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The Wiles Defect for Principal Series Deformation Rings

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The Wiles Defect for Principal Series Deformation Rings

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by

Ethan Alwaise

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ABSTRACT OF THE DISSERTATION

The Wiles Defect for Principal Series Deformation Rings

by

Ethan Alwaise

Doctor of Philosophy in Mathematics

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Professor Chandrashekhar Khare, Chair

Wiles' proof of Fermat's last theorem boils down to proving the existence of a ring isomorphism $R \to T$, where R is a Galois deformation ring and T is a Hecke algebra acting on a space of cusp forms. This relies on a numerical criterion for such a map to be an isomorphism of complete intersections.

In [3] and [4], the authors study contexts where R and T are not complete intersections, thus the Wiles numerical criterion cannot hold. They quantify the failure of the numerical criterion by computing the associated *Wiles defect* in terms of the local behavior of a global Galois representation ρ_f associated to a modular form f. We use the methods of [4] to compute the Wiles defect in the case where we demand that the given modular representation ρ_f is of *principal series type* at a fixed set of primes.

The dissertation of Ethan Alwaise is approved.

Haruzo Hida

Romyar Thomas Sharifi

William Drexel Duke

Chandrashekhar Khare, Committee Chair

University of California, Los Angeles

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Contents

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Vita

Ethan Alwaise graduated from Emory University in 2017 with a M.S. and B.S. degree in mathematics. He was a National Science Foundation Graduate Research Fellow from 2017 to 2022.

Chapter 1

Introduction

Fermat's last theorem states that if $n \geq 3$ is an integer and $a, b, c \in \mathbb{Z}$ are integers which are a solution to the Fermat equation of degree n

$$a^n + b^n = c^n,$$

then at least one of a, b, c is equal to 0. Although Pierre de Fermat originally stated his last theorem in the margin of a copy of *Arithmetica* around 1637, he did not provide proof, and it is widely agreed that he never found a proof. After over three centuries of effort by a number of mathematicians, Andrew Wiles proved Fermat's last theorem in [42] by showing that all semistable elliptic curves over $\mathbb{Q}$ are modular. Wiles' original proof in fact contained a gap, but with the help of his graduate student Richard Taylor, Wiles was able to remedy this gap in [38].

Today the more general modularity theorem is known, which guarantees that every elliptic curve $E/\mathbb{Q}$ is modular [5], meaning that for some integer $N \geq 1$, there exists a surjective morphism (defined over $\mathbb{Q}$) of algebriac curves $X_0(N) \to E$, where $X_0(N)$ is the modular curve obtained by compactifying the quotient of the upper half-plane in $\mathbb{C}$ by the action of the congruence subgroup $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$. The modularity theorem was first conjectured in an imprecise form by Yutaka Taniyama at the 1955 international symposium on algebraic number theory in Tokyo and Nikkō. Goro Shimura and Taniyama collaborated until 1957 to improve the rigor of the conjecture. André Weil further built upon the conjecture [41], adding the stipulation that the integer N could be taken to be the conductor of E. At the time of Wiles' proof, the Taniyama-Shimura-Weil conjecture was considered out of reach.

A priori, there is no obvious relationship between Fermat's last theorem and the theory of elliptic curves. In 1975, Yves Hellegouarch introduced in [18] the idea of associating a solution $(a, b, c) \in \mathbb{Z}^3$ to the Fermat equation of prime degree ℓ to the elliptic curve $E/\mathbb{Q}$ defined by the equation

$$y^2 = x(x - a^\ell)(x + b^\ell).$$

In 1986, Gerhard Frey, iterating upon this idea, had the insight that the above curve could link Fermat's last theorem to the Shimura-Taniyama-Weil conjeture [12]. More specifically, Frey observed that E is *semistable* (i.e. the conductor of E is square-free), and the number field obtained by adjoining the ℓ-torsion points of E to $\mathbb{Q}$ is ramified only at 2 and ℓ. Frey suggested that the exotic properties of the elliptic curve E would lead to a contradiction if E were known to be modular. The above curve is known as a Frey curve due to Frey's contribution.

In 1987, Serre studied mod ℓ Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ for primes $\ell > 3$ [35]. In particular, he was able to procure a precise list of properties of the mod ℓ representation $\overline{\rho}_{E,\ell} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ (where E is a Frey curve) which would lead to the contradiction imagined by Frey. More precisely, the representation $\overline{\rho}_{E,\ell}$ should be irreducible, unramified outside 2 and ℓ, finite flat at ℓ, and the image of inertia at 2 should have order ℓ. Serre predicted that $\overline{\rho}_{E,\ell}$ should arise from a cusp form of level 2 and weight 2 on $\Gamma_0(2)$. Such forms do not exist, however, as they correspond to holomorphic differentials of the modular curve $X_0(2)$, which has genus 0. Obtaining this contradiction, however, requires a result that one can "lower the level" of a newform $f \in \Gamma_0(qN)$ (where $N \geq 1$ is an integer and q is a prime not dividing N) to a newform $g \in \Gamma_0(N)$ such that the associated Galois representations ρ_g and ρ_g are equivalent modulo ℓ. Serre left this level-lowering result as a conjecture, which came to be known as the "ϵ-conjecture". In 1990, Ribet proved the ϵ-conjecture [32], thus putting the connection between Fermat's last theorem and the Taniyama-Shimura-Weil conjecture on solid ground and setting the stage for Wiles' 1995 proof.

Given an elliptic curve $E/\mathbb{Q}$, for each prime p one obtains a 2-dimensional p-adic representation $\rho_{E,p}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{Q}_p)$ from the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Tate module $T_p(E) = \varprojlim_n E[p^n]$ (with a choice of basis). A priori, the modularity of E means that $\rho_{E,p}$ is modular (i.e. is equivalent up to semisimplification to the p-adic representation attached to a newform) for all primes p. However, using Faltings' isogeny theorem, one can show that the modularity of E is equivalent to the modularity of $\rho_{E,p}$ for a single prime p. Wiles' proof relies on choosing a prime p such that $\overline{\rho}_{E,p}$ (the reduction of $\rho_{E,p}$ modulo p) is modular, and showing that under certain conditions, this implies that $\rho_{E,p}$ itself is modular.

Wiles' approach is to reframe the problem in a more general context. Let p be a prime and let $\mathcal{O}$ be the ring of integers in some finite extension of $\mathbb{Q}_p$ with uniformizer ϖ and residue field k. Suppose $\overline{\rho}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(k)$ is a continuous, absolutely irreducible Galois representation which is modular. Rather than studying a specific lift of $\overline{\rho}$, one wishes to consider lifts of $\overline{\rho}$ to rings in $\mathrm{CNL}_{\mathcal{O}}$, the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k. Given a finite set of primes Σ, one can consider the *deformation functor* $\mathcal{D}_{\overline{\rho}}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ which sends a ring $R \in \mathrm{CNL}_{\mathcal{O}}$ to all lifts $\rho\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(R)$ which reduce to $\overline{\rho}$ modulo the maximal ideal of R (up to an equivalence relation) and are of "type Σ" (meaning unramified outside $\Sigma \cup S$, where S is the set of primes at which $\overline{\rho}$ is ramified). An equivalence class of lifts is called a *deformation* of $\overline{\rho}$. Work of Mazur [29] shows that $\mathcal{D}_{\overline{\rho}}$ is representable. The representating object R_Σ is called the *universal deformation ring* of $\overline{\rho}$ and satisfies the universal property that there exists a *universal deformation* $\rho_\Sigma\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(R_\Sigma)$ such that if $\rho\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(R)$ is any deformation of $\overline{\rho}$ with $R \in \mathrm{CNL}_{\mathcal{O}}$, then ρ factors through ρ_Σ via a unique morphism $R_\Sigma \to R$.

Rather than show that a specific deformation of $\overline{\rho}$ is modular, as is the goal, Wiles shows that in fact all deformations of $\overline{\rho}$ are. To this end, Wiles constructs a universal "modular" deformation ring $\mathbb{T}_\Sigma \in \mathrm{CNL}_{\mathcal{O}}$ which parameterizes modular deformations of $\overline{\rho}$ of type Σ. The ring $\mathbb{T}_\Sigma$ is a Hecke algebra acting on a suitable space of cusp forms, localized at a maximal ideal corresponding to a cusp form f such that $\overline{\rho}_f = \overline{\rho}$ (recall that $\overline{\rho}$ is assumed to be modular). Wiles then constructs a deformation $\rho_{\Sigma,\mathrm{mod}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_\Sigma)$ of type Σ. If g is some other cusp form such that $\overline{\rho}_g = \overline{\rho}$, then g induces an augmentation $\mathbb{T}_\Sigma \to \mathcal{O}$ via which $\overline{\rho}_g$ factors through $\rho_{\Sigma,\mathrm{mod}}$. In this sense the ring $\mathbb{T}_\Sigma$ parameterizes modular deformations of $\overline{\rho}$. The universal property of R_Σ then induces a unique morphism $\varphi\colon R_\Sigma \to \mathbb{T}_\Sigma$. The task of showing that all deformations of $\overline{\rho}$ are modular thus becomes equivalent to showing that φ is an isomorphism. Such a result is often referred to as a "$R = T$" theorem.

One easily shows that φ is surjective, and thus one only needs to show injectivity. Wiles accomplishes this in the minimal level case ($\Sigma = \varnothing$) via an ingenius commutative algebra argument. Given an augmentation $\lambda\colon R \to \mathcal{O}$ for $R \in \mathrm{CNL}_{\mathcal{O}}$, one defines the invariants:

$$\Phi_R = \ker \lambda / (\ker \lambda)^2$$
$$\Psi_R = \mathcal{O}/\lambda(\mathrm{Ann}_R(\ker \lambda)).$$

The invariants Φ_λ and Ψ_λ are called the *cotangent space* and *congruence module*, respectively. The key to proving the $R = T$ theorem is the *Wiles numerical criterion*. As it turns out, that $R_\Sigma \cong \mathbb{T}_\Sigma$ is intertwined with the fact that both rings are complete intersections. Suppose we have surjective morphism $\phi\colon R \to T$ in $\mathrm{CNL}_{\mathcal{O}}$ such that T is a finitely generated torsion-free $\mathcal{O}$-module and $\Psi_R \neq 0$. Suppose further that there is an augmentation $\lambda\colon T \to \mathcal{O}$ (which defines an augmentation $R \to \mathcal{O}$ via composition with ϕ. Then the conditions that ϕ is an isomorphism and R and T are complete intersections are together equivalent to the numerical criterion $|\Phi_R| = |\Psi_T|$. In fact surjectivity alone implies that $|\Phi_R| \geq |\Psi_T|$, so one only needs $|\Phi_R| \leq |\Psi_T|$.

The proof that $R_\Sigma \cong \mathbb{T}_\Sigma$ proceeds by induction on the cardinality of Σ. The minimal level case ($\Sigma = \varnothing$) is proved via the *Taylor-Wiles patching method.* The remaining cases are proved by bounding the growth of the cotangent space and congruence module as the size of Σ is increased so as to show that the numerical criterion continues to hold. The Hecke algebra $\mathbb{T}_\varnothing$ acts on a space of cusp forms $M_\varnothing$, hence $R_\varnothing$ also acts on $M_\varnothing$. If one can show that the action of $R_\varnothing$ on $M_\varnothing$ is free, then since this action factors through the surjective map $\varphi\colon R_\varnothing \to \mathbb{T}_\varnothing$, it follows that φ is an isomorphism.

One considers auxillary sets Q of *Taylor-Wiles primes* at which ramification is allowed. For $q \in Q$ let Δ_q be the maximal p-power quotient of $\mathbb{F}_q^\times$ and let $\Delta_Q = \prod_{q\in Q} \Delta_q$. Local class field theory gives an action of the group ring $\mathcal{O}[\Delta_Q]$ on R_Q (and thus on $\mathbb{T}_Q$). Moreover, there is a commutative diagram

$$\begin{array}{ccc} R_Q & \longrightarrow & \mathbb{T}_Q \\ \downarrow & & \downarrow \\ R_\varnothing & \longrightarrow & \mathbb{T}_\varnothing, \end{array}$$

where all maps are surjective and the kernels of the vertical maps are $\mathfrak{a}_Q R_Q$ and $\mathfrak{a}_Q \mathbb{T}_Q$, respectively, where $\mathfrak{a}_Q$ is the augmentation ideal of $\mathcal{O}[\Delta_Q]$. For each $n \geq 1$, one chooses Taylor-Wiles sets Q_n with prescribed properties. One would like to take an "inverse limit" over the above diagrams as $n \to \infty$, but a priori there are no obvious maps between the rings for different choices of Taylor-Wiles sets. However, using a pigeonhole principle argument, one can extract a subsequence of the Q_n which gives a compatible system. This produces a "patched" ring R_∞ which is an algebra over a power series ring $S_\infty = \mathcal{O}[[y_1, \ldots, y_d]]$ for some integer d. Similarly, one patches T_{Q_n}-modules M_{Q_n}, which produces a patched R_∞-module M_∞. Critically, $R_\infty/\mathfrak{a}R_\infty$ acts freely on $M \cong M_\infty/\mathfrak{a}M_\infty \cong M_\varnothing$, where $\mathfrak{a} = (y_1, \ldots, y_d) \subseteq S_\infty$. Moreover, this action factors through a surjective map $R_\infty/\mathfrak{a}R_\infty \to R_\varnothing$, which gives that

$R_\varnothing$ acts freely on $M_\varnothing$ as desired.

Wiles' proof relied on the freeness of the action of the Hecke algebra T. In general, freeness can be difficult to establish. However, Diamond and showed that in fact one only needs that the action of T is faithful. In [23], Mark Kisin widely expanded the scope of patching techniques by patching spaces of modular forms rather than Hecke algebras and adopting the viewpoint that the global deformation ring R should be viewed as an algebra over a completed tensor product of local deformation rings. The patching method is often referred to as the Taylor-Wiles-Kisin patching method due to the significance of Kisin's contributions. The argument can be also be streamlined using the *ultrapatching method* introduced by Peter Scholze [34], which replaces the role of the pigeonhole principle.

The work of Wiles inspired a large body of working focused on proving "modularity lifting theorems", which aim to prove the modularity of certain lifts of a starting residual representation $\overline{\rho}$. Serre conjectured [35] that any absolutely irreducible odd representation $\overline{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$ over a finite field k should arise from a eigenform of specified weight and level. In 2006, Chandrashekhar Khare proved Serre's conjecture in the level 1 case [21], and in 2011 he and Jean-Pierre Wintenberger proved the general level N case [22]. Very recently, Ana Caraiani and James Newton proved the modularity of all elliptic curves over infinitely many imaginary quadratic fields [7].

In [3], Böeckle, Khare, and Manning investigate Hecke algebras T which act on the cohomology of Shimura curves arising from maximal orders in indefinite quaternion algebras over $\mathbb{Q}$. They consider a newform f contributing to cohomology of a Shimura curve, which gives an augmentation $\lambda\colon T \to \mathcal{O}$. One obtains a surjective map $R \to T$, where R is the universal deformation ring which parameterizes deformations of $\overline{\rho}_f$ which satisfy additional local conditions (namely, deformations which are *Steinberg* at a fixed set of primes). In this context, the Taylor-Wiles-Kisin patching method still gives that $R \to T$ is an isomorphism, but the rings are not complete intersections and the Wiles numerical criterion cannot hold. This leads the authors to define the *Wiles defect* to be the quotient $\delta_\lambda(R) = \delta_\lambda(T) = |\Phi_R|/|\Psi_T|$, which measures the degree to which the rings R and T fail to be complete intersections. The authors prove a theorem which gives a formula for the Wiles defect in terms of the behavior of the local behavior of the representation ρ_f at the primes at which the Steinberg condition is imposed.

In [4], Böeckle, Khare, and Manning significantly generalize the results of [3] and give a proof which shows a priori that the global Wiles defect $\delta_\lambda(R)$ is a sum of local Wiles defects. In an unpublished work [40], Akshay Venkatesh showed that when the ring R is of dimension 1, the Wiles defect $\delta_\lambda(R)$ is given in terms of two invariants. The authors of [4] extend the definition of the Wiles defect to rings to higher dimensional rings by quotienting by regular sequences to reduce to the 1-dimensional case. In this work, we use the methods of [4] to compute the Wiles defect of a deformation ring R which parameterizes *principal series* deformations of a modular residual representation $\overline{\rho}$. As in [4], we use the Taylor-Wiles-Kisin patching method to prove a $R = T$ theorem which allows us to show that the global Wiles defect $\delta_\lambda(R) = \delta_\lambda(T)$ is a sum of local Wiles defects. As in the Steinberg case, we find a formula for the local Wiles defect in terms of the local behavior of a representation ρ (arising from an augmetation $R \to T \to \mathcal{O}$) at a set of places Σ^{ps} at which the principal series condition is imposed. More precisely, we show that the Wiles defect is given by

$$\delta_\lambda(R) = \delta_\lambda(T) = \sum_{v \in \Sigma^{\text{ps}}} \frac{n_v}{e},$$

where e is the ramification index of $E/\mathbb{Q}_p$, and for each $v \in \Sigma^{\text{ps}}$, n_v is the greatest nonnegative integer such that $\rho \pmod{\varpi^{n_v}}$ is scalar.

1.1 Structure of the thesis

This thesis is organized as follows: in Chapter 2, we recall some background information about deformations of Galois representations and the representability of deformation functors, and we define the specific deformation rings we will consider in this work. In Chapter 3, we, as in [4], define the Wiles defect for rings not necessarily of dimension 1 and record some basic properties of the Wiles defect. In Chapter 4, we compute the Wiles defect of a local deformation ring corresponding to the principal series inertial type at an augmentation arising from a Hilbert modular form of parallel weight 2. In Chapter 5, we use the Taylor-Wiles-Kisin patching method to prove a $R = T$ theorem, where R is a global deformation ring parameterizing principal series deformations of a modular residual representation $\overline{\rho}$, and T is a Hecke algebra acting on a cohomology group. As a consequence, we deduce that the Wiles defect of R and T is a sum of the local defects computed in Chapter 4.

1.2 Notation

Here we fix some notation which we will use throughout this work. We will frequently remind the reader of much of this notation.

For a field L, we let $G_L = \mathrm{Gal}(\overline{L}/L)$ for some fixed separable closure $\overline{L}$ of L. We denote by F our base field, which will be a totally real number field, and fix an algebraic closure $\overline{F}$ of F. For a place v of F, we let F_v be the completion of F at v with $\overline{F}_v$ a fixed algebraic closure of F_v, and we let k_v be the residue field of F_v, with $q_v = |k_v|$. We fix embeddings $\overline{F} \to \overline{F}_v$, which determines embeddings $G_{F_v} \to G_F$. We let $P_{F_v} \subseteq I_{F_v}$ be the wild inertia and inertia subgroups of G_{F_v}, respectively. We let ϕ_v, ι_v be generators of the tame quotient $G^t_{F_v} = G_{F_v}/P_{F_v}$, where ϕ_v is a lift of Frobenius (Frob_v) and ι_v is a topological generator of I_{F_v}/P_{F_v} satisfying $\phi_v \iota_v \phi_v^{-1} = \iota_v^{q_v}$.

We fix an odd prime p, and we let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$. We let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. We denote by Σ_p the set of places of F above p.

We define $\mathrm{CNL}_{\mathcal{O}}$ to be the category of complete Noetherian local $\mathcal{O}$-algebras R with maximal ideal $\mathfrak{m}_R$ and a fixed isomorphism $R/\mathfrak{m}_R \to k$. For a ring $R \in \mathrm{CNL}_{\mathcal{O}}$, by an augmentation λ of R we mean a surjective $\mathcal{O}$-algebra homomorphism $\lambda\colon R \twoheadrightarrow \mathcal{O}$.

We let $\varepsilon_p\colon G_F \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character. By abuse of notation, we also write ε_p to mean any character obtained by composing ε_p with $G_{F_v} \to G_F$ on the left or $\mathbb{Z}_p^\times \to R^\times$ on the right, where v is a place of F and $\mathbb{Z}_p^\times \to R^\times$ is a map induced by a morphism $\mathbb{Z}_p \to R$ in $\mathrm{CNL}_{\mathbb{Z}_p}$.

Chapter 2

Galois deformation rings

In this chapter we recall the notion of a deformation functor of a residual Galois representation and show that under certain conditions, these functors are representable by objects which we call Galois deformation rings. We fix an odd prime p and a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k.

2.1 Deformation functors

In this section we recall the notion of a deformation functor which sends a ring R to the set of lifts of a fixed residual representation $\overline{\rho}$ to R (sometimes up to equivalency). This material can also be found in [17].

Fix a profinite group G (which in cases of interest to us will always be the Galois group of some extension of $\mathbb{Q}$ or $\mathbb{Q}_p$). We fix a continuous representation

$$\overline{\rho}\colon G \to \mathrm{GL}_2(k).$$

Let $\mathrm{CNL}_{\mathcal{O}}$ be the category of complete Noetherian local $\mathcal{O}$-algebras R with maximal ideal $\mathfrak{m}_R$ and residue field $R/\mathfrak{m}_R \cong k$. Technically speaking, objects in $\mathrm{CNL}_{\mathcal{O}}$ are pairs (R, ι_R) where R is a complete Noetherian local $\mathcal{O}$-algebra and $\iota_R\colon R/\mathfrak{m}_R \to k$ is a fixed $\mathcal{O}$-algebra isomorphism. Morphisms in $\mathrm{CNL}_{\mathcal{O}}$ are then continuous local $\mathcal{O}$-algebra homomorphisms $f\colon R \to S$ such that $\iota_S \circ (f \pmod{\mathfrak{m}_S}) = \iota_R$. We will usually simply identify the object (R, ι_R) with the ring R.

Given a ring $R \in \mathrm{CNL}_{\mathcal{O}}$, a *lift of* $\overline{\rho}$ *to* R is a continuous representation $\rho\colon G \to \mathrm{GL}_2(R)$ such that $\rho \pmod{\mathfrak{m}_R} = \overline{\rho}$, i.e. the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_2(R) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow{\scriptstyle \iota_R \circ \pi_R} \\ & & \mathrm{GL}_2(k) \end{array} \tag{2.1}$$

commutes, where $\pi_R\colon R \twoheadrightarrow R/\mathfrak{m}_R$ is the projection. We define the *framed deformation functor*

$$\mathcal{D}_{\overline{\rho}}^{\square}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$$

by $\mathcal{D}_{\overline{\rho}}^{\square}(R) = \{\text{lifts of } \overline{\rho} \text{ to } R\}$.

We need another deformation functor which only considers lifts of $\overline{\rho}$ up to a conjugacy relation. We say that two lifts $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(R)$ of $\overline{\rho}$ to R are *strictly equivalent* if there exists some $\gamma \in \ker(\mathrm{GL}_2(R) \twoheadrightarrow \mathrm{GL}_2(k))$ such that $\rho_2 = \gamma^{-1}\rho_1\gamma$. Note that if $\rho\colon G \to \mathrm{GL}_2(R)$ is a lift, then for any $\gamma \in \ker(\mathrm{GL}_2(R) \twoheadrightarrow \mathrm{GL}_2(k))$, the diagram (2.1) still commutes when ρ is replaced by $\gamma^{-1}\rho\gamma$, i.e. $\gamma^{-1}\rho\gamma$ is a lift. We call an equivalence class of lifts a *deformation of* $\overline{\rho}$ *to* R. We then define the *unframed deformation functor*

$$\mathcal{D}_{\overline{\rho}}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$$

by $\mathcal{D}_{\overline{\rho}}(R) = \{\text{deformations of } \overline{\rho} \text{ to } R\}$. Note that if $f\colon R \to S$ is a morphism in $\mathrm{CNL}_{\mathcal{O}}$, then for any $\gamma \in \ker(\mathrm{GL}_2(R) \twoheadrightarrow \mathrm{GL}_2(k))$ we have $f(\gamma) \in \ker(\mathrm{GL}_2(S) \twoheadrightarrow \mathrm{GL}_2(k))$, so $\mathcal{D}_{\overline{\rho}}$ is well-defined on morphisms.

The deformation functors $\mathcal{D}_{\overline{\rho}}^{\square}$ and $\mathcal{D}_{\overline{\rho}}$ satisfy the following continuity property

Proposition 2.1.1. *Let $\mathcal{D}$ be either of the functors $\mathcal{D}_{\overline{\rho}}^{\square}$ or $\mathcal{D}_{\overline{\rho}}$. Then for any $R \in \mathrm{CNL}_{\mathcal{O}}$ we* *have*

$$\mathcal{D}(R) = \varprojlim_n \mathcal{D}(R/\mathfrak{m}_R^n).$$

Proof. Since R is complete, we have

$$\varprojlim_n \mathrm{GL}_2(R/\mathfrak{m}_R^n) = \mathrm{GL}_2(R).$$

It follows that we have a canonical bijection

$$\mathcal{D}_{\overline{\rho}}^{\square}(R) \to \varprojlim_n \mathcal{D}_{\overline{\rho}}^{\square}(R/\mathfrak{m}_R^n)$$

given by sending a lift $\rho\colon G \to \mathrm{GL}_2(R)$ to the compatible sequence of lifts ($\rho \pmod{m_R^n}$). This establishes the proposition for $\mathcal{D} = \mathcal{D}_{\overline{\rho}}^{\square}$.

To establish the proposition for $\mathcal{D} = \mathcal{D}_{\overline{\rho}}$, let $\Gamma_n = \ker(\mathrm{GL}_2(R/\mathfrak{m}_R^n) \twoheadrightarrow \mathrm{GL}_2(k))$ for $n \geq 1$ and $\Gamma = \ker(\mathrm{GL}_2(R) \twoheadrightarrow \mathrm{GL}_2(k))$. If $\rho\colon G \to \mathrm{GL}_2(R)$ is a lift and $\gamma \in \Gamma$, each of the lifts $\rho \pmod{m_R^n}$ is equivalent to $\gamma^{-1}\rho\gamma \pmod{m_R^n}$ by $\gamma \pmod{m_R^n}$. Therefore we have a well-defined map

$$\mathcal{D}_{\overline{\rho}}(R) \to \varprojlim_n \mathcal{D}_{\overline{\rho}}(R/\mathfrak{m}_R^n).$$

If $\rho_1, \rho_2\colon G \to \mathrm{GL}_2(R)$ are lifts such that $\rho_1 \pmod{m_R^n}$ and $\rho_2 \pmod{m_R^n}$ are strictly equivalent for all n, then we can inductively choose $\gamma_n \in \Gamma_n$ such that $\gamma_{n+1} \equiv \gamma_n \pmod{m_R^n}$ and $\rho_1 \pmod{m_R^n}$ and $\rho_2 \pmod{m_R^n}$ are equivalent by γ_n. Since $\Gamma = \varprojlim_n \Gamma_n$, we see that (γ_n) defines an element $\gamma \in \Gamma$ such that ρ_1 and ρ_2 are equivalent by γ. This proves injectivity.

Suppose $(\rho_n\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^n))$ is a compatible sequence of deformations. Note that the equivalence class ρ_1 contains only $\overline{\rho}$. Assume by induction that for $1 \leq i \leq m$ we have lifts $r_i\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^i)$ such that $r_{i+1} \equiv r_k \pmod{\mathfrak{m}_R^i}$ and r_i is a representative of the strict equivalence class ρ_i. Let $r'_{m+1}\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^{m+1})$ be any lift representing ρ_{m+1}. Since

$(\rho_n\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^n))$ is a compatible sequence, there exists $\gamma_m \in \Gamma_m$ such that

$$\gamma_m^{-1}(r'_{m+1} \pmod{\mathfrak{m}_R^m})\gamma_m = r_m.$$

Choose a lift $\gamma_{m+1} \in \Gamma_{m+1}$ of γ_m and let $r_{m+1} = \gamma_{m+1}^{-1} r'_{m+1} \gamma_{m+1}$ so that $r_{m+1} \equiv r_m \pmod{\mathfrak{m}_R^m}$. By induction there exists a compatible sequence of lifts $(r_n\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^n))$ such that r_n represents ρ_n. The induced lift $r\colon G \to \mathrm{GL}_2(R)$ satisfies $r \pmod{\mathfrak{m}_R^n} \equiv r_n$, so the deformation $\rho\colon G \to \mathrm{GL}_2(R)$ represented by r maps to $(\rho_n\colon G \to \mathrm{GL}_2(R/\mathfrak{m}_R^n))$. This proves surjectivity. $\square$

2.2 Representability of deformation functors

In this section we show that the deformation functors $\mathcal{D}_{\overline{\rho}}^{\square}, \mathcal{D}_{\overline{\rho}}$ defined in the previous section are representable under certain conditions. Again this material can be found in [17].

Suppose the functor $\mathcal{D} = \mathcal{D}_{\overline{\rho}}^{\square}, \mathcal{D}_{\overline{\rho}}$ is representible by some object $R_{\overline{\rho}} \in \mathrm{CNL}_{\mathcal{O}}$, i.e. $\mathcal{D} \cong \mathrm{Hom}_{\mathrm{CNL}_{\mathcal{O}}}(R_{\overline{\rho}}, \cdot)$ naturally. Then the identity morphism on $R_{\overline{\rho}}$ corresponds to a unique lift (deformation) $\rho\colon G \to \mathrm{GL}_2(R)$ with the following universal property: given any $A \in \mathrm{CNL}_{\mathcal{O}}$ and a lift (deformation) $\rho \in \mathcal{D}(A)$, there exists a unique morphism $f\colon R_{\overline{\rho}} \to A$ such that $\rho = f \circ \rho$. The lift (deformation) ρ is thus universal in the sense that all lifts (deformations) of $\overline{\rho}$ factor uniquely through ρ, and so the ring R "parameterizes" all lifts (deformations) of $\overline{\rho}$.

Let $\mathrm{AL}_{\mathcal{O}}$ denote the full subcategory of $\mathrm{CNL}_{\mathcal{O}}$ whose objects are Artinian local $\mathcal{O}$-algebras. If $R \in \mathrm{CNL}_{\mathcal{O}}$, then $R/\mathfrak{m}_R^n \in \mathrm{AL}_{\mathcal{O}}$ for all n, so Proposition 2.1.1 tells us that the deformation functors are determined by their values on $\mathrm{AL}_{\mathcal{O}}$. This is crucial as showing that our functors are representable requires checking their behavior on fiber products. Unfortunately, $\mathrm{CNL}_{\mathcal{O}}$ does not admit fiber products, as the fiber product of two rings in $\mathrm{CNL}_{\mathcal{O}}$ is not necessarily Noetherian. However, the category $\mathrm{AL}_{\mathcal{O}}$ does have fiber products.

Lemma 2.2.1. *Fiber products exist in the category* $\mathrm{AL}_{\mathcal{O}}$*.*

Proof. Suppose $f\colon A \to C, g\colon B \to C$ are morphisms in $\mathrm{AL}_{\mathcal{O}}$. We need to show that the ring

$$A \times_C B = \{(a,b) \in A \times B : f(a) = g(b)\}$$

belongs to $\mathrm{AL}_{\mathcal{O}}$.

Since f and g are $\mathcal{O}$-algebra homomorphisms, the compositions $\mathcal{O} \to A \xrightarrow{f} C$ and $\mathcal{O} \to B \xrightarrow{g} C$ are both equal to the $\mathcal{O}$-algebra structure map $\mathcal{O} \to C$, so $A \times_C B$ is an $\mathcal{O}$-algebra.

Consider the ideal

$$\mathfrak{m} = \{(a,b) \in A \times_C B : f(a) \in \mathfrak{m}_C\}$$

in $A \times_C B$. If $(a,b) \in A \times_C B$ is not in $\mathfrak{m}$, then $f(a) = g(b)$ is a unit in C, thus a and b are units since f and g are local. Therefore $A \times_C B$ is a local ring with maximal ideal $\mathfrak{m}$. Moreover, projecting onto either coordinate and composing with $C \twoheadrightarrow C/\mathfrak{m}_C \cong k$ yields an injective map $(A \times_C B)/\mathfrak{m} \hookrightarrow k$. Since A is an Artinian local ring, the map $A \twoheadrightarrow A/\mathfrak{m}_A \cong k$ has a section $k \to A$ and likewise for B. These sections induce a morphism $k \to A \times_C B$ such that the composition $k \to A \times_C B \twoheadrightarrow (A \times_C B)/\mathfrak{m} \hookrightarrow k$ is the identity, thus k is the residue field of $A \times_C B$.

Since A is Artinian, i.e. a finite length A-module, there is a finite composition series

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A.$$

Then each quotient I_m/I_{m-1} is a simple A-module, which must be isomorphic to $A/\mathfrak{m}_R \cong k$ since A is local. Then we see that $I_m/I_{m-1} \cong k$ as an $\mathcal{O}$-module as well, hence I_m/I_{m-1} is also a simple $\mathcal{O}$-module. The above sequence is therefore a composition series for A as an $\mathcal{O}$-module, hence A has finite length as an $\mathcal{O}$-module, and likewise for B. Then $A \times B$ is a finite length $\mathcal{O}$-module, hence so is $A \times_C B \subset A \times B$. Therefore $A \times_C B$ is a finite length $(A \times_C B)$-module, i.e. $A \times_C B$ is Artinian. $\square$

Suppose $\mathcal{F}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ is a functor which is continuous in the sense of Proposition 2.1.1. As we alluded to earlier, in the cases of interest to us, proving that $\mathcal{F}$ is representable will involve examining its behavior on the subcategory $\mathrm{AL}_{\mathcal{O}}$. Although the restriction $\mathcal{F}|_{\mathrm{AL}_{\mathcal{O}}}$ may not be representable, a slightly weaker condition will suffice. We say that $\mathcal{F}|_{\mathrm{AL}_{\mathcal{O}}}$ is *pro-representable* if there exists an object $R_{\mathcal{F}} \in \mathrm{CNL}_{\mathcal{O}}$ such that

$$\mathcal{F}(A) = \mathrm{Hom}_{\mathrm{CNL}_{\mathcal{O}}}(R_{\mathcal{F}}, A)$$

naturally for all $A \in \mathrm{AL}_{\mathcal{O}}$. Note that the continuity of $\mathcal{F}$ implies that $\mathcal{F}$ is represented by $R_{\mathcal{F}}$ if $\mathcal{F}|_{\mathrm{AL}_{\mathcal{O}}}$ is pro-represented by $R_{\mathcal{F}}$, and the converse is clear. Thus $\mathcal{F}$ is representable if and only if $\mathcal{F}|_{\mathrm{AL}_{\mathcal{O}}}$ is pro-representable.

To state the criteria for $\mathcal{F}$ to be representable, we need to introduce a "tangent space" construction for the functor $\mathcal{F}$. Let $k[\varepsilon] = k[X]/(X^2)$ be the *ring of dual numbers*, where $\varepsilon = X \pmod{X}^2$. Assume that $\mathcal{F}(k)$ is a singleton. Since $\mathcal{F}$ is a functor, there is a natural map

$$\mathcal{F}(k[\varepsilon] \times_k k[\varepsilon]) \to \mathcal{F}(k[\varepsilon]) \times_{\mathcal{F}(k)} \mathcal{F}(k[\varepsilon]) = \mathcal{F}(k[\varepsilon]) \times \mathcal{F}(k[\varepsilon]), \tag{2.2}$$

where the equality holds by the assumption that $\mathcal{F}(k)$ is a singleton. We assume that the above map is a bijection. Note that if $\mathcal{F}$ is representable, then this assumption holds by the universal property of the fiber product.

We want to use our assumption that (2.2) is a bijection to endow $\mathcal{F}(k[\varepsilon])$ with a natural k-vector space structure. For scalar multiplication, note that if $x \in k$, then $a+b\varepsilon \mapsto a+xb\varepsilon$ is an automorphism of $k[\varepsilon]$ as an object in $\mathrm{CNL}_{\mathcal{O}}$. By functoriality this induces an automorphism of $\mathcal{F}(k[\varepsilon])$, which we define scalar multiplication by.

For addition, note that the map

$$\alpha\colon k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon]$$

defined by $\alpha(a+b\varepsilon, a+d\varepsilon) = a+(b+d)\varepsilon$ is a morphism in $\mathrm{CNL}_{\mathcal{O}}$. We then define vector addition by the map

$$\mathcal{F}(k[\varepsilon]) \times \mathcal{F}(k[\varepsilon]) \cong \mathcal{F}(k[\varepsilon] \times_k k[\varepsilon]) \xrightarrow{\mathcal{F}(\alpha)} \mathcal{F}(k[\varepsilon]).$$

Proposition 2.2.2. *The addition and multiplication operations defined above make $\mathcal{F}(k[\varepsilon])$ into a k-vector space.*

Proof. Addition and scalar multiplication are associative since $\mathcal{F}$ is a functor. The composition

$$k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\mathrm{rev}} k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\alpha} k[\varepsilon]$$

where rev is the coordinate-reversing map is equal to α, so applying $\mathcal{F}$ shows that the addition is commutative.

The composition

$$k[\varepsilon] \xrightarrow{\Delta} k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\alpha} k[\varepsilon]$$

where Δ is the diagonal map is the identity on $k[\varepsilon]$. Then applying $\mathcal{F}$ and noting that $\mathcal{F}(k[\varepsilon]) = \mathcal{F}(k[\varepsilon]) \times \mathcal{F}(k)$ since $\mathcal{F}(k)$ is a singleton, we see that $\mathcal{F}(k)$ (which we can identify with an element of $\mathcal{F}(k[\varepsilon])$ by applying $\mathcal{F}$ to $k \to k[\varepsilon]$) is the additive identity.

Let $\mathrm{inv}\colon k[\varepsilon] \to k[\varepsilon]$ be the map defined by $a + b\varepsilon \mapsto a - b\varepsilon$. Then the composition

$$k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\mathrm{id}\times\mathrm{inv}} k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\alpha} k[\varepsilon]$$

factors through k (and likewise if the first map is replaced by $\mathrm{inv} \times \mathrm{id}$), which shows that additive inverses exist. Namely, the map $\mathcal{F}(\mathrm{inv})\colon \mathcal{F}(k[\varepsilon]) \to \mathcal{F}(k[\varepsilon])$ takes elements to their additive inverses.

Given $x \in k$, the automorphism $a + b\varepsilon \mapsto a + xb\varepsilon$ of $k[\varepsilon]$ induces a coordinate-wise automorphism of $k[\varepsilon] \times_k k[\varepsilon]$ such that the compositions

$$k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{x} k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\alpha} k[\varepsilon]$$

$$k[\varepsilon] \times_k k[\varepsilon] \xrightarrow{\alpha} k[\varepsilon] \xrightarrow{x} k[\varepsilon]$$

are the same, so applying $\mathcal{F}$ shows that scalar multiplication distributes over addition.

Finally, $1 \in k$ clearly acts as the identity on $\mathcal{F}(k[\varepsilon])$ by functoriality since the automorphism of $k[\varepsilon]$ defined by 1 is the identity. □

Remark 2.2.3. *The k-vector space $\mathcal{F}(k[\varepsilon])$ is referred to as the tangent space of the functor $\mathcal{F}$ because if $\mathcal{F}$ is represented by $R_{\mathcal{F}} \in \mathrm{CNL}_{\mathcal{O}}$, then there is a natural bijection*

$$\mathrm{Hom}_k(\mathfrak{m}_{R_{\mathcal{F}}}/(\mathfrak{m}^2_{R_{\mathcal{F}}}, \varpi), k) \cong \mathrm{Hom}_{\mathrm{CNL}_{\mathcal{O}}}(R_{\mathcal{F}}, R[\varepsilon])$$

of k-vector spaces. In the representability theorem we present below, the assumption that $\mathcal{F}(k[\varepsilon])$ is finite-dimensional is necessary to ensure the representing object is Noetherian.

We need one last bit of terminology to state the representability criteria. We say that a morphism $A \to B$ in $\mathrm{CNL}_{\mathcal{O}}$ is *small* if it is surjective and its kernel is principal and annihilated by $\mathfrak{m}_A$.

Let $A \to C$ and $B \to C$ be morphisms in $\mathrm{AL}_{\mathcal{O}}$. Since $\mathcal{F}$ is a functor, we have a map

$$\mathcal{F}(A \times_C B) \to \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B). \tag{2.3}$$

We state four conditions (known as the "hull axioms") regarding the properties of the above map.

- **H1**: If $B \to C$ is small, then (2.3) is surjective.
- **H2**: If $B = k[\varepsilon]$ and $C = k$, then (2.3) is bijective.
- **H3**: The k-vector space $\mathcal{F}(k[\varepsilon])$ is finite-dimensional.

- **H4**: If $A = B$ and the maps $A \to C$ and $B \to C$ are identical and small, then (2.3) is bijective.

The following theorem due to Schlessinger [33] is the tool we use to prove that our deformation functors are representable:

Theorem 2.2.4. *Let $\mathcal{F}\colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ and assume that $\mathcal{F}(k)$ is a singleton. If $\mathcal{F}$ satisfies the hull axioms **H1** through **H4**, then $\mathcal{F}|_{\mathrm{AL}_{\mathcal{O}}}$ is pro-representable.*

We will now check using the above theorem that our deformation functors $\mathcal{D}_{\overline{\rho}}$ and $\mathcal{D}_{\overline{\rho}}^{\square}$ are representable under certain conditions. We check that the four hull axioms hold in a series of lemmas as in [17]. In these proofs we use the notation $\Gamma(R)$ to denote the kernel of $\mathrm{GL}_2(R) \to \mathrm{GL}_2(k)$ for a ring $R \in \mathrm{CNL}_{\mathcal{O}}$.

Lemma 2.2.5. *The functor $\mathcal{D}_{\overline{\rho}}^{\square}$ satisfies **H1**, **H2**, **H4**.*

Proof. Let $f\colon A \to C$ and $g\colon B \to C$ be morphisms in $\mathrm{AL}_{\mathcal{O}}$. Then the map

$$\mathcal{D}_{\overline{\rho}}^{\square}(A \times_C B) \to \mathcal{D}_{\overline{\rho}}^{\square}(A) \times_{\mathcal{D}_{\overline{\rho}}^{\square}(C)} \mathcal{D}_{\overline{\rho}}^{\square}(B)$$

sends a lift $\rho\colon G \to \mathrm{GL}_2(A \times_C B) = \mathrm{GL}_2(A) \times_{\mathrm{GL}_2(C)} \mathrm{GL}_2(B)$ to (ρ_A, ρ_B), where ρ_A and ρ_B are the compositions of ρ with the projections onto $\mathrm{GL}_2(A)$ and $\mathrm{GL}_2(B)$, respectively. But note that ρ_A and ρ_B determine ρ completely. The above map is thus injective. Conversely, if $(\rho_A, \rho_B) \in \mathcal{D}_{\overline{\rho}}^{\square}(A) \times_{\mathcal{D}_{\overline{\rho}}^{\square}(C)} \mathcal{D}_{\overline{\rho}}^{\square}(B)$, then this is to say that $f \circ \rho_A = g \circ \rho_B$. Then $\rho\colon G \to \mathrm{GL}_2(A \times_C B), \rho(g) = (\rho_A(g), \rho_B(g))$ is a well-defined lift in $\mathcal{D}_{\overline{\rho}}^{\square}(A \times_C B)$ which is sent to (ρ_A, ρ_B) by the above map. This proves surjectivity. The above map is thus always a bijection (without any additional hypotheses on the rings A, B, C or the morphisms f, g), which simultaneously shows that $\mathcal{D}_{\overline{\rho}}^{\square}$ satisfies **H1**, **H2**, **H4**. □

Lemma 2.2.6. *The functor $\mathcal{D}_{\overline{\rho}}$ satisfies **H1**.*

Proof. Let $f\colon A \to C$ and $g\colon B \to C$ be morphisms in $\mathrm{AL}_{\mathcal{O}}$ with g small. Let $(\rho_A, \rho_B) \in \mathcal{D}_{\overline{\rho}}(A) \times_{\mathcal{D}_{\overline{\rho}}(C)} \mathcal{D}_{\overline{\rho}}(B)$. This is to say that if we choose lifts $r_A\colon G \to \mathrm{GL}_2(A)$ and $r_B\colon G \to \mathrm{GL}_2(B)$ representing ρ_A and ρ_B, then there exists some $\gamma_C \in \Gamma(C)$ such that

$$(g \circ r_B) = \gamma_C^{-1}(f \circ r_A)\gamma_C.$$

Since g is small, it is surjective, so $\Gamma(B) \to \Gamma(C)$ is surjective. Therefore there exists $\gamma_B \in \Gamma(B)$ with $g(\gamma_B) = \gamma_C$, so $r\colon G \to \mathrm{GL}_2(A \times_C B)$ defined by $r(g) = (r_A(g), \gamma_B^{-1} r_B(g)\gamma_B)$ indeed has image in $\mathrm{GL}_2(A \times_C B)$. The strict equivalence class of r thus maps to (ρ_A, ρ_B) under (2.3), which shows the desired surjectivity. □

Lemma 2.2.7. *The functor $\mathcal{D}_{\overline{\rho}}$ satisfies **H2**.*

Proof. The map $k[\varepsilon] \to k$ is small, so by Lemma 2.2.6 the map (2.3) with $B = k[\varepsilon], C = k$ is surjective. To prove injectivity, suppose $\rho, \rho' \in \mathcal{D}_{\overline{\rho}}(A \times_k k[\varepsilon])$ are such that their images $(\rho_1, \rho_2), (\rho_1', \rho_2') \in \mathcal{D}_{\overline{\rho}}(A) \times_{\mathcal{D}_{\overline{\rho}}(k)} \mathcal{D}_{\overline{\rho}}(k[\varepsilon])$ are equal. This is to say that if we choose lifts $r = (r_1, r_2)\colon G \to \mathrm{GL}_2(A \times_k k[\varepsilon])$ and $r' = (r_1', r_2')\colon G \to \mathrm{GL}_2(A \times_k k[\varepsilon])$ representing ρ and ρ', then there exist $\gamma_1 \in \Gamma(A)$ and $\gamma_2 \in \Gamma(k[\varepsilon])$ such that

$$r_1' = \gamma_1^{-1} r_1 \gamma_1, \qquad r_2' = \gamma_2^{-1} r_2 \gamma_2.$$

Since γ_1 and γ_2 both reduce to the identity in $\mathrm{GL}_2(k)$, we have $(\gamma_1, \gamma_2) \in \mathrm{GL}_2(A \times_k k[\varepsilon])$, so r and r' are strictly equivalent by (γ_1, γ_2). This shows that $\rho = \rho'$, which proves that (2.3) is injective. □

We say that a profinite group G is p-finite if for every open compact subgroup $H \subseteq G$, there exist only finitely many continuous homomorphisms $H \to \mathbb{F}_p$. This condition is necessary in order to ensure that our functors satisfy **H3** (which is needed to guarantee that the representing object is Noetherian).

Lemma 2.2.8. *If G is p-finite, then the functors $\mathcal{D}^{\square}_{\overline{\rho}}$ and $\mathcal{D}_{\overline{\rho}}$ satisfiy **H3**.*

Proof. Let $\rho\colon G \to \mathrm{GL}_2(k[\varepsilon])$ be a lift. Since $\rho \pmod{\epsilon} = \overline{\rho}$, for all $g \in G$ we have

$$\rho(g) = \overline{\rho}(g)(1 + M_g)$$

for some $M_g \in \varepsilon M_2(k)$. Moreover, for $g_1, g_2 \in G$ we have

$$M_{g_1 g_2} = M_{g_1} + M_{g_2} + M_{g_1} M_{g_2}.$$

For $g \in G$, write $g = g_0 h$ for $h \in \ker \overline{\rho}$. Then the above relation shows that M_g, hence $\rho(g)$, is determined by the values of ρ on $\ker \overline{\rho}$ and a set of left coset representatives for $\ker \overline{\rho}$.

We have a group isomorphism

$$\Gamma(k[\varepsilon]) = \left\{ \begin{pmatrix} 1 + a\varepsilon & b\varepsilon \\ c\varepsilon & 1 + d\varepsilon \end{pmatrix} : a, b, c, d \in k \right\} \cong k^4 \cong \mathbb{F}_p^{4n},$$

where $|k| = p^n$, thus

$$\mathrm{Hom}_{\mathrm{cts}}(\ker \overline{\rho}, \Gamma(k[\varepsilon])) = \mathrm{Hom}_{\mathrm{cts}}(\ker \overline{\rho}, \mathbb{F}_p)^{4n}.$$

Now $\ker \overline{\rho}$ is an open compact subgroup of G since $\mathrm{GL}_2(k)$ is finite, so $\mathrm{Hom}_{\mathrm{cts}}(\ker \overline{\rho}, \Gamma(k[\varepsilon])$ is finite by the assumption that G is p-finite. Now $\rho(\ker \overline{\rho}) \subseteq \Gamma(k[\varepsilon])$ since $\rho \pmod{\epsilon} = \overline{\rho}$, and $\ker \overline{\rho}$ has finite index in G, so it follow that $\mathcal{D}^{\square}_{\overline{\rho}}(k[\varepsilon])$ is a finite set. Then $\mathcal{D}_{\overline{\rho}}(k[\varepsilon])$ is also finite since $\mathcal{D}_{\overline{\rho}}(k[\varepsilon])$ is just the set of strict equivalence classes in $\mathcal{D}^{\square}_{\overline{\rho}}(k[\varepsilon])$. ☐

Unlike $\mathcal{D}^{\square}_{\overline{\rho}}$, the functor $\mathcal{D}_{\overline{\rho}}$ need not satisfy **H4** in general. For this to be true, we need an additional assumption on $\overline{\rho}$. Mazur proved the representability of $\mathcal{D}_{\overline{\rho}}$ under the hypothesis that $\overline{\rho}$ is absolutely irreducible [29]. Note that in this case, Schur's lemma implies that $\mathrm{End}_{k[G]}(\overline{\rho}) = k$. Ramakrishna noted in [31] that this weaker assumption is sufficient.

Lemma 2.2.9. *If* $\mathrm{End}_{k[G]}(\overline{\rho}) = k$*, then the functor* $\mathcal{D}_{\overline{\rho}}$ *satisfies* ***H4****.*

Proof. Let $f\colon A \to C$ be a small morphism in $\mathrm{AL}_{\mathcal{O}}$. Then by Lemma 2.2.6 the map (2.3) with $B = A$ is surjective. To prove injectivity, suppose $\rho, \rho' \in \mathcal{D}_{\overline{\rho}}(A \times_C A)$ are such that their images $(\rho_1, \rho_2), (\rho_1', \rho_2') \in \mathcal{D}_{\overline{\rho}}(A) \times_{\mathcal{D}_{\overline{\rho}}(C)} \mathcal{D}_{\overline{\rho}}(A)$ are equal. This is to say that if we choose lifts $r = (r_1, r_2)\colon G \to \mathrm{GL}_2(A \times_C A)$ and $r' = (r_1', r_2')\colon G \to \mathrm{GL}_2(A \times_k A)$ representing ρ and ρ', then there exist $\gamma_1, \gamma_2 \in \Gamma(A)$ such that

$$r_1' = \gamma_1^{-1} r_1 \gamma_1, \qquad r_2' = \gamma_2^{-1} r_2 \gamma_2.$$

Since $(\rho_1, \rho_2), (\rho_1', \rho_2') \in \mathcal{D}_{\overline{\rho}}(A) \times_{\mathcal{D}_{\overline{\rho}}(C)} \mathcal{D}_{\overline{\rho}}(A)$, $f(r_1)$ (resp. $f(r_1')$) is strictly equivalent to $f(r_2)$ (resp. $f(r_2')$). Since $\Gamma(A) \to \Gamma(C)$ is surjective, by conjugating by suitable elements of $\Gamma(A)$, we can in fact choose the representatives r_1, r_2, r_1', r_2' such that

$$f(r_1) = f(r_2), \qquad f(r_1') = f(r_2').$$

Then

$$f(r_1) = f(\gamma_1) f(r_1') f(\gamma_1^{-1}) = f(\gamma_1) f(r_2') f(\gamma_1^{-1}) = f(\gamma_1) f(\gamma_2^{-1}) f(r_2) f(\gamma_2) f(\gamma_1^{-1}).$$

Since $f(r_1) = f(r_2)$, this shows that $\delta = f(\gamma_1) f(\gamma_2^{-1})$ commutes with the image of $f(r_1)$, hence $\overline{\delta} = \delta \pmod{\mathfrak{m}_C}$ commutes with the image of $\overline{\rho}$. But $\mathrm{End}_{k[G]}(\overline{\rho}) = k$ by assumption, so $\overline{\delta}$ is scalar, hence $\overline{\delta} = 1$ since $\delta \in \Gamma(C)$. Since the kernel of f is annihilated by $\mathfrak{m}_A$, we may multiply γ_2 by a suitable scalar lift of $\overline{\delta}$ so that $f(\gamma_1) = f(\gamma_2)$. Then $(\gamma_1, \gamma_2) \in \mathrm{GL}_2(A \times_C A)$, and we still have

$$r_1' = \gamma_1^{-1} r_1 \gamma_1, \qquad r_2' = \gamma_2^{-1} r_2 \gamma_2.$$

This shows that r and r' are equivalent, thus $\rho = \rho'$, which proves that (2.3) is injective. □

The above lemmas show that $\mathcal{D}_\rho^\square$ and $\mathcal{D}_\rho$ satisfy all four hull axioms under the stated hypotheses, and we know that $\mathcal{D}^{\square}_{\rho}(k)$ and $\mathcal{D}_{\rho}(k)$ are singletons. Therefore from Theorem 2.2.4 we obtain the following theorem:

Theorem 2.2.10. *Suppose G is p-finite. Then $\mathcal{D}^{\square}_{\overline{\rho}}$ is representable. If in addition $\mathrm{End}_{k[G]}(\overline{\rho}) = k$ (in particular if $\overline{\rho}$ is absolutely irreducible), then $\mathcal{D}_{\overline{\rho}}$ is also representable.*

When the conditions of the above theorem are saitisifed, we make the following definition:

Definition 2.2.11. *We denote by $R^{\square}_{\overline{\rho}}$ (resp. $R_{\overline{\rho}}$) the representing object for $\mathcal{D}^{\square}_{\overline{\rho}}$ (resp. $\mathcal{D}_{\overline{\rho}}$), which we call the universal lifting ring (resp. universal deformation ring) of $\overline{\rho}$.*

We record the following special cases of the above theorem:

Theorem 2.2.12. *Let F be a number field and S be a finite set of finite places of F. Let F_S/F be the maximal extension of F unramified outside S and let F_v be the completion of F at any finite place v of F. If $G = \mathrm{Gal}(F_S/F)$ or $G = \mathrm{Gal}(\overline{F_v}/F)$, then $\mathcal{D}^{\square}_{\overline{\rho}}$ is representable. If in addition $\mathrm{End}_{k[G]}(\overline{\rho}) = k$ (in particular if $\overline{\rho}$ is absolutely irreducible), then $\mathcal{D}_{\overline{\rho}}$ is also representable.*

Proof. For $G = \mathrm{Gal}(F_S/F)$, recall that the Hermite-Minkowski theorem says that there are only finitely many extensions of F of a given degree which unramified outside S. Any continuous homomorphism $G \to \mathbb{F}_p$ corresponds to a degree p extension of F which is unramified outside S, so G is p-finite. If $G = \mathrm{Gal}(\overline{F_v}/F)$, then G is p-finite since G is topologically finitely generated. Therefore Theorem 2.2.12 applies in both cases. □

2.3 Local deformation conditions

Fix a totally real number field F which is unramified at p and an algebraic closure $\overline{F}$ of F. Let $G_F = \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F, and let $\epsilon_p\colon G_F \to \mathcal{O}^\times$ be the cyclotomic character. We fix a continuous, absolutely irreducible residual representation

$$\overline{\rho}\colon G_F \to \mathrm{GL}_2(k)$$

such that $\det \overline{\rho} = \epsilon_p$. Assume that k contains the eigenvalues of all elements in the image of $\overline{\rho}$.

For a place v of F we let G_{F_v} denote the absolute Galois group of F_v and we let $P_{F_v} \subset I_{F_v}$ denote the wild inertia and inertia subgroups, respectively. We recall that the tame quotient $G^t_{F_v} = G_{F_v}/P_{F_v}$ is topologically generated by two elements ϕ_v, ι_v, where ϕ_v is a lift of Frobenius and ι_v is a topological generator of $I_{F_v}/P_{F_v} \cong \mathcal{O}_{F_v}$. Moreover, the tame quotient is characterized by the relation $\phi_v \circ \iota_v \circ \phi_v^{-1} = \iota_v^{q_v}$, where q_v is the order of the residue field of F_v.

In the last section we considered all lifts of $\overline{\rho}$ to objects in $\mathrm{CNL}_{\mathcal{O}}$. We often want to consider only lifts which satisfy certain conditions. To this end, we make the following definition:

Definition 2.3.1. *For a finite place v of F we define a local deformation condition to be a subfunctor $\mathcal{D}'_v \subseteq \mathcal{D}^{\square}_{\overline{\rho}|_{G_{F_v}}}$ such that*

- *$\mathcal{D}'_v$ is represented by a quotient R'_v of $R^{\square}_{\overline{\rho}|_{G_{F_v}}}$.*
- *If $A \in \mathrm{CNL}_{\mathcal{O}}$, then for all lifts $\rho \in \mathcal{D}'_v(A)$ and $\gamma \in \ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(k))$, we have $\gamma\rho\gamma^{-1} \in \mathcal{D}'_v(A)$.*

The representing object R'_v is called the restricted deformation ring associated to $\mathcal{D}'_v$.

In particular, if $\chi\colon G_F \to \mathcal{O}^\times$ is a character, then the condition $\det \rho = \chi$ for a lift ρ is a local deformation condition, since determinants are invariant under conjugation. If $\rho^{\square}\colon G_{F_v} \to R^{\square}_{\overline{\rho}|_{G_{F_v}}}$ is the universal lifting, then the deformation condition is represented by the quotient $R^{\square}_{\overline{\rho}|_{G_{F_v}}}/(\det \rho^{\square}(g) - \chi(g) : g \in G_{F_v})$. We will only ever consider lifts which have determinant ϵ_p, so we define $\mathcal{D}^{\square}_v$ to be the subfunctor of $\mathcal{D}^{\square}_{\overline{\rho}|_{G_{F_v}}}$ giving liftings whose determinant is ϵ_p. We also let $R^{\square}_v$ be the resulting restricted deformation ring and we let $\rho^{\square}_v\colon G_{F_v} \to \mathrm{GL}_2(R^{\square}_v)$ be the universal lifting.

The following lemma [39, Lemma 5.12] gives a sufficient condition for a quotient of $R^{\square}_v$ to be a local deformation condition:

Lemma 2.3.2. *Suppose $\pi\colon R_v^\square \twoheadrightarrow R_v$ is a surjective morphism in* $\mathrm{CNL}_\mathcal{O}$ *and $\rho_\pi\colon G_{F_v} \to \mathrm{GL}_2(R_v)$ composition of π with the universal lifting. Then the subfunctor of $\mathcal{D}_v^\square$ defined by R_v is a local deformation condition if the following conditions hold:*

- *The ring R_v is reduced and not isomorphic to k.*
- *For any $\gamma \in \ker(\mathrm{GL}_2(R_v) \twoheadrightarrow \mathrm{GL}_2(k))$, the morphism $R_v^\square \to R_v$ induced by the representation $\gamma^{-1}\pi\gamma\colon G_{F_v} \to \mathrm{GL}_2(R_v)$ via the universal property of $R_v^\square$ factors through π.*

We will need to consider several local deformation rings. Let Σ_p denote the set of places of F above p. We fix a finite set of finite places Σ of F disjoint from Σ_p which contains all places $v \notin \Sigma_p$ at which $\overline{\rho}$ is ramified. For each $v \in \Sigma$ we use a superscript $\tau_v \in \{\mathrm{fl}, \min, \mathrm{ps}, \square\}$ to indicate a local deformation condition, and we denote by $R_v^{\tau_v}$ the resulting restricted deformation ring. For all liftings ρ of $\overline{\rho}|_{G_{F_v}}$ we impose the condition that $\det\rho = \varepsilon_p$. All of the restrictied deformation rings considered will thus be quotients of $\mathcal{D}_v^\square$.

For $v \in \Sigma_p$ the extension $F_v/\mathbb{Q}_p$ is unramified by assumption, so Fontaine-Laffaile theory applies. For all $v \in \Sigma_p$ we assume that $\overline{\rho}|_{G_{F_v}}$ is flat and we define

- R_v^{fl} to be the quotient of $R_v^\square$ parameterizing flat lifts of $\overline{\rho}|_{G_{F_v}}$.
- $R_v^{\min}$ be the quotient of $R_v^\square$ parameterizing minimally ramified (semistable at places $v \mid p$ and unramified at places $v \nmid p$ such that $\overline{\rho}|_{G_{F_v}}$ is unramified) lifts of $\overline{\rho}|_{G_{F_v}}$. In the case when $\overline{\rho}|_{G_{F_v}}$ is unramified, $R_v^{\min}$ parameterizes unramified lifts.

We wish to define one more deformation condition for $v \in \Sigma^{\mathrm{ps}}$ which is of primary concern in this thesis. We must first recall some definitions. Fix a place finite place v of F. The Weil group W_{F_v} is of F_v is the inverse image of Frobenius under the surjective morphism $G_{F_v} \twoheadrightarrow G_{k_v}$. A Weil-Deligne representation is a pair (ρ_0, N) consisting of a continuous representation $\rho_0\colon W_{F_v} \to \mathrm{GL}_2(\overline{E})$ of W_{F_v} and a nilpotent operator (called the *monodromy* *operator*) such that

$$\rho_0 N \rho_0^{-1} = \|\cdot\| N,$$

where $\|\cdot\|$ is the composition of the isomorphism $W_{F_v}^{\mathrm{ab}} \to F_v^\times$ given by local class field theory with the valuation on $F_v^\times$. Two Weli-Deligne representations (ρ_0, N) and (ρ_0', N') are equivalent if there exists $\gamma \in \mathrm{GL}_2(\overline{E})$ such that $\rho_0' = \gamma\rho_0\gamma^{-1}$ and $N' = \gamma N \gamma^{-1}$. Grothendieck's monodromy theorem gives an equivalence of categories between the category of Weil-Deligne representations and the category of represention $G_{F_v} \to \mathrm{GL}_2(\overline{E})$. The local Langlands correspondence for GL_2 gives a bijection between semisimple Weil-Deligne representations and irreducible smooth representations of $\mathrm{GL}_2(F_v)$.

Next we recall the notion of inertial types, as defined in [36]. These types correspond to classes of irreducible smooth representations of $\mathrm{GL}_2(F_v)$. An *inertial type* τ is an equivalence class of pairs (r_τ, N_τ) such that

- $r_\tau \colon I_{F_v} \to \mathrm{GL}_2(\overline{E})$ is a continuous representation with open kernel.
- N_τ is a nilpotent matrix in $\mathrm{GL}_2(\overline{E})$.
- (r_τ, N_τ) extends to a Weil-Deligne representation of G_{F_v}.

Two pairs are equivalent if they are conjugate by an element of $\mathrm{GL}_2(\overline{E})$.

We say that a continuous representation $\rho \colon G_{F_v} \to \mathrm{GL}_2(\overline{E})$ has inertial type τ if the restriction of the associated Weil-Deligne representation to I_{F_v} is equivalent to τ. In other words, the representation of $\mathrm{GL}_2(F_v))$ given by the local Langlands correspondence belongs to the class of representations corresponding to τ.

Let $\zeta \in \mathcal{O}$ be a p^m-th root of unity, where m is such that $p^m \mid\mid (q_v - 1)$. We define the *principal series* inertial type $\tau_\zeta = (r_\tau, N_\tau)$ by

- $r_{\tau_\zeta}(\iota_v) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $N_{\tau_\zeta} = 0$.

Note here we are defining r_{τ_ζ} to be trivial on P_{F_v} so that r_{τ_ζ} factors through I_{F_v}/P_{F_v}, hence is determined by $r_{\tau_\zeta}(\iota_v)$. The term "principal series" refers to the principal series representations of $\mathrm{GL}_2(F_v)$. These are representations which are induced from representations of the subgroup of upper-triangular matrices in $\mathrm{GL}_2(F_v)$ given by $\left(\begin{smallmatrix} a & b \\ * & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$, where χ_1, χ_2 are characters of $F_v^\times$.

We now define our last deformation condition

Definition 2.3.3. *For $v \in \Sigma^{\mathrm{ps}}$, we define the local deformation ring R_v^{ps} to be the maximal reduced, p-torsion free quotient of $R_v^\square$ with the following universal property: if $\alpha\colon R_v^{\mathrm{ps}} \to \mathrm{GL}_2(\overline{E})$ is any continuous homomorphism such that the induced representation $\rho_\alpha\colon G_{F_v} \to \mathrm{GL}_2(\overline{E})$ has type τ_ζ, then α factors through R_v^{ps}.*

Note that R_v^{ps} is the maximal reduced, p-torsion free quotient of the ring $R_v^{\mathrm{ps},\circ}$ defined by the local deformation condition that the characteristic polynomial of ι_v is $(X-\zeta)(X-\zeta^{-1})$. The subfunctor of $\mathcal{D}_v^\square$ defined by R_v^{ps} is a local deformation condition by Lemma 2.3.2.

We will also need to consider modified deformation problems introduced in [6], which take into account a fixed eigenvalue $\overline{\alpha}$ of $\overline{\rho}(\mathrm{Frob}_v)$. Recall that $\overline{\alpha} \in k$ by hypothesis. We define a functor $\widetilde{\mathcal{D}}_v^\square\colon \mathrm{CNL}_\mathcal{O} \to \mathrm{Set}$ by which takes a ring $R \in \mathrm{CNL}_\mathcal{O}$ to the set of tuples (ρ, α), where $\rho \in \mathcal{D}_v^\square(R)$ is a lifting such that α is an eigenvalue of $\rho(\mathrm{Frob}_v)$ such that $\alpha \equiv \overline{\alpha} \pmod{\mathfrak{m}_R}$.

Note that there is a natural transformation $\widetilde{\mathcal{D}}_v^\square \to \mathcal{D}_v^\square$ given by forgetting the choice of eigenvalue α. We also see that $\widetilde{\mathcal{D}}_v^\square$ is represented by the localization $\widetilde{R}_v^\square$ of the ring $R_v^\square[X]/(X^2 - X \operatorname{tr} \rho_v^\square(\mathrm{Frob}_v) + \det \rho_v^\square(\mathrm{Frob}_v))$ at the maximal ideal $\mathfrak{m}_{R_v^\square} + (X - \alpha)$. We will only be interested in the case when $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, in which case we have the following result (see [6, Lemma 2.1]):

Lemma 2.3.4. *If $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, then the natural map $R_v^\square \to \widetilde{R}_v^\square$ is an isomorphism.*

Proof. Let α, β be the eigenvalues of $\rho_v^\square$. Since $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, $\alpha \not\equiv \beta \pmod{\mathfrak{m}_{R_v^\square}}$. Therefore $(X-\beta)$ is a unit in the localization $\widetilde{R}_v^\square$ of $R_v^\square[X]/(X^2 - X\,\mathrm{tr}\,\rho_v^\square(\mathrm{Frob}_v) + \det \rho_v^\square(\mathrm{Frob}_v))$ at $\mathfrak{m}_{R_v^\square} + (X-\alpha)$. Then $\widetilde{R}_v^\square$ is isomorphic to the localization of $R_v^\square[X]/(X-\alpha) \cong R_v^\square$ at the unique maximal ideal. This provides an isomorphism $\widetilde{R}_v^\square \to R_v^\square$ which is inverse to the natural map $R_v^\square \to \widetilde{R}_v^\square$. $\square$

We will need to know some ring-theoretic properties of our restricted local deformation rings (in particular the fact that the rings $R_v^{\mathrm{fl}}, R_v^{\min}, R_v^\square$ are all complete intersections).

Proposition 2.3.5. *We have the following:*

(a) For $v \in \Sigma_p$ *we have* $R_v^{\mathrm{fl}} \cong \mathcal{O}[[x_1, \ldots, x_{3+[F:\mathbb{Q}]}]]$*, and for* $v \in \Sigma$ *we have* $R_v^{\min} \cong \mathcal{O}[[x_1, x_2, x_3]]$.

(b) For $v \in \Sigma$*, the ring* $R_v^\square$ *is a complete intersection, reduced and flat over* $\mathcal{O}$*, and* $\dim R_v^\square = 4$.

(c) For $v \in \Sigma^{\mathrm{ps}}$*, the ring* R_v^{ps} *is Cohen-Macaulay and flat over* $\mathcal{O}$*, and* $\dim R_v^{\mathrm{ps}} = 4$.

Proof. (a) is from [9, Section 2.4.1, Section 2.4.4]. (b) follows from [36, Theorem 2.5], modifiying the statement and proof to include the fixed determinant condition we impose. As we will see from (the proof of) Theorem 4.1.1, imposing the fixed determinant condition results in a dimension drop of 1, which ensures that $R_v^\square$ is still a complete intersection by Theorem A.0.7, and $\dim R_v^\square = 4$. (c) will follow later from (the proof of) Theorem 4.1.1. $\square$

2.4 Global deformation rings

Our ultimate goal is to prove a theorem which gives the Wiles defect of a global deformation ring as a sum of Wiles defects of local deformation rings. In this section we introduce the global deformation rings we will consider.

To prove our main theorem, we will need to consider rings which are completed tensor products (over $\mathcal{O}$) of local deformation rings. Recall that if M and N are linearly topologized $\mathcal{O}$-modules, then the completed tensor product $M\widehat{\otimes}_{\mathcal{O}}N$ is defined to be the completion of $M \otimes_{\mathcal{O}} N$ with respect to the linear topology defined by declaring the images of $M_\mu \otimes_{\mathcal{O}} N$ and $M \otimes_{\mathcal{O}} N_\nu$ to be a fundamental system of open submodules, where $M_\mu \subseteq M$ and $N_\nu \subseteq N$ run through fundamental systems of open submodules of M and N, respectively. For each $v \in \Sigma$ fix $\tau_v \in \{\min, \mathrm{ps}, \square\}$ and let $\tau = (\tau_v)_{v\in\Sigma}$. Then we define

$$R^{\tau}_{\mathrm{loc}} = \left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\tau_v}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{fl}}\right),$$

where the tensor products taken over $\mathcal{O}$. When $\tau_v = \square$ for all $v \in \Sigma$, we simply write $R^{\tau}_{\mathrm{loc}} = R_{\mathrm{loc}}$.

By [3, Lemma 4.4], we have

Lemma 2.4.1. *The ring R_{loc} is a complete intersection, the ring R^{τ}_{loc} is Cohen-Macaulay, and both rings are reduced and flat over $\mathcal{O}$.*

Adopting the notation of the previous section, we let R (resp. $R^{\square}$) be the global unframed (resp. framed) deformation ring parameterizing deformations (resp. lifts) of $\overline{\rho}$ which are unramified outside $\Sigma \cup \Sigma_p$. We may fix a non-canonical isomorphism $R^{\square} = R[[X_1, \ldots, X_{4|\Sigma\cup\Sigma_p|-1}]]$ so that we may view R as a quotient of $R^{\square}$. We may thus define

$$R^{\tau} = R^{\tau}_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R \qquad \text{and} \qquad R^{\square,\tau} = R^{\tau}_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R^{\square}.$$

Chapter 3

The Wiles defect

In this chapter we define and discuss the Wiles defect, a numerical invariant which is the primary interest of this thesis. We closely follow the work of [4].

We fix a prime $p > 2$ and a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. We work in the category $\mathrm{CNL}_{\mathcal{O}}$ of complete Noetherian local $\mathcal{O}$-algebras R with maximal ideal $\mathfrak{m}_R$ and residue field k. The Wiles defect is defined with respect to an augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$, which by definition is a surjective homomorphism of $\mathcal{O}$-algebras. We need the augmentation to be formally smooth on the generic fiber of R. Furthermore, we must deal separately with the case $\dim R = 1$ and the case $\dim R > 1$.

3.1 The Wiles defect for 1-dimensional rings

Let R be a ring in $\mathrm{CNL}_{\mathcal{O}}$. Further assume that R is finite free over $\mathcal{O}$. We fix an augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$. If we assume further that R is has Krull dimension 1, then the Wiles defect of R can be defined in terms of invariants studied by Wiles. Moreover, this definition coincides with a formula given in terms of two invariants defined by Venkatesh [40].

For a finitely-generated R-module M and a set of generators $m_1, \ldots, m_n \in M$ inducing a surjection $R^n \twoheadrightarrow M$ we define the 0-*th fitting ideal* $\mathrm{Fitt}_R(M)$ to be the ideal of R generated by all elements of the form $\det(v_1, \ldots, v_n)$ where $v_i \in \ker(R^n \twoheadrightarrow M)$. We have $\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M)$ and moreover, $\mathrm{Fitt}_R(M)$ does not depend on the choice of generators of M. In the case where M is a cyclic module, we have $\mathrm{Fitt}_R(M) = \ker(R \twoheadrightarrow M) = \mathrm{Ann}_R(M)$.

We define the *cotangent space* and *congruence module* of R with respect to λ as

$$\Phi_\lambda(R) := (\ker\lambda)/(\ker\lambda)^2$$

and

$$\Psi_\lambda(R) := \mathcal{O}/\lambda(\mathrm{Ann}_R(\ker\lambda)),$$

respectively. We assume that $\Phi_\lambda(R)$ is finite. The Wiles defect is defined in terms of these two invariants.

Definition 3.1.1. *We define the* ***Wiles defect*** *of R as*

$$\delta_\lambda(R) = \frac{\log_p|\Phi_\lambda(R)| - \log_p|\Psi_\lambda(R)|}{\log_p|\mathcal{O}/p|}.$$

The Wiles defect is known to be a nonnegative rational number. Moreover, $\delta_\lambda(R) = 0$ if and only if R is a complete intersection. The Wiles defect can thus be understood as a numerical measurement of the degree to which R fails to be a complete intersection. In situations of interest to us, we typically need to replace the coefficient ring $\mathcal{O}$ by the ring of integers in a finite extension of E. The normalizing factor of $\log_p|\mathcal{O}/p|$ ensures that $\delta_\lambda(R)$ is invariant under such an extension.

Let R act on $E/\mathcal{O}$ via the augmentation. Venkatesh's first invariant [40] is the André-Quillen cohomology group $\mathrm{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})$. The André-Quillen cohomology groups arise from the derived functors of the derivation functor $\mathrm{Der}_{\mathcal{O}}(\cdot, E/\mathcal{O})$. The degree $0, 1$, and 2 cohomology groups were introduced by Lichtenbaum and Schlessinger [25]. The higher groups were defined independently by André [2] and Quillen [30].

Fix a surjection $\varphi\colon \widetilde{R} \twoheadrightarrow R$ where $\widetilde{R}$ is a complete intersection which is 1-dimensional and finite free over $\mathcal{O}$ such that $\Phi_{\lambda\circ\varphi}(\widetilde{R})$ is finite. Let $I = \ker\varphi\colon \widetilde{R} \twoheadrightarrow R$ so that we have a containment of $\widetilde{R}$-modules $\mathrm{Fitt}_{\widetilde{R}}(I) \subseteq \mathrm{Ann}_{\widetilde{R}}(I)$. We define Venkatesh's second invariant as

$$C_{1,\lambda}(R) = \lambda(\mathrm{Ann}_{\widetilde{R}}(I))/\lambda(\mathrm{Fitt}_{\widetilde{R}}(I)).$$

Although it appears that $C_{1,\lambda}(R)$ depends on the choice of complete intersection $\widetilde{R}$, in fact $C_{1,\lambda}(R)$ depends only on R and λ [4, Lemma A.5]. This is proved by rewriting the right-hand side above in terms of the homology of Koszul complex associated to a sequence of generators of $\ker\varphi$. Furthermore, we have the following theorem [4, Proposition A.6], which expresses the Wiles defect in terms of Venkatesh's invariants:

Theorem 3.1.2. *For a ring R and an augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$ as in this section, we have*

$$\frac{|\operatorname{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})|}{|C_{1,\lambda}(R)|} = \frac{|\Phi_\lambda(R)|}{|\Psi_\lambda(R)|}.$$

The Wiles defect $\delta_\lambda(R)$ can thus be expressed as

$$\delta_\lambda(R) = \frac{\log_p |\operatorname{Der}^1_{\mathcal{O}}(R, E/\mathcal{O})| - \log_p |C_{1,\lambda}(R)|}{\log_p |\mathcal{O}/p|}.$$

Remark 3.1.3. *If R is not torsion-free over $\mathcal{O}$, one can replace R by its maximal torsion-free quotient in the definition of $\Psi_\lambda(R)$. However, in all cases of interest to us, R will be $\mathcal{O}$-torsion free.*

3.2 The Wiles defect for higher-dimensional rings

The definition of the Wiles defect in the previous section relied crucially on the finiteness of the cotangent space $\Phi_\lambda(R)$. The following proposition shows that if R is Cohen-Macaulay, this assumption actually forces R to be 1-dimensional.

Proposition 3.2.1. *Suppose R is a ring in $\mathrm{CNL}_{\mathcal{O}}$ with an augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$ such* *that $|\Phi_\lambda(R)|$ is finite. If R is Cohen-Macaulay, then R is 1-dimensional.*

Proof. The localization of the augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$ at $\mathfrak{p} = \ker\lambda$ gives a surjection $R_\mathfrak{p} \twoheadrightarrow E$. The residue field of $R_\mathfrak{p}$ is thus isomorphic to E. By assumption $\mathfrak{p}/\mathfrak{p}^2$ is finite, so $\mathfrak{m}/\mathfrak{m}^2$ is finite, where $\mathfrak{m} = \mathfrak{p}R_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$. But $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over E, thus $\mathfrak{m} = \mathfrak{m}^2$. Nakayama's lemma then implies that $\mathfrak{m} = 0$. Therefore $R_\mathfrak{p}$ is a field, hence $\mathfrak{p}$ is a minimal prime of R. Since Cohen-Macaulay rings are equidimensional and $R/\mathfrak{p} \cong \mathcal{O}$, we see that all minimal primes of R are 1-dimensional. It follows that every nonmaximal prime ideal of R is minimal, i.e. R is 1-dimensional. □

We let $\mathrm{CNL}^a_\mathcal{O}$ be the category whose objects are pairs (R, λ_R) where

- R is a complete Noetherian local $\mathcal{O}$-algebra with maximal ideal m_R and residue field $R/m_R = k$
- R is flat over $\mathcal{O}$ and Cohen-Macaulay
- $\lambda_R\colon R \twoheadrightarrow \mathcal{O}$ is an augmentation which is smooth on the generic fiber of R.

The morphisms in $\mathrm{CNL}^a_\mathcal{O}$ are local $\mathcal{O}$-algebra homomorphisms $\varphi\colon R \to S$ which are compatible with the augmentations in the sense that $\lambda_R = \lambda_S \circ \varphi$. For convenience, we will often omit the subscript R from λ_R. We wish to extend the definition of the Wiles defect to objects in this category.

Let $R \in \mathrm{CNL}^a_\mathcal{O}$ and consider a power series ring $S = \mathcal{O}[[y_1, \ldots, y_d]]$. We will be interested in certain maps $S \hookrightarrow R$ which produce a quotient of R for which the cotangent space is finite.

Definition 3.2.2. *We say that a continous injective $\mathcal{O}$-algebra homomorphism $\theta\colon S \hookrightarrow R$ is a 1-**codimensional embedding** if the following hold:*

- *θ makes R into a finite free S-module.*
- $\theta(y_1), \ldots, \theta(y_d) \subseteq \ker\lambda$.

- *For $R_\theta = R/(\theta(y_1), \dots, \theta(y_d))$ and the induced augmentation $\lambda_\theta \colon R_\theta \twoheadrightarrow \mathcal{O}$, the cotangent space $\Phi_{\lambda_\theta}(R_\theta)$ is finite.*

As in the 1-dimensional case, in order to define Venkatesh's invariants for rings in $\mathrm{CNL}_{\mathcal{O}}^a$, we will need to consider an auxillary complete intersection which surjects onto R.

Definition 3.2.3. *We say that a continuous surjection of $\mathcal{O}$-algebras $\varphi \colon \widetilde{R} \twoheadrightarrow R$ is a **CI covering** if the following hold:*

- *$\widetilde{R}$ is a complete Noetherian local O-algebra which is flat over $\mathcal{O}$ and of the same dimension as R.*
- *$\widetilde{R}$ is a complete intersection.*
- *$\widetilde{R}[1/\varpi]$ is formally smooth (see [43, Definition 10.138.1]) at the augmentation $(\lambda \circ \varphi)[1/\varpi] \colon \widetilde{R}[1/\varpi] \to E$.*

In the last condition above, we consider $\lambda \circ \varphi$ as a map $R \to E$ via composing with the inclusion $\mathcal{O} \hookrightarrow E$. In order to prove that a 1-codimensional embedding and a CI covering exist, we need the following lemma:

Lemma 3.2.4. *Let A be a complete Noetherian local $\mathcal{O}$-algebra with an augmentation $\lambda \colon A \twoheadrightarrow \mathcal{O}$. Suppose there exist elements $f_1, \dots, f_d \in \ker \lambda$ such that $\varpi, f_1, \dots, f_d$ is a regular sequence in S and $A[1/\varpi]$ is formally smooth at λ of dimension $n \geq d$. Then there exist $h_1, \dots, h_d \in \ker \lambda \cap (f_1, \dots, f_d, \varpi)$ such that $\varpi, h_1, \dots, h_d$ is a regular sequence in A and for the quotient $B = A/(h_1, \dots, h_d)$ and the induced augmentation $\lambda_B \colon B \twoheadrightarrow \mathcal{O}$, the ring $B[1/\varpi]$ is formally smooth at $\lambda_B[1/\varpi]$ of dimension $n - d$.*

Proof. By replacing each f_i by f_i^2, we may assume that $f_1, \dots, f_d \in (\ker \lambda)^2$ [43, Lemma 10.68.9].

Let $\widehat{A[1/\varpi]}$ be the completion of $A[1/\varpi]$ at $(\ker \lambda)[1/\varpi]$ and let $\widehat{m}$ be its maximal ideal. By the formal smoothness hypothesis, $A[1/\varpi] \cong E[[y_1, \dots, y_n]]$ with $n \geq d$. Since $d \leq n$, by Nakayama's lemma we can choose $g_1, \dots, g_d \in \ker \lambda$ whose images in $\widehat{m}/\widehat{m}^2$ are linearly independent over E. Let $h_i = f_i + \varpi g_i$ for $1 \le i \le d$. Observe that $\varpi, h_1, \dots, h_d$ is still a regular sequence in A, and since each $f_i \in (\ker \lambda)^2$, the images of the h_i in $\widehat{m}/\widehat{m}^2$ are also linearly independent over E. This implies the desired formal smoothness for the quotient $A/(h_1, \dots, h_d)$. □

Using Lemma 3.2.4, we can show that a 1-codimensional embedding into R always exists.

Proposition 3.2.5. *A* 1*-codimensional embedding* $\theta\colon S \hookrightarrow R$ *exists.*

Proof. Since R is flat over $\mathcal{O}$, we have that ϖ is not a zero-divisor in R, thus $\dim R/(\varpi) = \dim R - 1 = d$. Choosing a system of parameters for $R/(\varpi)$ and lifting it to R, we see from Theorem A.0.7 that there exists a regular sequence $\varpi, f_1, \dots, f_d$ in R. We can assume that each $f_i \in \ker \lambda_R$, since adding an element of ϖR to each f_i preserves regularity and $\ker \lambda_R + \varpi R = \mathfrak{m}_R$.

By Lemma 3.2.4, there exist $h_1, \dots, h_d \in \ker \lambda_R$ such that $\varpi, h_1, \dots, h_d$ is a regular sequence in R and for the quotient $B = R/(h_1, \dots, h_d)$, the ring $B[1/\varpi]$ is formally smooth of dimension 0 at the induced augmentation $\lambda_B[1/\varpi]$. Since $\varpi, h_1, \dots, h_d$ is a regular sequence in R, $B/(\varpi) = R/(\varpi, h_1, \dots, h_d)$ is a finitely-generated algebra over $\mathcal{O}/(\varpi) = k$ of Krull dimension 0. Therefore $B/(\varpi)$ is finite, as it has finite dimension as a k-vector space. It follows that if we define $\theta\colon S = \mathcal{O}[[y_1, \dots, y_d]] \to R$ by $\theta(y_1) = h_i$, then R is finite over S. Then since $\varpi, y_1, \dots, y_d$ is a regular sequence in S generating its maximal ideal and is also R-regular, we have that R is free over S.

Now $R_\theta[1/\varpi] = B[1/\varpi]$ is a 0-dimensional E-algebra, and is therefore a direct product of Artinian E-algebras, where the components are the localizations of $R_\theta[1/\varpi]$ at its maximal ideals. In particular, since $R_\theta[1/\varpi]$ is formally smooth of dimension 0 at the localized augmentation, the component corresponding to $(\ker \lambda_\theta)[1/\varpi] \cong \ker(\lambda_\theta[1/\varpi])$ is equal to E. It follows that the localization $\Phi_{\lambda_\theta}(R_\theta)[1/\varpi]$ is 0, thus $\Phi_{\lambda_\theta}(R_\theta)$ is a torsion $\mathcal{O}$-module. Now

$R_\theta = R/(h_1, \dots, h_d)$ is finite over $\mathcal{O}$ for the same reason that R is finite over S, so $\Phi_{\lambda_\theta}(R_\theta)$ is also finite over $\mathcal{O}$. We conclude that $\Phi_{\lambda_\theta}(R_\theta)$ is finite.

Lastly note that R is S-torsion free since R is free over S, hence θ is injective. We have thus shown that $\theta\colon S \hookrightarrow R$ is a 1-codimensional embedding into R. □

Lemma 3.2.4 also implies that a CI covering of R exists.

Proposition 3.2.6. *A CI covering $\varphi\colon \widetilde{R} \twoheadrightarrow R$ exists.*

Proof. Choose a surjective ring homomorphism $\Pi\colon A = \mathcal{O}[[x_1, \dots, x_n]] \twoheadrightarrow R$ and let $\mathfrak{p} = \Pi^{-1}(\ker \lambda_R)$. Let $d + 1 = \dim R$ so that $m = n - d \geq 0$. Since A has no ϖ-torsion and $A/(\varpi) \cong k[[x_1, \dots, x_n]]$ is regular of dimension n, there exist $f_1, \dots, f_m \in \ker \Pi$ such that $\varpi, f_1, \dots, f_m$ is a regular sequence in S. Since $A[1/\varpi] \cong E[[x_1, \dots, x_n]]$ is regular of dimension n with maximal ideal $\mathfrak{p}[1/\varpi]$, the ring $S[1/\varpi]$ is formally smooth at $\mathfrak{p}[1/\varpi]$ of dimension n.

Let $\lambda = \lambda_R \circ \Pi\colon A \twoheadrightarrow \mathcal{O}$ and choose $h_1, \dots, h_m \in (\ker \lambda \cap (f_1, \dots, f_m, \varpi)) \subseteq \ker \Pi + \varpi A$ as in Lemma 3.2.4. Then $\varpi, h_1, \dots, h_m$ is a regular sequence in A, and for the quotient $\widetilde{R} = A/(h_1, \dots, h_m)$ and $\lambda_{\widetilde{R}}\colon \widetilde{R} \twoheadrightarrow \mathcal{O}$, we have that $\widetilde{R}[1/\varpi]$ is formally smooth at $\lambda_{\widetilde{R}}[1/\varpi]$ of dimension $n - m = d$. Since any permutation of a regular sequence in a Noetherian local ring is a regular sequence, the regularity of $\varpi, h_1, \dots, h_m$ in A implies that ϖ is not a zero-divisor in $\widetilde{R}$, thus $\widetilde{R}$ is flat over $\mathcal{O}$. We also see that $h_1, \dots, h_m$ is a regular sequence in A. Since A is a regular local ring, this implies that $\widetilde{R} = A/(h_1, \dots, h_m)$ is a complete intersection, and together with Krull's principal ideal theorem it also implies that

$$\dim \widetilde{R} = \dim A - m = n + 1 - m = d + 1 = \dim R.$$

The natural map $\varphi\colon \widetilde{R} \twoheadrightarrow R$ is thus a CI covering. □

We want to show that a 1-codimensional embedding into R can be lifted along a CI covering of R. We first need the following lemma:

Lemma 3.2.7. *Suppose A is a Noetherian ring and $I \subseteq A$ is an ideal. If y is an element of $B = A/I$ not contained in any minimal prime of B, then there exists a lift $\widetilde{y} \in A$ of x such that $\widetilde{y}$ is not contained in any minimal prime of A.*

Proof. Let $\widetilde{y}_0 \in A$ be an arbitary lift of y. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal prime ideals of A. Observe that none of the $\mathfrak{p}_i$ contain $(\widetilde{y}_0) + I$, for if $\widetilde{y}_0 \in \mathfrak{p}_i$, then $\mathfrak{p}_i$ does not contain I, or else the image of $\mathfrak{p}_i$ in B is a minimal prime ideal of B which contains y, a contradiction. By the prime avoidance lemma, there exists $x \in I$ such that $x + \widetilde{y}_0 \notin \mathfrak{p}_i$ for each i. Then $\widetilde{y} = x + \widetilde{y}_0$ is a lift of y not contained in any minimal prime of A. $\square$

Now we can be prove the following lemma:

Lemma 3.2.8. *Suppose $\theta\colon S = \mathcal{O}[[y_1, \ldots, y_d]] \hookrightarrow R$ is a 1-codimensional embedding and $\varphi\colon \widetilde{R} \twoheadrightarrow R$ is a CI covering. Then θ lifts to a map $\widetilde{\theta}\colon S \hookrightarrow \widetilde{R}$ which is a 1-codimensional embedding into $\widetilde{R}$.*

Furthermore, the quotient $\widetilde{R}_\theta = \widetilde{R}/(\widetilde{\theta}(y_1), \ldots, \widetilde{\theta}(y_d))$ is a complete intersection of dimension 1.

Proof. We identify each $y_i \in S$ with its image in R so that $\varpi, y_1, \ldots, y_d$ is a regular sequence in S and R. Let $d = \dim R - 1$. We show by induction that there exist $\widetilde{y}_1, \ldots, \widetilde{y}_d \in \widetilde{R}$ such that $\varphi(\widetilde{y}_i) = y_i$ for each i and $\dim \widetilde{R}/(\varpi, \widetilde{y}_1, \ldots, \widetilde{y}_j) = d - j = \dim R/(\varpi, y_1, \ldots, y_j)$ for all $0 \leq j \leq d$. We then set $\widetilde{\theta}(y_i) = \widetilde{y}_i$.

Since $\widetilde{R}$ and R are flat over $\mathcal{O}$, we have $\dim \widetilde{R}/(\varpi) = d = \dim R/(\varpi)$, so the base case holds. Now suppose for some $0 \leq j < d$ we have found $\widetilde{y}_1, \ldots, \widetilde{y}_j \in \widetilde{R}$ satisfying the desired properties. Since $\varpi, y_1, \ldots, y_d$ is a regular sequence in R, we have that y_{j+1} is not a zero-divisor in $B_j = R/(\varpi, y_1, \ldots, y_j)$, hence is not in any minimal prime of B. By Lemma 3.2.7, there exists a lift $\widetilde{y}_{j+1} \in A_j = \widetilde{R}/(\varpi, \widetilde{y}_1, \ldots, \widetilde{y}_j)$ such that $\widetilde{y}_{j+1}$ is not in any minimal prime of A_j. This implies that

$$\dim A_j/(\widetilde{y}_{j+1}) = \dim A_j - 1 = \dim B_j - 1 = \dim B_j/(y_{j+1}),$$

which completes the induction step.

Note that $\widetilde{\theta}(y_i)$ is necessarily in $\ker \widetilde{\lambda}$ since $\widetilde{\theta}$ lifts θ. Now observe that the sequence $\varpi, \widetilde{y}_1, \ldots, \widetilde{y}_d$ is a system of parameters in $\widetilde{R}$. It is therefore a regular sequence in $\widetilde{R}$ by Theorem A.0.7 since $\widetilde{R}$ is Cohen-Macaulay as it is a complete intersection. It follows that $\widetilde{R}_\theta = \widetilde{R}/(\widetilde{y}_1, \ldots, \widetilde{y}_d)$ is a complete intersection and the Krull principal ideal theorem implies $\dim \widetilde{R}_\theta = 1$. Defining $\widetilde{\theta}\colon S \to \widetilde{R}$ by $\widetilde{\theta}(y_i) = \widetilde{y}_i$ for each i, we see as in the proof of Proposition 3.2.5 that $\widetilde{\theta}$ makes $\widetilde{R}$ into a finite free S-module, $\widetilde{\theta}$ is injective, and $\widetilde{R}_\theta$ is finite free over $\mathcal{O}$.

For the finiteness of the cotangent space, let $R_\theta = R/(y_1, \ldots, y_d)$ and $\lambda_\theta\colon R_\theta \twoheadrightarrow \mathcal{O}$ be the induced augmentation and consider the following commutative diagram with exact rows (see the proof of [3, Theorem 7.16]):

$$\begin{array}{ccccccc}
\mathcal{O}^d & \xrightarrow{\widetilde{\Theta}} & \Phi_{\widetilde{\lambda}}(\widetilde{R}) & \longrightarrow & \Phi_{\widetilde{\lambda}_\theta}(\widetilde{R}_\theta) & \longrightarrow & 0 \\
\Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow & & \\
\mathcal{O}^d & \xrightarrow{\Theta} & \Phi_{\lambda}(R) & \longrightarrow & \Phi_{\lambda_\theta}(R_\theta) & \longrightarrow & 0
\end{array}$$

where the maps $\widetilde{\Theta}$ and Θ are given in terms of differentials by $\widetilde{\Theta}(e_i) = \mathrm{d}\widetilde{y}_i$ and $\Theta(e_i) = \mathrm{d}y_i$. Since $R[1/\varpi]$ and $\widetilde{R}[1/\varpi]$ are both equidimensional of dimension d and are formally smooth at λ and $\widetilde{\lambda}$, respectively, we have that $\Phi_\lambda(R)$ and $\Phi_{\widetilde{\lambda}}(\widetilde{R})$ both have rank d as $\mathcal{O}$-modules. Since $\Phi_{\lambda_\theta}(R_\theta)$ is finite, it follows from exactness that Θ is injective. The commutativity of the diagram then implies that $\widetilde{\Theta}$ is also injective, so by exactness $\Phi_{\widetilde{\lambda}_\theta}(\widetilde{R}_\theta)$ is also finite. $\square$

3.3 Independence of $C_{1,\lambda_\theta}(R_\theta)$ of θ

In this section we fix a $(R, \lambda) \in \mathrm{CNL}_{\mathcal{O}}^{a}$, a 1-codimensional embedding $\theta\colon S \hookrightarrow R$, and a CI covering $\varphi\colon \widetilde{R} \twoheadrightarrow R$ with $I = \ker \varphi$. We also fix a lift $\widetilde{\theta}\colon S \hookrightarrow \widetilde{R}$ which satisfies the conclusion of Lemma 3.2.8 and we identify S with its images in R and $\widetilde{R}$.

Let $R_\theta = R/(y_1, \ldots, y_d)$ and $\widetilde{R}_\theta = \widetilde{R}/(y_1, \ldots, y_d)$. Noting that $R_\theta \cong R \otimes_S \mathcal{O}$ and similarly $\widetilde{R}_\theta \cong \widetilde{R} \otimes_S \mathcal{O}$, we define $\varphi_\theta = \varphi \otimes \mathrm{id}_\mathcal{O} \colon \widetilde{R}_\theta \twoheadrightarrow R_\theta$ and $I_\theta = \ker \varphi_\theta$. We also let $\pi_\theta \colon \widetilde{R} \twoheadrightarrow \widetilde{R}_\theta$ be the quotient map and note that $\widetilde{\lambda} = \widetilde{\lambda}_\theta \circ \pi_\theta$. Since $\widetilde{R}$ is a free, hence flat, S-module, we also have $I_\theta = \pi_\theta(I)$.

We wish to define $C_{1,\lambda}(R)$ as an analog of Venkatesh's second invariant in the 1-dimensional case. This will require a few lemmas.

Lemma 3.3.1. *There exist isomorphisms* $\Lambda \colon \widetilde{R} \to \mathrm{Hom}_S(\widetilde{R}, S)$ *and* $\Lambda_\theta \colon \widetilde{R}_\theta \to \mathrm{Hom}_\mathcal{O}(\widetilde{R}_\theta, \mathcal{O})$ *such that the following diagram commutes*

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\Lambda} & \mathrm{Hom}_S(\widetilde{R}, S) \\ \downarrow{\scriptstyle \pi_\theta} & & \downarrow{\scriptstyle \alpha} \\ \widetilde{R}_\theta & \xrightarrow{\Lambda_\theta} & \mathrm{Hom}_\mathcal{O}(\widetilde{R}_\theta, \mathcal{O}) \end{array}$$

where α *sends a map* $\widetilde{R} \to S$ *to the map* $\widetilde{R}_\theta \to \mathcal{O}$ *induced by the quotient map* $S \twoheadrightarrow \mathcal{O}$.

Proof. The ring S is Gorenstein since it is a complete intersection, so $\omega_S \cong S$, where ω_S is the dualizing module of S. Likewise $\omega_{\widetilde{R}} \cong \widetilde{R}$ since $\widetilde{R}$ is also a complete intersection. Since S and $\widetilde{R}$ are Cohen-Macaulay and $\theta \colon S \hookrightarrow R$ is a finite local map, by [19, Theorem 3.3.7 (b)] we have

$$\omega_{\widetilde{R}} \cong \mathrm{Ext}_S^{\dim S - \dim \widetilde{R}}(\widetilde{R}, \omega_S) \cong \mathrm{Hom}_S(\widetilde{R}, S),$$

where the second isomorphism follows since $\dim S = \dim \widetilde{R}$. We let Λ be the composition $\widetilde{R} \cong \omega_{\widetilde{R}} \cong \mathrm{Hom}_S(\widetilde{R}, S)$.

Since $\widetilde{R}$ is free over S, we have

$$\ker \alpha = \mathrm{Hom}_S(\widetilde{R}, (y_1, \ldots, y_d)S) = \sum_{i=1}^{d} y_i \,\mathrm{Hom}_S(\widetilde{R}, S).$$

Since Λ is an isomorphism, we see that

$$\Lambda^{-1}(\ker \alpha) = (y_1, \ldots, y_d)\widetilde{R} = \ker \pi_\theta.$$

Then the map $\Lambda_\theta \colon \widetilde{R}_\theta \to \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta, \mathcal{O})$ given by $\Lambda_\theta(r) = (\alpha \circ \Lambda)(\widetilde{r})$ where $\widetilde{r} \in \widetilde{R}$ is a lift of $r \in \widetilde{R}_\theta$ is well-defined and makes the diagram commute. Since π_θ is surjective, Λ_θ is necessarily injective. Moreover, α is surjective since $\widetilde{R}$ is a projective S-module, thus Λ_θ is an isomorphism. □

Lemma 3.3.2. *Let* $\Lambda \colon \widetilde{R} \to \mathrm{Hom}_S(\widetilde{R}, S)$ *and* $\Lambda_\theta \colon \widetilde{R}_\theta \colon \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta, \mathcal{O})$ *be isomorphisms as in Lemma 3.3.1. Then*

(i) $\Psi(\mathrm{Ann}_{\widetilde{R}}(I)) = \mathrm{Hom}_S(\widetilde{R}/I, S)$

(ii) $\Psi_\theta(\mathrm{Ann}_{\widetilde{R}_\theta}(I_\theta)) = \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O})$.

Proof. We have $\Psi(\mathrm{Ann}_{\widetilde{R}}(I)) = \mathrm{Hom}_S(\widetilde{R}, S)[I]$ since Ψ is an isomorphism. Now

$$\begin{aligned}\mathrm{Hom}_S(\widetilde{R}, S)[I] &= \{f \colon \widetilde{R} \to S : r \cdot f = 0 \text{ for all } r \in I\} \\ &= \{f \colon \widetilde{R} \to S : f(rx) = 0 \text{ for all } r \in I, x \in R\} \\ &= \{f \colon \widetilde{R} \to S : f(r) = 0 \text{ for all } r \in I\} \\ &= \mathrm{Hom}_S(\widetilde{R}/I, S).\end{aligned}$$

This proves (i), and (ii) follows from the same argument. □

Theorem 3.3.3. *We have*

(i) $\mathrm{Ann}_{\widetilde{R}_\theta}(I_\theta) = \pi_\theta(\mathrm{Ann}_{\widetilde{R}}(I))$

(ii) $\mathrm{Fitt}_{\widetilde{R}_\theta}(I_\theta) = \pi_\theta(\mathrm{Fitt}_{\widetilde{R}}(I))$

Proof. Let $\Lambda \colon \widetilde{R} \to \mathrm{Hom}_S(\widetilde{R}, S)$, let $\Lambda_\theta \colon \widetilde{R}_\theta \to \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta, \mathcal{O})$, and let $\alpha \colon \mathrm{Hom}_S(\widetilde{R}, S) \to \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta, \mathcal{O})$ be as in Lemma 3.3.1. Since $\widetilde{R}/I \cong R$ is a projective S-module, α induces a surjective map $\mathrm{Hom}_S(\widetilde{R}/I, R) \twoheadrightarrow \mathrm{Hom}_S(\widetilde{R}/I, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O})$. Then using Lemma 3.3.2 (i), we see that

$$(\alpha \circ \Psi)(\mathrm{Ann}_{\widetilde{R}}(I)) = \alpha(\mathrm{Hom}_S(\widetilde{R}/I, S)) = \mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O}).$$

The commutativity of the diagram in Lemma 3.3.1 implies that

$$\pi_\theta(\mathrm{Ann}_{\widetilde{R}}(I)) = \Lambda_\theta^{-1}((\Lambda_\theta \circ \pi_\theta)(\mathrm{Ann}_{\widetilde{R}}(I))) = \Lambda_\theta^{-1}(\mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O}))$$

On the other hand, by Lemma 3.3.2 (ii) we have

$$\Lambda_\theta^{-1}(\mathrm{Hom}_{\mathcal{O}}(\widetilde{R}_\theta/I_\theta, \mathcal{O})) = \mathrm{Ann}_{\widetilde{R}_\theta}(I_\theta),$$

thus $\mathrm{Ann}_{\widetilde{R}_\theta}(I_\theta) = \pi_\theta(\mathrm{Ann}_{\widetilde{R}}(I))$ as desired. This proves (i).

Now we prove (ii). We have a short exact sequence

$$0 \longrightarrow I \longrightarrow \widetilde{R} \longrightarrow R \longrightarrow 0.$$

Tensoring with $\mathcal{O}$ over S gives an an exact sequence

$$\mathrm{Tor}_1^S(R, \mathcal{O}) \longrightarrow I \otimes_S \mathcal{O} \longrightarrow \widetilde{R}_\theta \xrightarrow{\varphi_\theta} R_\theta \longrightarrow 0.$$

But $\mathrm{Tor}_1^S(R, \mathcal{O}) = 0$ since R is a free S-module, thus $I_\theta \cong I \otimes_S \mathcal{O}$ as S-modules. It now follows from [43, Lemma 15.8.4 (3)] that

$$\pi_\theta(\mathrm{Fitt}_{\widetilde{R}}(I)) = \mathrm{Fitt}_{R_\theta}(I \otimes_S \mathcal{O}) = \mathrm{Fitt}_{\widetilde{R}_\theta}(I_\theta),$$

proving (ii). □

We can now define $C_{1,\lambda}(R)$ in the same way as in the case where $\dim R = 1$.

Definition 3.3.4. *We define*

$$C_{1,\lambda}(R) := C_{1,\widetilde{\lambda}}(\widetilde{R}) := \widetilde{\lambda}(\mathrm{Ann}_{\widetilde{R}}(I))/\widetilde{\lambda}(\mathrm{Fitt}_{\widetilde{R}}(I)).$$

Theorem 3.3.3 shows that

$$\begin{aligned}
\widetilde{\lambda}(\mathrm{Ann}_{\widetilde{R}}(I))/\widetilde{\lambda}(\mathrm{Fitt}_{\widetilde{R}}(I)) &= \widetilde{\lambda}_\theta(\pi_\theta(\mathrm{Ann}_{\widetilde{R}}(I)))/\widetilde{\lambda}_\theta(\pi_\theta(\mathrm{Fitt}_{\widetilde{R}}(I))) \\
&= \widetilde{\lambda}_\theta(\mathrm{Ann}_{\widetilde{R}_\theta}(I_\theta))/\widetilde{\lambda}_\theta(\mathrm{Fitt}_{\widetilde{R}_\theta}(I_\theta)) \\
&= C_{1,\lambda_\theta}(R_\theta).
\end{aligned}$$

By [4, Lemma A.5], the invariant $C_{1,\lambda_\theta}(R_\theta)$ is independent of $\widetilde{R}_\theta$, so $C_{1,\lambda}(R)$ is well-defined as in the 1-dimensional case.

3.4 Independence of $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$ of θ

Fix $(R,\lambda) \in \mathrm{CNL}^a_{\mathcal{O}}$ and a 1-codimensional embedding $S = \mathcal{O}[[y_1,\dots,y_d]] \hookrightarrow R$. We let $R_\theta = R/(y_1,\dots,y_d)$ as before. In this section we show that the André-Quillen cohomology group $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$ is independent of the choice of θ and is in fact isomorphic to the continuous cohomology group $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$ which we will use to define the Wiles defect for (R,λ).

Given any ring A, let Mod_A denote the category of A-modules, $D(\mathrm{Mod}_A)$ the derived category of Mod_A, and $D^-(\mathrm{Mod}_A)$ the subcategory of Mod_A whose objects are bounded above complexes.

Given any ring homomorphism $A \to B$, let $L_{B/A} \in D^-(\mathrm{Mod}_B)$ denote the relative cotangent complex [44, Definition 2.1]. For a ring $A \in \mathrm{CNL}_{\mathcal{O}}$, we let $\wedge\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ denote the $\mathfrak{m}_A$-adic completion functor. By an abuse of notation, we also let $\wedge\colon D^-(\mathrm{Mod}\,A) \to D^-(\mathrm{Mod}\,A)$ denote the left-derived functor of $\wedge$, defined as in [13, Chapter 7.1].

Given a continuous ring homomorphism $A \to B$, we define the analytic relative cotangent complex to be $L^{\mathrm{an}}_{B/A} = (L_{B/A})^\wedge$. We can now define the André-Quillen cohomology groups.

Definition 3.4.1. *Let $A \to B$ be a ring homomorphism and let M be a B-module. Then* *for $i \geq 0$ we define the i-th André-Quillen cohomology group to be*

$$\mathrm{Der}^i_A(B, M) = H^i(\mathrm{RHom}_B(L_{B/A}, M)).$$

Similarly, if $A \to B$ is continuous, then we define the i-th André-Quillen cohomology group to be

$$\widehat{\mathrm{Der}}^i_A(B, M) = H^i(\mathrm{RHom}_B(L^{\mathrm{an}}_{B/A}, M)).$$

We will need to make use of some of the basic properties of continuous André-Quillen cohomology to prove the main result of this section.

Proposition 3.4.2. *Given $A, B, C \in \mathrm{CNL}_{\mathcal{O}}$ and continuous ring homomorphisms $A \to B \to C$ and any C-module M, we have the following long exact sequence in André-Quillen cohomology:*

$$0 \longrightarrow \widehat{\mathrm{Der}}^0_B(C, M) \longrightarrow \widehat{\mathrm{Der}}^0_A(C, M) \longrightarrow \widehat{\mathrm{Der}}^0_A(B, M)$$

$$\longrightarrow \widehat{\mathrm{Der}}^1_B(C, M) \longrightarrow \widehat{\mathrm{Der}}^1_A(C, M) \longrightarrow \widehat{\mathrm{Der}}^1_A(B, M) \longrightarrow \cdots$$

Proof. This follows from the definition of the continuous André-Quillen cohomology groups and the distinguished triangle

$$C \otimes^{\mathrm{L}}_B L^{\mathrm{an}}_{B/A} \to L^{\mathrm{an}}_{C/A} \to L^{\mathrm{an}}_{C/B} \to C \otimes^{\mathrm{L}}_B L^{\mathrm{an}}_{B/A}[1].$$

from [13, Theorem 7.1.33]. □

Proposition 3.4.3. *If $A \to B$ is a continuous morphism in $\mathrm{CNL}_{\mathcal{O}}$ which makes B into a finite A-module, then $L^{\mathrm{an}}_{B/A} \cong L_{B/A}$, hence $\widehat{\mathrm{Der}}^i_A(B, M) \cong \mathrm{Der}^i_A(B, M)$ for all $i \geq 0$ and all B-modules M.*

Proof. Since the map $A \to B$ is of finite type, by [20, 6.11] we have that $L_{B/A}$ is quasi-isomorphic to a bounded above complex of finite free B-modules $L^\bullet \in \mathrm{Der}^-(\mathrm{Mod}_B)$. Then we have

$$L^{\mathrm{an}}_{B/A} = (L_{B/A})^\wedge \cong (L^\bullet)^\wedge \cong L_{B/A},$$

since finite free B-modules are $\mathfrak{m}_B$-adically complete. The isomorphism of André-Quillen cohomology groups now follows from their definition. □

Proposition 3.4.4. *Let $A, B \in \mathrm{CNL}_\mathcal{O}$ and let $A \to B$ be a continuous ring homomorphism. The module $\widehat{\Omega}_{B/A} = \varprojlim \Omega_{(B/\mathfrak{m}_B^n)/A}$ of continuous Kähler differentials is the $\mathfrak{m}_B$-adic completion of $\Omega_{B/A}$ and we have $\widehat{\mathrm{Der}}^0_A(B, M) \cong \mathrm{Hom}_B(\widehat{\Omega}_{B/A}, M)$ for any B-module M.*

Proof. For the first claim, we proceed as in the proof of [3, Lemma 7.1]. For $n > m$ we have

$$\Omega_{B/A}/\mathfrak{m}_B^m\Omega_{B/A} = \Omega_{B/A} \otimes_B B/\mathfrak{m}_B^m \cong \Omega_{(B/\mathfrak{m}_B^n)/A} \otimes B/\mathfrak{m}_B^m.$$

Taking inverse limits, we obtain

$$\begin{aligned}\Omega_{B/A}/\mathfrak{m}_B^m\Omega_{B/A} &\cong \varprojlim_n (\Omega_{(B/\mathfrak{m}_B^n)/A} \otimes B/\mathfrak{m}_B^m)\\ &\cong \varprojlim_n (\Omega_{(B/\mathfrak{m}_B^n)/A}) \otimes_B B/\mathfrak{m}_B^m\\ &= \widehat{\Omega}_{B/A} \otimes_B B/\mathfrak{m}_B^m.\end{aligned}$$

Since $\widehat{\Omega}_{B/A}$ is finite over B, it is $\mathfrak{m}_B$-adically complete, so taking inverse limits again gives

$$\widehat{\Omega}_{B/A} \cong \varprojlim_m (\widehat{\Omega}_{B/A} \otimes_B B/\mathfrak{m}_B^m) \cong \varprojlim_m \Omega_{B/A}/\mathfrak{m}_B^m\Omega_{B/A}.$$

This proves the first claim. Note that this shows $\widehat{\Omega}_{B/A}$ is the module $\widehat{\Omega}^{\mathrm{an}}_{B/A} = (\widehat{\Omega}_{B/A})^\wedge$ from [13], so the second claim follows from the definition of the André-Quillen cohomology groups and [13, Lemma 7.1.27(iii)]. □

We will need to know the continuous André-Quillen cohomology of power series rings, which we compute in the following lemma:

Lemma 3.4.5. *Given any $n \geq 0$, let $A = \mathcal{O}[[x_1, \dots, x_n]]$ and let M be an A-module. We have*

$$\widehat{\mathrm{Der}}^i_{\mathcal{O}}(A, M) = \begin{cases} M^n & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. By [13, Proposition 7.1.29] we have $L^{\mathrm{an}}_{A/\mathcal{O}} = \widehat{\Omega}_{A/\mathcal{O}}[0] = A^n[0]$, thus

$$\mathrm{RHom}_A(L^{\mathrm{an}}_{A/\mathcal{O}}, M) = \mathrm{RHom}_A(A^n[0], M) = \mathrm{RHom}_A(A^n, M) = M^n[0],$$

from which the lemma follows. □

Lastly we will need to know the degree 0 and 1 continuous André-Quillen cohomology groups $\widehat{\mathrm{Der}}^i_A(B, \cdot)$ for rings B which are quotients of A.

Lemma 3.4.6. *Let A be a ring and $B = A/I$ for some ideal $I \subseteq A$. Then $\widehat{\mathrm{Der}}^0_A(B, M) = 0$ and $\widehat{\mathrm{Der}}^1_A(B, M) = \mathrm{Hom}_B(I/I^2, M)$*

Proof. Clearly B is a finite A-module, so Proposition 3.4.3 gives $\widehat{\mathrm{Der}}^i_A(B, M) \cong \mathrm{Der}^i_A(B, M)$ for all $i \geq 0$ and all B-modules M. The lemma now follows from [20, 6.12]. □

We can now work toward proving the main result of this section. We first want to show that the continuous André-Quillen cohomology of R_θ depends only on R and not the choice of θ.

Lemma 3.4.7. *For any $i \geq 0$ and any R_θ-module M, we have*

$$\widehat{\mathrm{Der}}^i_S(R, M) \cong \mathrm{Der}^i_S(R, M) \cong \mathrm{Der}^i_{\mathcal{O}}(R_\theta, M) \cong \widehat{\mathrm{Der}}^i_{\mathcal{O}}(R_\theta, M).$$

Proof. The first and last isomorphisms follow from Proposition 3.4.3 since R is a finite S-module and R_θ is a finite $\mathcal{O}$-module. For the second isomorphism, recall that R is a finite free S-module, thus $0 \to R \to R \to 0$ is a projective resolution of R in $D(\mathrm{Mod}_S)$. Therefore $R \otimes^{\mathrm{L}}_S \mathcal{O} = R \otimes_S \mathcal{O} \cong R_\theta$, so applying [43, Lemma 91.6.2] with the commutative square

$$\begin{array}{ccc} R & \longrightarrow & R_\theta \\ {\scriptstyle\theta}\uparrow & & \uparrow \\ S & \longrightarrow & \mathcal{O} \end{array}$$

gives that $L_{R/S} \otimes^{\mathrm{L}}_R R_\theta \cong L_{R_\theta/\mathcal{O}}$. Then [43, Lemma 15.99.1] gives

$$\mathrm{RHom}_R(L_{R/S}, M) \cong \mathrm{RHom}_{R_\theta}(L_{R/S} \otimes^{\mathrm{L}}_R R_\theta, M) \cong \mathrm{RHom}_{R_\theta}(L_{R_\theta/\mathcal{O}}, M)$$

for all R_θ-modules M. The second isomorphism now follows from the definition of the André-Quillen cohomology groups. □

We now need to show that $\widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O})$ does not depend on the choice of $\theta\colon S \hookrightarrow R$. We view $E/\mathcal{O}$ as a module over R and R_θ via the augmentations $\lambda\colon R \twoheadrightarrow \mathcal{O}$ and $\lambda_\theta\colon R_\theta \twoheadrightarrow \mathcal{O}$.

Proposition 3.4.8. *We have* $\widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$.

Proof. We apply Proposition 3.4.2 to $\mathcal{O} \to S \xrightarrow{\theta} R$ with $M = E/\mathcal{O}$ to obtain the exact sequence

$$0 \longrightarrow \widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(S, E/\mathcal{O})$$

$$\longrightarrow \widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(S, E/\mathcal{O}).$$

By Proposition 3.4.4 we have

$$\widehat{\mathrm{Der}}^0_{\mathcal{O}}(R, E/\mathcal{O}) = \mathrm{Hom}_R(\widehat{\Omega}_{R/\mathcal{O}}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}, E/\mathcal{O}).$$

By assumption $R[1/\varpi]$ is smooth at the augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$, so
$\widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O} \cong \mathcal{O}^d \oplus T$ with T a finite torsion $\mathcal{O}$-module. Then

$$\mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R/\mathcal{O}} \otimes_\lambda \mathcal{O}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^d, E/\mathcal{O}) \oplus \mathrm{Hom}_{\mathcal{O}}(T, E/\mathcal{O}) \cong (E/\mathcal{O})^d \oplus T.$$

Similarly, since θ is a 1-codimensional embedding, $\Phi_{\lambda_\theta}(R_\theta) \cong \widehat{\Omega}_{R_\theta/\mathcal{O}} \otimes_{\lambda_\theta} \mathcal{O}$ is finite, so

$$\mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R_\theta/\mathcal{O}}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\Phi_{\lambda_\theta}(R_\theta), E/\mathcal{O}) \cong T',$$

where T' is some finite torsion $\mathcal{O}$-module. Using Proposition 3.4.8 and Proposition 3.4.4 we see that

$$\widehat{\mathrm{Der}}^0_S(R, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R_\theta, E/\mathcal{O}) = \mathrm{Hom}_{R_\theta}(\widehat{\Omega}_{R_\theta/\mathcal{O}}, E/\mathcal{O}).$$

We also have $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(S, E/\mathcal{O}) \cong (E/\mathcal{O})^d$ and $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(S, E/\mathcal{O}) = 0$ by Lemma 3.4.5.

The exact sequence now simplifies to

$$0 \longrightarrow T' \longrightarrow (E/\mathcal{O})^d \oplus T \longrightarrow (E/\mathcal{O})^d \longrightarrow \widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow 0$$

Now we see that $(E/\mathcal{O})^d \oplus T \to (E/\mathcal{O})^d$ has finite kernel and its image is a finite-index subgroup of $(E/\mathcal{O})^d$. Since $(E/\mathcal{O})^d$ is divisible, the only such subgroup is $(E/\mathcal{O})^d$, hence $(E/\mathcal{O})^d \oplus T \to (E/\mathcal{O})^d$ is surjective. Then $(E/\mathcal{O})^d \to \widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O})$ has trivial image, so $\widehat{\mathrm{Der}}^1_S(R, E/\mathcal{O}) \to \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$ is an isomorphism as desired. □

Lemma 3.4.7 and Proposition 3.4.8 now immediately imply the following theorem, which shows that $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$ is independent of the choice of θ.

Theorem 3.4.9. *We have* $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$.

To actually compute the Wiles defect of (R, λ), we require a method for computing $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$. For this we fix a CI covering $\widetilde{R} \twoheadrightarrow R$ with kernel I. We will need the following lemmas:

Lemma 3.4.10. *Let A be a Noetherian local ring and $J \subseteq A$ be an ideal generated by a regular sequence $f_1, \dots, f_n$. Then J/J^2 is free of rank n as an A/J-module.*

Proof. Clearly J/J^2 is generated over A/J by the images of $f_1, \dots, f_n$, so it suffices to show these are linearly independent. Suppose

$$a_1 f_1 + \cdots + a_n f_n \in J^2$$

for $a_1, \dots, a_n \in A$. Then there exist $b_1, \dots, b_n \in J$ such that

$$a_1 f_1 + \cdots + a_n f_n = b_1 f_1 + \cdots b_n f_n.$$

Let $c_i = a_i - b_i$ for each i. Then

$$c_1 f_1 + \cdots + c_n f_n = 0,$$

hence

$$-c_n f_n = c_1 f_1 + \cdots + c_{n-1} f_{n-1} \in (f_1, \dots, f_{n-1})A$$

Since $f_1, \dots, f_n$ is a regular sequence, this implies that $c_n \in J$, so $a_n \in J$. Since A is a Noetherian local ring, any permutation of $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n, f_i$ is also a regular sequence for each $i < n$, so the above argument shows that $a_i \in J$ for each i, hence the images of f_i in A/J are linearly independent as desired. □

Lemma 3.4.11. *Let $n \geq 1$ and let $M \subset (E/\mathcal{O})^n$ be an $\mathcal{O}$-submodule.*

(i) We have $M \cong (E/\mathcal{O})^n$ if and only if $M = (E/\mathcal{O})^n$.

(ii) If $|M|$ is finite, then $(E/\mathcal{O})^n / M \cong (E/\mathcal{O})^n$.

Proof. To prove (i), note that

$$(E/\mathcal{O})^n \cong \varinjlim_k (E/\mathcal{O})^n[\varpi^k],$$

where $(E/\mathcal{O})^n[\varpi^k]$ is the ϖ^k-torsion submodule of $(E/\mathcal{O})^n$. Therefore $M \cong (E/\mathcal{O})^n$ if and only if $M[\varpi^k] = (E/\mathcal{O})^n[\varpi^k]$ for all $k \geq 1$. But $(E/\mathcal{O})^n[\varpi^k] \cong (\mathcal{O}/(\varpi^k))^n$ has finite cardinality, so this occurs if and only if $M = (E/\mathcal{O})^n$, proving (i).

Now we show (ii). First suppose M is generated by a single element $x \in M$. Consider the natural map

$$E^n \twoheadrightarrow (E/\mathcal{O})^n \twoheadrightarrow (E/\mathcal{O})^n/M,$$

whose kernel is $L = \mathcal{O}^n + \mathcal{O}\widetilde{x}$ for some lift $\widetilde{x}$ of x. Note that E^n is torsion-free, so L is torsion-free and finitely-generated, hence free. Moreover, the rank of L is clearly at least n but also at most n since L is a submodule of E^n. If $v_1, \ldots, v_n \in L$ is an $\mathcal{O}$-basis of L, then it is an E-basis of E^n, so we have

$$(E/\mathcal{O})^n/M \cong E^n/L \cong \bigoplus_{i=1}^n Ev_i / \bigoplus_{i=1}^n Lv_i \cong (E/\mathcal{O})^n.$$

(ii) now follows by induction on the number of generators of M. □

Our last theorem of this section gives a method of computing $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$.

Theorem 3.4.12. *We have an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_R(\widehat{\Omega}_{R/\mathcal{O}}, E/\mathcal{O}) \longrightarrow \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, E/\mathcal{O})$$

$$\longrightarrow \mathrm{Hom}_R(I/I^2, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow 0$$

Proof. We apply Proposition 3.4.2 to $\mathcal{O} \to \widetilde{R} \xrightarrow{\varphi} R$ with $M = E/\mathcal{O}$ to obtain the exact sequence

$$0 \longrightarrow \widehat{\mathrm{Der}}^0_{\widetilde{R}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O})$$

$$\longrightarrow \widehat{\mathrm{Der}}^1_{\widetilde{R}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O})$$

Applying Proposition 3.4.4 to the first three terms and Lemma 3.4.6 to the fourth term, the exact sequence becomes

$$0 \longrightarrow \mathrm{Hom}_R(\widehat{\Omega}_{R/\widetilde{R}}, E/\mathcal{O}) \longrightarrow \mathrm{Hom}_R(\widehat{\Omega}_{R/\mathcal{O}}, E/\mathcal{O}) \longrightarrow \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, E/\mathcal{O})$$

$$\longrightarrow \mathrm{Hom}_R(I/I^2, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O})$$

But $\widehat{\Omega}_{R/\widetilde{R}} = 0$ since $\widetilde{R} \to R$ is surjective, thus $\mathrm{Hom}_R(\widehat{\Omega}_{R/\widetilde{R}}, E/\mathcal{O}) = 0$. The proof will thus be complete once we show that $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) = 0$.

Since $\widetilde{R}$ is a complete intersection, there exists a power series ring $P = \mathcal{O}[[x_1, \ldots, x_{n+d}]]$ and a surjection $P \twoheadrightarrow \widetilde{R}$ with kernel $J \subseteq P$ generated by a regular sequence $f_1, \ldots, f_n$. We apply Proposition 3.4.2 to $\mathcal{O} \to P \to \widetilde{R}$ with $M = E/\mathcal{O}$ to obtain the exact sequence

$$0 \longrightarrow \widehat{\mathrm{Der}}^0_{P}(\widetilde{R}, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^0_{\mathcal{O}}(P, E/\mathcal{O})$$

$$\longrightarrow \widehat{\mathrm{Der}}^1_{P}(\widetilde{R}, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(P, E/\mathcal{O})$$

As with $\widehat{\mathrm{Der}}^0_{\widetilde{R}}(R, E/\mathcal{O})$, we have $\widehat{\mathrm{Der}}^0_{P}(\widetilde{R}, E/\mathcal{O}) = 0$ since $P \to \widetilde{R}$ is surjective. By Lemma 3.4.5 we have $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(P, E/\mathcal{O}) = (E/\mathcal{O})^{n+d}$ and $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(P, E/\mathcal{O}) = 0$. As in the proof of Proposition 3.4.8, we see that $\widehat{\mathrm{Der}}^0_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) \cong (E/\mathcal{O})^d \oplus T$ for some finite torsion $\mathcal{O}$-module T since $\widetilde{R}[1/\varpi]$ is formally smooth at $\widetilde{\lambda}$. Lastly, by Lemma 3.4.10, we have that $J/J^2 \cong \widetilde{R}^n$ as an $\widetilde{R}$-module. Applying Lemma 3.4.6 again, we thus see that

$$\widehat{\mathrm{Der}}^1_{P}(\widetilde{R}, E/\mathcal{O}) \cong \mathrm{Hom}_{\widetilde{R}}(J/J^2, E/\mathcal{O}) \cong (E/\mathcal{O})^n.$$

Therefore the exact sequence above is

$$0 \longrightarrow (E/\mathcal{O})^d \oplus T \longrightarrow (E/\mathcal{O})^{n+d} \longrightarrow (E/\mathcal{O})^n \longrightarrow \widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) \longrightarrow 0$$

Consider the exact sequence

$$0 \longrightarrow (E/\mathcal{O})^d \longrightarrow (E/\mathcal{O})^{n+d} \longrightarrow M' \longrightarrow 0$$

where the injective map is the composition $(E/\mathcal{O})^d \hookrightarrow (E/\mathcal{O})^d \oplus T \to (E/\mathcal{O})^{n+d}$ and $M = (E/\mathcal{O})^{n+d}/(E/\mathcal{O})^d$ is its cokernel. Since $(E/\mathcal{O})^d$ is an injective $\mathcal{O}$-module, the above sequence splits, thus

$$(E/\mathcal{O})^{n+d} \cong (E/\mathcal{O})^d \oplus M'.$$

Now observe that for every $r, k \geq 1$, the ϖ^k-torsion submodule of $(E/\mathcal{O})^r$ is isomorphic to $(\mathcal{O}/(\varpi^k))^r$. We thus see from the above direct sum decomposition that the ϖ^k-torsion submodule of M' is isomorphic to $(\mathcal{O}/(\varpi^k))^n$, thus

$$M' \cong \varinjlim_k (\mathcal{O}/(\varpi^k))^n \cong (E/\mathcal{O})^n.$$

Let M be the image of $(E/\mathcal{O})^{n+d} \to (E/\mathcal{O})^n$, which is isomorphic to $(E/\mathcal{O})^{n+d}/((E/\mathcal{O})^d \oplus T)$. Then $M \cong M'/T'$ where $T' = ((E/\mathcal{O})^d + T)/(E/\mathcal{O})^d$. Since $|T'|$ has finite cardinality, we have $M \cong (E/\mathcal{O})^n$ by Lemma 3.4.11 (ii). Then M is a submodule of $(E/\mathcal{O})^n$ which is isomorphic to $(E/\mathcal{O})^n$, so $M = (E/\mathcal{O})^n$ by Lemma 3.4.11 (i), i.e. $(E/\mathcal{O})^{n+d} \to (E/\mathcal{O})^n$ is surjective. We now see from the exact sequence that $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(\widetilde{R}, E/\mathcal{O}) = 0$ as desired. $\square$

We now have a concrete method of computing $|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})|$, explained in the following corollary:

Corollary 3.4.13. *Let M be the cokernel of* $\mathrm{Hom}_R(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, E/\mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{R/\mathcal{O}}, E/\mathcal{O})$. *Then*

$$|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})| = |\operatorname{Hom}_R(I/I^2, E/\mathcal{O})|/|M|.$$

Proof. This follows straightforwardly from Theorem 3.4.12. $\square$

3.5 The Wiles defect for augmented rings in the category $\mathrm{CNL}^a_{\mathcal{O}}$

In the previous two sections, we defined the invariants $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$ which make sense for any augmented ring $(R, \lambda) \in \mathrm{CNL}_{\mathcal{O}}$, even when $\dim R > 1$. In this section, we define the Wiles defect for objects in $\mathrm{CNL}_{\mathcal{O}}$ in terms of these two invariants. We then prove some useful properties satisfied by the Wiles defect.

Definition 3.5.1. *Let $(R, \lambda_R) \in \mathrm{CNL}_{\mathcal{O}}$, i.e. R is a complete Noetherian local $\mathcal{O}$-algebra which is Cohen-Macaulay and flat over $\mathcal{O}$ together with an augmentation $\lambda\colon R \twoheadrightarrow \mathcal{O}$ such that $R[1/\varpi]$ is formally smooth at λ. Define the invariants*

$$d_{1,\lambda}(R) = \frac{\log_p |\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})|}{\log_p |\mathcal{O}/p|}$$

and

$$c_{1,\lambda}(R) = \frac{\log_p |C_{1,\lambda}(R)|}{\log_p |\mathcal{O}/p|},$$

where the definition of $C_{1,\lambda}(R)$ is given in Definition 3.3.4. Then we define the Wiles defect of R with respect to the augmentation λ to be

$$\delta_\lambda(R) = d_{1,\lambda}(R) - c_{1,\lambda}(R).$$

We fix an augmented ring $(R, \lambda) \in \mathrm{CNL}_{\mathcal{O}}$ for the remainder of this section so as not to repeat this in the statements of the following results. First we give our main theorem of this chapter, which shows that the Wiles defect of (R, λ) is the same as the Wiles defect of the augmented quotient ring $(R_\theta.\lambda_\theta)$ obtained from a 1-codimensional embedding:

Theorem 3.5.2. *Let $\theta\colon S = \mathcal{O}[[y_1, \ldots, y_d]] \hookrightarrow R$ be a 1-codimensional embedding into R, and let $R_\theta = R/(y_1, \ldots, y_d)$ with induced augmentation $\lambda_\theta\colon R_\theta \twoheadrightarrow \mathcal{O}$. Then the invariants*

$\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}), C_{1,\lambda_\theta}(R_\theta)$, *and* $\delta_{\lambda_\theta}(R_\theta)$ *depend only on* R *and* λ *and not on* θ.

Proof. Lemma 3.4.7 and Proposition 3.4.8 imply that $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) \cong \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$, so $\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$ and thus $d_{1,\lambda_\theta}(R_\theta)$ depends only on R and λ. As mentioned in Definition 3.3.4, it follows from Theorem 3.3.3 that $C_{1,\lambda_\theta} \cong C_{1,\lambda}(\widetilde{R})$ given a CI covering $\widetilde{R} \twoheadrightarrow R$. It thus follows from [40] that $C_{1,\lambda_\theta}(R_\theta)$ and thus $c_{1,\lambda}$ also only depends on R and λ. The same is thus true for $\delta_{\lambda_\theta}(R_\theta)$ by definition. □

We have the following useful reformulation of the above theorem, which shows that the Wiles defect remains invariant upon quotienting by a regular sequence annihilated by the augmentation:

Theorem 3.5.3. *Let* $r_1, \ldots, r_d, \varpi$ *be a regular sequence in* R *with* $(r_1, \ldots, r_d) \subseteq \ker\lambda$ *and let* $\lambda_\theta \colon R_\theta = R/(r_1, \ldots, r_d) \twoheadrightarrow \mathcal{O}$ *be the augmentation induced by* λ. *Then*

$$\delta_\lambda(R) = \delta_{\lambda_\theta}(R_\theta).$$

Proof. Define $\theta \colon S = \mathcal{O}[[y_1, \ldots, y_d]] \to R$ by $\theta(y_i) = r_i$. Note that the proof Proposition 3.2.5 depended only on choosing a regular sequence $f_1, \ldots, f_d, \varpi$ in R with $(f_1, \ldots, f_d) \subseteq \ker\lambda$, so θ is a 1-codimensional embedding. The result now follows from Theorem 3.5.2. □

The following proposition shows that our definition of the Wiles defect for augmented rings agrees with Definition 3.1.1 for rings of dimension one:

Proposition 3.5.4. *In the case when* $\dim R = 1$, *we have*

$$\delta_\lambda(R) = d_{1,\lambda}(R) - c_{1,\lambda}(R) = \frac{\log_p |\Phi_\lambda(R)| - \log_p |\Psi_\lambda(R)|}{\log_p |\mathcal{O}/p|}.$$

Proof. □

Proof. Choose a 1-codimensional embedding $\theta\colon S = \mathcal{O}[[y_1, \dots, y_d]] \hookrightarrow R$. Then by Theorem 3.5.2 we have

$$\delta_\lambda(R) = \delta_{\lambda_\theta}(R_\theta),$$

where $\lambda_\theta\colon R_\theta = R/(y_1, \dots, y_d) \twoheadrightarrow \mathcal{O}$ is the induced augmentation. Definition 3.3.4 for $C_{1,\lambda_\theta}(R_\theta)$ coincides with the definition given in Section 3.1 for 1-dimensional rings, and

$$\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) \cong \mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O})$$

by Lemma 3.4.7. The proposition now follows from Theorem 3.1.2. □

As in the 1-dimensional case, the Wiles defect measures the degree to which R fails to be a complete intersection.

Proposition 3.5.5. *We have $\delta_\lambda(R) = 0$ if and only if R is a complete intersection.*

Proof. If R is a complete intersection, then the identity map $\varphi\colon R \to R$ is a CI covering of R. Since $I = \ker\varphi = 0$, we have $\mathrm{Ann}_R(I) = \mathrm{Fitt}_R(I) = R$, thus $C_{1,\lambda}(R) = 0$, and the argument given in the proof of Theorem 3.4.12 shows that $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) = 0$. Therefore $\delta_\lambda(R) = 0$.

If $\delta_\lambda(R) = 0$, then choose a 1-codimensional embedding $\theta\colon S = \mathcal{O}[[y_1, \dots, y_d]] \hookrightarrow R$ so that $R_\theta = R/(y_1, \dots, y_d)$ has dimension 1. Letting $\lambda_\theta\colon R_\theta \twoheadrightarrow \mathcal{O}$ be the induced augmentation, we have

$$\delta_{\lambda_\theta}(R_\theta) = \delta_\lambda(R) = 0$$

by Theorem 3.5.2, so R_θ is a complete intersection by Proposition 3.5.4. Since R_θ is a quotient of R by a regular sequence, this implies that R is also a complete intersection. □

The last property of the Wiles defect we will need to know is additivity over completed tensor products.

Proposition 3.5.6. *Let* $(R_1, \lambda_1), (R_2, \lambda_2) \in \mathrm{CNL}^a_{\mathcal{O}}$. *Assume that* R_1 *and* R_2 *are reduced. If we let* $R = R_1 \widehat{\otimes}_{\mathcal{O}} R_2$ *and* $\lambda = \lambda_1 \widehat{\otimes} \lambda_2 \colon R \twoheadrightarrow \mathcal{O}$, *then we have*

$$\delta_\lambda(R) = \delta_{\lambda_1}(R_1) + \delta_{\lambda_2}(R_2).$$

Proof. The lemma will follow by definition once we show that

$$c_{1,\lambda}(R) = c_{1,\lambda_1}(R_1) + c_{1,\lambda_2}(R_2) \tag{3.1}$$

and

$$d_{1,\lambda}(R) = d_{1,\lambda_1}(R_1) + d_{1,\lambda_2}(R_2). \tag{3.2}$$

To prove (3.2), choose 1-codimensional embeddings $\theta_1 \colon S_1 = \mathcal{O}[[x_1, \ldots, x_{d_1}]] \hookrightarrow R_1$ and $\theta_2 \colon S_2 = \mathcal{O}[[y_1, \ldots, y_{d_2}]] \hookrightarrow R_2$. Then the map

$$\theta = \theta_1 \otimes \theta_2 \colon S_1 \widehat{\otimes}_{\mathcal{O}} S_2 = \mathcal{O}[[x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}]] \hookrightarrow R$$

is a 1-codimensional embedding into R. Let

$$\begin{aligned} R_{1,\theta_1} &= R_{1,\theta_1}/(x_1, \ldots, x_{d_1}) \\ R_{2,\theta_2} &= R_{2,\theta_2}/(y_1, \ldots, y_{d_2}) \\ R_\theta &= R/(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}). \end{aligned}$$

By Lemma 3.4.7 we have

$$\begin{aligned} \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_1, E/\mathcal{O}) &\cong \mathrm{Der}^1_{\mathcal{O}}(R_{1,\theta_1}, E/\mathcal{O}) \\ \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_2, E/\mathcal{O}) &\cong \mathrm{Der}^1_{\mathcal{O}}(R_{2,\theta_2}, E/\mathcal{O}) \\ \widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O}) &\cong \mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}), \end{aligned}$$

so it suffices to show that

$$\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) \cong \mathrm{Der}^1_{\mathcal{O}}(R_{1,\theta_1}, E/\mathcal{O}) \oplus \mathrm{Der}^1_{\mathcal{O}}(R_{2,\theta_2}, E/\mathcal{O}).$$

Per the proof of Proposition 3.2.5, R_{1,θ_1} and R_{2,θ_2} are free over $\mathcal{O}$, thus Tor independent $\mathcal{O}$-algebras, i.e. $\mathrm{Tor}_i^{\mathcal{O}}(R_{1,\theta_1}, R_{2,\theta_2}) = 0$ for all $i > 0$. Noting that $R_\theta \cong R_{1,\theta_1} \otimes_{\mathcal{O}} R_{2,\theta_2}$, we see by [43, Lemma 91.15.1] that

$$L_{R_\theta/\mathcal{O}} \cong L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{1,\theta_1}} R_\theta \oplus L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{2,\theta_2}} R_\theta.$$

Therefore

$$\begin{aligned}
\mathrm{Der}^1_{\mathcal{O}}(R_\theta, E/\mathcal{O}) &= H^1(\mathrm{RHom}_{R_\theta}(L_{R_\theta/\mathcal{O}}, E/\mathcal{O})) \\
&\cong H^1(\mathrm{RHom}_{R_\theta}(L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{1,\theta_1}} R_\theta \oplus L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{2,\theta_2}} R_\theta, E/\mathcal{O})) \\
&\cong H^1(\mathrm{RHom}_{R_\theta}(L_{R_{1,\theta_1}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{1,\theta_1}} R_\theta, E/\mathcal{O})) \\
&\quad \oplus H^1(\mathrm{RHom}_{R_\theta}(L_{R_{2,\theta_2}/\mathcal{O}} \otimes^{\mathrm{L}}_{R_{2,\theta_2}} R_\theta, E/\mathcal{O})) \\
&\cong H^1(\mathrm{RHom}_{R_{1,\theta_1}}(L_{R_{1,\theta_1}/\mathcal{O}}, E/\mathcal{O})) \oplus H^1(\mathrm{RHom}_{R_{2,\theta_2}}(L_{R_{2,\theta_2}/\mathcal{O}}, E/\mathcal{O})) \\
&= \mathrm{Der}^1_{\mathcal{O}}(R_{1,\theta_1}, E/\mathcal{O}) \oplus \mathrm{Der}^1_{\mathcal{O}}(R_{2,\theta_2}, E/\mathcal{O})
\end{aligned}$$

as desired.

To prove (3.1), choose CI coverings $\varphi_1 \colon \widetilde{R}_1 \to R_1$ and $\varphi_2 \colon \widetilde{R}_2 \to R_2$ with kernels I_1 and I_2, respectively. Let $\widetilde{R} = \widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2$ and $\widetilde{\lambda} = \widetilde{\lambda}_1 \otimes \widetilde{\lambda}_2 \colon \widetilde{R} \twoheadrightarrow \mathcal{O}$. Note that $\varphi = \varphi_1 \otimes \varphi_2 \colon \widetilde{R} \twoheadrightarrow R$ is a CI covering. Since R_1 and R_2 are flat over $\mathcal{O}$, we see that $I_1 \widehat{\otimes}_{\mathcal{O}} \widetilde{R}_2$ and $\widetilde{R}_1 \widehat{\otimes}_{\mathcal{O}} I_2$ are ideals in $\widetilde{R}$ and their sum is $I = \ker \varphi$. For $(R', \lambda') \in \mathrm{CNL}^a_{\mathcal{O}}$ and a CI covering $\varphi' \colon \widetilde{R}' \twoheadrightarrow R'$ with kernel I', we have

$$c_{1,\lambda'}(R') \log_p |\mathcal{O}/p| = \log_p |\lambda'(\mathrm{Ann}_{\widetilde{R}'}(I'))/\lambda'(\mathrm{Fitt}_{\widetilde{R}'}(I'))|,$$

so to prove (3.1), it suffices to show that

$$\widetilde{\lambda}(\mathrm{Ann}_{\widetilde{R}}(I)) = \widetilde{\lambda}_1(\mathrm{Ann}_{\widetilde{R}_1}(I_1))\widetilde{\lambda}_2(\mathrm{Ann}_{\widetilde{R}_2}(I_2))$$
$$\widetilde{\lambda}((\mathrm{Fitt}_{\widetilde{R}}(I)) = \widetilde{\lambda}_1((\mathrm{Fitt}_{\widetilde{R}_1}(I))\widetilde{\lambda}_2(\mathrm{Fitt}_{\widetilde{R}_2}(I)).$$

We have

$$\begin{aligned}
\mathrm{Ann}_{\widetilde{R}}(I) &= \mathrm{Ann}_{\widetilde{R}}(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2 + \widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2) \\
&= \mathrm{Ann}_{\widetilde{R}}(I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2) \cap \mathrm{Ann}_{\widetilde{R}}(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2) \\
&= \mathrm{Ann}_{\widetilde{R}_1}(I_1)\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2 \cap \widetilde{R}_1\widehat{\otimes}_{\mathcal{O}} \mathrm{Ann}_{\widetilde{R}_2}(I_2) \\
&= \mathrm{Ann}_{\widetilde{R}_1}(I_1)\widehat{\otimes}_{\mathcal{O}} \mathrm{Ann}_{\widetilde{R}_2}(I_2),
\end{aligned}$$

where the third line follows by taking a presentation of I_1 (resp. I_2) and tensoring it with $\widetilde{R}_2$ (resp. $\widetilde{R}_1$) and the fourth line follows from $\mathcal{O}$-flatness. Therefore

$$\widetilde{\lambda}(\mathrm{Ann}_{\widetilde{R}}(I)) = (\widetilde{\lambda}_1 \otimes \widetilde{\lambda}_2)(\mathrm{Ann}_{\widetilde{R}_1}(I_1)\widehat{\otimes}_{\mathcal{O}} \mathrm{Ann}_{\widetilde{R}_2}(I_2)) = \widetilde{\lambda}_1((\mathrm{Fitt}_{\widetilde{R}_1}(I))\widetilde{\lambda}_2(\mathrm{Fitt}_{\widetilde{R}_2}(I))$$

as desired.

For the statement above concerning fitting ideals, fix presentations

$$0 \longrightarrow K_1 \longrightarrow \widetilde{R}_1^m \xrightarrow{\alpha} I_1 \longrightarrow 0$$

$$0 \longrightarrow K_2 \longrightarrow \widetilde{R}_2^m \xrightarrow{\beta} I_2 \longrightarrow 0,$$

where K_i is a finitely-generated $\widetilde{R}_i$-module. Then α and β induce surjective maps $\alpha\otimes 1\colon \widetilde{R}^m = \widetilde{R}_1^m\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2 \to I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2$ and $1 \otimes \beta\colon \widetilde{R}^n = \widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2^n \to \widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2$. We obtain another surjective map

$$\gamma = (\alpha \otimes 1) - (1 \otimes \beta)\colon \widetilde{R}^{m+n} = \widetilde{R}^m \oplus \widetilde{R}^n \to I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2 + \widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2 = I$$

with kernel $K = \ker\gamma$. Identify elements in K_1, K_2, K with their images in $\widetilde{R}_1^m, \widetilde{R}_2^m, \widetilde{R}^{m+n}$, respectively. By definition $\mathrm{Fitt}_{\widetilde{R}_1}(I_1)$ is the ideal in $\widetilde{R}_1$ generated by elements of the form $\det(u_1,\ldots,u_m) \in \widetilde{R}_1$ for all $u_1,\ldots,u_m \in K_1$, and $\mathrm{Fitt}_{\widetilde{R}_2}(I_2)$ is likewise generated by all elements of the form $\det(v_1,\ldots,v_n) \in \widetilde{R}_2$ for all $v_1,\ldots,v_n \in K_2$. Given $u_1,\ldots,u_m \in K_1$ and $v_1,\ldots,v_m \in K_2$, we see that $\left(\begin{smallmatrix} u_i\otimes 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1\otimes v_j \end{smallmatrix}\right) \in K$ for all i and j, thus $\mathrm{Fitt}_{\widetilde{R}}(I)$ contains

$$\det\begin{pmatrix} u_1\otimes 1 & \cdots & u_m\otimes 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1\otimes v_1 & \cdots & 1\otimes v_n \end{pmatrix} = \det(u_1,\ldots,u_m)\otimes\det(v_1,\ldots,v_m).$$

This shows that $\mathrm{Fitt}_{\widetilde{R}_1}(I_1)\otimes_{\mathcal{O}}\mathrm{Fitt}_{\widetilde{R}_2}(I_2) \subseteq \mathrm{Fitt}_{\widetilde{R}}(I)$. Therefore

$$\widetilde{\lambda}_1(\mathrm{Fitt}_{\widetilde{R}_1}(I))\widetilde{\lambda}_2(\mathrm{Fitt}_{\widetilde{R}_2}(I)) \subseteq (\widetilde{\lambda}_1\otimes\widetilde{\lambda}_2)(\mathrm{Fitt}_{\widetilde{R}_1}(I_1)\otimes_{\mathcal{O}}\mathrm{Fitt}_{\widetilde{R}_2}(I_2)) \subseteq \widetilde{\lambda}(\mathrm{Fitt}_{\widetilde{R}}(I)).$$

For the reverse incluson, let $w = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \in K$ with $w_1 \in \widetilde{R}^m$ and $w_2 \in \widetilde{R}^n$. Since $w \in K$, we have

$$\gamma(w) = (\alpha\otimes 1)(w_1) - (1\otimes\beta)(w_2) = 0,$$

so we may let $r = (\alpha\otimes 1)(w_1) = (1\otimes\beta)(w_2) \in \widetilde{R}$. Note that

$$r \in (I_1\widehat{\otimes}_{\mathcal{O}}\widetilde{R}_2)\cap(\widetilde{R}_1\widehat{\otimes}_{\mathcal{O}}I_2) = I_1\widehat{\otimes}_{\mathcal{O}}I_2.$$

By definition $\widetilde{\lambda}_1(I_1) = \widetilde{\lambda}_2(I_2) = 0$, thus $(\widetilde{\lambda}_1\otimes 1)(r) = (1\otimes\widetilde{\lambda}_2)(r) = 0$. Let $u = (1\otimes\widetilde{\lambda}_2)(w_1) \in \widetilde{R}_1^m$ and $v = (\widetilde{\lambda}_1\otimes 1)(w_2) \in \widetilde{R}_2^n$ so that

$$\begin{aligned} \widetilde{\lambda}_1(u) &= \widetilde{\lambda}_1((\mathrm{id}\otimes\widetilde{\lambda}_2)(w_1)) = (\widetilde{\lambda}_1\otimes\widetilde{\lambda}_2)(w_1) = \widetilde{\lambda}(w_1) \\ \widetilde{\lambda}_2(u) &= \widetilde{\lambda}_2((\widetilde{\lambda}_1\otimes\mathrm{id})(w_2)) = (\widetilde{\lambda}_1\otimes\widetilde{\lambda}_2)(w_2) = \widetilde{\lambda}(w_2). \end{aligned}$$

and

$$\alpha(u) = (\alpha \otimes \mathrm{id})(\mathrm{id} \otimes \widetilde{\lambda}_2)(w_1) = (\mathrm{id} \otimes \widetilde{\lambda}_2)(\alpha \otimes \mathrm{id})(w_1) = (\mathrm{id} \otimes \widetilde{\lambda}_2)(r) \ = 0$$

$$\beta(u) = (\mathrm{id} \otimes \beta)(\widetilde{\lambda}_1 \otimes \mathrm{id})(w_2) = (\widetilde{\lambda}_1 \otimes \mathrm{id})(\mathrm{id} \otimes \beta)(w_1) = (\widetilde{\lambda}_1 \otimes \mathrm{id})(r) = 0.$$

Therefore $w_1 \in \ker \alpha$ and $w_2 \in \ker \beta$.

Given $w_1, \ldots, w_{m+n} \in K$, we can thus write $\widetilde{\lambda}(w_i) = \left(\begin{smallmatrix} \widetilde{\lambda}_1(u_i) \\ \widetilde{\lambda}_2(v_i) \end{smallmatrix}\right)$ for $u_i \in K_1, v_i \in K_2$. Then

$$\widetilde{\lambda}(\det(w_1, \ldots, w_{m+n})) = \det \begin{pmatrix} \widetilde{\lambda}_1(u_1) & \cdots & \widetilde{\lambda}_1(u_{m+n}) \\ \widetilde{\lambda}_2(v_1) & \cdots & \widetilde{\lambda}_2(v_{m+n}) \end{pmatrix}.$$

The above determinant can be expressed as an alternating sum of the form

$$\sum_{X,Y} (\pm 1) \det((\widetilde{\lambda}_1(u_i))_{i \in X}) \det((\widetilde{\lambda}_2(v_j))_{j \in Y}) = \sum_{X,Y} (\pm 1) \widetilde{\lambda}_1(\det((u_i)_{i \in X})) \widetilde{\lambda}_2(\det((v_j)_{j \in Y})),$$

where the sums are taken over disjoint partitions $X \cup Y$ of $\{1, \ldots, m+n\}$ with $|X| = m$ and $|Y| = n$. The above sum is clearly in $\widetilde{\lambda}_1(\mathrm{Fitt}_{\mathcal{O}}(I_1))\widetilde{\lambda}_2(\mathrm{Fitt}_{\mathcal{O}}(I_2))$, so

$$\widetilde{\lambda}(\mathrm{Fitt}_{\mathcal{O}}(I)) \subseteq \widetilde{\lambda}_1(\mathrm{Fitt}_{\widetilde{R}_1}(I))\widetilde{\lambda}_2(\mathrm{Fitt}_{\widetilde{R}_2}(I))$$

as required. □

Chapter 4

Local computations

Let F be a totally real number field and v a finite place of F. Let k_v denote the residue field of F_v and let $q_v = |k_v|$.

Let p be an odd prime which is unramified in F and not divisible by v such that $q_v \equiv 1 \pmod{p}$. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. We fix a nontrivial p^m-th root of unity $\zeta \in \mathcal{O}$, where $p^m \mid\mid (q_v - 1)$. We let $\varepsilon_p \colon G_F \to \mathcal{O}^\times$ be the cyclotomic character.

Fix a residual representation $\overline{\rho}_v = \overline{\rho}|_{G_{F_v}} : G_{F_v} \to \mathrm{GL}_2(k)$ which we assume is trivial. Fix an augmentation $\lambda \colon R_v^{\mathrm{ps}} \twoheadrightarrow \mathcal{O}$ such that the induced representation $\rho_\lambda \colon G_{F_v} \to \mathrm{GL}_2(\mathcal{O})$ is of the form

$$\begin{pmatrix} \varepsilon_p \chi & * \\ 0 & \chi^{-1} \end{pmatrix},$$

where $\chi \colon G_{F_v} \to \mathcal{O}^\times$ is a character with $\chi(\iota_v) = \zeta$.

Our main goal in this chapter is to compute the Wiles defect of $(R_v^{\mathrm{ps}}, \lambda) \in \mathrm{CNL}_{\mathcal{O}}^a$. By the results of Chapter 3, this can be done by finding a CI covering $\varphi \colon \widetilde{R} \twoheadrightarrow R_v^{\mathrm{ps}}$ with kernel I and computing the following: (a) the first two steps in a finite free resolution of I, (b) the $\widetilde{R}$-annihilator $\mathrm{Ann}_{\widetilde{R}}(I)$ of I, and (c) the cokernel of the map $\mathrm{Hom}_{R_v^{\mathrm{ps}}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}}, \mathcal{O}) \to \widehat{\Omega}_{\widetilde{R}/\mathcal{O}}$. We will begin by finding a presentation of R_v^{ps} as a quotient of a power series ring over $\mathcal{O}$.

Using this presentation, we find a CI covering $\varphi\colon \widetilde{R} \twoheadrightarrow R_v^{\mathrm{ps}}$. Using properties of regular sequences with respect to exact sequences, we produce (a), from which (b) can be computed. The calculation of (c) is reduced to linear algebra upon showing that the desired cokernel is a quotient of $\mathcal{O}$-lattices.

4.1 A presentation of R_v^{ps}

Our first task is to compute an explicit presentation for the ring R_v^{ps}. Since the image of inertia under $\overline{\rho}_v$ divides p and v does not divide p, the deformations parameterized by R_v^{ps} factor through the tame quotient $G_{F_v}^t$ of G_{F_v}. Recall that $G_{F_v}^t$ can be topologically generated by two elements ϕ_v and ι_v, where ϕ_v is a lift of Frobenius and ι_v is a topological generator of the inertia subgroup of G_{F_v}. Moreover, $G_{F_v}^t$ is characterized by the relation $\phi_v\iota_v\phi_v^{-1} = \iota_v^{q_v}$. Computing a presentation for R_v^{ps} thus amounts to elementary matrix calculations and checking the necessary ring-theoretic properties, as is done in [36].

Let $\rho_v^{\mathrm{ps}}\colon G_{F_v} \to \mathrm{GL}_2(R_v^{\mathrm{ps}})$ be the universal deformation and let

$$Y = \rho_v^{\mathrm{ps}}(\phi_v) = \begin{pmatrix} 1+A & B \\ C & 1+D \end{pmatrix}, \qquad Z = \rho_v^{\mathrm{ps}}(\iota_v) = \begin{pmatrix} 1+T & U \\ V & 1+W \end{pmatrix}.$$

$R_v^{\mathrm{ps},\circ}$ is thus the quotient of $\mathcal{R} = \mathcal{O}[[A,B,C,D,T,U,V,W]]$ by the one relation which arises from fixing the determinant of $\rho_v^{\mathrm{ps}}(\phi_v)$, the two relations which arises from the condition that $\rho_v^{\mathrm{ps}}(\iota_v)$ has characteristic polynomial $(X-\zeta)(X-\zeta^{-1})$, and the four relations which arise from the relation $YZY^{-1} = Z^{q_v}$.

Using that $\det(\rho_v^{\mathrm{ps}})$ is the cyclotomic character and that the characteristic polynomial of

Z is $(X-\zeta)(X-\zeta^{-1})$, we obtain the relations

$$\begin{aligned} r_1 &= \det(Y) - q_v &&= 1 - q_v + A + D + AD - BC \\ r_2 &= \det(Z) - 1 &&= T + W + TW - UV \\ r_3 &= \operatorname{tr}(Z) - (\zeta + \zeta^{-1}) &&= 2 - \zeta - \zeta^{-1} + T + W. \end{aligned}$$

Since the eigenvalues of Z are $(q_v - 1)$-th roots of unity, we have $Z^{q_v} = Z$, so the relation $YZY^{-1} = Z^{q_v} = Z$ implies that Y and Z commute. From this we obtain four relations r_4, r_5, r_6, r_7:

$$\begin{pmatrix} r_4 & r_5 \\ r_6 & r_7 \end{pmatrix} = YZ - ZY = \begin{pmatrix} BV - CU & U(A-D) + B(W-T) \\ C(T-W) + V(D-A) & CU - BV \end{pmatrix}.$$

We immediately note that $r_4 = -r_7$. This shows that $R_v^{\mathrm{ps},\circ} \cong \mathcal{R}/(r_1, r_2, r_3, r_4, r_5, r_6)$.

Theorem 4.1.1. *We have $R_v^{\mathrm{ps}} \cong \mathcal{R}/\mathcal{I}^{\mathrm{ps}}$, where $\mathcal{I}^{\mathrm{ps}} = (r_1, r_2, r_3, r_4, r_5, r_6)$. Moreover, R_v^{ps} is flat over $\mathcal{O}$ and Cohen-Macaulay.*

Proof. We have already shown that $\mathcal{R}/(r_1, r_2, r_3, r_4, r_5, r_6) \cong R_v^{\mathrm{ps},\circ}$. Since R_v^{ps} is the maximal reduced p-torsion free quotient of $R_v^{\mathrm{ps},\circ}$, it suffices to show that $R_v^{\mathrm{ps},\circ}$ is already reduced and p-torsion free.

Lemma 4.2.1, proved in the next section, shows that $R_v^{\mathrm{ps},\circ}$ is flat over $\mathcal{O}$, hence p-torsion free. Now observe that ideal $(r_4, r_5, r_6) \subset \mathcal{I}^{\mathrm{ps}}$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} B & -C & A-D \\ -U & V & W-T \end{pmatrix}.$$

The determinantal ring $\mathcal{R}/(r_4, r_5, r_6)$ is therefore a Cohen-Macaulay, non-Gorenstein domain and flat over $\mathcal{O}$ by [36, Proposition 2.7]. By Lemma 4.3.3, also proved in the next section, we have that r_1, r_2, r_3 is a regular sequence in $\mathcal{R}/(r_4, r_5, r_6)$. Thus $R_v^{\mathrm{ps},\circ}$ is the quotient of a

Cohen-Macaulay ring by a regular sequence, and therefore Cohen-Macaulay.

To show that $R_v^{\mathrm{ps},\circ}$ is reduced, let $\mathcal{X} = \mathrm{Spec}(R_v^{\mathrm{ps},\circ} \otimes_{\mathcal{O}} E)$. Since $2-\zeta-\zeta^{-1}+T+W \in \mathcal{I}^{\mathrm{ps}}$, we see that $T + W \neq 0$ on $\mathcal{X}$. Therefore the affine open subsets $\mathcal{U}_T = \{T \neq 0\}$ and $\mathcal{U}_W = \{W \neq 0\}$ cover $\mathcal{X}$. Consider the natural morphism

$$\alpha \colon \mathcal{X} \to \mathrm{Spec}\left(\frac{\mathcal{O}[[A,B,C,D,T,U,V,W]]}{(2-\zeta-\zeta^{-1}+T+W)} \otimes_{\mathcal{O}} E\right)$$

so that α maps $\mathcal{U}_T$ and $\mathcal{U}_W$ isomorphically onto open subschemes. Sending $W \mapsto \zeta + \zeta^{-1} - 2 - T$ defines an isomorphism

$$\frac{\mathcal{O}[[A,B,C,D,T,U,V,W]]}{(2-\zeta-\zeta^{-1}+T+W)} \cong \mathcal{O}[[A,B,C,D,T,U,V]],$$

thus the image of α is $\mathrm{Spec}(E[[A,B,C,D,T,U,V]])$, which is formally smooth. This shows that $\mathcal{X}$ is formally smooth, and thus reduced. Since $R_v^{\mathrm{ps},\circ}$ is flat over $\mathcal{O}$, this shows that $R_v^{\mathrm{ps},\circ}$ is generically reduced. Therefore by [16, Prop. 14.124], $R_v^{\mathrm{ps},\circ}$ is reduced since it is Cohen-Macaulay. □

We observe that R_v^{ps} is not a complete intersection.

Remark 4.1.2. *As mentioned in the proof of Theorem 4.1.1, the ring $\mathcal{R}/(r_4, r_5, r_6)$ is non-Gorenstein, hence not a complete intersection. It follows that $r_1, r_2, r_3, r_4, r_5, r_6$ is not a regular sequence in $\mathcal{R}$ and R_v^{ps} is not a complete intersection.*

We fix the notation defining our presentation of R_v^{ps} for the remainder of this chapter.

4.2 A CI covering of R_v^{ps}

In this section we find a CI covering of the ring R_v^{ps}, which we will use to compute the Wiles defect of $(R_v^{\mathrm{ps}}, \lambda)$ using the results of Chapter 3. One would hope that a CI covering can be obtained by taking the quotient map $\mathcal{R}/\mathcal{I} \twoheadrightarrow R_v^{\mathrm{ps}}$, where $\mathcal{I}$ is an ideal generated by some subset of $r_1, r_2, r_3, r_4, r_5, r_6$. If this is the case, then necessarily $\mathcal{I}$ cannot contain all of r_4, r_5, r_6 by Remark 4.1.2. Fortunately, it turns that one can in fact obtain a CI covering by simply dropping the relation r_4. The following lemma ensures that the resulting quotient of $\mathcal{R}$ is a complete interesecton:

Lemma 4.2.1. *The sequence $\varpi, D, A-U-V, C-B-T, r_1, r_2, r_3, r_5, r_6$ is regular in $\mathcal{R}$.*

Proof. Let $\mathcal{J} = (\varpi, D, A-U-V, C-B-T, r_1, r_2, r_3, r_5, r_6)$. By Theorem A.0.7, it suffices to show that $\dim \mathcal{R}/\mathcal{J} = 0$.

After quotienting by D and ϖ, the relation $r_3 = 2 - \zeta - \zeta^{-1} + T + W$ becomes $T + W$ since $\zeta \equiv 1 \pmod{\varpi}$, allowing us to eliminate W. Thus

$$\frac{\mathcal{R}}{\mathcal{J}} \cong \frac{k[[A,B,C,T,U,V]]}{(A-BC, T^2+UV, UA-2BT, -VA+2CT, A-U-V, C-B-T)}.$$

Now the relations $A-U-V = C-B-T = 0$ allow us to eliminate A and C, thus

$$\frac{\mathcal{R}}{\mathcal{J}} \cong \frac{k[[B,T,U,V]]}{(B^2+BT-U-V, T^2+UV, U^2+UV-2BT, V^2+VU-2T(T+B))}.$$

We use Macaulay2 to compute a Gröebner basis of the ideal

$$J = (B^2+BT-U-V, T^2+UV, U^2+UV-2BT, V^2+VU-2T(T+B))$$

in the ring $\mathbb{Z}[B,T,U,V]$:

$$U^2 - 2UV - V^2$$
$$T^2 + UV$$
$$2BT - 3UV - V^2$$
$$B^2 + BT - U - V$$
$$3TUV - 2BV^2 + TV^2 + 4TU + 4TV$$
$$BUV + BV^2 - 2TU - 2TV$$
$$10UV^3 + 4V^4 + 12UV^2 + 4V^3$$
$$2TV^3 - TUV - 2BV^2 + 13TV^2 + 4TU + 4TV$$
$$2BV^3 - 4TV^2$$
$$TUV^2 + TV^3 + 4TV^2$$
$$2V^5 - 4UV^3 + 12V^4$$
$$UV^4 + V^5 + 4V^4.$$

We want to show that $\dim \mathbb{F}_q[B,T,U,V]/J = 0$ for all odd primes q. Let $\mathcal{B}$ denote the above basis. Since the elements of $\mathcal{B}$ have coefficients divisible only by the primes $2, 3, 5, 13$, it follows that the image of $\mathcal{B}$ in $\mathbb{F}_q[B,T,U,V]$ is also a Gröebner basis for the image of J for any prime $q \neq 2, 3, 5, 13$. It thus suffices to check that $\dim \mathbb{F}_q[B,T,U,V]/J = 0$ for $q = 3, 5, 13$ and one another prime not in $\{2, 3, 5, 13\}$ (we choose $q = 7$). Using the above Gröebner basis, we use Macaulay2 to compute that $\dim \mathbb{F}_q[B,T,U,V]/J = 0$ for $q = 3, 5, 7, 13$. We conclude that $\dim \mathbb{F}_q[B,T,U,V]/J = 0$ for all odd primes q as desired. Now $\mathbb{F}_q[[B,T,U,V]]/J$ is the completion of $\mathbb{F}_q[B,T,U,V]/J$ at the maximal ideal (B,T,U,V), thus

$$\dim \mathbb{F}_q[[B,T,U,V]]/J \leq \dim \mathbb{F}_q[B,T,U,V]/J$$

and so $\dim \mathbb{F}_q[[B,T,U,V]]/J = 0$ for all odd primes q. In particular we have

$$\dim \mathcal{R}/\mathcal{J} = \dim k[[B,T,U,V]]/\mathcal{J} = \dim(k \otimes_{\mathbb{F}_p} \mathbb{F}_p[[B,T,U,V]]/\mathcal{J}) = 0$$

as desired. □

Let $\widetilde{R} = \mathcal{R}/(r_1, r_2, r_3, r_5, r_6)$ and let $\varphi\colon \widetilde{R} \twoheadrightarrow R_v^{\mathrm{ps}}$ be the quotient map. Then we have the following:

Lemma 4.2.2. *The map $\varphi\colon \widetilde{R} \twoheadrightarrow R_v^{\mathrm{ps}}$ is a CI covering.*

Proof. It follows from Lemma 4.2.1 that $r_1, r_2, r_3, r_5, r_6, \varpi$ is a regular sequence in $\mathcal{R}$, thus $\widetilde{R}$ is a complete intersection which is flat over $\mathcal{O}$ and $\dim \widetilde{R} = \dim \mathcal{R} - 5 = 4$. Since r_4, r_5, r_6 is not a regular sequence in $\mathcal{R}$ as mentioned in Remark 4.1.2, we have $\dim R_v^{\mathrm{ps}} = \dim \mathcal{R} = 4$ as well, hence $\dim \widetilde{R} = \dim R_v^{\mathrm{ps}}$.

To complete the proof, we need to check the formal smoothness condition. The Jacobian matrix for $(A,B,C,D,T,U,V,W) \mapsto (r_1, r_2, r_3, r_4, r_5, r_6)$, obtained by differentiating the expression for each relation with respect to each of the variables A,B,C,D,T,U,V,W, is

$$\begin{pmatrix} 1+D & -C & -B & 1+A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+W & -V & -U & 1+T \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ U & W-T & D & -U & -B & A-D & 0 & B \\ -V & 0 & T-W & V & C & 0 & D-A & -C \end{pmatrix}$$

Let $q = q_v, a = \chi(\phi_v)$ and let b, u denote the top-right entries of $\rho_\lambda(\phi_v)$ and $\rho_\lambda(\iota_v)$, respectively. Evaluating the above matrix at the augmentation $\widetilde{\lambda} = \lambda \circ \varphi$ and rearranging rows, we obtain

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} & 0 & -u & \zeta \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ u & \zeta^{-1}-\zeta & a^{-1}-1 & -u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \end{pmatrix}$$

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ u & \zeta^{-1}-\zeta & a^{-1}-1 & -u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} & 0 & -u & \zeta \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The above matrix has full rank since $\zeta \neq \pm 1$, so $\widetilde{R}[1/\varpi]$ is formally smooth at $\widetilde{\lambda}$ as desired.

We fix our CI covering $\varphi\colon \widetilde{R} \twoheadrightarrow R_v^{\mathrm{ps}}$ with $I = \ker\varphi$ for the remainder of this chapter. □

4.3 A partial finite free resolution of I

In order to calculate the André-Quillen cohomology group $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R, E/\mathcal{O})$, we need the first two terms of a finite free resolution of I as an $\widetilde{R}$-module. Since $I = (BV - CU)$ is generated by a single element, this amounts to calculating $\mathrm{Ann}_{\widetilde{R}}(I)$. We will do this by first calculating $\mathrm{Ann}_{\mathcal{S}}(BV - CU)$, where $\mathcal{S}$ is a larger quotient of $\mathcal{R}$. Properties of regular sequences will then allow us to leverage this calculation in order to compute $\mathrm{Ann}_{\widetilde{R}}(BV - CU)$.

Lemma 4.3.1. *Let $\mathcal{S} = \mathcal{R}/(r_5, r_6)$. Then*

$$\mathrm{Ann}_{\mathcal{S}}(BV - CU) = (D - A, W - T).$$

Proof. Let $\mathcal{J} = (r_5, r_6) = (U(A - D) + B(W - T), C(T - W) + V(D - A))$ and let $\mathcal{P} = (D - A, W - T)$. Clearly $J \subseteq \mathcal{P}$. Note that $\mathcal{P}$ is a prime ideal of $\mathcal{R}$ since $\mathcal{R}/\mathcal{J} \cong$

$\mathcal{O}[[A, B, C, T, U, V]]$ is a domain. If $r \in \mathcal{R}$ is such that its image $\overline{r}$ in $\mathcal{S}$ is in $\mathrm{Ann}_{\mathcal{S}}(BV - CU)$, then $r(BV - CU) \in \mathcal{J} \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal and does not contain $BV - CU$, we must have $r \in \mathcal{P}$. This shows that $\mathrm{Ann}_{\mathcal{S}}(BV - CU) \subseteq \mathcal{P}$.

Conversely, note that

$$\begin{aligned}(D-A)(BV-CU) &= BV(D-A) - CU(D-A) \\ &= C(U(A-D) + B(W-T)) + B(C(T-W) + V(D-A))\end{aligned}$$

and

$$\begin{aligned}(W-T)(BV-CU) &= BV(W-T) - CU(W-T) \\ &= V(U(A-D) + B(W-T)) + U(C(T-W) + V(D-A)),\end{aligned}$$

which shows that $\mathcal{P} \subseteq (r_5, r_6)$. □

Although the functor $M \mapsto M/JM$ in Mod_S for S a ring with $J \subseteq S$ an ideal is not exact in general, when applied to a short exact sequence such that J is generated by a sequence which is regular on the third term, one does obtain an exact sequence.

Lemma 4.3.2. *Let S be a ring and suppose*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of S-modules. If $J \subseteq S$ is an ideal generated by an M_3-regular sequence $s_1, \ldots, s_m \in S$, then

$$0 \longrightarrow M_1/JM_1 \longrightarrow M_2/JM_2 \longrightarrow M_3/JM_3 \longrightarrow 0$$

is also exact.

Proof. Consider the following diagram, where the vertical maps are multiplication by s_1:

$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & & \end{array}$$

The rows are exact by hypothesis. Since s_1 is not a zero-divisor in M_3, multiplication by s_1 on M_3 is injective. Then by the snake lemma

$$0 \longrightarrow M_1/s_1M_1 \longrightarrow M_2/s_1M_2 \longrightarrow M_3/s_3M_3 \longrightarrow 0$$

is exact. The lemma now follows by induction. □

We will need to use a specific regular sequence in the quotient $\mathcal{R}/(r_4, r_5, r_6)$ to compute our resolution of I.

Lemma 4.3.3. *The sequence r_1, r_2, r_3 is regular for $S = \mathcal{R}/(r_4, r_5, r_6)$.*

Proof. Note that the proof of Lemma 4.2.1 shows that

$$\frac{S}{(r_1, r_2, r_3, A-U-V, C-B-T, D, \varpi)} \cong \frac{\mathcal{R}}{(r_1, r_2, r_3, r_5, r_6, A-U-V, C-B-T, D, \varpi)}$$

is 0-dimensional. We have that r_5, r_6 is a regular sequence in $\mathcal{R}$ by Lemma 4.2.1, but r_4, r_5, r_6 is not a regular sequence in $\mathcal{R}$ as mentioned in Remark 4.1.2. Therefore $\dim S = \dim \mathcal{R} - 2 = 7$, so it follows from Theorem A.0.7 that r_1, r_2, r_3 is a regular sequence for S. □

Note that the above lemma does not contradict that $R_v^{\mathrm{ps}} \cong \mathcal{R}/(r_1, r_2, r_3, r_4, r_5, r_6)$ is not a complete intersection, as r_4, r_5, r_6 is not a regular sequence in $\mathcal{R}$.

The following lemma gives the first two terms of a finite free resolution of I as a $\widetilde{R}$-module:

Lemma 4.3.4. *We have an exact sequence of $\widetilde{R}$-modules*

$$\widetilde{R}^2 \longrightarrow \widetilde{R} \xrightarrow{BV-CU} I \longrightarrow 0$$

where the first map takes the standard $\widetilde{R}$-basis to $D-A, W-T$. In particular, the annihilator of I in $\widetilde{R}$ is

$$\mathrm{Ann}_{\widetilde{R}}(I) = (D-A, W-T).$$

Proof. Throughout this proof, we view ideals in quotients of $\mathcal{R}$ as $\mathcal{R}$-modules.

Let $\mathcal{S} = \mathcal{R}/(r_5, r_6)$ and $\mathcal{S}' = \mathcal{R}/(r_4, r_5, r_6)$. By Lemma 4.3.1, the sequence

$$\mathcal{S}^2 \longrightarrow \mathcal{S} \xrightarrow{BV-CU} \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow 0$$

is exact, where the image of the first map is $\mathcal{J} = (D - A, W - T)\mathcal{S}$ and the third map is the quotient map. Let $\mathcal{I} = (BV - CU)\mathcal{S}$ so that $\mathcal{S}' \cong \mathcal{S}/\mathcal{I}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{S} \xrightarrow{BV-CU} \mathcal{I} \longrightarrow 0.$$

We wish to show that the above exact sequence remains exact upon quotienting by (r_1, r_2, r_3). This will follow from Lemma 4.3.2 once we show that r_1, r_2, r_3 is a regular sequence for $\mathcal{I}$.

By Lemma 4.3.3 we have that r_1, r_2, r_3 is regular for $\mathcal{S}'$. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow 0.$$

Since r_1, r_2, r_3 is an exact sequence for $\mathcal{S}'$, it follows from Lemma 4.3.2 that the sequences

$$0 \longrightarrow \mathcal{I}/(r_1) \longrightarrow \mathcal{S}/(r_1) \longrightarrow \mathcal{S}'/(r_1) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}/(r_1, r_2) \longrightarrow \mathcal{S}/(r_1, r_2) \longrightarrow \mathcal{S}'/(r_1, r_2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}/(r_1, r_2, r_3) \longrightarrow \mathcal{S}/(r_1, r_2, r_3) \longrightarrow \mathcal{S}'/(r_1, r_2, r_3) \longrightarrow 0$$

are all exact. It follows that r_1, r_2, r_3 is a regular sequence for $\mathcal{I}$. To see this, note that r_1, r_2, r_3 is a regular sequence for $\mathcal{S}$ by Lemma 4.2.1, i.e. for each i we have that r_i is not a zero-divisor in $\mathcal{S}/(r_1, \dots, r_{i-1})$. Then r_i is also not a zero-divisor in $\mathcal{I}/(r_1, \dots, r_i) \subseteq \mathcal{S}/(r_1, \dots, r_{i-1})$. Furthermore, the exactness of the third sequence above shows that $\mathcal{I}/(r_1, r_2, r_3)$ is isomorphic to the ideal $(BV - CU)$ in $\mathcal{S}/(r_1, r_2, r_3)$.

Now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}/(r_1,r_2,r_3) & \longrightarrow & \mathcal{S}/(r_1,r_2,r_3) & \xrightarrow{BV-CU} & \mathcal{I}/(r_1,r_2,r_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow = & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{J}} & \longrightarrow & \mathcal{S}/(r_1,r_2,r_3) & \xrightarrow{BV-CU} & \overline{\mathcal{I}} & \longrightarrow & 0
\end{array}$$

where $\overline{\mathcal{J}}$ and $\overline{\mathcal{I}}$ are the ideals $(D-A, W-T)$ and $(BV-CU)$ in $\mathcal{S}/(r_1,r_2,r_3)$, respectively. We have shown that the third vertical map is an isomorphisms. The first vertical map is clearly surjective and is thus an isomorphism by the commutativity of the diagram. Additionally, we know that the top row is exact by Lemma 4.3.2. Therefore the bottom row is also exact. Noting that $\mathcal{S}/(r_1,r_2,r_3) \cong \mathcal{R}/(r_1,r_2,r_3,r_5,r_6) = \widetilde{R}$ and the images of $\overline{\mathcal{J}}$ and $\overline{\mathcal{I}}$ under this isomorphism are the ideals $(D-A, W-T)$ and $(BV-CU)$ in $\widetilde{R}$, respectively, we see that

$$\widetilde{R}^2 \longrightarrow \widetilde{R} \xrightarrow{BV-CU} I \longrightarrow 0$$

is an exact sequence of $\mathcal{R}$-modules, where first map takes the standard $\widetilde{R}$-basis to $D-A, W-T$. Since these maps are also homomorphisms of $\widetilde{R}$-modules, this proves the first statement of the lemma, from which the second statement is immediate. □

4.4 The cokernel of $\mathrm{Hom}_{R_v^{\mathrm{ps}}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}}, \mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, \mathcal{O})$

In this section we compute the size of the cokernel of $\mathrm{Hom}_{R_v^{\mathrm{ps}}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}}, \mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, \mathcal{O})$. This is needed to compute $|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{ps}}, E/\mathcal{O})|$ using Corollary 3.4.13.

For a differential $x \in \widehat{\Omega}_{\mathcal{R}/\mathcal{O}}$, we write $\mathrm{d}x\big|_{\widetilde{\lambda}} = \mathrm{d}x\big|_{\lambda}$ for the evaluation of x at the augmentation λ, i.e. the differential obtained by setting the variables $A, B, C, D, P, T, U, V, W$ equal to 0. Note that $\widehat{\Omega}_{\mathcal{R}/\mathcal{O}} \otimes_{\mathcal{R}}^{\widetilde{\lambda}} \mathcal{O} \cong \mathcal{O}^8$ is a free $\mathcal{O}$-module of rank 8 spanned by the differentials $\mathrm{d}A, \mathrm{d}B, \mathrm{d}C, \mathrm{d}D, \mathrm{d}P, \mathrm{d}T, \mathrm{d}U, \mathrm{d}V, \mathrm{d}W$.

Since $\widetilde{R} = \mathcal{R}/(r_1,r_2,r_3,r_5,r_6)$, the kernel of the natural surjection $\widehat{\Omega}_{\mathcal{R}/\mathcal{O}} \otimes_{\mathcal{R}}^{\widetilde{\lambda}} \mathcal{O} \twoheadrightarrow \widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes_{\widetilde{R}}^{\widetilde{\lambda}} \mathcal{O}$ is an $\mathcal{O}$-lattice spanned by $\mathrm{d}r_1\big|_{\lambda}, \mathrm{d}r_2\big|_{\lambda}, \mathrm{d}r_3\big|_{\lambda}, \mathrm{d}r_5\big|_{\lambda}, \mathrm{d}r_6\big|_{\lambda}$. Denote this lattice by $\widetilde{\Lambda}$. Similarly, let Λ^{ps} denote the kernel of $\widehat{\Omega}_{\mathcal{R}/\mathcal{O}} \otimes^{\lambda}_{\mathcal{R}} \mathcal{O} \twoheadrightarrow \widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}} \otimes^{\lambda}_{R_v^{\mathrm{ps}}} \mathcal{O}$, which is an $\mathcal{O}$-lattice spanned by $\mathrm{d}r_1\big|_{\lambda}, \mathrm{d}r_2\big|_{\lambda}, \mathrm{d}r_3\big|_{\lambda}, \mathrm{d}r_4\big|_{\lambda}, \mathrm{d}r_5\big|_{\lambda}, \mathrm{d}r_6\big|_{\lambda}$. Note that $\widetilde{\Lambda} \subset \Lambda^{\mathrm{ps}}$. We thus have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\Lambda} & \longrightarrow & \mathcal{O}^8 & \longrightarrow & \widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes^{\widetilde{\lambda}}_{\widetilde{R}} \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{\mathrm{ps}} & \longrightarrow & \mathcal{O}^8 & \longrightarrow & \widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}} \otimes^{\lambda}_{R_v^{\mathrm{ps}}} \mathcal{O} & \longrightarrow & 0
\end{array}$$

with exact rows, where the third vertical map is the surjection induced by φ. Applying the exact functor $\mathrm{Hom}_{\mathcal{O}}(\cdot, E/\mathcal{O})$, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}} \otimes^{\lambda}_{R_v^{\mathrm{ps}}} \mathcal{O}, E/\mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8, E/\mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\Lambda^{\mathrm{ps}}, E/\mathcal{O}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{=} & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes^{\widetilde{\lambda}}_{\widetilde{R}} \mathcal{O}, E/\mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8, E/\mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\widetilde{\Lambda}, E/\mathcal{O}) & \longrightarrow & 0
\end{array} \tag{4.1}$$

with exact rows, where the first vertical map is injective. Now

$$\mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}} \otimes^{\widetilde{\lambda}}_{\widetilde{R}} \mathcal{O}, E/\mathcal{O}) \cong \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, E/\mathcal{O})) \cong \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, E/\mathcal{O}),$$

and likewise $\mathrm{Hom}_{\mathcal{O}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}} \otimes^{\lambda}_{R_v^{\mathrm{ps}}} \mathcal{O}, E/\mathcal{O}) \cong \mathrm{Hom}_{R_v^{\mathrm{ps}}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}}, E/\mathcal{O})$, so the size of the cokernel of interest is the size of the cokernel of the first vertical map in (4.1). Note that by exactness of $\mathrm{Hom}_{\mathcal{O}}(\cdot, E/\mathcal{O})$, the kernels in the rows of (4.1) are $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8/\Lambda^{\mathrm{ps}}, E/\mathcal{O})$ and $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8/\widetilde{\Lambda}, E/\mathcal{O})$, so the cokernel of the first vertical map in (4.1) is given by

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8/\Lambda^{\mathrm{ps}}, E/\mathcal{O})/\,\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^8/\widetilde{\Lambda}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}, E/\mathcal{O}) \cong \Lambda^{\mathrm{ps}}/\widetilde{\Lambda}.$$

Here the first isomorphism follows again from the exactness of $\mathrm{Hom}_{\mathcal{O}}(\cdot, E/\mathcal{O})$ and the second isomorphism follows since $\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}$ is a finite torsion $\mathcal{O}$-module (as we will show in the following lemma). Our task is thus to calculate the size of the quotient $\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}$.

For the remainder of this section we set $q = q_v$ for convenience. Recall that the representation $\rho_\lambda\colon G_{F_v} \to \mathrm{GL}_2(\mathcal{O})$ induced by the augmentation λ is given by

$$\rho_\lambda(\phi_v) = \begin{pmatrix} qa & b \\ 0 & a^{-1} \end{pmatrix}, \qquad \rho_\lambda(\iota_v) = \begin{pmatrix} \zeta & u \\ 0 & \zeta^{-1} \end{pmatrix}$$

for some $u, b \in \mathcal{O}$. We see from the proof of Theorem 4.1.1 that the above two matrices must commute, from which we obtain the relation $b(\zeta - \zeta^{-1}) = u(qa - a^{-1})$.

Lemma 4.4.1. *The $\mathcal{O}$-module $\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}$ is given by*

$$\Lambda^{\mathrm{ps}}/\widetilde{\Lambda} \cong \begin{cases} \mathcal{O}/(u^{-1}(\zeta - \zeta^{-1})) & u \mid (\zeta - \zeta^{-1}) \\ 0 & (\zeta - \zeta^{-1}) \mid u \end{cases}$$

Note that u and $\zeta - \zeta^{-1}$ both divide each other if and only if $u^{-1}(\zeta - \zeta^{-1})$ is a unit, i.e. $\mathcal{O}/(u^{-1}(\zeta - \zeta^{-1})) = 0$.

Proof. The differentials of the relations $r_1, r_2, r_3, r_4, r_5, r_6$ are

$$\mathrm{d}r_1 = (1 + D)\,\mathrm{d}A + (1 + A)\,\mathrm{d}D - B\,\mathrm{d}C - C\,\mathrm{d}B$$

$$\mathrm{d}r_2 = (1 + W)\,\mathrm{d}T + (1 + T)\,\mathrm{d}W - U\,\mathrm{d}V - V\,\mathrm{d}U$$

$$\mathrm{d}r_3 = \mathrm{d}T + \mathrm{d}W$$

$$\mathrm{d}r_4 = B\,\mathrm{d}V + V\,\mathrm{d}B - C\,\mathrm{d}U - U\,\mathrm{d}C$$

$$\mathrm{d}r_5 = U\,\mathrm{d}A + (A - D)\,\mathrm{d}U - U\,\mathrm{d}D + B\,\mathrm{d}W + (W - T)\,\mathrm{d}B - B\,\mathrm{d}T$$

$$\mathrm{d}r_6 = C\,\mathrm{d}T + (T - W)\,\mathrm{d}C - C\,\mathrm{d}W + V\,\mathrm{d}D + (D - A)\,\mathrm{d}V - V\,\mathrm{d}A.$$

Evaluating at the augmentation λ, we get

$$
\begin{aligned}
\mathrm{d}r_1\big|_\lambda &= a^{-1}\,\mathrm{d}A + qa\,\mathrm{d}D - b\,\mathrm{d}C \\
\mathrm{d}r_2\big|_\lambda &= \zeta^{-1}\,\mathrm{d}T + \zeta\,\mathrm{d}W - u\,\mathrm{d}V \\
\mathrm{d}r_3\big|_\lambda &= \mathrm{d}T + \mathrm{d}W \\
\mathrm{d}r_4\big|_\lambda &= b\,\mathrm{d}V - u\,\mathrm{d}C \\
\mathrm{d}r_5\big|_\lambda &= u\,\mathrm{d}A + (qa - a^{-1})\,\mathrm{d}U - u\,\mathrm{d}D + b\,\mathrm{d}W + (\zeta^{-1} - \zeta)\,\mathrm{d}B - b\,\mathrm{d}T \\
\mathrm{d}r_6\big|_\lambda &= (\zeta - \zeta^{-1})\,\mathrm{d}C + (a^{-1} - qa)\,\mathrm{d}V.
\end{aligned}
$$

The $\mathcal{O}$-lattice Λ^{ps} is thus spanned by the rows of the matrix

$$
\begin{pmatrix}
a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta^{-1} & 0 & -u & \zeta \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & -u & 0 & 0 & 0 & b & 0 \\
u & \zeta^{-1} - \zeta & 0 & -u & -b & qa - a^{-1} & 0 & b \\
0 & 0 & \zeta - \zeta^{-1} & 0 & 0 & 0 & a^{-1} - qa & 0
\end{pmatrix}.
$$

We will row reduce the above matrix to obtain an $\mathcal{O}$-basis of Λ^{ps}. After a few steps, we obtain the matrix

$$
\begin{pmatrix}
a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\
0 & \zeta^{-1} - \zeta & bau & -u - qa^2u & -b & qa - a^{-1} & 0 & b \\
0 & 0 & -u & 0 & 0 & 0 & b & 0 \\
0 & 0 & \zeta - \zeta^{-1} & 0 & 0 & 0 & a^{-1} - qa & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta - \zeta^{-1}
\end{pmatrix}. \tag{4.2}
$$

To proceed further, we need to consider two cases depending on whether or not $\zeta - \zeta^{-1}$ divides u or vice versa.

Case $u \mid (\zeta - \zeta^{-1})$

Beginning with the matrix (4.2), we add $u^{-1}(\zeta - \zeta^{-1})$ times the third row to the fourth row to obtain:

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} - \zeta & bau & -u - qa^2u & -b & qa - a^{-1} & 0 & b \\ 0 & 0 & -u & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^{-1} - qa + bu^{-1}(\zeta - \zeta^{-1}) & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta - \zeta^{-1} \end{pmatrix}.$$

The relation $b(\zeta - \zeta^{-1}) = u(qa - a^{-1})$ implies that the fourth row is zero, and we see that the remaining rows are linearly independent since $\zeta \neq \pm 1$. Therefore an $\mathcal{O}$-basis of Λ^{ps} is given by the rows of the matrix

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1} - \zeta & bau & -u - qa^2u & -b & qa - a^{-1} & 0 & b \\ 0 & 0 & -u & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta - \zeta^{-1} \end{pmatrix}. \tag{4.3}$$

Case $(\zeta - \zeta^{-1}) \mid u$

Beginning with the matrix (4.2), we add $u(\zeta - \zeta^{-1})^{-1}$ times the fourth row to the third row to obtain:

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1}-\zeta & bau & -u-qa^2u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & b+\frac{(a^{-1}-qa)u}{\zeta-\zeta^{-1}} & 0 \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta-\zeta^{-1} \end{pmatrix}.$$

The relation $b(\zeta-\zeta^{-1}) = u(qa-a^{-1})$ implies that the third row is zero, and we see that the remaining rows are linearly independent since $\zeta \neq \pm 1$. Therefore an $\mathcal{O}$-basis of Λ^{ps} is given by the rows of the matrix

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1}-\zeta & bau & -u-qa^2u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta-\zeta^{-1} \end{pmatrix}. \tag{4.4}$$

Now we need to compute an $\mathcal{O}$-basis for $\widetilde{\Lambda}$. The lattice $\widetilde{\Lambda}$ is spanned by $\mathrm{d}r_1, \mathrm{d}r_2, \mathrm{d}r_3, \mathrm{d}r_5, \mathrm{d}r_6$, i.e. the rows of the matrix

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta^{-1} & 0 & -u & \zeta \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ u & \zeta^{-1}-\zeta & 0 & -u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \end{pmatrix}.$$

After row reducing, we obtain the matrix

$$\begin{pmatrix} a^{-1} & 0 & -b & qa & 0 & 0 & 0 & 0 \\ 0 & \zeta^{-1}-\zeta & bau & -u-qa^2u & -b & qa-a^{-1} & 0 & b \\ 0 & 0 & \zeta-\zeta^{-1} & 0 & 0 & 0 & a^{-1}-qa & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u & \zeta-\zeta^{-1} \end{pmatrix}. \tag{4.5}$$

The rows of the above matrix are linearly independent since $\zeta \neq \pm 1$, and thus form an $\mathcal{O}$-basis for $\widetilde{\Lambda}$.

To finish the proof, we need to calculate the coordinate matrix M which expresses our basis for $\widetilde{\Lambda}$ in terms of our basis for Λ^{ps}, i.e. the i-th row of M should be the coordinate vector of the i-th basis vector for $\widetilde{\Lambda}$ with respect to the basis for Λ^{ps} (where the basis vectors, as expressed in matrices above, are ordered from top to bottom). Then $\Lambda^{\mathrm{ps}}/\widetilde{\Lambda} \cong \mathcal{O}/(\det M)$. Again we have two cases

Case $u \mid (\zeta - \zeta^{-1})$

In this case we need to express the rows of (4.5) as linear combinations of the rows of (4.3). Using the relation $b(\zeta - \zeta^{-1}) = u(qa - a^{-1})$, we find that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & u^{-1}(\zeta^{-1}-\zeta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Case $(\zeta - \zeta^{-1}) \mid u$

In this case we need to express the rows of (4.5) as linear combinations of the rows of

(4.4). We note that these two matrices are identical, thus

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

4.5 The Wiles defect of $(R_v^{\mathrm{ps}}, \lambda)$

We have now gathered all of the information necessary to compute $c_{1,\lambda}(R_v^{\mathrm{ps}})$ and $d_{1,\lambda}(R_v^{\mathrm{ps}})$ and thus the Wiles defect of $(R_v^{\mathrm{ps}}, \lambda)$. First we observe that $c_{1,\lambda}(R_v^{\mathrm{ps}}) = 0$.

Corollary 4.5.1. *We have* $c_{1,\lambda}(R_v^{\mathrm{ps}}) = 0$.

Proof. By definition $c_{1,\lambda}(R_v^{\mathrm{ps}}) = \log_p |C_{1,\lambda}(R_v^{\mathrm{ps}})| / \log_p |\mathcal{O}/p|$. We have

$$C_{1,\lambda}(R_v^{\mathrm{ps}}) = C_{1,\widetilde{\lambda}}(\widetilde{R}) = \widetilde{\lambda}(\mathrm{Ann}_{\widetilde{R}}(I))/\widetilde{\lambda}(\mathrm{Fitt}_{\widetilde{R}}(I)).$$

But since $I = (BV - CU)$ is a cyclic $\widetilde{R}$-module, we have $\mathrm{Fitt}_{\widetilde{R}}(I) = \mathrm{Ann}_{\widetilde{R}}(I)$, so the corollary holds. □

Remark 4.5.2. *The proof of the above corollary only relied on the fact that I is a cyclic $\widetilde{R}$-module, and not the explicit computation of $\mathrm{Ann}_{\widetilde{R}}(I)$. However, we will still need the partial resolution of I given in Lemma 4.3.4 in order to compute $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{ps}}, E/\mathcal{O})$.*

To calculate $d_{1,\lambda}(R_v^{\mathrm{ps}})$, we first need to calculate $\mathrm{Hom}_{R_v^{\mathrm{ps}}}(I/I^2, E/\mathcal{O})$ as an $\mathcal{O}$-module.

Lemma 4.5.3. *As $\mathcal{O}$-modules, we have*

$$\mathrm{Hom}_{R_v^{\mathrm{ps}}}(I/I^2, E/\mathcal{O}) \cong \mathcal{O}/(qa - a^{-1}, \zeta - \zeta^{-1}).$$

Proof. Regarding E as an R_v^{ps}-module via the augmentation λ, we see that

$$\mathrm{Hom}_{R_v^{\mathrm{ps}}}(I/I^2, E/\mathcal{O}) \cong \mathrm{Hom}_{R_v^{\mathrm{ps}}}(I \otimes_{\widetilde{R}} R_v^{\mathrm{ps}}, E/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(I \otimes_{\widetilde{R}}^{\widetilde{\lambda}} \mathcal{O}, E/\mathcal{O}).$$

Now recall the partial resolution of I

$$\widetilde{R}^2 \xrightarrow{f} \widetilde{R} \longrightarrow I \longrightarrow 0$$

given by Lemma 4.3.4, where f sends the standard $\widetilde{R}$-basis of $\widetilde{R}^2$ to $D-A, W-T$. Tensoring the above sequence with $\mathcal{O}$ over $\widetilde{R}$ gives the right-exact sequence

$$\mathcal{O}^2 \xrightarrow{\widetilde{\lambda}(f)} \mathcal{O} \longrightarrow I \otimes_{\widetilde{R}}^{\widetilde{\lambda}} \mathcal{O} \longrightarrow 0$$

Noting that $\widetilde{\lambda}(D-A) = a^{-1} - qa$ and $\widetilde{\lambda}(W-T) = \zeta^{-1} - \zeta$, we see that

$$I \otimes_{\widetilde{R}}^{\widetilde{\lambda}} \mathcal{O} \cong \mathcal{O}/(\mathrm{Im}\, \widetilde{\lambda}(f)) \cong \mathcal{O}/(a^{-1} - qa, \zeta^{-1} - \zeta).$$

□

Now we calculate the invariant $d_{1,\lambda}(R_v^{\mathrm{ps}})$.

Proposition 4.5.4. *Let e be the ramification index of E over $\mathbb{Q}_p$. Then we have*

$$d_{1,\lambda}(R_v^{\mathrm{ps}}) = \frac{n_v}{e},$$

where n_v is the largest nonnegative integer such that $\rho_\lambda \pmod{\varpi^{n_v}}$ is scalar.

Proof. Recall that the size of the cokernel of $\mathrm{Hom}_{R_v^{\mathrm{ps}}}(\widehat{\Omega}_{R_v^{\mathrm{ps}}/\mathcal{O}}, E/\mathcal{O}) \to \mathrm{Hom}_{\widetilde{R}}(\widehat{\Omega}_{\widetilde{R}/\mathcal{O}}, E/\mathcal{O})$ is equal to $|\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}|$. Then by Corollary 3.4.13 the size of $\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{ps}}, E/\mathcal{O})$ is given by

$$|\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{ps}}, E/\mathcal{O})| = |\,\mathrm{Hom}_{R_v^{\mathrm{ps}}}(I/I^2, E/\mathcal{O})|/|\Lambda^{\mathrm{ps}}/\widetilde{\Lambda}|.$$

We have

$$\mathrm{Hom}_{R_v^{\mathrm{ps}}}(I/I^2, E/\mathcal{O}) \cong \mathcal{O}/(qa - a^{-1}, \zeta - \zeta^{-1}).$$

by Lemma 4.5.3, and by Lemma 4.4.1 we have

$$\Lambda^{\mathrm{ps}}/\widetilde{\Lambda} \cong \begin{cases} \mathcal{O}/(u^{-1}(\zeta - \zeta^{-1})) & u \mid (\zeta - \zeta^{-1}) \\ 0 & (\zeta - \zeta^{-1}) \mid u \end{cases}.$$

By definition $d_{1,\lambda}(R_v^{\mathrm{ps}}) = \log_p |\widehat{\mathrm{Der}}^1_{\mathcal{O}}(R_v^{\mathrm{ps}}, E/\mathcal{O})|/\log_p |\mathcal{O}/p|$. Analyzing the four possible cases depending on the valuations of $qa - a^{-1}, \zeta - \zeta^{-1}, u$ and applying the relation $b(\zeta - \zeta^{-1}) = u(qa - a^{-1})$, we find that

$$d_{1,\lambda}(R_v^{\mathrm{ps}}) \cong \begin{cases} \frac{1}{ef}\log_p |\mathcal{O}/(u)| & (\zeta - \zeta^{-1}) \mid (qa - a^{-1}), u \mid (\zeta - \zeta^{-1}) \\ \frac{1}{ef}\log_p |\mathcal{O}/(\zeta - \zeta^{-1})| & (\zeta - \zeta^{-1}) \mid (qa - a^{-1}), (\zeta - \zeta^{-1}) \mid u \\ \frac{1}{ef}\log_p |\mathcal{O}/(b)| & (\zeta - \zeta^{-1}) \mid (qa - a^{-1}), u \mid (\zeta - \zeta^{-1}) \\ \frac{1}{ef}\log_p |\mathcal{O}/(qa - qa^{-1})| & (qa - a^{-1}) \mid (\zeta - \zeta^{-1}), (\zeta - \zeta^{-1}) \mid u \end{cases},$$

where f is the order of residue field of $\mathcal{O}$. Examining the conditions in each of the four cases and using the relation $b(\zeta - \zeta^{-1}) = u(qa - a^{-1})$, we find that in all cases

$$d_{1,\lambda}(R_v^{\mathrm{ps}}) = \frac{1}{ef}\log_p |\mathcal{O}/(b, u, qa - a^{-1}, \zeta - \zeta^{-1})|.$$

Now note that $\frac{1}{f}\log_p |\mathcal{O}/(b, u, qa - a^{-1}, \zeta - \zeta^{-1})|$ is the greatest integer n such that

$b \equiv u \equiv 0 \pmod{\varpi^n}$ and $qa \equiv a^{-1}, \zeta \equiv \zeta^{-1} \pmod{\varpi^n}$. Recalling that

$$\rho_\lambda(\phi_v) = \begin{pmatrix} qa & b \\ 0 & a^{-1} \end{pmatrix}, \qquad \rho_\lambda(\iota_v) = \begin{pmatrix} \zeta & u \\ 0 & \zeta^{-1} \end{pmatrix},$$

we see that $n_v = n = \frac{1}{f} \log_p |\mathcal{O}/(b, u, qa - a^{-1}, \zeta - \zeta^{-1})|$, thus the corollary holds. □

We now obtain the following theorem which gives the Wiles defect of $(R_v^{\mathrm{ps}}, \lambda)$:

Theorem 4.5.5. *Let e be the ramification index of E over $\mathbb{Q}_p$. The Wiles defect of the ring R_v^{ps} with the augmentation $\lambda \colon R_v^{\mathrm{ps}} \twoheadrightarrow \mathcal{O}$ is given by*

$$\delta_\lambda(R_v^{\mathrm{ps}}) = \frac{n_v}{e},$$

where n_v is the largest integer such that $\rho_\lambda \pmod{\varpi^{n_v}}$ is scalar.

Proof. By definition

$$\delta_\lambda(R_v^{\mathrm{ps}}) = d_{1,\lambda}(R_v^{\mathrm{ps}}) - c_{1,\lambda}(R_v^{\mathrm{ps}}),$$

so this is immediate from Corollary 4.5.1 and Proposition 4.5.4. □

Chapter 5

Global computations

In this chapter, we fix a residual representation $\overline{\rho}$ which we assume to be modular. We define a global deformation ring R parameterizing deformations of $\overline{\rho}$ with prescribed local behavior. Using the Taylor-Wiles-Kisin patching method, we show that R is isomorphic to a (localized) Hecke algebra T (with an augmentation $\lambda\colon T \to \mathcal{O}$) acting on the cohomology of a Shimura curve (or Shimura set). Moreover, we show that R is the quotient by a regular sequence of a power series ring over a completed tensor product of local deformation rings. This leads us to our main theorem, which shows that the global Wiles defect $\delta_\lambda(R) = \delta_\lambda(T)$ is a sum of local defects.

5.1 The Taylor-Wiles-Kisin patching method

Let F be a totally real number field. Fix a finite set of finite places Σ of F and for each $v \in \Sigma$, fix a local deformation condition $\tau_v \in \{\min, \mathrm{ps}, \square\}$. Let $\tau = (\tau_v)_{v\in\Sigma}$ and for $\sigma \in \{\min, \mathrm{ps}, \square\}$ let $\Sigma^\sigma = \{v \in \Sigma : \tau_v = \sigma\}$. For each $v \in \Sigma$ we let F_v be the completion of F at v with ring of integers $\mathcal{O}_v$ and residue field k_v with $q_v = |k_v|$. We also let $P_{F_v} \subseteq I_{F_v}$ be the wild inertia and inertia subgroups of G_{F_v}, respectively, and we let ϕ_v, ι_v be topological generators for the tame quotient $G^t_{F_v}$, where ϕ_v is a lift of Frobenius and ι_v is a topological generator of I_{F_v}/P_{F_v} satisfying the relation $\phi_v \iota_v \phi_v^{-1} = \iota_v^{q_v}$.

Let p be an odd prime unramified in F and not divisible by any prime in Σ and let Σ_p be the set of places of F above p. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. For each $v \in \Sigma^{\mathrm{ps}}$, fix $\zeta_v \in \mathcal{O}$ such that ζ_v is a nontrivial p^{m_v}-th root of unity, where $p^{m_v} \mid\mid (q_v - 1)$. Let $\epsilon_p \colon G_F \to \mathcal{O}^\times$ be the cyclotomic character. Let $\rho \colon G_F \to \mathrm{GL}_2(\mathcal{O})$ be a Galois representation for which:

- ρ corresponds to a Hilbert modular form of parallel weight 2.
- $\det \rho = \epsilon_p$.
- ρ is unramified for all places $v \notin \Sigma \cup \Sigma_p$.
- For every place $v \mid p$, $\overline{\rho}|_{G_{F_v}}$ is finite flat.
- If $v \in \Sigma^{\min}$, then either $q_v \not\equiv -1 \pmod{p}$, $\overline{\rho}|_{G_{I_{F_v}}}$ is irreducible or $\overline{\rho}|_{G_{F_v}}$ is absolutely irreducible.
- If $v \in \Sigma^{\mathrm{ps}}$, then $\overline{\rho}|_{G_{F_v}}$ is trivial and $\rho|_{G_{F_v}}(\iota_v)$ has characteristic polynomial $(X - \zeta_v)(X - \zeta_v^{-1})$.
- The residual representation $\overline{\rho} \colon G_F \to \mathrm{GL}_2(k)$ is absolutely irreducible and satisfies the *Taylor-Wiles conditions*: $\overline{\rho}|_{G_{F(\zeta_p)}}$ is still absolutely irreducible, and in the case when $p = 5$, we have $\sqrt{5} \in F$ and the image of the projective representation $\mathrm{proj}\,\overline{\rho} \colon G_F \to \mathrm{PGL}_2(\overline{\mathbb{F}}_5)$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$.

Let D be a quaternion algebra over F unramified at all finite places of F and ramified at either all or all but one of the infinite places of F (depending on the parity of $[F : \mathbb{Q}]$). Define an open compact subgroup $K^\tau = \prod_v K_v^\tau \subseteq (D \otimes \mathbb{A}_{F,f})^\times$ by:

- $K_v^\tau = \mathrm{GL}_2(\mathcal{O}_v)$ if $v \notin \Sigma$.
- $K_v^\tau = U_1(v)$ if $v \in \Sigma^{\mathrm{ps}}$.

- $K_v^\tau = U_0(v^{a_v})$ if $v \in \Sigma^{\min}$, where a_v is the Artin conductor of $\overline{\rho}|_{G_{F_v}}$.

- $K_v^\tau = U_0(v^{a_v+2})$ if $v \in \Sigma^\square$.

Here $U_0(v^{a_v})$ (resp. $U_1(v)$) is the subgroup of $\mathrm{GL}_2(\mathcal{O}_v)$ consisting of matrices which are upper-triangular modulo v (resp. upper-triangular and unipotent modulo v). We will omit the τ from the notation for convenience.

In the case when $[F : \mathbb{Q}]$ is odd (resp. even), i.e. D is ramified at all (resp. all but one) infinite place of F, let X_K be the Shimura curve (resp. Shimura set) associated to K (see [8]). Let $\mathbb{T}^D(K)$ be the Hecke algebra acting on $H(K) = H^1(X_K, \mathcal{O})$ in the Shimura curve case and on $H^0(X_K, \mathcal{O})$ in the Shimura set case, generated as an $\mathcal{O}$-algebra by the operators T_v and S_v for all finite places $v \notin \Sigma$. Note that $\mathbb{T}^D(K)$ is a finite $\mathcal{O}$-module, as it is a submodule of $\mathrm{End}_{\mathcal{O}}(H(K))$.

Now let

$$\mathbb{T}^D(K)^{\mathrm{ps}} = \mathbb{T}^D(K)/((S_v - \epsilon_p(\mathrm{Frob}_v) : v \notin \Sigma), (S_v - \zeta_v : v \in \Sigma^{\mathrm{ps}}))$$

be the principal series fixed determinant Hecke algebra.

Since ρ corresponds to a Hilbert modular form of parallel weight 2 by assumption, we have the following:

Proposition 5.1.1. *There is an augmentation $\lambda\colon \mathbb{T}^D(K)^{\mathrm{ps}} \twoheadrightarrow \mathcal{O}$ such that $\rho(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - \lambda(T_v)X + \lambda(S_v)$ for any $v \notin \Sigma \cup \Sigma_p$. Moreover, $\Phi_\lambda(\mathbb{T}^D(K)^{\mathrm{ps}})$ is finite.*

Let $\lambda\colon \mathbb{T}^D(K)^{\mathrm{ps}} \twoheadrightarrow \mathcal{O}$ be an augmentation as in the above proposition, let $\mathfrak{m} = \lambda^{-1}(\varpi\mathcal{O}) \subseteq \mathbb{T}^D(K)^{\mathrm{ps}}$ be the maximal ideal of $\mathbb{T}^D(K)^{\mathrm{ps}}$ corresponding to $\overline{\rho}$, and let $\mathbb{T}^\tau$ be the localization of $\mathbb{T}^D(K)^{\mathrm{ps}}$ at $\mathfrak{m}$. Note that any ring homomorphism $x\colon \mathbb{T}^\tau \to \overline{E}$ corresponds to a Galois representation $\rho_x\colon G_F \to \mathrm{GL}_2(\overline{E})$ lifting $\overline{\rho}$ such that $\det \rho_x = \epsilon_p$ and $\mathrm{tr}\, \rho_x(\mathrm{Frob}_v) = x(T_v)$ for $v \notin \Sigma$.

Define $H^\tau = \mathrm{Hom}_{\mathcal{O}}(H(K), \mathcal{O})$ to be the $\mathcal{O}$-dual of $H(K)$ viewed as a $\mathbb{T}^D(K)$-module. Define

$$M^\tau = \mathbb{T}^\tau \otimes_{\mathbb{T}^D(K)^{\mathrm{ps}}} H^\tau = H^\tau/((S_v - \epsilon_p(\mathrm{Frob}_v) : v \notin \Sigma), (S_v - \zeta : v \in \Sigma^{\mathrm{ps}})).$$

Let

$$R_{\mathrm{loc}} = \left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\square}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{fl}}\right), \qquad R_{\mathrm{loc}}^\tau = \left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\tau_v}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{fl}}\right),$$

where the tensor products are taken over $\mathcal{O}$, as in Section 2.3. Note that R_{loc}^τ is naturally an R_{loc}-algebra since each $R_v^{\tau_v}$ for $v \in \Sigma$ is a quotient of $R_v^\square$.

We let R (resp. $R^\square$) denote the global unframed (resp. framed) deformation ring parameterizing lifts of $\overline{\rho}$ with determinant ϵ_p which are flat at all places $v \mid p$. Note that here we use the assumption that $\overline{\rho}$ is absolutely irreducible (and ramified at only finitely many places) to ensure that the ring R exists by Theorem 2.2.12.

We fix a noncanonical isomorphism $R^\square \cong R[[X_1, \ldots, X_{4j-1}]]$ for some j and thus treat R as a quotient of $R^\square$. Let $\rho_{\mathrm{univ}}^\square \colon G \to \mathrm{GL}_2(R^\square)$ be the universal lifting. For each $v \in \Sigma \cup \Sigma_p$, the restriction $\rho_{\mathrm{univ}}^\square\big|_{G_{F_v}}$ is a lifting with determinant ϵ_p which is flat if $v \mid p$, thus after composing with the inclusion $G_{F_v} \hookrightarrow G_F$, the universal property of $R_v^\square$ induces a map $R_v^\square \to R^\square$. We thus have a natural map $R_{\mathrm{loc}} \to R^\square$. Likewise, via the composition $R_v^\square \to R^\square \twoheadrightarrow R$, we have a map $R_{\mathrm{loc}} \to R$. We then define $R^{\square,\tau} = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}} R^\square$ and $R^\tau = R_{\mathrm{loc}}^\tau \otimes_{R_{\mathrm{loc}}} R$ as in Section 2.4.

We have the following standard lemma [23] concerning the existence of a surjective "$R \to T$" map, which we ultimately show is an isomorphism via the patching method:

Lemma 5.1.2. *There is a surjective map $R^\tau \twoheadrightarrow \mathbb{T}^\tau$ inducing a representation $\rho^\tau \colon G_F \to \mathrm{GL}_2(\mathbb{T}^\tau)$ such that $\rho^\tau(\mathrm{Frob}_v)$ has characteristic polynomial $X^2 - T_vX + S_v$ for all places $v \notin \Sigma \cup \Sigma_p$.*

Our goal is prove that the map $R^\tau \twoheadrightarrow \mathbb{T}^\tau$ is in fact an isomorphism. Since $\mathbb{T}^\tau$ acts faithfully on M^τ, this will follow if we show that $M^\tau \otimes_{\mathcal{O}} E$ is supported on all of $\operatorname{Spec} R^\tau[1/\varpi]$ and R^τ is free over $\mathcal{O}$. These statements are shown via the Taylor-Wiles-Kisin patching method. We will consider deformation rings where we impose additional deformation conditions at a finite set Q of "Taylor-Wiles" primes disjoint from $\Sigma \cup \Sigma_p$. The patching method allows us to take a "limit" of these deformation rings to produce a "patched" ring $\mathcal{R}^\tau_\infty$ and $\mathcal{R}^\tau_\infty$-module M^τ_∞. From the action of $\mathcal{R}^\tau_\infty$ on M^τ_∞, we deduce the required properties of R^τ. We use the notation and terminology of [26] in applying the "ultrapatching" method due to Scholze [34].

For each $v \in Q$ we fix an eigenvalue α_v of $\rho_v^\square(\mathrm{Frob}_v)$, and let $\widetilde{R}_v^\square$ be the modified local deformation corresponding to α_v as defined in Section 2.3. We then define

$$R_Q^{\square,\tau} = \left(\widehat{\bigotimes}_{v\in Q}\widetilde{R}_v^\square\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\tau_v}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{fl}}\right) \otimes_{R_{\mathrm{loc}}} R^\square$$
$$R_Q^\tau = R_Q^{\square,\tau} \otimes_{R^\square} R$$

analogous to the definitions of $R^{\square,\tau}$ and R^τ, and we let $\rho_Q^{\square,\tau} \colon G_F \to \mathrm{GL}_2(R_Q^{\square,\tau})$ and $\rho_Q^\tau \colon G_F \to \mathrm{GL}_2(R_Q^\tau)$ be the universal lifting and universal deformation, respectively.

We require the following standard lemma (see [27, Lemma 2.5], [23, Proposition 3.2.5], and [15, Proposition 5.10]), which guarantees we can produce a sequence of Taylor-Wiles sets satisfying additional conditions.

Lemma 5.1.3. *Let* $r = \max(\dim H^1(G_{F,\Sigma\cup\Sigma_p}, (\mathrm{ad}^0\,\overline{\rho})(1)), 1 + [F:\mathbb{Q}] - |\Sigma \cup \Sigma_p|)$. *For each* $n \geq 1$, *there exists a set* Q_n *of finite places of* F *such that*

- $Q_n \cap (\Sigma \cup \Sigma_p) = \varnothing$.
- *For each* $v \in Q_n$, $\overline{\rho}(\mathrm{Frob}_v)$ *has distinct eigenvalues.*
- $q_v \equiv 1 \pmod{p^n}$ *for all* $v \in Q_n$.
- $|Q_n| = r$.

- $R_n^{\square,\tau} = R_{Q_n}^{\square,\tau}$ *is topologically generated over* R_{loc}^τ *by* $g = |\Sigma \cup \Sigma_p| - 1 - [F : \mathbb{Q}] + r$ *elements.*

Lemma 5.1.4. *Let Q be a set of Taylor-Wiles primes as in Lemma 5.1.3. Then for each $v \in Q$, we have $\rho_Q^\tau\big|_{G_{F_v}} = \chi_1 \oplus \chi_2$ for some tamely ramified characters $\chi_1, \chi_2 \colon G_{F_v} \to R_Q^\tau$.*

Proof. Let $v \in Q$. Since Q is disjoint from $\Sigma \cup \Sigma_p$, we have that $\overline{\rho}|_{G_{F_v}}$ is unramified. Therefore $\rho_Q^\tau(I_v)$ is contained in the 1-units of R_Q^τ, which is a pro-p group. But P_{F_v} is pro-v and Q is disjoint from Σ_p, thus $\rho_Q^\tau(P_{F_v})$ is trivial. This proves the tamely ramified part of the assertion.

By assumption $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, so by Hensel's lemma we may choose a basis such that

$$\rho_Q^\tau(\phi_v) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

is diagonal. With respect to this basis, write

$$\rho_Q^\tau(\iota_v) = 1 + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $a, b, c, d \in \mathfrak{m} = \mathfrak{m}_{R_n^\tau}$. Using the relation $\phi_v \iota_v \phi^{-1} = \iota_v^{q_v}$, we find that

$$\begin{pmatrix} a & \alpha\beta^{-1}b \\ \alpha^{-1}\beta c & d \end{pmatrix} = \sum_{m=0}^{q_v} \binom{q_v}{m} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^m.$$

Comparing top-right and bottom-left entries modulo $\mathfrak{m}^2$, we see that

$$\alpha\beta^{-1}b \equiv q_v b \pmod{\mathfrak{m}^2}$$
$$\alpha^{-1}\beta c \equiv q_v c \pmod{\mathfrak{m}^2}$$

By assumption α, β are distinct modulo $\mathfrak{m}$ and $q_v \equiv 1 \pmod{p}$, so the above relations imply that $b, c \in \mathfrak{m}^2$. Assuming by induction that $b, c \in \mathfrak{m}^m$, we see that the above relations hold modulo $\mathfrak{m}^{m+1}$, thus $b, c \in \mathfrak{m}^{m+1}$. Therefore $b, c \in \mathfrak{m}^m$ for all $m \geq 1$, hence $b = c = 0$ by completeness. This completes the proof. $\square$

Let r and Q_n for $n \geq 1$ be as in Lemma 5.1.3, and let $\rho_n^{\square,\tau} = \rho_{Q_n}^{\square,\tau}$. For each $n \geq 1$, choose a place $v \in Q_n$. By Lemma 5.1.4, $\rho_n^\tau|_{G_{F_v}} = \chi_{v,1} \oplus \chi_{v,2}$ for some tamely ramified characters $\chi_{v,2}, \chi_{v,2} \colon G_{F_v} \to R_n^\tau$. Let χ_v be equal to either $\chi_{v,1}$ or $\chi_{v,2}$. By local class field theory, $\chi_v|_{I_{F_v}}$ determines a character $\widetilde{\chi}_v \colon \mathcal{O}_v^\times \to (R_n^\tau)^\times$ Moreover, $\chi_v(I_{F_v})$ is pro-p, being a closed subgroup of the 1-units of R_n^τ. Since Q_n is disjoint from Σ_p and the 1-units of $\mathcal{O}_v$ are pro-ℓ when $v \mid \ell$, it follows that $\chi_v|_{I_{F_v}}$ factors through Δ_v, where Δ_v is the maximal p-power quotient of $k_v^\times$. Let $\Delta_n = \prod_{v \in Q_n} \Delta_v$. Then the choices of the χ_v define an action of $\mathcal{O}[\Delta_n]$ on R_n^τ via the characters $\widetilde{\chi}_v$. Moreover, the congruence condition $|k_v| \equiv 1 \pmod{p^n}$ implies that $\Delta_{v,n}$ is cyclic of order divisible by p^n, so

$$\mathcal{O}[\Delta_n] \cong \frac{\mathcal{O}[[y_1, \dots, y_r]]}{((y_1+1)^{p^{m_1}} - 1, \dots, (y_r+1)^{p^{m_r}} - 1)} \tag{5.1}$$

for some integers $m_i \geq n$.

We need to define a few more objects in order to proceed with the patching argument. Let r and g be as in Lemma 5.1.3 and let $d = r + 4j - 1$, where $j = |\Sigma \cup \Sigma_p|$. Now define

$$R_\infty^\tau = R_{\text{loc}}^\tau[[x_1, \dots, x_g]],$$
$$S_\infty = \mathcal{O}[[y_1, \dots, y_d]].$$

By Proposition 2.3.5, $R_v^{\tau_v}$ is of dimension 4 for $v \in \Sigma$ while R_v^{fl} is of dimension $4 + [F_v : \mathbb{Q}_p]$ for $v \mid p$. Recalling that $g = |\Sigma \cup \Sigma_p| - 1 - [F : \mathbb{Q}] + r$, we see from the definition of R_{loc}^τ that

$$\dim R_{\text{loc}}^\tau = 3 \cdot |\Sigma \cup \Sigma_p| + \sum_{v \mid p} [F_v : \mathbb{Q}_p] + 1 = 3j + [F : \mathbb{Q}] + 1,$$

from which it follows that

$$\dim S_\infty = d + 1 = g + \dim R^\tau_{\mathrm{loc}} = \dim R^\tau_\infty.$$

For each $n \geq 1$, choose Q_n as in Lemma 5.1.3 and let $R^{\square,\tau}_n = R^{\square,\tau}_{Q_n}$ and $R^\tau_n = R^\tau_{Q_n}$. By the last statement in Lemma 5.1.3, there exists a surjection $R^\tau_\infty \twoheadrightarrow R^{\square,\tau}_n$. Now $R^{\square,\tau}_n \cong R^\tau_n[[y_{r+1}, \dots, y_d]]$, so by the preceding discussion regarding the action of $\mathcal{O}[\Delta_n]$ on R^τ_n, we see that $R^{\square,\tau}_n$ has a S_∞-module structure which factors through $\mathcal{O}[\Delta_n][[y_{r+1}, \dots, y_d]]$ (treated as a quotient of S_∞).

The construction in [27, Section 4.2] produces a compact open subgroup $K_n = \prod_v K_{n,v} \subset (D \otimes \mathbb{A}_{F,f})^\times$ (with $K_{n,v} = K_v$ for all $v \notin Q_n$). We define a Hecke algebra $\mathbb{T}^\tau_n$ and a Hecke-module M^τ_n analogously to $\mathbb{T}^\tau$ and M^τ (but at level K_n). Then we have surjections $R^\tau_n \twoheadrightarrow \mathbb{T}^\tau_n$, which makes M^τ_n an R^τ_n-module (hence an S_∞-module as well). Using this surjection and the fact that R^τ_n is naturally an $\mathcal{O}$-subalgebra of $R^{\square,\tau}_n$ since $\overline{\rho}$ is irreducible, we define a framed Hecke algebra and a framed Hecke module by

$$\mathbb{T}^{\square,\tau}_n = \mathbb{T}^\tau_n \otimes_{R^\tau_n} R^{\square,\tau}_n$$
$$M^{\square,\tau}_n = M^\tau_n \otimes_{R^\tau_n} R^{\square,\tau}_n$$

Next we show that $\mathscr{R}^{\square,\tau} = \{R^{\square,\tau}_n\}_{n=1}^\infty$ is a weak patching algebra and $\mathscr{M}^{\square,\tau} = \{M^{\square,\tau}_n\}_{n=1}^\infty$ is a weak patching $\mathscr{R}$-module as defined in [26]. Moreover, we show the stronger statement that $\mathscr{M}^{\square,\tau}$ is a free patching $\mathscr{R}$-module as defined in [26].

Lemma 5.1.5. *We have that $\mathscr{R}^{\square,\tau} = \{R^{\square,\tau}_n\}_{n=1}^\infty$ is a weak patching algebra covered by R^τ_∞, and $\mathscr{M}^{\square,\tau} = \{M^{\square,\tau}_n\}_{n=1}^\infty$ is a free patching $\mathscr{R}$-module.*

Proof. Consider the ideal $\mathfrak{n} = (y_1, \dots, y_d)$ and the subring $S'_\infty = \mathcal{O}[[y_1, \dots, y_r]]$ in S_∞. Let $\mathfrak{n}' = (y_1, \dots, y_r) = \mathfrak{n} \cap S'_\infty$. Then $R^{\square,\tau}_n/\mathfrak{n} \cong R^\tau$ and $M^{\square,\tau}_n/\mathfrak{n} \cong M^\tau$ and $M^{\square,\tau}_n$ is finite free as an $\mathcal{O}[\Delta_n][[y_{r+1}, \dots, y_d]]$-module (see [11], [10], [37], [23]). Now $R^{\square,\tau}_n \cong R^\tau_n \otimes_{S'_\infty} S_\infty$ and

$M_n^{\square,\tau} = M^\tau \otimes_{S'_\infty} S_\infty$ (the $d - r = 4j - 1$ variables $y_{r+1}, \ldots, y_d$ make the unframed objects non-canonical quotients of the framed objects), so

$$\operatorname{rank}_{S_\infty} R_n^{\square,\tau} = \operatorname{rank}_{S'_\infty} R_n^\tau = \operatorname{rank}_{\mathcal{O}} R^\tau$$
$$\operatorname{rank}_{S_\infty} M_n^{\square,\tau} = \operatorname{rank}_{S'_\infty} M_n^\tau = \operatorname{rank}_{\mathcal{O}} M^\tau.$$

This shows that the S_∞-ranks of the rings $R_n^{\square,\tau}$ and the modules $M_n^{\square,\tau}$ for $n \geq 1$ are constant, hence uniformly bounded. Therefore $\mathscr{R}^{\square,\tau} = \{R_n^{\square,\tau}\}_{n=1}^\infty$ is a weak patching algebra and $\mathscr{M}^{\square,\tau} = \{M_n^{\square,\tau}\}_{n=1}^\infty$ is a weak patching $\mathscr{R}$-module as defined in [26].

Since $M_n^{\square,\tau}$ is free over $\mathcal{O}[\Delta_n][[y_{r+1}, \ldots, y_d]]$, we have

$$I_n = \operatorname{Ann}_{S_\infty} M_n^{\square,\tau} = \operatorname{Ann}_{S_\infty} \mathcal{O}[\Delta_n][[y_{r+1}, \ldots, y_d]] = ((y_1+1)^{p^{m_{n,1}}} - 1, \ldots, (y_r+1)^{p^{m_{n,r}}} - 1),$$

where the $m_{n,i}$ are as in (5.1). Therefore $M_n^{\square,\tau}$ is free over $S_\infty / \operatorname{Ann}_{S_\infty} M_n^{\square,\tau}$.

Now let $\mathfrak{a} \subseteq S_\infty$ be an open ideal. Then $S_\infty/\mathfrak{a}$ is finite. Since $1+\mathfrak{m}_{S_\infty}$ is a pro-p group, the image of $1+\mathfrak{m}_{S_\infty}$ in $S/\mathfrak{a}$ is a finite p-group. For each $i = 1, \ldots, r$, we have $y_i + 1 \in 1 + \mathfrak{m}_{S_\infty}$, thus there exists some integer $N \geq 0$ such that $(y_i+1)^{p^N} \equiv 1 \pmod{\mathfrak{a}}$ for all $i = 1, \ldots, r$. Recall that $m_{n,i} \geq n$ for each $i = 1, \ldots, r$, thus if $n \geq N$, we have $(y_i+1)^{p^{m_{n,i}}} - 1 \in \mathfrak{a}$ for all $i = 1, \ldots, r$, i.e. $I_n \subseteq \mathfrak{a}$. This, together with the freeness of $M_n^{\square,\tau}$ over $S_\infty/\operatorname{Ann}_{S_\infty} M_n^{\square,\tau}$, shows that $\mathscr{M}^{\square,\tau}$ is a free patching $\mathscr{R}$-module as defined in [26]. Moreover, we see from the last condition of Lemma 5.1.3 that R_∞^τ covers the weak patching algebra $\mathscr{R}^{\square,\tau}$. $\square$

We are now ready to apply the patching method to show our map $R^\tau \to T^\tau$ is an isomorphism.

Theorem 5.1.6. *The rings R_∞^τ and S_∞ satisfy the following conditions:*

(a) $\dim S_\infty = \dim R_\infty^\tau$.

(b) There exists a continuous $\mathcal{O}$-algebra homomorphism $i\colon S_\infty \to R_\infty^\tau$ which makes R_∞^τ

into a finite free S_∞-module.

(c) *There exists an isomorphism $R^\tau_\infty \otimes_{S_\infty} \mathcal{O} \cong R^\tau$ and R^τ is finite free over $\mathcal{O}$.*

(d) *The surjective map in Lemma 5.1.2 is an isomorphism. Moreover, the rings R^τ and $\mathbb{T}^\tau$ are reduced.*

(e) *If $\lambda_{R^\tau_\infty} \colon R^\tau_\infty \to R^\tau \cong \mathbb{T}^\tau \xrightarrow{\lambda} \mathcal{O}$ is the augmentation induced by λ then $R^\tau_\infty[1/\varpi]$ is formally smooth at $\lambda_{R^\tau_\infty}$.*

Proof. Part (a) was shown in the preceding discussion in which we defined R^τ_∞ and S_∞.

By [27, Theorem 4.2] and Lemma 5.1.5, we obtain the following:

- $\mathcal{R}^\tau_\infty \cong \mathscr{P}(\mathscr{R}^{\square,\tau})$ is a finite type S_∞-algebra and $M^\tau_\infty = \mathscr{P}(\mathscr{M}^{\square,\tau})$ is a finite free S_∞-module. Here $\mathscr{P}$ denotes the patching functor defined in [26].
- M^τ_∞ is a maximal Cohen-Macaulay module over $\mathcal{R}^\tau_\infty$.
- $\mathcal{R}^\tau_\infty \otimes_{S_\infty} \mathcal{O} \cong R^\tau$ and $M^\tau_\infty \otimes_{S_\infty} \mathcal{O} \cong M^\tau$.
- There is a surjection $\pi_\infty \colon R^\tau_\infty \twoheadrightarrow \mathcal{R}^\tau_\infty$ through which the map $R^\tau_{\mathrm{loc}} \to R^\tau = R^\tau_{\mathrm{loc}} \otimes_{R_{\mathrm{loc}}} R$ factors.

Since $\mathcal{R}^\tau_\infty$ is a S_∞-algebra, R^τ_∞ is a complete local ring, and S_∞ is a power series ring, we can lift the structure map $S_\infty \to \mathcal{R}^\tau_\infty$ to a map $i \colon S_\infty \to R^\tau_\infty$, which makes π_∞ a surjection of S_∞-modules. Then M^τ_∞ is a maximal Cohen-Macaulay R^τ_∞-module. Therefore the support of M^τ_∞ is a union of irreducible components of $\operatorname{Spec} R^\tau_\infty$. Since $R^\tau_\infty = R^\tau_{\mathrm{loc}}[[x_1, \ldots, x_g]]$, the irreducible components of $\operatorname{Spec} R^\tau_\infty$ are in bijection with the irreducible components of $\operatorname{Spec} R^\tau_{\mathrm{loc}}$.

Using [14, Corollary 3.1.7], we see that each irreducible component of $\operatorname{Spec} R^\tau_\infty$ contains a point in the support of $M^\tau_\infty/(i(y_1), \ldots, i(y_d)) \otimes_{\mathcal{O}} E \cong M^\tau \otimes_{\mathcal{O}} E$ which is not contained in any other irreducible component. Now since the support of M^τ_∞ clearly contains the support of $M_\infty^\tau/(i(y_1),\dots,i(y_d))\otimes_{\mathcal{O}} E$, it follows that the support of M_∞^τ is all of $\operatorname{Spec} R_\infty^\tau$. Then $\operatorname{Ann}_{R_\infty^\tau}(M_\infty^\tau)$ is contained in the nilradical of R_∞^τ. But R_{loc}^τ, hence R_∞^τ, is reduced, thus the action of R_∞^τ on M_∞^τ is faithful. This implies that $\pi_\infty\colon R_\infty^\tau \twoheadrightarrow \mathcal{R}_\infty^\tau$ is injective and thus an isomorphism of S_∞-algebras, which together with the condition $\mathcal{R}_\infty^\tau\otimes_{S_\infty}\mathcal{O}\cong R^\tau$ proves the first part of (c).

It follows from Lemma 2.4.1 that R_∞^τ is Cohen-Macaulay, thus $\operatorname{depth}_{R_\infty^\tau} R_\infty^\tau = \dim R_\infty^\tau$. Since M_∞^τ is a maximal Cohen-Macaulay R_∞^τ-module supported on all of $\operatorname{Spec} R_\infty^\tau$, we have

$$\operatorname{depth}_{R_\infty^\tau} M_\infty^\tau = \dim(\operatorname{Supp}_{R_\infty^\tau} M_\infty^\tau) = \dim R_\infty^\tau.$$

Then by the Auslander–Buchsbaum formula, the projective dimension of M_∞^τ as a R_∞^τ-module is equal to

$$\operatorname{depth}_{R_\infty^\tau} R_\infty^\tau - \operatorname{depth}_{R_\infty^\tau} M_\infty^\tau = 0.$$

Therefore M_∞^τ is projective over R_∞^τ, hence free since R_∞^τ is a local ring. By definition the action of S_∞ on M_∞^τ factors through $i\colon S_\infty\to R_\infty^\tau$. Since M_∞^τ is free over S_∞ it follows that M_∞^τ is also free over R_∞^τ, proving (b).

Since M_∞^τ is supported on all of $\operatorname{Spec} R_\infty^\tau$, the module $M^\tau\cong M_\infty^\tau\otimes_{S_\infty}\mathcal{O}$ is supported on all of $\operatorname{Spec} R^\tau\cong\operatorname{Spec}(R_\infty^\tau\otimes_{S_\infty}\mathcal{O})$. Let $I=\ker(R^\tau\twoheadrightarrow\mathbb{T}^\tau)$. By definition the action of R^τ on M^τ factors through $R^\tau\twoheadrightarrow\mathbb{T}^\tau$, thus every prime ideal of R^τ contains $I=\ker(R^\tau\twoheadrightarrow\mathbb{T}^\tau)$ since M^τ is a faithful $\mathbb{T}^\tau$-module. Since $\mathbb{T}^\tau$ is reduced, it follows that I is the nilradical of R^τ, hence $(R^\tau)^{\mathrm{red}}\cong\mathbb{T}^\tau$. This implies that R^τ is finite over $\mathcal{O}$ since $\mathbb{T}^\tau$ is finite over $\mathcal{O}$. Therefore R^τ is also free over $\mathcal{O}$ since $R^\tau=R_\infty^\tau\otimes_{S_\infty}\mathcal{O}$ is free over S_∞, proving the second part of (c).

Since the action of $R^\tau[1/\varpi]$ on $M^\tau\otimes_{\mathcal{O}}E$ factors through $R^\tau[1/\varpi]\to\mathbb{T}^\tau[1/\varpi]$, it follows that $R^\tau[1/\varpi]\to\mathbb{T}^\tau[1/\varpi]$ is injective. Therefore $\ker(R^\tau\twoheadrightarrow\mathbb{T}^\tau)$ is a torsion $\mathcal{O}$-module. But we showed that R^τ is free over $\mathcal{O}$, thus $R^\tau\twoheadrightarrow\mathbb{T}^\tau$ is injective, which proves (d).

As in the proof of [3, Theorem 6.3], $\operatorname{Spec} R_\infty^\tau$ is formally smooth at every point in the support of $M^\tau\otimes_{\mathcal{O}}E$, in particular at the point corresponding to $\lambda_{R_\infty^\tau}\colon R_\infty^\tau\twoheadrightarrow\mathcal{O}$, which proves (e). The smoothness follows from the fact that cohomological Hilbert modular forms are generic in the sense of [1, Lemma 1.1.5], which in turn follows from the genericity of the corresponding automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ at all finite places and local-global compatibility as recorded in [1, Theorem 2.1.2]. □

5.2 Main theorem

We are now ready to prove our main theorem, which gives the Wiles defect of the global rings $R^\tau \cong \mathbb{T}^\tau$ with respect to the augmentation $\lambda\colon \mathbb{T}^\tau \twoheadrightarrow \mathcal{O}$ from Proposition 5.1.1. By making use of the properties of the Wiles defect recorded in Section 3.5 and the patching argument of the previous section, we show that the global Wiles defect is equal to a sum of local defects.

Theorem 5.2.1. *For the rings $R^\tau \cong \mathbb{T}^\tau$ and the augmentation $R^\tau \cong \mathbb{T}^\tau \xrightarrow{\lambda} \mathcal{O}$, the Wiles defect is given by*

$$\delta_\lambda(R^\tau) = \delta_\lambda(\mathbb{T}^\tau) = \sum_{v\in\Sigma^{\mathrm{ps}}} \frac{n_v}{e},$$

where e is the ramification index of $E/\mathbb{Q}_p$ and for each $v \in \Sigma^{\mathrm{ps}}$, n_v is the greatest integer such that $\rho_\lambda|_{G_{F_v}}$ is scalar.

Proof. Theorem 5.1.6 (b) implies that $S_\infty \to R^\tau_\infty$ is a 1-codimensional embedding. Then by Theorem 3.5.2 we have

$$\delta_\lambda(\mathbb{T}^\tau) = \delta_\lambda(R^\tau) = \delta_\lambda(R^\tau_\infty \otimes_{S_\infty} \mathcal{O}) = \delta_\lambda(R^\tau_\infty).$$

Now $x_1, \ldots, x_g \in \ker\lambda$ is a regular sequence in $R^\tau_\infty = R^\tau_{\mathrm{loc}}$, so by Theorem 3.5.3 we have

$$\delta_\lambda(R^\tau_\infty) = \delta_\lambda = \delta_\lambda(R^\tau_{\mathrm{loc}}) = \delta_\lambda\left(\left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\tau_v}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\mathrm{fl}}\right)\right).$$

Since the rings R_v^{τ} for $v \in \Sigma$ and R_v^{fl} for $v \mid p$ are all reduced, by Proposition 3.5.6 we have

$$\delta_\lambda\left(\left(\widehat{\bigotimes}_{v\in\Sigma} R_v^{\tau_v}\right) \widehat{\otimes} \left(\widehat{\bigotimes}_{v|p} R_v^{\text{fl}}\right)\right) = \sum_{v\in\Sigma} \delta_\lambda(R_v^{\tau_v}) + \sum_{v|p} \delta_\lambda(R_v^{\text{fl}})$$

By Proposition 2.3.5, we have that $R_v^{\text{fl}} \cong \mathcal{O}[[x_1, \ldots, x_{3+[F_v:\mathbb{Q}_p]}]]$ and $R_v^{\min}$ and $R_v^{\square}$ are complete intersections, thus $\delta_\lambda(R_v^{\text{fl}}) = \delta_\lambda(R_v^{\min}) = \delta_\lambda(R_v^{\square}) = 0$ by Proposition 3.5.5. By Theorem 4.5.5 we have $\delta_\lambda(R_v^{\text{ps}}) = n_v/e$. Therefore

$$\begin{aligned}\sum_{v\in\Sigma} \delta_\lambda(R_v^{\tau_v}) + \sum_{v|p} \delta_\lambda(R_v^{\text{fl}}) &= \sum_{v\in\Sigma^{\text{ps}}} \delta_\lambda(R_v^{\tau_v}) + \sum_{v\in\Sigma^{\min}} \delta_\lambda(R_v^{\tau_v}) + \sum_{v\in\Sigma^{\square}} \delta_\lambda(R_v^{\tau_v}) + \sum_{v|p} \delta_\lambda(R_v^{\text{fl}}) \\ &= \sum_{v\in\Sigma^{\text{ps}}} \frac{n_v}{e}.\end{aligned}$$

□

Appendices

Appendix A

Cohen-Macaulay modules and rings

In this appendix we recall the definition of and basic facts about Cohen-Macaulay modules M over a Noetherian local ring R. We are particularly intersted in Cohen-Macaulay rings, i.e. Noetherian local rings R which are Cohen-Macaulay modules over themselves.

We first recall the meaning of regular sequences and the depth of a module.

Definition A.0.1. *Let M be a module over a ring R. A sequence of elements $r_1, \ldots, r_n \in R$ is called a regular sequence on M (or a M-regular sequence) if $M/(r_1, \ldots, r_n) \neq M$ and for each $1 \leq i \leq n$, the element r_i is not a zero divisor on $M/(r_1, \ldots, r_{i-1})$. In particular, a regular sequence in R is a regular sequence on R considered as a module over itself.*

Definition A.0.2. *Let M be a module over a ring R and $I \subseteq R$ an ideal. We define* $\operatorname{depth}_I(M)$ *to be the supremum of lengths of M-regular sequences contained in I if $IM \neq M$. If $IM = M$, we define* $\operatorname{depth}_I(M) = \infty$. *If R is a local ring with maximal ideal $\mathfrak{m}$, we simply write* $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$.

Definition A.0.3. *Let M be a module over a Noetherian local ring R. We say M is Cohen-Macaulay if* $\operatorname{depth} M = \dim M$. *In particular, we say that R is a Cohen-Macaulay ring if* $\operatorname{depth} R = \dim R$ *when R is considered as a module over itself.*

In general the regularity of a sequence depends on order, but for finitely-generated modules over Noetherian local rings this is not the case (see [24, Lemma 1.4]).

Proposition A.0.4. *Let M be a finitely generated module over a Noetherian local ring R. If $r_1, \ldots, r_n$ is a regular sequence on M, then so is $r_{\sigma(1)}, \ldots, r_{\sigma(n)}$ for any permutation $\sigma \in S_n$.*

Proof. We prove the case $n = 2$. The result then follows by induction.

Suppose r_1, r_2 is a regular sequence on M. If $x \in M$ is such that $r_2 x = 0$, then we must have $x = r_1 x_1$ for some $x_1 \in M$ since r_1, r_2 is regular on M. Then

$$r_2 x = r_1 r_2 x_1 = 0,$$

so $r_2 x_1 = 0$ since r_1 is regular on M. Then $x_1 = r_1 x_2$ for some $x_2 \in M$. By induction we have a sequence $x = x_0, x_1, x_2, \ldots$ in M such that $x_n = r_1 x_{n-1}$ for $n \geq 1$. It follows that $x \in \bigcap_{n=1}^{\infty} r_1 M$. But this intersection is 0 by the Krull intersection theorem, thus $x = 0$. This shows that r_2 is regular on M.

If $r_1 x_1 = r_2 x_2$ for some $x_1, x_2 \in M$, then $x_2 = r_1 x_2'$ for some $x_2' \in M$ since r_1, r_2 is regular on M. Then

$$r_1 x_1 = r_1 r_2 x_2',$$

which implies $x_1 = r_2 x_2'$ since r_1 is regular on M. We conclude that r_2, r_1 is regular on M as desired. □

We want to prove an equivalent condition for a sequence in a Cohen-Macaulay ring to be regular. We require a theorem due to Ischebeck (see [28]):

Theorem A.0.5. *Let R be a Noetherian local ring. If M and N are finitely generated R-modules, then*

$$\operatorname{Ext}_R^i(N, M) = 0 \qquad \textit{for all } i < \operatorname{depth} M - \dim N.$$

Let R be a ring and let M be a R-module. Recall that the set of *associated primes of* M is the set $\operatorname{Ass}_R(M)$ of prime ideals $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} = \operatorname{Ann}_R(x)$ for some $x \in M$. Minimal elements of $\operatorname{Ass}_R(M)$ are called *isolated primes* while the remaining elements of $\operatorname{Ass}_R(M)$ are called *embedded primes*. The following theorem (see [24, Theorem 1.22]) asserts that all associated primes of a Cohen-Macaualy module are minimal and have the same dimension.

Theorem A.0.6. *Let M be a finitely generated Cohen-Macaulay module over a Noetherian local ring R. Then*

(a) We have $\dim R/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M)$.

(b) The module M has no embedded primes.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Then $\mathfrak{p} = \operatorname{Ann}_R(x)$ for some nonzero $x \in M$, so $r \mapsto rx$ defines a nonzero R-module homomorphism $R/\mathfrak{p} \to M$. Then

$$\operatorname{Ext}^0_R(R/\mathfrak{p}, M) = \operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0.$$

thus $\dim R/\mathfrak{p} \geq \operatorname{depth} M$ by Theorem A.0.5. On the other hand, since $\mathfrak{p} \in \operatorname{Ass}_R(M)$, we have $\mathfrak{p} \in \operatorname{Supp} M$, so $\dim R/\mathfrak{p} \leq \dim M$. Now $\operatorname{depth} M = \dim M$ since M is Cohen-Macaualy, thus we conclude that $\dim R/\mathfrak{p} = \dim M$.

For the second part, note that if $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ is a strict inclusion of primes in $\operatorname{Ass}_R(M)$, then

$$\dim R/\mathfrak{p}_2 < \dim R/\mathfrak{p}_1,$$

contradicting (a). □

The following theorem is the main result of this appendix. It provides a useful criterion for checking if a sequence in a Cohen-Macaualy ring is regular.

Theorem A.0.7. *Let R be a Noetherian local ring which is Cohen–Macaulay. A sequence $r_1, \ldots, r_n$ in R is regular if and only if*

$$\dim R/(r_1, \ldots, r_n) = \dim R - n.$$

Proof. The sequence r_1 is regular if and only if r_1 is not a unit or a zero divisor. If r_1 is not a unit or a zero divisor, then $\dim R/(r_1) = \dim R - 1$ since R is a Noetherian local ring. Conversely, suppose $\dim R/(r_1) = \dim R - 1$. Then r_1 is not a unit, and if r_1 is a zero divisor, then $\operatorname{Ann}_R(r_1) \neq 0$, thus $r_1 \in \mathfrak{p}$ for some associated prime $\mathfrak{p} \in \operatorname{Ass}_R(M)$. But Theorem A.0.6 implies that $\mathfrak{p}$ is a minimal prime, thus

$$\dim R = \dim R/\mathfrak{p} \leq \dim R/(r_1) = \dim R - 1,$$

a contradiction. Therefore r_1 is not a zero divisor.

Now assume by induction that a sequence $r_1, \ldots, r_n$ in R is regular if and only if $\dim R/(r_1, \ldots, r_n) = \dim R - n$. The sequence is regular if and only if $r_1, \ldots, r_n$ is a regular sequence in R and r_{n+1} is a regular sequence in $R/(r_1, \ldots, r_n)$. This is the case if and only if

$$\dim \frac{R}{(r_1, \ldots, r_{n+1})} = \dim \frac{R/(r_1, \ldots, r_n)}{(r_{n+1})} = \dim \frac{R}{(r_1, \ldots, r_n)} - 1 = \dim R - (n+1).$$

The theorem now follows by induction. □

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