	wavefunctions of harmonic oscillator	representations of SU(2)

Table 1.1: Examples of structures found in symplectic geometry and higher symplectic geometry (for the 2-plectic case). Comparisons of their roles in field theory, prequantization, and quantization are listed.

differential graded Lie algebras. They consist of a graded vector space concentrated in degrees $0, \ldots, n-1$, and are equipped with a collection of skew-symmetric k-ary brackets, for $1 \leq k \leq n+1$, that satisfy a generalized Jacobi identity. In particular, the $k = 2$ bilinear bracket behaves like a Lie bracket that only satisfies the ordinary Jacobi identity up to higher coherent chain homotopy. In Theorem 3.14, we prove that, given an n-plectic manifold, one can explicitly construct a Lie n-algebra on a complex consisting of Hamiltonian $(n-1)$-forms and arbitrary p-forms for $0 \leq p \leq n-2$. The bilinear bracket, as well as all higher k-ary brackets, are completely determined by the n-plectic structure.

We consider an important example of this construction in Chapter 4: the Lie 2-algebra arising from a compact simple Lie group. Every such Lie group has a 1-parameter family of canonical 2-plectic structures generated by the 'Cartan 3-form'. These 3-forms are used to build central extensions of, and line bundles on, the corresponding loop group [47]. They also play a key role in the theory of gerbes on Lie groups [43] and the quantization of conjugacy classes [46]. We show how the Lie 2-algebra of Hamiltonian 1-forms on a compact simple Lie group G relates to the 'string Lie 2-algebra' of G [4]. It is known that the string Lie 2-algebra can be integrated to a 'Lie 2-group' [28]. This Lie 2-group can be geometrically realized as a topological group which appears in the study of spin structures on loop spaces.

Since geometric quantization has seen so much success in symplectic geometry, we wish to extend it to the n-plectic setting. In symplectic geometry, prequantization involves equipping the manifold with a principal U(1)-bundle with a connection, whose curvature is the symplectic 2-form. Therefore, in Chapter 5 we consider 'stacks', the 2-plectic analogue of bundles. A stack on a manifold can be thought of as a categorified sheaf i.e. an assignment of a category to each open neighborhood of the manifold. In particular, the higher analogue of a principal U(1)-bundle is a special kind of stack called a 'U(1)-gerbe'. Just as a section of a U(1)-bundle locally looks like a U(1)-valued function, a section of a U(1)-gerbe locally looks like a principal U(1)-bundle.

We then review Brylinski's theory of '2-connections' for U(1)-gerbes [13]. To understand what a 2-connection is, first recall that a U(1)-bundle with connection can be described by local transition functions and 1-forms satisfying certain compatibility conditions. This local data represents a degree 1 class in 'Deligne cohomology', which can be thought of as a refinement of the usual classification of bundles by Čech cohomology. Similarly, a U(1)-gerbe equipped with a 2-connection can be described by local transition functions, 1-forms, and 2-forms. This local data gives a degree 2 class in Deligne cohomology. Just as the curvature of a connection on a principal bundle is a 2-form, the '2-curvature' of a 2-connection is a 3-form. In general, we define a prequantized n-plectic manifold to be an n-plectic manifold equipped a Deligne n-cocycle whose n-curvature is, up to sign, the n-plectic form. As in the symplectic case, we show in Propositions 5.20 and 5.21 that only those n-plectic manifolds which satisfy an integrality condition can be prequantized.

In the remainder of the thesis, we focus on developing a quantization scheme for 2-plectic manifolds. For prequantized symplectic manifolds, the prequantum Hilbert space is obtained by considering global sections of the Hermitian line bundle associated to the U(1)-bundle. We generalize this to 2-plectic manifolds by constructing the '2-line stack' associated to a U(1)-gerbe. Sections of the 2-line stack locally look like Hermitian vector bundles. In Section 5.5, we use some basic ideas from '2-bundle theory' to explain why 2-line stacks are a natural generalization of line bundles. We also present a formalism by Carey, Johnson, and Murray [17] which generalizes the notion of holonomy to U(1)-gerbes equipped with a 2-connection. We shall use this '2-holonomy' in our quantization procedure for 2-plectic manifolds.

In Chapter 6, we consider prequantization for 2-plectic manifolds in detail. In order to understand our results, it is, again, helpful to momentarily return to the symplectic case. For a prequantized symplectic manifold, the connection on the principal bundle determines a representation of the Poisson algebra as linear operators on the prequantum Hilbert space. This representation identifies the Poisson algebra with certain U(1)-invariant vector fields on the bundle's total space. These vector fields are characterized by the fact that their flows are connection-preserving automorphisms of the bundle. Therefore, the Poisson algebra acts as linear differential operators on the space of smooth complex-valued functions on the total space. The prequantum Hilbert space is built using global sections of the associated Hermitian line bundle, and there is a way to interpret these sections as functions on the total space of the principal bundle. Hence, the Poisson algebra acts as operators on this Hilbert space.

This process of representing the Poisson algebra as operators can be nicely explained in terms of the Atiyah sequence associated to a principal bundle. Over any prequantized symplectic manifold, there is a special kind of vector bundle called the 'Atiyah algebroid' [15]. The global sections of this vector bundle are the U(1)-invariant vector fields on the total space of the principal U(1)-bundle. Hence, the space of sections form a Lie algebra under the Lie bracket of vector fields. In fact, the Atiyah algebroid is an example of a more general structure called a 'Lie algebroid'. The representation we described in the previous paragraph corresponds to an injective Lie algebra morphism embedding the Poisson algebra into the global sections of the Atiyah algebroid.

We define a prequantized 2-plectic manifold to be an integral 2-plectic manifold equipped with a U(1)-gerbe with 2-connection. A construction given by Hitchin [29] associates to any such gerbe on a manifold, a vector bundle called a 'Courant algebroid'. Its space of global sections is equipped with a skew-symmetric bracket which gives it the structure of a Lie 2-algebra. Hence, the Courant algebroid can be understood as a 'Lie 2-algebroid'. This 'Courant bracket' plays an important role in generalized complex geometry [26] and Poisson geometry [40]. Beginning in Section 6.3, we show how the Courant algebroid associated to a U(1)-gerbe is the higher analogue of the Atiyah algebroid associated to a U(1)-bundle. Such an analogy was conjectured to exist by Bressler and Chervov [11] as well as others. Our main result in this chapter is Theorem 6.16. It implies that the 2-connection of a gerbe on a prequantized 2-plectic manifold induces an injective morphism from the Lie 2-algebra of Hamiltonian 1-forms into the Lie 2-algebra of global sections of the Courant algebroid. In this way, we obtain a prequantization of the Hamiltonian 1-forms, in complete analogy with the symplectic case.

Finally, in Chapter 7, we use the 2-plectic analogue of 'real polarizations' to fully geometrically quantize 2-plectic manifolds. A real polarization on a prequantized symplectic manifold is a certain kind of foliation. Over any leaf of the polarization, the prequantum bundle restricts to a flat bundle. The prequantum Hilbert space of global sections is cut down by considering only those sections covariantly constant along the leaves of the polarization. However, there are topological obstructions to obtaining a non-trivial Hilbert space from this process. For example, if the leaves of the polarization are not simply-connected, then we are forced to consider only the leaves on which the restricted bundle has trivial holonomy. The collection of all such leaves is called the 'Bohr-Sommerfeld variety' associated to the polarization [63]. The space of quantum states is built using certain sections which are covariantly constant on the leaves contained in the variety. As the name suggests, there is a relationship between this construction and the old Bohr-Sommerfeld quantization rules from physics.

Before we go to the 2-plectic case, we review a well-known example in symplectic geometry in Section 7.1.2. We quantize the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$, equipped with a volume-form $\omega = d\theta$, as the phase space of the 'simple harmonic oscillator'. Here θ is not the angular coordinate on M, but rather a global 1-form which is related to the energy of the oscillator. We prequantize M using the trivial principal U(1)-bundle with connection θ. The associated Hermitian line bundle is the trivial line bundle. We choose the polarization given by concentric circles about the origin. The corresponding Bohr-Sommerfeld variety is a countable subset of these circles. We find sections of the prequantum line bundle over the Bohr-Somerfeld variety which are covariantly constant along the circles contained in the variety. This is equivalent to finding solutions to the Schrödinger wave equation. After applying a small correction, the radii of the circles in the variety correspond to the discrete energy levels for the quantized oscillator.

We generalize this entire construction to the 2-plectic case in Section 7.2. We start with a prequantized 2-plectic manifold equipped with a Deligne 2-cocycle. We consider the associated 2-line stack with 2-connection whose 2-curvature is the 2-plectic structure. The 2-plectic analogue of the prequantum Hilbert space is the category of global sections of the 2-line stack, i.e. the category of twisted Hermitian vector bundles on the manifold.

We quantize the manifold by choosing a real polarization as defined in Chapter 2. Over any leaf of the polarization, the 2-line stack restricts to a 'flat stack' i.e. the 2-curvature vanishes. The Bohr-Sommerfeld variety associated to the polarization is made up of those leaves on which the restricted 2-line stack has trivial 2-holonomy. Here, we use the 2-holonomy formalism for Deligne 2-cocycles which we described in Chapter 5. The 2-plectic analogue of the space of quantum states is the category of quantum states. Its objects are twisted vector bundles over the Bohr-Sommerfeld variety whose restriction to each leaf in the variety is 'twisted-flat'. This twisted-flat condition replaces the covariantly constant condition used in the symplectic case.

As an example of 2-plectic quantization, we consider the space $M = \mathbb{R}^3 \setminus \{0\}$ equipped with a particular volume form $\omega = dB$. We prequantize the space using the trivial U(1)-gerbe whose 2-connection is given by the global 2-form B. The associated 2-line stack in this case is equivalent to the stack of Hermitian vector bundles equipped with connection over M. (There is no twisting since the Deligne 2-cocycle is just a global 2-form.) We choose the polarization given by concentric spheres about the origin.

A sphere centered about the origin in $\mathbb{R}^3$ is a coadjoint orbit of the Lie group SU(2). This can easily seen by identifying $\mathbb{R}^3$ with $\mathfrak{su}(2) \cong \mathfrak{su}(2)^*$. It turns out that the restriction of B to any such sphere gives the famous KKS symplectic form used in Kirillov's orbit method [34]. By definition, a sphere is included in the Bohr-Sommerfeld variety if the Deligne 2-cocycle given by B has trivial 2-holonomy. Requiring trivial 2-holonomy is equivalent to the KKS symplectic form satisfying an integrality condition, which further implies that it is the curvature of a line bundle. We use some basic facts about the orbit method to pass from bundles to representations. We show that, in this example, the category of quantum states obtained from our quantization process is closely related to the category of finite-dimensional representations of SU(2). This suggests that, in some sense, 2-plectic quantization categorifies Kirillov's orbit method. Interestingly, the process fails to produce representations whose decomposition into irreducibles contains the trivial representation of SU(2). However, this is somewhat expected, since it is well known that the analogous quantization procedure for the harmonic oscillator in symplectic geometry requires an additional correction in order to obtain the correct space of quantum states.

We conclude the thesis in Chapter8 by providing a technical summary of the main results, and by discussing some open problems and future directions for research.

Previous work

We have recently published some of the results presented here. Theorem 3.14 in Chapter 3 and Proposition A.3 in Appendix A appear in [50]. Theorem 4.7 in Chapter 4 appears in [6], which was co-authored with J. Baez. The other results in Chapter 4 generalize or improve upon those of [6]. Chapter 6 is based on a recent preprint [49], which has been submitted for publication. Finally, a different proof of Theorem A.10 in Appendix A appears in [5], which was co-authored with J. Baez and A. Hoffnung.

Chapter 2

n-Plectic geometry

Our basic geometric objects of interest are n-plectic manifolds: manifolds equipped with a closed, nondegenerate form of degree $n+1$. Hence, a 1-plectic manifold is a symplectic manifold. n-Plectic manifolds are also called multisymplectic manifolds. Multisymplectic geometry originated in covariant Hamiltonian formalisms for classical field theory, just as symplectic geometry originated in classical mechanics. However, multisymplectic manifolds can be found outside the context of classical field theory, and are interesting from a purely geometric point of view. A few different definitions for multisymplectic structures exist in the literature. We adopt the formalism developed by Cantrijn, Ibort, and de León [16], since it provides the simplest generalization of symplectic structures, and also encapsulates a wide variety of interesting examples.

2.1 Linear theory

We begin by introducing multisymplectic/n-plectic structures on vector spaces. For the most part, we only present those aspects of the theory needed for subsequent chapters. For more details, we refer the reader to [16].

Definition 2.1. *An $(n+1)$-form ω on a vector space is* **n-plectic** *iff it is nondegenerate:*

$$\forall v \in V \;\; \iota_v \omega = 0 \Rightarrow v = 0.$$

If ω is an n-plectic form on V, then we call the pair (V, ω) an **n-plectic vector space**.

Note that a 1-plectic vector space is simply a symplectic vector space. A straightforward exercise in linear algebra shows that n-plectic structures do not exist on vector spaces of dimension $n+2$. For the $n = 1$ case, there is the stronger result that every finite-dimensional symplectic vector space has even dimension. Conversely, any even-dimensional vector space V admits a symplectic form ω, which can be put into a normal form by choosing a particular basis. Hence, $\mathrm{GL}(V)$ acts transitively on the space of symplectic structures on a symplectic vector space (V, ω). In contrast, it has been shown that if $\dim V \geq 6$, then n-plectic structures on V are generic for $2 \leq n \leq \dim V - 4$ [42]. Furthermore, 2-plectic structures on real vector spaces V with $\dim V \leq 7$ have been classified. In these cases, the action of $\mathrm{GL}(V)$ is not transitive. If $\dim V = 6$, then there are 2 equivalence classes, and if $\dim V = 7$, then there are 8 classes [42]. In general, the classification of n-plectic structures remains an open problem [16].

Next, we consider several natural generalizations of the orthogonal complement associated to a bilinear form.

Definition 2.2 ([16])**.** *Let (V, ω) be an n-plectic vector space and $W \subseteq V$ be a subspace. The* ***k*-orthogonal complement of** *W is the subspace*

$$W^{\perp,k} = \{v \in V \mid \omega(v, w_1, w_2, \ldots, w_k) = 0 \ \forall w_1, w_2, \ldots, w_k \in W\}.$$

Hence, there is a filtration of orthogonal complements:

$$W^{\perp,1} \subseteq W^{\perp,2} \subseteq \cdots \subseteq W^{\perp,n}.$$

Definition 2.3 ([16])**.** *A subspace W of an n-plectic vector space (V, ω) is* ***k*-isotropic** *iff $W \subseteq W^{\perp,k}$, and* ***k*-Lagrangian** *iff $W = W^{\perp,k}$.*

For convenience, if W is an n-isotropic or n-Lagrangian subspace of an n-plectic vector space, then we will say W is **isotropic** or **Lagrangian**, respectively. The notion of a k-co-isotropic subspace exists as well, but we will not need it here.

Obviously, every 1-dimensional subspace of an n-plectic vector space is 1-isotropic. Hence, the next proposition guarantees the existence of k-Lagrangian subspaces for all $k \geq 1$.

Proposition 2.4. *Let (V, ω) be an n-plectic vector space. If $W \subseteq V$ is a k-isotropic subspace, then for all $k' \geq k$ there exists a k'-Lagrangian subspace containing W.*

Proof. See Proposition 3.4 (iii) in the paper by Cantrijn, Ibort, and de Léon [16]. □

In contrast with the symplectic case, two k-Lagrangian subspaces need not have the same dimension. However, if the n-plectic vector space is $(n+1)$-dimensional, then it is simply a vector space equipped with a volume form and we have:

Proposition 2.5. *If (V, ω) is an n-plectic vector space with $\dim V = n + 1$, then a subspace $W \subseteq V$ is n-Lagrangian if and only if $\dim W = n$.*

Proof. First suppose $W = W^{\perp,n}$. Then $\dim W = k \leq n$. Let $e_1, \ldots, e_k$ be a basis for W, and let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{n+1}$ be its extension to a basis for V. Let $\theta^1, \ldots, \theta^{n+1}$ be the dual basis with $\theta^i(e_j) = \delta^i_j$. The n-plectic form can be written as

$$\omega = r \cdot \theta^1 \wedge \cdots \wedge \theta^{n+1},$$

with $|r| > 0$. If $w_1, \ldots, w_n$ are elements of W, with $w_i = \sum_{j=1}^{k} c_{ij} e_j$, and $\dim W$ is strictly less than n, then

$$\omega(v, w_1, w_2, \ldots, w_n) = 0$$

for all $v \in V$. Hence, we must have $\dim W = n$.

Now suppose W has dimension n with basis $e_1, \ldots, e_n$. Let $e_1, \ldots, e_n, e_{n+1}$ be the extended basis of V. It is easy to see that $W \subseteq W^{\perp,n}$. If $v \in W^{\perp,n}$ is not an element in W, then its contraction with the dual basis element θ^{n+1} is non-zero. However, we have:

$$0 = \omega(v, e_1, e_2, \ldots, e_n) = \pm\omega(e_1, e_2, \ldots, e_n, v) = \pm r \cdot \theta^{n+1}(v),$$

giving a contradiction. Hence no such v exists, and therefore $W = W^{\perp,n}$. □

2.2 n-Plectic manifolds

We now turn to the global theory. Our first definition generalizes the definition of a symplectic manifold.

Definition 2.6. *An $(n+1)$-form ω on a smooth manifold M is* ***n*-plectic***, or more specifically an* ***n*-plectic structure***, if it is both closed:*

$$d\omega = 0,$$

and nondegenerate:

$$\forall x \in M \ \forall v \in T_x M, \ \iota_v \omega = 0 \Rightarrow v = 0$$

If ω is an n-plectic form on M we call the pair (M, ω) an ***n*-plectic manifold***.*

Remark 2.7. In general, n-plectic manifolds are much more abundant than symplectic manifolds. On a finite-dimensional manifold M, n-plectic structures are generic for $2 \leq n \leq \dim M - 4$ (i.e. the set of n-plectic structures is comeager in $\Gamma(\Lambda^{n+1} T^* M)$ by Thm. II 2.2 and Prop. II 4.2 in [42]). Also, the remarks made after Def. 2.1 imply that no Darboux-like theorem holds for n-plectic structures.

Clearly, an n-plectic structure on an $(n+1)$-dimensional manifold M is a non-vanishing section of the top-exterior power of the cotangent bundle. Hence, orientable manifolds equipped with a volume form provide simple examples of n-plectic manifolds. Below, we describe some other interesting examples of n-plectic manifolds.

Example 2.8 (Compact simple Lie groups)**.** Every compact simple Lie group admits a 1-parameter family of canonical 2-plectic structures. These structures have been discussed in the multisymplectic geometry literature [16, 30], and play an important role in several branches of mathematics connected to string theory.

Recall that if G is a compact Lie group, then its Lie algebra $\mathfrak{g}$ admits an inner product $\langle \cdot, \cdot \rangle$ that is invariant under the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. For any nonzero real number k, we can define a trilinear form

$$\omega_k(x, y, z) = k\langle x, [y, z]\rangle$$

for any $x, y, z \in \mathfrak{g}$. Since the inner product is invariant under the adjoint representation, it follows that the linear transformations $\mathrm{ad}_y\colon \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_y(x) = [y, x]$ are skew adjoint. That is, $\langle \mathrm{ad}_y(x), z\rangle = -\langle x, \mathrm{ad}_y(z)\rangle$ for all $x, y, z \in \mathfrak{g}$. Hence, ω_k is totally antisymmetric. Moreover, ω_k is invariant under the adjoint representation since $[\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)] = \mathrm{Ad}_g([x, y])$.

Let $L_g\colon G \to G$ and $R_g\colon G \to G$ denote left and right translation by g, respectively. Let $\theta_L \in \Omega^1(G, \mathfrak{g})$ denote the left-invariant Maurer-Cartan form, which sends a vector $v \in T_g G$ to $L_{g^{-1}*} v \in \mathfrak{g}$. Using left translation, we can extend ω_k to a left invariant 3-form ν_k on G:

$$\begin{aligned} \nu_k &= \omega_k(\theta_L, \theta_L, \theta_L) \\ &= k\langle \theta_L, [\theta_L, \theta_L]\rangle. \end{aligned}$$

It is straightforward to show that ν_k is also a right invariant 3-form. Indeed, since $\mathrm{Ad}_g = L_{g*} \circ R_{g^{-1}*}$, the invariance of ω_k under the adjoint representation implies $R_g^* \nu_k = \nu_k$. From the left and right invariance we can conclude

$$d\nu_k = 0,$$

since any p-form on a Lie group that is both left and right invariant is closed.

Now suppose that G is a compact simple Lie group. Then $\mathfrak{g}$ is simple, so it has a canonical invariant inner product: the Killing form (up to a choice of normalization). With this choice of inner product, the trilinear form ω_k is nondegenerate in the sense of Definition 2.1.

Proposition 2.9. *If G is a compact simple Lie group, then (G, ν_k) is a 2-plectic manifold.*

Proof. We just need to show that ω_k is nondegenerate i.e. if $x \in \mathfrak{g}$ and $\omega_k(x, y, z) = 0$ for all $y, z \in \mathfrak{g}$ then $x = 0$. Recall that if $\mathfrak{g}$ is simple, then it is equal to its derived algebra $\big[\mathfrak{g}, \mathfrak{g}\big]$. Hence we may write $x = \sum_{i=1}^{n}[y_i, z_i]$. Therefore

$$k\big\langle x, x\big\rangle = k\sum_{i=1}^{n}\big\langle x, [y_i, z_i]\big\rangle = \sum_{i=1}^{n}\omega_k(x, y_i, z_i) = 0,$$

implies $x = 0$ since $\big\langle \cdot, \cdot \big\rangle$ is an inner product. □

Example 2.10 (Exterior powers of cotangent bundles)**.** This next example generalizes the well-known fact that cotangent bundles are symplectic manifolds. Suppose M is a smooth manifold, and let $X = \Lambda^n T^*M$ be the n-th exterior power of the cotangent bundle of M. Then there is a canonical n-form θ on X given as follows:

$$\theta(v_1, \ldots, v_n)|_x = x(\pi_*(v_1), \ldots, \pi_*(v_n))$$

where $v_1, \ldots v_n$ are tangent vectors at the point $x \in X$, and $\pi \colon X \to M$ is the projection from the bundle X to the base space M.

We claim the $(n+1)$-form

$$\omega = d\theta$$

is n-plectic. This can be seen by explicit computation. Let $q^1, \ldots, q^d$ be coordinates on an open set $U \subseteq M$. Then there is a basis of n-forms on U given by $dq^I = dq^{i_1} \wedge \cdots \wedge dq^{i_n}$ where $I = (i_1, \ldots, i_n)$ ranges over multi-indices of length n. Corresponding to these n-forms there are fiber coordinates p_I which combined with the coordinates q^i pulled back from the base give a coordinate system on $\Lambda^n T^*U$. In these coordinates we have

$$\theta = p_I dq^I,$$

where we follow the Einstein summation convention to sum over repeated multi-indices of length n. It follows that

$$\omega = dp_I \wedge dq^I.$$

Using this formula one can check that ω is indeed n-plectic.

Example 2.11 (Hyper-Kähler manifolds)**.** Let (M, g) be a Riemannian manifold which admits two anti-commuting, almost complex structures $J_1, J_2 \colon TM \to TM$, i.e. $J_1^2 = J_2^2 = -\operatorname{id}$ and $J_1 J_2 = -J_2 J_1$. Then $J_3 = J_1 J_2$ is also an almost complex structure.

If J_1, J_2, J_3 preserve the metric g, then one can define the 2-forms $\theta_1, \theta_2, \theta_3$, where $\theta_i(v_1, v_2) = g(v_1, J_i v_2)$. If each θ_i is closed, then M is called a hyper-Kähler manifold [65]. Given such a manifold, one can construct the 4-form:

$$\omega = \theta_1 \wedge \theta_1 + \theta_2 \wedge \theta_2 + \theta_3 \wedge \theta_3.$$

Clearly, ω is closed. It is also straightforward to show nondegeneracy. Indeed, suppose there existed a vector field v such that $\omega(v, \cdot, \cdot, \cdot) = 0$. A calculation shows that $\omega(v, J_1 v, J_2 v, J_3 v) = 0$ implies that $g(v, v)^2 = 0$. Since g is Riemannian, we must have $v = 0$. Hence a hyper-Kähler manifold is a 3-plectic manifold.

2.3 k-Lagrangian submanifolds and k-polarizations

We return to our presentation of the general theory and describe some geometric structures that will play important roles in the geometric quantization of n-plectic manifolds.

Definition 2.12 ([16]). *A submanifold N of an n-plectic manifold (M, ω) is* **k-isotropic** *(***k-Lagrangian***) iff for all $x \in N$, $T_x N$ is a k-isotropic (k-Lagrangian) subspace of the n-plectic vector space $(T_x M, \omega|_x)$.*

As in the linear case, if N is an n-isotropic or n-Lagrangian submanifold of an n-plectic manifold, then we say N is **isotropic** or **Lagrangian**, respectively. Of course, we recover the usual definitions when $n = 1$.

In symplectic geometry, polarizations are defined as integrable maximally isotropic sub-bundles of the complexified tangent bundle of a symplectic manifold. They are used in geometric quantization to cut down the size of the Hilbert space associated to the symplectic manifold. Certain polarizations called "real polarizations" can be understood as integrable distributions living in the real tangent bundle rather than its complexification. We currently do not know what an "n-complex structure" should be. Therefore, we are only able to generalize real polarizations to the n-plectic case.

Definition 2.13. *A foliation F of an n-plectic manifold (M, ω) is a* **real k-polarization** *iff the leaves of F are immersed k-Lagrangian submanifolds of M.*

For brevity, we call a real n-polarization on an n-plectic manifold simply a **polarization**. We conclude with an example which we will use in Chapter 7.

Example 2.14. A volume form on $M = \mathbb{R}^{n+1} \setminus \{0\}$ is an n-plectic form. Let F be the foliation of M by n-spheres centered about the origin. Since each leaf has codimension 1, it follows from Prop. 2.5 that F is a real polarization of M.

Chapter 3

Algebraic structures on n-plectic manifolds

From the algebraic point of view, the fundamental object in symplectic geometry is the Poisson algebra of smooth functions whose bracket is induced by the symplectic form. The nondegeneracy of a symplectic 2-form on M induces an isomorphism from TM to T^*M. Hence, for every function f there exists a unique vector field v_f such that $df = -\omega(v_f, \cdot)$. This assignment gives the Poisson bracket:

$$\{f, g\} = \omega(v_f, v_g), \quad \forall f, g \in C^\infty(M). \tag{3.1}$$

This bracket is skew-symmetric and satisfies the Jacobi identity. Hence, the space of smooth functions on a symplectic manifold is a Lie algebra.[1] In classical mechanics, the functions play the role of the 'observables', or measurements, of a physical system of point particles. The Poisson bracket is used to describe how these measurements change as the system evolves in time.

Certain complications arise if we try to repeat the above construction for an arbitrary n-plectic manifold (M, ω). The nondegeneracy of the n-plectic form gives an injection $TM \to \Lambda^n T^*M$ that is not necessarily onto. Therefore, only a subspace of the $(n-1)$-forms on M have the property that there exists a unique vector field v_α such that

$$d\alpha = -\omega(v_\alpha, \cdots).$$

We call such $(n-1)$-forms 'Hamiltonian'. Hence, we can copy the definition of the Poisson bracket given above and define a skew-symmetric bracket on the Hamiltonian $(n-1)$-forms

$$\{\alpha, \beta\} = \omega(v_\alpha, v_\beta, \cdots).$$

[1] The Poisson bracket also obeys an additional Leibniz-like rule: $\{f, gh\} = \{f, g\}\, h + g\, \{f, h\}$.

However, as we will see in Lemma 3.6, this bracket only satisfies the Jacobi identity up to an exact form:

$$\{\alpha, \{\beta, \gamma\}\} - \{\{\alpha, \beta\}, \gamma\} - \{\beta, \{\alpha, \gamma\}\} = -d\big(\omega(v_\alpha, v_\beta, v_\gamma, \cdots)\big). \tag{3.2}$$

Therefore, it is not necessarily a Lie bracket for $n > 1$.

Roughly speaking, we can imagine the Hamiltonian forms as being part of a complex L whose boundary operator is the de Rham differential, and interpret the left-hand side of Eq. 3.2 as the difference of two chain maps:

$$\{\cdot, \{\cdot, \cdot\}\} : L \otimes L \otimes L \to L,$$

and

$$\{\{\cdot, \cdot\}, \cdot\} + \{\cdot, \{\cdot, \cdot\}\} : L \otimes L \otimes L \to L.$$

From this point of view, the right-hand side of Eq. 3.2 suggests that we interpret the evaluation of ω on three Hamiltonian vector fields as a chain homotopy. This leads us to consider an algebraic structure called a Lie n-algebra.

Lie n-algebras are higher analogs of differential graded Lie algebras (DGLAs). They consist of a graded vector space concentrated in degrees $0, \ldots, n-1$ and are equipped with a collection of skew-symmetric k-ary brackets, for $1 \leq k \leq n+1$, that satisfy a generalized Jacobi identity [38, 39]. In particular, the $k = 2$ bilinear bracket behaves like a Lie bracket that only satisfies the ordinary Jacobi identity up to 'higher coherent' chain homotopy. When $n = 1$, we recover the definition of an ordinary Lie algebra. For $n = \infty$, we obtain the more general notion of an L_∞-algebra, which was first discovered by Schlessinger and Stasheff [58]. The definition of a Lie n-algebra may seem at first rather artificial. However, they are ubiquitous in mathematical physics and in certain areas of algebraic topology. In fact, there is an alternative definition of an L_∞-algebra, based on a construction of Quillen [48], which shows that it is an obvious and quite natural generalization of a DGLA.

The main result of this chapter is Theorem 3.14. Given an n-plectic manifold, we explicitly construct a Lie n-algebra on a complex consisting of the Hamiltonian $(n-1)$-forms and arbitrary p-forms for $0 \leq p \leq n-2$. The bilinear bracket, as well as all higher k-ary brackets, are specified by the n-plectic structure. For $n = 1$, the Lie 1-algebra we obtain from this construction is the underlying Lie algebra of the Poisson algebra of a symplectic manifold. For a 2-plectic manifold representing the 'multi-phase' space of a bosonic string, we showed in our work with Baez and Hoffnung that the Lie 2-algebra of Hamiltonian 1-forms contains the physical observables used in string theory

[5]. Hence, we often refer to the Lie n-algebra arising from an n-plectic manifold as the "algebra of observables".

In Appendix A, we consider other algebraic structures which naturally arise in higher symplectic geometry: dg Leibniz algebras and Roytenberg's weak Lie 2-algebras.

3.1 Hamiltonian forms

In this section, we equip the space of Hamiltonian $(n-1)$-forms on an n-plectic manifold with a bilinear skew-symmetric bracket, and note some of its properties. In order to aid our computations, we introduce some notation and review the Cartan calculus involving multivector fields and differential forms. We follow the notation and sign conventions found in Appendix A of the paper by Forger, Paufler, and Römer [19].

Let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on a manifold M and let

$$\mathfrak{X}^{\wedge\bullet}(M) = \bigoplus_{k=0}^{\dim M} \Lambda^k \left(\mathfrak{X}(M)\right)$$

be the graded commutative algebra of multivector fields. On $\mathfrak{X}^{\wedge\bullet}(M)$ there is a $\mathbb{R}$-bilinear map $[\cdot,\cdot]\colon \mathfrak{X}^{\wedge\bullet}(M) \times \mathfrak{X}^{\wedge\bullet}(M) \to \mathfrak{X}^{\wedge\bullet}(M)$ called the **Schouten bracket**, which gives $\mathfrak{X}^{\wedge\bullet}(M)$ the structure of a Gerstenhaber algebra. This means the Schouten bracket is a degree -1 Lie bracket which satisfies the graded Leibniz rule with respect to the wedge product. The Schouten bracket of two decomposable multivector fields $u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n \in \mathfrak{X}^{\wedge\bullet}(M)$ is

$$\begin{aligned}[u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = \\ \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}[u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \\ \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n, \quad (3.3)\end{aligned}$$

where $[u_i, v_j]$ is the usual Lie bracket of vector fields.

Given a form $\alpha \in \Omega^\bullet(M)$, the **interior product** of a decomposable multivector field $v_1 \wedge \cdots \wedge v_n$ with α is

$$\iota(v_1 \wedge \cdots \wedge v_n)\alpha = \iota_{v_n} \cdots \iota_{v_1}\alpha, \tag{3.4}$$

where $\iota_{v_i}\alpha$ is the usual interior product of vector fields and differential forms. The interior product of an arbitrary multivector field is obtained by extending the above formula by $C^\infty(M)$-linearity.

The **Lie derivative** $\mathcal{L}_v$ of a differential form along a multivector field $v \in \mathfrak{X}^{\wedge\bullet}(M)$ is defined via the graded commutator of d and $\iota(v)$:

$$\mathcal{L}_v\alpha = d\iota(v)\alpha - (-1)^{|v|}\iota(v)d\alpha, \tag{3.5}$$

where $\iota(v)$ is considered as a degree $-|v|$ operator.

The last identity we will need involving multivector fields is for the graded commutator of the Lie derivative and the interior product. Given $u, v \in \mathfrak{X}^{\wedge\bullet}(M)$, it follows from Proposition A3 in [19] that

$$\iota([u,v])\alpha = (-1)^{(|u|-1)|v|}\mathcal{L}_u\iota(v)\alpha - \iota(v)\mathcal{L}_u\alpha. \tag{3.6}$$

We return now to n-plectic geometry. Our first definition is:

Definition 3.1. *Let (M,ω) be an n-plectic manifold. An $(n-1)$-form α is* **Hamiltonian** *iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that*

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say v_α is the **Hamiltonian vector field** *corresponding to α. The set of Hamiltonian $(n-1)$-forms and the set of Hamiltonian vector fields on an n-plectic manifold are both vector spaces and are denoted as $\Omega^{n-1}_{\mathrm{Ham}}(M)$ and $\mathfrak{X}_{\mathrm{Ham}}(M)$, respectively.*

The Hamiltonian vector field v_α is unique if it exists, but there may be $(n-1)$-forms having no Hamiltonian vector field. Note that if $\alpha \in \Omega^{n-1}(M)$ is closed, then it is Hamiltonian and its Hamiltonian vector field is the zero vector field.

An elementary, yet important, fact is that the flow of a Hamiltonian vector field preserves the n-plectic structure.

Lemma 3.2. *If v_α is a Hamiltonian vector field, then $\mathcal{L}_{v_\alpha}\omega = 0$.*

Proof.

$$\mathcal{L}_{v_\alpha}\omega = d\iota_{v_\alpha}\omega + \iota_{v_\alpha}d\omega = -dd\alpha = 0$$

□

We now formally define the bracket on $\Omega^{n-1}_{\mathrm{Ham}}(M)$, which we described earlier in the introduction. One motivation for considering this bracket comes from its appearance in the multisymplectic formulations of classical field theories [27, 32].

Definition 3.3. *Given $\alpha, \beta \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, the* **bracket** $\{\alpha,\beta\}$ *is the $(n-1)$-form given by*

$$\{\alpha,\beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega.$$

When $n = 1$, this bracket is the usual Poisson bracket of smooth functions on a symplectic manifold. These next propositions show that for $n > 1$ the bracket of Hamiltonian forms has several properties in common with the Poisson bracket. However, unlike the case in symplectic geometry, we see that the bracket $\{\cdot, \cdot\}$ does not need to satisfy the Jacobi identity for $n > 1$.

Proposition 3.4. *Let $\alpha, \beta \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ and v_α, v_β be their respective Hamiltonian vector fields. The bracket $\{\cdot, \cdot\}$ has the following properties:*

1. *The bracket is skew-symmetric:*

$$\{\alpha, \beta\} = -\{\beta, \alpha\}.$$

2. *The bracket of Hamiltonian forms is Hamiltonian:*

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega,$$

and in particular we have

$$v_{\{\alpha,\beta\}} = [v_\alpha, v_\beta].$$

Proof. The first statement follows from the antisymmetry of ω. To prove the second statement, we use Lemma 3.2:

$$\begin{aligned} d\{\alpha, \beta\} &= d\iota_{v_\beta}\iota_{v_\alpha}\omega \\ &= \left(\mathcal{L}_{v_\beta} - \iota_{v_\beta}d\right)\iota_{v_\alpha}\omega \\ &= \mathcal{L}_{v_\beta}\iota_{v_\alpha}\omega + \iota_{v_\beta}dd\alpha \\ &= \iota_{[v_\beta, v_\alpha]}\omega + \iota_{v_\alpha}\mathcal{L}_{v_\beta}\omega \\ &= -\iota_{[v_\alpha, v_\beta]}\omega. \end{aligned}$$

□

Proposition 3.5. *The bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity up to an exact $(n-1)$-form:*

$$\{\alpha_1, \{\alpha_2, \alpha_3\}\} - \{\{\alpha_1, \alpha_2\}, \alpha_3\} - \{\alpha_2, \{\alpha_1, \alpha_3\}\} = -d\iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3})\omega.$$

The proof of Proposition 3.5 follows from the next lemma. We will also use this lemma in the proof of Theorem 3.14 in Section 3.3.

Lemma 3.6. *If (M,ω) is an n-plectic manifold and $v_1,\dots,v_m \in \mathfrak{X}_{\mathrm{Ham}}(M)$ with $m \geq 2$ then*

$$d\iota(v_1 \wedge \cdots \wedge v_m)\omega = \\ (-1)^m \sum_{1\leq i<j\leq m} (-1)^{i+j}\iota([v_i,v_j]\wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m)\omega. \quad (3.7)$$

Proof. We proceed via induction on m. For $m = 2$:

$$d\iota(v_1 \wedge v_2)\omega = d\{\alpha_1,\alpha_2\},$$

where α_1,α_2 are any Hamiltonian $(n-1)$-forms whose Hamiltonian vector fields are v_1, v_2, respectively. Then Proposition 3.4 implies Eq. 3.7 holds.

Assume Eq. 3.7 holds for $m-1$. Since $\iota(v_1 \wedge \cdots \wedge v_m) = \iota_{v_m}\iota(v_1 \wedge \cdots \wedge v_{m-1})$, Eq. 3.5 implies:

$$d\iota(v_1 \wedge \cdots \wedge v_m)\omega = \mathcal{L}_{v_m}\iota(v_1 \wedge \cdots \wedge v_{m-1})\omega - \iota_{v_m}d\iota(v_1 \wedge \cdots \wedge v_{m-1})\omega. \quad (3.8)$$

Consider the first term on the right hand side. Using Eq. 3.6 we can rewrite it as

$$\begin{aligned}\mathcal{L}_{v_m}\iota(v_1 \wedge \cdots \wedge v_{m-1})\omega &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega \\ &\quad + \iota(v_1 \wedge \cdots \wedge v_{m-1})\mathcal{L}_{v_m}\omega \\ &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega,\end{aligned}$$

where the last equality follows from Lemma 3.2.

The definition of the Schouten bracket given in Eq. 3.3 implies

$$[v_m, v_1 \wedge \cdots \wedge v_{m-1}] = \sum_{i=1}^{m-1} (-1)^{i+1}[v_m,v_i] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}.$$

Therefore we have

$$\begin{aligned}\mathcal{L}_{v_m}\iota(v_1 \wedge \cdots \wedge v_{m-1})\omega &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega \\ &= \sum_{i=1}^{m-1} (-1)^{i}\iota([v_i,v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega.\end{aligned}$$

Combining this with the second term in Eq. 3.8 and using the inductive hypothesis gives

$$
\begin{aligned}
d\iota(v_1 \wedge \cdots \wedge v_m)\omega &= \sum_{i=1}^{m-1} (-1)^i \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \\
&- (-1)^{m-1} \sum_{1 \le i < j \le m-1} (-1)^{i+j} \iota_{v_m} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \\
&\wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{m-1})\omega \\
&= (-1)^m \Bigg(\sum_{i=1}^{m-1} (-1)^{i+m} \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \\
&+ \sum_{1 \le i < j \le m-1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m)\omega \Bigg) \\
&= (-1)^m \sum_{1 \le i < j \le m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m)\omega.
\end{aligned}
$$

□

Proof of Proposition 3.5. Apply Lemma 3.6 with $m = 3$, and use the fact that $v_{\{\alpha_i, \alpha_j\}} = [v_{\alpha_i}, v_{\alpha_j}]$. □

3.2 L_∞-algebras and Lie n-algebras

We begin this section by recalling some basic graded linear algebra. Let V be a graded vector space. Let $x_1, \ldots, x_n$ be elements of V and $\sigma \in \mathcal{S}_n$ a permutation. The **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ is defined by the equality

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which holds in the free graded commutative algebra generated by V. Given $\sigma \in \mathcal{S}_n$, let $(-1)^\sigma$ denote the usual sign of a permutation. Note that $\epsilon(\sigma)$ does not include the sign $(-1)^\sigma$.

We say $\sigma \in \mathcal{S}_{p+q}$ is a $(\mathbf{p}, \mathbf{q})$**-unshuffle** iff $\sigma(i) < \sigma(i+1)$ whenever $i \neq p$. The set of (p, q)-unshuffles is denoted by $\mathrm{Sh}(p, q)$. For example, $\mathrm{Sh}(2, 1) = \{(1), (23), (123)\}$.

If V and W are graded vector spaces, a linear map $f \colon V^{\otimes n} \to W$ is **skew-symmetric** iff

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma) f(v_1, \ldots, v_n),$$

for all $\sigma \in \mathcal{S}_n$. The degree of an element $x_1 \otimes \cdots \otimes x_n \in V^{\otimes \bullet}$ of the graded tensor algebra generated by V is defined to be $|x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^n |x_i|$.

Proposition 3.5 implies that we should not expect $\Omega^{n-1}_{\mathrm{Ham}}(M)$ to be a Lie algebra unless $n = 1$. However, the fact that the Jacobi identity is satisfied modulo boundary terms suggests we consider what are known as strongly homotopy Lie algebras, or L_∞-algebras [38, 39].

Definition 3.7. *An* $\boldsymbol{L_\infty}$**-algebra** *is a graded vector space* L *equipped with a collection*

$$\Big\{l_k\colon L^{\otimes k} \to L | 1 \le k < \infty\Big\}$$

of skew-symmetric linear maps with $|l_k| = k-2$ *such that the following identity holds for* $1 \le m < \infty$ *:*

$$\sum_{\substack{i+j=m+1,\\ \sigma\in \mathrm{Sh}(i,m-i)}} (-1)^\sigma \epsilon(\sigma)(-1)^{i(j-1)} l_j\big(l_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),x_{\sigma(i+1)},\dots,x_{\sigma(m)}\big) = 0. \tag{3.9}$$

Definition 3.8. *An* L_∞*-algebra* $(L,\{l_k\})$ *is a* **Lie** $\boldsymbol{n}$**-algebra** *iff the underlying graded vector space* L *is concentrated in degrees* $0,\dots,n-1$*.*

Note that if $(L,\{l_k\})$ is a Lie n-algebra, then by degree counting $l_k = 0$ for $k > n+1$.

The identity satisfied by the maps in Definition 3.7 can be interpreted as a 'generalized Jacobi identity'. Indeed, using the notation $d = l_1$ and $[\cdot,\cdot] = l_2$, Eq. 3.9 implies

$$d^2 = 0$$
$$d[x_1,x_2] = [dx_1,x_2] + (-1)^{|x_1|}[x_1,dx_2].$$

Hence the map $l_1\colon L \to L$ can be interpreted as a differential, while the map $l_2\colon L\otimes L \to L$ can be interpreted as a bracket. The bracket is, of course, skew symmetric:

$$[x_1,x_2] = -(-1)^{|x_1||x_2|}[x_2,x_1],$$

but does not need to satisfy the usual Jacobi identity. In fact, Eq. 3.9 implies:

$$\begin{aligned}(-1)^{|x_1||x_3|}[[x_1,x_2],x_3] &+ (-1)^{|x_2||x_3|}[[x_3,x_1],x_2] + (-1)^{|x_1||x_2|}[[x_2,x_3],x_1] \\ &= (-1)^{|x_1||x_3|+1}\big(dl_3(x_1,x_2,x_3) + l_3(dx_1,x_2,x_3) \\ &\qquad + (-1)^{|x_1|}l_3(x_1,dx_2,x_3) + (-1)^{|x_1|+|x_2|}l_3(x_1,x_2,dx_3)\big).\end{aligned}$$

Therefore one can interpret the traditional Jacobi identity as a null-homotopic chain map from $L\otimes L\otimes L$ to L. The map l_3 acts as a chain homotopy and is referred to as the **Jacobiator**. Eq. 3.9 also implies that l_3 must satisfy a coherence condition of its own. From the above discussion, it is easy to see that a Lie 1-algebra is an ordinary Lie algebra, while an L_∞-algebra with $l_k \equiv 0$ for all $k \ge 3$ is a differential graded Lie algebra.

Remark 3.9 (Morphisms of L_∞-algebras). There is a more elegant way to define an L_∞-algebra using the language of graded coalgebras. This is inspired by the Quillen construction [48] for DGLAs, which realizes any DGLA structure on a graded vector space V as a codifferential on the cofree, cocommutative coalgebra (without counit) generated by the suspension of V. One can then define an L_∞-structure on V to simply be *any* codifferential on this coalgebra [38]. The fact that a codifferential squares to zero is equivalent to Eq. 3.9. The reader unfamiliar with coalgebras is probably quite confused by these remarks. We only mention this alternative definition, since it provides a natural definition of morphism between L_∞-algebras. Such a morphism is just a morphism between the corresponding graded coalgebras which respects the codifferentials. In this thesis, we will only consider morphisms between Lie 2-algebras (Def. 3.11).

3.2.1 Lie 2-algebras

Since we will be focusing specifically on 2-plectic manifolds in later chapters, we discuss here the theory of Lie 2-algebras in more detail. As L_∞-algebras, Lie 2-algebras are relatively easy to work with, since the underlying complex is concentrated in only two degrees. In this case, one can write out the axioms explicitly using elementary homological algebra.

Proposition 3.10. *A* **Lie 2-algebra** *is a 2-term chain complex of vector spaces* $L = (L_1 \xrightarrow{d} L_0)$ *equipped with:*

- *skew-symmetric chain map* $[\cdot,\cdot]\colon L \otimes L \to L$ *called the* **bracket***;*
- *an skew-symmetric chain homotopy* $J\colon L \otimes L \otimes L \to L$ *from the chain map*

$$\begin{array}{ccc} L \otimes L \otimes L & \to & L \\ x \otimes y \otimes z & \longmapsto & [x,[y,z]], \end{array}$$

  to the chain map

$$\begin{array}{ccc} L \otimes L \otimes L & \to & L \\ x \otimes y \otimes z & \longmapsto & [[x,y],z] + [y,[x,z]] \end{array}$$

  called the **Jacobiator***,*

such that the following equation holds:

$$\begin{aligned} &[x, J(y,z,w)] + J(x,[y,z],w) + J(x,z,[y,w]) + [J(x,y,z),w] \\ &\quad +[z, J(x,y,w)] = J(x,y,[z,w]) + J([x,y],z,w) \\ &\quad +[y, J(x,z,w)] + J(y,[x,z],w) + J(y,z,[x,w]). \end{aligned} \tag{3.10}$$

Proof. See Lemma 33 in Baez and Crans [4]. Note that the Jacobiator J is the map l_3 in Definition 3.8. □

For Lie 2-algebras, it is easy to write down the definition of a morphism without using coalgebras. (See Remark 3.9.)

Definition 3.11 ([4]). *Given Lie 2-algebras $L = (L, [\cdot, \cdot], J)$ and $L' = (L', [\cdot, \cdot]', J')$ a* **morphism** *from L to L' consists of:*

- *a chain map $\phi\colon L \to L'$, and*
- *a chain homotopy $\Phi\colon L \otimes L \to L'$ from the chain map*

$$\begin{aligned} L \otimes L &\to L' \\ x \otimes y &\longmapsto \phi([x, y]) \end{aligned}$$

to the chain map

$$\begin{aligned} L \otimes L &\to L' \\ x \otimes y &\longmapsto [\phi(x), \phi(y)]', \end{aligned}$$

such that the following equation holds:

$$\begin{aligned} &\phi_1(J(x, y, z)) - J'(\phi_0(x), \phi_0(y), \phi_0(z)) = \\ &\Phi(x, [y, z]) - \Phi([x, y], z) - \Phi(y, [x, z]) - [\Phi(x, y), \phi_0(z)]' \\ &+[\phi_0(x), \Phi(y, z)]' - [\phi_0(y), \Phi(x, z)]'. \end{aligned} \tag{3.11}$$

We say a morphism is **strict** *iff $\Phi = 0$.*

Typically, isomorphism is too strong of an equivalence to use for L_∞-algebras. Instead we use:

Definition 3.12. *A Lie 2-algebra morphism $(\phi, \Phi)\colon L \to L'$ is a* **quasi-isomorphism** *iff the chain map ϕ induces an isomorphism on the homology of the underlying chain complexes of L and L'.*

Since every vector space is free, quasi-isomorphism in our case is the same thing as chain homotopy equivalence, or categorical equivalence in the sense of Baez and Crans [4].

3.3 Lie n-algebras from n-plectic manifolds

There are several clues that suggest that any n-plectic manifold gives an L_∞-algebra. Comparing Eq. 3.7 to the generalized Jacobi identity (3.9) suggests that, for an n-plectic manifold, we should look for Lie n-algebra structures on the chain complex

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{\mathrm{Ham}}(M), \tag{3.12}$$

with the l_1 map equal to d. We denote this complex as (L, d). Note that here we are using the de Rham differential as a degree -1 operator. Hence $L_0 = \Omega^{n-1}_{\mathrm{Ham}}(M)$, while $L_{n-1} = C^\infty(M)$.

Note that the bracket $\{\cdot,\cdot\}$ given in Definition 3.3 induces a well-defined bracket $[\cdot,\cdot]'$ on the quotient

$$\mathfrak{g} = \Omega^{n-1}_{\mathrm{Ham}}(M)\,/d\Omega^{n-2}(M),$$

where $d\Omega^{n-2}(M)$ is the space of exact $(n-1)$-forms. This is because the Hamiltonian vector field of an exact $(n-1)$-form is the zero vector field. It follows from Proposition 3.5 that $(\mathfrak{g}, [\cdot,\cdot]')$ is, in fact, a Lie algebra.

If M is contractible, then the homology of (L, d) is

$$\begin{aligned} H_0(L) &= \mathfrak{g}, \\ H_k(L) &= 0 \quad \text{for } 0 < k < n-1, \\ H_{n-1}(L) &= \mathbb{R}. \end{aligned}$$

Therefore, the augmented complex

$$0 \to \mathbb{R} \hookrightarrow C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{\mathrm{Ham}}(M) \tag{3.13}$$

is a resolution of $\mathfrak{g}$.

Barnich, Fulp, Lada, and Stasheff [9] showed that, in general, if (C, δ) is a resolution of a vector space $V \cong H_0(C)$ and C_0 is equipped with a skew-symmetric map $\tilde{l}_2\colon C_0 \otimes C_0 \to C_0$ that induces a Lie bracket on V, then $\tilde{l}_2$ extends to an L_∞-structure on (C, δ). Hence we have the following proposition:

Proposition 3.13. *Given a contractible n-plectic manifold (M, ω), there is an L_∞-algebra $(\tilde{L}, \{l_k\})$ with underlying graded vector space*

$$\tilde{L}_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M) & i = 0, \\ \Omega^{n-1-i}(M) & 0 < i \le n-1, \\ \mathbb{R} & i = n, \end{cases}$$

and $l_1\colon \tilde{L} \to \tilde{L}$ defined as

$$l_1(\alpha) = \begin{cases} \alpha, & \text{if } |\alpha| = n \\ d\alpha & \text{if } |\alpha| \neq n, \end{cases}$$

and all higher maps $\left\{ l_k\colon \tilde{L}^{\otimes k} \to \tilde{L} | 2 \le k < \infty \right\}$ are constructed inductively by using the bracket

$$\{\cdot,\cdot\} : \tilde{L}_0 \otimes \tilde{L}_0 \to \tilde{L}_0, \quad \{\alpha_1, \alpha_2\} = \iota_{v_{\alpha_2}} \iota_{v_{\alpha_1}} \omega,$$

where $v_{\alpha_1}, v_{\alpha_2}$ are the Hamiltonian vector fields corresponding to the Hamiltonian forms α_1, α_2. Moreover the maps $\{l_k\}$ may be constructed so that

$$l_k(\alpha_1, \ldots, \alpha_k) \neq 0 \quad \textit{only if all } \alpha_k \textit{ have degree 0},$$

for $k \geq 2$.

Proof. The proposition follows from Theorem 7 in the paper by Barnich, Fulp, Lada, and Stasheff [9]. Since for any n-plectic manifold,

$$\{\alpha, d\beta\} = 0 \quad \forall \alpha \in \Omega^{n-1}_{\mathrm{Ham}}(M) \ \ \forall \beta \in \Omega^{n-2}(M),$$

the second remark following Theorem 7 in [9] implies that the maps $\{l_k\}$ may be constructed so that they are trivial when restricted to the positive-degree part of the k-th tensor power of $\tilde{L}$. □

For an arbitrary n-plectic manifold (M, ω), Proposition 3.13 guarantees the existence of L_∞-algebras locally. We want, of course, a global result in which the higher l_k maps are explicitly constructed using only the n-plectic structure. Moreover, in our previous work on 2-plectic geometry [5], we were able to construct by hand a Lie 2-algebra on a 2-term complex consisting of functions and Hamiltonian 1-forms. We did not need to use a 3-term complex consisting of constants, functions, and Hamiltonian 1-forms. Hence in the general case, we'd expect an n-plectic manifold to give a Lie n-algebra whose underlying complex is (L, d), instead of a Lie $(n+1)$-algebra whose underlying complex is the $(n+1)$-term complex used in the above proposition.

We can get an intuitive sense for what the maps $l_k \colon L^{\otimes k} \to L$ should be by unraveling the identity given in Definition 3.7 for small values of m and momentarily disregarding signs and summations over unshuffles. For example, if $m = 2$, then Eq. 3.9 implies that the map $l_2 \colon L \otimes L \to L$ must satisfy:

$$l_1 l_2 + l_2 l_1 = 0. \tag{3.14}$$

Obviously we want l_1 to be the de Rham differential and l_2 to be equal to the bracket $\{\cdot, \cdot\}$ when restricted to degree 0 elements:

$$l_2(\alpha_1, \alpha_2) = \pm \iota_{v_{\alpha_2}} \iota_{v_{\alpha_1}} \omega = \{\alpha_1, \alpha_2\} \quad \forall \alpha_i \in L_0 = \Omega^{n-1}_{\mathrm{Ham}}(M).$$

Now consider elements of degree 1. For example, if $\alpha \in L_0$ and $\beta \in L_1 = \Omega^{n-2}(M)$, then $l_2(\alpha, d\beta) = \{\alpha, d\beta\} = 0$. Therefore Eq. 3.14 implies

$$dl_2(\alpha, \beta) = l_1 l_2(\alpha, \beta) = 0.$$

Hence, when restricted to elements of degree 1, $l_2(\alpha, \beta)$ must be a closed $(n-2)$-form. We will choose this closed form to be 0. In fact, we will choose l_2 to vanish on all elements with degree > 0, since, in general, we want the L_∞ structure to only depend on the de Rham differential and the n-plectic structure.

Now suppose l_2 is defined as above and let $m = 3$. Then Eq. 3.9 implies:

$$l_1 l_3 + l_2 l_2 + l_3 l_1 = 0. \tag{3.15}$$

On degree 0 elements, $l_1 = 0$. Therefore it is clear from Proposition 3.5 that the map $l_3 \colon L^{\otimes 3} \to L$ when restricted to degree 0 elements must be

$$l_3(\alpha_1, \alpha_2, \alpha_3) = \pm \iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3})\omega,$$

where v_{α_i} is the Hamiltonian vector field associated to α_i. Now consider a degree 1 element of $L \otimes L \otimes L$, for example: $\alpha_1 \otimes \alpha_2 \otimes \beta \in \Omega^{n-1}_{\mathrm{Ham}}(M) \otimes \Omega^{n-1}_{\mathrm{Ham}}(M) \otimes \Omega^{n-2}(M)$. Since $l_3(\alpha_1, \alpha_2, d\beta) = \pm\iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{d\beta})\omega = 0$, and l_2 vanishes on the positive-degree part of the k-th tensor power of L, Eq. 3.15 holds if and only if

$$dl_3(\alpha_1, \alpha_2, \beta) = 0.$$

Hence, when restricted to elements of degree 1, $l_3(\alpha_1, \alpha_2, \beta)$ must be a closed $(n-2)$-form. Again, we will choose this closed form to be 0 by forcing l_3 to vanish on all elements with degree > 0.

Observations like these bring us to our main theorem. In general, we will define the maps $l_k \colon L^{\otimes k} \to L$ on degree zero elements to be completely specified (up to sign) by the n-plectic structure ω:

$$l_k(\alpha_1, \ldots, \alpha_k) = \pm\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega \quad \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0,$$

and trivial otherwise:

$$l_k(\alpha_1, \ldots, \alpha_k) = 0 \quad \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| > 0.$$

Theorem 3.14. *Given an n-plectic manifold (M, ω), there is a Lie n-algebra $L_\infty(M, \omega) = (L, \{l_k\})$ with underlying graded vector space*

$$L_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M) & i = 0, \\ \Omega^{n-1-i}(M) & 0 < i \leq n-1, \end{cases}$$

and maps $\left\{ l_k \colon L^{\otimes k} \to L \mid 1 \leq k < \infty \right\}$ defined as

$$l_1(\alpha) = d\alpha,$$

if $|\alpha| > 0$ *and*

$$l_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 0 & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| > 0, \\ (-1)^{\frac{k}{2}+1}\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0 \text{ and } k \text{ even}, \\ (-1)^{\frac{k-1}{2}}\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0 \text{ and } k \text{ odd}, \end{cases} \tag{3.16}$$

for $k > 1$, *where* v_{α_i} *is the unique Hamiltonian vector field associated to* $\alpha_i \in \Omega^{n-1}_{\mathrm{Ham}}(M)$.

Proof of Theorem 3.14. We begin by showing the maps $\{l_k\}$ are well-defined skew symmetric maps with $|l_k| = k-2$. If $\alpha_1 \otimes \cdots \otimes \alpha_k \in L^{\otimes \bullet}$ has degree 0, then for all $\sigma \in \mathcal{S}_k$ the antisymmetry of ω implies

$$l_k(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) = (-1)^\sigma l_k(\alpha_1, \ldots, \alpha_k).$$

Since for each i, we have $|\alpha_i| = 0$, it follows that $\epsilon(\sigma) = 1$. Hence l_k is skew symmetric and well-defined. Since $\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega \in \Omega^{n+1-k}(M) = L_{k-2}$, we have $|l_k| = k-2$. We also have, by construction, $l_k = 0$ for $k > n+1$.

Now we prove the maps satisfy Eq. 3.9 in Definition 3.7. If $m = 1$, then it is satisfied since l_1 is the de Rham differential. If $m = 2$, then a direct calculation shows

$$l_1(l_2(\alpha_1, \alpha_2)) = l_2(l_1(\alpha_1), \alpha_2) + (-1)^{|\alpha_1|} l_2(\alpha_1, l_1(\alpha_2)).$$

Let $m > 2$. We will regroup the summands in Eq. 3.9 into two separate sums depending on the value of the index j and show that each of these is zero, thereby proving the theorem.

We first consider the sum of the terms with $2 \leq j \leq m-2$:

$$\sum_{j=2}^{m-2} \sum_{\sigma \in \mathrm{Sh}(i,m-i)} (-1)^\sigma \epsilon(\sigma)(-1)^{i(j-1)} l_j(l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}), \alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(m)}). \tag{3.17}$$

In this case we claim that for all $\sigma \in \mathrm{Sh}(i, m-i)$ we have

$$l_j(l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}), \alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(m)}) = 0.$$

Indeed, if there exists an unshuffle such that the above equality did not hold, then the definition of $l_j \colon L^{\otimes j} \to L$ implies

$$\left| l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}) \otimes \alpha_{\sigma(i+1)} \otimes \cdots \otimes \alpha_{\sigma(m)} \right| = 0,$$

which further implies

$$\left| l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}) \right| = \left| \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(i)} \right| + i - 2 = 0. \tag{3.18}$$

By assumption, $l_i(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(i)})$ must be non-zero and $j<m-1$ implies $i>1$. Hence we must have $\left|\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(i)}\right|=0$ and therefore, by Eq. 3.18, $i=2$. But this implies $j=m-1$, which contradicts our bounds on j. So no such unshuffle could exist, and therefore the sum (3.17) is zero.

We next consider the sum of the terms $j=1$, $j=m-1$, and $j=m$:

$$\begin{aligned} l_1(l_m(\alpha_1,\ldots,\alpha_m)) + &\sum_{\sigma\in \mathrm{Sh}(2,m-2)} (-1)^\sigma \epsilon(\sigma) l_{m-1}(l_2(\alpha_{\sigma(1)},\alpha_{\sigma(2)}),\alpha_{\sigma(3)},\ldots,\alpha_{\sigma(m)}) \\ &+ \sum_{\sigma\in \mathrm{Sh}(1,m-1)} (-1)^\sigma \epsilon(\sigma)(-1)^{m-1} l_m(l_1(\alpha_{\sigma(1)}),\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(m)}). \end{aligned} \tag{3.19}$$

Note that if $\sigma\in \mathrm{Sh}(1,m-1)$ and $\left|l_1(\alpha_{\sigma(1)})\right|>0$, then

$$l_m(l_1(\alpha_{\sigma(1)}),\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(m)})=0$$

by definition of the map l_m. On the other hand, if $\left|l_1(\alpha_{\sigma(1)})\right|=0$, then $l_1(\alpha_{\sigma(1)})=d\alpha_{\sigma(1)}$ is Hamiltonian and its Hamiltonian vector field is the zero vector field. Hence the third term in (3.19) is zero.

Since the map l_2 is degree 0, we only need to consider the first two terms of (3.19) in the case when $|\alpha_1\otimes\cdots\otimes\alpha_m|=0$. For the first term we have:

$$l_1(l_m(\alpha_1,\ldots,\alpha_m)) = \begin{cases} (-1)^{\frac{m}{2}+1} d\iota(v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_m})\omega & \text{if } m \text{ even,} \\ (-1)^{\frac{m-1}{2}} d\iota(v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_m})\omega & \text{if } m \text{ odd.} \end{cases}$$

Now consider the second term. If $\alpha_i,\alpha_j\in\Omega^{n-1}_{\mathrm{Ham}}(M)$ are Hamiltonian $(n-1)$-forms then by Definition 3.3, $l_2(\alpha_i,\alpha_j)=\{\alpha_i,\alpha_j\}$. By Proposition 3.4, $l_2(\alpha_i,\alpha_j)$ is Hamiltonian and its Hamiltonian vector field is $v_{\{\alpha_i,\alpha_j\}}=[v_{\alpha_i},v_{\alpha_j}]$. Therefore for $\sigma\in\mathrm{Sh}(2,m-2)$, we have

$$l_{m-1}(l_2(\alpha_{\sigma(1)},\alpha_{\sigma(2)}),\alpha_{\sigma(3)},\ldots,\alpha_{\sigma(m)}) = \begin{cases} (-1)^{\frac{m}{2}-1}\iota([v_{\alpha_{\sigma(1)}},v_{\alpha_{\sigma(2)}}]\wedge\cdots\wedge v_{\alpha_{\sigma(m)}})\omega & \text{if } m \text{ even,} \\ (-1)^{\frac{m+1}{2}}\iota([v_{\alpha_{\sigma(1)}},v_{\alpha_{\sigma(2)}}]\wedge\cdots\wedge v_{\alpha_{\sigma(m)}})\omega & \text{if } m \text{ odd.} \end{cases}$$

Since each α_i is degree 0, we can rewrite the sum over $\sigma\in\mathrm{Sh}(2,m-2)$ as

$$\sum_{\sigma\in\mathrm{Sh}(2,m-2)} (-1)^\sigma\epsilon(\sigma) l_{m-1}(l_2(\alpha_{\sigma(1)},\alpha_{\sigma(2)}),\alpha_{\sigma(3)},\ldots,\alpha_{\sigma(m)}) = \sum_{1\le i<j\le m} (-1)^{i+j-1} l_{m-1}(l_2(\alpha_i,\alpha_j),\alpha_1,\alpha_2,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_m).$$

Therefore, if m is even, the sum (3.19) becomes

$$(-1)^{\frac{m}{2}+1} d\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_m})\omega + (-1)^{\frac{m}{2}} \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_{\alpha_i}, v_{\alpha_j}] \wedge v_{\alpha_1} \wedge \cdots \wedge \hat{v}_{\alpha_i} \wedge \cdots \wedge \hat{v}_{\alpha_j} \wedge \cdots \wedge v_{\alpha_m})\omega$$

and, if m is odd:

$$(-1)^{\frac{m-1}{2}} d\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_m})\omega + (-1)^{\frac{m-1}{2}} \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_{\alpha_i}, v_{\alpha_j}] \wedge v_{\alpha_1} \wedge \cdots \wedge \hat{v}_{\alpha_i} \wedge \cdots \wedge \hat{v}_{\alpha_j} \wedge \cdots \wedge v_{\alpha_m})\omega.$$

It then follows from Lemma 3.6 that, in either case, (3.19) is zero. □

It is clear that in the $n = 1$ case, $L_\infty(M, \omega)$ is the underlying Lie algebra of the usual Poisson algebra of smooth functions on a symplectic manifold. In the $n = 2$ case, $L_\infty(M, \omega)$ is the Lie 2-algebra obtained in our previous work with Baez and Hoffnung [5].

For the $n = 2$ case, it will be convenient for us in later chapters to express the Lie 2-algebra $L_\infty(M, \omega)$ in the language of Prop. 3.10:

Proposition 3.15. *If (M, ω) is a 2-plectic manifold, then there is a Lie 2-algebra $L_\infty(M, \omega) = (L, [\cdot, \cdot], J)$ where:*

- $L_0 = \Omega^1_{\mathrm{Ham}}(M)$,
- $L_1 = C^\infty(M)$,
- *the differential $L_1 \xrightarrow{d} L_0$ is the de Rham differential,*
- *the bracket $[\cdot, \cdot]$ is $\{\cdot, \cdot\}$ in degree 0 and trivial otherwise,*
- *the Jacobiator is given by the linear map $J \colon \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \to C^\infty$, where $J(\alpha, \beta, \gamma) = \iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega$.*

Proof. This follows from the fact that d, $[\cdot, \cdot]$, and J are the structure maps l_1, l_2, and l_3, respectively, described in Thm. 3.14. □

Finally, we mention that the equality

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega$$

given in Proposition 3.4 implies the existence of a bracket-preserving chain map

$$\phi \colon L_\infty(M, \omega) \to \mathfrak{X}_{\mathrm{Ham}}(M),$$

which in degree 0 takes a Hamiltonian $(n-1)$-form α to its vector field v_α. Here we consider the Lie algebra of Hamiltonian vector fields as a Lie 1-algebra whose underlying complex is concentrated in degree 0:

$$\ldots \to 0 \to 0 \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

Hence ϕ is trivial in all higher degrees. In light of Theorem 3.14, ϕ becomes a strict morphism of L_∞-algebras. See the paper by Lada and Markl [38] for the definition of strict L_∞-algebra morphisms.

Chapter 4

Lie 2-algebras from compact simple Lie groups

Here we consider some Lie 2-algebras which arise on an important class of 2-plectic manifolds: compact simple Lie groups. Recall from Example 2.8 that such a group admits a 1 parameter family of 2-plectic structures given by a non-zero constant times the Cartan 3-form:

$$\nu_k = k\langle \theta_L, [\theta_L, \theta_L] \rangle, \quad k \neq 0,$$

where θ_L is the left-invariant Maurer-Cartan form, and $\langle \cdot, \cdot \rangle$ is the inner product on the corresponding Lie algebra, whose bracket is $[\cdot, \cdot]$. This 3-form plays an important role in the theory of affine Lie algebras, central extensions of loop groups, and gerbes [8, 13, 47].

Baez and Crans showed that the Lie algebra of a compact simple Lie group G can be used to build a Lie 2-algebra called the 'string Lie 2-algebra' [4]. This Lie 2-algebra can be integrated to a special kind of category called a Lie 2-group. For $G = \mathrm{Spin}(n)$, the geometric realization of this Lie 2-group is homotopy equivalent to the topological group $\mathrm{String}(n)$ [8, 28]. The group $\mathrm{String}(n)$ naturally arises in the study of spin structures on loop spaces [69].

The structure of the string Lie 2-algebra associated to G closely resembles the structure of the Lie 2-algebra $L_\infty(G, \nu_k)$ of Hamiltonian 1-forms on the 2-plectic manifold (G, ν_k). In a private communication, D. Stevenson asked if these Lie 2-algebras are quasi-isomorphic. As we show in Section 4.3, this turns out not to be true. However, we prove that the string Lie 2-algebra is isomorphic to a particular sub Lie-2 algebra of $L_\infty(G, \nu_k)$, consisting of left-invariant Hamiltonian 1-forms. This gives a new geometric construction of the string Lie 2-algebra. For another construction, based on central extensions of loop groups, see the paper by Baez, Crans, Schreiber and Stevenson [8].

It will be interesting to see what can be learned from comparing these approaches.

4.1 Group actions on n-plectic manifolds

We begin by giving some basic results concerning group actions on n-plectic manifolds. Suppose we have a Lie group acting on an n-plectic manifold (M,ω), preserving the n-plectic structure. In this situation the Lie n-algebra $L_\infty(M,\omega)$ constructed in Thm. 3.14 has a sub-n-algebra consisting of invariant differential forms.

More precisely, let $\mu\colon G\times M\to M$ be a left action of the Lie group G on the n-plectic manifold (M,ω), and assume this action preserves the n-plectic structure:

$$\mu_g^*\omega=\omega,$$

for all $g\in G$. Denote the subspace of invariant Hamiltonian $(n-1)$-forms by

$$\Omega^{n-1}_{\mathrm{Ham}}(M)^G=\left\{\alpha\in\Omega^{n-1}_{\mathrm{Ham}}(M)\ \mid \forall g\in G\ \mu_g^*\alpha=\alpha\right\}.$$

The Hamiltonian vector field of an invariant Hamiltonian $(n-1)$-form is itself invariant under the action of G:

Proposition 4.1. *If $\alpha\in\Omega^{n-1}_{\mathrm{Ham}}(M)^G$ and v_α is the Hamiltonian vector field associated with α, then $\mu_{g*}v_\alpha=v_\alpha$ for all $g\in G$.*

Proof. The exterior derivative commutes with the pullback of the group action. Therefore if $v_1,\dots,v_n$ are smooth vector fields, then $d\alpha\left(\mu_{g*}v_1,\dots,\mu_{g*}v_n\right)=d\alpha\left(v_1,\dots,v_n\right)$, since we are assuming α is G-invariant. Since $\alpha\in\Omega^{n-1}_{\mathrm{Ham}}(M)$, we have $d\alpha=-\iota_{v_\alpha}\omega$, so

$$\omega\left(v_\alpha,\mu_{g*}v_1,\dots,\mu_{g*}v_n\right)=\omega\left(v_\alpha,v_1,\dots,v_n\right)=\omega\left(\mu_{g*}v_\alpha,\mu_{g*}v_1,\dots,\mu_{g*}v_n\right),$$

where the last equality follows from ${\mu_g}^*\omega=\omega$. Therefore

$$\omega\left(v_\alpha-\mu_{g*}v_\alpha,\mu_{g*}v_1,\dots,\mu_{g*}v_n\right)=0.$$

Since ω is nondegenerate, and $v_1,\dots,v_n$ are arbitrary, it follows that $\mu_{g*}v_\alpha=v_\alpha$. □

Let $\Omega^k(M)^G$ denote the subspace of invariant k-forms on M:

$$\Omega^k(M)^G=\left\{\alpha\in\Omega^k(M)\mid\forall g\in G\ \mu_g^*\alpha=\alpha\right\},$$

and let (L^G,d) denote the n-term complex

$$C^\infty(M)^G\xrightarrow{d}\Omega^1(M)^G\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{n-2}(M)^G\xrightarrow{d}\Omega^{n-1}_{\mathrm{Ham}}(M)^G.$$

Clearly, this is a subcomplex of the underlying complex of the Lie n-algebra $L_\infty(M,\omega)$. Moreover, the invariant differential forms on M form a graded subalgebra that is stable under exterior derivative and interior product with an invariant vector field [23][Sec. III.4]. Since both the bracket introduced in Def. 3.3 and the proof of Lemma 3.6 depend only on compositions of these operations, the Lie n-algebra structure described in Theorem 3.14 restricts to a Lie n-algebra structure on the subcomplex L^G. Hence, we have the following theorem:

Theorem 4.2. *Given an n-plectic manifold (M,ω) equipped with group action $G\times M \to M$ preserving the n-plectic structure, there is a Lie n-algebra $L_\infty(M,\omega)^G = (L^G, \{l_k\})$ with underlying graded vector space*

$$L_i^G = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M)^G & i=0,\\ \Omega^{n-1-i}(M)^G & 0<i\leq n-1,\end{cases}$$

and maps $\left\{l_k\colon \left(L^G\right)^{\otimes k} \to L^G | 1\leq k<\infty\right\}$ defined as

$$l_1(\alpha)=d\alpha,$$

if $|\alpha|>0$ and

$$l_k(\alpha_1,\dots,\alpha_k) = \begin{cases} 0 & \text{if } |\alpha_1\otimes\cdots\otimes\alpha_k|>0,\\ (-1)^{\frac{k}{2}+1}\iota(v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_k})\omega & \text{if } |\alpha_1\otimes\cdots\otimes\alpha_k|=0 \text{ and } k \text{ even},\\ (-1)^{\frac{k-1}{2}}\iota(v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_k})\omega & \text{if } |\alpha_1\otimes\cdots\otimes\alpha_k|=0 \text{ and } k \text{ odd},\end{cases} \tag{4.1}$$

for $k>1$, where v_{α_i} is the unique invariant Hamiltonian vector field associated to $\alpha_i\in\Omega^{n-1}_{\mathrm{Ham}}(M)^G$.

4.2 Compact simple Lie groups as 2-plectic manifolds

Recall from Example 2.8 that for any compact simple Lie group G, the 2-plectic structure $\nu_k = k\langle\theta_L,[\theta_L,\theta_L]\rangle$ is left-invariant. Hence, Thm. 4.2 implies there exists a Lie 2-algebra whose underlying 2-term chain complex is composed of left-invariant Hamiltonian 1-forms $\Omega^1_{\mathrm{Ham}}(G)^L$ on G in degree 0, and left-invariant functions $C^\infty(G)^L$ in degree 1.

If $f\in C^\infty(G)^L$, then by definition $f=f\circ L_g$ for all $g\in G$. Hence f must be a constant function, so $C^\infty(G)^L$ may be identified with $\mathbb{R}$. Denote the space of all left invariant 1-forms as $\Omega^1(G)^L \cong \mathfrak{g}^*$, and left invariant vector fields as $\mathfrak{X}(G)^L \cong \mathfrak{g}$. The following theorem characterizes the left invariant Hamiltonian 1-forms.

Theorem 4.3. *Every left invariant 1-form on (G, ν_k) is Hamiltonian. That is, $\Omega^1_{\mathrm{Ham}}(G)^L = \Omega^1(G)^L$.*

Proof. Recall that if α is a 1-form and v_0, v_1 are vector fields, then

$$d\alpha(v_0, v_1) = v_0(\alpha(v_1)) - v_1(\alpha(v_0)) - \alpha([v_0, v_1]).$$

Suppose now that α is a left invariant 1-form on G and v_0, v_1 are left invariant vector fields. Then the smooth functions $\alpha(v_1)$ and $\alpha(v_0)$ are also left invariant and therefore constant. Therefore the right hand side of the above equality simplifies and we have

$$d\alpha(v_0, v_1) = -\alpha([v_0, v_1]).$$

Let $\alpha \in \Omega^1(G)^L$ and let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{g}$ used in the construction of ν_k. Note we have two isomorphisms

$$\mathfrak{g} \xrightarrow{k\langle \cdot, \cdot \rangle} \mathfrak{g}^*, \quad \mathfrak{X}(G)^L \xrightarrow{\theta_L} \mathfrak{g}.$$

Therefore, there exists a left invariant vector field $v_\alpha \in \mathfrak{X}(G)^L$ such that $\alpha(v') = k\langle \theta_L(v_\alpha), \theta_L(v') \rangle$ for all left invariant vector fields $v' \in \mathfrak{X}(G)^L$. Combining this with the above expression for $d\alpha$ gives

$$d\alpha(v_0, v_1) = -k\langle \theta_L(v_\alpha), [\theta_L(v_0), \theta_L(v_1)] \rangle,$$

which implies

$$d\alpha = -\iota_{v_\alpha} \nu_k.$$

Hence $\alpha \in \Omega^1_{\mathrm{Ham}}(G)$, and $\Omega^1_{\mathrm{Ham}}(G)^L = \Omega^1_{\mathrm{Ham}}(G) \cap \Omega^1(G)^L = \Omega^1(G)^L$. □

The most important application of Thm. 4.3 is that it allows us to use Thm. 4.2 and the isomorphism $\Omega^1_{\mathrm{Ham}}(G)^L = \Omega^1(G)^L \cong \mathfrak{g}^*$ to construct a Lie 2-algebra having $\mathfrak{g}^*$ as its space of 0-chains, for any compact simple Lie group. Recalling the simpler definition of a Lie 2-algebra given in Prop. 3.10, we summarize these facts in the following corollary.

Corollary 4.4. *If G is a compact simple Lie group with Lie algebra $\mathfrak{g}$ and 2-plectic structure ν_k, then there is a Lie 2-algebra $L_\infty(G, \nu_k)^L$ where:*

- *the space of 0-chains is $\mathfrak{g}^*$,*
- *the space of 1-chains is $\mathbb{R}$,*

- *the differential is the exterior derivative* $d\colon \mathbb{R} \to \mathfrak{g}^*$ *(i.e.* $d = 0$*),*
- *the bracket is* $\{\alpha, \beta\} = \nu_k(v_\alpha, v_\beta, \cdot)$ *in degree 0, and trivial otherwise,*
- *the Jacobiator is the linear map* $J\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathbb{R}$ *defined by* $J(\alpha, \beta, \gamma) = -\nu_k(v_\alpha, v_\beta, v_\gamma)$*.*

In the statement of the above corollary, we are abusing notation slightly by viewing $\alpha \in g^*$ as a left-invariant Hamiltonian 1-form. Note that the corollary implies that we have a 1-parameter family of Lie 2-algebras:

$$\left\{L_\infty(G, \nu_k)^L\right\}_{k \neq 0}.$$

Also, we see from the proof of Thm. 4.3 that there is a simple correspondence between left invariant Hamiltonian 1-forms and left invariant Hamiltonian vector fields which relies on the isomorphism between $\mathfrak{g}$ and its dual space via the inner product $\langle \cdot, \cdot \rangle$. As a result, we have the following proposition which will be useful in the next section.

Proposition 4.5. *If* G *is a compact simple Lie group with 2-plectic structure* ν_k *and* $\langle \cdot, \cdot \rangle$ *is the inner product on the Lie algebra* $\mathfrak{g}$ *of* G *used in the construction of* ν_k*, then there is an isomorphism of vector spaces*

$$\varphi\colon \mathfrak{X}(G)^L \xrightarrow{\sim} \Omega^1_{\mathrm{Ham}}(G)^L$$

such that $\varphi(v) = k\langle \theta_L(v), \theta_L(\cdot) \rangle$ *is the unique left-invariant Hamiltonian 1-form whose Hamiltonian vector field is* v*.*

Proof. We show only uniqueness since the rest of the proposition follows immediately from the arguments made in the proof of Thm. 4.3. Let α and β be left invariant 1-forms. The arguments made in the aforementioned proof imply $d\alpha = -\iota_{v_\alpha}\nu_k$ and $d\beta = -\iota_{v_\beta}\nu_k$, where v_α and v_β are the unique left-invariant vector fields such that $\alpha = k\langle \theta_L(v_\alpha), \cdot \rangle$ and $\beta = k\langle \theta_L(v_\beta), \cdot \rangle$. If $v_\alpha = v_\beta$ is the Hamiltonian vector field for both α and β, then the nondegeneracy of the inner product implies $\alpha = \beta$. □

Remark 4.6. In general, if α and β are Hamiltonian 1-forms sharing the same Hamiltonian vector field, then $d(\alpha - \beta) = 0$. Hence, Prop. 4.5 implies that there are no non-trivial left invariant closed 1-forms. Since the left-invariant de Rham cohomology of G is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$, Prop. 4.5 is equivalent to the well-known fact that $H^1_{\mathrm{CE}}(\mathfrak{g}, \mathbb{R}) = 0$ for any simple Lie algebra.

4.3 The string Lie 2-algebra

We have described how to construct a Lie 2-algebra of left-invariant forms, from any compact simple Lie group G, and any nonzero real number k, using the 2-plectic structure ν_k. Now we show that this Lie 2-algebra is isomorphic to the 'string Lie 2-algebra' of G.

It was shown in previous work by Baez and Crans [4] that Lie 2-algebras can be classified up to equivalence by data consisting of:

- a Lie algebra $\mathfrak{g}$,
- a vector space V,
- a representation $\rho\colon \mathfrak{g} \to \mathrm{End}\,(V)$,
- an element $[j] \in H^3\,(\mathfrak{g}, V)$ of the Lie algebra cohomology of $\mathfrak{g}$.

A Lie 2-algebra L is constructed from this data by setting the space of 0-chains L_0 equal to $\mathfrak{g}$, the space 1-chains L_1 equal to V, and the differential to be the zero map: $d = 0$. The bracket $[\cdot,\cdot] : L \otimes L \to L$ is defined to be the Lie bracket on $\mathfrak{g}$ in degree 0, and defined in degrees 1 and 2 by:

$$[x, a] = \rho_x(a), \qquad [a, x] = -\rho_x(a), \qquad [a, b] = 0,$$

for all $x \in L_0$ and $a, b \in L_1$. The Jacobiator is taken to be any 3-cocycle j representing the cohomology class $[j]$.

From this classification we can construct the **string Lie 2-algebra** $\mathfrak{g}_k$ of a compact simple Lie group G by taking $\mathfrak{g}$ to be the Lie algebra of G, V to be $\mathbb{R}$, ρ to be the trivial representation, and

$$j(x, y, z) = k\big\langle x, [y, z]\big\rangle$$

where $k \in \mathbb{R}$. When $k \neq 0$, the 3-cocycle j represents a nontrivial cohomology class. Note that since ρ is trivial, the bracket of $\mathfrak{g}_k$ is trivial in all degrees except 0.

It is natural to expect that the string Lie 2-algebra is closely related to the Lie 2-algebra $L_\infty(G, \nu_k)^L$ described in Corollary 4.4, since both are built using solely the trilinear form $k\langle\cdot, [\cdot,\cdot]\rangle$ on $\mathfrak{g}$. Indeed, this turns out to be the case:

Theorem 4.7. *If G is a compact simple Lie group with Lie algebra $\mathfrak{g}$ and 2-plectic structure ν_k, then the string Lie 2-algebra $\mathfrak{g}_k$ is isomorphic to the Lie 2-algebra $L_\infty(G, \nu_k)^L$ of left-invariant Hamiltonian 1-forms.*

Proof. The underlying chain complex of $\mathfrak{g}_k$ is $\mathbb{R} \xrightarrow{0} \mathfrak{g}$, while the underlying chain complex of $L_\infty(G, \nu_k)^L$ is $\mathbb{R} \xrightarrow{0} \mathfrak{g}^*$. The isomorphism given Prop. 4.5:

$$\varphi\colon \mathfrak{X}(G)^L \xrightarrow{\sim} \Omega^1_{\mathrm{Ham}}(G)^L, \quad \varphi(v) = k\langle\theta_L(v), \theta_L(\cdot)\rangle$$

induces an isomorphism of complexes

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{0} & \mathfrak{g} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{R} & \xrightarrow{0} & \mathfrak{g}^* \end{array}$$

Note we implicitly used the identifications $\mathfrak{g} \cong \mathfrak{X}(G)$ and $\mathfrak{g}^* \cong \Omega^1_{\mathrm{Ham}}(G)^L$. Let $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ be the brackets of $\mathfrak{g}_k$ and $L_\infty(G, \nu_k)^L$, respectively. According to Def. 3.11, we must show that the maps $\{\cdot,\cdot\} \circ (\varphi \otimes \varphi)$ and $\varphi \circ [\cdot,\cdot]$ are chain homotopic. They are, in fact, equal.

Indeed, if $v_1, v_2 \in \mathfrak{g}$, then it follows from Proposition 4.5 that $\varphi(v_1)$, $\varphi(v_2)$, and $\varphi([v_1, v_2])$ are the unique left invariant Hamiltonian 1-forms whose Hamiltonian vector fields are v_1, v_2, and $[v_1, v_2]$, respectively. But Proposition 3.4 implies

$$d\{\varphi(v_1), \varphi(v_2)\} = -\iota_{[v_1,v_2]}\nu_k.$$

Hence $[v_1, v_2]$ is also the Hamiltonian vector field of $\{\varphi(v_1), \varphi(v_2)\}$. It then follows from uniqueness that $\{\varphi(\cdot), \varphi(\cdot)\} = \varphi([\cdot,\cdot])$. □

We conclude this chapter by showing that $L_\infty(G, \nu_k)$ and $\mathfrak{g}_k$ are not equivalent.

Proposition 4.8. *If G is a compact simple Lie group with Lie algebra $\mathfrak{g}$, then the Lie 2-algebra of Hamiltonian 1-forms $L_\infty(G, \nu_k)$ and the string Lie 2-algebra $\mathfrak{g}_k$ are not quasi-isomorphic.*

Proof. By definition, any quasi-isomorphism of Lie 2-algebras must induce an isomorphism on homology. Hence, to prove the statement, it is sufficient to show that the homology of the complex

$$L_\bullet = C^\infty(G) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(G),$$

is not isomorphic to the complex $\mathbb{R} \xrightarrow{0} \mathfrak{g}$. We will prove this by showing that the degree 0 homology of $L_\bullet$ has dimension greater than $\dim \mathfrak{g} = \dim \mathfrak{X}(G)^L$.

Let $\theta_R \in \Omega^1(G, \mathfrak{g})$ be the right-invariant Maurer-Cartan form. At any point $g \in G$, it can be written as

$$\theta_R|_g(v) = R_{g^{-1}*}v, \quad v \in T_gG.$$

Therefore, $\theta_R|_g = \mathrm{Ad}_g\,\theta_L|_g$. Since the 2-plectic form ν_k is left and right invariant, we have the equalities:

$$\begin{aligned}\nu_k &= k\langle \theta_L, [\theta_L, \theta_L]\rangle \\ &= k\langle \mathrm{Ad}_g\,\theta_L, \mathrm{Ad}_g[\theta_L, \theta_L]\rangle \\ &= k\langle \mathrm{Ad}_g\,\theta_L, [\mathrm{Ad}_g\,\theta_L, \mathrm{Ad}_g\,\theta_L]\rangle \\ &= k\langle \theta_R, [\theta_R, \theta_R]\rangle.\end{aligned}$$

The last equality implies that we can use the proof of Thm. 4.3 to show that every right invariant form is Hamiltonian.

Since the Lie algebra $\mathfrak{g}$ is simple, it is not abelian. Therefore, there exists $x, y \in \mathfrak{g}$ such that $[x, y] \neq 0$. Let v^x be the right invariant vector field equal to x at the identity. That is,

$$v^x|_g = R_{g*}x.$$

Note that v^x is the Hamiltonian vector field corresponding to the right invariant Hamiltonian 1-form $k\langle \theta_R(v^x), \theta_R\rangle$. We claim v^x is not left invariant. Indeed, if it was then the equality

$$L_{g*}x = v^x|_g = R_{g*}x$$

would hold for all g. In particular, this implies

$$\mathrm{Ad}_{\exp(ty)}\,x = x,$$

and therefore

$$[y, x] = \frac{d}{dt}\mathrm{Ad}_{\exp(ty)}\,x\Big|_{t=0} = 0,$$

which contradicts our choice of x and y. Hence

$$\mathfrak{X}(G)^L \cap \mathrm{span}_{\mathbb{R}} v^x = 0 \tag{4.2}$$

The kernel of the surjection $\Omega^1_{\mathrm{Ham}}(G) \twoheadrightarrow \mathfrak{X}_{\mathrm{Ham}}(G)$ which sends a Hamiltonian 1-form to its vector field is the space of closed 1-forms. Since G is compact, its de Rham cohomology is isomorphic to the Chevalley-Eilenberg cohomology of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, its first cohomology group vanishes. Hence every closed 1-form on G is exact. Therefore,

$$H_0(L_\bullet) = \Omega^1_{\mathrm{Ham}}(G)/dC^\infty(G) \cong \mathfrak{X}_{\mathrm{Ham}}(G).$$

The left invariant vector fields $\mathfrak{X}(G)^L \cong \mathfrak{g}$ are all Hamiltonian by Prop. 4.5. Since v^x is Hamiltonian, (4.2) implies

$$\dim \mathfrak{g} < \dim \mathfrak{X}_{\mathrm{Ham}}(G) = \dim H_0(L_\bullet).$$

□

Chapter 5

Stacks, gerbes, and Deligne cohomology

In this chapter, we begin the passage from the classical to the quantum by introducing the technical machinery needed to geometrically quantize n-plectic manifolds.

A principal U(1)-bundle P over a manifold M can be specified by giving U(1)-valued transition functions with respect to an open cover of M. A connection on P is given by specifying local 1-forms on M that satisfy a compatibility condition with the transition functions. The exterior derivative of these 1-forms gives a global 2-form on M called the curvature of the connection. Conversely, if M is equipped with a closed 2-form ω satisfying a certain integrality condition, then one can show that there exists a principal U(1)-bundle, with connection, on M whose curvature is ω. When ω is also non-degenerate, the bundle or, equivalently, its associated Hermitian line bundle, plays a major role in the geometric quantization of the symplectic manifold (M, ω).

Our goal is to generalize these facts to n-plectic geometry. We begin by observing that the word "bundle" can be replaced by the word "sheaf". From any fiber bundle $E \to M$, one can construct a sheaf of sections, which assigns to an open set $U \subseteq M$ the set of local sections $\sigma \colon U \to E$. In particular, the sheaf of sections of a principal U(1)-bundle is what is known as a '$\underline{\mathrm{U}(1)}$-torsor', where $\underline{\mathrm{U}(1)}$ denotes the sheaf of sections of the trivial U(1)-bundle. These torsors can be equipped with extra structure which gives a connection on the corresponding bundle.

The higher analogue of a sheaf is what is known as a 'stack'. In particular, the higher analogue of a $\underline{\mathrm{U}(1)}$-torsor is a special kind of stack called a U(1)-gerbe. Just as the transition functions of a $\underline{\mathrm{U}(1)}$-torsor give a 1-cocycle, the transition functions of a U(1)-gerbe give a 2-cocycle. Stacks and gerbes were originally developed within the context of algebraic geometry by Grothendieck [25] and Giraud [22], respectively. More recent work demonstrates that they naturally arise in differential geometry as well. Brylinski [13] showed that U(1)-gerbes on manifolds can be equipped with additional structures, which we call '2-connections'. These are the higher analogues of connections on U(1)-bundles. More precisely, a 2-connection on a U(1)-gerbe over M is specified by local 1-forms and 2-forms on M satisfying various compatibility conditions. The exterior derivative of the 2-forms give a global closed 3-form called the '2-curvature'. Conversely, if M is equipped with a closed 3-form ω satisfying an integrality condition, then one can show that there exists a U(1)-gerbe with 2-connection on M whose 2-curvature is ω. As we will see, in analogy with the symplectic case, U(1)-gerbes with 2-connections play an important role in the quantization of 2-plectic manifolds.

Brylinski's results rely heavily on a formalism called 'Deligne cohomology', which can be thought of as a refinement of the usual Čech cohomology that classifies principal bundles. In degree one, Deligne cohomology classifies principal U(1)-bundles equipped with a connection. Similarly, in degree two, it classifies U(1)-gerbes equipped with a 2-connection. It is easy to describe the higher degree groups as well. However, geometric structures [21] that are classified by these groups are, in general, more difficult to work with.

Let us conclude this introduction by briefly outlining the main results found in the chapter. We first review the basic theory of stacks and gerbes. We then give a somewhat detailed description of Deligne cohomology, and we provide proofs of some statements not easily found in the literature. After presenting Brylinski's construction for equipping a gerbe with a 2-connection, we introduce what we call a '2-line stack'. This stack categorifies the concept of a Hermitian line bundle . We show that every U(1)-gerbe with 2-connection has an associated 2-line stack with 2-connection. In the final section, we present Carey, Johnson, and Murray's formalism [17] for computing the holonomy of a 2-connection, which we will use in our quantization procedure for 2-plectic manifolds in Chapter 7.

5.1 Stacks

When introducing sheaf theory, one begins by first defining a presheaf on a topological space M as a contravariant functor $\mathsf{Open}(M) \to \mathsf{Set}$. The objects of the category $\mathsf{Open}(M)$ are open sets of M and the morphisms are inclusion maps. Similarly, in the theory of stacks, we begin by defining fibered categories and prestacks. Just as a presheaf assigns a set to each open set $U \subseteq M$, a fibered category assigns a category to each such set.

Definition 5.1 ([45])**.** *A* **fibered category** F *over* M *consists of:*

- *a category* $\mathsf{F}(U)$ *for each open set* $U \subseteq M$*,*
- *a functor* $i^*\colon \mathsf{F}(V) \to \mathsf{F}(U)$ *for each inclusion* $i\colon U \hookrightarrow V$ *of open sets,*
- *a natural isomorphism* $t_{i,j}\colon (ij)^* \xrightarrow{\sim} j^*i^*$ *for each pair of composable inclusions*

$$W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U,$$

such that for any triple of composable inclusions

$$Y \overset{k}{\hookrightarrow} W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$$

the following diagram commutes:

$$\begin{array}{ccc} (ijk)^* & \xrightarrow{t_{ij,k}} & k^*(ij)^* \\ {\scriptstyle t_{i,jk}}\downarrow & & \downarrow{\scriptstyle k^*t_{i,j}} \\ (jk)^*i^* & \xrightarrow{t_{j,k}i^*} & k^*j^*i^* \end{array}$$

The above definition implies that a fibered category is a contravariant 'pseudo-functor' $\mathsf{F}\colon \mathsf{Open}(M) \to \mathsf{Cat}$. The following example of a fibered category is perhaps the most important one for us.

Example 5.2 (Sheaves on a manifold)**.** Let M be a manifold. To each open set $U \subseteq M$, assign the category $\mathsf{Sh}(U)$, whose objects are sheaves on U. To each inclusion of open sets $V \overset{i}{\hookrightarrow} U$ assign the functor

$$\mathsf{Sh}(U) \xrightarrow{i^*} \mathsf{Sh}(V)$$
$$F \mapsto F|_V,$$

where $F|_V$ is the restriction of the sheaf F to the open set V. For any open set $W \subseteq V$, we have $F|_V(W) = F(W)$. Hence, given $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$, the functors $(ij)^*$ and j^*i^* are equal. Therefore, the natural isomorphisms $t_{i,j}$ may be taken to be the identity.

Definition 5.3 ([45])**.** *A* **morphism** *between fibered categories* F *and* G *over* M *consists of*

- *a functor* $\phi_U\colon \mathsf{F}(U) \to \mathsf{G}(U)$ *for every open set* $U \subseteq M$*,*

- *a natural isomorphism* $\alpha_i\colon \phi_V i^* \xrightarrow{\sim} i^*\phi_U$ *for every inclusion* $V \overset{i}{\hookrightarrow} U$*, such that for every pair of composable inclusions* $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$ *the diagram*

$$\begin{array}{ccccc} \phi_W(ij)^* & & \xrightarrow{\quad\alpha_{ij}\quad} & & (ij)^*\phi_U \\ \Big\downarrow{\scriptstyle \phi_W\tau_{i,j}} & & & & \Big\downarrow{\scriptstyle \tau_{i,j}\phi_U} \\ \phi_W j^* i^* & \xrightarrow{\alpha_j i^*} & j^*\phi_V i^* & \xrightarrow{j^*\alpha_i} & j^* i^* \phi_U \end{array}$$

  commutes.

Recall that an isomorphism of presheaves is given by local isomorphisms of sets. The corresponding notion for fibered categories is slightly weaker. It incorporates equivalences of categories, rather than isomorphisms of categories.

Definition 5.4. *A morphism* $(\phi, \alpha)\colon \mathsf{F} \to \mathsf{G}$ *is an* **equivalence** *iff every functor* ϕ_U *is an equivalence of categories.*[1]

If F is a fibered category over M, and $U \subseteq M$ is an open set, then given any objects $x, y \in \mathsf{F}(U)$, one can construct a presheaf on U by assigning to an open set $V \overset{i}{\hookrightarrow} U$ the set $\mathrm{Hom}_{\mathsf{F}(V)}(i^*x, i^*y)$. We denote this presheaf $\underline{\mathrm{Hom}}_{\mathsf{F}}(x, y)$.

Definition 5.5 ([45]). *A fibered category* F *over* M *is a* **prestack** *iff for every open set* $U \subseteq M$ *and objects* $x, y \in \mathsf{F}(U)$*, the presheaf* $\underline{\mathrm{Hom}}_{\mathsf{F}}(x, y)$ *is a sheaf.*

Our definition of a stack will, again, come from Moerdijk [45]. However, it is more convenient to give his definition using nerves of open covers, which we will explain below. This makes our notation appear more like Brylinski's [13] Def. 5.2.1. However, we warn the reader that Brylinski's definition of a fibered category uses a "larger" source category than $\mathsf{Open}(M)$. Its objects are arbitrary local homeomorphisms into M. For what we need to do, it is not necessary to use this larger category.

Given an open cover $\mathcal{U} = \{U_a\}$ of an open set $V \subseteq M$, we consider the disjoint union $\mathcal{U}^{[0]} = \coprod_a U_a$, and the n-fold fiber product:

$$\mathcal{U}^{[n]} = \underbrace{\mathcal{U}^{[0]} \times_V \cdots \times_V \mathcal{U}^{[0]}}_{n+1} = \coprod_{a_1, a_2, \cdots, a_{n+1}} U_{a_1} \cap U_{a_2} \cap \cdots \cap U_{a_{n+1}}. \tag{5.1}$$

There is a map $p_0\colon \mathcal{U}^{[0]} \to V$ given by the inclusion maps $U_a \hookrightarrow V$. Similarly, there exists $n+1$ maps $p_{1,\ldots,\hat{k},\ldots,n+1}\colon \mathcal{U}^{[n]} \to \mathcal{U}^{[n-1]}$ determined by inclusion maps of the form

$$U_{a_1} \cap \cdots \cap U_{a_{n+1}} \hookrightarrow U_{a_1} \cap \cdots \cap U_{a_{k-1}} \cap \widehat{U_{a_k}} \cap U_{a_{k+1}} \cap \cdots \cap U_{a_{n+1}}, \tag{5.2}$$

[1] An equivalence in the sense of Def. 5.4 is called a 'strong equivalence' in [45].

Putting these all together, we obtain the following diagram in the category of manifolds:

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathcal{U}^{[2]} \underset{p_{23}}{\overset{p_{12}}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} \mathcal{U}^{[1]} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{U}^{[0]} \xrightarrow{p_0} V. \tag{5.3}$$

This is called the **nerve** of the cover $\mathcal{U}$. In particular, the maps p_1, p_2 are the projections from the first and second factor, respectively, and p_{12}, p_{13}, p_{23} are the projections from the first and second, first and third, and second and third factors, respectively. We sometimes will slightly abuse notation by writing the compositions $p_1 p_{ij}$ and $p_2 p_{ij}$ as p_i and p_j, respectively. The nerve of a cover is useful for expressing the various gluing properties of both sheaves and stacks.

Let us establish just a bit more notation. If F is a presheaf on M, then we define the product

$$F(\mathcal{U}^{[n]}) := \prod_{a_1,\dots,a_{n+1}} F(U_{a_1} \cap \dots \cap U_{a_{n+1}}). \tag{5.4}$$

Then applying F to the diagram (5.3) gives, for example,

$$F(V) \xrightarrow{p_0^*} F(\mathcal{U}^{[0]}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} F(\mathcal{U}^{[1]}) \cdots, \tag{5.5}$$

where p_i^* are *maps between sets* corresponding to restriction of sections. Now, if F is a fibered category on M, we define the category:

$$\mathsf{F}(\mathcal{U}^{[n]}) := \prod_{a_1,\dots,a_{n+1}} \mathsf{F}(U_{a_1} \cap \dots \cap U_{a_{n+1}}),$$

where the product on the right-hand side is the product of categories. We apply F to (5.3) and obtain:

$$\mathsf{F}(V) \xrightarrow{p_0^*} \mathsf{F}(\mathcal{U}^{[0]}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathsf{F}(\mathcal{U}^{[1]}) \underset{p_{23}^*}{\overset{p_{12}^*}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}} \mathsf{F}(\mathcal{U}^{[2]}) \cdots$$

Here, p_i^*, p_{ij}^* are *functors between categories*, which are determined by the functors corresponding to the inclusions (5.2). Similarly, there are natural isomorphisms $t_{p_i,p_{jk}} \colon (p_i p_{jk})^* \to p_{jk}^* p_i^*$.

We can now give a relatively concise definition of a stack.

Definition 5.6. *A prestack* F *over* M *is a* **stack** *if and only if given the data:*

- *an open cover* $\mathcal{U}$ *of an open set* $V \subseteq M$*,*
- *an object* $x \in \mathsf{F}(\mathcal{U}^{[0]})$*,*

- *an isomorphism*

$$\phi\colon p_2^*x \xrightarrow{\sim} p_1^*x$$

in $\mathsf{F}(\mathcal{U}^{[1]})$ *such that the following diagram in* $\mathsf{F}(\mathcal{U}^{[2]})$ *commutes:*

$$\begin{array}{ccccc}
p_{23}^*p_2^*x & \xrightarrow{\phi} & p_{23}^*p_1^*x & \xrightarrow{t^{-1}_{p_1,p_{23}}} & p_2^*x \\
{\scriptstyle t^{-1}_{p_2,p_{23}}}\downarrow & & & & \downarrow{\scriptstyle t_{p_2,p_{12}}} \\
p_3^*x & & & & p_{12}^*p_2^*x \\
{\scriptstyle t_{p_2,p_{13}}}\downarrow & & & & \downarrow{\scriptstyle \phi} \\
p_{13}^*p_2^*x & & & & p_{12}^*p_1^*x \\
{\scriptstyle \phi}\downarrow & & & & \downarrow{\scriptstyle t^{-1}_{p_1,p_{12}}} \\
p_{13}^*p_1^*x & & \xrightarrow{t^{-1}_{p_1,p_{13}}} & & p_1^*p_1^*x
\end{array}$$

there exists an object $\tilde{x} \in \mathsf{F}(V)$*, unique up to isomorphism, together with an isomorphism*

$$\psi\colon p_0^*\tilde{x} \xrightarrow{\sim} x$$

in $\mathsf{F}(\mathcal{U}^{[0]})$ *such that the following diagram in* $\mathsf{F}(\mathcal{U}^{[1]})$ *commutes:*

$$\begin{array}{ccccccc}
p_2^*p_0^*\tilde{x} & \xrightarrow{t_{p_0,p_2}} & (p_0p_2)^*\tilde{x} & = & (p_0p_1)^*\tilde{x} & \xrightarrow{t^{-1}_{p_0,p_1}} & p_1^*p_0^*\tilde{x} \\
{\scriptstyle \psi}\downarrow & & & & & & \downarrow{\scriptstyle \psi} \\
p_2^*x & & & \xrightarrow{\phi} & & & p_1^*x
\end{array}$$

Hence, just as sections of a sheaf can glue together in a unique way, objects in a stack can glue together uniquely up to isomorphism. In addition, note that the prestack condition implies that morphisms between objects can be glued together as well. A morphism between stacks is simply a morphism between the underlying fibered categories.

Proposition 5.7. *Let* M *be a manifold. The fibered category which assigns to an open set* $U \subseteq M$ *the category* $\mathsf{Sh}(U)$ *of sheaves on* U*, as defined in Example 5.2, is a stack.*

Proof. We refer the reader to Sec. 5.1 in [13] for the proof. □

Finally, we mention that if F is a stack over M, then we will often refer to the objects of the category $\mathsf{F}(M)$ as the **global sections** of F.

5.2 Gerbes

Roughly, gerbes are to stacks, as principal bundles are to fiber bundles. To see this, let us first give the precise definition for a torsor.

Definition 5.8. *Let $\underline{G}$ be the sheaf of smooth functions with values in the Lie group G. A* $\underline{G}$**-torsor** *over a manifold M is a sheaf F together with an action $\underline{G} \times F \to F$ such that for each $x \in M$, there exists an open neighborhood U of x with the property that for each open $V \subseteq U$, the set $F(V)$ is a principal homogeneous $G(V)$-space.*

The sheaf $\underline{G}$ itself is the **trivial** $\underline{G}$**-torsor**. Note the definition implies that if F is a $\underline{G}$-torsor on M, then F is **locally isomorphic** to $\underline{G}$. That is, for all $x \in M$ there exists an open neighborhood $U \ni x$, such that restricted sheaves F_U and $\underline{G}_U$ are isomorphic. Morphisms between $\underline{G}$-torsors are morphisms of the underlying sheaves which respect the $\underline{G}$-action. As mentioned in the introduction to the chapter, the sheaf of sections of a principal G-bundle is a $\underline{G}$-torsor. Conversely, every $\underline{G}$-torsor is isomorphic to such a sheaf of sections.

We can construct a fibered category on a manifold M which assigns to every open set U, the category of $\underline{G}$-torsors over U. Using the fact that Sh is a stack, it is not difficult to see that this fibered category is also a stack, which we denote as Tor_G. Just as $\underline{G}$-torsors are special kinds of sheaves, Tor_G is a special kind of stack. For example, for any open set U, the morphisms in the category $\mathsf{Tor}_G(U)$ are all isomorphisms. Hence, $\mathsf{Tor}_G(U)$ is a groupoid. In fact, it is a non-empty groupoid, since we always have the trivial $\underline{G}$-torsor over every open set U. Also, since every $\underline{G}$-torsor is locally isomorphic to $\underline{G}$, any two $\underline{G}$-torsors in $\mathsf{Tor}_G(U)$ will become isomorphic when pulled back to the category $\mathsf{Tor}_G(V)$, if V is a "small enough" open subset of U. By axiomatizing these facts, one arrives at the definition of a G-gerbe. Tor_G itself is called the **trivial G-gerbe**. In fact, as we will see, the definition implies that a G-gerbe is a stack that is locally isomorphic to the stack Tor_G.

Definition 5.9 ([13, 22])**.** *Let G be a Lie group. A stack G over M is a G***-gerbe** *iff:*

1. *for every open set $U \subseteq M$, the category $\mathsf{G}(U)$ is a groupoid,*
2. *there exists an open cover $\mathcal{U}$ of M such that the groupoid $\mathsf{G}(\mathcal{U}^{[0]})$ is non-empty,*
3. *for every open set $V \subseteq M$ and every pair of objects $P, Q \in \mathsf{G}(V)$, there exists an open cover $\mathcal{U}$ of V such that p_0^*P and p_0^*Q are isomorphic as objects in $\mathsf{G}(\mathcal{U}^{[0]})$,*
4. *for every open set $U \subseteq M$ and every object $P \in \mathsf{G}(U)$, there exists a local isomorphism between the sheaf of groups $\underline{\mathrm{Aut}}_{\mathsf{G}}(P) = \underline{\mathrm{Hom}}_{\mathsf{G}}(P, P)$ and the sheaf $\underline{G}_U$. This local isomorphism is unique up to inner automorphisms of G.*

Roughly, a morphism between G-gerbes is a morphism between the underlying stacks, which respects the local isomorphisms between the sheaves $\underline{\mathrm{Aut}}_{\mathsf{G}}(P)$ and $\underline{G}_U$. See the definition following Prop. 5.2.7 in [13] for the precise details.

The classification of U(1)-gerbes

From here on we shall only consider the case $G = \mathrm{U}(1)$. As we shall see, U(1)-gerbes are classified by the group $H^3(M,\mathbb{Z})$, just as $H^2(M,\mathbb{Z})$ classifies U(1)-bundles.

We first review the classification of principal U(1)-bundles using sheaf cohomology. We will always be working with paracompact manifolds, therefore we canonically identify sheaf cohomology with its corresponding Čech cohomology. Let us recall some basic facts concerning Čech cohomology. Let F be a sheaf of abelian groups on M, and let $\mathcal{U} = \{U_i\}$ be an open cover. The space of **Čech *k*-cochains** with values in F is the abelian group

$$C^k(\mathcal{U}, F) = \prod_{a_1 < a_2 < \ldots < a_{k+1}} F(U_{a_1} \cap \cdots \cap U_{a_{k+1}}) \subseteq F(\mathcal{U}^{[k]}). \tag{5.6}$$

The **Čech coboundary**:

$$C^k(\mathcal{U}, F) \xrightarrow{\delta} C^{k+1}(\mathcal{U}, F)$$

is given, component-wise, by

$$\delta(g)_{a_1,\ldots,a_{k+1}} = \sum_{j=1}^{k+1} (-1)^j g_{a_1,\ldots,\widehat{a_j},\ldots,a_{k+1}}|_{U_{a_1}\cap\cdots\cap U_{a_{k+2}}}.$$

The set of open covers of M is a directed set, with the order given by refinement. Therefore, the cohomology groups $H^\bullet(\mathcal{U}, F)$ of the complexes $(C^\bullet(\mathcal{U}, F), \delta)$ form a direct system. The **Čech cohomology** of M with values in F is the direct limit of these groups:

$$H^\bullet(M, F) = \varinjlim_{\mathcal{U}} H^\bullet(\mathcal{U}, F).$$

Recall that an open cover $\mathcal{U} = \{U_i\}$ of M is **good** iff every non-empty intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ is contractible. Every manifold admits a good cover, and such covers are cofinal in the aforementioned directed set. Hence, the direct limit above can be computed by just considering good covers.

Let $P \to M$ be a principal U(1)-bundle and $\mathcal{U} = \{U_i\}$ an open cover of M admitting local trivializations of P. The corresponding transition functions $g_{ij} \colon U_i \cap U_j \to \mathrm{U}(1)$ satisfy the cocycle condition $g_{jk} g_{ik}^{-1} g_{ij} = 1$ on $U_i \cap U_j \cap U_k$, and hence give a class in $H^1(M, \underline{\mathrm{U}(1)})$, the degree 1 cohomology group with values in the sheaf of smooth U(1)-valued functions. It is well-known that $H^1(M, \underline{\mathrm{U}(1)})$ is in one-to-one correspondence with isomorphism classes of principal U(1)-bundles on the manifold M.

Let $\mathbb{Z}(1)$ denote the sheaf whose sections are locally-constant functions with values in $2\pi\sqrt{-1}\cdot\mathbb{Z}$, and let C^∞_{Im} denote the sheaf of smooth imaginary-valued functions on M. There is a short exact sequence

$$0 \to \mathbb{Z}(1) \hookrightarrow C^\infty_{\mathrm{Im}} \xrightarrow{\exp} \underline{\mathrm{U}(1)} \to 0, \tag{5.7}$$

giving a long exact sequence in cohomology. Since C^∞_{Im} is a soft sheaf, the long exact sequence gives the isomorphisms:

$$H^k(M, \underline{\mathrm{U}(1)}) \cong H^{k+1}(M, \mathbb{Z}(1)) \cong H^{k+1}(M, \mathbb{Z}). \tag{5.8}$$

For $k = 1$, the isomorphism (5.8) associates to a principal U(1)-bundle its Chern class.

Now we consider the $k = 2$ case, and explain how to obtain a U(1)-valued 2-cocycle from a U(1)-gerbe G. By the second axiom in Def. 5.9, there exists an open cover $\mathcal{U} = \{U_i\}$ of the manifold M, such that for all i, there exists an object $P_i \in \mathsf{G}(U_i)$. By pulling back along refinements, we may assume the following: $\mathcal{U}$ is a good cover, there exists isomorphisms of sheaves $\underline{\mathrm{Aut}}_{\mathsf{G}}(P_i) \cong \underline{\mathrm{U}(1)}|_{U_i}$ for all P_i (by axiom 4), and there exists isomorphisms

$$u_{ij} \colon P_j|_{U_{ij}} \xrightarrow{\sim} P_i|_{U_{ij}},$$

where $P_i|_{U_{ij}}$ and $P_j|_{U_{ij}}$ are the pullbacks of P_i and P_j to $\mathsf{G}(U_i \cap U_j)$. Therefore, by pulling back objects P_i, P_j, P_k to $U_i \cap U_j \cap U_k$, we have the commuting diagram

$$\begin{array}{ccc} P_k|_{U_{ijk}} & \xrightarrow{\;u_{jk}\;} & P_j|_{U_{ijk}} \\ & {\scriptstyle u_{ik}^{-1}} \nwarrow \quad \swarrow {\scriptstyle u_{ij}} & \\ & P_i|_{U_{ijk}} & \end{array}$$

giving a morphism $u_{ik}^{-1} u_{ij} u_{jk} \in \underline{\mathrm{Aut}}_{\mathsf{G}}(P_k)(U_i \cap U_j \cap U_k)$. Since $\underline{\mathrm{Aut}}_{\mathsf{G}}(P_k)(U_i \cap U_j \cap U_k) \cong \underline{\mathrm{U}(1)}(U_i \cap U_j \cap U_k)$, this automorphism corresponds to a map $g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1)$. It is easy to see that g_{ijk} satisfies the cocycle condition on intersections $U_i \cap U_j \cap U_k \cap U_l$, and therefore gives a class $[g] \in H^2(M, \underline{\mathrm{U}(1)})$.

Conversely, suppose $g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1)$ is a 2-cocycle on a good open cover $\mathcal{U} = \{U_i\}$. Recall from the discussion preceding Def. 5.9 that $\mathsf{Tor}_{\mathrm{U}(1)}$ is a gerbe. We construct a new gerbe G by "twisting" $\mathsf{Tor}_{\mathrm{U}(1)}$ by h_{ijk}. Given an open set $V \subseteq M$, an object (P_i, u_{ij}) in $\mathsf{G}(V)$ is defined to be a collection of objects $P_i \in \mathsf{Tor}_{\mathrm{U}(1)}(V \cap U_i)$, together with isomorphisms

$$u_{ij} \colon P_j|_{V \cap U_i \cap U_j} \xrightarrow{\sim} P_i|_{V \cap U_i \cap U_j}$$

in $\mathsf{Tor}_{\mathrm{U}(1)}(V \cap U_i \cap U_j)$, such that $u_{ik}^{-1} u_{ij} u_{jk} = g_{ijk} \in \underline{\mathrm{U}(1)}(V \cap U_i \cap U_j \cap U_k)$. A morphism $(P_i, u_{ij}) \to (P_i', u_{ij}')$ consists of a family of morphisms of U(1)-torsors $P_i \to P_i'$ whose pullbacks in $\mathsf{Tor}_{\mathrm{U}(1)}(V \cap U_i \cap U_j)$ commute with the morphisms u_{ij}, u_{ij}'. It is straightforward to show that by using the pullback functors defined for $\mathsf{Tor}_{\mathrm{U}(1)}$, we obtain a stack G in this way. To see that $\mathrm{Aut}_{\mathsf{G}}(P_i, u_{ij})$ is locally isomorphic to $\underline{\mathrm{U}(1)}$, note that such an automorphism must be given by a collection of morphisms $P_i \xrightarrow{\sim} P_i$ corresponding to sections in $\underline{\mathrm{U}(1)}(V \cap U_i)$, which must agree when pulled back to $V \cap U_i \cap U_j$. These glue to give a section in $\underline{U(1)}(V)$, thereby establishing an isomorphism $\mathrm{Aut}_{\mathsf{G}}(P_i, u_{ij})(V) \cong \underline{\mathrm{U}(1)}(V)$. To show that the other axioms in Def. 5.9 hold, one may show that the categories $\mathsf{G}(U_i)$ and $\mathsf{Tor}_{\mathrm{U}(1)}(U_i)$ are equivalent for all U_i. (This follows from the fact that g_{ijk} restricted to U_i is a 2-coboundary since $H^2(U_i, \underline{\mathrm{U}(1)}) = 0$. See Sec. 5.2 in [13].) This construction, combined with the isomorphism (5.8) leads to the following theorem:

Theorem 5.10 ([13, 22])**.** *There is a one-to-one correspondence between equivalence classes of* U(1)*-gerbes on a manifold* M *and classes in* $H^3(M, \mathbb{Z})$*.*

In fact, one can go further and define the product of two U(1)-gerbes, which is similar to the contracted product of principal U(1)-bundles. The set of equivalence classes of U(1)-gerbes therefore form an abelian group, and the bijection in the above theorem lifts to an isomorphism of groups.

U(1)-gerbes can be equipped with structures that are the higher analogs of connections and curvature. To classify these, we need to introduce a more sophisticated cohomology theory.

5.3 Deligne cohomology

To motivate this section, let us return to the familiar case of principal bundles. If $P \to M$ is a principal U(1)-bundle equipped with a connection, then, in addition to the transition functions g_{ij}, we have local 1-forms $\theta_i \in \Omega^1(U_i)$ satisfying a cocycle-like condition $\sqrt{-1} \cdot (\theta_i - \theta_j) = g_{ij}^{-1} dg_{ij}$ on $U_i \cap U_j$. The curvature of the connection is the global 2-form ω on M satisfying $\omega|_{U_i} = d\theta_i$.

The classification of principal U(1)-bundles equipped with connection requires a refinement of the Čech cohomology group $H^1(M, \underline{U(1)})$. The purpose of real Deligne cohomology is to make this notion precise. In fact, as we will see, Deligne cohomology provides such a refinement for any geometric objects classified by $H^k(M, \mathbb{Z})$ for arbitrary k.

The primary reference for what follows is Sec. 1.5 of Brylinski [13]. However, Brylinski works with the group $\mathbb{C}^\times$ instead of U(1). What we call real Deligne cohomology is presented, without proofs, in Sec. 3 of Carey, Johnson, and Murray [17].

Let Ω^k denote the sheaf of smooth differential k-forms on a manifold M, and let $d_{\log}\colon \underline{\mathrm{U}(1)} \to \Omega^1$ be the differential operator

$$d_{\log} := \frac{1}{\sqrt{-1}} d\log .$$

Definition 5.11 ([17]). *The* **real Deligne cohomology** $H^\bullet(M, D_n^\bullet)$ *of* M *is the Čech hyper-cohomology the exact sequence of sheaves:*

$$D_n^\bullet := \underline{\mathrm{U}(1)} \xrightarrow{d_{\log}} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n, \quad n \geq 1.$$

We compute $H^\bullet(M, D_n^\bullet)$ in the following way. Let $\mathcal{U} = \{U_i\}$ be an open cover of M. We consider the double complex of abelian groups:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & & & \uparrow \delta \\
C^2(\mathcal{U}, \underline{\mathrm{U}(1)}) & \xrightarrow{d_{\log}} & C^2(\mathcal{U}, \Omega^1) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^2) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^n) \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & & & \uparrow \delta \\
C^1(\mathcal{U}, \underline{\mathrm{U}(1)}) & \xrightarrow{d_{\log}} & C^1(\mathcal{U}, \Omega^1) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^2) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^n) \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & & & \uparrow \delta \\
C^0(\mathcal{U}, \underline{\mathrm{U}(1)}) & \xrightarrow{d_{\log}} & C^0(\mathcal{U}, \Omega^1) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^2) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^n)
\end{array} \tag{5.9}$$

where δ is the usual Čech co-boundary operator, and $C^p(\mathcal{U}, \underline{\mathrm{U}(1)})$ and $C^p(\mathcal{U}, \Omega^k)$ denote the Čech p-cochains (as defined in Eq. 5.6). The total complex of the double complex (5.9) is

$$C^0(\mathcal{U}, \underline{\mathrm{U}(1)}) \xrightarrow{\mathbf{d}} C^1(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^0(\mathcal{U}, \Omega^1) \xrightarrow{\mathbf{d}} C^2(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^1(\mathcal{U}, \Omega^1) \oplus C^0(\mathcal{U}, \Omega^2) \xrightarrow{\mathbf{d}} \cdots,$$

with total differential

$$\mathbf{d}g = \delta g + (-1)^p \tfrac{1}{\sqrt{-1}} d\log g, \quad g \in C^p(\mathcal{U}, \underline{\mathrm{U}(1)})$$
$$\mathbf{d}\theta^k = \delta\theta^k + (-1)^p d\theta^k, \quad \theta^k \in C^p(\mathcal{U}, \Omega^k).$$

Let $H^\bullet(\mathcal{U}, D_n^\bullet)$ denote the cohomology of the above total complex. The Čech hyper-cohomology of $D_n^\bullet$ is, by definition, the direct limit of the groups $H^\bullet(\mathcal{U}, D_n^\bullet)$ over all covers

$$H^\bullet(M, D_n^\bullet) = \varinjlim_{\mathcal{U}} H^\bullet(\mathcal{U}, D_n^\bullet).$$

If an open cover $\mathcal{U} = \{U_i\}$ of M is good, then it is well known that there is an isomorphism

$$H^\bullet(M, D_n^\bullet) \cong H^\bullet(\mathcal{U}, D_n^\bullet).$$

We will be particularly interested in the groups $H^n(M, D_n^\bullet)$, which can be thought of as a refinement of the usual Čech cohomology groups $H^\bullet(M, \underline{U(1)})$.

Definition 5.12. *A* **Deligne n-cocycle on** M *is a representative of a class in* $H^n(M, D_n^\bullet)$

Hence, a Deligne n-cocycle is given by a cover $\mathcal{U}$ of M and a collection $(g, \theta^1, \theta^2, \cdots, \theta^n)$ with

$$g \in C^n(\mathcal{U}, \underline{\mathrm{U}(1)}), \quad \theta^k \in C^{n-k}(\mathcal{U}, \Omega^k),$$

satisfying

$$\begin{aligned} \delta g &= 1, \\ \delta\theta^1 &= \frac{1}{\sqrt{-1}}(-1)^{n-1} d\log g, \\ \delta\theta^k &= (-1)^{n-k} d\theta^{k-1}, \text{ for } 2 \leq k \leq n. \end{aligned} \tag{5.10}$$

We consider examples for $n = 1$ and $n = 2$ later on. The projection

$$D_n^\bullet \to D_n^0 = \underline{\mathrm{U}(1)}$$

gives a surjection in cohomology

$$\begin{gathered} H^n(M, D_n^\bullet) \twoheadrightarrow H^n(M, \underline{U(1)}) \\ \left[g, \theta^1, \cdots, \theta^n\right] \mapsto [g]. \end{gathered}$$

Hence, via the isomorphism $H^p(M, \underline{U(1)}) \cong H^{p+1}(M, \mathbb{Z}(1))$, we have a surjection

$$c\colon H^n(M, D_n^\bullet) \twoheadrightarrow H^{n+1}(M, \mathbb{Z}(1)). \tag{5.11}$$

We call $c([g, \theta^1, \cdots, \theta^n])$ the **Chern class** of $[g, \theta^1, \cdots, \theta^n]$.

There is also a map of complexes

$$\begin{array}{ccccccccc} \underline{U(1)} & \xrightarrow{d_{\log}} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow d \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}, \end{array}$$

given by the de Rham differential d. The induced map on the corresponding Čech resolutions sends an n-cocycle $(g, \theta^1, \cdots, \theta^n)$ to $d\theta^n \in C^0(\mathcal{U}, \Omega^{n+1})$. The equalities in (5.10) give $\delta\theta^n = d\theta^{n-1}$. Hence, $\delta d\theta^n = 0$, which implies $d\theta^n$ is the restriction of a globally defined closed form. This gives a map

$$\begin{gathered} \kappa\colon H^n(M, D_n^\bullet) \to Z^{n+1}(M) \\ \kappa([g, \theta^1, \cdots, \theta^n]) = (-1)^n d\theta^n, \end{gathered} \tag{5.12}$$

where $Z^{n+1}(M)$ are the closed $(n+1)$-forms on M. The forthcoming examples will make it clear why the sign $(-1)^n$ appears in the definition of κ.

Definition 5.13. *The* **n-curvature** *of a Deligne n-cocycle $(g, \theta^1, \cdots, \theta^n)$ on a manifold M is the closed $(n+1)$-form*

$$\kappa([g, \theta^1, \cdots, \theta^n]).$$

Let us consider some examples of Deligne n-cocycles and their n-curvatures.

Example 5.14 (Principal U(1)-bundles)**.** For $n = 1$, a class in $H^1(M, D_1^\bullet)$ is represented by maps $g_{ij}: U_i \cap U_j \to \mathrm{U}(1)$, and 1-forms $\theta_i \in \Omega^1(U_i)$, satisfying the cocycle conditions

$$g_{jk} g_{ik}^{-1} g_{ij} = 1, \quad \text{on } U_i \cap U_j \cap U_k$$
$$\sqrt{-1} \cdot (\theta_j - \theta_i) = g_{ij}^{-1} dg_{ij}, \quad \text{on } U_i \cap U_j$$

The 1-curvature is the closed 2-form ω on M satisfying

$$\omega = -d\theta_i \quad \text{on } U_i.$$

Let us consider two equivalent ways of realizing the above local data as a geometric object. (Our convention follows Section 2.2 of [13].) First, it gives us a Hermitian line bundle $L \to M$, equipped with a connection ∇. The local trivializations $s_i: U_i \to \mathrm{U}(1)$ of L satisfy

$$s_i = g_{ij} s_j, \text{ on } U_i \cap U_j.$$

The connection ∇ is locally determined by the 1-forms $-\theta_i$:

$$\frac{\nabla(s_i)}{s_i} = -\sqrt{-1} \cdot \theta_i,$$

which satisfy

$$-\sqrt{-1} \cdot (\theta_i - \theta_j) = g_{ij}^{-1} dg_{ij},$$

because (g, θ) is a cocycle. The curvature of the bundle is given by the global 2-form $-d\theta_i$.

Equivalently, the Deligne 1-cocycle gives a principal U(1)-bundle $P \to M$ equipped with a connection, i.e. a $\mathfrak{u}(1)$-valued 1-form θ on P. L is the line bundle associated to P. Using a trivialization $s: U_i \to P$, the connection 1-form on P can be expressed locally as

$$s_i^* \theta = -\sqrt{-1} \cdot \theta_i.$$

Hence, the Deligne class $[g, \theta]$ corresponds to an isomorphism class of principal U(1)-bundles equipped with connection whose curvature is equal to ω, the 1-curvature of $[g, \theta]$.

This leads to the following theorem:

Theorem 5.15 ([13])**.** *The group of isomorphism classes of principal* U(1)*-bundles with connection, on a manifold* M*, and the degree one Deligne cohomology group* $H^1(M, D_1^\bullet)$ *are isomorphic.*

Example 5.16 (U(1)-gerbes)**.** The $n = 2$ case will be particularly relevant for our work in the subsequent chapters. A class $[g, A, B] \in H^2(M, D_2^\bullet)$ is represented by maps $g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1)$, 1-forms $A_{ij} \in \Omega^1(U_i \cap U_j)$, and 2-forms $B_i \in \Omega^2(U_i)$ satisfying the cocycle conditions:

$$\begin{aligned} g_{jkl} g_{ikl}^{-1} g_{ijl} g_{ijk}^{-1} &= 1 \text{ on } U_i \cap U_j \cap U_k \cap U_l, \\ \sqrt{-1} \cdot (A_{jk} - A_{ik} + A_{ij}) &= -g_{ijk}^{-1} dg_{ijk} \text{ on } U_i \cap U_j \cap U_k, \\ B_j - B_i &= dA_{ij} \text{ on } U_i \cap U_j. \end{aligned} \tag{5.13}$$

The 2-curvature is the closed 3-form ω on M satisfying

$$\omega = dB_i \quad \text{on } U_i.$$

We will see in Section 5.4 that $\big[g, A, B\big]$ corresponds to an isomorphism class of a U(1)-gerbe equipped with a 2-connection whose 2-curvature is ω.

Integral differential forms

In the remainder of this section, we determine which closed differential forms can be realized as the n-curvature of a Deligne cocycle. Let $\mathbb{R}(1)$ denote the sheaf whose sections are locally-constant functions with values in $\sqrt{-1} \cdot \mathbb{R}$.

Definition 5.17. *A closed differential form* $\omega \in \Omega^k(M)$ *is* **integral** *iff the class* $\sqrt{-1} \cdot [\omega]$ *lies in the image of the composition*

$$H^k(M, \mathbb{Z}(1)) \to H^k(M, \mathbb{R}(1)) \xrightarrow{\sim} \sqrt{-1} \cdot H^k_{\mathrm{dR}}(M). \tag{5.14}$$

We denote by $\boldsymbol{Z^k(M)_{\mathrm{int}}}$ *the subspace of all closed integral* k*-forms on* M*.*

Our goal is to show that the n-curvature of a Deligne n-cocycle is an integral $(n+1)$-form, and conversely, every integral $(n + 1)$-form is the curvature of some Deligne n-cocycle.

We begin by introducing some necessary technical machinery. Let $\Omega^{1 \leq \bullet \leq k}$ denote the complex of sheaves $\Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k$ on a manifold M. Let $\mathbb{R}$ be the sheaf of locally constant $\mathbb{R}$-valued functions. Let $\dim M = m$. We consider the hyper-cohomology of the complex $C^\infty \xrightarrow{d} \Omega^{1\le\bullet\le m}$ via the double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & & & \vdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^2(\mathcal{U}, C^\infty) & \xrightarrow{d} & C^2(\mathcal{U}, \Omega^1) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^m(\mathcal{U}, \Omega^m) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^1(\mathcal{U}, C^\infty) & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^1) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^1(\mathcal{U}, \Omega^m) \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & & & \uparrow{\scriptstyle\delta} \\
C^0(\mathcal{U}, C^\infty) & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^1) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^0(\mathcal{U}, \Omega^m),
\end{array}$$

where $\mathcal{U} = \{U_i\}$ is a good cover. The total differential is:

$$\mathbf{d}\theta^k = \delta\theta^k + (-1)^p d\theta^k, \quad \theta^k \in C^p(\mathcal{U}, \Omega^k),\ 0 \le k \le m.$$

Suppose ω is a closed $(n+1)$-form on M, with $n < m$. Let $p_0 \colon \coprod U_i \to M$ be the usual inclusion map. Then $p_0^*\omega$ is in the group $C^0(\mathcal{U}, \Omega^{n+1})$, and gives a class

$$[0, \dots, p_0^*\omega, \dots 0] \in H^{n+1}\big(M, C^\infty \xrightarrow{d} \Omega^{1\le\bullet\le m}\big).$$

We also consider the augmented complex

$$\mathbb{R} \xrightarrow{\iota} C^\infty \xrightarrow{d} \Omega^{1\le\bullet\le m}.$$

If $r \in C^{n+1}(\mathcal{U}, \mathbb{R})$ represents a class $[r] \in H^{n+1}(M, \mathbb{R})$, then it also gives a class in the total cohomology

$$[r, \dots, 0] \in H^{n+1}\big(M, C^\infty \xrightarrow{d} \Omega^{1\le\bullet\le m}\big).$$

The following proposition essentially gives the well-known isomorphism: $H^\bullet(M, \mathbb{R}) \cong H^\bullet_{\mathrm{dR}}(M)$, which was implicitly used in Def. 5.17.

Proposition 5.18. *The $(n+1)$-cocycles $(r, \dots, 0)$ and $(0, \dots, p_0^*\omega, \dots, 0)$ are cohomologous if and only if there exists differential forms $\theta^k \in C^{n-k}(\mathcal{U}, \Omega^k)$ for $k = 0, \dots, n$ such that*

$$\begin{aligned}
&d\theta^n = p_0^*\omega \\
&\delta\theta^k = (-1)^{n-k} d\theta^{k-1} \text{ for } 1 \le k \le n, \\
&\delta\theta^0 = (-1)^n r.
\end{aligned}$$

Proof. The conditions given for the differential forms θ^k are equivalent to the statement

$$(r, \dots, 0) + \mathbf{d}(\theta^0, \dots, \theta^n) = (0, \dots, p_0^*\omega, \dots, 0),$$

where $\mathbf{d}$ is the total differential of the above double complex. □

One can always find a unique class $[r] \in H^{n+1}(M, \mathbb{R})$ such that $[r, \ldots, 0] = [0, \ldots, p_0^* \omega, \ldots, 0]$. Moreover, ω is integral if and only if $\sqrt{-1} \cdot [r] \in H^{n+1}(M, \mathbb{Z}(1))$.

Let Z^k denote the sheaf of closed k-forms. We will need the following lemma:

Lemma 5.19. *For $n \geq 1$, the complex $\underline{\mathrm{U}(1)} \xrightarrow{d_{\log}} \Omega^{1 \leq \bullet \leq n-1} \xrightarrow{d} Z^n$ is quasi-isomorphic to the constant sheaf* $\mathrm{U}(1)$*.*

Proof. We proceed via induction, starting with $\underline{\mathrm{U}(1)} \xrightarrow{d_{\log}} Z^1$. Consider the short exact sequence of complexes of sheaves:

$$\begin{array}{ccccc}
\mathrm{U}(1) & \xrightarrow{\text{incl}} & \underline{\mathrm{U}(1)} & \xrightarrow{d_{\log}} & Z^1 \\
\downarrow & & \downarrow{\scriptstyle d_{\log}} & & \downarrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & Z^1 & \xrightarrow{\mathrm{id}} & Z^1
\end{array}$$

Since $H^\bullet(M, Z^1 \stackrel{\mathrm{id}}{\to} Z^1) = 0$, and $H^\bullet(M, \mathrm{U}(1) \to 0) = H^\bullet(M, \mathrm{U}(1))$, the long exact sequence in cohomology gives:

$$H^\bullet(M, \underline{\mathrm{U}(1)} \xrightarrow{d_{\log}} Z^1) \cong H^\bullet(M, \mathrm{U}(1)).$$

Now assume $n > 1$ and

$$H^\bullet(M, \underline{\mathrm{U}(1)} \xrightarrow{d_{\log}} \Omega^{1 \leq \bullet \leq n-1} \xrightarrow{d} Z^n) \cong H^\bullet(M, \mathrm{U}(1)).$$

Again, we have a short exact sequence of complexes:

$$\begin{array}{ccccc}
\underline{\mathrm{U}(1)} & \xrightarrow{\mathrm{id}} & \underline{\mathrm{U}(1)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle d_{\log}} & & \downarrow{\scriptstyle d_{\log}} & & \downarrow \\
\Omega^1 & \xrightarrow{\mathrm{id}} & \Omega^1 & \longrightarrow & 0 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
\Omega^{n-1} & \xrightarrow{\mathrm{id}} & \Omega^{n-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
Z^n & \xrightarrow{\text{incl}} & \Omega^n & \xrightarrow{d} & Z^{n+1} \\
\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & Z^{n+1} & \xrightarrow{\mathrm{id}} & Z^{n+1}
\end{array}$$

The long exact sequence in cohomology combined with the induction hypothesis gives the desired result. □

We now prove:

Proposition 5.20. *The curvature $(n+1)$-form of a Deligne n-cocycle is integral.*

Proof. We consider the short exact sequence of complexes of sheaves

$$\begin{array}{ccccc}
\underline{\mathrm{U}(1)} & \xrightarrow{\mathrm{id}} & \underline{\mathrm{U}(1)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle d_{\log}} & & \downarrow{\scriptstyle d_{\log}} & & \downarrow \\
\Omega^1 & \xrightarrow{\mathrm{id}} & \Omega^1 & \longrightarrow & 0 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
\Omega^{n-1} & \xrightarrow{\mathrm{id}} & \Omega^{n-1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow \\
Z^n & \xrightarrow{\mathrm{incl}} & \Omega^n & \xrightarrow{(-1)^n d} & Z^{n+1}
\end{array} \tag{5.15}$$

The complex on the left is $\underline{\mathrm{U}(1)} \stackrel{d_{\log}}{\to} \Omega^{1\le\bullet\le n-1} \stackrel{d}{\to} Z^n$, while the middle complex is $D_n^\bullet$. The complex on the right is the shifted complex $Z^{n+1}[-n]$. Note that:

$$\begin{aligned}
H^{n-1}\big(M, Z^{n+1}[-n]\big) &= 0 \\
H^{n}\big(M, Z^{n+1}[-n]\big) &= H^0(M, Z^{n+1}) = Z^{n+1}(M).
\end{aligned}$$

This, in combination with Lemma 5.19, implies we have a long exact sequence

$$0 \to H^n(M, \mathrm{U}(1)) \to H^n(M, D_n^\bullet) \stackrel{\kappa}{\to} Z^{n+1}(M) \stackrel{f}{\to} H^{n+1}(M, \mathrm{U}(1)), \tag{5.16}$$

where $\kappa = (-1)^n d$ is the curvature map given in (5.12), and f is the composition of the connecting homomorphism

$$Z^{n+1}(M) \stackrel{\partial}{\to} H^{n+1}(M, \underline{\mathrm{U}(1)} \stackrel{d_{\log}}{\to} \Omega^{1\le\bullet\le n-1} \stackrel{d}{\to} Z^n),$$

with the isomorphism given by Lemma 5.19. The proposition is proven if we can show that $f(\omega) = 0$ implies ω is integral.

We proceed by working through the definition of ∂. Let $\mathcal{U} = \{U_i\}$ be a good cover of M, and take the Čech resolution of the complexes corresponding to the 3 columns in (5.15). Let $A^\bullet$, $B^\bullet$, and $K^\bullet$ be the total complexes associated to the resolutions of the left, middle, and right columns, respectively, of (5.15). In particular, we have

$$\begin{aligned}
A^n &= C^n(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^{n-1}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^1(\mathcal{U}, \Omega^{n-1}) \oplus C^0(\mathcal{U}, Z^n) \\
A^{n+1} &= C^{n+1}(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^{n}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^2(\mathcal{U}, \Omega^{n-1}) \oplus C^1(\mathcal{U}, Z^n),
\end{aligned}$$

$$B^n = C^n(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^{n-1}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^1(\mathcal{U}, \Omega^{n-1}) \oplus C^0(\mathcal{U}, \Omega^n)$$
$$B^{n+1} = C^{n+1}(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^n(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^2(\mathcal{U}, \Omega^{n-1}) \oplus C^1(\mathcal{U}, \Omega^n),$$

and

$$K^n = C^0(\mathcal{U}, Z^{n+1}), \quad K^{n+1} = C^1(\mathcal{U}, Z^{n+1}).$$

The connecting homomorphism is defined using the diagram

$$\begin{array}{ccccc} A^n & \longrightarrow & B^n & \xrightarrow{\kappa} & K^n \\ \downarrow{\mathbf{d}} & & \downarrow{\mathbf{d}} & & \downarrow{\delta} \\ A^{n+1} & \longrightarrow & B^{n+1} & \xrightarrow{\kappa} & K^{n+1} \end{array}$$

Given $\omega \in Z^{n+1}(M)$, we have $p_0^*(\omega)$ in the group K^n, where $p_0\colon \coprod U_i \to M$ is the inclusion. We next find an n-chain in B^n which maps to $p_0^*(\omega)$, via the map $\kappa = (-1)^n d$. Proposition 5.18, in combination with the isomorphism between Čech and de Rham cohomology, implies there exists $r \in C^{n+1}(\mathcal{U}, \mathbb{R})$ representing a class $[r] \in H^{n+1}(M, \mathbb{R})$ and differential forms $\theta^0, \ldots, \theta^n$ with $\theta^k \in C^{n-k}(\mathcal{U}, \Omega^k)$ such that

$$\begin{aligned} d\theta^n &= (-1)^n p_0^*(\omega) \\ \delta\theta^k &= (-1)^{n-k} d\theta^{k-1} \text{ for } 1 \leq k \leq n, \\ \delta\theta^0 &= -r. \end{aligned}$$

Setting $g = \exp(\sqrt{-1} \cdot \theta^0)$ gives the n-chain $(g, \theta^1, \ldots, \theta^n) \in B^n$, which, by construction, is mapped to $p_0^*(\omega) \in K^n$ by κ. We then apply the total differential $\mathbf{d}$ to $(g, \theta^1, \ldots, \theta^n)$. The conditions on the forms θ^k imply

$$\mathbf{d}(g, \theta^1, \ldots, \theta^n) = (\delta g, 0 \ldots, 0, \delta\theta^n).$$

The quasi-isomorphism in Lemma 5.19 sends the $(n+1)$-cocycle $(\delta g, 0 \ldots, 0, \delta\theta^n) \in B^{n+1}$ to

$$\delta g = \exp\big(\sqrt{-1} \cdot \delta\theta^0\big) = \exp\big(-\sqrt{-1} \cdot r\big) \in C^{n+1}(\mathcal{U}, U(1)).$$

Hence, we have determined f:

$$f(\omega) = \big[\exp\big(-\sqrt{-1} \cdot r\big)\big] \in H^{n+1}(M, \mathrm{U}(1)).$$

Finally, recall that the short exact sequence $0 \to \mathbb{Z}(1) \to \mathbb{R}(1) \overset{\exp}{\to} \mathrm{U}(1) \to 0$ gives the long exact sequence

$$\cdots \to H^{n+1}(M, \mathbb{Z}(1)) \to H^{n+1}(M, \mathbb{R}(1)) \to H^{n+1}(M, \mathrm{U}(1)) \to \cdots.$$

Therefore, if $f(\omega) = 0$, then we have $\sqrt{-1} \cdot [r] \in H^{n+1}(M, \mathbb{Z}(1))$, which implies ω is integral. □

The converse statement is:

Proposition 5.21. *If ω is a closed integral $(n+1)$-form, then there exists a Deligne n-cocycle whose n-curvature is ω.*

Proof. The statement follows from the exactness of (5.16) and the definition of the map $Z^{n+1}(M) \xrightarrow{f} H^{n+1}(M, \mathrm{U}(1))$ given in the proof of the previous proposition. $\square$

5.4 2-Connections on U(1)-gerbes

Here we present Brylinski's formalism [13, Sec. 5.3] which describes how to equip a U(1)-gerbe with a '2-connection', and how such a structure is related to a Deligne 2-cocycle. Recall that the set of connections on a U(1)-principal bundle over M forms an affine space modeled on the vector space $\sqrt{-1} \cdot \Omega^1(M)$. We can think of connections on P as global sections of a sheaf, which we denote as $\mathrm{Co}(P)$. Given an open set $U \subseteq M$, $\mathrm{Co}(P)(U)$ is the set of connections on the restriction of the bundle P to U. Since each set $\mathrm{Co}(P)(U)$ is equipped with a principal homogeneous $\Omega^1(U)$-space, the sheaf $\mathrm{Co}(P)$ is a Ω^1-torsor.

The above discussion implies that given an object $P \in \mathsf{Tor}_{\mathrm{U}(1)}$, we can assign to it a $\Omega^1|_U$-torsor $\mathrm{Co}(P)$. This sheaf satisfies some compatibility conditions that correspond to familiar facts about connections on bundles:

- Given an inclusion $V \overset{i}{\hookrightarrow} U$, we have an equality of sheaves on V: $i^*\mathrm{Co}(P) = \mathrm{Co}(i^*P)$.
- Given an isomorphism of U(1)-torsors $\phi \colon P_1 \xrightarrow{\sim} P_2$ on U, we have an obvious isomorphism of $\Omega^1|_U$-torsors $\phi_* \colon \mathrm{Co}(P_1) \xrightarrow{\sim} \mathrm{Co}(P_2)$.
- If the isomorphism in (2) is an automorphism $g \colon P \xrightarrow{\sim} P$ corresponding to a section $g \in \underline{\mathrm{U}(1)}(U)$, then we have the "gauge transformation"

$$g_*(\nabla) = \nabla - g^{-1}dg, \quad \forall \nabla \in \mathrm{Co}(P).$$

Any U(1)-gerbe is locally isomorphic to $\mathsf{Tor}_{\mathrm{U}(1)}$, therefore it makes sense to axiomatize the above construction for arbitrary gerbes.

Definition 5.22 ([13])**.** *Let G be a* U(1)*-gerbe over M. A* **connective structure** *on G is an assignment to every object $P \in \mathsf{G}(U)$ for every open set $U \subseteq M$, a $\Omega^1|_U$-torsor $\mathrm{Co}(P)$ equipped with the following data:*

1. *For every inclusion* $V \overset{i}{\hookrightarrow} U$*, an isomorphism of* $\Omega^1|_V$*-torsors*

$$\alpha_i \colon i^*\mathrm{Co}(P) \xrightarrow{\sim} \mathrm{Co}(i^*P),$$

where $i^*\mathrm{Co}(P)$ *is the pullback of* $\mathrm{Co}(P)$ *as an object in* $\mathsf{Sh}(U)$*, such that for any composable pair* $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$ *the diagram*

$$\begin{array}{ccccc} j^*i^*\mathrm{Co}(P) & \xrightarrow{j^*\alpha_i} & j^*\mathrm{Co}(i^*P) & \xrightarrow{\alpha_j} & \mathrm{Co}(j^*i^*P) \\ \| & & & & \downarrow t_{i,j*} \\ (ij)^*\mathrm{Co}(P) & & \xrightarrow{\alpha_{ij}} & & \mathrm{Co}((ij)^*P) \end{array}$$

commutes.

2. *For any isomorphisms* $\phi \colon P_1 \xrightarrow{\sim} P_2$ *and* $\psi \colon P_2 \xrightarrow{\sim} P_3$ *in* $\mathsf{G}(U)$*, isomorphisms of* $\Omega^1|_U$*-torsors*

$$\phi_* \colon \mathrm{Co}(P_1) \xrightarrow{\sim} \mathrm{Co}(P_2), \quad \psi_* \colon \mathrm{Co}(P_2) \xrightarrow{\sim} \mathrm{Co}(P_3),$$

such that $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ *and the diagram*

$$\begin{array}{ccc} i^*\mathrm{Co}(P_1) & \xrightarrow{i^*\phi_*} & i^*\mathrm{Co}(P_2) \\ \downarrow \alpha_{1,i} & & \downarrow \alpha_{2,i} \\ \mathrm{Co}(i^*P_1) & \xrightarrow{(i^*\phi)_*} & \mathrm{Co}(i^*P_2). \end{array}$$

commutes. Moreover, if $\underline{\mathrm{Aut}}_{\mathsf{G}}(P)(U) \cong \underline{\mathrm{U}(1)}(U)$ *and* $g \in \underline{\mathrm{U}(1)}(U)$*, then* $g_* \colon \mathrm{Co}(P) \xrightarrow{\sim} \mathrm{Co}(P)$ *is the map*

$$\nabla \mapsto \nabla - g^{-1}dg.$$

If $\mathrm{Co}(P)$ is the sheaf of connections on a principal U(1)-bundle $P \to M$, then to each section $\nabla \in \mathrm{Co}(P)$, we can assign a 2-form $K(\nabla)$ on M corresponding to its curvature. This fact motivates the next definition.

Definition 5.23 ([13])**.** *Let* G *be a* U(1)*-gerbe over* M *equipped with a connective structure* $P \mapsto \mathrm{Co}(P)$*. A* **curving** *of the connective structure is an assignment to every object* $P \in \mathsf{G}(U)$*, and every section* $\nabla \in \mathrm{Co}(P)(U)$*, for every open set* $U \subseteq M$*, a 2-form* $K(\nabla) \in \Omega^2(U)$ *with the following properties:*

1. *Given an inclusion* $V \overset{i}{\hookrightarrow} U$ *of open sets, and the associated isomorphism* $\alpha_i \colon i^*\mathrm{Co}(P) \xrightarrow{\sim} \mathrm{Co}(i^*P)$*, the equality*

$$K(\alpha_i(i^*\nabla)) = i^*K(\nabla)$$

holds, where $i^*K(\nabla)$ *is the usual pullback of differential forms.*

2. *Given an isomorphism* $\phi\colon P \xrightarrow{\sim} P'$ *in* $\mathsf{G}(U)$ *and the associated isomorphism* $\phi_*\colon \mathrm{Co}(P) \to \mathrm{Co}(P')$*, the equality*

$$K(\nabla) = K(\phi_*(\nabla))$$

holds.

3. *If* θ *is a 1-form on* U*, then* $K(\nabla + \sqrt{-1}\cdot\theta) = K(\nabla) + d\theta$*.*

We say G *is* U(1)*-gerbe equipped with a* **2-connection** *iff it is equipped with a connective structure and a curving.*

Finally, let us describe how 2-connections are related to Deligne 2-cocycles. Let G be a U(1)-gerbe on M equipped with a 2-connection. As we described in Section 5.2, we may choose a cover $\{U_i\}$ such that there exists objects $P_i \in \mathsf{G}(U_i)$, isomorphisms $u_{ij}\colon P_j|_{U_{ij}} \xrightarrow{\sim} P_i|_{U_{ij}}$ in $\mathsf{G}(U_i \cap U_j)$, and a 2-cocycle $g_{ijk} = u_{ik}^{-1}u_{ij}u_{jk} \in \underline{\mathrm{Aut}}_{\mathsf{G}}(P_k)(U_i \cap U_j \cap U_k) \cong \underline{\mathrm{U}(1)}(U_i \cap U_j \cap U_k)$. We choose a section $\nabla_i \in \mathrm{Co}(P_i)(U_i)$ for each i. The restriction of ∇_i to $U_i \cap U_j$ gives a section of $\mathrm{Co}(P_i|_{U_{ij}})$ by axiom 1 of Def. 5.22, which we will also denote as ∇_i. The isomorphisms u_{ij} induce isomorphisms $u_{ij*}\colon \mathrm{Co}(P_j|_{U_{ij}}) \xrightarrow{\sim} \mathrm{Co}(P_i|_{U_{ij}})$ of $\Omega^1|_{U_{ij}}$-torsors. Hence, ∇_i and $u_{ij*}\nabla_j$ are both sections of $\mathrm{Co}(P_i|_{U_{ij}})$. This implies that there exists 1-forms A_{ij} on $U_i \cap U_j$ such that

$$\sqrt{-1}\cdot A_{ij} = \nabla_i - u_{ij*}\nabla_j. \tag{5.17}$$

Restricting the above equalities to $U_i \cap U_j \cap U_k$ gives

$$\sqrt{-1}\cdot(A_{jk} - A_{ik} + A_{ij}) = \nabla_i - (u_{ik}^{-1}u_{ij}u_{jk})_*\nabla_i.$$

Axiom 2 of Def. 5.22 implies that the right-hand side of this equation is $g_{ijk}dg_{ijk}$. Hence,

$$\sqrt{-1}\cdot(A_{jk} - A_{ik} + A_{ij}) = g_{ijk}dg_{ijk}.$$

The curving on G assigns a 2-form $B_i = K(\nabla_i)$ on each U_i. On the intersections $U_i \cap U_j$, axiom 1 of Def. 5.23 implies that $K(\nabla_i)$ is just the restriction of B_i. It follows from axiom 2 of the same definition that

$$B_j = K(\nabla_j) = K(u_{ij*}\nabla_j),$$

and, by applying K to Eq. 5.17, we obtain

$$B_i - B_j = dA_{ij}.$$

By comparing these calculations with Eqs. 5.13 in Example 5.16, we see that we've obtained from G a Deligne 2-cocycle $(g, -A, B)$ whose 2-curvature is given by the 3-form $\omega = dB_i$. This leads to the following theorem.

Theorem 5.24 ([13]). *There is a one-to-one correspondence between the set of equivalence classes of* $\mathrm{U}(1)$*-gerbes with 2-connection on a manifold* M *and the degree two Deligne cohomology group* $H^2(M, D_2^\bullet)$.

5.5 2-Line stacks

The category of principal U(1)-bundles with connection over a manifold is equivalent to the category of Hermitian line bundles with connection. This equivalence sends a principal bundle to its associated line bundle. The goal of this section is to construct an analogous associated object to a U(1)-gerbe with 2-connection. We call this the 'associated 2-line stack'. In the subsequent chapters on quantization, it will be convenient to consider both the principal bundle/gerbe perspective and the line bundle/2-line stack perspective.

Twisted vector bundles

We begin by introducing the concept of twisting a Hermitian vector bundle by a $\underline{\mathrm{U}(1)}$-valued Čech 2-cocycle. Hermitian vector bundles on a manifold M are equivalent to certain locally free sheaves with extra structure. It is well-known that these vector bundles form a stack Bund over M, which inherits its structure as a fibered category from the stack of sheaves Sh. Let $\mathcal{U} = \{U_i\}$ be a cover of M. Assume we have a vector bundle $E_i \in \mathsf{Bund}(U_i)$ for each U_i and isomorphisms of vectors bundles preserving the Hermitian structure $\phi_{ij} \colon E_j|_{U_i \cap U_j} \xrightarrow{\sim} E_i|_{U_i \cap U_j}$ such that the composition $\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk}$ is the identity automorphism of the vector bundle $E_k|_{U_i \cap U_j \cap U_k} \in \mathsf{Bund}(U_i \cap U_j \cap U_k)$. Comparing this data with Def. 5.6, we see that we are giving an object $(E_i) \in \mathsf{Bund}(\mathcal{U}^{[0]})$, and an isomorphism $(\phi_i) \colon p_2^*(E_i) \xrightarrow{\sim} p_1^*(E_i)$, which satisfies the necessary gluing conditions to give a global vector bundle $E \to M$ in $\mathsf{Bund}(M)$. The restriction of E to each U_i is isomorphic to the bundle E_i.

Now let $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ be a 2-cocycle given by the functions $g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1)$. If $E \in \mathsf{Bund}(U_i \cap U_j \cap U_k)$ is a Hermitian vector bundle, then g induces an automorphism of E (preserving the Hermitian structure), which corresponds to multiplying sections of E by g_{ijk}. We consider, as above, an object $(E_i) \in \mathsf{Bund}(\mathcal{U}^{[0]})$, and an isomorphism $(\phi_i) \colon p_2^*(E_i) \xrightarrow{\sim} p_1^*(E_i)$. However, this time we require $\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = g_{ijk}$, instead of the identity. Unless g_{ijk} is a co-boundary, this twisting prevents us from gluing the E_i's together to form a global Hermitian vector bundle. Hence, we have the following definition:

Definition 5.25. *Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover of M and $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ a 2-cocycle. A* ***g*-twisted Hermitian vector bundle** *over M consists of the following data:*

- *on each U_i, a Hermitian vector bundle*

$$(E_i, \langle\cdot,\cdot\rangle_i),$$

- *on each $U_{ij} = U_i \cap U_j$, an isomorphism of Hermitian vector bundles*

$$\phi_{ij} \colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}},$$

  such that for all i, j, k in $\mathcal{I}$:

$$\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = g_{ijk}\cdot$$

  where $g_{ijk}\cdot$ is the automorphism of $E_k|_{U_{ijk}}$ corresponding to multiplication by

$$g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1).$$

A **morphism** *$\psi\colon (E_i, \phi_{ij}) \to (E'_i, \phi'_{ij})$ of g-twisted Hermitian vector bundles over U consists of a collection of morphisms of Hermitian vector bundles*

$$\psi_i \colon E_i \to E'_i,$$

for each $i \in \mathcal{I}$ such that

$$\psi_i \circ \phi_{ij} = \phi'_{ij} \circ \psi_j.$$

Notice that the definition of a twisted vector bundle mimics the construction we described in Sec. 5.2 for obtaining a gerbe from a 2-cocycle.

Let $\mathsf{Bund}^g(M)$ denote the category of g-twisted Hermitian vector bundles over M. We first consider the case when g is the trivial cocycle.

Proposition 5.26. *If $g = 1 \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ is the trivial 2-cocycle, then $\mathsf{Bund}^g(M)$ is equivalent to the category $\mathsf{Bund}(M)$.*

Proof. If g is trivial, then the data which describes a twisted bundle is the same data needed to glue local objects of a stack into a global object (Def. 5.6). Hence, given a trivially twisted bundle (E_i, ϕ_{ij}), there exists a global vector bundle E whose restriction to each U_i is isomorphic to E_i. Indeed, the category $\mathsf{Bund}^{g=1}(M)$ is a category of 'descent data' for the stack Bund. (See Appendix B.) The fact that Bund is a stack implies $\mathsf{Bund}(M)$ is equivalent to this category of descent data [45]. □

The next proposition implies that, up to equivalence, $\mathsf{Bund}^g(M)$ depends only on the class $[g] \in H^2(\mathcal{U}, \underline{\mathrm{U}(1)})$.

Proposition 5.27. *If g, $g' \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ are cohomologous 2-cocycles, then the categories $\mathsf{Bund}^g(M)$ and $\mathsf{Bund}^{g'}(M)$ are equivalent.*

Proof. Let $h \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ be a 2-cochain such that $g = g' + \delta h$. If (E_i, ϕ_{ij}) is an object of $\mathsf{Bund}^g(M)$, then we can define Hermitian vector bundle automorphisms

$$h_{ij} \colon E_i|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}}$$

over each open set $U_{ij} = U_i \cap U_j$ corresponding to multiplying the sections of $E_i|_{U_{ij}}$ by $h_{ij} \colon U_{ij} \to \mathrm{U}(1)$. This gives new isomorphisms

$$\psi_{ij} = h_{ij} \circ \phi_{ij} \colon E_j \xrightarrow{\sim} E_i.$$

Since the ϕ_{ij}'s are $\mathbb{C}$-linear, the morphisms ψ_{ij} satisfy on U_{ijk}:

$$\begin{aligned} \psi_{ik}^{-1} \circ \psi_{ij} \circ \psi_{jk} &= (h_{ik}^{-1} h_{ij} h_{jk}) g_{ijk} \\ &= g_{ijk} + \delta h \\ &= g'_{ijk}. \end{aligned}$$

Hence, there is a functor from $\mathsf{Bund}^g(M)$ to $\mathsf{Bund}^{g'}(M)$, determined by the map $(E_i, \phi_{ij}) \mapsto (E_i, \psi_{ij})$ on objects, and the identity map on morphisms. This functor gives the desired equivalence of categories. □

If $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ and $g' \in C^2(\mathcal{U}', \underline{\mathrm{U}(1)})$ are 2-cocycles related by a refinement, then one can show that the categories $\mathsf{Bund}^g(M)$ and $\mathsf{Bund}^{g'}(M)$ are equivalent. (See, for example, Lemma 1.2.3 in [14].) Hence, up to equivalence, we can uniquely associate the category $\mathsf{Bund}^g(M)$ to the class $[g] \in H^2(M, \underline{\mathrm{U}(1)})$.

The next proposition implies that g-twisted Hermitian vector bundles are the global sections of certain a stack which we think of as being associated to the $\mathrm{U}(1)$-gerbe whose equivalence class is determined by $[g]$.

Proposition 5.28. *Given a 2-cocycle $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ on a manifold M, there exists a stack over M whose category of global sections is equivalent to the category $\mathsf{Bund}^g(M)$ of g-twisted Hermitian vector bundles over M.*

Proof. The fact that twisted vector bundles or, more generally, twisted coherent sheaves, form a stack is a known result in complex algebraic geometry [3][Sec. 2.2], [57][Cor. 5.4.8]. The idea of the proof is simple. We construct the stack by gluing together the local stacks $\mathsf{Bund}|_{U_i}$ of Hermitian vector bundles over U_i, using the 2-cocycle g. However, the proof requires us to introduce additional technology for stacks, so we give the details in Appendix B. $\square$

The stack described in Prop. 5.28 is unique up to equivalence of stacks. We slightly abuse notation and denote it Bund^g, so that we may identify the global sections with twisted bundles in $\mathsf{Bund}^g(M)$.

Twisted bundles as sections of a 2-bundle

The sheaf of sections of a complex line bundle is constructed by using the transition functions to glue together local smooth functions $U \to \mathbb{C}$. There is a formalism known as '2-bundle theory' which categorifies this idea [7, 10]. The total space of a smooth 2-bundle over a manifold is, roughly, a category whose objects and morphisms are themselves manifolds.[2] In this context, the complex line is replaced by $\mathsf{Vect}_{\mathbb{C}}$, the category of finite-dimensional complex vector spaces. This category was interpreted by Kapranov and Voevodsky [31] as a rank 1 '2-vector space'. A complex 2-line bundle is therefore a 2-bundle whose fibers are categories equivalent to $\mathsf{Vect}_{\mathbb{C}}$. A section of the 2-bundle is determined locally by a particular kind of functor $U \to \mathsf{Vect}_{\mathbb{C}}$, where the open set $U \subseteq M$ is given the structure of a trivial category. Roughly speaking, such a functor assigns a vector space to each point in U in a smooth way, and hence determines a vector bundle over U. These local sections can be glued together using 2-cocycles (cf. Def. 5.25), in analogy with the line bundle case. Bartels' work [10] implies that the "sheaf of sections" of a 2-bundle over M is indeed a stack over M. We will not use 2-bundle theory in this work. However, this rough sketch provides the motivation for interpreting the stack Bund^g as the higher analog of a Hermitian line bundle.

Definition 5.29. *Let $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ be a 2-cocycle on M, and let G be the corresponding* U(1)*-gerbe whose equivalence class is $[g] \in H^2(M, \underline{\mathrm{U}(1)})$. The* **2-line stack associated to G** *is the stack Bund^g.*

Note that Thm. 5.10, Prop. 5.27, and Lemma 1.2.3 in [14] imply that the 2-line stack associated to a gerbe is unique up to equivalence.

2-Connections on 2-line stacks

If we equip a U(1)-gerbe with a 2-connection, then it is reasonable to expect that this extra structure can be transferred to its associated 2-line stack. Hence, we next

[2] This is an example of what is called a smooth '2-space' which is a slight generalization of the more familiar concept of a Lie groupoid.

consider twisting a Hermitian vector bundle, equipped with connection, by a Deligne 2-cocycle.

Definition 5.30. *Let* $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ *be an open cover of* M *and* $\xi = (g, A, B) \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^1(\mathcal{U}, \Omega^1) \oplus C^0(\mathcal{U}, \Omega^2)$ *a Deligne 2-cocycle. A* **$\boldsymbol{\xi}$-twisted Hermitian vector bundle with connection** *over* M *consists of the following data:*

- *on each* U_i*, a Hermitian vector bundle equipped with a Hermitian connection*

$$(E_i, \langle\cdot,\cdot\rangle_i, \nabla_i),$$

- *on each* $U_{ij} = U_i \cap U_j$*, an isomorphism of Hermitian vector bundles*

$$\phi_{ij} \colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}},$$

  such that

$$\phi_{ij}\nabla_j - \nabla_i\phi_{ij} = \sqrt{-1} \cdot A_{ij} \otimes \phi_{ij},$$

  and for all i, j, k *in* $\mathcal{I}$*:*

$$\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = g_{ijk} \cdot$$

  where $g_{ijk}\cdot$ *is the automorphism of* $E_k|_{U_{ijk}}$ *corresponding to multiplication by*

$$g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1).$$

A **morphism** $\psi\colon (E_i, \nabla_i, \phi_{ij}) \to (E'_i, \nabla'_i, \phi'_{ij})$ *of* ξ*-twisted Hermitian vector bundles with connection consists of a collection of connection-preserving morphisms of Hermitian vector bundles*

$$\psi_i \colon (E_i, \nabla_i) \to (E'_i, \nabla'_i)$$

for each $i \in \mathcal{I}$ *such that*

$$\psi_i \circ \phi_{ij} = \phi'_{ij} \circ \psi_j.$$

The above definition and the cocycle conditions on (g, A, B) force a compatibility between the curvatures ∇_i^2 of the vector bundles (E_i, ∇_i). More precisely, for all $i, j \in \mathcal{I}$, we have

$$\phi_{ij} \circ (\nabla_j^2 - \sqrt{-1} \cdot B_j \otimes \mathrm{id}) = (\nabla_i^2 - \sqrt{-1} \cdot B_i \otimes \mathrm{id}) \circ \phi_{ij}. \tag{5.18}$$

Definition 5.31. *We say a* ξ*-twisted Hermitian vector bundle with connection* $(E_i, \nabla_i, \phi_{ij})$ *is* **twisted-flat** *iff for all* $i \in \mathcal{I}$

$$\nabla_i^2 - \sqrt{-1} \cdot B_i \otimes \mathrm{id} = 0.$$

We interpret a twisted-flat section of a 2-line stack to be the 2-plectic analogue of a covariant constant section of a Hermitian line bundle. We will use this analogy in our quantization procedure for 2-plectic manifolds in Chapter 7. Not surprisingly, bundles twisted by a Deligne 2-cocycle are global sections of a stack.

Proposition 5.32. *Given a Deligne 2-cocycle ξ on a manifold M, there exists a stack Bund^{ξ} over M whose category of global sections is equivalent to the category of ξ-twisted Hermitian vector bundles with connections over M.*

Proof. The proof is essentially identical to the one given in Appendix B for Proposition 5.28. □

The last definition of this section completes the analogy between 2-line stacks and line bundles.

Definition 5.33. *Let ξ be a Deligne 2-cocycle on M and let G be the corresponding* U(1)*-gerbe with 2-connection whose equivalence class is $[\xi] \in H^2(M, D_2^{\bullet})$. The* **2-line stack equipped with 2-connection** *associated to G is the stack Bund^{ξ}.*

5.6 Holonomy of Deligne classes

Suppose we have a trivial principal U(1)-bundle $P \to M$ equipped with connection. The connection in this case is given by a 1-form θ on M. The holonomy of this connection is the function

$$S^1 \xrightarrow{\gamma} M \mapsto \exp\bigl(i \oint_{S^1} \gamma^* \theta\bigr), \tag{5.19}$$

from loops in M to U(1). If $\mathcal{U} = \{U_i\}$ is a cover of M, and $p_0 \colon \mathcal{U}^{[0]} \to M$ is the inclusion (5.1) then $p_0^* \theta$ is a 1-form on $\mathcal{U}^{[0]} = \coprod_i U_i$. Therefore, the Deligne 1-cocycle corresponding to the bundle P with connection θ is $(1, p_0^* \theta)$. It is reasonable to define the holonomy of this Deligne 1-cocycle to be the function given in (5.19). Locally, every bundle with connection is isomorphic to the trivial bundle equipped with a 1-form. Therefore, by gluing the local functions (5.19) together, we can compute the holonomy of any Deligne 1-cocycle, which would correspond to the usual notion of the holonomy of a bundle with connection. Carey, Johnson, and Murray [17] give a construction that does precisely this for both Deligne 1-cocycles and Deligne 2-cocycles. This allows one to define the '2-holonomy' of a U(1)-gerbe equipped with a 2-connection. We will use their construction in our quantization of 2-plectic manifolds in Chapter 7.

The construction begins by first observing that if α is an n-form on M and $\mathcal{U} = \{U_i\}$ is a good cover, then we can construct a Deligne n-cocycle $(1, 0, \ldots, 0, p_0^*\alpha)$ by generalizing the $n = 1$ case described in the previous paragraph. We therefore have an inclusion of groups

$$\Omega^n(M) \xrightarrow{\iota} H^n(M, D_n^\bullet) \cong H^n(\mathcal{U}, D_n^\bullet)$$
$$\alpha \mapsto [1, 0, \ldots, 0, p_0^*(\alpha)].$$

We also have the sequence

$$\Omega^n(M) \xrightarrow{\iota} H^n(M, D_n^\bullet) \xrightarrow{c} H^{n+1}(M, \mathbb{Z}(1)).$$

Here, c is the map (5.11) which sends a Deligne class to its Chern class. Clearly, the image of ι projects to the trivial class in $H^n(M, \underline{\mathrm{U}(1)}) \cong H^{n+1}(M, \mathbb{Z}(1))$. Therefore, $c \circ \iota = 0$. Moreover, we have the following proposition, which is given without proof in [17].

Proposition 5.34. *Let $Z^n(M)_{\mathrm{int}}$ be the subspace of all closed integral n-forms on a manifold M. The sequence of groups:*

$$0 \to Z^n(M)_{\mathrm{int}} \hookrightarrow \Omega^n(M) \xrightarrow{\iota} H^n(M, D_n^\bullet) \xrightarrow{c} H^{n+1}(M, \mathbb{Z}(1)) \to 0 \tag{5.20}$$

is exact.

Proof. We have already discussed the surjectivity of the map c in Sec. 5.3. To show $\ker c \subseteq \operatorname{im} \iota$, suppose

$$(g, \theta^1, \ldots, \theta^n) \in C^n(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^{n-1}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^1(\mathcal{U}, \Omega^{n-1}) \oplus C^0(\mathcal{U}, \Omega^n)$$

is a Deligne n-cocycle relative to a good open cover $\mathcal{U}$ such that $c([g, \theta^1, \ldots, \theta^n]) = 0$. Then, the isomorphism $H^n(M, \underline{\mathrm{U}(1)}) \cong H^{n+1}(M, \mathbb{Z}(1))$ implies there exists a cochain $h \in C^{n-1}(\mathcal{U}, \underline{\mathrm{U}(1)})$ such that $\delta h = g$. Since $\mathcal{U}$ is good, a staircase construction in the double complex (5.9) shows there exists k-forms

$$\eta^k \in C^{n-k-1}(\mathcal{U}, \Omega^k), \quad 1 \le k \le n-1$$

such that

$$\begin{aligned} \theta^1 &= \delta\eta^1 + (-1)^{n-1} \frac{1}{\sqrt{-1}} d\log h, \\ \theta^k &= \delta\eta^k + (-1)^{n-k} d\eta^{k-1}, \quad 2 \le k \le n-1. \end{aligned} \tag{5.21}$$

In particular, for $k = n-1$, we have $\theta^{n-1} = \delta\eta^{n-1} - d\eta^{n-2}$, and hence

$$d\theta^{n-1} = \delta d\eta^{n-1}.$$

The fact that $(g, \theta^1, \dots, \theta^n)$ is a cocycle implies

$$\delta\theta^n - d\theta^{n-1} = 0.$$

Combining the two equalities gives $\delta(\theta^n - d\eta^{n-1}) = 0$. Hence $\theta^n - d\eta^{n-1}$ is a cocycle in $C^0(\mathcal{U}, \Omega^n)$. Therefore there exists a global n-form $\alpha \in \Omega^n(M)$ such that $p_0^*(\alpha) = \theta^n - d\eta^{n-1}$. This result, combined with the Eqs. 5.21 imply

$$(g, \theta^1, \dots, \theta^n) - \mathbf{d}(h, \eta^1, \dots, \eta^{n-1}) = (1, 0, \dots, 0, p_0^*\alpha),$$

where $\mathbf{d}$ is the total differential of the double complex (5.9). Hence $\ker c = \operatorname{im} \iota$.

Next we show $Z^n(M)_{\text{int}} \subseteq \ker \iota$. Suppose α is a closed integral n-form. Then Prop. 5.21 implies there exists a Deligne $(n-1)$-cocycle $(h, \eta^1, \dots, \eta^{n-1})$ representing a class in $H^{n-1}(M, D^\bullet_{n-1})$ whose $(n-1)$-curvature is α. By definition of the curvature, this means

$$p_0^*\alpha = (-1)^{n-1} d\eta^{n-1}.$$

Embeding this cocycle in the complex $D^\bullet_n$ and applying the total differential gives:

$$\mathbf{d}(h, \eta^1, \dots, \eta^{n-1}) = (1, 0, \dots, 0, d\eta^{n-1}) = (1, 0, \dots, 0, (-1)^{n-1} p_0^*\alpha).$$

Hence

$$(1, 0, \dots, 0, p_0^*\alpha) - (-1)^{n-1}\mathbf{d}(h, \eta^1, \dots, \eta^{n-1}) = (1, 0, \dots, 0),$$

which implies $\iota(\alpha) = 0$.

Finally, we show $\ker \iota \subseteq Z^n(M)_{\text{int}}$. Let α be a n-form on M such that

$$\iota(\alpha) = [1, 0, \dots, p_0^*\alpha] = [1, 0, \dots, 0].$$

Hence the curvature of the cocycle $(1, 0, \dots, p_0^*\alpha)$ is zero. By definition of the curvature, this implies $p_0^* d\alpha = 0$. Therefore α is closed. Furthmore, by assumption, there exists a cochain

$$(h, \eta^1, \dots, \eta^{n-1}) \in C^{n-1}(\mathcal{U}, \underline{\mathrm{U}(1)}) \oplus C^{n-2}(\mathcal{U}, \Omega^1) \oplus \cdots \oplus C^0(\mathcal{U}, \Omega^{n-1})$$

such that

$$(1, 0, \dots, p_0^*\alpha) - \mathbf{d}(h, \eta^1, \dots, \eta^{n-1}) = (1, 0, \dots, 0).$$

By definition of the differential $\mathbf{d}$, this implies

$$p_0^*\alpha = d\eta^{n-1},$$

and:

$$\begin{aligned}
\delta h &= 1,\\
\delta\eta^1 &= (-1)^n \frac{1}{\sqrt{-1}} d\log h,\\
\delta\eta^k &= (-1)^{n-k-1} d\eta^{k-1}, \text{ for } 2 \le k \le n-1.
\end{aligned}$$

Hence, $(h, \eta^1, \dots, \eta^{n-1})$ is a Deligne $(n-1)$-cocycle representing a class in $H^{n-1}(M, D^\bullet_{n-1})$ whose $(n-1)$-curvature is $(-1)^{n-1}\alpha$. Therefore Prop. 5.20 implies that α is integral. $\square$

Let $[g, \theta^1, \dots, \theta^n] \in H^n(M, D^\bullet_n)$ be a degree n class relative to an open cover $\mathcal{U} = \{U_i\}$ of M. Let $\sigma\colon \Sigma^n \to M$ be a map from a compact, oriented n-dimensional manifold into M. It is easy to see that the pullback $[\sigma^* g, \sigma^*\theta^1, \dots, \sigma^*\theta^n]$ is a degree n class in $H^n(\sigma^{-1}\mathcal{U}, D^\bullet_n)$ relative to the open cover $\sigma^{-1}\mathcal{U} = \{\sigma^{-1}(U_i)\}$ of Σ^n. Since $H^{n+1}(\Sigma^n, \mathbb{Z}(1)) \cong H^{n+1}(\Sigma^n, \mathbb{Z}) = 0$, the sequence (5.20) implies there exists an n-form α on Σ^n such that

$$\iota(\alpha) = [\sigma^* g, \sigma^*\theta^1, \dots, \sigma^*\theta^n]. \tag{5.22}$$

Hence, we can integrate α and take the exponential

$$\exp\big(i \int_{\Sigma^n} \alpha\big) \tag{5.23}$$

to obtain an element of U(1). Note that if α' is any other n-form satisfying $\iota(\alpha') = [\sigma^* g, \sigma^*\theta^1, \dots, \sigma^*\theta^n]$, then the sequence (5.20) implies $\alpha - \alpha'$ is integral, which further implies

$$\int_{\Sigma^n} (\alpha - \alpha') \in 2\pi\,\mathbb{Z}.$$

Therefore the element (5.23) only depends on the class $[\sigma^* g, \sigma^*\theta^1, \dots, \sigma^*\theta^n]$, which allows us to give the following definition:

Definition 5.35 ([17]). *Let $[g, \theta^1, \dots, \theta^n] \in H^n(M, D^\bullet_n)$ be a degree n Deligne class. The n-**holonomy** of a map $\sigma\colon \Sigma^n \to M$ is the element*

$$\mathrm{hol}([g, \theta^1, \dots, \theta^n], \sigma) := \exp\big(i \int_{\Sigma^n} \alpha\big)$$

of U(1)*, where $\alpha \in \Omega^n(\Sigma^n)$ is the n-form defined in Eq. 5.22.*

It is straightforward to verify for $n = 1$, that $\mathrm{hol}([(g, \theta], \sigma)$ is the usual holonomy of a principal U(1)-bundle with transition functions and connection 1-forms representing the class $[g, \theta]$. Similarly, for gerbes we have:

Definition 5.36. *The* **2-holonomy of a 2-connection** *on a* U(1)*-gerbe corresponding to the Deligne 2-cocycle* (g, A, B) *is the assignment to every map* $\sigma : \Sigma^n \to M$*, the element*

$$\mathrm{hol}([g, A, B], \sigma) \in \mathrm{U}(1).$$

Since the 2-holonomy of the gerbe depends only on the Deligne class, we can just as easily define the 2-holonomy for the associated 2-line stack with 2-connection.

Chapter 6

Prequantization of 2-plectic manifolds

In Chapter 3, we showed that any n-plectic manifold gives rise to a Lie n-algebra. This generalizes the well-known fact that the functions on a symplectic manifold (M, ω) form a Poisson algebra. In the symplectic case, the geometric quantization procedure of Kirillov [34], Kostant [37], and Souriau [64] (KKS) involves constructing faithful representations of this algebra using structures that naturally arise on M. The first step of this procedure is called prequantization. Our goal in this chapter is to generalize this to 2-plectic manifolds, and prequantize the Lie 2-algebra of Hamiltonian 1-forms.

6.1 Overview of prequantization

In symplectic geometry, prequantization itself begins by assigning to a symplectic manifold either a principal U(1)-bundle, or a Hermitian line bundle, with connection whose curvature corresponds to the symplectic 2-form. In this chapter, we will use principal bundles.

Definition 6.1 ([64])**.** *A* **prequantized symplectic manifold** *is a symplectic manifold (M, ω) equipped with a principal* U(1)*-bundle $P \to M$ with connection, such that the curvature of the connection is ω.*

Definition 5.12 and Example 5.14 in the previous chapter imply that a prequantized symplectic manifold is a symplectic manifold equipped with a Deligne 1-cocycle whose 1-curvature is ω. This observation allows us to generalize Def. 6.1 to the n-plectic case.

Definition 6.2. *A* **prequantized n-plectic manifold** *is an n-plectic manifold (M,ω) equipped with a Deligne n-cocycle ξ whose n-curvature is ω.*

Not every n-plectic manifold can be prequantized. Indeed, Propositions 5.20 and 5.21 imply:

Proposition 6.3. *An n-plectic manifold (M,ω) is prequantizable if and only if ω is integral.*

In Prop. 5.20, we considered the long exact sequence

$$0 \to H^n(M,\mathrm{U}(1)) \to H^n(M,D_n^\bullet) \xrightarrow{\kappa} Z^{n+1}(M) \xrightarrow{f} H^{n+1}(M,\mathrm{U}(1)),$$

where $Z^{n+1}(M)$ is the space of closed $(n+1)$-forms, and κ is the curvature map. The sequence shows that the manifold may have several non-equivalent prequantizations. Indeed, the prequantizations of (M,ω) are classified by the Deligne cohomology group $H^n(M,D_n^\bullet)$.

Let (M,ω,ξ) be a prequantized symplectic manifold and let $P \xrightarrow{\pi} M$ be the $\mathrm{U}(1)$-bundle with connection corresponding to the Deligne 1-cocycle ξ. From this geometric data, the KKS procedure for prequantization gives a faithful representation of the Poisson algebra $(C^\infty(M),\{\cdot,\cdot\})$ as unitary operators on a Hilbert space. This representation can be constructed by using the 'Atiyah algebroid' associated to P. The Atiyah algebroid is an example of a Lie algebroid: roughly, a vector bundle $A \to M$ equipped with a bundle map to the tangent bundle of M, and a Lie algebra structure on its space of global sections. The total space of the Atiyah algebroid is the quotient $A = TP/U(1)$. Sections of A are $U(1)$-invariant vector fields on P. A connection on P is equivalent to a splitting $s\colon TM \to A$ of the short exact sequence

$$0 \to \mathbb{R}\times M \to A \xrightarrow{\pi_*} TM \to 0$$

where the map $\mathbb{R}\times M \to A$ corresponds to identifying the vertical subspace of T_pP with the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$. As we will see, those sections of A which act as infinitesimal symmetries preserving the connection (or splitting) form a Lie subalgebra that is isomorphic to the Poisson algebra. This implies that the Poisson algebra acts as linear differential operators on the $\mathbb{C}$-valued functions on P. In particular, the algebra acts on functions $f\colon P \to \mathbb{C}$ with the property

$$f(pg) = g^{-1}f(p), \quad g \in \mathrm{U}(1).$$

A simple calculation shows that such functions correspond to global sections of the Hermitian line bundle associated to P. Compactly supported global sections of this line bundle form a vector space equipped with a Hermitian inner product. The L^2-completion of this space is called the 'prequantum Hilbert space.' We shall consider this Hilbert space in more detail in the next chapter.

If the symplectic manifold is connected, then the Poisson algebra gives what is known as the 'Kostant-Souriau central extension' of the Lie algebra of Hamiltonian vector fields [37]. The symplectic form, evaluated at a point, gives a representative of the degree 2 class corresponding to this extension in the Lie algebra cohomology of the Hamiltonian vector fields. The fact that this central extension is quantized, rather than the Hamiltonian vector fields themselves, is the reason why the concept of 'phase' is introduced in quantum mechanics.

The goal of this chapter is to generalize the above prequantization procedure to 2-plectic manifolds. We already know from Chapter 5 that, for a prequantized 2-plectic manifold, a U(1)-gerbe with 2-connection plays the role of the U(1)-principal bundle. But what is the 2-plectic analogue of the Atiyah algebroid? We answer this question in this chapter by considering a more general problem: understanding the relationship between 2-plectic geometry and the theory of 'Courant algebroids.' Roughly, a Courant algebroid is a vector bundle that generalizes the structure of a Lie algebroid equipped with a symmetric nondegenerate bilinear form on the fibers. They were first used by Courant [18] to study generalizations of pre-symplectic and Poisson structures in the theory of constrained mechanical systems. Curiously, many of the ingredients found in 2-plectic geometry are also found in the theory of 'exact' Courant algebroids. An exact Courant algebroid is a Courant algebroid whose underlying vector bundle $C \to M$ is an extension of the tangent bundle by the cotangent bundle:

$$0 \to T^*M \to C \to TM \to 0.$$

In a letter to Weinstein, Ševera [66] described how exact Courant algebroids arise in 2-dimensional variational problems (e.g. bosonic string theory), and showed that they are classified up to isomorphism by the degree 3 de Rham cohomology of M. From any closed 3-form on M, one can explicitly construct an exact Courant algebroid equipped with an 'isotropic' splitting of the above short exact sequence, using local 1-forms and 2-forms that satisfy cocycle conditions [11, 29, 26].

Ševera's classification implies that every 2-plectic manifold (M, ω) gives a unique exact Courant algebroid (up to isomorphism) whose class is represented by the 2-plectic structure. However, there are more interesting similarities between 2-plectic structures and exact Courant algebroids. Roytenberg and Weinstein [55] showed that the bracket on the space of global sections of a Courant algebroid induces an L_∞ structure. If we are considering an exact Courant algebroid, then the global sections can be identified with vector fields and 1-forms on the base space. Roytenberg and Weinstein's results imply that these sections, when combined with the smooth functions on the base space, form a Lie 2-algebra [54]. Moreover, the Jacobiator of the Lie 2-algebra encode a closed 3-form representing the Ševera class [61].

The first new result we present in this chapter is that there exists a Lie 2-algebra morphism which embeds the Lie 2-algebra of Hamiltonian 1-forms on a 2-plectic manifold (M, ω) into the Lie 2-algebra of global sections of the corresponding exact Courant algebroid C equipped with an isotropic splitting. Moreover, this morphism gives an isomorphism between the Lie 2-algebra of Hamiltonian 1-forms and the sub Lie 2-algebra consisting of those sections of C which preserve the splitting via a particular kind of adjoint action. This result holds without any integrality condition on the 2-plectic structure. However, its meaning becomes clear in the context of prequantization: It is the higher analogue of the isomorphism between the underlying Lie algebra of the Poisson algebra on a prequantized symplectic manifold, and the Lie sub-algebra of sections of the Atiyah algebroid that preserve the connection on the associated principal bundle. Hence, we see that the 2-plectic analogue of the Atiyah algebroid associated to a principal $U(1)$-bundle is an exact Courant algebroid associated to a $U(1)$-gerbe. This idea that exact Courant algebroids are higher Atiyah algebroids has been discussed previously in the literature [11, 26]. However, this is the first time the analogy has been understood using Lie n-algebras within the context of prequantization.

The second result presented here involves identifying the 2-plectic analogue of the Kostant-Souriau central extension. On a 2-plectic manifold, associated to every Hamiltonian 1-form is a Hamiltonian vector field. These vector fields form a Lie algebra, which we can view as a trivial Lie 2-algebra, whose underlying chain complex is concentrated in degree 0, and whose bracket satisfies the Jacobi identity on the nose. For any 1-connected (i.e. connected and simply connected) 2-plectic manifold, we show that the Lie 2-algebra of Hamiltonian 1-forms is quasi-isomorphic to a 'strict central extension' of the trivial Lie 2-algebra of Hamiltonian vector fields by the abelian Lie 2-algebra $\mathbb{R} \to 0$. Furthermore, we show that this extension corresponds to a degree 3 class in the Lie algebra cohomology of the Hamiltonian vector fields with values in the trivial representation. In analogy with the symplectic case, a 3-cocycle representing this class can be constructed by using the 2-plectic form. It follows from the aforementioned results relating a 2-plectic manifold (M, ω) to the Courant algebroid C, that the sub Lie 2-algebra of sections of C that preserve the splitting is also quasi-isomorphic to this central extension, and can be interpreted as the prequantization of the Lie 2-algebra of

Hamiltonian 1-forms.

6.2 Prequantization of symplectic manifolds

In this section, we briefly review the construction of Lie algebroids on symplectic manifolds and describe an embedding of the Poisson algebra into the Lie algebra of sections of the algebroid. We emphasize the role played by the Atiyah algebroid in prequantization and the construction of the Kostant-Souriau central extension.

We begin by reviewing the construction of a Lie algebroid, which ultimately will describe how phases arise in the prequantization of symplectic manifolds. A section of this Lie algebroid is a vector field on the base manifold together with a 'phase', or more precisely, a real-valued function.

Definition 6.4 ([41])**.** *A* **Lie algebroid** *over a manifold M is a real vector bundle $A \to M$ equipped with a bundle map (called the anchor) $\rho\colon A \to TM$, and a Lie algebra bracket $[\cdot,\cdot]_A \colon \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ such that the induced map*

$$\Gamma(\rho)\colon \Gamma(A) \to \mathfrak{X}(M)$$

is a morphism of Lie algebras, and for all $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(A)$ we have the Leibniz rule

$$[e_1, fe_2]_A = f\,[e_1, e_2]_A + \rho(e_1)(f)e_2.$$

A Lie algebroid with surjective anchor map is called a **transitive Lie algebroid***.*

The main ideas of the following construction are presented in Sec. 17 of Cannas da Silva and Weinstein [15]. We provide the details here in order to compare to the 2-plectic case in Sec. 6.4. Let (M,ω) be a manifold equipped with a closed 2-form, e.g. a pre-symplectic manifold. By a **trivialization** of ω, we mean a cover $\{U_i\}$ of M, equipped with 1-forms $\theta_i \in \Omega^1(U_i)$, and smooth functions $g_{ij} \in C^\infty(U_i \cap U_j)$, such that

$$\omega|_{U_i} = d\theta_i \qquad (6.1)$$

$$(\theta_j - \theta_i)|_{U_{ij}} = dg_{ij}, \qquad (6.2)$$

where $U_{ij} = U_i \cap U_j$. Every manifold admits a good cover, hence every closed 2-form admits a trivialization. Given such a trivialization of ω, we can construct a transitive Lie algebroid over M. Over each U_i we consider the Lie algebroid

$$A_i = TU_i \oplus \mathbb{R} \to U_i,$$

with bracket

$$[v_1 + f_1, v_2 + f_2]_i = [v_1, v_2] + v_1(f_2) - v_2(f_1),$$

for all $v_i + f_i \in \mathfrak{X}(U_i) \oplus C^\infty(U_i)$, and anchor ρ given by the projection onto TU_i. From the 1-forms $dg_{ij} \in \Omega^1(U_{ij})$, we can construct transition functions

$$G_{ij} \colon U_{ij} \to GL(n+1),$$
$$G_{ij}(x) = \begin{pmatrix} 1 & 0 \\ dg_{ij}|_x & 1 \end{pmatrix},$$

which act on a point $v_x + r \in A_i|_{U_{ij}}$ by

$$G_{ij}(x)(v_x + r) = v_x + r + dg_{ij}(v_x).$$

Clearly, each G_{ij} satisfies the cocycle conditions on U_{ijk} by virtue of Eq. 6.2. Therefore, we have over M the vector bundle

$$A = \coprod_{x \in M} T_x U_i \oplus \mathbb{R}/\sim,$$

where the equivalence is defined via the functions G_{ij} in the usual way. For any sections $v_i + f_i$ of $A_i|_{U_{ij}}$, a direct calculation shows that

$$[G_{ij}(v_1 + f_1), G_{ij}(v_2 + f_2)]_i = G_{ij}([v_1, v_2] + v_1(f_2) - v_2(f_1)).$$

Hence, the local bracket descends to a well-defined bracket $[\cdot,\cdot]_A$ on the quotient. Henceforth, $(A, [\cdot,\cdot]_A, \rho)$ will denote this transitive Lie algebroid associated to the closed 2-form ω.

It is easy to see that the above Lie algebroid is an extension of the tangent bundle

$$0 \to M \times \mathbb{R} \to A \xrightarrow{\rho} TM \to 0.$$

Moreover, the 1-forms $\theta_i \in \Omega^1(U_i)$ induce a splitting

$$s \colon TM \to A$$

of the above sequence defined as

$$s(v_x) = v_x - \theta_i(v_x), \quad \forall\, v_x \in TU_i. \tag{6.3}$$

By a slight abuse of notation, we denote the horizontal lift $\Gamma(s) \colon \mathfrak{X}(M) \to \Gamma(A)$ also by s. Hence every section $e \in \Gamma(A)$ is of the form $e = s(v) + f$, for some $v \in \mathfrak{X}(M)$ and

$f \in C^\infty(M)$. Using the local definition of the splitting and the fact that $\omega|_{U_i} = d\theta_i$, a direct calculation shows that

$$[s(v_1) + f_1, s(v_2) + f_2]_A = s\big([v_1, v_2]\big) + v_1(f_2) - v_2(f_1) - \iota_{v_2}\iota_{v_1}\omega, \tag{6.4}$$

for all sections $s(v_i) + f_i$. The failure of the splitting $s\colon TM \to A$ to preserve the Lie bracket on sections is measured by the 2-form ω:

$$[s(v_1), s(v_2)]_A = s([v_1, v_2]) - \omega(v_1, v_2), \quad \forall v_1, v_2 \in \mathfrak{X}(M).$$

It is a simple exercise to show that a different choice of trivialization gives a Lie algebroid equipped with a splitting that is isomorphic to A equipped with the splitting given in Eq. 6.3.

The Poisson algebra

Let (M, ω) be a symplectic manifold. Here $\{f, g\} = \omega(v_f, v_g)$ denotes the Poisson bracket on smooth functions. The vector field v_f, satisfying the equality $df = -\iota_{v_f}\omega$, is the unique Hamiltonian vector field corresponding to the function f. We denote the Lie algebra of Hamiltonian vector fields by $\mathfrak{X}_{\mathrm{Ham}}(M)$. Let $(A, [\cdot,\cdot]_A, \rho)$ be the Lie algebroid associated to ω and $s\colon TM \to A$ be the splitting defined in Eq. 6.3. We are interested in a particular Lie sub-algebra of $\Gamma(A)$ acting on the subspace $s(\mathfrak{X}(M)) \subseteq \Gamma(A)$ via the adjoint action.

Definition 6.5. *A section $a = s(v) + f \in \Gamma(A)$* **preserves the splitting** $s\colon TM \to A$ *iff $\forall v' \in \mathfrak{X}(M)$*

$$\big[a, s(v')\big]_A = s([v, v']).$$

The subspace of sections that preserve the splitting is denoted as $\Gamma(A)^s$.

Proposition 6.6. *$\Gamma(A)^s$ is a Lie subalgebra of $\Gamma(A)$.*

Proof. Follows directly from the fact that the bracket on $\Gamma(A)$ and the bracket on $\mathfrak{X}(M)$ both satisfy the Jacobi identity. □

It is easy to show that a section $s(v) + f$ preserves the splitting if and only if $v = v_f$. In fact:

Proposition 6.7. *The underlying Lie algebra of the Poisson algebra $\big(C^\infty(M), \{\cdot,\cdot\}\big)$ is isomorphic to the Lie algebra $\big(\Gamma(A)^s, [\cdot,\cdot]_A\big)$.*

Proof. For any vector field $v' \in \mathfrak{X}(M)$, it follows from Eq. 6.4 that we have $[s(v)+f, s(v')]_A = s([v,v'])$ if and only if

$$v'(f) + \omega(v', v) = 0,$$

and hence $df = -\iota_v \omega$. Therefore the injective map

$$\phi\colon C^\infty(M) \to \Gamma(A)^s, \quad \phi(f) = s(v_f) + f$$

is also surjective. If v_f and v_g are Hamiltonian vector fields corresponding to the functions f and g, respectively, then

$$\begin{aligned}[\phi(f), \phi(g)]_A &= [s(v_f) + f, s(v_g) + g]_A \\ &= s([v_f, v_g]) + \big(v_f(g) - v_g(f)\big) - \iota_{v_g}\iota_{v_f}\omega \\ &= s([v_f, v_g]) + \omega(v_f, v_g) \\ &= \phi([v_f, v_g]).\end{aligned}$$

□

Prequantization and Atiyah algebroids

Definition 6.2 implies that a prequantized symplectic manifold is an integral symplectic manifold equipped with Deligne 1-cocycle. By definition, this 1-cocycle corresponds to a collection of 1-forms $\theta_i \in \Omega^1(U_i)$, and $U(1)$-valued functions $g_{ij}\colon U_{ij} \to U(1)$ defined on a good cover $\{U_i\}$ such that

$$\begin{gathered}\omega = d\theta_i \quad \text{on } U_i, \\ \sqrt{-1}(\theta_j - \theta_i) = g_{ij}^{-1} dg_{ij} \quad \text{on } U_{ij}, \\ g_{jk} g_{ik}^{-1} g_{ij} = 1 \quad \text{on } U_{ijk}.\end{gathered}$$

The Deligne 1-cocycle also gives, of course, a trivialization of the 2-form ω, and therefore the transitive Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ over M equipped with the splitting $s\colon TM \to A$. However in this case, the functions $g_{ij}\colon U_{ij} \to U(1)$ are the transition functions of a principal U(1)-bundle P with connection. Therefore, by identifying $\mathfrak{u}(1)$ with $\sqrt{-1}\cdot\mathbb{R}$, we see that A is isomorphic to the **Atiyah algebroid** $TP/U(1)$. A point in A corresponds to a vector field along the fiber $\pi^{-1}(x)$ that is invariant under the right $U(1)$ action. Hence a global section of A corresponds to a $U(1)$-invariant vector field on P.

Splittings of $0 \to M \times \mathbb{R} \to A \to TM \to 0$ correspond to connection 1-forms on P. The connection 1-form $\theta \in \Omega^1(P)$ corresponding to the local forms θ_i induces a 'left-splitting' $\hat{\theta}\colon A \to M \times \mathbb{R}$ such that $\hat{\theta} \circ s = 0$. It is straightforward to show that $a \in \Gamma(A)^s$ if and only if

$$\mathcal{L}_a \theta = 0.$$

That is, a section of the Atiyah algebroid preserves the splitting if and only if it preserves the corresponding connection on P. For a prequantized symplectic manifold, the Lie algebra $\Gamma(A)^s$ is a Lie sub-algebra of derivations on $C^\infty(P)_\mathbb{C}$ and therefore on the global sections of the associated line bundle of P. Proposition 6.7 then implies that we have a faithful representation of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$.

The Kostant-Souriau central extension

If (M, ω) is a connected symplectic manifold, then we have a short exact sequence of Lie algebras

$$0 \to \mathfrak{u}(1) \to C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M) \to 0 \tag{6.5}$$

The underlying Lie algebra of the Poisson algebra is known as the Kostant-Souriau central extension of the Lie algebra of Hamiltonian vector fields [37]. If $\sigma\colon \mathfrak{X}_{\mathrm{Ham}}(M) \to C^\infty(M)$ is a splitting of the underlying sequence of vector spaces, then the failure of σ to be a strict (i.e. bracket-preserving) Lie algebra morphism is measured by the difference

$$\{\sigma(v_1), \sigma(v_2)\} - \sigma([v_1, v_2])$$

which represents a degree 2 class in the Chevalley-Eilenberg cohomology $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$. This class can be represented by using the symplectic form. More specifically, pick a point $x \in M$ and let $c \in \mathrm{Hom}(\Lambda^2 \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ be the cochain given by:

$$c(v, v') = -\omega(v, v')|_x, \quad \forall v, v' \in \mathfrak{X}_{\mathrm{Ham}}(M).$$

The fact that c is a cocycle follows from the bracket $\{\cdot,\cdot\}$ satisfying the Jacobi identity. One can show that the class $[c]$ does not depend on the choice of $x \in M$.

If (M, ω) is a prequantized connected symplectic manifold, then Prop. 6.7 implies that the 'quantized Poisson algebra' gives an isomorphic central extension

$$0 \to \mathfrak{u}(1) \to \Gamma(A)^s \to \mathfrak{X}_{\mathrm{Ham}}(M) \to 0.$$

This central extension is responsible for introducing phases into the quantized system. Two functions f and f' differing by a constant $r \in \mathfrak{u}(1)$ will have the same Hamiltonian vector fields and therefore give the same flows on M. However, their quantizations will give unitary transformations which differ by a phase $\exp(2\pi\sqrt{-1}r)$.

6.3 Courant algebroids

Now, we begin our investigation of the 2-plectic case. First, we recall some basic facts and examples of Courant algebroids and then we proceed to describe Ševera's classification of exact Courant algebroids. The Courant algebroid will act as the 2-plectic analogue of the Atiyah algebroid.

There are several equivalent definitions of a Courant algebroid found in the literature. The following definition, due to Roytenberg [52], is equivalent to the original definition given by Liu, Weinstein, and Xu [40].

Definition 6.8. *A* **Courant algebroid** *is a vector bundle $C \to M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]_C$ on $\Gamma(C)$, and a bundle map (called the* **anchor***) $\rho\colon C \to TM$ such that for all $e_1, e_2, e_3 \in \Gamma(C)$ and for all $f, g \in C^\infty(M)$ the following properties hold:*

1. $[e_1,[e_2,e_3]_C]_C - [[e_1,e_2]_C, e_3]_C - [e_2,[e_1,e_3]_C]_C = -DT(e_1,e_2,e_3)$,
2. $\rho([e_1,e_2]_C) = [\rho(e_1),\rho(e_2)]$,
3. $[e_1, fe_2]_C = f\,[e_1,e_2]_C + \rho(e_1)(f)e_2 - \frac{1}{2}\langle e_1,e_2\rangle Df$,
4. $\langle Df, Dg\rangle = 0$,
5. $\rho(e_1)\left(\langle e_2,e_3\rangle\right) = \langle [e_1,e_2]_C + \frac{1}{2}D\langle e_1,e_2\rangle, e_3\rangle + \langle e_2, [e_1,e_3]_C + \frac{1}{2}D\langle e_1,e_3\rangle\rangle$,

where $[\cdot,\cdot]$ is the Lie bracket of vector fields, $D\colon C^\infty(M) \to \Gamma(C)$ is the map defined by $\langle Df, e\rangle = \rho(e)f$, and

$$T(e_1,e_2,e_3) = \frac{1}{6}\left(\langle [e_1,e_2]_C, e_3\rangle + \langle [e_3,e_1]_C, e_2\rangle + \langle [e_2,e_3]_C, e_1\rangle\right).$$

The bracket in Definition 6.8 is skew-symmetric, but the first property implies that it needs only to satisfy the Jacobi identity "up to DT". Note that the vector bundle $C \to M$ may be identified with $C^* \to M$ via the bilinear form $\langle \cdot,\cdot\rangle$ and therefore we have the dual map

$$\rho^*\colon T^*M \to C.$$

Hence the map D is simply the pullback of the de Rham differential by ρ^*.

There is a commonly used alternate definition given by Ševera [66] for Courant algebroids which involves a bracket operation on sections that satisfies a Jacobi identity but is not skew-symmetric.

Definition 6.9. *A* **Courant algebroid** *is a vector bundle $C \to M$ together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a bilinear operation $[\![\cdot,\cdot]\!]_C$ on $\Gamma(C)$, and a bundle map $\rho\colon C \to TM$ such that for all $e_1, e_2, e_3 \in \Gamma(C)$ and for all $f \in C^\infty(M)$ the following properties hold:*

1. $[\![e_1, [\![e_2, e_3]\!]_C]\!]_C = [\![[\![e_1, e_2]\!]_C, e_3]\!]_C + [\![e_2, [\![e_1, e_3]\!]_C]\!]_C$,
2. $\rho([\![e_1, e_2]\!]_C) = [\rho(e_1), \rho(e_2)]$,
3. $[\![e_1, fe_2]\!]_C = f\,[\![e_1, e_2]\!]_C + \rho(e_1)(f)e_2$,
4. $[\![e_1, e_1]\!]_C = \frac{1}{2}D\langle e_1, e_1\rangle$,
5. $\rho(e_1)\left(\langle e_2, e_3\rangle\right) = \langle [\![e_1, e_2]\!]_C, e_3\rangle + \langle e_2, [\![e_1, e_3]\!]_C\rangle$,

where $[\cdot,\cdot]$ is the Lie bracket of vector fields, and $D\colon C^\infty(M) \to \Gamma(C)$ is the map defined by $\langle Df, e\rangle = \rho(e)f$.

Roytenberg [52] showed that $C \to M$ is a Courant algebroid in the sense of Definition 6.8 with bracket $[\cdot,\cdot]_C$, bilinear form $\langle \cdot,\cdot\rangle$ and anchor ρ if and only if $C \to M$ is a Courant algebroid in the sense of Definition 6.9 with the same anchor and bilinear form but with bracket $[\![\cdot,\cdot]\!]_C$ given by

$$[\![e_1, e_2]\!]_C = [e_1, e_2]_C + \frac{1}{2}D\langle e_1, e_2\rangle. \tag{6.6}$$

All Courant algebroids in this chapter are considered to be Courant algebroids in the sense of Definition 6.8. We introduced Definition 6.9 mainly to connect our discussion here with previous results in the literature.

Example 6.10. An important example of a Courant algebroid is the **standard Courant algebroid** $C = TM \oplus T^*M$ over any manifold M equipped with the **standard Courant bracket**:

$$[v_1 + \alpha_1, v_2 + \alpha_2]_C = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \mathcal{L}_{v_2}\alpha_1 - \frac{1}{2}d\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle^-, \tag{6.7}$$

where

$$\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle^- = \iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1 \tag{6.8}$$

is the **standard skew-symmetric pairing**. The bilinear form is given by the **standard symmetric pairing**:

$$\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle^+ = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1. \tag{6.9}$$

The anchor $\rho\colon C \to TM$ is the projection map, and $D = d$ is the de Rham differential. The bracket $[\cdot,\cdot]_C$ is the skew-symmetrization of the **standard Dorfman bracket**:

$$[\![v_1 + \alpha_1, v_2 + \alpha_2]\!]_C = [v_1, v_2] + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1, \tag{6.10}$$

which plays the role of the bracket given in Definition 6.9.

The standard Courant algebroid is the prototypical example of an **exact Courant algebroid** [11].

Definition 6.11. *A Courant algebroid $C \to M$ with anchor map $\rho\colon C \to TM$ is* **exact** *iff*

$$0 \to T^*M \xrightarrow{\rho^*} C \xrightarrow{\rho} TM \to 0$$

is an exact sequence of vector bundles.

The Ševera class of an exact Courant algebroid

Ševera's classification [66] originates in the idea that a particular kind of splitting of the above short exact sequence corresponds to defining a connection.

Definition 6.12. *A* **splitting** *of an exact Courant algebroid C over a manifold M is a map of vector bundles $s\colon TM \to C$ such that*

1. $\rho \circ s = \mathrm{id}_{TM}$,
2. $\langle s(v_1), s(v_2)\rangle = 0$ *for all* $v_1, v_2 \in TM$,

where $\rho\colon C \to TM$ and $\langle \cdot,\cdot\rangle$ are the anchor and bilinear form, respectively.

In other words, a splitting of an exact Courant algebroid is an isotropic splitting of the sequence of vector bundles. Bressler and Chervov call splittings 'connections' [11]. If s is a splitting and $B \in \Omega^2(M)$ is a 2-form then one can construct a new splitting:

$$(s + B)(v) = s(v) + \rho^* B(v, \cdot). \tag{6.11}$$

Furthermore, one can show that any two splittings on an exact Courant algebroid must differ by a 2-form on M in this way. Hence the space of splittings on an exact Courant algebroid is an affine space modeled on the vector space of 2-forms $\Omega^2(M)$ [11].

The failure of a splitting to preserve the bracket gives a suitable notion of 'curvature'. Given vector fields v_1, v_2, v_3 on M, it can be shown that the function

$$\omega(v_1, v_2, v_3) = \langle [s(v_1), s(v_2)]_C, s(v_3)\rangle$$

defines a closed 3-form on M [11]. This is the curvature 3-form of an exact Courant algebroid over M. It gives a well-defined cohomology class in $H^3_{\mathrm{DR}}(M)$, independent of the choice of splitting.

Definition 6.13 ([26])**.** *The* **Ševera class** *of an exact Courant algebroid with bracket* $[\cdot,\cdot]_C$ *and bilinear form* $\langle\cdot,\cdot\rangle$ *is the cohomology class* $[-\omega] \in H^3_{\mathrm{DR}}(M)$*, where*

$$\omega(v_1, v_2, v_3) = \big\langle [s(v_1), s(v_2)]_C, s(v_3)\big\rangle.$$

6.4 Courant algebroids and 2-plectic geometry

In this section, we describe a relationship between Courant algebroids and 2-plectic manifolds which can be understood as the higher analogue of the relationship between Atiyah algebroids and symplectic manifolds.

We begin by recalling how to explicitly construct an exact Courant algebroid with Ševera class $[\omega]$. This is the 3-form version of the construction that gives a transitive Lie algebroid over a pre-symplectic manifold, which was previously discussed in Sec. 6.2. The approach given here is essentially identical to those given by Gualtieri [26], Hitchin [29], and Ševera [66] .

Let (M,ω) be a manifold equipped with a closed 3-form. A trivialization of ω is an open cover$\{U_i\}$ of M equipped with 2-forms $B_i \in \Omega^2(U_i)$, and 1-forms $A_{ij} \in \Omega^1(U_{ij})$ on intersections such that

$$\begin{aligned} \omega|_{U_i} &= dB_i \\ (B_j - B_i)|_{U_{ij}} &= dA_{ij}. \end{aligned} \tag{6.12}$$

Given such a trivialization, over each open set U_i consider the bundle $C_i = TU_i \oplus T^*U_i \to U_i$ equipped with the standard pairing

$$\big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle_i^+ = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1, \tag{6.13}$$

$v_1, v_2 \in \mathfrak{X}(U_i)$, $\alpha_1, \alpha_2 \in \Omega^1(U_i)$, which has signature (n,n). On double intersections, it is easy to see that

$$\big\langle v_1 + \iota_{v_1} dA_{ij} + \alpha_1, v_2 + \iota_{v_2} dA_{ij} + \alpha_2 \big\rangle_i^+ = \big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle_i^+.$$

Hence the 2-forms $\{dA_{ij}\}$ generate transition functions

$$\begin{aligned} G_{ij} \colon U_{ij} &\to SO(n,n), \\ G_{ij}(x) &= \begin{pmatrix} 1 & 0 \\ dA_{ij}|_x & 1 \end{pmatrix}, \end{aligned}$$

which satisfy the cocycle conditions on U_{ijk} by virtue of Eq. 6.12. Therefore, we have over M the vector bundle

$$C = \coprod_{x \in M} T_x U_i \oplus T_x^* U_i / \sim,$$

equipped with a bilinear form denoted as $\langle \cdot, \cdot \rangle^+$. C sits in the exact sequence

$$0 \to T^*M \xrightarrow{\jmath} C \xrightarrow{\rho} TM \to 0,$$

where the anchor ρ is induced by the projection $T^*U_i \oplus TU_i \to TU_i$, and $\jmath$ is the inclusion.

The 2-forms B_i induce a bundle map $s \colon TM \to C$

$$s(v_x) = v_x - B_i(v_x) \quad \text{if } x \in U_i, \tag{6.14}$$

It follows from Eq. 6.12 that s is well-defined when $x \in U_{ij}$. It is easy to see that this map is an isotropic splitting (Def. 6.12). Hence every section $e \in \Gamma(C)$ can be uniquely expressed as

$$e = s(v) + \alpha,$$

for some $v \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$. As before, we use s to also denote the map $\Gamma(s) \colon \mathfrak{X}(M) \to \Gamma(C)$. The anchor map is just

$$\rho\big(s(v) + \alpha\big) = v. \tag{6.15}$$

Given sections $s(v_1) + \alpha_1$, $s(v_2) + \alpha_2 \in \Gamma(C)$, a local calculation using Eq. 6.14 gives

$$\begin{aligned} \big\langle s(v_1) + \alpha_1, s(v_2) + \alpha_2 \big\rangle^+ &= \iota_{v_1}\alpha_2 - \iota_{v_1}\iota_{v_2}B_i + \iota_{v_2}\alpha_1 - \iota_{v_2}\iota_{v_1}B_i \\ &= \big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle^+. \end{aligned} \tag{6.16}$$

The above equality holds, in fact, for any splitting $s' \colon TM \to C$, since $s - s'$ is a 2-form on M and therefore skew-symmetric. The bracket on $\Gamma(C)$ is defined over the open set U_i by:

$$[s(v_1) + \alpha_1, s(v_2) + \alpha_2]_C \,|_{U_i} = [s(v_1) + \alpha_1, s(v_2) + \alpha_2]_i$$

where $[\cdot, \cdot]_i$ is the standard Courant bracket (6.7) on C_i. Since the 2-forms $\{dA_{ij}\}$ are closed, it follows by direct computation that on double intersections U_{ij}:

$$[G_{ij}(v_1 + \alpha_1), G_{ij}(v_2 + \alpha_2)]_i = G_{ij}\big([v_1 + \alpha_1, v_2 + \alpha_2]_i\big).$$

Hence the bracket $[\cdot, \cdot]_C$ is indeed globally well-defined. Using the local definition of the bracket and the splitting, as well as the fact that $dB_i = \omega$, it is easy to show that

$$\begin{aligned} [s(v_1) + \alpha_1, s(v_2) + \alpha_2]_C = s\big([v_1, v_2]\big) &+ \mathcal{L}_{v_1}\alpha_2 - \mathcal{L}_{v_2}\alpha_1 \\ &- \frac{1}{2} d\big\langle v_1 + \alpha_1, v_2 + \alpha_2 \big\rangle^- - \iota_{v_2}\iota_{v_1}\omega. \end{aligned} \tag{6.17}$$

The bracket $[\cdot,\cdot]_C$ is called the **twisted Courant bracket**. A analogous construction using the standard Dorfman bracket (6.10) on C_i gives the **twisted Dorfman bracket**:

$$[\![s(v_1)+\alpha_1, s(v_2)+\alpha_2]\!]_C = s\big([v_1,v_2]\big) + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1 - \iota_{v_2}\iota_{v_1}\omega. \tag{6.18}$$

These brackets were studied in detail by Ševera and Weinstein [61, 66].

It is straightforward to check that $C \to M$ equipped with the aforementioned bilinear form, anchor, and bracket $[\cdot,\cdot]_C$ is an exact Courant algebroid (Definition 6.8). Just as in Lie algebroid case, the construction of C is independent of the choice of trivialization up to a splitting-preserving isomorphism.

A direct calculation shows that

$$-\omega(v_1,v_2,v_3) = \big\langle [s(v_1), s(v_2)]_C, s(v_3) \big\rangle^+.$$

Hence, the Courant algebroid C has Ševera class $[\omega]$. Of course, we are interested in the special case when ω is a 2-plectic structure. We summarize the above discussion with the following proposition:

Proposition 6.14. *Let (M,ω) be a 2-plectic manifold. Up to isomorphism, there exists a unique exact Courant algebroid C over M, with bilinear form $\langle\cdot,\cdot\rangle^+$, anchor map ρ, and bracket $[\cdot,\cdot]_C$ given in Eqs. 6.13, 6.15, and 6.17, respectively, and equipped with a splitting whose curvature is $-\omega$.*

Lie 2-algebras from Courant algebroids

Next, we describe how a Courant algebroid gives a Lie 2-algebra. From here on, we shall describe a Lie 2-algebra using the terminology given in Prop. 3.10, i.e. as a 2-term chain complex, equipped with a bracket and a Jacobiator.

Recall that the space of global sections of a transitive Lie algebroid associated to a closed 2-form gives a Lie algebra. As we shall see, the global sections of a Courant algebroid form a Lie 2-algebra. Given any Courant algebroid $C \to M$ with bilinear form $\langle\cdot,\cdot\rangle$, bracket $[\cdot,\cdot]_C$, and anchor $\rho\colon C \to TM$, one can construct a 2-term chain complex

$$L \quad = \quad C^\infty(M) \xrightarrow{D} \Gamma(C),$$

with differential $D = \rho^* d$ where d is the de Rham differential. The bracket $[\cdot,\cdot]_C$ on global sections can be extended to a chain map $[\cdot,\cdot] : L \otimes L \to L$. If e_1, e_2 are degree 0 chains then $[e_1,e_2]$ is the original bracket. If e is a degree 0 chain and f,g are degree 1 chains, then we define:

$$\begin{aligned}
&[e,f] = -[f,e] = \frac{1}{2}\langle e, Df\rangle \\
&[f,g] = 0.
\end{aligned}$$

It was shown by Roytenberg and Weinstein [55] that this extended bracket gives a L_∞-algebra. Roytenberg's later work [53, 54] implies that a brutal truncation of this L_∞-algebra is a Lie 2-algebra whose underlying complex is L. For the Courant algebroid C associated to a 2-plectic manifold, their result implies:

Theorem 6.15. *If C is the exact Courant algebroid given in Proposition 6.14 then there is a Lie 2-algebra $L_\infty(C) = (L, [\cdot,\cdot], J)$ where:*

- $L_0 = \Gamma(C)$,
- $L_1 = C^\infty(M)$,
- *the differential* $L_1 \xrightarrow{d} L_0$ *is the de Rham differential*
- *the bracket* $[\cdot,\cdot]$ *is*

$$[e_1, e_2] = [e_1, e_2]_C \quad \text{in degree 0}$$

and

$$[e, f] = -[f, e] = \frac{1}{2}\langle e, df\rangle^+ \quad \text{in degree 1,}$$

- *the Jacobiator is the linear map* $J\colon \Gamma(C) \otimes \Gamma(C) \otimes \Gamma(C) \to C^\infty(M)$ *defined by*

$$\begin{aligned} J(e_1, e_2, e_3) &= -T(e_1, e_2, e_3) \\ &= -\frac{1}{6}\Big(\langle [e_1, e_2]_C, e_3\rangle^+ + \langle [e_3, e_1]_C, e_2\rangle^+ \\ &\quad + \langle [e_2, e_3]_C, e_1\rangle^+\Big). \end{aligned}$$

More precisely, the theorem follows from Example 5.4 of [54] and Section 4 of [53]. On the other hand, the original construction of Roytenberg and Weinstein gives a L_∞-algebra on the complex:

$$0 \to \ker D \xrightarrow{\iota} C^\infty(M) \xrightarrow{D} \Gamma(C),$$

with trivial structure maps l_n for $n > 3$. Moreover, the map l_2 (corresponding to the bracket $[\cdot,\cdot]$ given above) is trivial in degree > 1 and the map l_3 (corresponding to the Jacobiator J) is trivial in degree > 0. Hence these maps induce the above Lie 2-algebra structure on $C^\infty(M) \xrightarrow{D} \Gamma(C)$.

The algebraic relationship between 2-plectic and Courant

Associated to any 2-plectic manifold (M, ω), is a Lie 2-algebra $L_\infty(M, \omega)$ (Thm. 3.14). In Prop. 3.15, we described this Lie 2-algebra as a 2-term chain complex $L = (L_1 \xrightarrow{d} L_0)$ equipped with a bracket $[\cdot,\cdot]$ and Jacobiator J where:

- $L_0 = \Omega^1_{\text{Ham}}(M)$ is the space of Hamiltonian 1-forms,
- $L_1 = C^\infty(M)$,
- the differential $L_1 \xrightarrow{d} L_0$ is the de Rham differential,
- the bracket $[\cdot,\cdot]$ is $\{\alpha,\beta\} = \omega(v_\alpha, v_\beta, \cdot)$ in degree 0 and trivial otherwise,
- the Jacobiator is given by the linear map $J\colon \Omega^1_{\text{Ham}}(M) \otimes \Omega^1_{\text{Ham}}(M) \otimes \Omega^1_{\text{Ham}}(M) \to C^\infty$, where $J(\alpha,\beta,\gamma) = \omega(v_\gamma, v_\beta, v_\alpha)$.

Also associated to (M,ω), is the exact Courant algebroid $(C, [\cdot,\cdot]_C, \langle\cdot,\cdot\rangle^+, \rho)$ described in Prop. 6.14, equipped with a splitting $s\colon TM \to C$ whose curvature is $-\omega$. From this Courant algebroid, we obtain the Lie 2-algebra $L_\infty(C)$ described in Thm. 6.15.

We now describe the relationship between $L_\infty(M,\omega)$ and $L_\infty(C)$. We understand this as the 2-plectic analogue of the relationship described in Sec. 6.2 between the Poisson algebra of a symplectic manifold and the Lie algebra associated to the transitive Lie algebroid over the manifold.

Theorem 6.16. *Let (M,ω) be a 2-plectic manifold and let C be its corresponding Courant algebroid. Let $L_\infty(M,\omega)$ and $L_\infty(C)$ be the Lie 2-algebras corresponding to (M,ω) and C, respectively. There exists a morphism of Lie 2-algebras embedding $L_\infty(M,\omega)$ into $L_\infty(C)$.*

Before we prove the theorem, we introduce some technical lemmas to ease the calculations. Recall from Eq. 6.8 that the formula for the standard skew-symmetric pairing on $\mathfrak{X}(M) \oplus \Omega^1(M)$:

$$\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle^- = \iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1.$$

In what follows, by the symbol "c.p" we mean cyclic permutations of the symbols α, β, γ.

Lemma 6.17. *If $\alpha, \beta \in \Omega^1_{\text{Ham}}(M)$ with corresponding Hamiltonian vector fields v_α, v_β, then $\mathcal{L}_{v_\alpha}\beta = \{\alpha,\beta\} + d\iota_{v_\alpha}\beta$.*

Proof. Since $\mathcal{L}_v = \iota_v d + d\iota_v$,

$$\mathcal{L}_{v_\alpha}\beta = \iota_{v_\alpha} d\beta + d\iota_{v_\alpha}\beta = -\iota_{v_\alpha}\iota_{v_\beta}\omega + d\iota_{v_\alpha}\beta = \{\alpha,\beta\} + d\iota_{v_\alpha}\beta.$$

□

Lemma 6.18. *If* $\alpha, \beta, \gamma \in \Omega^1_{\mathrm{Ham}}(M)$ *with corresponding Hamiltonian vector fields* $v_\alpha, v_\beta, v_\gamma$*, then*

$$\begin{aligned}\iota_{[v_\alpha, v_\beta]}\gamma + \text{c.p} = -3\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega + \iota_{v_\alpha} d\langle v_\beta + \beta, v_\gamma + \gamma\rangle^- \\ + \iota_{v_\gamma} d\langle v_\alpha + \alpha, v_\beta + \beta\rangle^- + \iota_{v_\beta} d\langle v_\gamma + \gamma, v_\alpha + \alpha\rangle^-.\end{aligned}$$

Proof. The identity $\iota_{[v_\alpha, v_\beta]} = \mathcal{L}_{v_\alpha}\iota_{v_\beta} - \iota_{v_\beta}\mathcal{L}_{v_\alpha}$ and Lemma 6.17 imply:

$$\begin{aligned}\iota_{[v_\alpha, v_\beta]}\gamma &= \mathcal{L}_{v_\alpha}\iota_{v_\beta}\gamma - \iota_{v_\beta}\mathcal{L}_{v_\alpha}\gamma \\ &= \mathcal{L}_{v_\alpha}\iota_{v_\beta}\gamma - \iota_{v_\beta}\left(\{\alpha, \gamma\} + d\iota_{v_\alpha}\gamma\right) \\ &= \iota_{v_\alpha} d\iota_{v_\beta}\gamma - \iota_{v_\beta}\iota_{v_\gamma}\iota_{v_\alpha}\omega - \iota_{v_\beta} d\iota_{v_\alpha}\gamma,\end{aligned}$$

where the last equality follows from the definition of the bracket.

Therefore we have:

$$\begin{aligned}\iota_{[v_\gamma, v_\alpha]}\beta &= \iota_{v_\gamma} d\iota_{v_\alpha}\beta - \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega - \iota_{v_\alpha} d\iota_{v_\gamma}\beta, \\ \iota_{[v_\beta, v_\gamma]}\alpha &= \iota_{v_\beta} d\iota_{v_\gamma}\alpha - \iota_{v_\gamma}\iota_{v_\alpha}\iota_{v_\beta}\omega - \iota_{v_\gamma} d\iota_{v_\beta}\alpha,\end{aligned}$$

and Eq. 6.8 implies

$$\iota_{v_\alpha} d\iota_{v_\beta}\gamma - \iota_{v_\alpha} d\iota_{v_\gamma}\beta = \iota_{v_\alpha} d\langle v_\beta + \beta, v_\gamma + \gamma\rangle^-.$$

The statement then follows. □

Lemma 6.19. *If* $\alpha, \beta \in \Omega^1_{\mathrm{Ham}}(M)$ *with corresponding Hamiltonian vector fields* v_α, v_β*, then*

$$\mathcal{L}_{v_\alpha}\beta - \mathcal{L}_{v_\beta}\alpha = 2\{\alpha, \beta\} + d\langle v_\alpha + \alpha, v_\beta + \beta\rangle^-.$$

Proof. Follows immediately from Lemma 6.17 and Eq. 6.8. □

We have all we need to give a proof of Thm. 6.16.

Proof of Theorem 6.16. Let

$$\begin{aligned}L = C^\infty(M) &\xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M), \\ [\cdot, \cdot]_L &: L \otimes L \to L, \\ J_L &: L \otimes L \otimes L \to L\end{aligned}$$

denote the underlying chain complex, bracket, and Jacobiator of the Lie 2-algebra $L_\infty(M, \omega)$. Similarly,

$$\begin{aligned}L' = C^\infty(M) &\xrightarrow{d} \Gamma(C), \\ [\cdot, \cdot]_{L'} &: L' \otimes L' \to L', \\ J_{L'} &: L' \otimes L' \otimes L' \to L'\end{aligned}$$

denotes the underlying chain complex, bracket, and Jacobiator of the Lie 2-algebra $L_\infty(C)$.

We construct a Lie 2-algebra morphism from $L_\infty(M,\omega)$ to $L_\infty(C)$. Recall from Def. 3.11, that such a morphism consists of

- a chain map $\phi\colon L \to L'$, and
- a chain homotopy $\Phi\colon L\otimes L \to L'$ from the chain map

$$\begin{array}{ccc} L\otimes L & \to & L' \\ x\otimes y & \longmapsto & \phi\left([x,y]\right) \end{array}$$

to the chain map

$$\begin{array}{ccc} L\otimes L & \to & L' \\ x\otimes y & \longmapsto & [\phi(x),\phi(y)]', \end{array}$$

such that the following equation holds:

$$\begin{aligned} &\phi_1(J(x,y,z)) - J'(\phi_0(x),\phi_0(y),\phi_0(z)) = \\ &\Phi(x,[y,z]) - \Phi([x,y],z) - \Phi(y,[x,z]) - [\Phi(x,y),\phi_0(z)]' \\ &+[\phi_0(x),\Phi(y,z)]' - [\phi_0(y),\Phi(x,z)]'. \end{aligned} \tag{6.19}$$

Let $s\colon TM \to C$ be the splitting. Let $\phi_0\colon \Omega^1_{\mathrm{Ham}}(M) \to \Gamma(C)$ be given by

$$\phi_0(\alpha) = s(v_\alpha) + \alpha,$$

where v_α is the Hamiltonian vector field corresponding to α. Let $\phi_1\colon C^\infty(M) \to C^\infty(M)$ be the identity. Then $\phi\colon L \to L'$ is a chain map, since the Hamiltonian vector field of an exact 1-form is zero. Let $\Phi\colon \Omega^1_{\mathrm{Ham}}(M)\otimes\Omega^1_{\mathrm{Ham}}(M) \to C^\infty(M)$ be given by

$$\Phi(\alpha,\beta) = -\frac{1}{2}\langle v_\alpha+\alpha, v_\beta+\beta\rangle^-.$$

Now we show Φ is a well-defined chain homotopy in the sense of Def. 3.11. We have

$$\begin{aligned} [\phi_0(\alpha),\phi_0(\beta)]_{L'} &= [s(v_\alpha)+\alpha, s(v_\beta)+\beta]_C \\ &= s([v_\alpha,v_\beta]) + \mathcal{L}_{v_\alpha}\beta - \mathcal{L}_{v_\beta}\alpha - \iota_{v_\beta}\iota_{v_\alpha}\omega \\ &\quad - \frac{1}{2}d\langle v_\alpha+\alpha, v_\beta+\beta\rangle^- \\ &= s([v_\alpha,v_\beta]) + \{\alpha,\beta\} + \frac{1}{2}d\langle v_\alpha+\alpha, v_\beta+\beta\rangle^- \\ &= s([v_\alpha,v_\beta]) + [\alpha,\beta]_L - d\Phi(\alpha,\beta). \end{aligned} \tag{6.20}$$

The second line above is just the definition of the twisted Courant bracket (Eq. 6.17), while the second to last line follows from Lemma 6.19 and Def. 3.3 of the bracket $\{\cdot,\cdot\}$. By Prop. 3.4, the Hamiltonian vector field of $\{\alpha,\beta\}$ is $[v_\alpha, v_\beta]$. Hence we have:

$$\phi_0([\alpha,\beta]_L) - [\phi_0(\alpha),\phi_0(\beta)]_{L'} = d\Phi(\alpha,\beta).$$

In degree 1, the bracket $[\cdot,\cdot]_L$ is trivial. It follows from the definition of $[\cdot,\cdot]_{L'}$ that

$$\phi_1([\alpha,f]_L) - [\phi_0(\alpha),\phi_1(f)]_{L'} = -\frac{1}{2}\langle s(v_\alpha)+\alpha, df\rangle^+.$$

From Eq. 6.16, we have

$$\langle s(v_\alpha)+\alpha, df\rangle^+ = \langle s(v_\alpha)+\alpha, s(0)+df\rangle^+ = \iota_{v_\alpha} df.$$

Therefore

$$\phi_1([\alpha,f]_L) - [\phi_0(\alpha),\phi_1(f)]_{L'} = \Phi(\alpha, df),$$

and similarly

$$\phi_1([f,\alpha]_L) - [\phi_1(f),\phi_0(\alpha)]_{L'} = \Phi(df,\alpha).$$

Therefore Φ is a chain homotopy.

It remains to show the coherence condition (Eq. 6.19 in Definition 3.11) is satisfied. First we rewrite the Jacobiator $J_{L'}$ using the second to last line of (6.20):

$$\begin{aligned} J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma)) &= -\frac{1}{6}\langle [\phi_0(\alpha),\phi_0(\beta)]_{L'}, \phi_0(\gamma)\rangle^+ + \text{c.p} \\ &= -\frac{1}{6}\langle s([v_\alpha,v_\beta]) + \{\alpha,\beta\} - d\Phi(\alpha,\beta), s(v_\gamma)+\gamma\rangle^+ \\ &\quad + \text{c.p}\,. \end{aligned}$$

From the definition of the bracket $\{\cdot,\cdot\}$ and the symmetric pairing, we have

$$J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma)) = -\frac{1}{2}\iota_{v_\gamma}\iota_{v_\beta}\iota_{v_\alpha}\omega - \frac{1}{6}\big(\iota_{[v_\alpha,v_\beta]}\gamma - \iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\big). \quad (6.21)$$

Lemma 6.18 implies

$$\iota_{[v_\alpha,v_\beta]}\gamma + \text{c.p} = -3\iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega - \big(2\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\big), \quad (6.22)$$

so Eq. 6.21 becomes

$$J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma)) = \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega + \big(\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\big).$$

By definition, $J_L(\alpha,\beta,\gamma) = \iota_{v_\alpha}\iota_{v_\beta}\iota_{v_\gamma}\omega$. Therefore, in this case, the left-hand side of Eq. 6.19 is

$$\phi_1(J_L(\alpha,\beta,\gamma)) - J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma)) = -\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,. \quad (6.23)$$

Since the brackets and homotopy Φ are skew-symmetric, the right-hand side of Eq. 6.19 can be rewritten as:

$$\big(\Phi(\alpha,[\beta,\gamma]_L) + \text{c.p}\big) - \big([\Phi(\alpha,\beta),\phi_0(\gamma)]_{L'} + \text{c.p}\big). \tag{6.24}$$

Consider the first term in Eq. 6.24. The Hamiltonian vector field corresponding to $[\beta,\gamma]_L = \{\beta,\gamma\}$ is $[v_\beta, v_\gamma]$. Therefore the definition of Φ implies

$$\Phi(\alpha,[\beta,\gamma]_L) + \text{c.p} = -\frac{3}{2}\iota_{v_\gamma}\iota_{v_\beta}\iota_{v_\alpha}\omega + \frac{1}{2}\big(\iota_{[v_\beta,v_\gamma]}\alpha + \text{c.p}\big).$$

It then follows from Lemma 6.18 (see Eq. 6.22) that

$$\Phi(\alpha,[\beta,\gamma]_L) + \text{c.p} = -\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,.$$

By definition of the bracket $[\cdot,\cdot]_{L'}$, the second term in Eq. 6.24 can be written as

$$[\Phi(\alpha,\beta),\phi_0(\gamma)]_{L'} + \text{c.p} = -\frac{1}{2}\iota_{v_\gamma} d\Phi(\alpha,\beta) + \text{c.p}\,.$$

Hence the coherence condition:

$$\phi_1(J_L(\alpha,\beta,\gamma)) - J_{L'}(\phi_0(\alpha),\phi_0(\beta),\phi_0(\gamma)) = \Phi(\alpha,[\beta,\gamma]_L) - [\Phi(\alpha,\beta),\phi_0(\gamma)]_{L'} + \text{c.p}$$

is satisfied, and $(\phi,\Phi)\colon L_\infty(M,\omega) \to L_\infty(C)$ is a morphism of Lie 2-algebras. □

We now focus on a particular sub-Lie 2-algebra of $L_\infty(C)$. The following definition is due to Ševera [66] and is a generalization of Def. 6.5:

Definition 6.20. *Let C be the exact Courant algebroid given in Prop. 6.14 equipped with a splitting $s\colon TM \to C$. We say a section $e = s(v) + \alpha$* **preserves the splitting** *iff $\forall v' \in \mathfrak{X}(M)$*

$$[\![e, s(v')]\!]_C = s([v,v']).$$

The subspace of sections that preserve the splitting is denoted as $\Gamma(C)^s$.

Note that the twisted Dorfman bracket is used in the above definition rather than the twisted Courant bracket. Since it satisfies the Jacobi identity, it gives a 'strict' adjoint action on sections of C. The 2-plectic analogue of Proposition 6.6 is:

Proposition 6.21. *If C is the exact Courant algebroid given in Proposition 6.14 equipped with the splitting $s\colon TM \to C$, then there is a Lie 2-algebra $L_\infty(C)^s = (L,[\cdot,\cdot],J)$ where:*

- $L_0 = \Gamma(C)^s$,
- $L_1 = C^\infty(M)$,

- *the differential* $L_1 \xrightarrow{d} L_0$ *is the de Rham differential*
- *the bracket* $[\cdot,\cdot]$ *is*
$$[e_1, e_2] = [e_1, e_2]_C \quad \textit{in degree 0}$$
and
$$[e, f] = -[f, e] = \frac{1}{2}\langle e, df\rangle^+ \quad \textit{in degree 1,}$$
- *the Jacobiator is the linear map* $J\colon \Gamma(C)^s \otimes \Gamma(C)^s \otimes \Gamma(C)^s \to C^\infty(M)$ *defined by*
$$\begin{aligned} J(e_1, e_2, e_3) &= -T(e_1, e_2, e_3) \\ &= -\frac{1}{6}\Big(\langle [e_1, e_2]_C, e_3\rangle^+ + \langle [e_3, e_1]_C, e_2\rangle^+ \\ &\quad + \langle [e_2, e_3]_C, e_1\rangle^+\Big). \end{aligned}$$

Proof. Let v' be a vector field on M. By the definition of the twisted Dorfman bracket (Eq. 6.18), it follows that $[\![df, s(v')]\!]_C = 0\ \forall f \in C^\infty(M)$. Hence the complex L is well-defined. We now show that $\Gamma^s(C)$ is closed under the twisted Courant bracket. Suppose e_1 and e_2 are sections preserving the splitting. Let $e_i = s(v_i) + \alpha_i$. Since the twisted Dorfman bracket and the Lie bracket of vector fields satisfy the Jacobi identity, we have:
$$[\![[\![e_1, e_2]\!]_C, s(v')]\!]_C = s([[v_1, v_2], v']).$$
From Eq. 6.6, we have the identity:
$$[e_1, e_2]_C = [\![e_1, e_2]\!]_C - \frac{1}{2}d\langle e_1, e_2\rangle^+.$$
Therefore:
$$\begin{aligned} [\![[e_1, e_2]_C, s(v')]\!]_C &= [\![[\![e_1, e_2]\!]_C, s(v')]\!]_C - \frac{1}{2}\left[\!\left[d\langle e_1, e_2\rangle^+, s(v')\right]\!\right]_C \\ &= s([[v_1, v_2], v']). \end{aligned}$$
It follows from Theorem 6.15 that the Lie 2-algebra axioms are satisfied. □

This next result is essentially a corollary of Thm. 6.16. However, it is important since it is the 2-plectic analogue of Prop. 6.7.

Theorem 6.22. *$L_\infty(M, \omega)$ and $L_\infty(C)^s$ are isomorphic as Lie 2-algebras.*

Proof. Recall that in Theorem 6.16 we constructed a morphism of Lie 2-algebras given by a chain map $\phi\colon L_\infty(M, \omega) \to L_\infty(C)$:
$$\phi_0(\alpha) = s(v_\alpha) + \alpha, \quad \phi_1 = \mathrm{id},$$

and a homotopy $\Phi\colon \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \to C^\infty(M)$:

$$\Phi(\alpha, \beta) = -\frac{1}{2}\langle v_\alpha + \alpha, v_\beta + \beta \rangle^-.$$

Let $v' \in \mathfrak{X}(M)$ and $e = s(v) + \alpha$. By definition of the twisted Dorfman bracket, $[\![e, s(v')]\!]_C = s[v, v']$ if and only if $\iota_{v'}\big(d\alpha + \iota_v\omega\big) = 0$. Hence a section of C preserves the splitting if and only if it lies in the image of the chain map ϕ. Since this map is also injective, the statement follows. □

Theorem 6.22 suggests that we interpret the Lie 2-algebra $L_\infty(C)^s$ as the prequantization of the Lie 2-algebra of "observables" $L_\infty(M, \omega)$. Clearly, these results further support the idea that exact Courant algebroids play the role of higher Atiyah algebroids [11, 26]. However, interpreting $L_\infty(C)^s$ as 'operators' or as infinitesimal symmetries of a U(1)-gerbe with 2-connection is still a work in progress. It is likely that significant progress would be made by solving the larger problem of how to integrate an exact Courant algebroid to a Lie 2-groupoid.

6.5 Central extensions of Lie 2-algebras

In this section, we constructing the 2-plectic version of the Kostant-Souriau central extension, which we discussed in Sec. 6.2. First some preliminary definitions:

Definition 6.23. *A Lie 2-algebra* $(L, [\cdot,\cdot], J)$ *is* **trivial** *iff* $L_1 = 0$.

Any Lie algebra $\mathfrak{g}$ gives a trivial Lie 2-algebra whose underlying complex is

$$0 \to \mathfrak{g}.$$

In particular, the Lie algebra of Hamiltonian vector fields $\mathfrak{X}_{\mathrm{Ham}}(M)$ is a trivial Lie 2-algebra.

Definition 6.24. *A Lie 2-algebra* $(L, [\cdot,\cdot], J)$ *is* **abelian** *iff* $[\cdot,\cdot] = 0$ *and* $J = 0$.

Hence an abelian Lie 2-algebra is just a 2-term chain complex.

Definition 6.25. *If* L*,* L'*, and* L'' *are Lie 2-algebras, then* L' *is a* **strict extension** *of* L'' *by* L *iff there exists Lie 2-algebra morphisms*

$$(\phi, \Phi)\colon L \to L', \quad (\phi', \Phi')\colon L' \to L''$$

such that the chain maps ϕ*,* ϕ' *give a short exact sequence of the underlying chain complexes*

$$L \xrightarrow{\phi} L' \xrightarrow{\phi'} L''.$$

We say L' is a **strict central extension** *of L'' iff L' is an extension of L'' by L and*

$$\left[\mathrm{im}\,\phi, L'\right]' = 0.$$

Remark 6.26. These definitions will be sufficient for our work here. However, they are, in general, too strict. For example, one can have homotopies between morphisms between Lie 2-algebras, and therefore we should consider sequences that are only exact up to homotopy as "exact". Fully weak extensions for degree-wise finite-dimensional Lie n-algebras have recently been described as particular homotopy pushouts in the closed model category of differential graded (dg) algebras [59]. The opposite of this model structure is taken to be, by definition, a presentation of the $(\infty, 1)$-category of degree-wise finite-dimensional L_∞-algebras. For infinite-dimensional Lie n-algebras, such as the ones we consider here, it is likely that one can find a suitable definition in a similar manner by using a closed model category structure on the category of dg co-algebras.

We would like to understand how $L_\infty(M, \omega)$ is a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$ as a Lie 2-algebra. Our first two results are quite general and hold for any 2-plectic manifold (M, ω).

Proposition 6.27. *If (M, ω) is a 2-plectic manifold, then the Lie 2-algebra $L_\infty(M, \omega)$ is a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by the abelian Lie 2-algebra*

$$C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M),$$

consisting of smooth functions and closed 1-forms.

Proof. Consider the following short exact sequence of complexes:

$$\begin{array}{ccccc} \Omega^1_{\mathrm{cl}}(M) & \xrightarrow{\jmath} & \Omega^1_{\mathrm{Ham}}(M) & \xrightarrow{p} & \mathfrak{X}_{\mathrm{Ham}}(M) \\ \uparrow d & & \uparrow d & & \uparrow \\ C^\infty(M) & \xrightarrow{\mathrm{id}} & C^\infty(M) & \longrightarrow & 0 \end{array} \qquad (6.25)$$

The map $\jmath\colon \Omega^1_{\mathrm{cl}}(M) \to \Omega^1_{\mathrm{Ham}}(M)$ is the inclusion, and

$$p\colon \Omega^1_{\mathrm{Ham}}(M) \to \mathfrak{X}_{\mathrm{Ham}}(M), \quad p(\alpha) = v_\alpha$$

takes a Hamiltonian 1-form to its corresponding vector field. It follows from Prop. 3.4 that p preserves the bracket. In fact, all of the horizontal chain maps give strict Lie 2-algebra morphisms (i.e. all homotopies are trivial). The Hamiltonian vector field corresponding to a closed 1-form is zero. Thus, if α is closed, then for all $\beta \in \Omega^1_{\mathrm{Ham}}(M)$ we have $[\alpha, \beta]_{L_\infty(M,\omega)} = \{\alpha, \beta\} = 0$. Hence $L_\infty(M, \omega)$ is a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$. ∎

Proposition 6.28. *Let (M,ω) be a 2-plectic manifold. Given $x \in M$, there is a Lie 2-algebra $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x) = (L,[\cdot,\cdot],J_x)$ where*

- $L_0 = \mathfrak{X}_{\mathrm{Ham}}(M)$,
- $L_1 = \mathbb{R}$,
- *the differential* $L_1 \xrightarrow{d} L_0$ *is trivial* ($d = 0$),
- *the bracket* $[\cdot,\cdot]$ *is the Lie bracket on* $\mathfrak{X}_{\mathrm{Ham}}(M)$ *in degree 0 and trivial in all other degrees*
- *the Jacobiator is the linear map*

$$J_x \colon \mathfrak{X}_{\mathrm{Ham}}(M) \otimes \mathfrak{X}_{\mathrm{Ham}}(M) \otimes \mathfrak{X}_{\mathrm{Ham}}(M) \to \mathbb{R}$$

  defined by

$$J_x(v_1,v_2,v_3) = \iota_{v_1}\iota_{v_2}\iota_{v_3}\omega|_x.$$

Moreover, J_x is a 3-cocycle in the Chevalley-Eilenberg cochain complex $\mathrm{Hom}(\Lambda^\bullet \mathfrak{X}_{\mathrm{Ham}}(M),\mathbb{R})$.

Proof. We have a bracket defined on a complex with trivial differential that satisfies the Jacobi identity "on the nose". Hence to show $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ is a Lie 2-algebra, it sufficient to show that the Jacobiator $J_x(v_1,v_2,v_3)$ satisfies Eq. 3.10 in Def. 3.10 for $x \in M$. This follows immediately from Thm. 3.15. The classification theorem of Baez and Crans (Thm. 55 in [4]) implies that J_x satisfying Eq. 3.10 in the definition of a Lie 2-algebra is equivalent to J_x being a 3-cocycle with values in the trivial representation. □

Recall that in the symplectic case, if the manifold is connected, then the Poisson algebra is a central extension of the Hamiltonian vector fields by the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$. The categorified analog of the Lie algebra $\mathfrak{u}(1)$ is the abelian Lie 2-algebra $b\mathfrak{u}(1)$ whose underlying chain complex is simply

$$\mathbb{R} \to 0.$$

It is natural to suspect that, under suitable topological conditions, the abelian Lie algebra $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M)$ introduced in Prop. 6.27 is related to $b\mathfrak{u}(1)$.

Let us first assume that the 2-plectic manifold is connected. Note that the Jacobiator J_x of the Lie 2-algebra $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$ introduced in Prop. 6.28 depends explicitly on the choice of $x \in M$. However, if M is connected, then the cohomology class J_x represents as a 3-cocycle does not depend on x. This fact has important implications for $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M),x)$:

Proposition 6.29. *If (M, ω) is a connected 2-plectic manifold and J_x is the 3-cocycle given in Prop. 6.28, then the cohomology class $[J_x] \in H^3_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ is independent of the choice of $x \in M$. Moreover, given any other point $y \in M$, the Lie 2-algebras $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), y)$ are quasi-isomorphic.*

Proof. To prove that $[J_x]$ is independent of x, we use a construction similar to the proof of Prop. 4.1 in [12]. The Chevalley-Eilenberg differential

$$\delta\colon \mathrm{Hom}(\Lambda^n \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R}) \to \mathrm{Hom}(\Lambda^{n+1} \mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$$

is defined by

$$(\delta c)(v_1, \ldots, v_{n+1}) = \sum_{1 \leq i < j \leq n} (-1)^{i+j} c([v_i, v_j], v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \ldots, v_{n+1}).$$

Note that if c is an arbitrary 2-cochain then

$$(\delta c)(v_\alpha, v_\beta, v_\gamma) = -c([v_\alpha, v_\beta], v_\gamma) + c([v_\alpha, v_\gamma], v_\beta) - c([v_\beta, v_\gamma], v_\alpha).$$

Now let $y \in M$. Let $\Gamma\colon [0,1] \to M$ be a path from x to y. Given $v_\alpha, v_\beta \in \mathfrak{X}_{\mathrm{Ham}}(M)$, define

$$c(v_\alpha, v_\beta) = \int_\Gamma \omega(v_\alpha, v_\beta, \cdot).$$

Clearly, c is a 2-cochain. We claim

$$J_y(v_\alpha, v_\beta, v_\gamma) - J_x(v_\alpha, v_\beta, v_\gamma) = (\delta c)(v_\alpha, v_\beta, v_\gamma)$$

The failure of $\{\cdot, \cdot\}$ to satisfy the Jacobi identity implies

$$d\iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega = \{\alpha, \{\beta, \gamma\}\} - \{\{\alpha, \beta\}, \gamma\} - \{\beta, \{\alpha, \gamma\}\},$$

and, from the definition of $\{\cdot, \cdot\}$, we have

$$d\iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega = -\omega([v_\alpha, v_\beta], v_\gamma, \cdot) + \omega([v_\alpha, v_\gamma], v_\beta, \cdot) - \omega([v_\beta, v_\gamma], v_\alpha, \cdot).$$

Integrating both sides of the above equation gives

$$\begin{aligned}
\int_\Gamma d\iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega &= J_y(v_\alpha, v_\beta, v_\gamma) - J_x(v_\alpha, v_\beta, v_\gamma) \\
&= -\int_\Gamma \omega([v_\alpha, v_\beta], v_\gamma, \cdot) + \int_\Gamma \omega([v_\alpha, v_\gamma], v_\beta, \cdot) - \int_\Gamma \omega([v_\beta, v_\gamma], v_\alpha, \cdot) \\
&= (\delta c)(v_\alpha, v_\beta, v_\gamma).
\end{aligned}$$

It follows from Thm. 57 in Baez and Crans [4] that $[J_x] = [J_y]$ implies $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), y)$ are quasi-isomorphic (or 'equivalent' in their terminology). $\square$

Now we impose further conditions on our 2-plectic manifold. From here on, we assume (M, ω) is 1-connected (i.e. connected and simply connected). This is the 2-plectic analogue of the requirement that the symplectic manifold in Sec. 6.2 be connected. It will allow us to construct several elementary, yet interesting, quasi-isomorphisms of Lie 2-algebras.

Proposition 6.30. *If M is a 1-connected manifold, then the abelian Lie 2-algebra $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{cl}}(M)$ is quasi-isomorphic to $b\mathfrak{u}(1)$.*

Proof. Let $x \in M$. The chain map

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Omega^1_{\mathrm{cl}}(M) \\ {\scriptstyle \mathrm{ev}_x}\downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism. □

Proposition 6.31. *If (M, ω) is a 1-connected 2-plectic manifold and $x \in M$, then the Lie 2-algebras $L_\infty(M, \omega)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ are quasi-isomorphic.*

Proof. We construct a quasi-isomorphism from $L_\infty(M, \omega)$ to $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$. There is a chain map

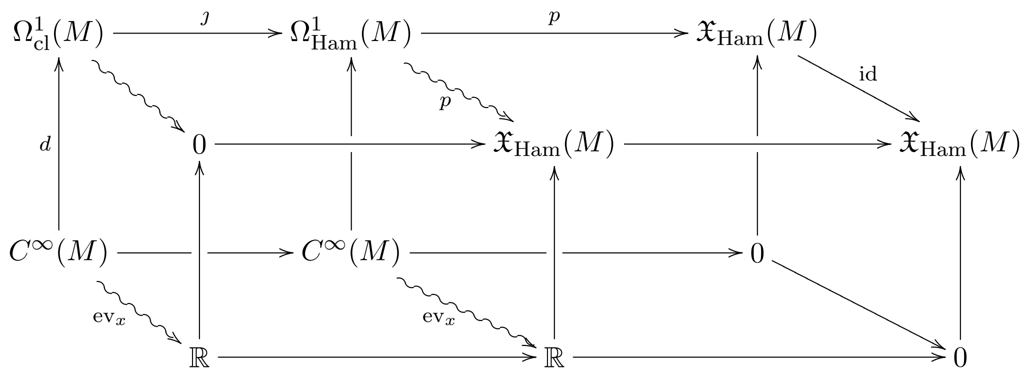
$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Omega^1_{\mathrm{Ham}}(M) \\ {\scriptstyle \mathrm{ev}_x}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{R} & \xrightarrow{0} & \mathfrak{X}_{\mathrm{Ham}}(M) \end{array}$$

with $\mathrm{ev}_x(f) = f(x)$ and $p(\alpha) = v_\alpha$. Since p preserves the bracket, we take Φ in Def. 3.11 to be the trivial homotopy. Eq. 6.19 holds since:

$$\mathrm{ev}_x(\omega(v_\gamma, v_\beta, v_\alpha)) = J_x(v_\alpha, v_\beta, v_\gamma),$$

and therefore we have constructed a Lie 2-algebra morphism. Since M is connected, the homology of the complex $C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M)$ is just $\mathbb{R}$ in degree 1 and $\Omega^1_{\mathrm{Ham}}(M)/dC^\infty(M)$ in degree 0. The kernel of the surjective map p is the space of closed 1-forms, which is $dC^\infty(M)$ since M is simply connected. □

We can summarize the results given in Props. 6.27 6.28 6.30, and 6.31 with the following commutative diagram:



The back of the diagram shows $L_\infty(M,\omega)$ as the central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$. The front shows $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ as a central extension of $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$. The morphisms going from back to front are all quasi-isomorphisms. Thus we have the 2-plectic analogue of the Kostant-Souriau central extension:

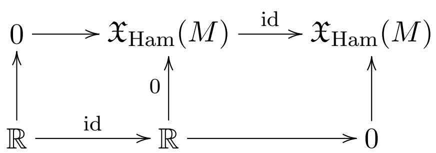
Theorem 6.32. *If (M,ω) is a 1-connected 2-plectic manifold, then $L_\infty(M,\omega)$ is quasi-isomorphic to a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$.*

Also, from Prop. 6.22 we know that $L_\infty(M,\omega)$ is isomorphic to the Lie 2-algebra $L_\infty(C)^s$ consisting of sections of the Courant algebroid C which preserve a chosen splitting $s\colon TM \to C$. Therefore:

Corollary 6.33. *If (M,ω) is a 1-connected 2-plectic manifold, then $L_\infty(C)^s$ is quasi-isomorphic to a central extension of the trivial Lie 2-algebra $\mathfrak{X}_{\mathrm{Ham}}(M)$ by $b\mathfrak{u}(1)$.*

A comparison of the above corollary to the results discussed in Sec. 6.2 suggests that $L_\infty(C)^s$ be interpreted as the quantization of $L_\infty(M,\omega)$ with $b\mathfrak{u}(1)$ giving rise to the quantum phase.

Finally, note that a splitting of the short exact sequence of complexes



is the identity map in degree 0 and the trivial map in degree 1. Obviously the splitting preserves the bracket but does not preserve the Jacobiator. Indeed, the failure of the splitting to be a strict Lie 2-algebra morphism between $\mathfrak{X}_{\mathrm{Ham}}(M)$ and $L_\infty(\mathfrak{X}_{\mathrm{Ham}}(M), x)$ is due to the presence of the 3-cocycle J_x.

Summary

Many new results have been given in this chapter, so we conclude with a brief summary. We defined a prequantized n-plectic manifold to be an integral n-plectic manifold equipped with Deligne n-cocycle. If (M, ω) is a 0-connected, prequantized symplectic manifold, then there exists a principal $U(1)$-bundle over M equipped with a connection whose curvature is ω, and a corresponding Atiyah algebroid $A \to M$ equipped with a splitting such that the Lie algebra of sections of A which preserve the splitting is isomorphic to a central extension of the Lie algebra of Hamiltonian vector fields:

$$\mathfrak{u}(1) \to C^{\infty}(M) \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

This central extension gives a cohomology class in $H^2_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ which can be represented by the symplectic form evaluated at a point in M.

Analogously, if (M, ω) is a 1-connected, prequantized 2-plectic manifold, then there exists a $U(1)$-gerbe over M equipped with a connection and curving whose 3-curvature is ω, and a corresponding exact Courant algebroid $C \to M$ equipped with a splitting such that the Lie 2-algebra of sections of C which preserve the splitting is quasi-isomorphic to a central extension of the (trivial) Lie 2-algebra of Hamiltonian vector fields:

$$b\mathfrak{u}(1) \to L_{\infty}(\mathfrak{X}_{\mathrm{Ham}}(M)) \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

This central extension gives a cohomology class in $H^3_{\mathrm{CE}}(\mathfrak{X}_{\mathrm{Ham}}(M), \mathbb{R})$ which can be represented by the 2-plectic form evaluated at a point in M.

Chapter 7

Geometric quantization of 2-plectic manifolds

In the previous chapter, we first considered prequantization for symplectic manifolds, and then generalized the procedure to 2-plectic manifolds. We were primarily concerned with prequantizing the algebra of observables, i.e. the Poisson algebra in the symplectic case, and the Lie 2-algebra of Hamiltonian 1-forms in the 2-plectic case. In this chapter, we switch our focus from quantizing observables to quantizing states.

Prequantization is a simple and elegant construction. However, numerous examples in symplectic geometry show that it is only the first step of a two-part process. Full quantization involves using additional structures in order to construct the correct space of quantum states. This process was developed over time by considering particular examples. We suspect that the development of a complete geometric quantization procedure for 2-plectic manifolds will follow a similar path. In this chapter, we generalize aspects of the quantization process for symplectic manifolds to the 2-plectic case by using the higher geometric structures introduced in earlier chapters. The result is a simple procedure for quantizing 2-plectic manifolds, which we apply to a particular example of interest. To the best of our knowledge, this is the first geometric quantization procedure ever developed for such manifolds.

Let us provide some motivation for why additional work beyond prequantization is needed in order to obtain the correct quantum states. In the last chapter, we described a prequantized symplectic manifold as a symplectic manifold equipped with principal U(1)-bundle with connection. A natural choice for the quantum state space is the space of square-integrable global sections of the Hermitian line bundle associated to the principal bundle. This is often called the 'prequantum Hilbert space'. It comes equipped with an inner product given by integrating the fiber-wise Hermitian inner product of sections with respect to the symplectic volume form. However, from the physicist's point of view, this space is too large to be the space of quantum states of a physical system.

For example, recall that the cotangent bundle of a manifold is a symplectic manifold, equipped with its canonical symplectic structure $\omega = \sum_i dp_i \wedge dq^i$. It is, in fact, an integral symplectic manifold since ω is exact. The sections in the prequantized Hilbert space locally look like functions $f(q^i, p_i)$ of $2n$ variables corresponding to the "position" coordinates q^i of the base manifold and the "momentum" coordinates p_i of the fibers. These functions can have arbitrarily small support, and hence, when interpreted as wavefunctions on a classical phase space, give probability densities which violate the Heisenberg uncertainty condition. To get around this problem, one reduces the size of the Hilbert space by taking the subspace consisting of those sections satisfying $\partial f/\partial p_i = 0$. Hence, the number of "variables" is reduced from $2n$ to n, by only considering those sections constant along the fibers.

Consider another example that is perhaps more mathematically interesting. The coadjoint orbits of the Lie group SU(2) correspond to 2-spheres centered about the origin in $\mathfrak{su}(2)^* \cong \mathbb{R}^3$. Each orbit is a symplectic manifold equipped with what is known as the 'KKS symplectic form'. This 2-form is integral if the radius of the sphere is one-half of a non-negative integer. On each integral orbit, we have the prequantized Hilbert space, consisting of global square-integrable sections of a Hermitian line bundle. This Hilbert space is infinite dimensional. However, we can equip the orbit with a complex structure and consider only holomorphic sections i.e. those sections which locally are functions $f(z^i, \bar{z}^i)$ satisfying $\partial f/\partial \bar{z}^i = 0$. This smaller space of holomorphic sections is much more interesting. First, it is finite-dimensional. Moreover, it is an irreducible representation of SU(2). This way of obtaining representations from coadjoint orbits by geometric quantization is quite general, and is known as Kirillov's orbit method [34]. Note that, again, the size of the prequantum space is reduced by decreasing the number of variables.

Hence, it is important to consider prequantized symplectic manifolds equipped with additional structure in order to cut down the number of admissible sections in the prequantum Hilbert space. In both of the above examples, the extra structure corresponds to a special integrable distribution called a 'polarization'. We introduced real k-polarizations for n-plectic manifolds in Chapter 2 precisely for this reason, and we see that real 1-polarizations appeared in our first example. The second example employed the use of a 'complex polarization'. These structures certainly play an important role in symplectic geometry [70, Chap. 5]. Unfortunately, it is not yet clear how to generalize them to the n-plectic case. Hence, we only will consider real k-polarizations for $n > 1$.

For symplectic manifolds, the output from quantization is a Hilbert space of quantum states. As we will see, the output from quantizing a 2-plectic manifold is a category of quantum states. In the last section of this chapter, we consider in detail an example in which the states correspond to objects in a representation category. This suggests that 2-plectic quantization can categorify Kirillov's orbit method.

7.1 Geometric quantization of symplectic manifolds

As usual, it is instructive to consider the symplectic case first. Consider a prequantized symplectic manifold (M, ω, ξ), where ξ is a Deligne 1-cocycle. Recall from Example 5.14 in Chap. 5 that $\xi = (g, \theta)$ is specified by an open cover $\{U_i\}$ of M, local 1-forms $\theta_i \in \Omega^1(U_i)$, and U(1)-valued functions $g_{ij} \colon U_i \cap U_j \to \mathrm{U}(1)$ satisfying certain cocycle conditions. In this chapter, we realize this 1-cocycle as the transition functions and local connection forms of a Hermitian line bundle $(L, \langle \cdot, \cdot \rangle)$ equipped with a connection ∇. We let $\Gamma(L)_c$ denote the smooth sections of L with compact support. The prequantum Hilbert space is defined to be the completion of $\Gamma(L)_c$ with respect to the inner product $(\sigma_1, \sigma_2) = \int_M \langle \sigma_1, \sigma_2 \rangle \omega^n$.

Recall from Def. 2.13 that a real polarization on M is a foliation F of M whose leaves are immersed Lagrangian submanifolds.

Definition 7.1. *A* **quantized symplectic manifold** *is a prequantized symplectic manifold (M, ω, ξ) equipped with a real polarization F.*

7.1.1 The Bohr-Sommerfeld variety

We use Deligne cocycles in some parts of this section, rather than the more traditional language of bundles, in order to make the analogy with the 2-plectic case as clear as possible. In the 2-plectic case, we use Deligne cocycles, rather than stacks directly, since the cocycles behave better under pullbacks and restrictions.

Given a quantized symplectic manifold (M, ω, ξ, F), let $D_F \subseteq TM$ denote the corresponding involutive distribution. A good candidate for the quantum Hilbert space is the space constructed from those sections of $\Gamma(L)_c$ which are covariantly constant along each leaf of F:

$$H = \{\sigma \in \Gamma(L)_c \mid \nabla_v \sigma = 0 \ \forall v \in \Gamma(D_F)\}.$$

Unfortunately, the topology of the leaves will often force this space to be trivial. For example, if the leaves of the foliation are not compact, then we must have $\sigma = 0$ for all $\sigma \in H$. Otherwise, the integral of $\langle \sigma, \sigma \rangle \omega^n$ will diverge.

There are additional topological obstructions which are more interesting. Let $\Lambda \subseteq M$ be a leaf of the foliation F. Since the restriction $(L|_\Lambda, \nabla|_\Lambda)$ is a flat Hermitian line bundle, it is completely determined by its holonomy representation

$$\oint \nabla|_\Lambda \colon \pi_1(M) \to \mathrm{U}(1).$$

If σ is a section of L which is covariantly constant along F, then $\sigma|_\Lambda$ is a covariantly constant global section of $(L|_\Lambda, \nabla|_\Lambda)$. Hence $\sigma|_\Lambda$ is either zero, or $(L|_\Lambda, \nabla|_\Lambda)$ is the trivial bundle with trivial connection, i.e. $\oint \nabla|_\Lambda = 1$.

So, we should consider only the leaves on which the restricted bundle has trivial holonomy. In the language of Section 5.6, these are the leaves $\Lambda \xrightarrow{i} M$ with the property that given a map $\sigma \colon S^1 \to \Lambda$, the corresponding holonomy (Def. 5.35) of the Deligne 1-cocycle $\xi|_\Lambda = i^*\xi$ is trivial: $\mathrm{hol}(\xi|_\Lambda, \sigma) = 1$. This leads us to the following definition.

Definition 7.2. *Let (M, ω, ξ, F) be a quantized symplectic manifold. The* **Bohr-Sommerfeld variety V_{BS}** *associated to F is the union of all leaves Λ of F which satisfy*

$$\mathrm{hol}(\xi|_\Lambda, \sigma) = 1$$

for all maps $\sigma \colon S^1 \to \Lambda$.

The relation with the Bohr-Sommerfeld conditions from physics comes from the fact that Λ is contained in the Bohr-Sommerfeld variety if and only if for every loop γ in $\Lambda \cap U_i$:

$$\exp\left(\sqrt{-1}\oint_\gamma \theta_i\right) = 1 \Leftrightarrow \oint_\gamma \theta_i = 2\pi n_\gamma, \quad n_\gamma \in \mathbb{Z},$$

where θ_i is the local connection 1-form on U_i.

The use of Bohr-Sommerfeld varieties in geometric quantization was developed considerably by Śniatycki [62]. He showed that the correct quantum Hilbert space is the completion of the space of sections of $L|_{V_{\mathrm{BS}}}$ which are covariantly constant along each leaf contained in the variety. In general, such a section will not be the pullback of a global smooth section of $L \to M$. Instead, it corresponds to a 'distributional section' of L [62][Sec. 5]. Śniatycki's work motivates the next definition.

Definition 7.3. *Let (M, ω, ξ, F) be a quantized symplectic manifold, V_{BS} be the corresponding Bohr-Sommerfeld variety, and $L|_{V_{\mathrm{BS}}}$ be Hermitian line bundle associated to the Deligne 1-cocycle $\xi|_{V_{\mathrm{BS}}}$. The* **quantum state space** *$Q(V_{\mathrm{BS}})$is the space of sections of $L|_{V_{\mathrm{BS}}}$ which are covariantly constant along each leaf contained in V_{BS}.*

7.1.2 Example: $\mathbb{R}^2 \setminus \{0\}$

In this example, we will construct the quantum state space associated to the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$ equipped with the 2-form

$$\omega = rdr \wedge dt,$$

with $0 < r < \infty,\ 0 \leq t < 2\pi$. Since $\omega = d\theta$, where

$$\theta = Hdt, \quad H = \frac{1}{2}r^2,$$

we see (M, ω) is an integral symplectic manifold. Hence θ is a connection 1-form on the trivial Hermitian line bundle $L = M \times \mathbb{C}$.

There is an obvious foliation F of M whose leaves are concentric circles of radius $R > 0$ about the origin. Since ω is a volume form on M, our discussion in Example 2.14 implies F is a polarization. The corresponding distribution D_F is the vector field $\partial/\partial t$.

Let us first consider global sections of L covariantly constant along the leaves of F in order to see why the Bohr-Sommerfeld variety enters the picture. Such a section ψ must satisfy:

$$\nabla_{\partial/\partial t}\psi = 0.$$

Since $\nabla = d + \sqrt{-1} \cdot \theta$, this is equivalent to ψ satisfying the differential equation

$$\frac{\partial \psi}{\partial t} = -\frac{\sqrt{-1}}{2} r^2 \psi,$$

which has solutions of the form

$$\psi(r,t) = \exp(-\frac{\sqrt{-1}}{2} r^2 t) g(r).$$

However, such a solution must also satisfy:

$$\psi(r,t) = \psi(r, t + 2\pi).$$

Hence, $\psi(r,t)$ must vanish if $\frac{r^2}{2}$ is not an integer, and therefore no non-trivial smooth solution exists.

Now let us consider the Bohr-Sommerfeld variety associated to F. Let the leaf S^1_R correspond to a circle of radius R. The Bohr-Sommerfeld condition implies

$$\oint_{S^1_R} \theta = \frac{1}{2}R^2 \int_0^{2\pi} dt \in 2\pi\mathbb{Z}.$$

Hence, the variety corresponds to the integer level sets of H:

$$V_{\mathrm{BS}} = \bigcup_{n \in \mathbb{N}^+} H^{-1}(\{n\}),$$

and the quantum state space $Q(V_{\mathrm{BS}})$ consists of linear combinations of functions of the form

$$\psi_n(t) = \exp(-nt\sqrt{-1})g(\sqrt{2n}).$$

This quantized symplectic manifold is closely related to the quantization of the simple harmonic oscillator. We can interpret M as the classical phase space of the oscillator, and H as a Hamiltonian function which measures the energy of the oscillator. It takes the familiar form $H = \frac{1}{2}(p^2 + q^2)$ in cartesian coordinates. The level sets of H are the leaves of the foliation and correspond to the classically allowed states in phase-space with constant energy $\frac{1}{2}R^2$. The Bohr-Sommerfeld condition restricts the allowed states of the oscillator to those in V_{BS} thereby quantizing the energy of the oscillator. The quantum values for the energy are the non-negative integers. The sections $\psi_n(t)$ represent the quantum states which satisfy the Schrödinger equation

$$\sqrt{-1} \cdot \frac{\partial \psi}{\partial t} = \hat{H}\psi.$$

Strictly speaking, this is not the correct quantization of the simple harmonic oscillator, since its quantum energy states are actually $n + 1/2$. Obtaining these shifted values for the energy requires using a more sophisticated approach involving the 'metaplectic correction' [62], [70][Ch. 10].

7.2 Categorified geometric quantization

Now we present the 2-plectic analogue of the previously discussed quantization process. We start with a prequantized 2-plectic manifold (M, ω, ξ), where ξ is a Deligne 2-cocycle. From Example 5.16 in Chap. 5, we know that $\xi = (g, A, B)$ is specified by an open cover $\{U_i\}$ of M, local 2-forms $B_i \in \Omega^2(U_i)$, local 1-forms $A_{ij} \in \Omega^1(U_i \cap U_j)$, and U(1)-valued functions $g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1)$ satisfying certain cocycle conditions. Recalling Definition 5.33, we realize this cocycle as the 2-line stack Bund^{ξ} equipped with a 2-connection.

We defined real k-polarizations for n-plectic manifolds in Def. 2.13. Recall that, unlike the symplectic case, there are several ways to define orthogonal complements for n-plectic manifolds. Hence, there are different ways to generalize the notion of Lagrangian submanifold, and therefore real polarization, to the n-plectic case. For the 2-plectic case, we can consider either 1-polarizations or 2-polarizations. Regardless, the definitions in the previous section for symplectic manifolds naturally generalize:

Definition 7.4. *A* **quantized 2-plectic manifold** *is a prequantized 2-plectic manifold* (M, ω, ξ) *equipped with a real* k*-polarization* F.

The next definition uses the notion of 2-holonomy for a Deligne 2-cocycle (Def. 5.36).

Definition 7.5. *Let* (M, ω, ξ, F) *be a quantized 2-plectic manifold. The* **Bohr-Sommerfeld variety** $\boldsymbol{V_{\mathrm{BS}}}$ *associated to* F *is the union of all leaves* Λ *of* F *which satisfy*

$$\mathrm{hol}(\xi|_{\Lambda}, \sigma) = 1$$

for all maps $\sigma \colon \Sigma^2 \to \Lambda$, *where* Σ^2 *is a compact, oriented 2-manifold.*

The Bohr-Sommerfeld variety is, by construction, a disjoint union of immersed submanifolds in M. The inclusion map $V_{\mathrm{BS}} \xrightarrow{i} M$ is smooth, and we can pull-back the Deligne 2-cocycle ξ to V_{BS}. If ξ is defined with respect to an open cover $\{U_i\}$ of M, then $\xi|_{V_{\mathrm{BS}}}$ is a 2-cocycle with respect to the cover $\{U_i \cap V_{\mathrm{BS}}\}$. In analogy with the symplectic case, we consider global sections of the 2-line stack Bund^{ξ} over V_{BS}, where by ξ we mean $\xi|_{V_{\mathrm{BS}}}$. Proposition 5.32 implies that the category of such global sections is equivalent to the category of $\xi|_{V_{\mathrm{BS}}}$-twisted Hermitian vector bundles over V_{BS}. In Definition 5.31, we described what it means for a twisted bundle to be twisted-flat. We interpret twisted-flatness to be the 2-plectic analogue of covariantly constant.

Let $(E_i, \nabla_i, \phi_{ij})$ be a $\xi|_{V_{\mathrm{BS}}}$-twisted Hermitian vector bundle over the Bohr-Sommerfeld variety. Recall from Def. 5.30, that such a bundle is given by the following data: Over each open set $V_i = U_i \cap V_{\mathrm{BS}}$, a Hermitian vector bundle with connection (E_i, ∇_i), and, over each intersection $V_i \cap V_j$, an isomorphism ϕ_{ij} between the pullbacks of bundles E_j and E_i. The isomorphisms ϕ_{ij} are required to satisfy compatibility relations with the 1-forms $A_{ij}|_{V_{\mathrm{BS}}}$ on $V_i \cap V_j$, and with the U(1)-valued functions $g_{ijk}|_{V_{\mathrm{BS}}}$ on $V_i \cap V_j \cap V_k$.

We can pull this twisted bundle back to any leaf $\Lambda \subseteq V_{\mathrm{BS}}$ in the obvious way, resulting in a bundle twisted by $\xi|_{\Lambda} = (g|_{\Lambda}, A|_{\Lambda}, B|_{\Lambda})$. It is twisted-flat iff the equality

$$\nabla_i^2|_{\Lambda} - \sqrt{-1} \cdot B_i|_{\Lambda} \otimes \mathrm{id} = 0.$$

holds for all i. Twisted bundles satisfying the above for all leaves $\Lambda \subseteq V_{\mathrm{BS}}$ form a full subcategory of $\mathsf{Bund}^{\xi}(V_{\mathrm{BS}})$. Hence, we have a categorified analogue of the quantum state space:

Definition 7.6. *Let (M, ω, ξ, F) be a quantized 2-plectic manifold and V_{BS} be the corresponding Bohr-Sommerfeld variety. The* **quantum state category** $\mathsf{Quant}(V_{\mathrm{BS}})$ *is the subcategory of* $\mathsf{Bund}^{\xi}(V_{\mathrm{BS}})$ *consisting of twisted Hermitian vector bundles that are twisted-flat along each leaf contained in* V_{BS}.

7.2.1 Example: $\mathbb{R}^3 \setminus \{0\}$

In this section, we consider an example in detail which will reveal several interesting aspects of our quantization procedure for 2-plectic manifolds. We construct the quantum state category associated to the manifold $M = \mathbb{R}^3 \setminus \{0\}$ equipped with the 2-plectic form

$$\omega = \frac{1}{r^2} dx^1 \wedge dx^2 \wedge dx^3,$$

where r is given by the usual Euclidean norm. In analogy with the example involving the symplectic manifold $\mathbb{R}^2 \setminus \{0\}$, we will see how the Bohr-Sommerfeld variety is used to overcome certain topological obstructions.

One reason for considering the 3-form ω is because $\omega = dB$, where

$$B = \frac{1}{r^2}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy).$$

Restricting the 2-form B to a sphere centered about the origin gives the famous KKS symplectic form. We mentioned this symplectic structure and the role it plays in representation theory in the introduction to the chapter. We shall make use of this fact later on in Sec. 7.3.

We prequantize (M, ω) with the Deligne 2-cocycle $\xi = (1, 0, B)$. More precisely, we choose a good open cover $\{U_i\}$ of M and consider ξ as the restriction of ξ to this cover.

Global sections of Bund^{ξ}

Before we proceed further, let us characterize the global sections of Bund^{ξ} i.e. ξ-twisted Hermitian vector bundles over M. Let $(E_i, \nabla_i, \phi_{ij})$ be such a bundle. Since $\xi = (1, 0, B)$ projects to the trivial class in $H^2(M, \underline{U(1)})$, we are dealing with trivially twisted vector bundles with connection. Let $\mathsf{Bund}_{\nabla}(M)$ denote the category whose objects are Hermitian vector bundles over M equipped with connection. The following proposition says we can identify trivially twisted bundles with ordinary bundles.

Proposition 7.7. *The categories* $\mathsf{Bund}^{\xi}(M)$ *and* $\mathsf{Bund}_{\nabla}(M)$ *are equivalent.*

Proof. Since $\xi = (1, 0, \sqrt{-1} \cdot B)$, Def. 5.30 implies that an object of $\mathsf{Bund}^{\xi}(M)$ is given by a Hermitian vector bundle with connection (E_i, ∇_i) on each U_i, an isomorphism $\phi_{ij} \colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}}$, which preserves the connection $\phi_{ij}\nabla_j = \nabla_i \phi_{ij}$ on U_{ij}, such that $\phi_{ik}^{-1}\phi_{ij}\phi_{jk} = 1$ on U_{ijk} A morphism $(E_i, \nabla_i, \phi_{ij}) \to (E'_i, \nabla'_i, \phi'_{ij})$ is given by a collection of bundle morphisms $E_i \xrightarrow{f_i} E'_i$ which preserve the connection $\nabla'_i f_i = f_i \nabla_i$, satisfying $f_i \phi_{ij} = \phi'_{ij} f_j$ on U_{ij}.

Now, consider the functor $F \colon \mathsf{Bund}_{\nabla}(M) \to \mathsf{Bund}^{\xi}(M)$ which sends a vector bundle (E, ∇) to the trivially twisted bundle $(E|_{U_i}, \nabla|_{U_i}, \phi_{ij} = \mathrm{id})$, and a morphism f to its restriction on each U_i. We shall show F is full, faithful, and essentially surjective, and hence gives an equivalence of categories. For essential surjectivity, we must show that given $(E_i, \nabla_i, \phi_{ij})$ there exists an object (E, ∇) such that $F(E, \nabla)$ is isomorphic to $(E_i, \nabla_i, \phi_{ij})$. By unraveling Def. 5.6 for a stack, we see that the above data for a trivially twisted bundle implies there exists a Hermitian vector bundle with connection (E, ∇) on M and connection preserving isomorphisms $E|_{U_i} \xrightarrow{\psi_i} E_i$ on U_i such that $\phi_{ij}\psi_j = \psi_i$ on U_{ij}. Hence, the ψ_i give an isomorphism in $\mathsf{Bund}^{\xi}(M)$ between $F(E, \nabla)$ and $(E_i, \nabla_i, \phi_{ij})$.

It's clear that F is faithful (i.e. injective on morphisms). For fullness, we must show $F \colon \mathrm{Hom}(E, E') \to \mathrm{Hom}(F(E), F(E'))$ is surjective. Let $E|_{U_i} \xrightarrow{f_i} E'|_{U_i}$ denote a morphism between $F(E)$ and $F(E')$. Since $\phi_{ij} = \phi'_{ij} = \mathrm{id}$, it follows from the definition of morphism that $f_i = f_j$ on each U_{ij}. Since morphisms between bundles form a sheaf, there exists a unique global morphism $E \xrightarrow{f} E'$ such that $f|_{U_i} = f_i$. Hence, the proposition is proven. □

Topological considerations

There is an obvious foliation F of M whose leaves S^2_R are concentric spheres of radius $R > 0$ about the origin. Since ω is a volume form on M, our discussion in Example 2.14 implies that F is a 2-polarization. Hence, (M, ω, ξ, F) is a quantized 2-plectic manifold.

To see why the Bohr-Sommerfeld variety is needed, let us consider global sections of Bund^{ξ} which are twisted-flat along the leaves of F. By Prop. 7.7, any global section can be thought of as a Hermitian vector bundle $E \to M$ with connection ∇. Let $E|_R$ denote the restriction of this bundle to a leaf S^2_R. By definition, $E|_R$ is twisted-flat if its curvature satisfies $\nabla^2|_R = \sqrt{-1} \cdot B|_R \otimes \mathrm{id}$.

The next proposition implies $B|_R$ must be an integral 2-form.

Proposition 7.8. *If E is a rank n Hermitian vector bundle with connection ∇ on $S^2|_R$*

with curvature $\nabla^2 = \sqrt{-1} \cdot B|_R \otimes \mathrm{id}$*, then there is an isomorphism of bundles*

$$E \xrightarrow{\sim} L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

where L_i *is a Hermitian line bundle with connection whose curvature 2-form is* $B|_R$*. Moreover, the* L_i*'s are all isomorphic as line bundles with connection.*

The proposition can be proven using classical results from differential geometry. Let P be a principal G-bundle over a connected manifold M equipped with a $\mathfrak{g}$-valued connection 1-form θ. Such a bundle is said to be **reducible** to a principal G'-bundle $P' \xrightarrow{\iota} P$ iff G' is a subgroup of G, and the inclusion map ι commutes with the group action of G'. The connection θ reduces to a connection on P' iff its pullback along the inclusion takes values in the Lie algebra of G'.

Given $p \in P$, let $H(p)$ denote the set of points in P which are joined by a piece-wise smooth horizontal path in P. Let $\mathrm{Hol}_p(\theta)$ be the holonomy group based at $p \in P$ i.e. the subgroup of G consisting of elements g such that p and pg are joined by a piece-wise smooth horizontal loop in P. Similarly, let $\mathrm{Hol}^0_p(\theta)$ be the subgroup consisting of those g such that p and pg are connected by a contractible horizontal loop. Both of these subgroups are, in fact, Lie subgroups. The following is Theorem 7.1 in Kobayashi-Nomizu [35].

Theorem 7.9 (Reduction Theorem)**.** *A principal* G*-bundle* P *with connection* θ *is reducible to a principal bundle with total space* $H(p)$ *and structure group* $\mathrm{Hol}_p(\theta)$*. Furthermore,* θ *reduces to a connection on* $H(p)$*.*

Next, we recall the Ambrose-Singer Theorem.

Theorem 7.10 ([1])**.** *If* Ω *is the curvature 2-form of a principal* G*-bundle* P *with connection* θ*, then the Lie algebra of* $\mathrm{Hol}_p(\theta)$ *is the subspace of* $\mathfrak{g}$ *spanned by all elements of the form* $\Omega_q(v_1, v_2)$*, where* $q \in H(p)$ *and* v_1, v_2 *are horizontal tangent vectors at* q*.*

Now we give the proof of our proposition.

Proof of Proposition 7.8. Let (P, θ) be the principal $\mathrm{U}(n)$-bundle with connection whose associated bundle is E. Let $p \in P$. Since the curvature of E is $\sqrt{-1} \cdot B|_R \otimes \mathrm{id}$, the Ambrose-Singer Theorem implies the Lie algebra of $\mathrm{Hol}_p(\theta)$ is

$$\underbrace{\mathfrak{u}(1) \times \cdots \times \mathfrak{u}(1)}_{n}, \tag{7.1}$$

where $n = \mathrm{rank}(E)$. The reduced holonomy group $\mathrm{Hol}^0_p(\theta)$ is the connected component of $\mathrm{Hol}_p(\theta)$ containing the identity. Therefore its Lie algebra is also (7.1) and hence

$$\mathrm{Hol}^0_p(\theta) = \underbrace{\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)}_{n}.$$

Since S^2 is simply connected, $\mathrm{Hol}_p(\theta) = \mathrm{Hol}^0_p(\theta)$. Therefore, by the Reduction Theorem, P reduces to a $\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ bundle, which implies that E is isomorphic to a direct sum of line bundles

$$E' = L_1 \oplus \cdots \oplus L_n.$$

Indeed, if $g_{ab}\colon U_{ab} \to \mathrm{U}(n)$ are local transition functions for E, the above isomorphism implies that there exists local functions $f\colon U_a \to \mathrm{U}(n)$ and transition functions

$$\begin{aligned} h_{ab}\colon U_{ab} &\to \mathrm{U}(1) \times \cdots \times \mathrm{U}(1) \\ x &\mapsto (h^1_{ab}(x), \ldots, h^n_{ab}(x)), \end{aligned}$$

such that $h_{ab} = f_a g_{ab} f_b^{-1}$. If Ω'_a and Ω_a are the local curvature 2-forms for E' and E, respectively, then

$$\Omega'_a = f_a \Omega_a f_a^{-1} = \sqrt{-1} f_a B|_R \cdot I f_a^{-1} = \sqrt{-1} B|_R \cdot I,$$

where I is the identity matrix. Hence, the connection ∇_i on the line bundle L_i induced by the reduction has curvature $B|_R$.

Finally, we show that all the line bundles (L_i, ∇_i) are isomorphic. We do so by showing that their local data of transition functions and 1-forms all represent the same class in the degree 1 Deligne cohomology of S^2. In the proof of Prop. 5.20, we showed that the sequence (5.16) is exact. Hence, the following sequence is exact:

$$0 \to H^1(S^2, \mathrm{U}(1)) \to H^1(S^2, D^\bullet_1) \xrightarrow{\kappa} Z^2(S^2) \xrightarrow{f} H^2(S^2, \mathrm{U}(1)),$$

which relates cohomology with $\mathrm{U}(1)$-coefficients to the Deligne cohomology group $H^1(S^2, D^\bullet_1)$. The map κ sends a Deligne class to the closed 2-form corresponding to its curvature. The Universal Coefficient Theorem implies:

$$H^1(S^2, \mathrm{U}(1)) \cong \mathrm{Hom}(H_1(S^2, \mathbb{Z}), \mathrm{U}(1)) = 0.$$

Hence the curvature map κ is injective. Therefore line bundles with the same curvature are isomorphic. This completes the proof. □

Constructing the Bohr-Sommerfeld variety

The quantum state category is the subcategory of sections of the stack Bund^{ξ} which are twisted-flat over the leaves contained in the Bohr-Sommerfeld variety. Proposition 7.8 implies that we have no hope of finding such sections if the 2-form B is not integral, since it must be the curvature of a line bundle. Remarkably, the 2-plectic Bohr-Sommerfeld variety, obtained by categorifying the symplectic definition, resolves this issue.

Proposition 7.11. *The 2-form B restricts to an integral 2-form on a leaf of the foliation F if and only if the leaf is contained in the Bohr-Sommerfeld variety.*

Proof. Let S^2_R be a leaf and assume $B|_R$ is integral. Recall from Def. 5.17 this means that the class $[B|_R]$ is in the image of the map

$$H^2(S^2_R, 2\pi\mathbb{Z}) \to H^2(S^2_R, \mathbb{R}) \xrightarrow{\sim} H^2_{\mathrm{dR}}(S^2_R).$$

There are canonical isomorphisms which identify singular cohomology with smooth singular cohomology for arbitrary coefficients, and $\mathbb{R}$-valued smooth singular cohomology with de Rham cohomology [67][Sec. 5.34]. Using these isomorphisms, $B|_R$ is integral if and only if

$$\int_{\Delta^2} s^* B|_R \in 2\pi\mathbb{Z}$$

for all smooth simplicies $s\colon \Delta^2 \to S^2_R$. By Def. 7.5, S^2_R is contained in the Bohr-Sommerfeld variety if and only if

$$\mathrm{hol}(\xi, \sigma) = 1$$

for all maps $\sigma\colon \Sigma^2 \to S^2_R$, where Σ^2 is a compact oriented 2-manifold. Definitions 5.35 and 5.36 imply $\mathrm{hol}(\xi, \sigma) = 1$ if and only if

$$\int_{\Sigma^2} \sigma^* B|_R \in 2\pi\mathbb{Z}.$$

Since $B|_R$ is integral, $\sigma^* B|_R$ is integral for any such map σ. Let $\sum_i n_i s_i$ represent the fundamental class in $H_2(\Sigma^2) \cong \mathbb{Z}$, where $s_i\colon \Delta^2 \to \Sigma^2$ are smooth simplicies. Then

$$\int_{\Sigma^2} \sigma^* B|_R = \sum_i n_i \int_{\Delta^2} s_i^* B|_R \in 2\pi\mathbb{Z}.$$

Hence, S^2_R is contained in the variety.

Conversely, assume S^2_R is a leaf in the Bohr-Sommerfeld variety. Then, by taking $\sigma = \mathrm{id}$, we have

$$\int_{S^2_R} B|_R \in 2\pi\mathbb{Z}. \tag{7.2}$$

We claim that this implies $B|_R$ is integral. Indeed, since S^2 is simply connected, the Universal Coefficient Theorem implies we have a commuting diagram

$$\begin{array}{ccc} H^2(S^2, 2\pi\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H_2(S^2), 2\pi\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(S^2, \mathbb{R}) & \xrightarrow{\sim} & \mathrm{Hom}(H_2(S^2), \mathbb{R}). \end{array}$$

Hence, $B|_R$ is integral if and only if for all classes $[s] \in H_2(S^2_R)$, we have

$$\int_{\Delta^2} s^* B|_R \in 2\pi\mathbb{Z}.$$

Any such class is an integer multiple of the fundamental class representing S^2_R. Therefore the integral (7.2) gives the desired result. □

Corollary 7.12. *A sphere with radius R is contained in the Bohr-Sommerfeld variety if and only if*

$$R \in \frac{1}{2}\mathbb{Z}.$$

Proof. Such a sphere is contained in the variety if and only if $B|_R$ is integral, i.e. if and only if

$$\int_{S^2_R} B|_R = 4\pi R \in 2\pi\mathbb{Z}.$$

□

Hence the variety is, precisely, the subspace

$$V_{\mathrm{BS}} = \coprod_{n=1}^{\infty} S^2_{n/2}.$$

The quantum state category

Now we can characterize the quantum state category $\mathsf{Quant}(V_{\mathrm{BS}})$, i.e. the subcategory of $\mathsf{Bund}^{\xi}(V_{\mathrm{BS}})$ whose objects are those ξ-twisted Hermitian bundles over V_{BS} which are twisted-flat along each leaf. The results obtained in the previous sections imply:

Theorem 7.13. *There is a one-to-one correspondence between isomorphism classes of objects in $\mathsf{Quant}(V_{\mathrm{BS}})$ and isomorphism classes of Hermitian vector bundles (with connection) over the Bohr-Sommerfeld variety whose restriction to any leaf $S^2_{n/2}$ is of the form*

$$L \oplus L \oplus \cdots \oplus L$$

where L is a line bundle with curvature $B|_{n/2}$.

Proof. Let $(E_i, \nabla_i, \phi_{ij})$ be an object in $\mathsf{Quant}(V_{\rm BS})$. It is a bundle twisted by the trivial cocycle $(1, 0, \sqrt{-1} \cdot B)$ on the cover $\{V_i\}$, where

$$V_i = U_i \cap V_{\rm BS} = \coprod_{n=1}^{\infty} U_i \cap S^2_{n/2}$$

Hence, the proof of Prop. 7.7 can be used to show there exists a Hermitian vector bundle E over $V_{\rm BS}$, unique up to isomorphism, with connection ∇ such that

$$(E|_{V_i}, \nabla|_{V_i}, \mathrm{id}) \cong (E_i, \nabla_i, \phi_{ij}).$$

Since $(E_i, \nabla_i, \phi_{ij})$ is twisted flat, the restriction of the curvature of the bundle (E, ∇) to a leaf $S^2_{n/2}$ satisfies

$$\nabla^2|_{n/2} = \sqrt{-1} \cdot B|_{n/2} \otimes \mathrm{id}\,.$$

Hence, Prop. 7.8 implies that the restriction of E to $S^2_{n/2}$ is isomorphic to direct sum of line bundles

$$\bigoplus_{i=1}^{k} L_i,$$

Here, $k \geq 0$, and each L_i is the line bundle, unique up to isomorphism, with curvature $B|_{n/2}$.

By reversing this argument, any Hermitian vector bundle over $V_{\rm BS}$ whose restriction to a leaf is isomorphic to the direct sum above represents a unique isomorphism class of trivially twisted bundles in $\mathsf{Quant}(V_{\rm BS})$ □

7.3 Applications to representation theory

As previously mentioned, some of the most important applications of geometric quantization lie in representation theory. Here we present evidence that the categorified geometric quantization of 2-plectic manifolds has similar uses. Roughly, the idea is the following: In ordinary geometric quantization, sections in the quantum space $Q(V_{\rm BS})$ correspond to vectors in a representation of a Lie group. In categorified geometric quantization, sections in the quantum category $\mathsf{Quant}(V_{\rm BS})$ correspond to representations i.e. objects in a representation category of a Lie group.

In particular, we describe this correspondence in detail for the example $M = \mathbb{R}^3 \setminus \{0\}$ considered in the previous section. The 2-spheres in the associated Bohr-Sommerfeld variety are special coadjoint orbits of the Lie group SU(2), via the identification $\mathfrak{su}(2)^* \cong \mathbb{R}^3$. These 2-spheres equipped with the restriction of the 2-form B are sympletic manifolds, and, through ordinary geometric quantization, they give irreducible representations of SU(2). As we will see, these facts imply that the quantum state category, obtained via the categorified quantization of M, is closely related to the category of finite dimensional representations of SU(2).

Let $S^2_{n/2}$ be the 2-sphere of radius $n/2$, i.e. a leaf in the Bohr-Sommerfeld variety. By identifying S^3 with the unit sphere in $\mathbb{C}^2$, we use the Hopf fibration $S^3 \to S^2_{n/2}$:

$$\big(Z_0, Z_1\big) \mapsto \frac{n}{2}\big(Z_1\bar{Z}_0 + Z_0\bar{Z}_1,\ iZ_1\bar{Z}_0 - iZ_0\bar{Z}_1,\ Z_0\bar{Z}_0 - Z_1\bar{Z}_1\big)$$

to identify $S^2_{n/2}$ with $\mathbb{C}\mathrm{P}^1$. Choosing the affine coordinate $w = Z_1/Z_0$, the 2-form

$$B|_{n/2} = \frac{4}{n^2}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$$

becomes

$$B|_{n/2} = n\sqrt{-1}\frac{dw \wedge d\bar{w}}{(1 + w\bar{w})^2}.$$

(See Woodhouse [70] Sections 3.5, 8.4, and 9.2 for details.)

Recall that the hyperplane bundle $H \to \mathbb{C}\mathrm{P}^1$ is the holomorphic line bundle whose fiber over each point $[Z_0, Z_1] \in \mathbb{C}\mathrm{P}^1$ is the dual space of the corresponding line in $\mathbb{C}^2$. The curvature of this bundle is $B|_{1/2}$. Hence, $B|_{n/2}$ is the curvature of the tensor product

$$H^{\otimes n} \to \mathbb{C}\mathrm{P}^1.$$

In fact, up to isomorphism, $H^{\otimes n}$ is the unique holomorphic line bundle with this curvature. Let (ζ^0, ζ^1) be the coordinates on the dual space $\mathbb{C}^{2^*}$. It can be shown using standard complex analysis that the global holomorphic sections $\Gamma(H^{\otimes n})_{\mathrm{h}}$ are the degree n homogeneous polynomials in the variables (ζ^0, ζ^1) [24][Sec. 1.3].

There is an action of the group $\mathrm{SU}(2) \subseteq \mathrm{SL}(2, \mathbb{C})$ on the polynomials $\Gamma(H^{\otimes n})_{\mathrm{h}}$ which is induced by its obvious action on $\mathbb{C}^2$. In fact, for each n, $\Gamma(H^{\otimes n})_{\mathrm{h}}$ is an irreducible representation, which represents the unique isomorphism class of irreducible representations of dimension $n+1$ [34][Sec. A3.2]. Moreover, any finite dimensional representation of SU(2) is isomorphic to a finite direct sum of irreducibles. Hence, any such representation is isomorphic to the holomorphic global sections of a direct sum

$$H^{\otimes n_1} \oplus H^{\otimes n_2} \oplus \cdots \oplus H^{\otimes n_k}$$

of line bundles over $\mathbb{C}\mathrm{P}^1$. Note that the trivial bundle over $\mathbb{C}P^1$ is the line bundle $H^{\otimes n}$ with $n = 0$. Its global sections are the holomorphic functions on $\mathbb{C}\mathrm{P}^1$, i.e. the constants $\mathbb{C}$.

Now we show how all of this is related to the quantization of the 2-plectic manifold $\mathbb{R}^3 \setminus \{0\}$. Theorem 7.13 implies that an isomorphism class of objects in the quantum state category $\mathsf{Quant}(V_{\mathrm{BS}})$ can be identified with a collection of bundles:

$$\begin{aligned}
k_1 \cdot L_1 &\to S^2_{1/2} \\
k_2 \cdot L_2 &\to S^2_{1} \\
k_3 \cdot L_3 &\to S^2_{3/2} \\
\vdots \quad & \quad \vdots \\
k_n \cdot L_n &\to S^2_{n/2} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

which are unique up to isomorphism. Here, k_n is a non-negative integer, and $k_n \cdot L_n$ is the direct sum of line bundles

$$k_n \cdot L_n = \underbrace{L_n \oplus L_n \oplus \cdots \oplus L_n}_{k_n},$$

where L_n is the line bundle with curvature $B|_{n/2}$. By identifying each sphere with $\mathbb{C}\mathrm{P}^1$, the above discussion implies we can identify each line bundle with a tensor power of the hyperplane bundle $H^{\otimes n}$. By taking global holomorphic sections, each copy of $H^{\otimes n}$ is then identified with $\mathrm{Sym}^n(\mathbb{C}^{2^*})$, the space of degree n homogeneous polynomials in 2 variables, which is a $(n+1)$-dimensional irreducible representation of SU(2). (See Figure 7.3.)

Note this procedure gives all finite-dimensional representations of SU(2) except for those built using the 1-dimensional trivial representation. This is because the sphere of radius 0 (the origin) is not contained in the Bohr-Sommerfeld variety. Hence, we have proven:

Theorem 7.14. *There is a one-to-one correspondence between isomorphism classes of objects in the quantum state category* $\mathsf{Quant}(V_{\mathrm{BS}})$ *and isomorphism classes of finite-dimensional representations of* SU(2) *whose decomposition into irreducibles does not contain the trivial representation.*

$$
\begin{array}{ccccc}
k_1 \cdot L_1 \longrightarrow S^2_{1/2} & \longmapsto & k_1 \cdot H \longrightarrow \mathbb{C}\mathrm{P}^1 & \overset{\Gamma_{\mathrm{h}}}{\longmapsto} & k_1 \cdot \mathrm{Sym}^1(\mathbb{C}^{2^*}) \\
k_2 \cdot L_2 \longrightarrow S^2_{1} & \longmapsto & k_2 \cdot H^{\otimes 2} \longrightarrow \mathbb{C}\mathrm{P}^1 & \overset{\Gamma_{\mathrm{h}}}{\longmapsto} & k_2 \cdot \mathrm{Sym}^2(\mathbb{C}^{2^*}) \\
k_3 \cdot L_3 \longrightarrow S^2_{3/2} & \longmapsto & k_3 \cdot H^{\otimes 3} \longrightarrow \mathbb{C}\mathrm{P}^1 & \overset{\Gamma_{\mathrm{h}}}{\longmapsto} & k_3 \cdot \mathrm{Sym}^3(\mathbb{C}^{2^*}) \\
\vdots \qquad \vdots & & \vdots \qquad \vdots & & \vdots \\
k_n \cdot L_n \longrightarrow S^2_{n/2} & \longmapsto & k_n \cdot H^{\otimes n} \longrightarrow \mathbb{C}\mathrm{P}^1 & \overset{\Gamma_{\mathrm{h}}}{\longmapsto} & k_n \cdot \mathrm{Sym}^{\mathrm{n}}(\mathbb{C}^{2^*}) \\
\vdots \qquad \vdots & & \vdots \qquad \vdots & & \vdots
\end{array}
$$

Figure 7.1: The quantum state given by the collection of vector bundles $\{k_1 \cdot L_1 \to S^2_{1/2}, k_2 \cdot L_2 \to S^2_1, \ldots\}$ is identified with the representation $k_1 \cdot \mathrm{Sym}^1(\mathbb{C}^{2^*}) \oplus k_2 \cdot \mathrm{Sym}^2(\mathbb{C}^{2^*}) \oplus \cdots$ of SU(2).

Chapter 8

Summary and future work

We conclude this thesis by providing a summary which ties together the main results of the previous chapters. Along the way, we make some brief remarks regarding open problems and possible directions for future research.

Polarizations on n-plectic manifolds

In Chapter 2, we presented the basic geometric facts needed for our study of n-plectic manifolds. In particular, we considered the n-plectic analogues for Lagrangian submanifolds and real polarizations. There are at least n different ways to generalize the definition of a Lagrangian submanifold to n-plectic geometry. This is due to the fact that there are n different ways to define the notion of orthogonal complement on an n-plectic vector space (Def. 2.2). Since real polarizations in symplectic geometry are foliations whose leaves are Lagrangian submanifolds, we have at least n different kinds of real polarizations on an n-plectic manifold (Def. 2.13). Polarizations play an important role in geometric quantization, but it is not clear which definition of polarization for n-plectic manifolds is "best" in this context.

Moreover, it is unknown if an n-plectic analogue of a complex polarization exists. It is possible that one could use ideas from generalized complex geometry [26] and the theory of 'higher Dirac structures' [71] to help develop such polarizations.

Lie n-algebras from n-plectic manifolds

In Chapter 3, we showed that an n-plectic structure on M induces a bracket on the space of Hamiltonian $(n-1)$-forms. The bracket is skew-symmetric, but only satisfies the Jacobi identity up to homotopy. We proved that this bracket gives a Lie n-algebra $L_{\infty}(M,\omega)$, whose underlying n-term chain complex consists of Hamiltonian

$(n-1)$-forms and all differential forms of lower degrees (Thm. 3.14). When $n = 1$, the Hamiltonian forms are the smooth functions, and the Lie 1-algebra is just the underlying Lie algebra of the usual Poisson algebra of a symplectic manifold. For certain 2-plectic manifolds, our previous work with Baez and Hoffnung implies that the associated Lie 2-algebra can be used to describe the "observable algebra" of the classical bosonic string [5].

In Appendix A, we showed that an n-plectic manifold also gives a dg Leibniz algebra $\mathrm{Leib}(M, \omega)$ on the same complex (Prop. A.3). Its bracket satisfies Jacobi, but is skew-symmetric only up to homotopy. For the 2-plectic case, we showed $L_\infty(M, \omega)$ and $\mathrm{Leib}(M, \omega)$ are isomorphic in Roytenberg's category of weak Lie 2-algebras (Thm. A.10). The objects of this category are 2-term L_∞-algebras whose structure maps are skew-symmetric only up to homotopy. In general, we would like to conjecture that some sort of equivalence such as this holds for $n > 2$. Unfortunately, it is not clear in what category this should occur. Indeed, developing a theory of weak Lie n-algebras is an open problem. Perhaps by studying the relationships between the structures specifically on $L_\infty(M, \omega)$ and $\mathrm{Leib}(M, \omega)$ for arbitrary n one could get a sense of what explicit coherence conditions would be needed to give a good definition.

On the other hand, there are structures known as 'Loday-∞ algebras' (or sh Leibniz algebras) [2] that generalize the definition of an L_∞-algebra by, again, relaxing the skew symmetry condition on the structure maps. However, this time the skew symmetry is not required to hold up to homotopy. Hence any dg Leibniz algebra is a Loday-∞ algebra. Any L_∞-algebra is as well. Therefore there may be an isomorphism between $L_\infty(M, \omega)$ and $\mathrm{Leib}(M, \omega)$ in this category for $n \geq 2$.

Lie 2-algebras from compact simple Lie groups

In Chapter 4, we considered compact simple Lie groups as 2-plectic manifolds. Any compact simple Lie group G admits a 1 parameter family of canonical 2-plectic structures $\{\nu_k\}$, given by non-zero multiples of Cartan 3-form (Ex. 2.8). We proved that the associated Lie 2-algebra $L_\infty(G, \nu_k)$ contains a sub Lie 2-algebra consisting of left invariant Hamiltonian 1-forms (Cor. 4.4). We showed that this sub-algebra is not equivalent to $L_\infty(G, \nu_k)$, however, it is isomorphic to the so-called string Lie 2-algebra associated to (G, ν_k) (Thm. 4.7). The string Lie 2-algebra plays an important role in string theory and in the theory of loop groups.

Our results suggest a close link between these areas and 2-plectic geometry. There are many possible directions for future work here. In particular, it would be interesting to understand the relationship between $L_\infty(G, \nu_k)$ and the algebra of observables for certain string theory models called 'WZW models'.

Gerbes, 2-line stacks, and 2-bundles

In Chapter 5, we presented the technical tools needed to develop a geometric quantization theory for 2-plectic manifolds. The work of Brylinski [13] implies that if (M, ω) is a 2-plectic manifold and ω is an integral 3-form, then ω can be realized as the 2-curvature of a U(1)-gerbe equipped with a 2-connection. If $\{U_i\}$ is an open cover of M, then locally, a U(1)-gerbe with 2-connection is determined by a Deligne 2-cocycle i.e a collection of U(1)-valued transition functions g_{ijk} on $U_i \cap U_j \cap U_k$, 1-forms A_{ij} on $U_i \cap U_j$, and 2-forms B_i on U_i, with $dB_i = \omega$, satisfying certain compatibility conditions (Ex. 5.16).

Every principal U(1)-bundle with connection has an associated Hermitian line bundle with connection. Similarly, we showed that every U(1)-gerbe with 2-connection over a 2-plectic manifold has an associated 2-line stack with 2-connection (Prop. 5.32). The category of global sections of the 2-line stack is equivalent to the category of Hermitian vector bundles twisted by the gerbe's Deligne 2-cocycle $\xi = (g_{ijk}, A_{ij}, B_i)$. Such a twisted bundle is given locally by a collection of Hermitian vector bundles E_i with connection ∇_i (Def. 5.30). The twisting by ξ characterizes the obstruction to gluing these bundles together into a global bundle over M. A ξ-twisted Hermitian vector bundle is twisted-flat if, for each E_i, the curvature ∇_i^2 is equal to $\sqrt{-1} \cdot B_i \otimes \mathrm{id}$ (Def. 5.31). This is the 2-plectic analogue of a flat section of a Hermitian line-bundle. We also showed that there is a good notion of holonomy (Def. 5.36) for 2-line stacks equipped with 2-connection, given by Carey, Johnson, and Murray's formula for the 2-holonomy of the Deligne class $[\xi]$ [17]. The 2-holonomy plays an important role in our quantization procedure for 2-plectic manifolds. In particular, it is used in our definition for the 2-plectic version of the Bohr-Sommerfeld variety (Def. 7.5).

It is not obvious, at first glance, why twisted Hermitian vector bundles are the 2-plectic analogues of sections of a Hermitian line bundle. In the same chapter, we sketch an argument supporting this point of view using Bartels' work in 2-bundle theory [10]. It becomes clear that twisted Hermitian vector bundles should be understood as sections of a 2-vector bundle of rank 1. It would be very interesting to make this argument more precise, and perhaps recast our results within the context of 2-vector bundles. For example, for line bundles, one can consider different kinds of sections e.g. smooth, square-integrable, etc. Similarly, the 2-bundle approach might suggest that we consider 2-lines stacks whose sections are more general than twisted bundles, e.g. twisted coherent sheaves. This could have important consequences for the output of our geometric quantization procedure for 2-plectic manifolds.

2-Plectic prequantization and Courant algebroids

We defined a prequantized 2-plectic manifold to be a 2-plectic manifold equipped with a Deligne 2-cocycle (Def. 6.2). This 2-cocycle can be realized geometrically as a U(1)-gerbe with 2-connection, or as its associated 2-line stack. This is in complete analogy with the symplectic case, where we prequantize using either a principal U(1)-bundle or a Hermitian line bundle. In Section 6.2 , we first recall how to prequantize the Poisson algebra on a symplectic manifold equipped with a principal U(1)-bundle P with connection. By prequantizing, we mean faithfully representing the Poisson algebra as linear differential operators. This is done by considering the Atiyah algebroid A associated to P. There is an injective Lie algebra morphism from the Poisson algebra to the Lie algebra of global sections of A, which identifies the Poisson algebra with those invariant vector fields on P whose flows preserve the connection (Prop. 6.7).

For the 2-plectic case, we described a known construction which gives a Courant algebroid C over a prequantized 2-plectic manifold (M, ω) equipped with a U(1)-gerbe with 2-connection (Sec. 6.4). In this case, C is a vector bundle over M whose sections are locally given by vector fields and 1-forms on M. Its space of global sections form a Lie 2-algebra. There is a short exact sequence of vector bundles over M

$$T^*M \to C \to TM,$$

whose splittings $TM \to C$ correspond to 2-connections on the U(1)-gerbe over M. We prove the existence of an injective Lie 2-algebra morphism from the Lie 2-algebra $L_\infty(M, \omega)$ of observables on M to the Lie 2-algebra $L_\infty(C)$ of global sections of C (Thm. 6.16). This morphism identifies $L_\infty(M, \omega)$ with a sub-Lie 2-algebra of $L_\infty(C)$, which, in a certain sense, preserves the 2-connection of the gerbe (Thm. 6.22). We interpret this sub-algebra as the prequantization of $L_\infty(M, \omega)$. Also, we show that this construction gives the higher analogue of the well known Kostant-Souriau central extension in symplectic geometry (Sec. 6.5).

This prequantization process gives an interesting relationship between Courant algebroids and prequantized 2-plectic manifolds. Let us give two possible directions for future work based on these results.

1. Sections of the Atiyah algebroid A over a prequantized symplectic manifold equipped with the principal U(1)-bundle are differential operators on a Hilbert space. This

Hilbert space is constructed from global sections of the associated Hermitian line bundle. The higher analogue of this Hilbert space is the category of global sections of the 2-line stack associated to a U(1)-gerbe. In what way, if at all, do sections of the Courant algebroid over a prequantized 2-plectic manifold act as "operators" on this higher analogue of a Hilbert space?

2. Recall that sections of the Atiyah algebroid are infinitesimal $U(1)$-equivariant symmetries of the corresponding principal $U(1)$-bundle. Integration gives the 'gauge groupoid' over M, whose elements correspond to the equivariant automorphisms of the principal bundle [15][Sec. 17.1]. Our results suggest that the Courant algebroid is the higher analogue of the Atiyah algebroid. So, how can we understand sections of the Courant algebroid on a prequantized 2-plectic manifold as infinitesimal automorphisms of the corresponding $U(1)$-gerbe? In other words, what is the Lie 2-groupoid that integrates this Courant algebroid, and how does it act as the 'gauge 2-groupoid' of the $U(1)$-gerbe?

2-plectic quantization and representation theory

In the last chapter, we categorifed Śniatycki's [62] quantization procedure for symplectic manifolds, which employs Bohr-Sommerfeld varieties to overcome topological obstructions that arise when using real polarizations (Sec. 7.2). This categorification gives a simple procedure for quantizing a 2-plectic manifold, and the resulting output is a category of quantum states (Def. 7.6). An object of this category is a twisted Hermitian vector bundle over the Bohr-Sommerfeld variety (Def. 7.5) whose restriction to each leaf contained in the variety is twisted-flat.

In Section 7.2.1, we considered an interesting example: $M = \mathbb{R}^3 \setminus \{0\}$ equipped with a volume form $\omega = dB$. We quantized M by equipping it with the trivial Deligne 2-cocycle $\xi = (1, 0, B)$, and a 2-polarization whose leaves are spheres centered about the origin. The restriction of the 2-form B to such a sphere is the KKS symplectic form, which arises in Kirillov's orbit method for constructing representations of Lie groups. This is not surprising, since $\mathbb{R}^3$ is isomorphic to the dual of the Lie algebra $\mathfrak{su}(2)$, and each sphere is isomorphic to a coadjoint orbit. We then showed that in this example, a leaf of the polarization is contained in the Bohr-Sommerfeld variety if and only if it is a sphere of radius $n/2$, where n is an integer (Cor. 7.12). The orbit method identifies such a sphere with the irreducible representation of SU(2) whose dimension is $n + 1$.

Next, we proved that any twisted bundle in the associated category of quantum states is isomorphic to a direct sum of line bundles over spheres contained in the variety

(Thm. 7.13). The fact that the twisted bundle is twisted-flat on each sphere implies that each of these line bundles must be isomorphic to a tensor power of the hyperplane bundle over $\mathbb{C}\mathrm{P}^1$. This allowed us to identify a quantum state with a representation of SU(2). More precisely, we proved that isomorphism classes of objects in the quantum state category are in one-to-one correspondence with isomorphism classes of finite-dimensional representations of SU(2) whose decomposition into irreducibles does not contain the trivial representation (Thm. 7.14).

It is unfortunate that we are unable to obtain the trivial representation via our quantization procedure. However, it is not surprising. Our procedure identifies spheres of radius $n/2$ with irreducibles of dimension $n + 1$. Hence, the trivial representation corresponds to the origin in $\mathfrak{su}(2)^*$, which is not in M. In some sense, this identification needs to be shifted so that the sphere of radius $n/2$ is identified with the irreducible representation of dimension n. This is very similar to the 1/2 shift which arises in the usual geometric quantization of the simple harmonic oscillator (Sec. 7.1.2).

We believe we have just scratched the surface of a deeper relationship between representation theory and the geometric quantization of 2-plectic manifolds. Indeed, our example suggests that 2-plectic quantization can give a categorifed analogue of the orbit method. We conclude by mentioning two related directions for future work along these lines.

1. It is well known that closed integral forms on a manifold M can be mapped to closed integral forms on LM, the space of free loops of M, by a process called 'transgression'. Moreover, this process sends a U(1)-gerbe equipped with 2-connection on M to a principal U(1)-bundle with connection on LM [13][Ch. 6]. This suggests that the categorifed geometric quantization of a 2-plectic manifold may, in some way, correspond to ordinary geometric quantization on LM. (We are overlooking subtleties here, such as the fact that transgression need not preserve non-degeneracy.) For example, perhaps there is some 2-plectic structure on $\mathfrak{su}(2)$ whose quantization gives a category of quantum states, with objects corresponding to certain representations of the loop group $L\mathrm{SU}(2)$ obtained by applying the orbit method to the loop algebra $L\mathfrak{su}(2)$.

2. Much work has been done on quantizing the conjugacy classes of compact simple Lie groups via a variety of methods, all of which rely on the Cartan 3-form in some way [44, 46]. The output of these quantization procedures gives information about the representation theory of the corresponding loop group. Every compact simple Lie group, equipped with the Cartan 3-form, is a 2-plectic manifold. Hence, it is

natural to suspect that 2-plectic quantization of Lie *groups* is also related to the representation theory of loop groups. We have preliminary results which suggest that such a relationship exists, although, even in the simple case of SU(2), many issues remain unresolved.

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Appendix A

Other algebraic structures on n-plectic manifolds

There are other structures besides Lie n-algebras which can generalize the Poisson bracket to n-plectic manifolds. Here we show that any n-plectic manifold gives rise to another kind of algebraic structure known as a differential graded (dg) Leibniz algebra. A dg Leibniz algebra is a graded vector space equipped with a degree -1 differential and a bilinear bracket that satisfies a Jacobi-like identity, but does not need to be skew-symmetric. There is an interesting relationship between the bilinear bracket on the Lie n-algebra and the bracket on the corresponding dg Leibniz algebra. When $n = 2$, these algebras can be compared directly as objects in Roytenberg's category of 'weak Lie 2-algebras' [54]. A weak Lie 2-algebra is a Lie 2-algebra whose $k = 2$ bracket satisfies skew-symmetry only up to a chain homotopy. This homotopy must satisfy compatibility relations with the homotopy controlling the failure of the Jacobi identity. We show that the Lie 2-algebra and the 2-term dg Leibniz algebra arising from a 2-plectic manifold are isomorphic as weak Lie 2-algebras. We are unable to extend this result to the $n > 2$ case, since there is currently no definition available for weak L_∞-algebras.

A.1 dg Leibniz algebras

In symplectic geometry, every function $f \in C^\infty(M)$ is Hamiltonian. We also have the equality:

$$\{f, g\} = \iota_{v_f} dg = \mathcal{L}_{v_f} g \tag{A.1}$$

for all $f, g \in \Omega^0_{\text{Ham}}(M) = C^\infty(M)$. Hence $\{f, \cdot\}$ is a degree zero derivation on $\Omega^0_{\text{Ham}}(M)$, which makes $(\Omega^0_{\text{Ham}}(M), [\cdot,\cdot])$ a Poisson algebra. In general, for $n > 1$, an equality such as Eq. A.1 does not hold, and Hamiltonian forms are obviously not closed under wedge product. Therefore, we shouldn't expect the Lie n-algebra $L_\infty(M,\omega)$ to behave like a Poisson algebra. But we do have the following simple lemma:

Lemma A.1. *Let (M,ω) be an n-plectic manifold. If $\alpha, \beta \in \Omega^{n-1}_{\text{Ham}}(M)$ are Hamiltonian forms, then*

$$\mathcal{L}_{v_\alpha}\beta = \{\alpha,\beta\} + d\iota_{v_\alpha}\beta.$$

Proof. Definitions 3.1 and 3.3 imply:

$$\begin{aligned}\mathcal{L}_{v_\alpha}\beta &= \iota_{v_\alpha} d\beta + d\iota_{v_\alpha}\beta \\ &= -\iota_{v_\alpha}\iota_{v_\beta}\omega + d\iota_{v_\alpha}\beta \\ &= \{\alpha,\beta\} + d\iota_{v_\alpha}\beta.\end{aligned}$$

□

Lemma A.1 suggests that we interpret the $(n-1)$-form $\mathcal{L}_{v_\alpha}\beta$ as a type of bracket on $\Omega^{n-1}_{\text{Ham}}(M)$, equal to the bracket $\{\cdot,\cdot\}$ modulo boundary terms. To this end, we consider an algebraic structure known as a differential graded (dg) Leibniz algebra.

Definition A.2. *A* **differential graded Leibniz algebra** $(L, \delta, [\![\cdot,\cdot]\!])$ *is a graded vector space L equipped with a degree -1 linear map $\delta\colon L \to L$ and a degree 0 bilinear map $[\![\cdot,\cdot]\!] : L \otimes L \to L$ such that the following identities hold:*

$$\delta \circ \delta = 0 \tag{A.2}$$

$$\delta\,[\![x,y]\!] = [\![\delta x, y]\!] + (-1)^{|x|}\,[\![x, \delta y]\!] \tag{A.3}$$

$$[\![x, [\![y,z]\!]]\!] = [\![[\![x,y]\!], z]\!] + (-1)^{|x||y|}\,[\![y, [\![x,z]\!]]\!], \tag{A.4}$$

for all $x, y, z \in L$.

In the literature, dg Leibniz algebras are also called dg Loday algebras. This definition presented here is equivalent to the one given by Ammar and Poncin [2]. Note that the second condition given in the definition above can be interpreted as the Jacobi identity. Hence if the bilinear map $[\![\cdot,\cdot]\!]$ is skew-symmetric, then a dg Leibniz algebra is a DGLA.

We now show that every n-plectic manifold gives a dg Leibniz algebra.

Proposition A.3. *Given an n-plectic manifold (M, ω), there is a differential graded Leibniz algebra* $\mathrm{Leib}(M, \omega) = (L, \delta, [\![\cdot, \cdot]\!])$ *with underlying graded vector space*

$$L_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M) & i = 0, \\ \Omega^{n-1-i}(M) & 0 < i \leq n-1, \end{cases}$$

and maps $\delta \colon L \to L$, $[\![\cdot, \cdot]\!] : L \otimes L \to L$ *defined as*

$$\delta(\alpha) = d\alpha,$$

if $|\alpha| > 0$ *and*

$$[\![\alpha, \beta]\!] = \begin{cases} \mathcal{L}_{v_\alpha} \beta & \textit{if } |\alpha| = 0, \\ 0 & \textit{if } |\alpha| > 0, \end{cases}$$

where v_α *is the Hamiltonian vector field associated to* α.

Proof. If $\alpha, \beta \in L_0 = \Omega^{n-1}_{\mathrm{Ham}}(M)$ are Hamiltonian, then Lemma A.1 implies $d[\![\alpha, \beta]\!] = d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega$. Hence $[\![\alpha, \beta]\!]$ is Hamiltonian. For $|\beta| > 0$, we have $|\mathcal{L}_{v_\alpha}\beta| = |\beta|$, since the Lie derivative is a degree zero derivation. Hence $[\![\cdot, \cdot]\!]$ is a bilinear degree 0 map.

We next show that Eq. A.3 of Definition A.2 holds. If $|\alpha| > 1$, then it holds trivially. If $|\alpha| = 1$, then $[\![\alpha, \beta]\!] = [\![\alpha, \delta\beta]\!] = 0$ for all $\beta \in L$ by definition, and $[\![\delta\alpha, \beta]\!] = 0$ since the Hamiltonian vector field associated to $d\alpha$ is zero. If $|\alpha| = 0$ and $|\beta| = 0$, then $|[\![\alpha, \beta]\!]| = 0$. Hence all terms in (A.3) vanish by definition. The last case to consider is $|\alpha| = 0$ and $|\beta| > 0$. We have

$$\delta [\![\alpha, \beta]\!] = d\mathcal{L}_{v_\alpha}\beta = \mathcal{L}_{v_\alpha} d\beta = [\![\alpha, \delta\beta]\!].$$

Finally, we show the Jacobi identity (A.4) holds. Let $\alpha, \beta, \gamma \in L$. Then the left hand side of (A.4) is $[\![\alpha, [\![\beta, \gamma]\!]]\!]$, while the right hand side is $[\![[\![\alpha, \beta]\!], \gamma]\!] + (-1)^{|\alpha||\beta|} [\![\beta, [\![\alpha, \gamma]\!]]\!]$. Note equality holds trivially if $|\alpha| > 0$ or $|\beta| > 0$. Otherwise, we use the identity

$$\mathcal{L}_{[v_1, v_2]} = \mathcal{L}_{v_1}\mathcal{L}_{v_2} - \mathcal{L}_{v_2}\mathcal{L}_{v_1},$$

and the fact that $d[\![\alpha, \beta]\!] = -\iota_{[v_\alpha, v_\beta]}\omega$ to obtain the following equalities:

$$\begin{aligned} [\![\alpha, [\![\beta, \gamma]\!]]\!] &= \mathcal{L}_{v_\alpha}\mathcal{L}_{v_\beta}\gamma \\ &= \mathcal{L}_{[v_\alpha, v_\beta]}\gamma + \mathcal{L}_{v_\beta}\mathcal{L}_{v_\alpha}\gamma \\ &= [\![[\![\alpha, \beta]\!], \gamma]\!] + [\![\beta, [\![\alpha, \gamma]\!]]\!]. \end{aligned}$$

□

One interesting aspect of the dg Leibniz structure is that it interprets the bracket of Hamiltonian $(n-1)$-forms geometrically as the change of an observable along the flow of a Hamiltonian vector field. Leibniz algebras, in fact, naturally arise in a variety of geometric settings e.g. in Courant algebroid theory and, more generally, in the derived bracket formalism [36]. It would be interesting to compare $\mathrm{Leib}(M,\omega)$ to the Leibniz algebras that appear in these other formalisms.

A.2 Weak Lie 2-algebras

When (M,ω) is a symplectic manifold, $L_\infty(M,\omega)$ and $\mathrm{Leib}(M,\omega)$ give the same Lie algebra: the Poisson algebra of functions. It would be nice if we could show that for any n-plectic manifold, $L_\infty(M,\omega)$ and $\mathrm{Leib}(M,\omega)$ are also "the same", i.e. equivalent as objects in some category containing both L_∞-algebras and dg Leibniz algebras. This may seem unlikely at first since the brackets which induce these structures have different properties. For example, $\{\cdot,\cdot\}$ is skew-symmetric, while $[\![\cdot,\cdot]\!]$, in general, is not. However, we have the following proposition.

Proposition A.4. *Let (M,ω) be an n-plectic manifold, and $\{\cdot,\cdot\}$ and $[\![\cdot,\cdot]\!]$ be the brackets given in Def. 3.3 and Prop. A.3, respectively. If α and β are Hamiltonian $(n-1)$-forms, then*

$$[\![\alpha,\beta]\!] + [\![\beta,\alpha]\!] = d\left(\iota_{v_\alpha}\beta + \iota_{v_\beta}\alpha\right).$$

Proof. The statement follows from the formula $\mathcal{L}_v = \iota_v d + d\iota_v$. □

So, we seek a category whose objects originate from weakening, up to homotopy, both the skew-symmetric axiom and the Jacobi identity. Unfortunately, no such category exists, unless $n=2$. In this case, by extending the work of Baez and Crans [4], Roytenberg [54] developed what are known as 2-term weak L_∞-algebras, or 'weak Lie 2-algebras'. In a weak Lie 2-algebra, the skew symmetry condition on the maps given in Definition 3.7 is relaxed. In particular, the bilinear map $l_2\colon L\otimes L\to L$ is skew-symmetric only up to homotopy. This homotopy must satisfy a coherence condition, as well as compatibility conditions with the homotopy that controls the failure of the Jacobi identity. The goal of this section is to show that if (M,ω) is a 2-plectic manifold, then $L_\infty(M,\omega)$ and $\mathrm{Leib}(M,\omega)$ are isomorphic as weak Lie 2-algebras.

Definition A.5 ([54])**.** *A* **weak Lie 2-algebra** *is a 2-term chain complex of vector spaces $L=(L_1\xrightarrow{d}L_0)$ equipped with the following structure:*

- *a chain map $[\cdot,\cdot]\colon L\otimes L\to L$ called the* **bracket***;*

- *a chain homotopy* $S\colon L\otimes L\to L$ *from the chain map*

$$\begin{array}{ccc} L\otimes L & \to & L \\ x\otimes y & \longmapsto & [x,y] \end{array}$$

to the chain map

$$\begin{array}{ccc} L\otimes L & \to & L \\ x\otimes y & \longmapsto & -[y,x] \end{array}$$

called the **alternator***;*

- *a chain homotopy* $J\colon L\otimes L\otimes L\to L$ *from the chain map*

$$\begin{array}{ccc} L\otimes L\otimes L & \to & L \\ x\otimes y\otimes z & \longmapsto & [x,[y,z]] \end{array}$$

to the chain map

$$\begin{array}{ccc} L\otimes L\otimes L & \to & L \\ x\otimes y\otimes z & \longmapsto & [[x,y],z]+[y,[x,z]] \end{array}$$

called the **Jacobiator***.*

In addition, the following equations are required to hold:

$$\begin{aligned} &[x,J(y,z,w)]+J(x,[y,z],w)+J(x,z,[y,w])+[J(x,y,z),w] \\ &\quad +[z,J(x,y,w)] = J(x,y,[z,w])+J([x,y],z,w) \\ &\quad +[y,J(x,z,w)]+J(y,[x,z],w)+J(y,z,[x,w]), \end{aligned} \tag{A.5}$$

$$J(x,y,z)+J(y,x,z) = -[S(x,y),z], \tag{A.6}$$

$$J(x,y,z)+J(x,z,y) = [x,S(y,z)]-S([x,y],z)-S(y,[x,z]), \tag{A.7}$$

$$S(x,[y,z]) = S([y,z],x). \tag{A.8}$$

A weak Lie 2-algebra homomorphism is a chain map between the underlying chain complexes that preserves the bracket up to coherent chain homotopy. More precisely:

Definition A.6 ([54])**.** *Given Lie 2-algebras* L *and* L' *with bracket, alternator and Jacobiator* $[\cdot,\cdot]$*,* S*,* J *and* $[\cdot,\cdot]'$*,* S'*,* J' *respectively, a* **homomorphism** *from* L *to* L' *consists of:*

- *a chain map* $\phi=(\phi_0,\phi_1):L\to L'$*, and*

- *a chain homotopy* $\Phi\colon L\otimes L\to L'$ *from the chain map*

$$\begin{array}{ccc} L\otimes L & \to & L' \\ x\otimes y & \longmapsto & [\phi(x),\phi(y)]', \end{array}$$

to the chain map

$$\begin{array}{ccc} L\otimes L & \to & L' \\ x\otimes y & \longmapsto & \phi([x,y]) \end{array}$$

such that the following equations hold:

$$S'(\phi_0(x),\phi_0(y)) - \phi_1(S(x,y)) = \Phi(x,y)+\Phi(y,x), \tag{A.9}$$

$$\begin{aligned} &J'(\phi_0(x),\phi_0(y),\phi_0(z)) - \phi_1(J(x,y,z)) \\ &= [\phi_0(x),\Phi(y,z)]' - [\phi_0(y),\Phi(x,z)]' - [\Phi(x,y),\phi_0(z)]' \\ &-\Phi([x,y],z) - \Phi(y,[x,z]) + \Phi(x,[y,z]). \end{aligned} \tag{A.10}$$

The details involved in composing Lie 2-algebra homomorphisms are given by Roytenberg [54]. We say a Lie 2-algebra homomorphism with an inverse is an **isomorphism**.

Lie 2-algebras in the sense of Prop. 3.10 are weak Lie 2-algebras that satisfy skew-symmetry on the nose. They are called semi-strict Lie 2-algebras in this context, since the Jacobi identity may still fail to hold. More precisely:

Definition A.7 ([54]). *A weak Lie 2-algebra* $(L,[\cdot,\cdot],S,J)$ *is* **semi-strict** *iff* $S=0$, *and* **hemi-strict** *iff* $J=0$.

Note that the bracket of a hemi-strict Lie 2-algebra satisfies a Jacobi identity of the form

$$[x,[y,z]] - [[x,y],z] - [y,[x,z]] = 0,$$

but it is not necessarily skew-symmetric. In fact, any hemi-strict Lie 2-algebra is a 2-term dg Leibniz algebra. For 2-plectic manifolds, we have a converse:

Proposition A.8. *If* (M,ω) *is a 2-plectic manifold, then* $\mathrm{Leib}(M,\omega)$ *is a hemi-strict Lie 2-algebra with:*

- *underlying complex*

$$L = C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M),$$

- *bracket given by*

$$[\alpha,\beta] = [\![\alpha,\beta]\!] = \mathcal{L}_{v_\alpha}\beta,$$

in degree 0, and

$$[\alpha, f] = [\![\alpha, f]\!] = \mathcal{L}_{v_\alpha} f,$$
$$[f, \alpha] = [\![f, \alpha]\!] = 0$$

in degree 1,

- *alternator given by:*

$$S(\alpha, \beta) = \iota_{v_\alpha}\beta + \iota_{v_\beta}\alpha,$$

- *Jacobiator given by:*

$$J(\alpha, \beta, \gamma) = 0.$$

Proof. The axioms for a weak Lie 2-algebra given in Def. A.5 are verified by straightforward calculations using the Cartan calculus. In particular, the fact that Eq. A.7 is satisfied follows from the identity:

$$\mathcal{L}_v \iota_w \alpha = \iota_{[v,w]}\alpha + \iota_w \mathcal{L}_v \alpha.$$

□

For a 2-plectic manifold, we view $L_\infty(M, \omega)$ as a weak Lie 2-algebra with trivial alternator.

Proposition A.9. *If (M, ω) is a 2-plectic manifold, then $L_\infty(M, \omega)$ is a semi-strict Lie 2-algebra with:*

- *underlying complex*

$$L = C^\infty(M) \xrightarrow{d} \Omega^1_{\mathrm{Ham}}(M),$$

- *bracket given by*

$$[\alpha, \beta] = \{\alpha, \beta\} = \omega(v_\alpha, v_\beta, \cdot),$$

in degree 0, and

$$[\alpha, f] = 0$$
$$[f, \alpha] = 0$$

in degree 1,

- *alternator given by:*

$$S(\alpha, \beta) = 0,$$

- *Jacobiator given by:*

$$J(\alpha, \beta, \gamma) = \omega(v_\gamma, v_\beta, v_\alpha).$$

Proof. By setting $S = 0$, in Def. A.5, we recover the usual notion of a Lie 2-algebra Def. 3.8. Hence, the statement follows from Prop. 3.15. $\square$

The main result of this section is the following theorem:

Theorem A.10. *If* (M, ω) *is a 2-plectic manifold, then* $L_\infty(M, \omega)$ *and* $\mathrm{Leib}(M, \omega)$ *are isomorphic as weak Lie 2-algebras.*

Proof. Since the underlying chain complexes of $L_\infty(M, \omega)$ and $\mathrm{Leib}(M, \omega)$ are the same, we build a weak Lie 2-algebra isomorphism (Def. A.6) using the identity chain map

$$\phi_0 = \mathrm{id}, \quad \phi_1 = \mathrm{id}\,.$$

Let $\Phi\colon \Omega^1_{\mathrm{Ham}}(M) \otimes \Omega^1_{\mathrm{Ham}}(M) \to C^\infty(M)$ be the map:

$$\Phi(\alpha, \beta) = \iota_{v_\alpha}\beta.$$

Proposition A.4 and a straightforward calculation show that Φ gives a chain homotopy:

$$\begin{array}{ccc} L_0 \otimes L_1 \oplus L_1 \otimes L_0 & \longrightarrow & L_0 \otimes L_0 \\ \big\downarrow {\scriptstyle [\cdot,\cdot]-[\cdot,\cdot]'} & \overset{\Phi}{\swarrow} & \big\downarrow {\scriptstyle [\cdot,\cdot]-[\cdot,\cdot]'} \\ L_1' & \xrightarrow{\quad d \quad} & L_0' \end{array}$$

where $L_1 = L_1' = C^\infty(M)$, $L_0 = L_0' = \Omega^1_{\mathrm{Ham}}(M)$, $[\cdot,\cdot]$ is the bracket on $\mathrm{Leib}(M, \omega)$, and $[\cdot,\cdot]'$ is the bracket on $L_\infty(M, \omega)$.

The alternator for $\mathrm{Leib}(M, \omega)$ is

$$S(\alpha, \beta) = \iota_{v_\alpha}\beta + \iota_{v_\beta}\alpha = \Phi(\alpha, \beta) + \Phi(\beta, \alpha).$$

Since the alternator for $L_\infty(M, \omega)$ is trivial, the above equality implies Eq. A.9 in Def. A.6 holds.

Since the Jacobiator of $\mathrm{Leib}(M, \omega)$ is trivial, the left hand side of Eq. A.10 only involves the Jacobiator $J'(\alpha, \beta, \gamma) = \omega(v_\gamma, v_\beta, v_\alpha)$ of $L_\infty(M, \omega)$. Using the definition of the brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]'$, the right hand side of Eq. A.10 becomes:

$$[\![\alpha, \Phi(\beta, \gamma)]\!] - [\![\beta, \Phi(\alpha, \gamma)]\!] - [\![\Phi(\alpha, \beta), \gamma]\!] - \Phi(\{\alpha, \beta\}, \gamma) - \Phi(\beta, \{\alpha, \gamma\}) + \Phi(\alpha, \{\beta, \gamma\}).$$

By expanding the above using $\Phi(\alpha, \beta) = \iota_{v_\alpha}\beta$, $[\![\Phi(\alpha, \beta), \gamma]\!] = 0$, $[\![\alpha, \Phi(\beta, \gamma)]\!] = \mathcal{L}_{v_\alpha}\Phi(\beta, \gamma)$, $[\![\alpha, \beta]\!] = \omega(v_\alpha, v_\beta, \cdot)$, and the identity

$$\mathcal{L}_v \iota_w \alpha = \iota_{[v,w]}\alpha + \iota_w \mathcal{L}_v \alpha,$$

the right hand side of Eq. A.10 becomes:

$$\iota_{v_\beta} d\iota_{v_\alpha}\gamma + \iota_{v_\beta}\iota_{v_\alpha} d\gamma - \iota_{v_\beta} d\iota_{v_\alpha}\gamma + 2\omega(v_\alpha, v_\beta, v_\gamma).$$

Since $\omega(v_\alpha, v_\beta, v_\gamma) = -\iota_{v_\beta}\iota_{v_\alpha} d\gamma$, the above expression simplifies to:

$$\omega(v_\alpha, v_\beta, v_\gamma) = -\omega(v_\gamma, v_\beta, v_\alpha) = -J'(\alpha, \beta, \gamma),$$

which is the left hand side of Eq. A.10. Hence, (ϕ_0, ϕ_1, Φ) satisfies the axioms for an isomorphism of weak Lie 2-algebras. $\square$

Appendix B

Twisted bundles and the proof of Proposition 5.28

Recall Def. 5.25 of a Hermitian vector bundle twisted by a 2-cocycle $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ on an open cover $\mathcal{U} = \{U_i\}$ of a manifold M. Such an object is given by the following data:

- on each U_i, a Hermitian vector bundle
$$(E_i, \langle \cdot, \cdot \rangle_i),$$
- on each $U_{ij} = U_i \cap U_j$, an isomorphism of Hermitian vector bundles
$$\phi_{ij} \colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}},$$
such that $\forall i, j, k \in \mathcal{I}$:
$$\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = g_{ijk} \cdot$$
where $g_{ijk} \cdot$ is the automorphism of $E_k|_{U_{ijk}}$ corresponding to multiplication by
$$g_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1).$$

Also, recall that a morphism $f \colon (E_i, \phi_{ij}) \to (E'_i, \phi'_{ij})$ of g-twisted Hermitian vector bundles over M consists of a collection of morphisms of Hermitian vector bundles
$$f_i \colon E_i \to E'_i,$$
for each $i \in \mathcal{I}$ such that
$$f_i \circ \phi_{ij} = \phi'_{ij} \circ f_j.$$
In this section, we will prove Prop. 5.28 from Chapter 5:

Proposition. *Given a 2-cocycle* $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ *on a manifold* M*, there exists a stack over* M *whose category of global sections is equivalent to the category* $\mathsf{Bund}^g(M)$ *of* g*-twisted Hermitian vector bundles over* M*.*

As we will see, the proof follows from the fact that locally defined stacks can be glued together to form a stack over M, in analogy with the well-known result for sheaves.

We need to introduce some more machinery for stacks. First, just as we have natural transformations between functors, we can define fibered transformations between morphisms of fibered categories:

Definition ([45]). *Let* $(\phi, \alpha), (\psi, \beta) \colon \mathsf{F} \to \mathsf{G}$ *be morphisms between fibered categories. A* **fibered transformation** $\mu \colon (\phi, \alpha) \to (\psi, \beta)$ *consists of natural transformations*

$$\mu_U \colon \phi_U \to \psi_U,$$

for each $U \subseteq M$*, such that given an inclusion* $i \colon V \to U$ *of open sets, the diagram of natural transformations*

$$\begin{array}{ccc} \phi_V i^* & \xrightarrow{\alpha_i} & i^*\phi_U \\ {\scriptstyle \mu_V i^*}\downarrow & & \downarrow{\scriptstyle i^*\mu_U} \\ \psi_V i^* & \xrightarrow{\beta_i} & i^*\psi_U \end{array}$$

commutes. We say μ *is a* **fibered isomorphism** *if each* μ_U *is a natural isomorphism.*

Next, we describe the category of descent data associated to a fibered category over M and an open cover of M. One can think of this as the data needed to glue together locally defined sections into a global section.

Definition B.1 ([45]). *Let* F *be a fibered category over* M *and let* $\mathcal{U} = \{U_i\}$ *be an open cover of* M*. The category* $\mathsf{Des}(M, \mathcal{U})$ *of* **descent data** *has:*

- *As objects, collections* (x_i, ψ_{ij}) *where each* x_i *is an object of* $\mathsf{F}(U_i)$*, and each*

  $$\psi_{ij} \colon x_j|_{U_{ij}} \xrightarrow{\sim} x_i|_{U_{ij}}$$

  is an isomorphism in $\mathsf{F}(U_{ij})$ *required to satisfy the conditions*

  $$\psi_{ik}^{-1} \circ \psi_{ij} \circ \psi_{jk} = \mathrm{id} \tag{B.1}$$

  in $\mathsf{F}(U_{ijk})$*.*

- *As morphisms,* $(x_i, \psi_{ij}) \xrightarrow{f} (x_i', \psi_{ij}')$*, a collection of morphisms*

  $$x_i \xrightarrow{f_i} x_i'$$

in $\mathsf{F}(U_i)$ *such that the diagram*

$$\begin{array}{ccc} x_j|_{U_{ij}} & \xrightarrow{f_j} & x'_j|_{U_{ij}} \\ \psi_{ij}\downarrow & & \downarrow\psi'_{ij} \\ x_i|_{U_{ij}} & \xrightarrow{f_i} & x'_i|_{U_{ij}} \end{array}$$

commutes in $\mathsf{F}(U_{ij})$.

Categories of descent data are sometimes used directly in the definitions for pre-stack and stack. We observe that if F is a fibered category, for any open cover $\mathcal{U}$, there is a functor $D\colon \mathsf{F}(M) \to \mathsf{Des}(\mathsf{F},\mathcal{U})$ which sends an object $x \in \mathsf{F}(M)$ to $(x|_{U_i}, \psi_{ij} = \mathrm{id})$ in the descent category. If F is a prestack, then this functor is fully faithful i.e. a bijection on morphisms. We have used variations of the next proposition in Chapters 5 and 7.

Proposition B.2. *If* F *is a stack over* M *and* $\mathcal{U}$ *is an open cover of* M*, then the above functor*

$$\mathsf{F}(M) \xrightarrow{D} \mathsf{Des}(\mathsf{F},\mathcal{U})$$

gives an equivalence of categories.

Proof. Def. 5.6 implies that the objects $x_i \in \mathsf{F}(U_i)$ given in the descent data can be glued together into a global object which is unique up to isomorphism. This implies that D is essentially surjective, and hence an equivalence. □

Let F be a stack over M, and $U \subseteq M$ an open set. It is easy to see that we can construct a new stack $\mathsf{F}|_U$ on U which assigns to the open set $V \subseteq U$, the category $\mathsf{F}(V)$. We say $F|_U$ is the stack F **restricted** to U. The following theorem describes how stacks themselves glue together.

Theorem B.3 ([57])**.** *Let* $\{U_i\}$ *be a cover of* M*. Given the following data:*

1. *for each* U_i*, a stack* S_i*,*
2. *for each* $U_{ij} = U_i \cap U_j$*, an equivalence of stacks* $\varphi_{ij}\colon \mathsf{S}_j|_{U_{ij}} \xrightarrow{\sim} \mathsf{S}_i|_{U_{ij}}$*,*
3. *for each* U_{ijk}*, a fibered isomorphism* $\mu_{ijk}\colon \varphi_{ij} \circ \varphi_{jk} \xrightarrow{\sim} \varphi_{ik}$*, such that, for each* $U_{ijk} = U_i \cap U_j \cap U_k$*, the diagram*

$$\begin{array}{ccc} \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} & \xrightarrow{\mu_{jkl}} & \varphi_{ij} \circ \varphi_{jl} \\ \mu_{ijk}\downarrow & & \downarrow\mu_{ijl} \\ \varphi_{ik} \circ \varphi_{kl} & \xrightarrow{\mu_{ikl}} & \varphi_{il} \end{array} \tag{B.2}$$

commutes,

there exists a stack S *on* M*, equivalences of stacks* $\varphi_i \colon \mathsf{S}|_{U_i} \xrightarrow{\sim} \mathsf{S}_i$*, and fibered isomorphisms* $\eta_{ij} \colon \varphi_{ij} \xrightarrow{\sim} \varphi_i \circ \varphi_j^{-1}$ *satisfying*

$$\begin{array}{ccc} \varphi_{ij} \circ \varphi_{jk} & \xrightarrow{\eta_{jk}} & \varphi_{ij} \circ (\varphi_j \circ \varphi_k^{-1}) \\ \Big\downarrow{\scriptstyle \mu_{ijk}} & & \Big\downarrow{\scriptstyle \eta_{ij}} \\ \varphi_{ik} & \xrightarrow{\eta_{ik}} & \varphi_i \circ \varphi_k^{-1} \end{array} \tag{B.3}$$

The data $(\mathsf{S}, \varphi_i, \eta_{ij})$ *are unique up to equivalence of stacks. Moreover, this equivalence is unique up to unique fibered isomorphism.*

Constructing the stack Bund^g

Recall that Bund is the stack on M which assigns to each open set V, the category of Hermitian vector bundles on V. Given an inclusion $V \to U$, the corresponding functor $\mathsf{Bund}(U) \to \mathsf{Bund}(V)$ is just the pull-back of bundles. The natural isomorphisms $(ij)^* \simeq j^* i^*$ described in Def. 5.1 of fibered catgory are given by the identity.

Let us now construct the stack Bund^g described in the statement of Prop. 5.28. Let $g \in C^2(\mathcal{U}, \underline{\mathrm{U}(1)})$ be a 2-cocycle defined on an open cover $\mathcal{U} = \{U_i\}$ of M. For each i, let

$$\mathsf{Bund}_i = \mathsf{Bund}|_{U_i}$$

be the stack of Hermitian bundles on M restricted to the open set U_i. By definition of restriction, we have an equality of stacks

$$\mathsf{Bund}_j|_{U_{ij}} = \mathsf{Bund}_i|_{U_{ij}}$$

for each i and j, and therefore, an identity functor

$$\mathsf{Bund}_j|_{U_{ij}} \xrightarrow{\varphi_{ij} = \mathrm{id}} \mathsf{Bund}_i|_{U_{ij}}.$$

For any open subset V of U_{ijk}, we define a natural transformation between identity functors

$$\mathrm{id} = \varphi_{ij_V} \circ \varphi_{jk_V} \xrightarrow{\mu_{ijk_V}} \varphi_{ik_V} = \mathrm{id}$$

which sends a bundle $E \in \mathsf{Bund}(V)$ to the automorphism

$$E \xrightarrow{g_{ijk}|_V \cdot} E.$$

Here, $g_{ijk}|_V \cdot$ corresponds to multiplying sections of E by $g_{ijk}|_V \colon V \cap U_i \cap U_j \cap U_k \to \mathrm{U}(1)$. It is easy to see that this gives a fibered isomorphism

$$\varphi_{ij} \circ \varphi_{jk} \xrightarrow{\mu_{ijk}} \varphi_{ik}.$$

The fact that g satisfies the cocycle condition on each U_{ijkl} implies μ_{ijk} satisfies Eq. B.2.

Hence, it follows from Theorem B.3 that there exists a stack Bund^g on M with equivalences of stacks

$$\varphi_i \colon \mathsf{Bund}^g|_{U_i} \xrightarrow{\sim} \mathsf{Bund}_i = \mathsf{Bund}|_{U_i},$$

and fibered isomorphisms

$$\eta_{ij} \colon \varphi_{ij} = \mathrm{id} \xrightarrow{\sim} \varphi_i \circ \varphi_j^{-1}$$

satisfying Eq. B.3.

Global sections of Bund^g as twisted bundles

Now we prove Prop. 5.28 by showing that the category $\mathsf{Bund}^g(M)$ of global sections of the stack Bund^g is equivalent to the category C of g-twisted Hermitian vector bundles over M. Since Bund^g is a stack, Prop. B.2 implies $\mathsf{Bund}^g(M)$ is equivalent to the category of descent data $\mathsf{Des}(\mathsf{Bund}^g, \mathcal{U})$, where $\mathcal{U}$ is the open cover used in defining the cocycle g. Hence, it is sufficent to show that $\mathsf{Des}(\mathsf{Bund}^g, \mathcal{U})$ is equivalent to C.

We build a functor $\mathsf{Des}(\mathsf{Bund}^g, \mathcal{U}) \to \mathsf{C}$ in the following way. Let (x_i, ψ_{ij}) be an object in the category of descent data. We use the stack morphisms $\varphi \colon \mathsf{Bund}^g|_{U_i} \xrightarrow{\sim} \mathsf{Bund}_i$ to send the objects $x_i \in \mathsf{Bund}^g(U_i)$ to Hermitian vector bundles

$$E_i = \varphi_i(x_i) \in \mathsf{Bund}(U_i).$$

The fibered isomorphisms $\eta_{ij} \colon \mathrm{id} \xrightarrow{\sim} \varphi_i \circ \varphi_j^{-1}$ assign an isomorphism in $\mathsf{Bund}(U_{ij})$ to every object in $\mathsf{Bund}(U_{ij})$. Given the objects $\varphi_j(x_j), \varphi_j(x_i) \in \mathsf{Bund}(U_j)$, let the corresponding isomorphisms be denoted

$$E_j = \varphi_j(x_j) \xrightarrow{\eta_{ij}(x_j)} \varphi_i(x_j)$$
$$\varphi_j(x_i) \xrightarrow{\eta_{ij}(x_i)} \varphi_i(x_i) = E_i.$$

We have suppressed the restrictions to keep the notation under control. This will not cause any problems, since the morphisms and fibered transformations we are considering commute with the restriction functors "on the nose". We define isomorphisms

$$\phi_{ij} \colon E_j \xrightarrow{\sim} E_i, \quad \text{in } \mathsf{Bund}(U_{ij}),$$

by using the descent data $\psi_{ij} \colon x_j|_{U_{ij}} \xrightarrow{\sim} x_i|_{U_{ij}}$, and the commutative diagram

$$\begin{array}{ccc} E_j & \xrightarrow{\eta_{ij}(x_j)} & \varphi_i(x_j) \\ {\scriptstyle \varphi_j(\psi_{ij})}\downarrow & \overset{\phi_{ij}}{\searrow} & \downarrow{\scriptstyle \varphi_i(\psi_{ij})} \\ \varphi_j(x_i) & \xrightarrow{\eta_{ij}(x_i)} & E_i \end{array}$$

in $\mathsf{Bund}(U_{ij})$, which is given by the naturality of η_{ij}.

We claim the isomorphisms of bundles ϕ_{ij} satisfy

$$\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = g_{ijk} \cdot$$

on U_{ijk}. To show this, we write out convenient expressions for ϕ_{ij}, ϕ_{jk}, and ϕ_{ik}:

$$\begin{aligned} \phi_{ij} &= \eta_{ij}(x_i)\varphi_j(\psi_{ij}) \\ \phi_{jk} &= \varphi_j(\psi_{jk})\eta_{jk}(x_k) \\ \phi_{ik} &= \varphi_i(\psi_{ik})\eta_{ik}(x_k). \end{aligned}$$

We then consider the following commutative diagram of bundle isomorphisms:

$$\begin{array}{ccccc} E_k & \xrightarrow{\eta_{jk}(x_k)} & \varphi_j(x_k) & \xrightarrow{\varphi_j(\psi_{ij})} & \varphi_j(x_i) \\ {\scriptstyle \mu_{ijk}}\downarrow & & \downarrow{\scriptstyle \eta_{ij}} & & \downarrow{\scriptstyle \eta_{ij}(x_i)} \\ E_k & \xrightarrow{\eta_{ik}(x_k)} & \varphi_i(x_k) & \xrightarrow{\varphi_i(\psi_{ij})} & E_i. \end{array}$$

The first square on the left-hand side follows from the fact that η_{ij} satisfies Eq. B.3, while the second square follows from naturality. The commutativity of the diagram, combined with the equality $\varphi_j(\psi_{ij} \circ \psi_{jk}) = \varphi_j(\psi_{ik})$ given by Def. B.1, implies

$$\begin{aligned} \phi_{ij} \circ \phi_{jk} &= \eta_{ij}(x_i)\varphi_j(\psi_{ik})\eta_{jk}(x_k) \\ &= \varphi_i(\psi_{ik})\eta_{ik}(x_k)\mu_{ijk} \\ &= \phi_{ik} g_{ijk} \cdot, \end{aligned}$$

where the last line follows by definition of ϕ_{ik} and μ_{ijk}.

Hence, we have a functor

$$\mathsf{Des}(\mathsf{Bund}^g, \mathcal{U}) \xrightarrow{F} \mathsf{C},$$

which sends an object (x_i, ψ_{ij}) to the g-twisted bundle (E_i, ϕ_{ij}), as defined above. On morphisms, F sends

$$(x_i, \psi_{ij}) \xrightarrow{f} (x_i', \psi_{ij}')$$

to

$$(E_i, \phi_{ij}) \xrightarrow{\varphi_i(f)} (E_i', \phi_{ij}').$$

The fact that $\varphi_i(f)$ satisfies the axioms for a twisted bundle morphism follow from the naturality of η_{ij}. Finally, it is easy to see that F gives an equivalence of categories, since each φ_i is an equivalence of stacks.

This completes the proof of Prop. 5.28.