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THE HOMOGENIZED LINIAL ARRANGEMENT AND GENOCCHI NUMBERS

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Abstract. We study the intersection lattice of a hyperplane arrangement recently introduced by Hetyei who showed that the number of regions of the arrangement is a median Genocchi number. Using a different method, we refine Hetyei's result by providing a combinatorial interpretation of the coefficients of the characteristic polynomial of the intersection lattice of this arrangement. The Genocchi numbers count a class of permutations known as Dumont permutations and the median Genocchi numbers count the derangements in this class. We show that the signless coefficients of the characteristic polynomial count Dumont-like permutations with a given number of cycles. This enables us to derive formulas for the generating function of the characteristic polynomial, which reduce to known formulas for the generating functions of the Genocchi numbers and the median Genocchi numbers. As a byproduct of our work, we obtain new models for the Genocchi and median Genocchi numbers.

Keywords. Hyperplane arrangement, characteristic polynomial, Genocchi numbers, Dumont permutations, Ferrers graphs, surjective staircases

Mathematics Subject Classifications. 52C35 (Primary), 05A05, 05A15, 05B35, 06A07, 11B68 (Secondary)

1. Introduction

The *braid arrangement* (or *type A Coxeter arrangement*) is the hyperplane arrangement in $\mathbb{R}^n$, $n \geqslant 1$, defined by

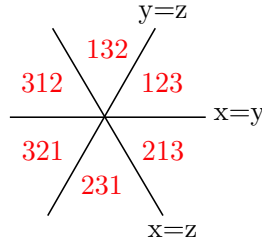$$\mathcal{A}_{n-1} := \{x_i - x_j = 0 : 1 \leqslant i < j \leqslant n\}.$$

Note that the hyperplanes of $\mathcal{A}_{n-1}$ divide $\mathbb{R}^n$ into open cones of the form

$$R_\sigma := \{\mathbf{x} \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\},$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$. Hence the braid arrangement $\mathcal{A}_{n-1}$ has $|\mathfrak{S}_n| = n!$ regions. The regions of the arrangement obtained by intersecting $\mathcal{A}_2$ with the plane $x + y + z = 0$ are shown in the figure below.

A classical formula of Zaslavsky [Zas75] gives the number of regions of any real hyperplane arrangement $\mathcal{A}$ in terms of the Möbius function of its intersection (semi)lattice $\mathcal{L}(\mathcal{A})$. Indeed, given any finite, ranked poset P of length ℓ, with a minimum element $\hat{0}$, the *characteristic polynomial* of P is defined to be

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{\ell - \mathrm{rk}(x)}, \tag{1.1}$$

where μ_P is the Möbius function of P and $\mathrm{rk}(x)$ is the rank of x. Zaslavsky's formula for the number of regions $r(\mathcal{A})$ of $\mathcal{A}$ is

$$r(\mathcal{A}) = (-1)^\ell \chi_{\mathcal{L}(\mathcal{A})}(-1). \tag{1.2}$$

It is well known and easy to see that the lattice of intersections of the braid arrangement $\mathcal{A}_{n-1}$ is isomorphic to the lattice Π_n of partitions of the set $[n] := \{1, 2, \ldots, n\}$. It is also well known that the characteristic polynomial of Π_n is given by

$$\chi_{\Pi_n}(t) = \sum_{k=1}^n s(n, k) t^{k-1}, \tag{1.3}$$

where $s(n, k)$ is the Stirling number of the first kind, which is equal to $(-1)^{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with exactly k cycles; see [Sta12, Example 3.10.4]. Hence $\chi_{\Pi_n}(-1) = (-1)^{n-1}|\mathfrak{S}_n|$. Therefore, from (1.2), we recover the result observed above that the number of regions of $\mathcal{A}_{n-1}$ is $n!$.

Deformations of the braid arrangement are often studied in the literature, and in many cases their number of regions is known. Among these deformations are the Linial arrangement, Shi arrangement, Catalan arrangement, semiorder arrangement, generic braid arrangement, and semigeneric braid arrangement; see [Sta07] for a discussion of these arrangements and for a general introduction to the combinatorics of hyperplane arrangements. The *Linial arrangement* is the hyperplane arrangement in $\mathbb{R}^n$ defined by,

$$\{x_i - x_j = 1 : 1 \leqslant i < j \leqslant n\}.$$

In [PS00] Postnikov and Stanley show that the number of regions of this arrangement is the number of alternating trees on node set $[n+1]$, where a tree is *alternating* if each node is either greater than all its neighbors or smaller than all its neighbors.

Motivated by a problem on enumerating a certain class of tournaments, Hetyei [Het19] introduced the *homogenized Linial arrangement*, which is the hyperplane arrangement in

$$\{(x_1,\ldots,x_n,y_1,\ldots,y_{n-1}) : x_i \in \mathbb{R} \ \forall i \in [n], \text{ and } y_i \in \mathbb{R} \ \forall i \in [n-1]\} = \mathbb{R}^{2n-1}$$

given by[1]

$$\mathcal{H}_{2n-3} := \{x_i - x_j = y_i \mid 1 \leqslant i < j \leqslant n\}.$$

Note that by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_{n-1} = 0$, one gets the braid arrangement $\mathcal{A}_{n-1}$. Similarly by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \ldots = y_{n-1} = 1$, one gets the Linial arrangement in $\mathbb{R}^n$.

In [Het19], Hetyei proves that

$$r(\mathcal{H}_{2n-1}) = h_n, \tag{1.4}$$

where h_n is a median Genocchi number,[2] or equivalently by (1.2) that

$$-\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1) = h_n. \tag{1.5}$$

There are numerous characterizations of the median Genocchi numbers in the literature. One such characterization is given by the following formula for the generating function, which was obtained by Barsky and Dumont [BD81],

$$\sum_{n \geqslant 1} h_n x^n = \sum_{n \geqslant 1} \frac{n!(n+1)!x^n}{\prod_{k=1}^n (1 + k(k+1)x)}. \tag{1.6}$$

A combinatorial characterization in terms of a class of permutations now called Dumont derangements was also given by Barsky and Dumont in [BD81]. Another was given by Randrianarivony in [Ran94] in terms of a class of objects called surjective staircases.

Hetyei's proof of (1.5) relies on the finite field method of Athanasiadis [Ath96], which is a method for computing the characteristic polynomial by counting points in the complement of an associated arrangement over a finite field. Hetyei obtains a recurrence for a refinement of the associated point count polynomial and relates it to a recurrence of Andrews, Gawronsky, and Littlejohn [AGL11] for Legandre–Stirling numbers. This yields a formula for $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)$ in terms of the Legandre–Stirling numbers, which is identical to a formula of Claesson, Kitaev, Ragnarsson, and Tenner [CKRT10] for the median Genocchi numbers.

In this paper we further study the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ and its characteristic polynomial $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$ using an approach quite different from Hetyei's. The first few characteristic polynomials and their values at $t = 1$ and $t = 0$ are given in the table below.

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[1] Our indexing is justified by the fact that the length of the intersection lattice of the arrangement is $2n - 3$. This is an immediate consequence of Theorem 3.2.

[2] In the literature the median Genocchi number h_n is usually denoted H_{2n+3} or $|H_{2n+3}|$,

n	$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$	$t=-1$	$t=0$
1	$t-1$	-2	-1
2	$t^3 - 3t^2 + 3t - 1$	-8	-1
3	$t^5 - 6t^4 + 15t^3 - 19t^2 + 12t - 3$	-56	-3
4	$t^7 - 10t^6 + 45t^5 - 115t^4 + 177t^3 - 162t^2 + 81t - 17$	-608	-17

We begin by showing that the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice of a certain bipartite graph, which belongs to the class of Ferrers graphs introduced by Ehrenborg and van Willigenburg [EvW04]. This enables us to refine Hetyei's result by deriving a combinatorial formula for the Möbius invariant of the lower intervals of $\mathcal{L}(\mathcal{H}_{2n-1})$ in terms of a class of permutations that we call D-permutations, which are similar to the Dumont permutations introduced in [Dum74]. We obtain the following analog of (1.3):

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = \sum_{k=1}^{2n} s_D(2n, k) t^{k-1}, \tag{1.7}$$

where $(-1)^k s_D(2n, k)$ is equal to the number of D-permutations on $[2n]$ with exactly k cycles.

The D-permutations have a simple description. A permutation σ on $[2n]$ is a *D-permutation* if $i \leqslant \sigma(i)$ whenever i is odd, and $i \geqslant \sigma(i)$ whenever i is even. A consequence of (1.7) and (1.5) is that the median Genocchi number h_n is equal to the number of D-permutations on $[2n]$. A particularly interesting feature of this permutation model for the median Genocchi numbers is that the (nonmedian) Genocchi number[3] g_n enumerates a subset of the set of D-permutations on $[2n]$; namely the set of D-cycles on $[2n]$. This follows from a simple bijection with a set of permutations on $[2n-1]$ shown by Dumont in [Dum74] to have cardinality equal to g_n. This feature is in contrast with the Dumont permutation model in which the Genocchi numbers enumerate the full set of Dumont permutations on $[2n]$ and the median Genocchi numbers enumerate a subset of the these.

Our formula (1.7) therefore implies

$$\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0}, \hat{1}) = \chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(0) = -g_n, \tag{1.8}$$

where $\hat{0}$ and $\hat{1}$ are the minimum and maximum elements of $\mathcal{L}(\mathcal{H}_{2n-1})$, respectively. Hence the Genocchi numbers, as well as the median Genocchi numbers, play a fundamental role in the study of the homogenized Linial arrangement.

We recover Hetyei's result given in equation (1.5) by constructing a bijection from the full set of D-permutations on $[2n]$ to a set of surjective staircases shown by Randrianarivony in [Ran94] to have cardinality equal to the median Genocchi number h_n. Moreover, this bijection and the theory of surjective staircases enable us to prove two generating function formulas for the characteristic polynomials. One such formula is

$$\sum_{n \geqslant 1} \chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)\, x^n = \sum_{n \geqslant 1} \frac{(t-1)_{n-1}(t-1)_n\, x^n}{\prod_{k=1}^n (1 - k(t-k)x)}, \tag{1.9}$$

---

[3] In the literature the Genocchi number g_n is usually denoted G_{2n} or $|G_{2n}|$.

where $(a)_n$ denotes the falling factorial $a(a - 1) \cdots (a - n + 1)$. Note that (1.9) reduces to the Barsky–Dumont formula (1.6) for $\sum_{n \geqslant 1} h_n x^n$ when t is set equal to -1, and to a similar formula of Barsky and Dumont in [BD81] for $\sum_{n \geqslant 1} g_n x^n$ when t is set equal to 0.

In addition to the D-permutation model, our study of the characteristic polynomial leads to other interesting combinatorial models for the median Genocchi numbers and the Genocchi numbers. For instance, using our bond lattice result and Hetyei's formula (1.5), we show that a formula of Chung and Graham [CG95] for chromatic polynomials of incomparability graphs yields another nice permutation model for the median Genocchi numbers: h_n is the number of permutations σ on $[2n]$ such that $i > \sigma(i)$ only if i is even and $\sigma(i)$ is odd. This leads to the conjectural interpretation of $-\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-t)$ as the enumerator of such permutations by their number of cycles, which would imply that the Genocchi number g_n is equal to the number of such permutations that are cycles.

The paper is organized as follows. In Section 2 we review some basic material on hyperplane arrangements, bond lattices, and Genocchi numbers. Our result that the intersection lattice of the homogenized Linial arrangement is isomorphic to a bond lattice is proved in Section 3. This bond lattice has a nice description as the induced subposet of the partition lattice Π_{2n} consisting of partitions all of whose nonsingleton blocks have odd minimum and even maximum. Using Whitney's NBC theorem, we show that the signless Möbius invariant of each lower interval $[\hat{0}, \pi]$ of the bond lattice enumerates a certain class of alternating forests on $[2n]$.

In Section 4, we prove (1.7) and a more general formula for chromatic polynomials of general Ferrers graphs by constructing a bijection between the alternating forests of Section 3 and the D-permutations on $[2n]$. We also give some alternative formulas for the characteristic polynomials in terms of D-permutations and we present some of their consequences. We observe that the characteristic polynomial $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$ is divisible by $(t - 1)^3$ and we give combinatorial and geometric interpretations of the polynomial $(t - 1)^{-3} \chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$.

Section 5 contains the bijection from the D-permutations on $[2n]$ to the set of surjective staircases whose cardinality is equal to h_n. This bijection, the formulas of Section 4 for the characteristic polynomial, and the theory of surjective staircases are used to prove (1.9) and a similar generating function formula.

In Section 6, we present the permutation models for the Genocchi numbers and median Genocchi numbers that arise from the Chung–Graham result on chromatic polynomials. Further work on a type B analog and a Dowling arrangement generalization, which will appear in a forthcoming paper, is discussed in Section 7.

Most of the results of this paper and of the forthcoming paper on the Dowling generalization were announced in the extended abstract [LW19].

2. Preliminaries

2.1. Hyperplane Arrangements

Let k be a field (here k is $\mathbb{R}$ or $\mathbb{C}$). A *hyperplane arrangement* $\mathcal{A} \subseteq k^n$ is a finite collection of affine codimension-1 subspaces of k^n. The *intersection poset* of $\mathcal{A}$ is the poset $\mathcal{L}(\mathcal{A})$ of intersections of hyperplanes in $\mathcal{A}$ (viewed as affine subspaces of k^n), partially-ordered by reverse

inclusion. If $\bigcap_{H \in \mathcal{A}} H \neq \varnothing$ then the intersection poset is a geometric lattice, otherwise it's a geometric semilattice; cf. [WW86].

If $\mathcal{A}$ is a real hyperplane arrangement in $\mathbb{R}^n$, then its complement $\mathbb{R}^n \setminus \mathcal{A}$ is disconnected. By the number of regions $r(\mathcal{A})$ of $\mathcal{A}$ we mean the number of connected components of $\mathbb{R}^n \setminus \mathcal{A}$. This number can be detected solely from $\mathcal{L}(A)$ as Zaslavsky's formula (1.2) shows. See [Sta07] or [Sta12, Section 3.11] for further discussion of combinatorics of hyperplane arrangements.

2.2. The Bond Lattice of a Graph

Let $G = (V, E)$ be a graph. Given a subset B of V, let $G|_B$ denote the induced subgraph of G with vertex set B. Let Π_V denote the lattice of partitions of the set V ordered in the usual way by reverse refinement. If $V = [n] := \{1, 2, \ldots, n\}$ we write Π_n for Π_V. The *bond lattice of* G is the induced subposet Π_G of Π_V consisting of partitions $\pi = B_1 | \cdots | B_k$ such that $G|_{B_i}$ is connected for all i. A formula of Whitney [Whi32] for the chromatic polynomial $\mathrm{ch}_G(t)$ of G is given by
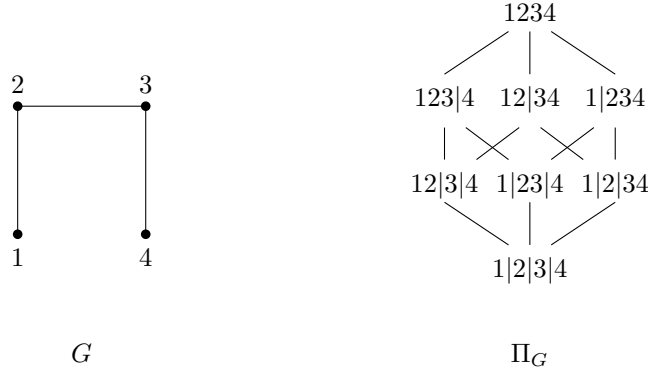
$$\mathrm{ch}_G(t) = t \chi_{\Pi_G}(t). \tag{2.1}$$

If $V = [n]$, we can also associate to G the *graphic hyperplane arrangement*

$$\mathcal{A}_G = \{x_i - x_j = 0 \mid \{i, j\} \in E(G)\} \subseteq \mathbb{R}^n.$$

It is well known and easy to see that

$$\Pi_G \cong \mathcal{L}(\mathcal{A}_G). \tag{2.2}$$

Note that Π_n is the bond lattice of the complete graph K_n and that the braid arrangement $\mathcal{A}_{n-1}$ is the graphical arrangement associated with K_n. Another example is given below.

$$G \qquad\qquad\qquad \Pi_G$$

Broken circuits provide a useful means of computing the Möbius function of the bond lattice of a graph $G = (V, E)$ (or more generally, of a geometric lattice). Fix a total ordering of E and let S be a subset of E. Then S is called a *broken circuit* if it is the edge set of a cycle of G with its smallest edge (with respect to this ordering) removed. If S does not contain a broken circuit, we say that S is a *non-broken circuit set* or *NBC* set. Clearly the edge set of a cycle always contains a broken circuit. Hence if S is an NBC set, (V, S) is a forest. We call (V, S) for which S is an NBC set an *NBC forest* of G.

Given any $S \subseteq E$, let π_S be the partition of V whose blocks are the vertex sets of the connected components of the graph (V, S). The following formula is due to Whitney [Whi32, Section 7]; a generalization for geometric lattices is due to Rota [Rot64, Pg. 359]. For $\pi \in \Pi_G$,

$$(-1)^{\mathrm{rk}(\pi)} \mu(\hat{0}, \pi) = \#\{\text{NBC forests } (V, S) \text{ of } G : \pi_S = \pi\}. \tag{2.3}$$

Given a tree T whose node set is a subset of $\mathbb{Z}_{>0}$, we say the tree is *increasing* if when T is rooted at its smallest node, each nonroot vertex is larger than its parent. A forest on a subset of $\mathbb{Z}_{>0}$ is said to be *increasing* if it consists of increasing trees. Note that if G is K_n then by ordering the edges lexicographically with the smallest element as the first component, the NBC forests of G are exactly the increasing forests on $[n]$. It therefore follows from (2.3) and (1.2) that $r(\mathcal{A}_{n-1})$ equals the number of increasing forests on $[n]$, which can easily be shown to be equal to $n!$.

For any tree T (rooted or unrooted), let $|T|$ denote the number of nodes of T. Recall that a *plane tree* is a rooted tree in which the children of each node are linearly ordered. Let T be a plane tree on a subset of $\mathbb{Z}_{>0}$. When T is drawn in the plane, the children of each vertex v are drawn from left to right according to their linear order.

Now let T be a plane tree on a subset of $\mathbb{Z}_{>0}$. If $|T| > 1$, let $T_1, T_2, \ldots, T_k$ be the subtrees rooted at the children of the root r of T ordered from "left to right". The *postorder word* $\mathrm{pw}(T)$ of T is defined recursively as the concatenation

$$\mathrm{pw}(T) = \mathrm{pw}(T_1) \cdot \mathrm{pw}(T_2) \cdot \cdots \cdot \mathrm{pw}(T_k) \cdot r,$$

if $|T| > 1$ and by $\mathrm{pw}(T) = r$ if $|T| = 1$.

Every increasing tree T on node set $V \subset \mathbb{Z}_{>0}$ can be viewed as a plane tree on V by rooting it at its smallest node and then ordering the children of each node in increasing order. It is not difficult to prove that the map that sends an increasing tree T on V to the permutation whose cycle form is $(\mathrm{pw}(T))$, is a bijection from the set of increasing trees to the set of cycles in $\mathfrak{S}_V$. This bijection extends in the obvious way to a bijection from the set of increasing forests on V to the set of permutations in $\mathfrak{S}_V$ whose cycles correspond to the trees of the forest. Thus this bijection and (2.3) yield another way of proving (1.3).

2.3. Genocchi and median Genocchi Numbers

The Genocchi numbers and median Genocchi numbers are classical sequences of numbers, which have been defined in many ways. Below we take a characterization of Dumont [Dum74, Section 6] as our definition of the Genocchi numbers and a characterization of Barsky and Dumont [BD81] as our definition of the median Genocchi numbers.

A *Dumont permutation* is a permutation $\sigma \in \mathfrak{S}_{2n}$ such that $2i > \sigma(2i)$ and $2i-1 \leqslant \sigma(2i-1)$ for all $i = 1, \ldots, n$. A *Dumont derangement* is a Dumont permutation without fixed points, i.e., $2i > \sigma(2i)$ and $2i - 1 < \sigma(2i - 1)$ for all $i = 1, \ldots, n$.

Example 2.1. When $n = 2$, the Dumont permutations on $[4]$ (in cycle form) are

$$(1, 2)(3, 4) \qquad (1, 3, 4, 2) \qquad (1, 4, 2)(3).$$

When $n = 3$, the Dumont derangements on $[6]$ are:

$$(1, 3, 5, 6, 4, 2) \quad (1, 3, 4, 2)(5, 6) \quad (1, 2)(3, 4)(5, 6) \quad (1, 2)(3, 5, 6, 4)$$
$$(1, 4, 3, 5, 6, 2) \quad (1, 5, 6, 3, 4, 2) \quad (1, 5, 6, 2)(3, 4) \quad (1, 4, 2)(3, 5, 6).$$

For $n \geqslant 1$, the (signless) *Genocchi number* g_n is defined to be the number of Dumont permutations on $[2n - 2]$, and for $n \geqslant 0$, the (signless) *median Genocchi number* h_n is defined to be the number of Dumont derangements on $[2n + 2]$. We list the Genocchi numbers and the median Genocchi numbers for small values of n in the table below.

n	0	1	2	3	4	5	6
g_n		1	1	3	17	155	2073
h_n	1	2	8	56	608	9440	198272

Our notation is nonstandard in that g_n is usually denoted G_{2n} or $|G_{2n}|$, while h_n is usually denoted H_{2n+3} or $|H_{2n+3}|$. The Genocchi numbers and median Genocchi numbers are also known as Genocchi numbers of the first and second kind, respectively.

Another permutation characterization of the Genocchi numbers, obtained by Dumont in [Dum74, Section 6], is given by

$$g_n = |\{\sigma \in \mathfrak{S}_{2n-1} : \forall i \in [2n - 2], \, \sigma(i) > \sigma(i + 1) \text{ if and only if } \sigma(i) \text{ is even}\}|. \tag{2.4}$$

We also mention the following exponential generating function formula for the Genocchi numbers (see [Dum74, page 305])

$$\sum_{n \geqslant 1} g_n \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}.$$

In [BD81], two formulas for the generating function of the Genocchi numbers and two formulas for the generating function of the median Genocchi numbers are given. We list these four generating function formulas here.

$$\sum_{n \geqslant 1} g_n z^n = \sum_{n \geqslant 1} \frac{(n - 1)! \, n! \, z^n}{\prod_{k=1}^{n}(1 + k^2 z)} \tag{2.5}$$

$$\sum_{n \geqslant 0} h_n z^n = \sum_{n \geqslant 0} \frac{n! \, (n + 1)! \, z^n}{\prod_{k=1}^{n}(1 + k(k + 1)z)} \tag{2.6}$$

$$\sum_{n \geqslant 0} g_{n+1} z^n = \sum_{n \geqslant 0} \frac{(n!)^2 \, z^n}{\prod_{k=1}^{n}(1 + k^2 z)} \tag{2.7}$$

$$\sum_{n \geqslant 1} h_{n-1} z^n = \sum_{n \geqslant 1} \frac{(n!)^2 z^n}{\prod_{k=1}^{n}(1 + k(k + 1)z)}. \tag{2.8}$$

Equation (2.7) is due to Carlitz [Car71] and to Riordan and Stein [RS73], and the rest are due to Barsky and Dumont [BD81].

3. The intersection lattice is a bond lattice

In this section we show that $\mathcal{L}(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice of a bipartite graph. This enables us to give a characterization of the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ as an induced subposet of Π_{2n} and to compute its Möbius function by counting NBC sets.

Let Γ_{2n} be the bipartite graph on $\{1, 3, 5, \ldots, 2n-1\} \sqcup \{2, 4, 6, \ldots, 2n\}$ with an edge $\{2i - 1, 2j\}$ whenever $1 \leqslant i \leqslant j \leqslant n$. The graph Γ_6 is shown below.
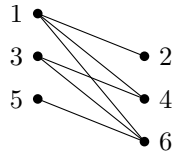
More generally for any finite subset V of $\mathbb{Z}_{>0}$, let Γ_V be the bipartite graph on V with edge set

$$E = \{\{u, v\} : u < v \in V, \ u \text{ is odd and } v \text{ is even}\}.$$

Remark 3.1. The graphs Γ_V are easily seen to belong to the class of *Ferrers graphs*, which were introduced by Ehrenborg and van Willegenburg in [EvW04]. In fact, it is not much more difficult to show that all Ferrers graphs are of this form. Ferrers graphs have been further studied in (among others) [CKRT10] and [SSS18].

Theorem 3.2. *The poset isomorphism $\mathcal{L}(\mathcal{H}_{2n-1}) \cong \Pi_{\Gamma_{2n}}$ holds for all $n \geqslant 1$.*

Proof. Let $(e_1, e_2, \ldots, e_{2n+1})$ be the standard basis for $\mathbb{R}^{2n+1}$. For $1 \leqslant i \leqslant j \leqslant n$, let

$$H_{i,j} = \{v \in \mathbb{R}^{2n+1} : (e_i - e_{n+1+i} - e_{j+1}) \cdot v = 0\}$$

and

$$K_{i,j} = \{v \in \mathbb{R}^{2n+1} : (e_{2i-1} - e_{2j}) \cdot v = 0\}.$$

Clearly $\{H_{i,j} : 1 \leqslant i \leqslant j \leqslant n\}$ is precisely the arrangement $\mathcal{H}_{2n-1}$. Note that $\mathcal{K}_{2n} := \{K_{i,j} : 1 \leqslant i \leqslant j \leqslant n\}$ is the graphical arrangement $\mathcal{A}_G$, where G is the graph on $[2n + 1]$ whose edge set is equal to $E(\Gamma_{2n})$. In other words, G is Γ_{2n} with an appended isolated node $2n + 1$. Since adding an isolated node to a graph yields an isomorphic bond lattice, by (2.2) we have

$$\mathcal{L}(\mathcal{K}_{2n}) \cong \Pi_{\Gamma_{2n}}.$$

We will prove the result by producing a vector space isomorphism $\psi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ that takes $\mathcal{K}_{2n}$ to $\mathcal{H}_{2n-1}$. Indeed, such a map will induce an isomorphism from $\mathcal{L}(\mathcal{K}_{2n})$ to $\mathcal{L}(\mathcal{H}_{2n-1})$.

First consider the linear operator $\phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ defined on the standard basis by letting $\phi(e_{2i-1}) = e_i - e_{n+1+i}$ and $\phi(e_{2i}) = e_{i+1}$ for all $i \in [n]$ and letting $\phi(e_{2n+1}) = e_1$. Let A be the matrix of ϕ with respect to the standard basis. One can easily check that $|\det A| = 1$. Now let $\psi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ be the linear operator whose matrix with respect to the standard basis is the transpose of A^{-1}. Clearly ψ is an isomorphism.

We claim that ψ takes the hyperplane $K_{i,j}$ to the hyperplane $H_{i,j}$ for all $1 \leqslant i \leqslant j \leqslant n$. To prove the claim, let $v \in K_{i,j}$, so $(e_{2i-1} - e_{2j}) \cdot v = 0$. We have

$$\phi(e_{2i-1} - e_{2j}) \cdot \psi(v) = (e_i - e_{n+1+i} - e_{j+1}) \cdot \psi(v).$$

We also have

$$\begin{aligned}
\phi(e_{2i-1} - e_{2j}) \cdot \psi(v) &= (A(e_{2i-1} - e_{2j})) \cdot ((A^{-1})^T v) \\
&= (A^{-1} A(e_{2i-1} - e_{2j})) \cdot v \\
&= (e_{2i-1} - e_{2j}) \cdot v \\
&= 0.
\end{aligned}$$

It follows that $(e_i - e_{n+1+i} - e_{j+1}) \cdot \psi(v) = 0$. Hence $\psi(v)$ is in $H_{i,j}$, which proves the claim. It follows from the claim that ψ takes $\mathcal{K}_{2n}$ to $\mathcal{H}_{2n-1}$ as desired. $\qquad\square$

Hence, to study $\mathcal{L}(\mathcal{H}_{2n-1})$, it suffices to study $\Pi_{\Gamma_{2n}}$.

Proposition 3.3. *For all finite subsets V of $\mathbb{Z}_{>0}$, the bond lattice Π_{Γ_V} is the induced subposet of Π_V consisting of those partitions π for which each nonsingleton block of π has an odd minimum and an even maximum.*

Proof. Suppose that $\pi \in \Pi_{\Gamma_V}$. Let B be a nonsingleton block of π and let $u = \min B$. Since $\Gamma_V|_B$ is connected, there must be an element $v \in B$ such that $\{u, v\}$ is an edge of Γ_V. By definition of Γ_V, since $u < v$, the node u must be odd. A similar argument tells us that $\max B$ is even.

Conversely, suppose that we have a partition $\pi = B_1 | \cdots | B_k \in \Pi_V$ such that each nonsingleton B_i has an odd minimum and an even maximum. We wish to show that $\Gamma_V|_{B_i}$ is connected. Indeed, let $M_i = \max B_i$ and $m_i = \min B_i$. By the definition of Γ_V, each odd vertex in B_i shares an edge with M_i and each even vertex of B_i shares an edge with m_i. Since, in particular, this means that m_i and M_i share an edge, $\Gamma_V|_{B_i}$ is connected. Hence $\pi \in \Pi_{\Gamma_V}$. $\qquad\square$

We will use the Rota–Whitney formula (2.3) to evaluate the Möbius function on each lower interval $[\hat{0}, \pi]$ of Π_{Γ_V}. Let T be a tree whose nodes are in $\mathbb{Z}_{>0}$. We say that T is *increasing-decreasing (ID)* if, when T is rooted at its largest node, each internal node v satisfies

(1) if v is odd then v is less than all its descendants and all its children are even,

(2) if v is even then v is greater than all its descendants and all its children are odd.

An an *increasing-decreasing (or ID) forest* is a forest whose components are all increasing-decreasing trees. An example of an ID forest is given in the figure below.
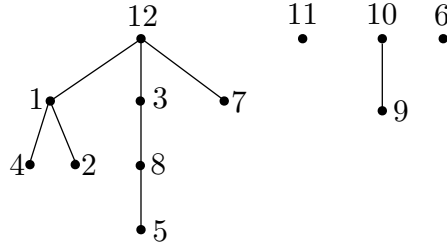
Proposition 3.4. *Let T be a tree whose nodes are in $\mathbb{Z}_{>0}$. Then T is an ID tree if and only if its internal nodes satisfy conditions (1) and (2) of the definition of ID tree when T is rooted at its smallest node.*

Proof. Assume $|T| > 1$. Let T be an ID tree with odd smallest node s and even largest node t. Root T at t. Let $T_1, \ldots, T_k$ be the subtrees rooted at the children of t. Clearly, s must be the root of one of these subtrees say T_1. Let $S_1, \ldots, S_j$ be the subtrees rooted at the children of s. Let T' be the tree obtained by removing T_1 from T. When we root T at s we get the rooted tree in which the subtrees rooted at the children of s are $S_1, \ldots, S_j, T'$. Since T is an ID tree, these subtrees $S_1, \ldots, S_j, T'$ must be ID trees rooted at their largest nodes, which are even. Hence the internal nodes of T rooted at s satisfy (1) and (2) of the definition of ID tree.

The proof of the converse is analogous with the roles of even and odd exchanged, and the roles of larger and smaller exchanged. $\square$

The following result is implicit in the proof of Theorem 24 of [SSS18]. We include a proof for the sake of completeness.

Theorem 3.5. *For all $\pi \in \Pi_{\Gamma_V}$, we have that $(-1)^{|\pi|}\mu(\hat{0}, \pi)$ equals the number of ID forests on V whose trees have node sets equal to the blocks of π. Consequently, $-\mu_{\Pi_{\Gamma_V}}(\hat{0}, \hat{1})$ is equal to the number of ID trees on V and*

$$\chi_{\Pi_{\Gamma_V}}(t) = \sum_{k=1}^{2n} (-1)^k |\mathcal{F}_{V,k}| t^{k-1}, \tag{3.1}$$

where $\mathcal{F}_{V,k}$ is the set of ID forests on V with exactly k trees.

Proof. We prove this result by showing that under an appropriate ordering of the edges of Γ_V, the NBC forests of Γ_V are precisely the ID forests on V. The total order on the edges of Γ_V we use is as follows: First order the odd vertices in *increasing* order and the even vertices in *decreasing* order, and then fix any linear extension $\prec$ of the product order of the two linear orders. By associating the edge $\{2i - 1, 2j\}$ with the ordered pair $(2i - 1, 2j)$, we get a total order on the edges of Γ_V. For instance, we have $\{1, 8\} \prec \{1, 6\} \prec \{3, 2\}$.

Since adding an edge between two different components of a forest can never complete a cycle, a forest is an NBC forest of Γ_V if and only if its trees are NBC trees of the subgraph of Γ_V induced by the node set of the tree. By definition, a forest is an ID forest if and only if its trees are ID trees. Hence, it suffices to prove that the NBC trees of Γ_V are precisely the ID trees on V. We can assume $|V| > 1$, $\min V$ is odd and $\max V$ is even.

Let T be an ID tree on V. Suppose that $e = \{2i - 1, 2j\}$ is an edge of Γ_V that is not in T. If we add e to T, we create a unique cycle ρ, which contains e. To show that T is NBC, it suffices to find an edge of ρ that strictly precedes e in the order $\prec$. Root T at its largest node and let h be the youngest common ancestor of $2i - 1$ and $2j$ (that is, the furthest from the root). Clearly h is a node of ρ.

Suppose h is odd. Then $h \leqslant 2i - 1$ since T is an ID tree. If h is the parent of $2j$ then h cannot be $2i - 1$ since $\{2i - 1, 2j\}$ is not an edge of T. Therefore $h < 2i - 1$, which implies that $\{h, 2j\} \prec \{2i - 1, 2j\} = e$. Thus $\{h, 2j\}$ is the edge of ρ that we seek. If h is not the parent

of $2j$ then let $2k$ be the child of h that is a proper ancestor of $2j$. We thus have $2k > 2j$, which implies $\{h, 2k\} \prec \{2i-1, 2j\} = e$. Thus in this case, $\{h, 2k\}$ is the edge of ρ that we seek.

The case that h is even is handled analogously with the role of even and odd exchanged and the role of $<$ and $>$ exchanged. We can now conclude that T is an NBC tree of Γ_V.

Conversely, let T be an NBC tree of Γ_V. Suppose that T is not an ID tree. Then T rooted at its largest node has a node x with a grandchild y satisfying

$$x > y \text{ if } x \text{ is odd and } x < y \text{ if } x \text{ is even.}$$

Choose x so that its distance in T from the root is minimal. We claim that x cannot be the root of T. Indeed, if x is even then it is not the largest node of T. Thus x is not the root. If x is odd then it is not the root since the largest node of T must be even. Let u be the parent of x and let v be the parent of y, which implies that x is that parent of v. Thus, u, x, v, y forms a path in T.

We claim that $\{u, y\}$ is an edge of Γ_V. Indeed, if u is even then x is odd. We thus have $u > x > y$. Also y is odd since it is a grandchild of x. It follows that $\{u, y\}$ is an edge of Γ_V. The case when u is odd is handled analogously.

Since $\{u, y\}$ is an edge of Γ_V we can use it to complete the cycle induced by the node set $\{u, x, v, y\}$. Clearly $\{u, y\} \prec \{u, x\}$. By the minimality of the choice of x, since u is closer to the root than x is, we have $u < v$ if u is odd and $u > v$ if u is even. Hence $\{u, y\} \prec \{u, x\}, \{v, x\}, \{v, y\}$. This implies that the subgraph of T induced by $\{u, x, v, y\}$ is a broken circuit which contradicts the hypothesis that T is an NBC tree. Therefore T is indeed an ID tree. $\qquad\square$

By Theorem 3.2, we have the following consequence of Theorem 3.5.

Corollary 3.6. *For all $n \geqslant 1$,*

$$\chi_{\mathcal{L}(H_{2n-1})}(t) = \sum_{k=1}^{2n} (-1)^k |\mathcal{F}_{2n,k}| t^{k-1},$$

where $\mathcal{F}_{2n,k}$ is the set of ID forests on $[2n]$ with exactly k trees.

Using Hetyei's formula (1.5) and Corollary 3.6, we obtain the following combinatorial interpretation of the median Genocchi numbers. In Section 5, we give an alternative proof that does not rely on these results.

Corollary 3.7. *For all $n \geqslant 1$, the number of ID forests on $[2n]$ is equal to h_n.*

In the next section we prove the following analogous result; see Remark 4.7.

Theorem 3.8. *For all $n \geqslant 1$, the number of ID trees on $[2n]$ is equal to g_n.*

From this we see that the nonmedian Genocchi numbers also play an important role in the study of the intersection lattice of the homogenized Linial arrangement.

Corollary 3.9. *For all $n \geqslant 1$,*

$$\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0}, \hat{1}) = -g_n. \tag{3.2}$$

4. From ID forests to Dumont-like permutations

Our next step is to introduce a class of permutations similar to the Dumont permutations discussed in Section 2.3 and then give a bijection between these permutations and the ID forests. This will enable us to express the characteristic polynomial in terms of D-permutations in several ways.

4.1. D-permutations

Let A be a finite subset of $\mathbb{Z}_{>0}$. We say $\sigma \in \mathfrak{S}_A$ is a *D-permutation* on A if $i \leqslant \sigma(i)$ whenever i is odd and $i \geqslant \sigma(i)$ whenever i is even. We denote by $\mathcal{D}_A$ the set of D-permutations on A and by $\mathcal{DC}_A$ the set of D-cycles on A. If $A = [n]$, we write $\mathcal{D}_n$ and $\mathcal{DC}_n$.

Example 4.1. The D-permutations on $[4]$ (in cycle form) are

$$(1)(2)(3)(4) \quad (1,2)(3)(4) \quad (1,4)(2)(3) \quad (3,4)(1)(2)$$
$$(1,4,2)(3) \quad (1,3,4)(2) \quad (1,2)(3,4) \quad (1,3,4,2).$$

The D-cycles on $[6]$ are:

$$(1,3,5,6,4,2) \qquad (1,4,3,5,6,2) \qquad (1,5,6,3,4,2).$$

Note that all Dumont permutations are D-permutations, but not conversely. Indeed, a D-permutation can have both even and odd fixed points, while a Dumont permutation can only have odd fixed points. It follows immediately from the definitions that

$$\mathcal{DC}_{2n} \subseteq \{\text{Dumont derange. in } \mathfrak{S}_{2n}\} \subseteq \{\text{Dumont permut. in } \mathfrak{S}_{2n}\} \subseteq \mathcal{D}_{2n}.$$

Recall that the two sets in the middle of this chain are enumerated by median Genocchi number h_{n-1} and Genocchi number g_{n+1}, respectively. It turns out that the sets on the ends of the chain are also enumerated by Genocchi and median Genocchi numbers, respectively. The count for $\mathcal{DC}_{2n}$ is an easy consequence of (2.4). We will see at the end of the next subsection that the count for $\mathcal{D}_{2n}$ can be proved using Hetyei's formula (1.5). In Section 5 we will prove it without relying on Hetyei's result.

Theorem 4.2. *For all $n \geqslant 1$,*

(1) $|\mathcal{DC}_{2n}| = g_n$

(2) $|\mathcal{D}_{2n}| = h_n$.

Proof of (1). There is an easy bijection between the set of D-cycles on $[2n]$ and the set S of permutations given in (2.4). Indeed, we send a D-cycle $(a_1, a_2, \ldots, a_{2n})$, where $a_{2n} = 2n$, to the permutation $a_1, a_2, \ldots, a_{2n-1}$ (in one line notation), which is clearly in S. Hence the result follows from (2.4). □

Proof of (2). Delayed to the end of Section 4.2. □

4.2. The bijection

Let $\mathcal{T}$ be the set of ID trees. For $T \in \mathcal{T}$, let $\hat{T}$ be the plane tree obtained by rooting T at its largest node (which is even when $|T| > 1$) and fixing the following total ordering on the children of each node v of T:

- if v is even, place its children (all of which are odd) in increasing order from left to right

- if v is odd, place its children (all of which are even) in decreasing order from left to right.

Also let $\tilde{T}$ be the plane tree obtained by rooting T at its smallest node (which is odd when $|T| > 1$) and fixing the above total ordering on the children of each node v of T.

Recall from Section 2.2, that $\mathrm{pw}(T)$ denotes the postorder word of a plane tree T. For the ID tree T below, $\mathrm{pw}(\hat{T}) = 42156378$ and $\mathrm{pw}(\tilde{T}) = 56378421$.
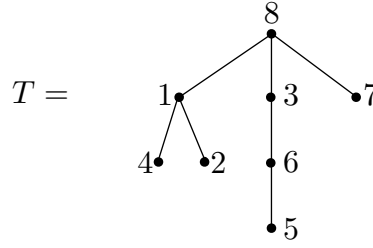
Let $\mathcal{W}$ be the set of all words $w_1 w_2 \ldots w_m$, where $m \geqslant 1$, with distinct letters in $\mathbb{Z}_{>0}$ that satisfy the following conditions:

(1) for all $i \in [m-1]$, if w_i is odd then $w_i < w_{i+1}$

(2) for all $i \in [m-1]$, if w_i is even then $w_i > w_{i+1}$

(3) if w_m is odd then it is the smallest letter of w

(4) if w_m is even then it is the largest letter of w.

Lemma 4.3. *Let $T \in \mathcal{T}$. Then $\mathrm{pw}(\hat{T}) \in \mathcal{W}$ and $\mathrm{pw}(\tilde{T}) \in \mathcal{W}$.*

Proof. Suppose $T \in \mathcal{T}$. We prove the result for $\hat{T}$ and $\tilde{T}$ simultaneously by induction on $|T|$. The result is obviously true for $|T| = 1$, so assume $|T| > 1$. Let $v_1 < \cdots < v_k$ be the children of the root r of $\hat{T}$. For each i, the subtree of $\hat{T}$ rooted at v_i is of the form $\tilde{T}_i$ for some ID tree T_i by Proposition 3.4. We have

$$\mathrm{pw}(\hat{T}) = \mathrm{pw}(\tilde{T}_1) \cdot \; \cdots \; \cdot \mathrm{pw}(\tilde{T}_k) \cdot r,$$

where $\cdot$ denotes concatenation. Since the last letter of $\mathrm{pw}(\hat{T})$ is the root of $\hat{T}$, it is even and is larger than all the other letters.

By induction on $|T|$, each $\mathrm{pw}(\tilde{T}_i)$ is in $\mathcal{W}$. We also know that the last letter of $\mathrm{pw}(\tilde{T}_i)$ is the root of $\tilde{T}_i$, which is odd. Hence to prove that $\mathrm{pw}(\hat{T}) \in \mathcal{W}$ we need only show that the last letter v_i of $\mathrm{pw}(\tilde{T}_i)$ is less than the first letter of $\mathrm{pw}(\tilde{T}_{i+1})$ for all $i \in [k-1]$. (Note that the last letter of $\mathrm{pw}(\tilde{T}_k)$ is less than r). The first letter of $\mathrm{pw}(\tilde{T}_{i+1})$ is greater than the root v_{i+1} of $\tilde{T}_{i+1}$. Since $v_i < v_{i+1}$, we have that v_i is less than the first letter of $\mathrm{pw}(\tilde{T}_{i+1})$. Thus $\mathrm{pw}(\hat{T}) \in \mathcal{W}$. An analogous argument yields $\mathrm{pw}(\tilde{T}) \in \mathcal{W}$. $\qquad\square$

Given a word $w = w_1 w_2 \cdots w_m$ over alphabet $\mathbb{Z}_{>0}$, we say that w_j is a *right-to-left minimum* if $w_j < w_i$ for all $i > j$. Similarly we say that w_j is a *right-to-left maximum* if $w_j > w_i$ for all $i > j$. For example, the right-to-left minima of 2156437 are $1, 3, 7$. We also let w' denote w with its last letter removed, that is, $w' = w_1 \cdots w_{m-1}$.

Lemma 4.4. *Let $T \in \mathcal{T}$. Then the children of the root of $\hat{T}$ are exactly the right-to-left minima of $\mathrm{pw}(\hat{T})'$, and the children of the root of $\tilde{T}$ are exactly the right-to-left maxima of the word $\mathrm{pw}(\tilde{T})'$.*

Proof. Let $v_1 < \cdots < v_k$ be the children of the root r of $\hat{T}$. Each v_j is less than all its descendants in $\hat{T}$, so for all $i < j$, we have that v_i is less than all descendents of v_j. Hence, v_i, which is the last letter of the postorder word of the subtree rooted at v_i, is less than all subsequent letters of $\mathrm{pw}(\hat{T})'$. Conversely, if v is not a child of r then it has an odd anscestor u, which must be less than v. Since u appears after v in the postorder word, v is not a right-to-left minimum. The proof for $\tilde{T}$ is completely analogous. $\square$

Lemma 4.5. *The map* $\mathrm{pw} : \{\hat{T} : T \in \mathcal{T}\} \cup \{\tilde{T} : T \in \mathcal{T}\} \to \mathcal{W}$ *is a bijection.*

Proof. To show that the map is a bijection, we construct a map

$$\gamma : \mathcal{W} \to \{\hat{T} : T \in \mathcal{T}\} \cup \{\tilde{T} : T \in \mathcal{T}\}$$

and show that it is the inverse of pw.

We define γ recursively. Let $w = w_1 \cdots w_m \in \mathcal{W}$. If $m = 1$, let $\gamma(w)$ be the tree with the single node w_1.

Now suppose $m > 1$. To define γ, we need two cases:

Case 1: Assume w_m is even. Then it is the largest letter of w. Let $j_1 < \cdots < j_k$ be such that $w_{j_1} \cdots, w_{j_k}$ are the right-to-left minima of the word $w' := w_1 w_2 \ldots w_{m-1}$. Note that $j_k = m - 1$. For each $i = 1, \ldots, k$, let $\alpha_i(w)$ be the segment $w_{j_{i-1}+1} w_{j_{i-1}+2} \cdots w_{j_i}$, where $j_0 = 0$. Hence w' is the concatenation of the words $\alpha_1(w), \alpha_2(w), \ldots, \alpha_k(w)$.

For example, given the word $w = 21564378$ in $\mathcal{W}$, we have, $w' = 2156437$, and w' breaks up into $\alpha_1(w) = 21$, $\alpha_2(w) = 5643$, and $\alpha_3(w) = 7$.

Since for all i, $\alpha_i(w)$ is a segment of $w \in \mathcal{W}$, it satisfies conditions (1) and (2) of the definition of $\mathcal{W}$. We claim that conditions (3) and (4) hold as well. Indeed, the last letter w_{j_i} of $\alpha_i(w)$ is odd since it is a right-to-left minimum of w' and therefore is followed by a larger letter in w. The last letter of $\alpha_i(w)$ is smaller than all the other letters of $\alpha_i(w)$ since none of the other letters are right-to-left minima. Hence (3) holds and (4) holds vacuously. Thus $\alpha_i(w) \in \mathcal{W}$.

Now, for each i, we can recursively apply γ to $\alpha_i(w)$ to obtain a plane tree $\gamma(\alpha_i(w))$ in $\{\tilde{T} : T \in \mathcal{T}\}$. Let $\gamma(w)$ be the plane tree constructed as follows:

(1) The root is w_m, which is even and largest.

(2) The children of w_m are the right-to-left minima $w_{j_1} < \cdots < w_{j_k}$, which are odd.

(3) The subtree rooted at w_{j_i} is $\gamma(\alpha_i(w))$ for each i.

Clearly $\gamma(w) \in \{\hat{T} : T \in \mathcal{T}\}$.

Case 2: Assume w_m is odd. An analogous argument allows us to decompose

$$w = w_1 \cdots w_{m-1} \in \mathcal{W}$$

into segments $\alpha_i(w) \in \mathcal{W}$, whose last letter is the ith right-to-left maximum of w. We define $\gamma(w)$ to be the plane tree constructed as follows:

(1) The root is w_m, which is odd.

(2) The children of w_m are the right-to-left maxima $w_{j_1} > \cdots > w_{j_k}$, which are even.

(3) The subtree rooted at w_{j_i} is $\gamma(\alpha_i(w))$ for each i.

We have $\gamma(w) \in \{\tilde{T} : T \in \mathcal{T}\}$.

Now that we have shown that γ is well-defined, it remains to check that the maps pw and γ are inverses of each other. We prove $\gamma(\mathrm{pw}(\hat{T})) = \hat{T}$ and $\gamma(\mathrm{pw}(\tilde{T})) = \tilde{T}$, for all $T \in \mathcal{T}$, by induction on $|T|$. If $|T| = 1$, this is clear, so suppose $|T| > 1$.

Case 1: $\hat{T}$. By Lemma 4.4, the children $v_1 < v_2 < \cdots < v_k$ of the root r of $\hat{T}$ are exactly the right-to-left minima of the word $\mathrm{pw}(\hat{T})' = \mathrm{pw}(\tilde{T}_1) \cdot \mathrm{pw}(\tilde{T}_2) \cdot \cdots \cdot \mathrm{pw}(\tilde{T}_k)$, where $\tilde{T}_i$ is the subtree of $\hat{T}$ rooted at v_i. Since v_i is the last letter of $\mathrm{pw}(\tilde{T}_i)$, we have $\alpha_i(\mathrm{pw}(\hat{T})) = \mathrm{pw}(\tilde{T}_i)$ for all i. By induction, $\gamma(\alpha_i(\mathrm{pw}(\hat{T}))) = \gamma(\mathrm{pw}(\tilde{T}_i)) = \tilde{T}_i$.

Now we have that $\gamma(\mathrm{pw}(\hat{T}))$ is the plane tree whose root is the last letter of $\mathrm{pw}(\hat{T})$, which is r. The children of r are the right-to-left minima of $\mathrm{pw}(\hat{T})'$ which are $v_1 < \cdots < v_k$. The subtree rooted at v_i is $\gamma(\alpha_i(\mathrm{pw}(\hat{T})))$ which is $\tilde{T}_i$. Hence $\gamma(\mathrm{pw}(\hat{T})) = \hat{T}$, as desired.

Case 2: $\tilde{T}$. A completely analogous argument gives $\gamma(\mathrm{pw}(\tilde{T})) = \tilde{T}$.

Hence $\gamma \circ \mathrm{pw}$ is the identity map on $\{\hat{T} : T \in \mathcal{T}\} \cup \{\tilde{T} : T \in \mathcal{T}\}$. A similar argument can be used to prove that the other composition $\mathrm{pw} \circ \gamma$ equals the identity map on $\mathcal{W}$. $\square$

For any finite subset A of $\mathbb{Z}_{>0}$, let $\mathcal{T}_A = \{T \in \mathcal{T} : \text{the node set of } T \text{ is } A\}$ and let $\mathcal{W}_A = \{w \in \mathcal{W} : \text{the set of letters of } w \text{ is } A\}$. Recall that $\mathcal{DC}_A$ is the set of D-cycles on A.

Theorem 4.6. *For any finite subset A of $\mathbb{Z}_{>0}$, let $\psi : \mathcal{T}_A \to \mathcal{DC}_A$ be the map defined by letting $\psi(T)$ be the cycle $(\mathrm{pw}(\hat{T}))$ in $\mathfrak{S}_A$. Then ψ is a well defined bijection. Consequently $|\mathcal{T}_A| = |\mathcal{DC}_A|$.*

Proof. Consider the following maps:

(1) the map from $\mathcal{T}_A$ to $\{\hat{T} : T \in \mathcal{T}_A\}$ defined by $T \mapsto \hat{T}$

(2) the map from $\{\hat{T} : T \in \mathcal{T}_A\}$ to $\{w \in \mathcal{W}_A : \text{last letter of } w \text{ is largest}\}$ defined by $\hat{T} \mapsto \mathrm{pw}(\hat{T})$

(3) the map from $\{w \in \mathcal{W}_A : \text{last letter of } w \text{ is largest}\}$ to $\mathcal{DC}_A$ defined by $w \mapsto (w)$.

It is clear that the first map is a bijection. It follows from Lemma 4.5 that the second is also bijection. The third map is a well defined bijection since the last letter of w must be even. Since ψ is the composition of three bijections, it is also a bjection. $\square$

Remark 4.7. Theorem 3.8 is now proved since it follows immediately from Theorems 4.2(1) and 4.6.

An analogous argument can be used to show that the map $\mathcal{T}_A \to \mathcal{DC}_A$ that takes T to the cycle $(\mathrm{pw}(\tilde{T}))$ is also a well defined bijection. The following result shows that this map is, in fact, identical to ψ.

Proposition 4.8. *For all $T \in \mathcal{T}_A$, the cycles $(\mathrm{pw}(\hat{T}))$ and $(\mathrm{pw}(\tilde{T}))$ in $\mathfrak{S}_A$ are equal.*

Proof. Let s be the smallest element of A and t the largest element. Then s is the root of $\tilde{T}$ and t is its leftmost child. Let $t_1 < \cdots < t_k$ be the children of t in $\tilde{T}$ and let $t > s_1 > \cdots > s_j$ be the children of s. For each $i \geqslant 1$, let T_i be the subtree of $\tilde{T}$ rooted at t_i and let S_i be the subtree of $\tilde{T}$ rooted at s_i. Then $\mathrm{pw}(\tilde{T})$ is the concatenation

$$\mathrm{pw}(T_1) \cdots \mathrm{pw}(T_k) \cdot t \cdot \mathrm{pw}(S_1) \cdots \mathrm{pw}(S_j) \cdot s.$$

Now let us look at $\mathrm{pw}(\hat{T})$. Clearly t is the root of $\hat{T}$ and s is its leftmost child. The children of t are $s < t_1 < \cdots < t_k$. The children of s are $s_1 < \cdots < s_j$. For each $i \geqslant 1$, the subtree of $\hat{T}$ rooted s_i is S_i and the subtree rooted at t_i is T_i. Then $\mathrm{pw}(\hat{T})$ is the concatenation

$$\mathrm{pw}(S_1) \cdots \mathrm{pw}(S_j) \cdot s \cdot \mathrm{pw}(T_1) \cdots \mathrm{pw}(T_k) \cdot t.$$

From this we see that $\mathrm{pw}(\tilde{T})$ and $\mathrm{pw}(\hat{T})$ are related by a cyclic shift. Hence the cycles are the same. $\square$

The *cycle support* of $\sigma \in \mathfrak{S}_n$ is the partition $\mathrm{cyc}(\sigma) \in \Pi_n$ whose blocks are comprised of the elements of the cycles of σ. For example,

$$\mathrm{cyc}((1,7,2,4)(5)(6,8,9,3)) = 1247|5|3689.$$

The bijection in Theorem 4.6 extends to a bijection between ID forests and D-permutations. Under this bijection, the blocks of the cycle support of the image of an ID forest are the node sets of the trees of the ID forest. Hence we have the following consequence of Theorem 4.6.

Corollary 4.9. *Let π be a partition of a finite subset of $Z_{>0}$. Then the ID forests whose trees have node sets equal to the blocks of π are in bijection with the D-permutations whose cycle support is π.*

As a consequence of Theorem 3.5 and Corollary 4.9, we have the following result.

Theorem 4.10. *Let A be a finite subset of $\mathbb{Z}_{>0}$. For all $\pi \in \Pi_{\Gamma_A}$,*

$$(-1)^{|\pi|} \mu_{\Pi_{\Gamma_A}}(\hat{0}, \pi) = |\{\sigma \in \mathcal{D}_A \mid \mathrm{cyc}(\sigma) = \pi\}|.$$

Consequently,

$$\chi_{\Pi_{\Gamma_A}}(t) = \sum_{k=1}^{2n} s_D(A, k) t^{k-1}, \tag{4.1}$$

where $(-1)^k s_D(A, k)$ is equal to the number of D-permutations on A with exactly k cycles.

Equation (4.1) is equivalent to the following result on the chromatic polynomial $\mathrm{ch}_G(t)$ of a general Ferrers graph G (see (2.1) and Remark 3.1).

Corollary 4.11. *For all finite subsets A of $\mathbb{Z}_{>0}$,*

$$\mathrm{ch}_{\Gamma_A}(t) = \sum_{k=1}^{2n} s_D(A, k) t^k.$$

By Theorem 3.2, we have now proved (1.7), which is restated here.

Corollary 4.12. *For all $n \geqslant 1$,*

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})} = \sum_{k=1}^{2n} s_D(2n, k) t^{k-1},$$

where $(-1)^k s_D(2n, k)$ is equal to the number of D-permutations on $[2n]$ with exactly k cycles.

Corollary 4.13. *For all $n \geqslant 1$,*

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1) = -|\mathcal{D}_{2n}|$$
$$\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0}, \hat{1}) = -|\mathcal{DC}_{2n}|.$$

Note that it follows from Corollary 4.13 that Part (2) of Theorem 4.2 is equivalent to Hetyei's formula (1.5). So Part (2) is now proved. In Section 5.2, we will prove Part (2) without relying on (1.5), thereby recovering Hetyei's result.

4.3. Alternative formulas

We now give two alternative ways of expressing the characteristic polynomial in terms of D-permutations (equivalently ID forests), which will be used in Section 5.3. As a consequence of these formulas, we get a decomposition of the Genocchi numbers and the median Genocchi numbers into a sum of powers of 2.

Theorem 4.14. *For all $n \geqslant 1$,*

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = (t-1) \sum_{\sigma \in \mathcal{D}_{2n-2}} (-t)^{\#\{\text{even fixed points of } \sigma\}} (1-t)^{\#\{\text{other cycles of } \sigma\}}. \qquad (4.2)$$

Proof. Let $\mathcal{F}_V$ be the set of ID forests on node set V. For $V = [2n]$, equation (3.1) can be restated as

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = - \sum_{F \in \mathcal{F}_{[2n]}} (-t)^{m(F)-1},$$

where $m(F)$ denotes the number of trees of the forest F. For each $F \in \mathcal{F}_{[2n]}$, let F' be the forest obtained by removing node $2n$ from F. Clearly $F' \in \mathcal{F}_{[2n-1]}$. We thus have

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = - \sum_{G \in \mathcal{F}_{[2n-1]}} \sum_{\substack{F \in \mathcal{F}_{[2n]} \\ F' = G}} (-t)^{m(F)-1}.$$

Fix $G \in \mathcal{F}_{[2n-1]}$ and let T be a tree of the forest G. For all $F \in \mathcal{F}_{[2n]}$ for which $F' = G$, the tree T is either a tree of the forest F or it is attached to $2n$ in F. Hence to obtain a forest $F \in \mathcal{F}_{[2n]}$ for which $F' = G$, we have the option of not attaching T to $2n$ or attaching T at its smallest node, which can be done if and only if T is not an even single node. Hence if T is not an even single node then it contributes a factor of $1 - t$ to the inner sum, and if T is an even single node then it contributes a factor of $-t$ to the inner sum. It follows that

$$\sum_{\substack{F \in \mathcal{F}_{[2n]} \\ F' = G}} (-t)^{m(F)-1} = (-t)^{\#\{\text{even isolated nodes of } G\}} (1-t)^{\#\{\text{other trees of } G\}}.$$

We now have

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = - \sum_{G \in \mathcal{F}_{[2n-1]}} (-t)^{\#\{\text{even isolated nodes of } G\}} (1-t)^{\#\{\text{other trees of } G\}}$$

$$= -(1-t) \sum_{G \in \mathcal{F}_{[2n-2]}} (-t)^{\#\{\text{even isolated nodes of } G\}} (1-t)^{\#\{\text{other trees of } G\}},$$

with the second equality following from the fact that every $G \in \mathcal{F}_{[2n-1]}$ consists of the isolated node $2n - 1$ and an ID forest on $[2n - 2]$. The result now follows from Corollary 4.9. $\square$

Theorem 4.15. *For all $n \geqslant 2$,*

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = (t-1)^3 \sum_{\sigma \in \mathcal{D}_{2n-4}} (1-t)^{\#\{\text{fixed points of } \sigma\}} (2-t)^{\#\{\text{other cycles of } \sigma\}}. \tag{4.3}$$

Proof. Each $F \in \mathcal{F}_{[2n]}$ can be obtained from a forest in $\mathcal{F}_{[2n]\setminus\{2\}}$ either by adding an isolated node 2 or by attaching the node 2 to 1. This implies, again by Theorem 3.5, that

$$\chi_{\Pi_{\Gamma_{[2n]}}} = (1-t)\chi_{\Pi_{\Gamma_{[2n]\setminus\{2\}}}}. \tag{4.4}$$

Similarly,

$$\chi_{\Pi_{\Gamma_{[2n]\setminus\{2\}}}} = (1-t)\chi_{\Pi_{\Gamma_{[2n]\setminus\{2,2n-1\}}}}. \tag{4.5}$$

Now let $\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}$ be the subset of $\mathcal{F}_{[2n]\setminus\{2,2n-1\}}$ consisting of forests in which the nodes 1 and $2n$ are in distinct trees. Since each forest in $\mathcal{F}_{[2n]\setminus\{2,2n-1\}}$ is either a forest in $\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}$

or is obtained from one in $\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}$ by adding an edge between the nodes 1 and $2n$, we have by Theorem 3.5 that

$$\chi_{\Pi_{\Gamma_{[2n]\setminus\{2,2n-1\}}}}(t) = -(1-t)\sum_{F\in\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}}(-t)^{m(F)-1}, \tag{4.6}$$

where $m(F)$ is the number of trees of the forest F.

For each $F \in \overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}$, let $\hat{F}$ be the forest obtained by removing the nodes 1 and $2n$ from F. Clearly $\hat{F} \in \mathcal{F}_{\{3,\dots,2n-2\}}$. It therefore follows from (4.4), (4.5), and (4.6) that

$$\chi_{\Pi_{\Gamma_{[2n]}}}(t) = (t-1)^3 \sum_{F\in\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}}(-t)^{m(F)-1} \tag{4.7}$$

$$= (t-1)^3 \sum_{G\in\mathcal{F}_{\{3,\dots,2n-2\}}}\ \sum_{\substack{F\in\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}\\ \hat{F}=G}}(-t)^{m(F)-1}.$$

Fix $G \in \mathcal{F}_{\{3,\dots,2n-2\}}$ and let T be a tree of the forest G. For all $F \in \overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}$ for which $\hat{F} = G$, the tree T is either a tree of the forest F or it is attached to either 1 or $2n$ in F. If T is a single node it can be attached to only one of the nodes in $\{1, 2n\}$ depending on its parity. Hence singleton trees of G contribute a factor of $1 - t$ to the inner sum. If T has more than one node then there are exactly two ways to attach it; with an edge between its largest node and 1, or with an edge between its smallest node and $2n$. Thus, nonsingleton trees of G contribute a factor $2 - t$ to the inner sum. It follows that

$$\sum_{\substack{F\in\overline{\mathcal{F}}_{[2n]\setminus\{2,2n-1\}}\\ \hat{F}=G}}(-t)^{m(F)-1} = (1-t)^{\#\{\text{isolated nodes of } G\}}(2-t)^{\#\{\text{other trees of } G\}}.$$

We now have

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = (t-1)^3 \sum_{G\in\mathcal{F}_{\{3,\dots,2n-2\}}}(1-t)^{\#\{\text{isolated nodes of } G\}}(2-t)^{\#\{\text{other trees of } G\}}$$

$$= (t-1)^3 \sum_{G\in\mathcal{F}_{\{1,\dots,2n-4\}}}(1-t)^{\#\{\text{isolated nodes of } G\}}(2-t)^{\#\{\text{other trees of } G\}}$$

$$= (t-1)^3 \sum_{\sigma\in\mathcal{D}_{2n-4}}(1-t)^{\#\{\text{fixed points of } \sigma\}}(2-t)^{\#\{\text{other cycles of } \sigma\}},$$

with the last equality following from Corollary 4.9. $\square$

Note that when t is set equal to 0, equation (4.2) reduces to (3.2). Indeed, the only terms to survive on right hand side of (4.2) are the ones corresponding to D-permutations with no even fixed points. But these are the Dumont permutations, which were used in Section 2.3 to define the Genocchi numbers.

In the following corollary, we obtain the decomposition of the median Genocchi numbers by setting $t = -1$ in (4.2) and using (1.5). We obtain the decomposition of the Genocchi numbers by setting $t = 0$ in (4.3) and using (3.2).

Corollary 4.16. *For all $n \geqslant 2$,*

$$h_n = \sum_{j=1}^{n-1} h_{n-1,j} 2^{j+1}$$

$$g_{n+1} = \sum_{j=0}^{n-1} g_{n-1,j} 2^j,$$

where $h_{n,j}$ is the number of D-permutations on $[2n]$ with exactly j cycles that are not even fixed points and $g_{n,j}$ is the number of D-permutations on $[2n]$ with exactly j cycles that are not fixed points.

We expect that the coefficients $g_{n-1,j}$ are equal to the coefficients in Sundaram's decomposition [Sun95, Proposition 3.5] of the Genocchi numbers into powers of 2.

4.4. Reducing the characteristic polynomial

From Theorem 4.15, we can see that $\chi_{\Pi_{\Gamma_{2n}}}(t)$ is divisible by $(t-1)^3$ when $n \geqslant 2$. In this subsection we give combinatorial and geometric interpretations of the polynomial $(t-1)^{-3}\chi_{\Pi_{\Gamma_{2n}}}(t)$.

For $n \geqslant 2$, let L_{2n} be the geometric semilattice (see [WW86]) obtained from $\Pi_{\Gamma_{[2n]\backslash\{2,2n-1\}}}$ by removing the order filter generated by the atom of $\Pi_{\Gamma_{[2n]\backslash\{2,2n-1\}}}$ whose only nonsingleton block is $\{1, 2n\}$. That is,

$$L_{2n} = \left\{\pi \in \Pi_{\Gamma_{[2n]\backslash\{2,2n-1\}}} : 1 \text{ and } 2n \text{ are in distinct blocks of } \pi\right\}.$$

Geometrically, L_{2n} is the intersection semilattice of the affine hyperplane arrangement obtained by deconing the graphic arrangement of $\Gamma_{[2n]\backslash\{2,2n-1\}}$. Indeed, for $n \geqslant 2$, let $\mathcal{P}_{2n}$ be the affine arrangement obtained by intersecting the graphical arrangement of the graph $\Gamma_{[2n]\backslash\{2,2n-1\}}$ with the affine hyperplane $x_1 = x_{2n} + 1$. It is easy to see that the intersection semilattice $\mathcal{L}(\mathcal{P}_{2n})$ of $\mathcal{P}_{2n}$ is L_{2n}.

From (4.7), we can see that for all $n \geqslant 2$,

$$\chi_{L_{2n}}(t) = (t-1)^{-3}\chi_{\Pi_{\Gamma_{2n}}}(t). \tag{4.8}$$

By dividing both sides of (4.3) by $(t-1)^3$ and then setting $t = 1$, the median Genocchi numbers appear again.

Corollary 4.17. *For all $n \geqslant 2$,*

$$\chi_{L_{2n}}(0) = g_n \tag{4.9}$$

$$\chi_{L_{2n}}(1) = h_{n-3}, \tag{4.10}$$

where $h_{-1} := 1$. Consequently, the number of relatively bounded regions of $\mathcal{P}_{2n}$ is h_{n-3}.

Proof. Equation (4.9) is an immediate consequence of (4.8) and Corollary 3.9.

For $n = 2$, equation (4.10) holds since $\chi_{L_4}(t) = 1$. Now let $n \geqslant 3$. It follows from Theorem 4.15 that $\chi_{L_{2n}}(1)$ equals the number of D-permutations on $[2n - 4]$ with no fixed points. But these are precisely the Dumont derangements on $[2n - 4]$, which by the definition of the median Genocchi numbers given in Section 2.3, is equal to h_{n-3}. Hence (4.10) holds.

The consequence follows from another well known result of Zaslavsky [Zas75], namely that the number of relatively bounded regions of any affine arrangement $\mathcal{A}$ is equal to $|\chi_{L(\mathcal{A})}(1)|$. $\square$

5. From D-permutations to surjective staircases

In this section we prove (1.9) along with a similar formula for the generating function of the characteristic polynomial of the geometric semilattice L_{2n} defined in Section 4.4. The proofs rely on the theory of surjective staircases introduced by Dumont [Dum74]. We first construct a bijection from the D-permutations on $[2n]$ to a certain class of surjective staircases known to be enumerated by h_n, thereby proving Theorem 4.2 (2) and completing our path to Hetyei's formula (1.5).

5.1. Surjective staircases

An *excedent function* is a map $f : [m] \to [m]$ such that for all $i \in [m]$, $f(i) \geqslant i$. It is convenient to visualize excedent functions by associating them with fillings of the Ferrers diagram of shape $(m, m - 1, \ldots, 1)$. We write the diagram in English notation with the row lengths decreasing from top to bottom. The rows are labeled from top to bottom with the numbers m down to 1, and the columns are labeled from left to right with the numbers 1 up to m. Let $C_{i,j}$ be the cell in the row labeled i and the column labeled j. The tableau $T(f)$ corresponding to excedent function f has an X in cell $C_{i,j}$ whenever $f(j) = i$ and blanks in the other cells. For example, the tableau in the figure below represents the excedent function $f : [5] \to [5]$ with $f(1) = 1, f(2) = 4, f(3) = 5, f(4) = 4, f(5) = 5$.

Clearly this correspondence is a bijection between the set $\mathcal{X}_m$ of excedent functions $f : [m] \to [m]$ and the set of tableaux T of shape $(m, m - 1, \ldots, 1)$ filled with X's and blanks so that each column has exactly one X. The image of f corresponds to the set of nonempty rows of $T(f)$, that is, the rows with an X. The *fixed points* of f are the $j \in [m]$ for which $f(j) = j$. Clearly the fixed points of f correspond to the rows of $T(f)$ that have an X in the rightmost cell of the row. An *isolated fixed point* of f is a fixed point j of f such that $f^{-1}(j) = \{j\}$. Clearly

the isolated fixed points correspond to the rows of $T(f)$ that have only one X, which is in the rightmost cell of the row.

It is not difficult to see that $|\mathcal{X}_m| = m!$. For our purposes, we will need a bijection from $\mathfrak{S}_m$ to $\mathcal{X}_m$ constructed in [DR94].

Proposition 5.1 (Dumont and Randrianarivony [DR94, Proposition 1.3]). *There exists a bijection $\tau : \mathfrak{S}_m \to \mathcal{X}_m$ such that for all $\sigma \in \mathfrak{S}_m$ and $j \in [m]$, the following properties hold:*

(1) *j is a cycle maximum of σ if and only if it is a fixed point of $\tau(\sigma)$,*

(2) *j is a fixed point of σ if and only if it is an isolated fixed point of $\tau(\sigma)$,*

(3) *$\sigma(j) \leqslant j$ if and only if j is in the image of $\tau(\sigma)$.*

An excedent function $f : [2n] \to [2n]$ is said to be a *surjective staircase*[4] if its image is $\{2, 4, \ldots, 2n\}$. Since the odd labeled rows in the tableau $T(f)$ corresponding to a surjective staircase f are empty, we can delete these rows from the tableau. Let $T'(f)$ denote the tableau obtained from $T(f)$ by deleting the odd labeled rows. For example, let $f : [10] \to [10]$ be defined by

$$f(1) = 2, f(2) = 8, f(3) = 4, f(4) = 4, f(5) = 10,$$
$$f(6) = 6, f(7) = 10, f(8) = 8, f(9) = 10, f(10) = 10.$$

Then $T'(f)$ is the tableau given in the figure below.

	1	2	3	4	5	6	7	8	9	10
10					X		X		X	X
8		X						X		
6						X				
4			X	X						
2	X									

Let $\mathcal{E}_{2n}$ be the set of surjective staircases in $\mathcal{X}_{2n}$. For $f \in \mathcal{E}_{2n}$, we say that j is a *maximum* of f if $j \in [2n-2]$ and $f(j) = 2n$, that is, if f achieves its maximum value at j and $j \neq 2n-1, 2n$. So $j \in [2n-2]$ is a maximum of f if, in $T'(f)$, the top row has an X in column j. For the surjective staircase shown in the figure above, the maxima are $\{5, 7\}$.

5.2. The bijection

We use the Dumont–Randrianarivony bijection (Proposition 5.1) to prove the following result.

Lemma 5.2. *There is a bijection*

$$\phi : \mathcal{D}_{2n} \to \{f \in \mathcal{E}_{2n+2} : f \text{ has no even maxima}\}$$

such that for all $\sigma \in \mathcal{D}_{2n}$ and $j \in [2n]$, the following properties hold:

---

[4]These are also known as surjective pistols in the literature.

(1) j *is an even cycle maximum of* σ *if and only if it is a fixed point of* $\phi(\sigma)$,

(2) j *is an even fixed point of* σ *if and only if it is an isolated fixed point of* $\phi(\sigma)$,

(3) j *is an odd fixed point of* σ *if and only if it is an odd maximum of* $\phi(\sigma)$.

Proof. Let $\mathcal{G}_{2n}$ be the set of all excedent functions $g \in \mathcal{X}_{2n}$ that satisfy

- the image of g contains $\{2, 4, \ldots, 2n\}$

- for all $j \in [2n]$, $g(j)$ is odd if and only if j is an odd isolated fixed point of g.

Let τ be the bijection of Proposition 5.1. We claim that $\tau(\mathcal{D}_{2n}) = \mathcal{G}_{2n}$. Indeed, let $\sigma \in \mathcal{D}_{2n}$. The image of the excedent function $\tau(\sigma)$ is $\{2, 4, \ldots, 2n\} \cup \{\text{odd fixed points of } \sigma\}$ by property (3) of Proposition 5.1, since σ is a D-permutation. Hence $\tau(\sigma)(j)$ is odd if and only if it is a fixed point of σ. By property (2) of Proposition 5.1, the odd fixed points of σ are the odd isolated fixed points of $\tau(\sigma)$.

To prove the result we construct a bijection

$$\gamma : \mathcal{G}_{2n} \to \{f \in \mathcal{E}_{2n+2} : f \text{ has no even maxima}\}.$$

Then we show that the map $\gamma \circ \tau$ restricted to $\mathcal{D}_{2n}$ has the desired properties.

Let $g \in \mathcal{G}_{2n}$. We define an excedent function $f : [2n + 2] \to [2n + 2]$ by

$$f(j) = \begin{cases} g(j), & j \in [2n] \text{ and } g(j) \text{ is even} \\ 2n + 2, & j \in [2n] \text{ and } g(j) \text{ is odd} \\ 2n + 2, & j \in \{2n+1, 2n+2\}. \end{cases}$$

It is clear that $f \in \mathcal{E}_{2n+2}$. We claim that f has no even maxima. Indeed, the maxima of f are the j for which $g(j)$ is odd. By the definition of $\mathcal{G}_{2n}$, such a j must be a fixed point of g, so all the maxima of f are odd. We can now let

$$\gamma(g) = f.$$

It is helpful to visualize the map γ in terms of tableaux. First γ adds an empty row labeled $2n + 2$ of length $2n + 2$ to the top of the tableau $T(g)$, where $g \in \mathcal{G}_{2n}$. An X is placed in both of its two rightmost cells. An X in an odd labeled row of $T(g)$ represents an isolated fixed point of g, so it is in the rightmost cell of its row. Next γ slides each such X to the top of its column. At this point, the odd labeled rows become empty. We remove these rows to get $T'(\gamma(g))$. The figure below illustrates the map γ.
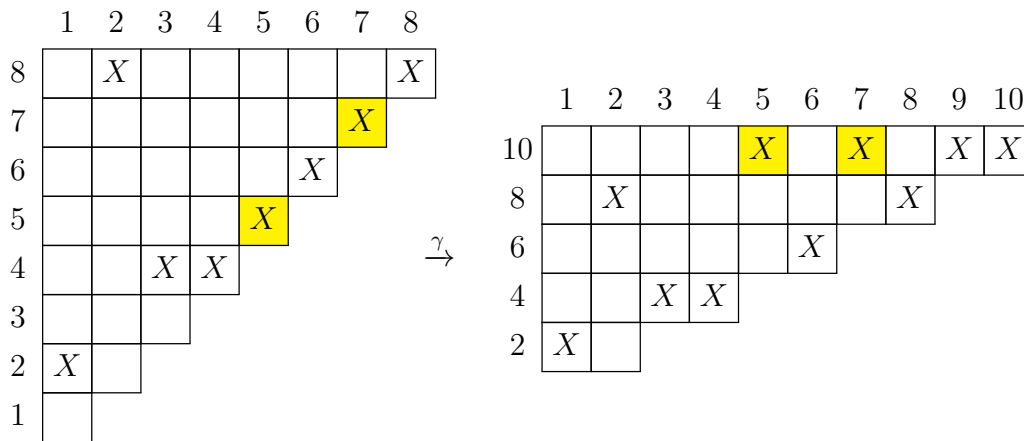
We claim γ is bijective. Indeed, given a surjective staircase $f \in \mathcal{E}_{2n+2}$ with no even maxima, we can obtain an excedent function $g : [2n] \to [2n]$ by inserting empty odd labeled rows between the even labeled rows in $T'(f)$ and then sliding the X's from the top row down to the bottom of their columns. After deleting the top row we obtain $T(g)$. Clearly $g \in \mathcal{G}_{2n}$. It is not difficult to see that the sliding down process and sliding up process are inverses of each other.

The sliding process also makes it easy to see that for all $g \in \mathcal{G}_{2n}$ and $j \in [2n]$,

(A) j is an even fixed point of g if and only if j is a fixed point of $\gamma(g)$,

(B) j is an even isolated fixed point of g if and only if j is an isolated fixed point of $\gamma(g)$,

(C) j is an odd isolated fixed point of g if and only if j is a maximum of $\gamma(g)$.

Now let ϕ be the composition $\gamma \circ \tau|_{\mathcal{D}_{2n}}$. Clearly in combination with condition (1) of Proposition 5.1, condition (A) implies condition (1) of the desired result. Similarly, in combination with condition (2) of Proposition 5.1, conditions (B) and (C) imply conditions (2) and (3), respectively, of the desired result. $\qquad\square$

We are now ready to give a proof of Theorem 4.2 (2) that does not rely on Hetyei's formula (1.5). This proof completes our path to (1.5).

Proof of Theorem 4.2, Part (2). By Lemma 5.2, we need only show that

$$|\{f \in \mathcal{E}_{2n+2} : f \text{ has no even maxima}\}| = h_n. \tag{5.1}$$

But this is implicit in the work of Dumont and Randrianarivony [DR95] and stated explicitly in Corollary 10 (iii) of Randrianarivony [Ran94]. $\qquad\square$

5.3. The generating function formulas

In this subsection, we will use the bijections of the previous sections to derive generating function formulas for the characteristic polynomial of the homogenized Linial arrangement. To do so, we use a formula (due independently to Randrianarivony [Ran94] and Zeng [Zen96]) for the

generating function of a multivariate polynomial that enumerates surjective staircases according to six statistics.

Let $f \in \mathcal{E}_{2n}$. Recall that a *fixed point* of f is a $j \in [2n-2]$ such that $f(j) = j$. A *surfixed point* of f is a $j \in [2n-2]$ such that $f(j) = j+1$. A *maximum* of f is a $j \in [2n-2]$ such that $f(j) = 2n$. Consider the following six statistics on f:

- $\mathrm{fd}(f)$, the number of fixed points j of f such that $f^{-1}(j) \setminus \{j\}$ is nonempty (*doubled fixed point*)

- $\mathrm{fi}(f)$, the number of fixed points j of f such that $f^{-1}(j) = \{j\}$ (*isolated fixed point*)

- $\mathrm{sd}(f)$, the number of surfixed points j of f such that $f^{-1}(j+1) \setminus \{j\}$ is nonempty (*doubled surfixed point*)

- $\mathrm{si}(f)$, the number of surfixed points j of f such that $f^{-1}(j+1) = \{j\}$ (*isolated surfixed point*)

- $\mathrm{mo}(f)$, the number of maxima j of f that are odd (*odd maximum*)

- $\mathrm{me}(f)$, the number of maxima j of f that are even (*even maximum*).

The *generalized Dumont–Foata polynomial* $\Lambda_{2n}(x, y, z, \bar{x}, \bar{y}, \bar{z})$ (introduced by Dumont in [Dum95]) is defined by

$$\Lambda_{2n}(x, y, z, \bar{x}, \bar{y}, \bar{z}) = \sum_{f \in \mathcal{E}_{2n}} x^{\mathrm{mo}(f)} y^{\mathrm{fd}(f)} z^{\mathrm{si}(f)} \bar{x}^{\mathrm{me}(f)} \bar{y}^{\mathrm{fi}(f)} \bar{z}^{\mathrm{sd}(f)}.$$

Theorem 5.3 (Randrianarivony [Ran94, Theorem 4] and Zeng [Zen96, Theorem 5])**.**

$$\sum_{n \geqslant 1} \Lambda_{2n} u^n = \sum_{n \geqslant 1} \frac{(x + \bar{z})^{(n-1)}(y + \bar{x})^{(n-1)} u^n}{\prod_{k=0}^{n-1}(1 - [(x+k)(\bar{y}-y) - (\bar{x}+k)(\bar{z}-z) - (x+k)(\bar{x}+k)]u)},$$

where $a^{(n)} = a(a+1) \cdots (a+n-1)$.

Lemma 5.4. *For all* $n \geqslant 1$,

$$\sum_{\sigma \in \mathcal{D}_{2n}} t^{c(\sigma)} = \Lambda_{2n+2}(t, t, 1, 0, t, 1),$$

where $c(\sigma)$ *is the number of cycles of* σ.

Proof. Let ϕ be the bijection of Lemma 5.2. By Lemma 5.2,

$$\sum_{\sigma \in \mathcal{D}_{2n}} t^{c(\sigma)} = \sum_{\sigma \in \mathcal{D}_{2n}} t^{\mathrm{fi}(\phi(\sigma)) + \mathrm{fd}(\phi(\sigma)) + \mathrm{mo}(\phi(\sigma))}$$

$$= \sum_{\substack{f \in \mathcal{E}_{2n+2} \\ \mathrm{me}(f) = 0}} t^{\mathrm{fi}(f) + \mathrm{fd}(f) + \mathrm{mo}(f)}$$

$$= \Lambda_{2n+2}(t, t, 1, 0, t, 1). \qquad \square$$

We finally have the tools needed to prove (1.9), which is restated here.

Theorem 5.5. *We have*

$$\sum_{n \geqslant 1} \chi_{\Pi_{\Gamma_{2n}}}(t)\, u^n = \sum_{n \geqslant 1} \frac{(t-1)_n (t-1)_{n-1} u^n}{\prod_{k=1}^n (1 - k(t-k)u)}, \tag{5.2}$$

where $(a)_n = a(a-1)\cdots(a-(n-1))$.

Proof. By Lemma 5.4, we have that

$$\sum_{n \geqslant 1} \sum_{\sigma \in \mathcal{D}_{2n}} t^{c(\sigma)} u^n = \sum_{n \geqslant 2} \Lambda_{2n}(t, t, 1, 0, t, 1) u^{n-1}.$$

Since $\Lambda_2(t, t, 1, 0, t, 1) = 1$, it therefore follows from Theorem 5.3 that

$$\begin{aligned}
\sum_{n \geqslant 1} \sum_{\sigma \in \mathcal{D}_{2n}} t^{c(\sigma)} u^n &= -1 + \sum_{n \geqslant 1} \frac{(t+1)^{(n-1)} t^{(n-1)} u^{n-1}}{\prod_{k=1}^{n-1}(1 + (t+k)ku)} \\
&= \sum_{n \geqslant 2} \frac{(t+1)^{(n-1)} t^{(n-1)} u^{n-1}}{\prod_{k=1}^{n-1}(1 + (t+k)ku)} \\
&= \sum_{n \geqslant 1} \frac{(t+1)^{(n)} t^{(n)} u^n}{\prod_{k=1}^{n}(1 + (t+k)ku)}.
\end{aligned}$$

Now by Theorem 4.10, we have

$$\begin{aligned}
\sum_{n \geqslant 1} \chi_{\Pi_{\Gamma_{2n}}}(t)\, u^n &= -\sum_{n \geqslant 1} \sum_{\sigma \in \mathcal{D}_{2n}} (-t)^{c(\sigma)-1} u^n \\
&= t^{-1} \sum_{n \geqslant 1} \frac{(-t+1)^{(n)}(-t)^{(n)} u^n}{\prod_{k=1}^{n}(1 + (-t+k)ku)} \\
&= \sum_{n \geqslant 1} \frac{(t-1)_n (t-1)_{n-1} u^n}{\prod_{k=1}^{n}(1 - (t-k)ku)}. \qquad \square
\end{aligned}$$

Remark 5.6. Theorem 4.14 can also be used to prove Theorem 5.5. Indeed, by Theorem 4.14 and Lemma 5.2,

$$\chi_{\Pi_{\Gamma_{2n}}}(t) = (t-1)\Lambda_{2n}(1-t, 1-t, 1, 0, -t, 1).$$

Equation (5.2) now follows from the Randrianarivony–Zeng formula given in Theorem 5.3.

It follows from (3.2) and (1.5) that equation (5.2) can be viewed as a unifying generalization of two of the four generating function formulas stated in Section 2.3. Indeed, when t is set equal to 0, equation (5.2) reduces to the Barsky–Dumont formula (2.5) for the Genocchi numbers. When t is set equal to -1, equation (5.2) reduces to the first Barsky–Dumont formula (2.6) for the median Genocchi numbers. The next result is a unifying generalization of the other two generating function formulas stated in Section 2.3.

Theorem 5.7. *We have*

$$\sum_{n \geqslant 1} \chi_{\Pi_{\Gamma_{2n+2}}}(t) \, u^n = (t-1) \sum_{n \geqslant 1} \frac{((t-1)_n)^2 \, u^n}{\prod_{k=1}^{n} (1 - k(t-k)u)}. \tag{5.3}$$

Equivalently,

$$\sum_{n \geqslant 1} \chi_{L_{2n+2}}(t) \, u^n = \sum_{n \geqslant 1} \frac{((t-2)_{n-1})^2 \, u^n}{\prod_{k=1}^{n} (1 - k(t-k)u)}, \tag{5.4}$$

where L_{2n} is the geometric semilattice defined in Section 4.4.

Proof. By Theorem 4.15 and Lemma 5.2 we have that for all $n \geqslant 1$,

$$\chi_{\Pi_{\Gamma_{2n+2}}}(t) = (t-1)^3 \Lambda_{2n}(1-t, 2-t, 1, 0, 1-t, 1).$$

The result now follows from the Randrianarivony–Zeng formula given in Theorem 5.3.

The equivalency of (5.3) and (5.4) follows from (4.8). □

By Corollary 4.17, when t is set equal to 0, equation (5.4) reduces to the Carlitz-Riordan-Stein formula (2.7) for the Genocchi numbers. When t is set equal to 1, equation (5.4) reduces to the second Barsky–Dumont formula (2.8) for the median Genocchi numbers. It is interesting that the generating function formula (5.2) for the characteristic polynomial of $\Pi_{\Gamma_{2n}}$ reduces to one set of generating function formulas given in Section 2.3, while the generating function formula (5.4) for the characteristic polynomial of L_{2n} reduces to the other set.

When we set $t = -1$ in (5.3) we obtain a formula for the generating function of the median Genocchi numbers that is not listed in Section 2.3, namely

$$\sum_{n \geqslant 0} h_n u^n = 1 + 2 \sum_{n \geqslant 1} \frac{(n!)^2 u^n}{\prod_{k=1}^{n-1}(1 + k(k+1)u)}.$$

This formula might be known, but we have not seen it in the literature.

Remark 5.8. Many of the results in this section for $V = [2n]$ are extended to general Ferrers graphs Γ_V in [Laz21, Laz20].

6. Other permutation models

In this section, we show that a formula of Chung and Graham [CG95] for chromatic polynomials of incomparablity graphs yields an interpretation of the median Genocchi numbers in terms of yet another class of Dumont like permutations. This leads us to conjecture an analogous interpretation of the Genocchi numbers.

A *drop* of a permutation σ on a finite set of positive integers is a pair $(i, \sigma(i))$ for which $i > \sigma(i)$. We say a drop $(i, \sigma(i))$ is an *even-odd drop* if i is even and $\sigma(i)$ is odd. Let $d(n, k)$ be the number of permutations in $\mathfrak{S}_n$ with exactly k drops that are *not* even-odd. For example the cycle $(1, 3, 2) \in \mathfrak{S}_3$ has two drops $(2, 1)$ and $(3, 2)$; the first drop is even-odd and the second drop is not. Hence this cycle counts towards $d(3, 1)$. One can check that $d(3, 0) = 2$, $d(3, 1) = 4$, and $d(3, 2) = 0$.

Theorem 6.1. *For all $n \geqslant 0$,*

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = \frac{1}{(2n)!} \sum_{k=0}^{2n-1} d(2n,k)(t+1)^{(k)}(t-1)_{2n-1-k}, \tag{6.1}$$

where $x^{(m)} = x(x+1)\cdots(x+m-1)$ and $x_m := x(x-1)\cdots(x-m+1)$.

Before proving this theorem we state a few consequences. By setting $t = -1$ in (6.1) and using Hetyei's result (1.5), we obtain yet another combinatorial interpretation of the median Genocchi numbers.

Corollary 6.2. *For all $n \geqslant 1$,*

$$h_n = |\{\sigma \in \mathfrak{S}_{2n} : \sigma \text{ has only even-odd drops}\}|.$$

By setting $t = 0$ in (6.1) and applying (3.2) we obtain yet another formula for the Genocchi numbers.

Corollary 6.3. *For all $n \geqslant 1$,*

$$g_n = \frac{1}{(2n)!} \sum_{k=0}^{2n-1} (-1)^k d(2n,k) k! (2n-1-k)!.$$

A much nicer combinatorial formula for the Genocchi numbers analogous to that of Corollary 6.2 is given in the following conjecture.[5]

Conjecture 6.4. For all $n \geqslant 1$, g_n is equal to the number of cycles on $[2n]$ with only even-odd drops.

We have verified this conjecture by computer for $n \leqslant 6$. The following conjecture[6] along with Theorem 4.2 implies both Corollary 6.2 and Conjecture 6.4. The first consequence follows by Theorem 4.10.

Conjecture 6.5. Let A be a finite subset of $Z_{>0}$. Then the number of cycles on A with only even-odd drops is equal to the number of D-cycles on A. Consequently, for all $n \geqslant 1$,

(1) if $\pi \in \Pi_{\Gamma_{2n}}$ then $|\mu_{\Pi_{\Gamma_{2n}}}(\hat{0}, \pi)|$ is equal to the number of permutations on $[2n]$ with only even-odd drops and cycle support π.

(2) for all j, the number of permutations on $[2n]$ with j cycles and with only even-odd drops is equal to the number of D-permutations with j cycles.

We have verified (2) by computer for $n \leqslant 6$.

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[5]Independent proofs of Conjecture 6.4 have since appeared in [PZ21] and [LY21].

[6]A proof of Conjecture 6.5 has since appeared in [LY21].

Proof of Theorem 6.1. The *incomparability graph* $\mathrm{inc}(P)$ of a finite poset P is the graph with vertex set P and edge set

$$E = \{\{x, y\} : x \text{ and } y \text{ are incomparable in } P\}.$$

Given a poset P, we say that a permutation σ of the vertices of P has a *P-drop* $(x, \sigma(x))$ if $x >_P \sigma(x)$, and write $d(P, k)$ for the number of permutations of P with exactly k P-drops.

Next we observe that Γ_{2n} is the incomparability graph of a poset. Indeed, Γ_{2n} is the incomparability graph of the poset P_{2n} on $[2n]$ with order relation given by $x \leqslant_{P_{2n}} y$ if:

- $x \leqslant y$ and x and y have the same parity

- $x < y$, x is even, and y is odd.

The Hasse diagram of P_6 and its incomparability graph Γ_6 are given below.
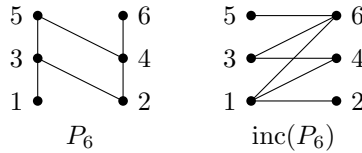
Recall from Theorem 3.2 and Section 2.2 that

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = \chi_{\Pi_{\Gamma_{2n}}}(t) = t^{-1}\mathrm{ch}_{\Gamma_{2n}}(t),$$

where $\mathrm{ch}_G(t)$ denotes the chromatic polynomial of a graph G. Chung and Graham [CG95, Corollary 7] prove that for any finite poset P, the chromatic polynomial $\mathrm{ch}_{\mathrm{inc}(P)}$ has the following expansion:

$$\mathrm{ch}_{\mathrm{inc}(P)}(t) = \sum_{k=0}^{|P|-1} d(P, k)\binom{t + k}{|P|}. \tag{6.2}$$

Hence since $\Gamma_{2n} = \mathrm{inc}(P_n)$,

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = \frac{1}{(2n)!}\sum_{k=0}^{2n-1} d(P_{2n}, k)(t + 1)^{(k)}(t - 1)_{2n-1-k}.$$

Now note that $(i, \sigma(i))$ is a P_{2n}-drop of $\sigma \in \mathfrak{S}_{2n}$ if and only if it is a drop of σ that is not even-odd. Hence $d(P_{2n}, k) = d(2n, k)$ and the result holds. $\qquad\square$

We define an *even-odd descent* of a permutation σ to be a descent $\sigma(i) > \sigma(i + 1)$ such that $\sigma(i)$ is even and $\sigma(i + 1)$ is odd. Since Foata's first fundamental transformation takes drops to descents, the word "drop" can be replaced by the word "descent" in the definition of $d(2n, k)$ without changing the validity of Theorem 6.1 and Corollary 6.3. Similarly, Corollary 6.2 remains true if we replace the word "drop" with "descent".

There is a known permutation model for the Genocchi numbers that resembles the one we have been discussing. Kitaev and Remmel [KR07, KR06] conjectured and Burstein, Josuat-Vergès, and Stromquist [BJVS10] proved that the set of permutations in $\mathfrak{S}_{2n}$ with only even-even descents has cardinality equal to g_{2n-2}, where an even-even descent is defined in the obvious way.

7. Further results: Type B and Dowling arrangements

In a forthcoming paper [LW22] (cf. [Laz20]), we will present a type B analog of our results and a Dowling arrangement generalization that unifies our results in types A and B. We define the *type B homogenized Linial arrangement* to be the hyperplane arrangement in $\mathbb{R}^{2n}$,

$$\mathcal{H}^B_{2n-1} = \{x_i \pm x_j = y_i : 1 \leqslant i < j \leqslant n\} \cup \{x_i = y_i : i = 1 \dots, n\}.$$

Our type B analog of (1.9) yields the following generating function formula for the number of regions $r(\mathcal{H}^B_{2n-1})$ of $\mathcal{H}^B_{2n-1}$:

$$\sum_{n \geqslant 1} r(\mathcal{H}^B_{2n-1}) x^n = \sum_{n \geqslant 1} \frac{(2n)! x^n}{\prod_{k=1}^n (1 + 2k(2k+1)x)}. \tag{7.1}$$

For $m \geqslant 1$, let ω_m be the primitive mth root of unity $e^{\frac{2\pi i}{m}}$. For $m, n \geqslant 1$, we define the homogenized *Linial–Dowling arrangement* to be the hyperplane arrangement in $\mathbb{C}^{2n}$ given by

$$\mathcal{H}^m_{2n-1} := \{x_i - \omega_m^l x_j = y_i : 1 \leqslant i < j \leqslant n, \, 0 \leqslant l < m\} \cup \{x_i = y_i : 1 \leqslant i \leqslant n\}.$$

By intersecting $\mathcal{H}^m_{2n-1}$ with the subspace $y_1 = y_2 = \cdots = y_n = 0$, one gets the *Dowling arrangement* $\mathcal{A}^m_n$ in $\mathbb{C}^n$. The intersection lattice of $\mathcal{A}^m_n$ is isomorphic to Π_{n+1} when $m = 1$ and to the type B partition lattice Π^B_n when $m = 2$. By introducing this Dowling analog of the homogenized Linial arrangement, we obtain unifying generalizations of the types A and B results. For instance, we obtain the m-analog of (1.9),

$$\sum_{n \geqslant 1} \chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t) \, x^n = \sum_{n \geqslant 1} \frac{(t-1)_{n,m}(t-m)_{n-1,m} \, x^n}{\prod_{k=1}^n (1 - mk(t-mk)x)},$$

where $(a)_{n,m} = a(a-m)(a-2m) \cdots (a-(n-1)m)$. Note that this reduces to (7.1) when we set $m = 2$ and $t = -1$.

We also obtain an m-analog of the formula $g_n = |\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0}, \hat{1})|$ involving a well studied polynomial analog of the Genocchi numbers known as the Gandhi polynomials [Gan70, DR95].

For a synopsis of some of our results on the homogenized Linial–Dowling arrangement, see the extended abstract [LW19, Section 4].

Acknowledgements

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