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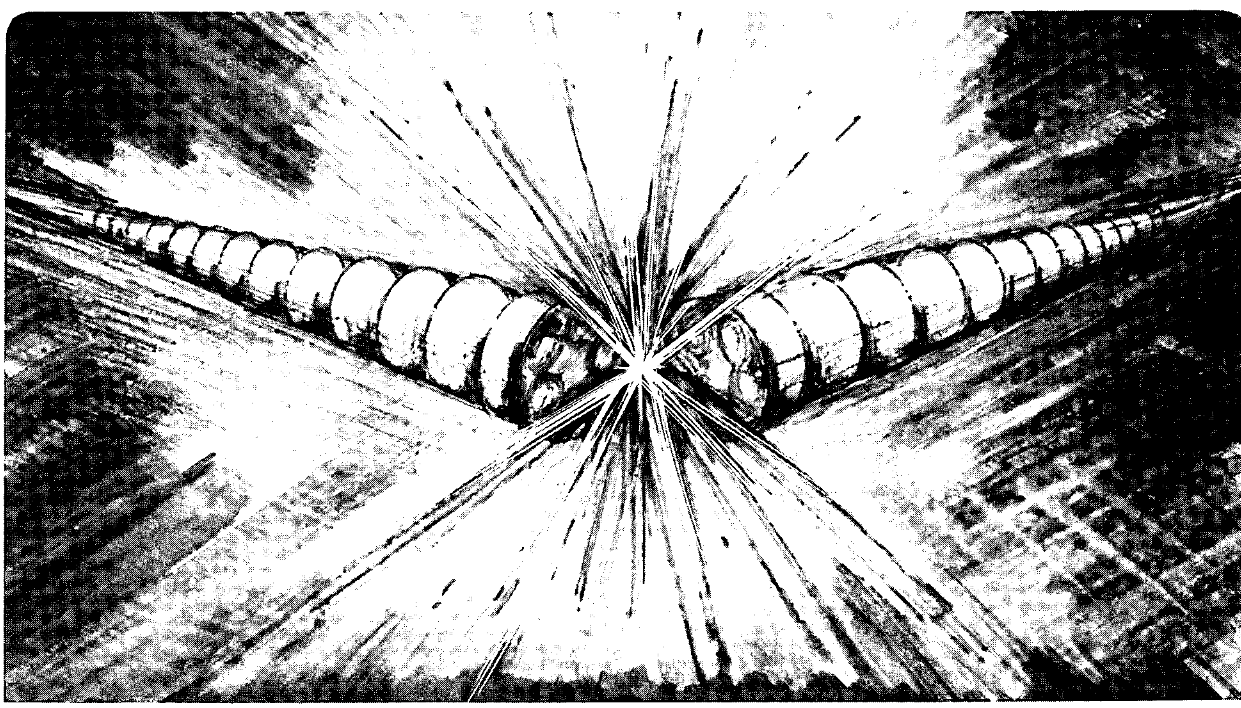

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A Concise Expression for Calculating Classical Radiation

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Abstract

In this paper we present a very simple formula for calculating the classical electromagnetic radiation of a moving charge. It is shown to be generally equivalent to the often used and much more complicated formula derived from the Lienard-Wiechert potentials. This formula will be of practical value in numerical calculations of the spectral output of accelerated charges. The advantages of this formula for analytical and numerical applications are discussed and the bending magnet synchrotron radiation spectrum is calculated.

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1 Introduction

The desire to use the electromagnetic radiation of moving charges has led to the construction of many synchrotron radiation light sources[1,2]. To utilize effectively this radiation one must be able to calculate its characteristics in detail[3–8], especially in synchrotron radiation research, where specially designed magnetic structures are used to control electron motion[9,1–4]. The geometry of this kind of problem and the symbols used are shown in Fig. 1. One of the basic properties of interest is the spectrum, defined by the energy radiated into an unit frequency interval and unit area or solid angle. Equations(1) and (2) are used to calculate this spectrum. They are well known[10] and widely used in spite of their complexity.

$$\frac{d^2 I}{d\omega dA} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\left\{\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2 R}+\frac{(\mathbf{n}-\boldsymbol{\beta})c}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^2 R^2}\right\}e^{i\omega(\tau+\frac{R(\tau)}{c})}d\tau\right|^2 \quad (1)$$

and

$$\frac{d^2 I}{d\omega d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\,e^{i\omega(\tau-\frac{\mathbf{n}\cdot\mathbf{x}_e}{c})}d\tau\right|^2. \quad (2)$$

Equation(1) is exact and Eq.(2) is a simplified form valid only for the infinitely far-field region. Equation(2) is the most widely used form because of its simplicity. However, in this paper we present the following expression:

$$\frac{d^2 I}{d\omega dA} = \frac{\alpha\hbar\,\omega^4}{4\pi^2 c^2}\left|\int_{-\infty}^{+\infty}\mathbf{n}(ret)\,e^{i\omega t}dt\right|^2 \quad (3)$$

where t is the observer time and "ret" means $\mathbf{n}$ should be calculated at the retarded time τ. This form is much simpler than but generally equivalent to Eq.(1).

As a related exercise, we present a proof of a concise expression of the electric field of a moving charge that appeared in the Feynman lectures[11]

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0}\left\{\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right\}_{ret}. \quad (4)$$

This form is remarkable because the third term, which describes the main radiation field, is simply a second derivative of the direction vector from the radiating charge to the observer. The formula Eq.(3), which has not been described in the literature, can be derived from Feynman's expression. In section 2, we show that Eq.(4) is identical to the more complicated, but widely used expression:[10]

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\{\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3Rc}+\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3R^2}\}_{ret}. \tag{5}$$

In section 3 we derive Eq.(3) and discuss its validity and potential advantages in analytical and numerical calculations. Finally, in section 4, Eq.(3) is used to calculate the well-known synchrotron radiation spectrum from a bending magnet[10,12].

A brief historical review of Eq.(4) may be interesting. Though rarely used, It was first obtained by Heaviside in 1902 and rediscovered by Feynman in 1950. A proof of Eq.(4) directly from the four-vector potential of electromagnetic field was given in Ref. 13.

2 The Heaviside-Feynman expression of electric field of moving charge

Here we begin with Eq.(4) from Feynman and show that it is equivalent to Eq.(5). For brevity, we drop the constant factor $\frac{e}{4\pi\epsilon_0}$ in this section.

The relationship between the observer's time t and the particle time τ is:

$$t=\tau+R(\tau)/c. \tag{6}$$

Differentiating this equation we get:

$$\frac{dt}{d\tau}=1-\mathbf{n}\cdot\boldsymbol{\beta}. \tag{7}$$

Since $1-\mathbf{n}\cdot\boldsymbol{\beta}$ is always positive, Eq.(7) guarantees there is an one-to-one mapping between t and τ. Thus we can change between them whenever necessary. From the definition of $\mathbf{n}$

and Eq.(7) we find that:

$$\frac{d\mathbf{n}}{d\tau} = \frac{c}{R}\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}). \tag{8}$$

Using this equation, the first two terms of Eq.(4) can be changed into:

$$\begin{aligned}\frac{\mathbf{n}}{R^2} + \frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2} &= \frac{\mathbf{n}}{R^2}\frac{1-\mathbf{n}\cdot\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}} + \frac{R}{c}\frac{d\tau}{dt}\left(\frac{d\mathbf{n}}{d\tau}\frac{1}{R^2} - \frac{2\mathbf{n}}{R^3}\frac{dR}{d\tau}\right) \\ &= \frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) + \mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})]\end{aligned} \tag{9}$$

The third term in Eq.(4) may be expressed in particle time τ as:

$$\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2} = \frac{1}{c}\frac{d\tau}{dt}\frac{d}{d\tau}\left[\frac{1}{c}\frac{d\tau}{dt}\frac{d\mathbf{n}}{d\tau}\right] = \frac{1}{c}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})}\frac{d}{d\tau}\left[\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\right] \tag{10}$$

Calculating this derivative is tedious. It is accomplished by using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ repeatedly, by differentiating term-by-term, and by collecting all $\dot{\boldsymbol{\beta}}$ terms together. Via this process we obtain:

$$\frac{d}{d\tau}\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}} = \frac{\mathbf{n} \times [(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} + \frac{1}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\left[(\dot{\mathbf{n}}\cdot\boldsymbol{\beta})\mathbf{n} + (\mathbf{n}\cdot\boldsymbol{\beta})\dot{\mathbf{n}} + \frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}(\dot{\mathbf{n}}\cdot\boldsymbol{\beta})\right] \tag{11}$$

The term $\dot{\mathbf{n}}\cdot\boldsymbol{\beta}$ in Eq.(11) can be calculated by using Eq.(8) as:

$$\begin{aligned}-\tfrac{R}{c}\dot{\mathbf{n}}\cdot\boldsymbol{\beta} &= [(\mathbf{n} \times \boldsymbol{\beta}) \times \mathbf{n}]\cdot\boldsymbol{\beta} \\ &= (\mathbf{n} \times \boldsymbol{\beta})^2 \\ &= \beta^2 - (\mathbf{n}\cdot\boldsymbol{\beta})^2\end{aligned} \tag{12}$$

With these relations we can finish the differentiation in Eq.(11) and obtain:

$$\frac{d}{d\tau}\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}} = \frac{\mathbf{n} \times [(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2} - \frac{c}{R}\left[\frac{(\mathbf{n} \times \boldsymbol{\beta})^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\mathbf{n} + \frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}(\beta^2 - \mathbf{n}\cdot\boldsymbol{\beta})\right] \tag{13}$$

Thus, the third term in Eq.(4) is:

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2} &= \frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})c}\left[\frac{1}{R}\frac{d}{d\tau}\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}} - \frac{1}{R^2}\frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\frac{dR}{d\tau}\right] \\ &= \frac{\mathbf{n} \times [(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3 Rc} - \frac{1}{R^2}\left\{\frac{(\mathbf{n} \times \boldsymbol{\beta})^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\mathbf{n} + \frac{\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\left[1 - \frac{1}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\right]\right\}\end{aligned} \tag{14}$$

Combining Eq.(9) and Eq.(14), and using Eq.(12), we obtain the relationship:

$$\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}=\frac{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]}{(1-\mathbf{n}\cdot\beta)^3Rc}+\frac{\mathbf{n}-\beta}{\gamma^2(1-\mathbf{n}\cdot\beta)^3R^2} \tag{15}$$

which shows that Feynman's expression, Eq.(4), is exactly equal to the generally used expression, Eq.(5). Feynman interestingly interpreted the first two terms of Eq.(4) as the static Coulomb field and the first-order correction to it. The third term gives the radiation field. Though remarkably simple, it is difficult to interpret. In the next section we will present a simple but important application of the Heaviside-Feynman expression.

3 New Radiation Spectrum Calculation Formula

The primary goal of most radiation calculations is to obtain the energy spectrum. It is well known that the spectral distribution of the total energy radiated into an unit area and unit frequency range by a moving charge is given by:

$$\frac{d^2I}{d\omega dA}=\frac{\epsilon_0 c}{\pi}\left|\int_{-\infty}^{+\infty}\mathbf{E}\,e^{i\omega t}dt\right|^2 \tag{16}$$

The traditional way to calculate the spectral distribution is to use Eq.(5), with τ as the variable, which leads to Eq.(1), the classic expression used for radiation calculations. However, Eq.(1) is rather complicated in form. Thus, the simpler form, Eq.(2), which is based on a far-field approximation, is widely used. Both Eq.(1) and (2) are based on Eq.(5). The expression in Eq.(4) can also be used for the electric field in Eq.(16). So a new way to do this is to use Eq.(4) and the observer time t as the independent variable, that is:

$$\frac{d^2I}{d\omega dA}=\frac{1}{4\pi\epsilon_0}\frac{e^2c}{4\pi^2}\left|\int_{-\infty}^{+\infty}[\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}]e^{i\omega t}dt\right|^2. \tag{17}$$

To show that the first two terms in Eq.(17) are negligible, we first use Eq.(8),(9) and the derivative theorem of the Fourier transform:[14]

$$\int_{-\infty}^{+\infty}\frac{d^m\mathbf{n}}{dt^m}e^{i\omega t}dt=(-i\omega)^m\int_{-\infty}^{+\infty}\mathbf{n}\,e^{i\omega t}dt \tag{18}$$

to obtain two different expressions for the integration part of Eq.(17):

$$\int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right]e^{i\omega t}dt$$

$$= \int_{-\infty}^{+\infty}\{\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})]-\frac{i\omega}{Rc}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\}e^{i\omega t}dt \qquad (19a)$$

$$= \int_{-\infty}^{+\infty}\{\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})]-\frac{\omega^2}{c^2}\mathbf{n}\}e^{i\omega t}dt \qquad (19b)$$

From Eq.(19a), if $R\gg\lambda=c/\omega$, we can omit the first term because it has the same direction as the third term and has a negligible magnitude. Similarly, from Eq.(19b), we see that the second term will be negligible also if the ratio of the second term to the third term,

$$\frac{1+\mathbf{n}\cdot\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}}(\frac{c}{R\omega})^2\leq\frac{2}{1-\beta}(\frac{\lambda}{R})^2\leq(\frac{2\gamma\lambda}{R})^2, \qquad (20)$$

is small, which is true when $R\gg 2\gamma\lambda$. From these arguments we conclude that if the observation distance satisfies the condition:

$$R\gg\gamma\lambda, \qquad (21)$$

then the first two terms in Eq.(17) are negligible. The third term is a complete second derivative of the vector $\mathbf{n}$, according to Eq.(18), we obtain the rather simple form[15], Eq.(3).

As a preliminary check of Eq.(3), we integrate it over frequency and get:

$$\begin{aligned}
\frac{dI}{dA} &= \int_0^{\infty}\frac{d^2I}{dAd\omega}d\omega \\
&= \frac{\alpha\hbar}{4\pi^2c^2}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dt'\mathbf{n}(t)\cdot\mathbf{n}(t')\frac{1}{2}\int_{-\infty}^{+\infty}d\omega\omega^4e^{i\omega(t-t')} \\
&= \frac{\alpha\hbar}{8\pi^2c^2}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}d\tau\mathbf{n}(t)\cdot\mathbf{n}(t+\tau)\int_{-\infty}^{+\infty}\omega^4e^{-i\omega\tau}d\omega \\
&= \frac{\alpha\hbar}{8\pi^2c^2}\int_{-\infty}^{+\infty}dt\mathbf{n}(t)\cdot[2\pi\frac{d^4}{d\tau^4}\mathbf{n}(t+\tau)]_{\tau=0} \\
&= \frac{\alpha\hbar}{4\pi c^2}\int_{-\infty}^{+\infty}dt\mathbf{n}\cdot\frac{d^4\mathbf{n}}{dt^4} \\
&= \frac{\alpha\hbar}{4\pi c^2}\int_{-\infty}^{+\infty}dt\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}} \\
&= \int_{-\infty}^{+\infty}(c\epsilon_0\mathbf{E}^2)dt, \qquad (22)
\end{aligned}$$

which is the expected result, the electromagnetic flux density integrated over time. Ref. 8 gives numerical verification of Eq.(3) and the related integration boundary problem concerned in numerical applications.

It is obvious that Eq.(21) is satisfied in most practical applications. Thus Eq.(3) is as good as the classic expression Eq.(1) in general. However, Eq.(3) is even simpler than Eq.(2), which is only valid for infinitely-far-field case. Moreover, the physical meaning of the direction vector $\mathbf{n}$ is quiet clear and, according to Eq.(3), the radiation spectrum is just the Fourier transform of $\mathbf{n}$. So, we can obtain a great deal of insight into the properties of radiation from knowledge of the Fourier transform and the particle trajectory. For example, in an undulator, electrons undulate periodically along a straight orbit[4]. According to Eq.(3), it is evident that the radiation will be linearly polarized if the electron moves in a plane and elliptically polarized if the electron moves in a spiral. Moreover, from the properties of the Fourier transform we know that the spectrum will consist of peaks having the same width, which is inversely proportional to the number of peroids of the device.

An important advantage of Eq.(3) for numerical calculations is that the spectrum can be calculated efficiently with the fast Fourier transform[14]. Another advantage is that Eq.(3) is applicable to the near-field case[5,6], where numerical methods are usually necessary because of the mathematical complexity. In addition, only the trajectory $\mathbf{n}$ is needed in Eq.(3) instead of $\mathbf{n}, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}$ in Eq.(1), so the requirement for storage of velocity and acceleration is removed when using Eq.(3) to calculate spectrum. Therefore, we believe that the new expression has fundamental importance for efficient radiation spectrum calculation algorithms. A detailed discussion of the numerical aspects of the new expression and its application in insertion device synchrotron radiation calculations appears in the Ref. 8. As an illustration of the method we will obtain the well known bending magnet synchrotron radiation spectrum in the next section using Eq.(3).

4 Bending Magnet Synchrotron Radiation Spectrum

The synchrotron radiation from a bending magnet is produced by a charged particle moving along a circular trajectory under the influence of an uniform magnetic field. We use the Cartesian coordinate system shown in Fig. 2 and assume that the charge moves in the x-z plane with a trajectory radius R_0 and a circular frequency ω_0. So we can write the trajectory as:

$$\begin{cases} x = R_0(\cos\omega_0\tau - 1) \\ y = 0 \\ z = R_0 \sin\omega_0\tau \\ \beta = \omega_0 R_0/c \simeq 1. \end{cases} \tag{23}$$

If the observer is located at $(0, r\theta, r)$, we can write the direction vector as:

$$\mathbf{n} = \frac{1}{R(\tau)}(\, R_0(1 - \cos\omega_0\tau),\ r\theta,\ r - R_0\sin\omega_0\tau\,), \tag{24}$$

where the distance between the observer and charge is:

$$\begin{aligned} R(\tau) &= \sqrt{1+\theta^2}\left\{(r - \frac{R_0\sin\omega_0\tau}{1+\theta^2})^2 + (\frac{2R_0\sin\frac{\omega_0\tau}{2}}{\sqrt{1+\theta^2}})^2 - (\frac{R_0\sin\omega_0\tau}{1+\theta^2})^2\right\} \\ &= \sqrt{1+\theta^2}(r - \frac{R_0\sin\omega_0\tau}{1+\theta^2}) + o(\frac{R_0\omega_0\tau}{r}). \end{aligned} \tag{25}$$

Because $\omega_0\tau \ll 1$ and $R_0 \leq r$, the higher order terms can be omitted. Therefore, to the lowest order of $\omega_0\tau$, Eq.(24) can be written as:

$$\mathbf{n} = (\frac{R_0}{r}\frac{1}{2}(\omega_0\tau)^2,\ \theta + \frac{R_0}{r}\theta\,\omega_0\tau,\ 1). \tag{26}$$

To get the function $\tau(t)$ used in Eq.(3), we have to solve Eq.(6). Usually it is difficult to derive an analytical function for $\tau(t)$. This is the main factor limiting the use of Eq.(3) in analytical calculations. However, in the present case we are able to get a sufficiently accurate solution. Using the above expression for $R(\tau)$ and dropping the constant term we get:

$$
\begin{aligned}
t &= \tau - \frac{R_0}{c}(1+\theta^2)^{-1/2}\sin(\omega_0\tau) \\
&= \frac{1}{2\gamma^2}[1+(\gamma\theta)^2]\tau + \frac{1}{6\omega_0}(\omega_0\tau)^3 + o((\omega_0\tau)^3),
\end{aligned} \tag{27}
$$

where the last identity of Eq.(23) is used. As previous derivations[10], we just keep up to the third order term of $\omega_0\tau$,

$$
(\omega_0\tau)^3 + 3\gamma^{-2}[1+(\gamma\theta)^2]\omega_0\tau - 6\omega_0 t = 0 \tag{28}
$$

yielding:

$$
\begin{aligned}
\omega_0\tau &= \gamma^{-1}[1+(\gamma\theta)^2]^{\frac{1}{2}}[(\sqrt{\eta^2+1}+\eta)^{\frac{1}{3}} - (\sqrt{\eta^2+1}-\eta)^{\frac{1}{3}}] \\
\eta &= 3\omega_0\gamma^3[1+(\gamma\theta)^2]^{-\frac{3}{2}}\, t.
\end{aligned} \tag{29}
$$

Defining

$$
\begin{aligned}
\omega_c &= \tfrac{3}{2}\gamma^3\omega_0, \\
\xi &= \tfrac{\omega}{2\omega_c}[1+(\gamma\theta)^2]^{\frac{3}{2}},
\end{aligned} \tag{30}
$$

we have $\omega t = \xi\eta$. Using Eqs.(30), (23), Eq.(26) and Eq.(29), and dropping constant terms, Eq.(3) becomes:

$$
\frac{d^2 I}{d\omega d\Omega} = \frac{\alpha\hbar}{4\pi^2}\left|\frac{\omega}{\omega_0}\xi\int_{-\infty}^{+\infty}(\frac{1}{2}(\omega_0\tau)^2,\ \theta\,\omega_0\tau)\, e^{i\xi\eta}d\eta\right|^2 \tag{31}
$$

The x component in the absolute square is:

$$
\begin{aligned}
& \frac{\omega}{\omega_0}\xi\frac{1}{2}\int_{-\infty}^{+\infty}(\omega_0\tau)^2\, e^{i\xi\eta}d\eta \\
= & \frac{\omega}{\omega_0}\xi\gamma^{-2}[1+(\gamma\theta)^2]\int_0^{\infty}[(\sqrt{\eta^2+1}+\eta)^{\frac{1}{3}} - (\sqrt{\eta^2+1}-\eta)^{\frac{1}{3}}]^2\cos\xi\eta d\eta \\
= & \frac{\omega}{\omega_c}\gamma[1+(\gamma\theta)^2]\frac{3}{2}\xi\cdot\frac{2}{\sqrt{3}}\frac{1}{\xi}K_{\frac{2}{3}}(\xi)
\end{aligned} \tag{32}
$$

So,

$$
\frac{d^2 I_x}{d\omega d\Omega} = \frac{3\alpha\hbar}{4\pi^2}\gamma^2(\frac{\omega}{\omega_c})^2[1+(\gamma\theta)^2]^2 K_{\frac{2}{3}}^2(\xi) \tag{33}
$$

Similarily, the y component is:

$$
\begin{aligned}
& \frac{\omega}{\omega_0}\xi\theta \int_{-\infty}^{+\infty} \omega_0 \tau \, e^{i\xi\eta} d\eta \\
= & \frac{\omega}{\omega_0}\theta\gamma^{-1}\sqrt{1+(\gamma\theta)^2}\xi \, 2i \int_0^{\infty} [(\sqrt{\eta^2+1}+\eta)^{\frac{1}{3}} - (\sqrt{\eta^2+1}-\eta)^{\frac{1}{3}}] \sin\xi\eta d\eta \quad (34) \\
= & -i\sqrt{3}\frac{\omega}{\omega_c}\gamma\gamma\theta\sqrt{1+(\gamma\theta)^2} \cdot K_{\frac{1}{3}}(\xi)
\end{aligned}
$$

and

$$
\frac{d^2 I_y}{d\omega d\Omega} = \frac{3\alpha\hbar}{4\pi^2}\gamma^2(\frac{\omega}{\omega_c})^2(\gamma\theta)^2[1+(\gamma\theta)^2]K^2_{\frac{1}{3}}(\xi) \quad (35)
$$

We see that the y component of the electric field is retarded in phase by $\pi/2$ relative to the x component when $\theta > 0$, i.e. above the orbit plane. Eqs.(33) and (35) are the same as the standard results. Detail calculation of the Fourier transform in Eq.(32) and Eq.(34) are shown in the Appendix.

5 Summary and Conclusions

In this paper we developed a new method of calculating the classical radiation of moving charges. It is a clearer and concise relationship between radiation properties and the trajectory of the radiating charge. The new approach may not make analytical calculations easier because of the difficulty to get an analytical function $\tau(t)$. But, it does simplify numerical calculations significantly when the FFT is applicable.

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Appendix

To calculate the Fourier transforms in Eq.(32) and Eq.(34), we introduce two symbols:

$$\eta_{\pm} = \sqrt{\eta^2+1} \pm \eta \tag{36}$$

and notice that $\eta_+\eta_- = 1$ and $\eta_+ + \eta_- = 2\sqrt{\eta^2+1}$. The Fourier transform in Eq.(32) is:

$$\int_0^\infty (\eta_+^{\frac{1}{3}} - \eta_-^{\frac{1}{3}})^2 \cos\xi\eta \, d\eta \tag{37}$$

$$\begin{aligned}
&= \int_0^\infty (\eta_+^{\frac{2}{3}} + \eta_-^{\frac{2}{3}} - 2)\cos\xi\eta \, d\eta \\
&= \int_0^\infty \frac{(\eta_+^{\frac{2}{3}} + \eta_-^{\frac{2}{3}})(\eta_+ + \eta_-)}{2\sqrt{\eta^2+1}} \cos\xi\eta \, d\eta - 2\pi\delta(\xi) \\
&= \frac{1}{2}\int_0^\infty \left(\frac{\eta_+^{\frac{5}{3}} + \eta_-^{\frac{5}{3}}}{\sqrt{\eta^2+1}} + \frac{\eta_+^{\frac{1}{3}} + \eta_-^{\frac{1}{3}}}{2\sqrt{\eta^2+1}}\right)\cos\xi\eta \, d\eta - 2\pi\delta(\xi) \\
&= \cos\frac{\pi}{6}\left[K_{\frac{2}{3}-1}(\xi) - K_{\frac{2}{3}+1}(\xi)\right] - 2\pi\delta(\xi) \\
&= \frac{2}{\sqrt{3}}\frac{1}{\xi}K_{\frac{2}{3}}(\xi) - 2\pi\delta(\xi),
\end{aligned} \tag{38}$$

(39)

where the identities[16,17]

$$\int_0^\infty \frac{(\sqrt{x^2+\beta^2}+x)^\nu + (\sqrt{x^2+\beta^2}-x)^\nu}{\sqrt{x^2+\beta^2}} \cos\alpha x \, dx = 2\beta^\nu \cos\frac{\nu\pi}{2} K_\nu(\alpha\beta), \tag{40}$$

$$K_{\nu-1}(\xi) - K_{\nu+1}(\xi) = \frac{2\nu}{\xi}K_\nu(\xi), \tag{41}$$

$$K_{-\nu}(\xi) = K_\nu(\xi), \tag{42}$$

are used. Similarly, the Fourier transform in Eq.(34) is found with the identity[16]

$$\int_0^\infty \frac{(\sqrt{x^2+\beta^2}+x)^\nu - (\sqrt{x^2+\beta^2}-x)^\nu}{\sqrt{x^2+\beta^2}} \sin\alpha x\, dx = 2\beta^\nu \sin\frac{\nu\pi}{2} K_\nu(\alpha\beta). \qquad (43)$$

Reference

1. H. Winick and S. Doniach, Eds, Synchrotron Radiation Research (Plenum, New York, 1979)
2. International Conference on Synchrotron Radiation Instrumentation (4th, 1991, Chester, U.K.), Review of Scientific Instruments, V.63,no.1,11A
3. International Conference on Insertion Devices for Synchrotron Radiation Source, SPIE Proc. 582(1985)
4. H. Kitamura, Jpn. J. Appl. Phys. 19(4)(1980)L185
5. Y. Hirai, A. Luccio and L. Yu, J. Appl. Phys. 55(1984)25
6. R. P. Walker, Nucl. Instr. and Meth. A267(1988)537
7. C. Wang and D. Xian, Nucl. Instr. and Meth. A288(1990)649
8. C. Wang and Y. Xiao, Proceedings of the International Conference on Synchrotron Radiation Sources, (Indore, 1992)
9. R. Bonifacio etc. eds, Undulator Magnets for Synchrotron Radiation and Free Electron Lasers, (World Scientific, Teanech, N.J. 1988)
10. J.D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975)
11. R. P. Feynman, The Feynman Lectures on Physics, (Addison-Wesley)
12. J. Schwinger, Phys. Rev. 75(1949)1912

13. J. J. Monaghan, J. Phys. A (Proc. Phys. Soc.), 2(1)(1968)112

14. Ronald N. Bracewell, The Fourier Transform and its Applications, 2nd ed. (Mc Graw-Hill Publishing Company, 1986)

15. It is interesting to notice that $\mathbf{n}$ can be viewed as the momentum direction of the photons received by the observer while $\hbar\omega/c$ is the magnitude of the momentum. So we can express Eq.(3) as:

$$\frac{d^2 I}{d\omega dA} = \frac{\alpha}{\hbar} \left|\mathrm{F}[\dot{\mathbf{p}}]\right|^2 ,$$

where $\dot{\mathbf{p}}$ is the observed rate of change of the photon momentum, and $\mathrm{F}[\cdots]$ represents the Fourier transform.

16. I.S. Gradshteyn, Table of Integrals, Series and Products, (Academic Press, New York, 1980)

17. M. Abramowitz, Handbook of Mathematical Functions (Dover, New York, 1971)

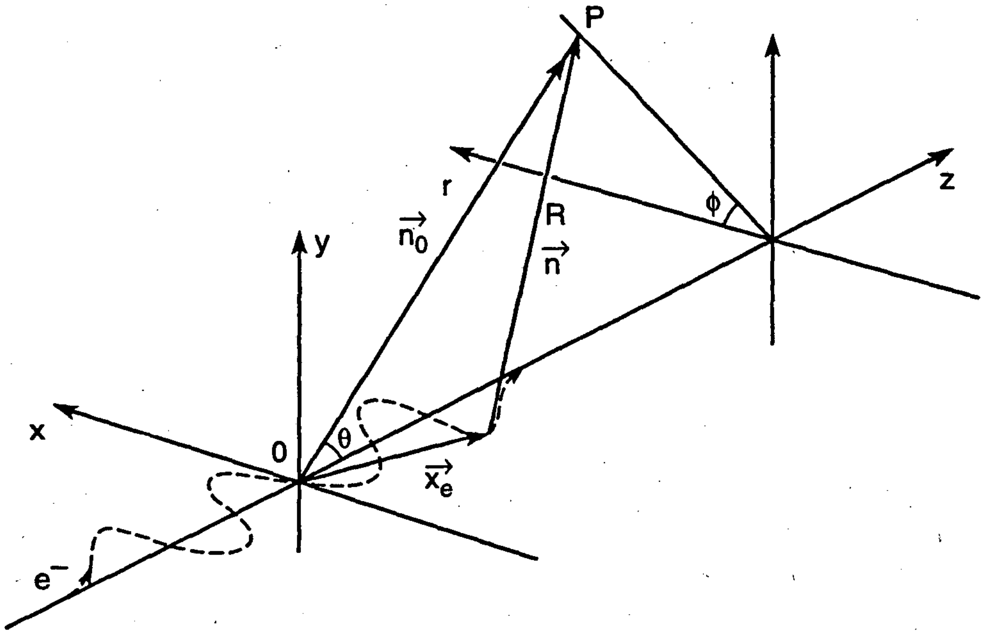

Fig.1 Reference frame and symbols used in analysis

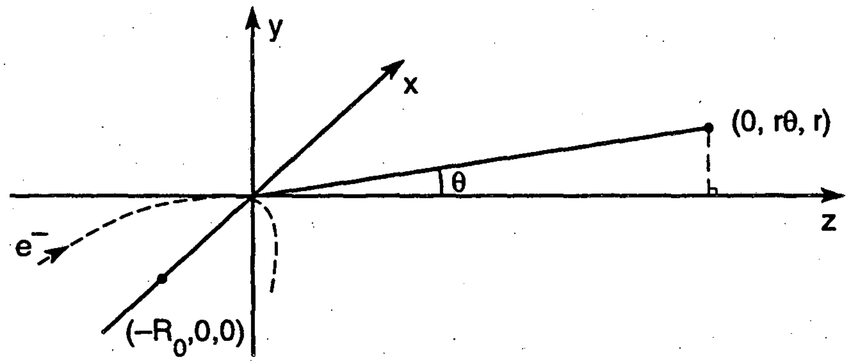

Fig.2 Coordinate system used in section 4

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