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Nonlinear structures, spectral features, and correlations in a nearly incompressible hydrodynamic fluid

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Turbulence simulations of a hydrodynamic fluid are performed to explore various nonlinear aspects of a nearly incompressible (NI) fluid in a regime where temperature fluctuations dominate. The NI model was developed primarily to understand weak compressive effects in interplanetary and interstellar media. Nonlinear structures generated by turbulent relaxation are shown to exist in our two-dimensional fluid simulations. Dynamically weak compressive effects are associated with passively convected thermal fluctuations which enhance the rate of selective decay in decaying NI turbulence. Turbulent relaxation leads to self-organization in thermally dominated NI velocity fluctuations and predicts the formation of large-scale steady-state coherent structures via an inverse cascade mechanism. In agreement with theoretical predictions, density fluctuations are slaved to the incompressible velocity fluctuations and exhibit a Kolmogorov-type power law. Thermal and density fluctuations are found to be anticorrelated in an adiabatic fluid. This suggests that a large fraction of the high plasma-β fluid departs from a thermal equilibrium. Furthermore, compressional effects in nearly incompressible turbulence enhance decay rates significantly and lead to the formation of coherent vortices on much faster time scales when compared with incompressible turbulence. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2191878]

I. INTRODUCTION

Observations of interstellar scintillations at radio wavelengths reveal a Kolmogorov-type scaling of the electron density spectrum with a spectral slope of $-5/3$ over many decades in wavenumber space[1,2] (from an outer scale of a few parsecs to scales of 200 km or less). These fluctuations are detected with great sensitivity by very long baseline interferometer (VLBI) phase scintillation measurements. In interstellar plasma turbulence, the plasma density fluctuates randomly in time and space. As the radio refractive index is proportional to the plasma density, there will be corresponding variations in the refractive index. That density irregularities exhibit a definite power-law spectrum is essentially a characteristic of a fully developed isotropic and statistically homogeneous incompressible fluid turbulence, described by Ref. 3 for hydrodynamic and Ref. 4 for magnetohydrodynamic fluids. In both scenarios, a fully developed isotropic and statistically homogeneous incompressible fluid turbulence possesses energy in the large scales which cascades towards smaller scales through nonlinear interactions.[6] The spectral transfer of the energy through an inertial range regime continues until it is dissipated by collisional or some other damping processes. This suggests that interstellar fluid motions possess structures that are possibly related to turbulence. The actual origin of interstellar turbulence remains unresolved. Several hypotheses suggest mechanisms for how the kinetic energy of large-scale interstellar fluid motions is converted into turbulent energy. These, among many others, include supernovae-induced turbulence, the Parker instability, and Kelvin-Helmholtz and other fluid instabilities. On the other hand, sources of small-scale interstellar turbulence, such as the observed electron density fluctuations, have been attributed to the reflection of shock waves from molecular clouds, energy cascades from large-scale turbulent eddies, and others.

Turbulence, manifested by interstellar plasma fluid motions, therefore plays a major role in the evolution of the interstellar medium (ISM) plasma density, velocity, magnetic fields, and the pressure. Radio wave scintillation data indicates that the rms fluctuations in the ISM and interplanetary medium density, of possibly turbulent origin and exhibiting Kolmogorov-type behavior, are only about 10% of the mean density.[5] This suggests that ISM density fluctuations are only weakly compressible.

Although interstellar fluid motion is not fully incompressible, it nevertheless exhibits a Kolmogorov power spectrum that is described by an incompressible fluid theory, suggesting that the observed density fluctuations are related to incompressible fluid models. For example, Montgomery *et al.*[7] suggested that the density fluctuations in the solar wind and interstellar medium might be "slaved" to the incompressible magnetic field in the context of a MHD model. This model describes fluctuations in the density to a first-order accuracy only, and essentially ignores the dynamics due to short-scale and high-frequency modes that could, in principle, be critical to the compressible interstellar fluid and to the ubiquitous $k^{-5/3}$ spectra. On the other hand, Zank and Matthaeus[8,9] predicted, within the context of a nearly incompressible (NI) fluid model, that the solar wind and interstellar medium density fluctuations might convect passively in the field of background incompressible fluid motion and exhibit

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a $k^{-5/3}$ spectrum. The NI fluid theory also includes the essential dynamics due to short-scale and high-frequency compressive modes.

The theory of nearly incompressible fluids proposed by Zank *et al.*[8,9,11] possesses weak compressional effects in the fluid density fluctuations, unlike purely incompressible (IN) hydrodynamics, which are included by first-order corrections to the leading-order purely incompressible fluctuations in the low turbulent Mach number hydrodynamic fluid equations. The compressible density fluctuations are coupled through thermal fluctuations which are passively advected by incompressible velocity fields. The resulting equations, thus, comprise the familiar incompressible hydrodynamical (and MHD) equations at leading order, together with a modified set of compressible hydrodynamical equations.[5,8–17] The initial motivation for NI theory was, as described above, to understand compressible effects such as density fluctuation spectra in the solar wind and interstellar medium.[1] Recently, we have integrated the equations of NI hydrodynamics numerically to determine the density spectrum.[15–17] Rather remarkably, we find that the two classes of NI hydrodynamic description admit a $k^{-5/3}$ (for the driven case) and a k^{-3} (for decaying simulations) spectrum for the density fluctuations and, moreover, the density spectrum follows the IN velocity fluctuations spectrum. Interestingly enough, the spectral law is a consequence of the passive convection of density fluctuations in the field of background incompressible velocity fluctuations,[15] rather than a pseudosound correlation between density fluctuations and incompressible pressure fluctuations through the square of the sound speed.[7] It therefore appears, at least for the high plasma beta limit, that NI models can explain the observed Kolmogorov-type density fluctuation spectrum. Of perhaps equally compelling interest is how "turbulence affects the structure and motions of nearly all temperature and density regimes in the interstellar gas."[18] Since density and temperature are included self-consistently within the NI models, we address this question directly in this paper. Specifically, we investigate (1) spectral characteristics of the NI model in a heat-fluctuation-dominated (HFD) regime; (2) the formation of large-scale structure; and (3) the relationship between density and temperature fluctuations.

The basic equations describing the HFD model and the underlying physics are discussed in Sec. II. Section III presents our nonlinear simulation results showing the emergence of nonlinearly saturated coherent vortices through the turbulent relaxation of weakly compressible fluctuations. The emergence of the saturated vortex structure observed in our nonlinear fluid simulations can be understood using a Larichev-Reznik technique.[19] This analysis demonstrates how nonlinear interactions are responsible for the generation of a large-scale stationary vortex. The large-scale analytic vortex solutions agree well with the numerical solutions. Section IV contains spectral studies of the HFD model in both decaying and driven cases that lead to a Kolmogorov-type power spectrum. Finally, the underlying HFD model exhibits fluctuations in density and temperature. These fluctuations follow a certain relationship. We illustrate the density-temperature correlations that result from our simulations in Sec. V, showing that they are in agreement with the analytic predictions of the HFD model. Conclusions are drawn in Sec. VI.

II. THE HFD EQUATIONS

The NI model indicates two distinct approaches to compressibility. This depends largely upon the relative magnitudes of the compressible pressure, the temperature, and the density fluctuations. When all the perturbations are of equal magnitude, a heat-fluctuation-modified (HFM) model emerges for which the compressible fluctuations behave quasiadiabatically. On the other hand, when temperature fluctuations dominate the pressure and density fluctuations, such a description of the compressibility is referred to as a heat-fluctuation-dominated (HFD) model. The density-temperature anticorrelation coefficient, predicted by HFD hydrodynamics, was observed in a new class of solar wind pressure-balanced structures, using Voyager 2 data.[11] This analysis, based on the ordering of the temperature and the density fluctuations, suggests that the solar wind may admit two classes of NI fluid description, of which the thermally dominated nearly incompressible fluid model is one. Although the two classes describe different regimes, they admit various correlations between fluctuating fluid quantities, many of which have been tested successfully in the solar wind,[20–23] although subtleties in interpretation still remain.[24,25] However, only recently[15–17] has a detailed theoretical study of the spectral characteristics and relaxation processes in HFM and HFD been undertaken, and we believe that this will provide a better framework for placing certain solar wind observations in a NI context.

The NI theory[9–11] describes compressive effects in hydrodynamics as well as magnetofluids using a singular expansion technique within the framework of "nearly incompressible" (NI) fluids. In particular, the set of hydrodynamical fluid equations derived by Zank *et al.*[9] couples convective fluid motion with high-frequency acoustic fluctuations in which leading-order fluctuations represent incompressible modes. The idea of using the incompressible hydrodynamic equations as a source for the acoustic density fluctuations dates back to a seminal paper by Lighthill.[26] The weakly perturbed compressive fluctuations about the incompressible modes (denoted by superscript ∞) for velocity and pressure variables are represented by $\mathbf{U}=\mathbf{U}^{\infty}+\epsilon\mathbf{U}_1$, $p=1+\epsilon^2(p^{\infty}+p^*)$, respectively. Here, ϵ is a small parameter associated with the turbulent fluid Mach number M_s through the relation $\epsilon^2=\gamma M_s^2$, and $M_s=u_0/C_s$, γ is the ratio of the specific heats, u_0 is the characteristic speed of the turbulent fluid, and C_s is the acoustic speed associated with sound waves, $C_s^2=\gamma p_0/\rho_0$. The p_0 and ρ_0 are typical amplitudes of the fluid pressure and density fluctuations. The high-frequency and short-wavelength component in this expansion describes effects due to the compressibility of the fluid. Due to a lack of uniqueness in the representation of the fluid density and temperature fields, either of the choices $\rho=1+\epsilon\rho_1$, $T=T_0+\epsilon T_1$, or $\rho=1+\epsilon^2\rho_1$, $T=T_0+\epsilon^2 T_1$ is consistent. The first choice corresponds to a state where temperature fluctuations dominate both the incompressible (p^{∞}) as well as compressible pressures (p^*) and is referred to as the HFD regime. On the other hand, the second choice in which all the variables are of similar order represents the HFM regime. The resulting equations, nonetheless, comprise the familiar incompressible hydrodynamical and MHD equations at leading order, together with a modified set of compressible hydrodynamical equations.[5,8–17] The background incompressible fluid, in both regimes, can be described by the usual equations of incompressible hydrodynamics,

$$\frac{\partial}{\partial t}\mathbf{U}^{\infty} + \mathbf{U}^{\infty}\cdot\nabla\mathbf{U}^{\infty} = -\nabla p^{\infty} + \mu\nabla^2\mathbf{U}^{\infty}, \tag{1}$$

$$\nabla\cdot\mathbf{U}^{\infty} = 0. \tag{2}$$

Here, the superscript ∞ indicates that the variables satisfy the incompressible fluid equations, i.e., Eqs. (1) and (2). The parameter μ is associated with viscous damping of small-scale velocity fluctuations in the leading-order incompressible fluid. The constant density of the incompressible fluid is expressed through Eq. (2), while pressure satisfies the nonlinear Poisson equation,

$$\nabla^2 p^{\infty} = -\nabla\cdot(\mathbf{U}^{\infty}\cdot\nabla\mathbf{U}^{\infty}). \tag{3}$$

The nonlinear fluid equations describing the dynamical evolution of the compressible fluctuations in the HFD NI hydrodynamical description (Refs. 8 and 9) contain the compressible fluid velocity, $\mathbf{U}_1$, the compressible density ρ_1, and the temperature T_1, and satisfy

$$\frac{\partial}{\partial t}\mathbf{U}_1 + \mathbf{U}^{\infty}\cdot\nabla\mathbf{U}_1 + \mathbf{U}_1\cdot\nabla\mathbf{U}^{\infty} = \rho_1\nabla p^{\infty} + \nu\nabla^2\mathbf{U}_1 + \left(\xi + \frac{1}{3}\right)\nabla(\nabla\cdot\mathbf{U}_1), \tag{4}$$

$$\frac{\partial}{\partial t}\rho_1 + \mathbf{U}^{\infty}\cdot\nabla\rho_1 + \nabla\cdot\mathbf{U}_1 = 0, \tag{5}$$

$$\frac{\partial}{\partial t}T_1 + \mathbf{U}^{\infty}\cdot\nabla T_1 = \frac{1}{\mathrm{Pr}}\nabla^2 T_1, \tag{6}$$

$$\nabla\cdot\mathbf{U}_1 = \frac{1}{\mathrm{Pr}}\frac{\gamma-1}{\gamma}\nabla^2 T_1. \tag{7}$$

The above equations are normalized, and correspond to their respective unnormalized variables as follows: $\mathbf{U}_1/M_s = \bar{\mathbf{U}}_1$, $\gamma^{1/2}\rho_1/\rho_0 = \bar{\rho}_1, \rho_0 C_p T_1/(\gamma^{1/2}p_0) = \bar{T}_1$. The time and space coordinates are normalized by characteristic time and length scales, respectively, $L\nabla = \bar{\nabla}$ and $u_0 t/L = \bar{t}$. The bars are absent from all the normalized variables for the sake of convenience. Here, Pr is the Prandtl number, C_v is the specific heat at constant volume, ν and ξ are the normalized viscosity. Equations (1)–(7) describe the dynamical evolution of the decaying NI system with no linear instabilities. It is noteworthy that, unlike the background IN velocity, the divergence of the compressible velocity [i.e., Eq. (7)] is modified by thermal conduction. The compressive velocity fluctuations in Eq. (4) modify the effective dissipation and are themselves generated explicitly through thermal fluctuations. The latter are governed by the scalar temperature which is convected passively by IN velocity fluctuations. Thus, compressional effects play a significant role in the relaxation mechanism amidst complex nonlinear interactions, which we will explore further below.

Although the NI fluid model has been successful in elucidating many nonlinear turbulent features of the solar wind and interstellar medium, the basic nonlinear aspects of NI hydrodynamics and MHD remain completely unexplored. This paper, therefore, presents a fully self-consistent investigation of thermally dominated NI hydrodynamic turbulence which is applicable to high-β interstellar plasmas (β is the ratio of thermal to magnetic pressure). NI MHD will be explored in a subsequent article. Our simulations indicate the emergence of steady-state coherent structures in decaying NI turbulence through a turbulent relaxation mechanism in which selective decay (as defined by the ratio of enstrophy and energy) rates are enhanced dramatically by compressive effects. Turbulent relaxation is one of the most remarkable features of hydrodynamic systems (Refs. 6, 27, and 28, and references therein). As exhibited in numerous two-dimensional decaying turbulence models, hydrodynamic (and other) systems try to "self-organize" themselves into coherent states. Such processes often lead to the formation of large-scale coherent vortex structures, essentially through an inverse cascade mechanism associated with at least one of the quadratic invariants of the system. In this work we describe nonlinear structures that are generated by turbulent relaxation processes in a two-dimensional nearly incompressible (NI) hydrodynamics model. Unlike the pseudosound approximation,[7] thermally induced compressible density fluctuations are found to be anticorrelated with thermal fluctuations, in agreement with the theoretical prediction of Zank and Matthaeus[8,9] and as observed sometimes in Voyager 2 data. This has also been suggested as an explanation for the observed interstellar density spectrum.[9,14,15] While the weakly compressive small-scale velocity fluctuations relax towards large-scale structures in a thermally dominated regime, it is of interest to know what happens to the density and the temperature fluctuations. Do they exhibit a Kolmogorov-type turbulent spectra? We consider these important issues here. Additionally, we concentrate on the possible formation of large-scale nonlinear coherent structures in the compressible interstellar fluid within a NI paradigm.

III. NONLINEAR SIMULATIONS

The nonlinear evolution of Eqs. (1)–(7) is investigated using a 2/3 dealiased Fourier spectral code [our Heat Fluctuation Dominated Hydro (HFD) code] with 256^2 or 512^2 Fourier modes in a two-dimensional box of size $15\pi\times15\pi$ with periodic boundary conditions along the x and the y directions. The time integration uses a second-order predictor-corrector method. The HFD code is different from our NIH code,[15–17] in which the thermal transport equation is not included. The latter model was used recently to show that a Kolmogorov-type power spectrum and anisotropy described weak density fluctuations, and this has been advanced as an explanation for related observations in the solar wind and the interstellar medium.[1,2,15,16,29] All fluctuations in the simula-

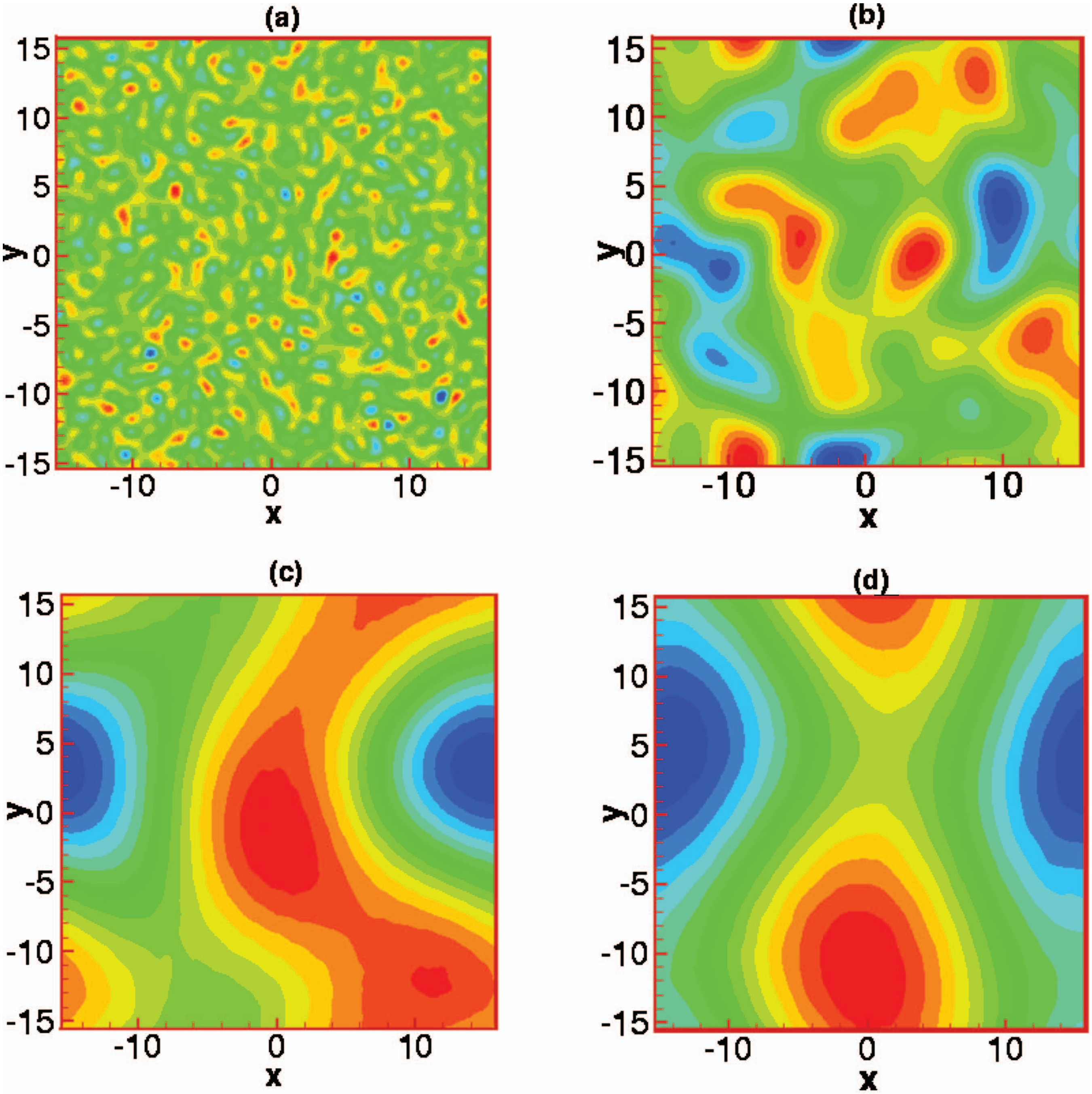

FIG. 1. (Color) Decaying turbulence in a NI fluid coupled to an IN fluid through nonlinear interactions. (a) Initial condition ($t=0$) specified on the x component of NI velocity field shows random fluctuations in a two-dimensional box. (b), (c), and (d) show fields at $t=5,15,40$, respectively. Box size is $10\pi \times 10\pi$. $k_{\min}=0.2$, $k_{\max}=17.06$. Other constants are $\gamma=5/3$, Pr=1000, $\mu=1.e-3$, $\xi=1.e-5$.

tions are initialized with a Gaussian random number generator to ensure that the Fourier modes are all spatially uncorrelated and randomly phased. The initially normalized energy spectrum, peaked at $k_{\min}$, is chosen to lie within the wavenumber band $k_{\min}<k<k_{\max}/2$. This is shown in Fig. 1(a) for the x component of the NI velocity field [i.e., $u_{1_x}(x,y)$] in configuration space. During the evolution of the simulations, the turbulence eventually decays through vortex merging in which like-signed smaller length-scale fluctuations merge to form relatively large-scale fluctuations. The process continues until all merging has occurred to finally form the largest scale coherent vortex dominated by the minimum allowed k in the simulation. The formation of large-scale structure through nonlinear interactions can be attributed to an inverse cascade phenomenon. In this process, small-scale turbulent fluctuations, in an inertial range Fourier space, transfer their spectral energy primarily amongst neighboring Fourier modes. While the energy cascade towards the smaller scales in the simulations is terminated essentially by viscous damping, which is efficient at the smaller scales, the largest allowed scale is determined typically by the $k_{\min}$ Fourier mode. The inertial range turbulent cascades in such a

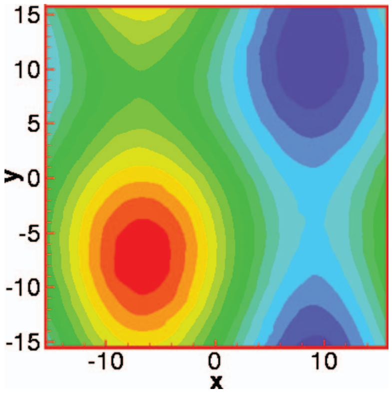

FIG. 2. (Color) Coherent structure formation corresponding to the y component of the NI fluid velocity.

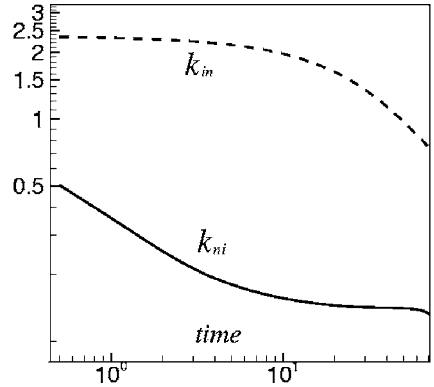

FIG. 4. Evolution of mean Fourier mode associated with selective decay rates. Dashed and solid curves are $k_{\rm in}=\sqrt{\Omega_{\rm in}/E_{\rm in}}$ and $k_{\rm ni}=\sqrt{\Omega_{\rm ni}/E_{\rm ni}}$ rates for IN and NI turbulence, respectively. Decay rates are stronger in NI turbulence than in IN hydrodynamics.

manner that it then leads to the formation of large-scale structures. Various stages in the simulations are shown in Fig. 1, with the final stage being a large-scale (comparable to computational box) coherent vortex. A similar configuration for the y component of the NI fluid velocity [i.e., $u_{1_y}(x,y)$] is shown in Fig. 2, when the turbulence has reached its saturated state. The evolving relaxation of the turbulent fluid is independent of spatial and temporal resolution as well as higher turbulent Reynolds numbers. The latter (i.e., higher Reynolds number) only slows the rate of relaxation, while the qualitative physics remains more or less unaltered. The passively convected temperature, the density, and the background incompressible velocity fluctuations, however, remain turbulent during the entire course of the simulation time, and their evolution depends critically upon the magnitude of the Prandtl number, Pr. We shall return to this point below.

Inspection of the kinetic energy (KE) as the NI and IN fluids relax [shown in Fig. 3] reveals that the KE of the NI fluid $E_{\rm ni}=\frac{1}{2}\Sigma_{\mathbf{k}}|\mathbf{U}_1(\mathbf{k},t)|^2$ (solid curve) decays more rapidly than $E_{\rm in}=\frac{1}{2}\Sigma_{\mathbf{k}}|\mathbf{U}^\infty(\mathbf{k},t)|^2$ (the IN fluid, dashed curve). The decay rate $(dE_{\rm ni}/dt > dE_{\rm in}/dt)$ is very large during an initial phase in which almost all the KE of the NI turbulence is dissipated, while there is only a small change in the KE of the IN fluid. Correspondingly, the selective decay rate in NI turbulence $(\sqrt{\Omega_{\rm ni}/E_{\rm ni}})$ is faster than that $(\sqrt{\Omega_{\rm in}/E_{\rm in}})$ in IN turbulence, where $\Omega_{\rm ni}=\frac{1}{2}\Sigma_{\mathbf{k}}|\nabla\times\mathbf{U}_1(\mathbf{k},t)|^2$ and $\Omega_{\rm in}=\frac{1}{2}\Sigma_{\mathbf{k}}|\nabla\times\mathbf{U}^\infty(\mathbf{k},t)|^2$ are, respectively, NI and IN turbulence vorticities. This is shown in Fig. 4. It is to be noted that the energy associated with nearly incompressible fluctuations decays rapidly, but not to zero. Its magnitude is small but remains finite as shown in Fig. 3. The strong dissipation is not only associated with the damping term proportional to $\nabla^2\mathbf{U}_1$ in Eq. (4), but is also influenced by source terms associated with the IN fluid. The decay rates for NI and IN turbulence, nevertheless, imply that the characteristic time associated with the decay of enstrophy is much shorter than that of energy throughout the simulation. A comparison of the two curves (dashed and solid lines) in Fig. 4 shows that compressional effects in NI turbulence enhance decay rates significantly and lead to the formation of coherent vortices on much faster time scales when compared with IN turbulence. Rapid decay rates observed here in low Mach number NI turbulence are consistent with the perturbative expansion that is used to obtain the high-frequency component, i.e., NI modes, against the slow IN modes.[8] These fast-time-scale compressional modes, dominated by thermal fluctuations, accelerate nonlinear selective decay processes to consequently generate coherent vortical structures on much faster time scales (when compared with the background IN turbulence) through turbulent relaxation.

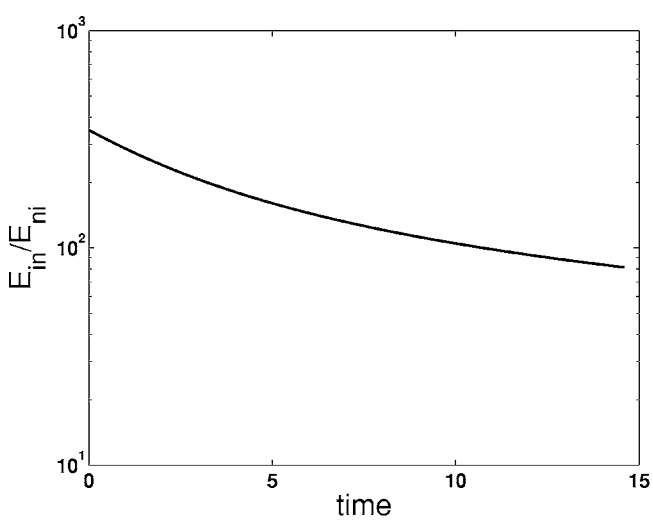

FIG. 3. The ratio of kinetic energies of incompressible and weakly compressible fluid, i.e., $E_{\rm IN}/E_{\rm NI}$. The ratio shows a finite value at long times.

An analytic understanding of the emergence of coherent vortices observed in our fluid simulation can be achieved by considering exact nonlinear wave solutions of the NI and IN velocity fields, i.e., Eqs. (1) and (4). The underlying mechanism on which such an analysis is based is that the coherent structures emerge from the nonlinear interaction processes to eventually become stationary in space and time, thereby forming long-lived stable entities.[19] This is what we observe in our simulations. We may first simplify Eqs. (1) and (4) by expressing the IN and NI velocities in terms of their respective fluxes, $\mathbf{U}^\infty=\hat{z}\times\nabla\phi$ and $\mathbf{U}_1=\hat{z}\times\nabla\phi+\nabla\psi$. The representation of the two flux functions in this fashion is not an arbitrary choice, but is motivated by several factors. First, the flux functions describing the potentials of IN and NI fluids strictly obey the respective conditions of incompressibility and weak compressibility. Second, the governing nonlinear equations describing the evolution of NI and IN velocity fluctuations can be transformed formally into the respective vorticity equations, and finally, the nonlinear interaction terms can be expressed in a Poisson bracket form to yield the exact nonlinear solutions from the vorticity equations.

We next use a flux representation of $\mathbf{U}^{\infty}$ and $\mathbf{U}_1$ in Eqs. (1)–(4) and take their curl. This yields, respectively, vorticity equations for the incompressible and weakly compressible flows. The NI vorticity equation then reads $\partial_t \nabla^2 \psi + [\phi, \nabla^2 \psi] = 0$, and $\partial_t \nabla^2 \phi + [\phi, \nabla^2 \phi] = 0$ for the incompressible fluid. Here, $[A,B] = \partial_x A \partial_y B - \partial_x B \partial_y A$ is the Poisson bracket describing nonlinear terms wherein the compressive velocity potential is convected by the incompressible velocity potential. Small-scale viscous terms (ineffective on vortex length scales) and gradients associated with the NI vorticity terms (small compared to corresponding IN terms) in Eq. (4) are omitted. This is also consistent with what we observed in our simulations. In a moving (with velocity u) vortex frame, the local coordinate transformation can be chosen as $x = x', y = y' - ut$. In this moving frame, the time derivative in the vorticity equations can be replaced by the convective derivative such that $\partial/\partial t = -u \partial/\partial y$. The NI vorticity equation can then be written in terms of the Poisson bracket as $[\phi - ux, \nabla^2 \psi] = 0$. This then readily yields $\nabla^2 \psi = G(\phi - ux)$ due to the vanishing property of the Poisson brackets {i.e., when $[A,B]=0$ then B can be expressed as an arbitrary function of A such that $B = f(A)$}. G is an arbitrary function of its argument and is continuous and differential within and outside the region of vortex boundaries. For G linear, $\nabla^2 \psi = \alpha(\phi - ux)$. This is Bessel's differential equation whose solutions are Bessel functions. We thus obtain finite radius vortex solutions in the form of Bessel functions,

$$\psi = A K_1(r) y, \quad r > r_0 \tag{8}$$

and

$$\psi = \left[\frac{B}{r} J_1(\alpha r) + \frac{C}{r} J_1(\beta r) + D \right] y, \quad r < r_0. \tag{9}$$

Equations (8) and (9) are, respectively, outer and inner vortex solutions where the potential ϕ was determined from the IN vorticity equation. The continuity and the differentiability of the NI velocity potential function allows one to connect the outer $(r > r_0)$ and the inner $(r < r_0)$ nonlinear wave solutions to form a continuous structure across the vortex boundary. Here, J_1, K_1 are, respectively, a Bessel function of the first kind and a modified Bessel function of the second kind, $x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$, and r_0 is the radius of vortex. The constants A, B, C, D, α, and β can be determined by matching the function ψ and its derivatives across the vortex radius r_0. Note that the solutions, Eqs. (8) and (9), represent a dipole vortex whose radius is r_0. To seek a physical understanding of these structures and their relevance to the numerical simulations, we compare them with

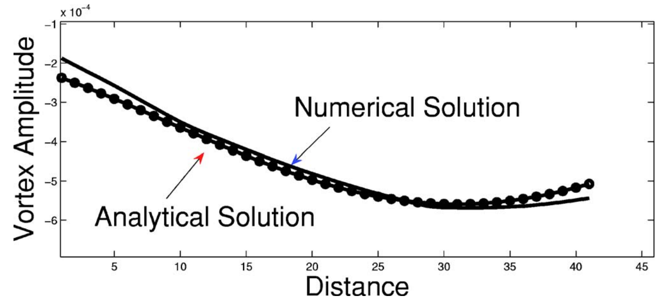

FIG. 5. (Color online) Comparison of analytic and numerical results of the vortex amplitude distribution in space. Shown is a 1D cut along the vorticity distribution. Clearly, the analytic and numerical solutions show excellent agreement.

each other. For this purpose, a 1D spatial distribution of the vortex amplitude, computed from Eqs. (8) and (9), is plotted in Fig. 5 that shows a close agreement with the steady-state large-scale vortex in the simulations. We find that the amplitude of the vortices in the steady state matches very well. In principle there may exist many classes of vortex solutions depending upon the choice of the arbitrary function G (which is linear for a class of vortex solution given above). We will not however discuss other form of the vortex solution here, as they are beyond the scope of the present work.

In brief, we find that the nonlinear structures in NI turbulence, observed in our fluid simulations, are influenced by the background IN turbulence and that they are generated by nonlinear interaction processes.

IV. SPECTRAL FEATURES

Consider now the spectral features of the density and temperature fluctuations in the HFD regime. In two-dimensional turbulence, the background incompressible (hydrodynamic) fluid admits two invariants (constants of motion), namely the energy and the mean-squared vorticity (i.e., irrotational velocity field). The two invariants, under the action of an external forcing, experience a simultaneous cascade, thereby exhibiting a dual cascade that is commonly observed in numerous 2D turbulence systems. In these processes, the energy cascades towards longer length scales, while the fluid vorticity transfers spectral power towards shorter length scales. Usually a dual cascade is observed in a driven turbulence simulation, i.e., one in which certain modes are excited externally through random turbulent forces in spectral space. The randomly excited Fourier modes transfer the spectral energy by conserving the constants of motion in k space. On the other hand, in freely decaying turbulence, the energy contained in the large-scale eddies is transferred to smaller scales, yielding a statistically stationary inertial regime associated with the cascade of one of the invariants. Decaying turbulence often leads to the formation of coherent structures as the turbulence relaxes, thus making the nonlinear interactions rather inefficient when they are saturated. Theoretical turbulence models further suggest that any physical quantity, convecting passively in the field of the background turbulence velocity fluctuations, tends to develop eventually a similar energy spectrum.[28] This

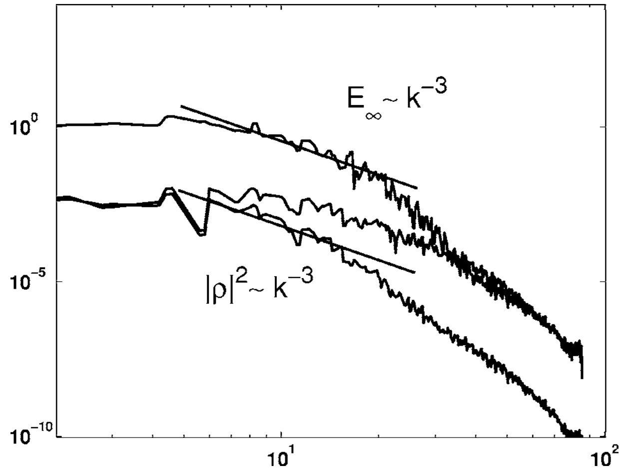

FIG. 6. Density power spectrum $|\rho_k|^2$ is plotted as a function of k (along the horizontal x axis) from a decaying NI hydrodynamics simulation. The incompressible velocity fluctuations (E_∞) follow a Kolmogorov spectrum close to k^{-3} (with an error ±0.08) in a forward (or enstrophy) cascade regime of decaying turbulence. The density fluctuations are passively convected by the incompressible velocity fluctuations and exhibit nearly the same spectrum. The intermediate curve represents the temperature spectrum.

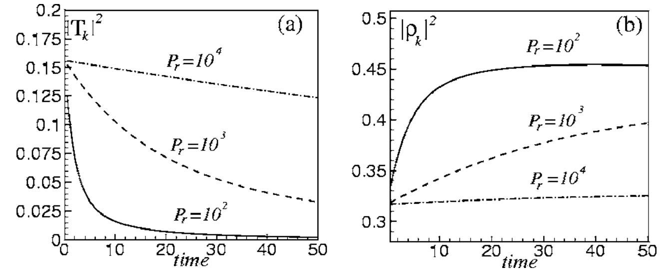

FIG. 8. Energy associated with temperature and density fluctuations in NI turbulence are shown, respectively, in (a) and (b). Shown here are solid ($\mathrm{Pr}=10^2$), dashed ($\mathrm{Pr}=10^3$), and dashed-dot ($\mathrm{Pr}=10^4$) curves. The decay rates depends critically upon the Prandtl number Pr. The density and the temperature fluctuations are clearly anticorrelated in agreement with the prediction of Ref. 8.

characteristic behavior of the passive scalars provides useful information about its evolution and has been used as a prime diagnostic in our investigations since our NI model possesses a passive, scalar-like density equation. We therefore follow the spectrum of the density fluctuations together with the background velocity in our nonlinear NI fluid simulations. On the basis of the HFD model, we see that the weakly compressible density spectrum in a 2D simulation follows the IN velocity fluctuation spectrum and yields a Kolmogorov-type $k^{-5/3}$ spectrum in the decaying case [see Fig. 6] and a k^{-3} spectrum in the driven case, shown in Fig. 7. While the temperature spectrum appears flatter in the decaying case, it is much steeper in driven turbulence. Although turbulence in 2D and 3D possesses distinct spectral features characterized essentially by the number of the inviscid quadratic invariants, our two-dimensional NI hydrodynamics simulations offers a plausible approach to the understanding of the origin of ISM density spectrum.

Thus, in agreement with the NI prediction that a passive scalar explanation emerges as a natural and self-consistent description of the weakly compressible density fluctuation spectrum in NI fluid theory, our NI fluid simulation results demonstrate such characteristics in both the forward and the inverse cascade regimes of HFD model. Our HFD results, in combination with HFM simulations,[15] demonstrate that the density fluctuation spectrum in the weakly compressible hydrodynamic limit can be an omnidirectional Kolmogorov power law and thus offer a possible explanation of the observed density fluctuation spectrum in interstellar medium.[1]

V. DENSITY TEMPERATURE CORRELATIONS

Dynamically finite compressional effects in the NI fluid are induced through thermal fluctuations which depend upon the Prandtl number. Thus, the passively convected thermal transport and density fluctuations mutually influence each other in a rather subtle manner, and both depend critically on the Prandtl number [see Eqs. (4)–(7)]. The dependence can be determined by computing the energy associated with the density and the temperature fluctuations for different values of Pr [shown in Fig. 8]. Clearly, a larger Pr implies a smaller thermal fluctuation dissipation rate [Fig. 8(a)]. Correspondingly, the density fluctuations [shown in Fig. 8(b)] grow as rapidly as thermal fluctuations dissipate. This relationship thus suggests that density and thermal fluctuations are anticorrelated in thermally driven NI hydrodynamics. This further supports Zank and Matthaeus' prediction,[8]

$$-\rho_1=\left(\frac{\gamma-1}{\gamma}\right)T_1, \tag{10}$$

that the density and the temperature fluctuations are anticorrelated at an order $\mathcal{O}(\epsilon)$, where ϵ is an expansion parameter associated with the turbulent Mach number. This relation-

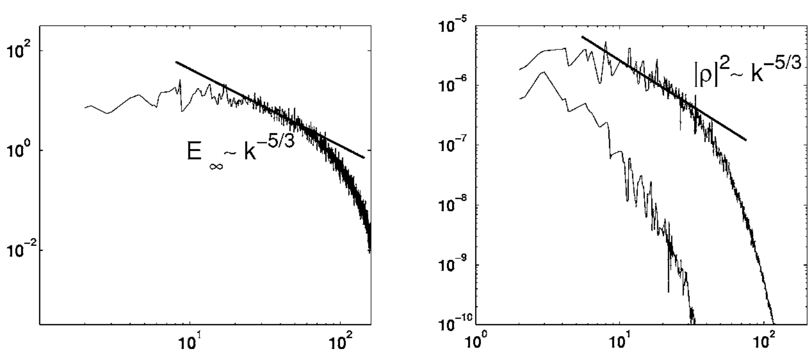

FIG. 7. A driven turbulence NI hydrodynamic simulation yields a Kolmogorov-type spectrum close to $k^{-5/3}$ (with an error ±0.1) in the inverse (or energy) cascade regime for incompressible velocity fluctuations (left). Turbulence is driven by a random forcing in space and time. The compressible density fluctuations (right) follow the incompressible velocity spectrum closely in the inertial regime of turbulence. In the right panel, the temperature spectrum is shown below the density spectrum. The horizontal x axis represents the modes k.

ship, i.e., Eq. (10), can also be derived directly from the nearly incompressible equations[9] and has been tested qualitatively for solar wind fluctuations observed by Voyager 2.[11,25,30] Our simulation results also show such behavior qualitatively.

VI. CONCLUSIONS

In summary, we have investigated NI hydrodynamics in the heat-fluctuation-dominated regime. The NI theory was developed primarily to understand the observed interstellar medium (ISM) density fluctuations which exhibit a Kolmogorov-type power spectrum (i.e., $k^{-5/3}$) in three dimensions.[1]

One of the important outcomes of the NI theory[8–11] is that the density fluctations in the ISM emerge as a result of weak compressibility in the gas and are convected predominantly passively in the field of background incompressible fluid flow. This hypothesis can be verified numerically by investigating the density spectra which should be slaved to the incompressible velocity spectra, as done in our earlier paper,[15] in the heat fluctuation modified (HFM) regime. In this paper, we explore another regime of the NI theory that corresponds to the heat fluctuation dominated (HFD) regime. Such a regime has been observed in the solar wind and has been suggested as an explanation for the observed interstellar medium (ISM) density spectrum. Several important results have emerged from our simulations. First, beginning with arbitrary initial incompressible and NI data, the compressible velocity fluctuations exhibit a Kolmogorov-type power spectra and the density fluctuations exhibit a passive convection. Second, the velocity fluctuations organize themselves into coherent structures, eventually forming a single vortex. The corresponding density and temperature fluctuations remained turbulent (due to turbulent convective flow). We showed theoretically that vortices emerge from nonlinear driving by the incompressible turbulent velocity field. The third important result is that the NI energy and selective decay rates are higher than the IN case (as might be expected with the inclusion of nonlinear compressive modes), which leads to a more rapid formation of coherent vortices for NI components. Finally, the theoretical NI model in the HFD regime, Eqs. (4)–(7), predicts the existence of anticorrelated density and temperature fluctuations. Our simulations demonstrate this explicitly and we find that it is true for all the Prandtl numbers simulated here. The HFD regime is distinct from the pseudosound regime since, unlike the latter, density and temperature behave as passive scalars. The pseudosound approach may be justified in a regime of NI hydrodynamics where acoustic modes, predominantly due to pressure fluctuations, are present.

Our analysis and the simulation of HFD hydrodynamics in the high-β limit may have several important implications for the ISM. These may be enumerated as follows:

(1) The most important result is the demonstration that density fluctuations couple advectively to the incompressible velocity field in weakly compressible hydrodynamic turbulence and can exhibit an omnidirectional Kolmogorov power law. Although the results presented in this paper correspond to two-dimensional turbulence in a high plasma-beta limit, we expect a similar characteristic behavior of the passive density field in 3D in that compressible turbulence in three dimensions will be described by a Kolmogorov-type power law for the density fluctuations. Thus, low-frequency incompressible velocity fluctuations can drive high-frequency compressible fluctuations, which, when fully developed, behave like a driven passive scalar and consequently exhibit a Kolmogorov-type spectrum. This, we suggest, may be the origin of the ubiquitous $k^{-5/3}$ power-law density spectrum observed in the diffuse ISM.

(2) Turbulent motions associated with a low Mach number, subsonic, compressible ISM fluid may lead to the formation of large-scale coherent vortical flow fields through an inverse cascade of energy in the decaying turbulence. These quasiadiabatic compressible structures are driven by purely incompressible fluid fluctuations that vary on rapid time scales. Long-time-scale slowly varying ISM flows emerge from the saturation of nonlinear interactions, which hinder further spectral transfer of the energy in an inertial range ISM turbulence. The ISM density and the temperature fluctuations, on the other hand, remain turbulent and convect passively.

(3) The density-temperature anticorrelations in our simulations further suggest that a large fraction of the ISM fluid is not in a thermodynamic equilibrium. This could lead to thermal instabilities that are believed to be one possible source of ISM turbulence.[31]

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[1] J. W. Armstrong, J. M. Cordes, and B. J. Rickett, "Density power spectrum in the local interstellar medium," Nature (London) **291**, 561 (1981).
[2] J. W. Armstrong, W. A. Coles, M. Kojima, and B. J. Rickett, "Observations of field-aligned density fluctuations in the inner solar wind," Astrophys. J. **358**, 685 (1990).
[3] A. N. Kolmogorov, "On degeneration of isotropic turbulence in an incompressible viscous liquid," Dokl. Akad. Nauk SSSR **31**, 538 (1941).
[4] R. H. Kraichnan, "Inertial range spectrum in hydromagnetic turbulence," Phys. Fluids **8**, 1385 (1965).
[5] W. H. Matthaeus, L. Klein, S. Ghosh, and M. Brown, "Nearly incompressible magnetohydrodynamics, pseudosound, and solar wind fluctuations," J. Geophys. Res. **96**, 5421 (1991).
[6] G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, England, 1970).
[7] D. C. Montgomery, M. R. Brown, and W. H. Matthaeus, "Density fluctuation spectra in magnetohydrodynamic turbulence," J. Geophys. Res. **92**, 282 (1987).
[8] G. P. Zank and W. H. Matthaeus, "Nearly incompressible hydrodynamics and heat conduction," Phys. Rev. Lett. **64**, 1243 (1990).
[9] G. P. Zank and W. H. Matthaeus, "The equations of nearly incompressible fluids. I. Hydrodynamics, turbulence, and waves," Phys. Fluids A **3**, 69 (1991).
[10] G. P. Zank and W. H. Matthaeus, "Nearly incompressible fluids. II. Magnetohydrodynamics, turbulence, and waves," Phys. Fluids A **5**, 257 (1993).
[11] G. P. Zank, W. H. Matthaeus, and L. W. Klein, "Temperature and density anti-correlations in solar wind fluctuations," Geophys. Res. Lett. **17**, 1239 (1990).
[12] W. H. Matthaeus, J. W. Bieber, and G. P. Zank, "Unquiet on any front:

Anisotropic turbulence in the solar wind," Rev. Geophys. Space Phys. **33**, 609 (1995).
[13] G. P. Zank and W. H. Matthaeus, "Waves and turbulence in the solar wind," J. Geophys. Res. **97**, 17189 (1992).
[14] B. J. Bayly, C. D. Levermore, and T. Passot, "Density variations in weakly compressible flows," Phys. Fluids A **4**, 945 (1992).
[15] S. Dastgeer and G. P. Zank, "Density spectrum in the diffuse interstellar medium and solar wind," Astrophys. J. Lett. **602**, L29 (2004).
[16] S. Dastgeer and G. P. Zank, "Anisotropic density fluctuations in a nearly incompressible hydrodynamic fluid," Astrophys. J. Lett. **604**, L125 (2004).
[17] S. Dastgeer and G. P. Zank, "Nonlinear flows in nearly incompressible hydrodynamic fluids," Phys. Rev. E **69**, 066309 (2004).
[18] B. G. Elmegreen and J. Scalo, "Interstellar turbulence. I. Observations and processes," Annu. Rev. Astron. Astrophys. **42**, 211 (2004).
[19] V. D. Larichev and G. M. Reznik, "Two-dimensional solitary Rossby waves," Dokl. Akad. Nauk SSSR **231**, 37603 (1976).
[20] E. Marsch and C.-Y. Tu, "Spectral and spatial evolution of compressible turbulence in the inner solar wind," J. Geophys. Res. **95**, 11945 (1990).
[21] W. H. Matthaeus, M. L. Goldstein, and D. A. Roberts, "Evidence for the presence of quasi-two-dimensional nearly incompressible fluctuations in the solar wind," J. Geophys. Res. **95**, 20673 (1990).
[22] R. Grappin, A. Mangeney, and E. Marsch, "On the origin of solar wind MHD turbulence—HELIOS data revisited," J. Geophys. Res. **95**, 8197 (1990).
[23] C.-Y. Tu, E. Marsch, and H. Rosenbauer, "Temperature fluctuation spectra in the inner solar wind," Adv. Geophys. **9**, 748 (1991).
[24] C.-Y. Tu and E. Marsch, "On the nature of compressive fluctuations in the solar wind," J. Geophys. Res. **99**, 21481 (1994).
[25] B. Bavassano, R. Bruno, and L. W. Klein, "Density-temperature correlation in solar wind magnetohydrodynamic fluctuations: A test for nearly incompressible models," J. Geophys. Res. **100**, 5871 (1995).
[26] M. J. Lighthill, "On sound generated aerodynamically. I. General theory," Proc. R. Soc. London, Ser. A **211**, 564 (1952).
[27] R. H. Kraichnan and D. Montgomery, "Two-dimensional turbulence," Rep. Prog. Phys. **43**, 547 (1980).
[28] M. Lesieur, *Turbulence in Fluids*, 2nd ed. (Kluwer Academic, Dordrecht, 1990).
[29] S. R. Spangler, "Multi-scale plasma turbulence in the diffuse interstellar medium," Space Sci. Rev. **99**, 261 (2001).
[30] L. Klein, R. Bruno, B. Bavassano, and H. Rosenbauer, "Scaling of density fluctuations with Mach number and density-temperature anticorrelations in the inner heliosphere," J. Geophys. Res. **98**, 7837 (1993).
[31] A. G. Kritsuk and M. L. Norman, "Thermal instability-induced interstellar turbulence," Astrophys. J. Lett. **569**, L127 (2002).