Title
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Publication Date
1996-08-28

LBNL-39275
UC-414

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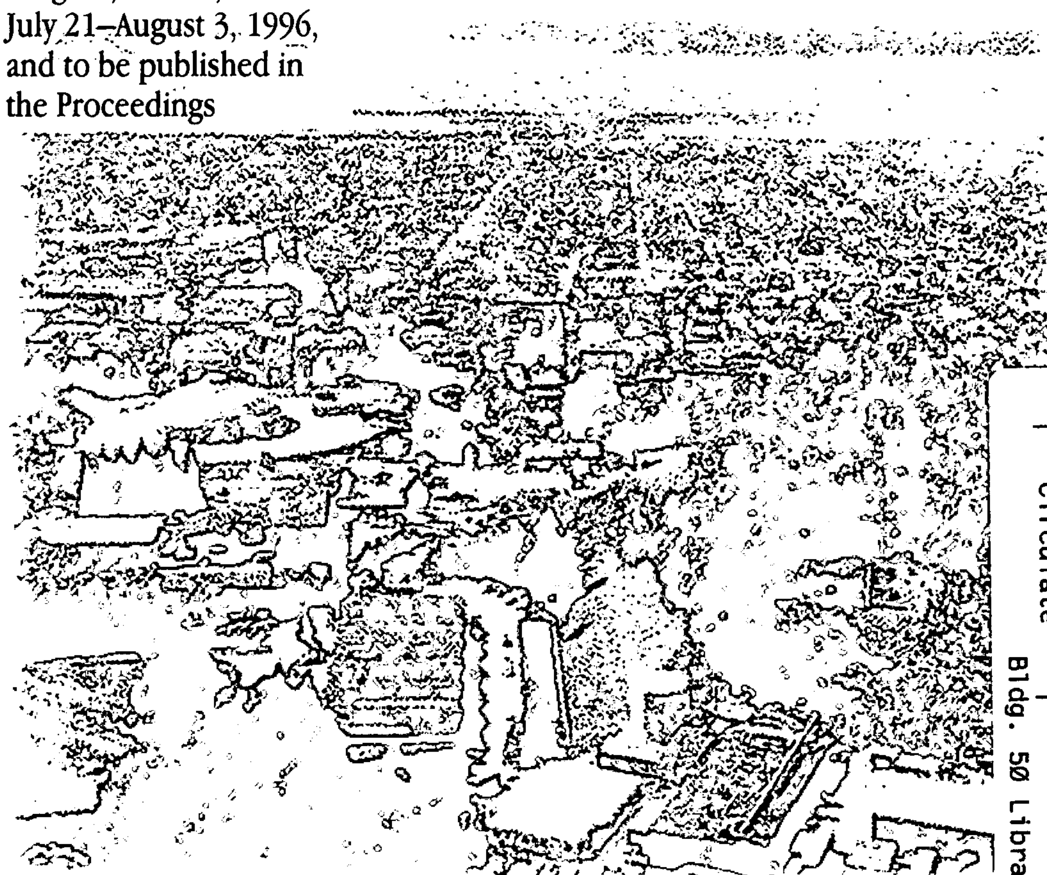
August 1996
Presented at the
Meeting on Quantum Fields and Quantum Space Time,
Cargese, France,
July 21–August 3, 1996,
and to be published in
the Proceedings



LBNL-39275
UCB-PTH-96/38

SOME COMPLEX QUANTUM MANIFOLDS AND THEIR GEOMETRY [1] [2]

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Abstract

After recalling briefly some basic properties of the quantum group $GL_q(2)$, we study the quantum sphere S_q^2, quantum projective space $CP_q(N)$ and quantum Grassmannians as examples of complex (Kähler) quantum manifolds. The differential and integral calculus on these manifolds are discussed. It is shown that many relations of classical projective geometry generalize to the quantum case. For the case of the quantum sphere a comparison is made with A. Connes' method.

Contents

[1]Lectures presented at the meeting on "Quantum Fields and Quantum Space Time", NATO Advanced Study Institute, Institut D'Etudes Scientifiques De Cargèse, July 1996.

[2]This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-9514797.

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1 INTRODUCTION

Quantum spheres can be defined in any number of dimensions by normalizing a vector of quantum Euclidean space [1]. The differential calculus on quantum Euclidean space [2] induces a calculus on the quantum sphere. The case of two-spheres in three-space is special in that there are many more possibilities. These have been studied by P. Podleś [3, 4, 5, 6] who has defined quantum spheres as quantum spaces on which quantum $SU_q(2)$ coacts. He has also developed a noncommutative differential calculus on them. In these lectures we consider, following [7], a special case of Podleś spheres which is one of those special to three space dimensions. In this case the quantum sphere S_q^2 is the analogue of the classical sphere defined as $SU(2)/U(1)$ or as isomorphic to $CP(1)$. We also define a stereographic projection and describe the coaction of $SU_q(2)$ on the sphere by fractional transformations on the complex variable in the plane analogous to the classical ones. The quantum sphere appears then as the quantum deformation of the classical two-sphere described as a complex Kähler manifold. We discuss the differential and integral calculus on S_q^2 and the action of $SU_q(2)$ vector fields on it. Finally, following [8], we show that one can define on braided copies of S_q^2 invariant anharmonic cross ratios analogous to the classical ones. All this is done in Sec.3, after recalling briefly in Sec.2 the basic properties of $GL_q(2)$ and $SU_q(2)$.

The above results are generalized in Secs.4 and 5 to quantum $CP_q(N)$ and in Sec.6 to quantum Grassmannians [9]. These quantum spaces appear as complex Kähler quantum manifolds which can be described in terms of homogeneous or inhomogeneous coordinates. Differential and integral calculus can be defined on them as well as the quantum analogues of projective invariants. For the general case of Grassmannians, we do not give explicit formulas for the integral and for the projective invariants. They should not be hard to derive by analogy with the $CP_q(N)$ case.

The type of quantization described here has the property that there exists a special quantum (connection) one-form which generates the differential calculus by taking commutators or anticommutators of it with functions or forms (see Eq.(4.70) below). This one-form is closely related to the Kähler form which can be obtained from it by differentiation. In the Poisson limit our quantization gives Poisson brackets not only between functions but also between functions and forms and between forms. The special one-form generates the calculus by taking Poisson brackets with functions or forms and the Kähler form can still be obtained from it by differentiation. Our Poisson structure on the manifold is singular and is not the standard one which is obtained by taking the Kähler form as symplectic form. Nevertheless, our Poisson structure is intimately related to the Kähler form, as just explained. For the algebra of functions on the sphere, this singular Poisson structure was already considered in [10].

All formulas and derivations of Sec.3 can be easily modified, with a few changes of signs, to describe the quantum unit disk and the coaction of quantum $SU_q(1,1)$ on it, as well as the corresponding invariant anharmonic ratios. This provides a quantum deformation of the Bolyai-Lobachevskiĭ non-Euclidean plane and of the differential and integral calculus on it. The modified equations can be guessed very easily and will not be given here. It should be mentioned that the commutation relations between z and $\bar{z}$ for the unit disk are consistent with a representation of z and $\bar{z}$ as bounded operators in a Hilbert space. This is to be contrasted with the case of the quantum sphere where z and $\bar{z}$ must be unbounded operators. The developments of Sec.4 and 6 can similarly be modified, again with some changes of signs, to describe a quantum deformation of various higher dimensional non-Euclidean geometries.

Finally, in Appendix A, we try to re-formulate the differential and integral calculus on the quantum sphere in a way as close as possible to Connes' formulation of quantum Riemannian geometry [27].

We will use the following notations throughout the paper:

$$\lambda = q - q^{-1} \tag{1.1}$$

and

$$[n]_q = \frac{q^{2n} - 1}{q^2 - 1}. \tag{1.2}$$

2 $GL_q(2)$ AS EXAMPLE OF A QUANTUM GROUP

The simplest example of a matrix quantum group [11] as a Hopf algebra [12] is $GL_q(2)$ [1]. It is a one-parameter deformation of the classical group $GL(2)$. The algebra of functions on $GL(2)$ is generated by the elements $\alpha, \beta, \gamma, \delta$ in the fundamental representation

$$\pi(g) = \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix} \tag{2.1}$$

for $g \in GL(2)$. The algebra $\mathcal{A}$ of functions on $GL_q(2)$ has the following Hopf algebra structure.

1. *Algebra:*

   The multiplication in $\mathcal{A}$ is noncommutative and the commutation relations are given compactly as

   $$\hat{R}^{ij}_{kl} T^k_m T^l_n = T^i_k T^j_l \hat{R}^{kl}_{mn} \tag{2.2}$$

   in terms of the quantum matrix

   $$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.3}$$

   and the $\hat{R}$-matrix

   $$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{2.4}$$

   where q is a complex number.

The classical limit is obtained when the deformation parameter $q \to 1$, the $\hat{R}$-matrix becomes the permutation matrix.

Explicitly the commutation relations are

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \tag{2.5}$$

$$\beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \tag{2.6}$$

$$\beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = \lambda\beta\gamma. \tag{2.7}$$

The self-consistency of the commutation relations is guaranteed by the quantum Yang-Baxter relation

$$\hat{R}^{ij}_{i'j'}\hat{R}^{j'k}_{j''n}\hat{R}^{i'j''}_{lm} = \hat{R}^{jk}_{j'k'}\hat{R}^{ij'}_{lj''}\hat{R}^{j''k'}_{mn}. \tag{2.8}$$

2. *Coproduct:*

The coproduct of a generator is defined by the matrix multiplication

$$\begin{pmatrix} \Delta(\alpha) & \Delta(\beta) \\ \Delta(\gamma) & \Delta(\delta) \end{pmatrix} = \begin{pmatrix} \alpha\otimes\alpha+\beta\otimes\gamma & \alpha\otimes\beta+\beta\otimes\delta \\ \gamma\otimes\delta+\delta\otimes\gamma & \gamma\otimes\beta+\delta\otimes\delta \end{pmatrix}. \tag{2.9}$$

This formula is the same as the classical one. The coproduct is a linear map and an algebra homomorphism. Another equivalent way to say that the coproduct is an algebra homomorphism is to say that the algebra is *covariant* under the left transformation

$$T \to T'' = TT' \tag{2.10}$$

or the right transformation

$$T \to T'' = T'T, \tag{2.11}$$

where T' is another quantum matrix satisfying (2.2) whose entries commute with the entries of T. The left- or right-covariance of the algebra means that the commutation relations among the entries of T'' are the same as those for the corresponding entries of T.

3. *Counit:*

The definition of the counit on generators also coincides with the classical case:

$$\begin{pmatrix} \epsilon(\alpha) & \epsilon(\beta) \\ \epsilon(\gamma) & \epsilon(\delta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.12}$$

In short, $\epsilon(T) = I$, where I is the identity matrix. For the counit to be an algebra homomorphism, we need I to be a quantum matrix. This is true since $I^i_j = \delta^i_j$ and the RTT relation (2.2) is trivially satisfied.

4. *Coinverse:*

The coinverse of the generators is defined so that they form the inverse matrix T^{-1}. It is

$$\begin{pmatrix} S(\alpha) & S(\beta) \\ S(\gamma) & S(\delta) \end{pmatrix} = (det_q(T))^{-1} \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = T^{-1}, \tag{2.13}$$

where $det_q(T) = \alpha\delta - q\beta\gamma$ is called the quantum determinant of T. The quantum determinant is central in $\mathcal{A}$ (it commutes with everything in $\mathcal{A}$) and is assumed not to vanish.

This concludes our brief description of the quantum group $GL_q(2)$ as a Hopf algebra.

Because the quantum determinant is central, it is consistent with the algebra to impose an additional condition

$$det_q(T) = 1. \tag{2.14}$$

What we obtain after imposing (2.14) is the deformation of $SL(2)$, naturally named $SL_q(2)$.

A further step can be taken to get $SU_q(2)$. We define the $*$-involution on $SL_q(2)$ for real q by

$$T^\dagger = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = T^{-1}. \tag{2.15}$$

The commutation relations are covariant under this $*$-involution.

The $*$-involution reverses a product: $(ff')^* = f'^* f^*$ and is complex conjugation on complex numbers. It corresponds to the Hermitian conjugation when one realizes the algebra as the algebra of operators on a Hilbert space.

Everything we mentioned in this section can be generalized to $GL_q(N)$, $SL_q(N)$ and $SU_q(N)$. These and the q-deformation for other classical groups are given in [1].

3 THE COMPLEX QUANTUM MANIFOLD S_q^2

In [3] Podleś described a family of quantum spheres. They are compact [5] quantum spaces with the quantum symmetry $SU_q(2)$. That is, the algebra X of functions on the quantum spheres is covariant under an $SU_q(2)$ transformation.

By studying the representations of the universal enveloping algebra of $SU_q(2)$ as in the classical case, one finds the quantum Clebsch-Gordan coefficients [13] which one must use to compose or decompose representations.

A classical sphere can be specified in terms of Cartesian coordinates as $x^2+y^2+z^2 = r^2$. The vector $(e_+, e_0, e_-) = (\frac{1}{\sqrt{2}}(x+iy), z, \frac{1}{\sqrt{2}}(x-iy))$ transforms as a spin-1 representation under $SU(2)$. In the deformed case we can use the quantum Clebsch-Gordan

[5] Their classical limit is compact.

coefficients to find commutation relations covariant under the linear transformation of the vector (e_+, e_0, e_-) as a $j=1$ representation of $SU_q(2)$. It is [3]

$$e_+e_- - e_-e_+ + \lambda e_0^2 = \mu e_0, \tag{3.1}$$

$$qe_0e_+ - q^{-1}e_+e_0 = \mu e_+, \tag{3.2}$$

$$qe_-e_0 - q^{-1}e_0e_- = \mu e_- \tag{3.3}$$

and

$$e_0^2 + qe_-e_+ + q^{-1}e_+e_- = s, \tag{3.4}$$

where $\mu \in \mathbf{R}$ and $s > 0$. We have in addition to q two free parameters μ and s, where s can always be scaled to a fixed number. Only μ labels inequivalent quantum spheres. With the $*$-involution $q^* = q$, $e_+^* = e_-$ and $e_0^* = e_0$, it gives a C^*-algebra.

A particularly interesting case is when this algebra is equivalent to the quotient $SU_q(2)/U(1)$ [14, 15]. The classical $U(1)$ [6] is represented as a subgroup of $SU_q(2)$ by

$$\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \tag{3.5}$$

where $U^* = U^{-1}$. It can be checked that this is an $SU_q(2)$-matrix. $SU_q(2)$ transforms under right multiplication by this matrix as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \tag{3.6}$$

which is again an $SU_q(2)$-matrix by taking U to commute with $\alpha, \beta, \gamma, \delta$.

The algebra of functions X on the quantum sphere $S_q^2 = SU_q(2)/U(1)$ is the subalgebra of the algebra of functions on $SU_q(2)$ which is invariant under this $U(1)$ transformation. It is generated by $\alpha\beta$, $\beta\gamma$ and $\gamma\delta$, which can be related to e_+, e_-, e_0 for $\mu = \lambda$ and $s = 1$ by

$$e_0 = 1 + q^{-1}[2]_q\beta\gamma, \quad e_+ = q^{-3/2}[2]_q^{1/2}\alpha\beta \tag{3.7}$$

and

$$e_- = -q^{-1/2}[2]_q^{1/2}\gamma\delta. \tag{3.8}$$

One can construct a stereographic projection of the sphere. Define

$$\begin{aligned} z &= -q^{1/2}[2]_q^{1/2}e_+(1-e_0)^{-1} = \alpha\gamma^{-1}, \\ \bar{z} &= -q^{1/2}[2]_q^{1/2}(1-e_0)^{-1}e_- = -\delta\beta^{-1}, \end{aligned} \tag{3.9}$$

which classically is the projection from the north pole of the sphere to the plane tangent to the south pole with coordinates $z, \bar{z}$. Using (3.9) and the properties of $SU_q(2)$, one obtains easily the commutation relation

$$z\bar{z} = q^{-2}\bar{z}z + q^{-2} - 1, \tag{3.10}$$

[6] Since $U(1)$ is one-dimensional there are no commutation relations to deform.

or equivalently

$$(1 + z\bar{z}) = q^{-2}(1 + \bar{z}z); \tag{3.11}$$

and the $*$-involution

$$z^* = \bar{z}. \tag{3.12}$$

An equivalent description of S_q^2 can be obtained from the $SU_q(2)$ left-covariant complex quantum plane with coordinates $\{x, \bar{x}, y, \bar{y}\}$ satisfying

$$\begin{aligned} xy &= qyx, \qquad y\bar{y} = \bar{y}y, \\ x\bar{y} &= q\bar{y}x, \quad x\bar{x} = \bar{x}x - q\lambda\bar{y}y \end{aligned} \tag{3.13}$$

and their $*$-involutions, by considering the subalgebra generated by the inhomogeneous coordinates

$$z = xy^{-1}, \quad \bar{z} = \bar{y}^{-1}\bar{x}. \tag{3.14}$$

It is easy to obtain the inverse relation of (3.9). It is

$$\begin{aligned} e_0 &= 1 - [2]_q \rho^{-1}, \\ e_+ &= -q^{-1/2}[2]_q^{1/2} z \rho^{-1}, \\ e_- &= -q^{-1/2}[2]_q^{1/2} \rho^{-1} \bar{z}, \end{aligned} \tag{3.15}$$

where $\rho = 1 + \bar{z}z$.

The $SU_q(2)$ transformation on $SU_q(2)$ induces rotations on the sphere. In terms of the coordinates $z, \bar{z}$ it is the fractional transformation:

$$z \to (az + b)(cz + d)^{-1}, \quad \bar{z} \to -(c - d\bar{z})(a - b\bar{z})^{-1}, \tag{3.16}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2)$ and a, b, c, d commute with z and $\bar{z}$. Eq.(3.10) is covariant under this fractional transformation.

We will denote the $*$-algebra generated by z and $\bar{z}$ as C^+. Classically C^+ is the algebra of functions on the plane. Notice that (3.10) for this plane differs from the usual quantum plane by an additional inhomogeneous constant term.

3.1 Differential Calculus

In Refs.[4, 5, 6], differential structures on S_q^2 are studied and classified. In this section, we give a differential calculus on the patch C^+ in terms of the complex coordinates z and $\bar{z}$. Just as the algebra of functions on C^+ can be inferred from that of $SU_q(2)$, so can the differential calculus.

For $SU_q(2)$ there are several forms of differential calculus[16, 17, 18]: the 3D left- or right-covariant differential calculus, and the $4D_+, 4D_-$ bi-covariant calculi. The 4D bi-covariant calculi have one extra dimension in their space of one-forms compared with the classical case. The right-covariant calculus will not give a projection on C^+ in a closed form in terms of z, $\bar{z}$, which are defined to transform from the left. Therefore we shall choose the left-covariant differential calculus.

It is straightforward to obtain the following relations from those for $SU_q(2)$:

$$zdz = q^{-2}dzz, \quad \bar{z}dz = q^2 dz\bar{z}, \tag{3.17}$$
$$zd\bar{z} = q^{-2}d\bar{z}z, \quad \bar{z}d\bar{z} = q^2 d\bar{z}\bar{z}, \tag{3.18}$$
$$(dz)^2 = (d\bar{z})^2 = 0 \tag{3.19}$$

and

$$dzd\bar{z} = -q^{-2}d\bar{z}dz. \tag{3.20}$$

We can also define derivatives ∂, $\bar{\partial}$ such that on functions

$$d = dz\partial + d\bar{z}\bar{\partial}. \tag{3.21}$$

From the requirement $d^2 = 0$ and the undeformed Leibniz rule for d together with Eqs.(3.17) to (3.19) it follows that:

$$\partial z = 1 + q^{-2}z\partial, \quad \partial\bar{z} = q^2\bar{z}\partial, \tag{3.22}$$
$$\bar{\partial}z = q^{-2}z\bar{\partial}, \quad \bar{\partial}\bar{z} = 1 + q^2\bar{z}\bar{\partial} \tag{3.23}$$

and

$$\partial\bar{\partial} = q^{-2}\bar{\partial}\partial. \tag{3.24}$$

It can be checked explicitly that these commutation relations are covariant under the transformation (3.16) and

$$dz \to (dz)(q^{-1}cz + d)^{-1}(cz + d)^{-1}, \tag{3.25}$$
$$\partial \to (cz + d)(q^{-1}cz + d)\partial, \tag{3.26}$$

which follow from (3.16) and the fact that the differential d is invariant. We hope that there is no confusion: (dz) is the differential of z rather than the quantum group element d times z.

The $*$-structure also follows from that of $SU_q(2)$:

$$(dz)^* = d\bar{z}, \tag{3.27}$$
$$\partial^* = -q^{-2}\bar{\partial} + (1 + q^{-2})z\rho^{-1}, \tag{3.28}$$
$$\bar{\partial}^* = -q^2\partial + (1 + q^2)\rho^{-1}\bar{z}, \tag{3.29}$$

where the $*$-involution inverts the order of factors in a product.

The inhomogeneous terms on the RHS of the Eqs.(3.28) and (3.29) reflect the fact that the sphere has curvature. Incidentally, all the commutation relations in this section admit another possible involution:

$$(dz)^* = d\bar{z}, \tag{3.30}$$
$$\partial^* = -q^2\bar{\partial}, \tag{3.31}$$
$$\bar{\partial}^* = -q^{-2}\partial. \tag{3.32}$$

This involution is not covariant under the fractional transformations and cannot be used for the sphere. However, it can be used when we have a quantum plane defined by the same algebra of functions and calculus. We shall take Eqs.(3.17) to (3.29) as the definition of the differential calculus on the patch $\mathcal{C}^+$.

It is interesting to note that there exist two different types of symmetries in the calculus. The first symmetry is that if we put a bar on all unbarred variables (z, dz, ∂), take away the bar from any barred ones and at the same time replace q by $1/q$ in any statement about the calculus, the statement is still true.

The second symmetry is the consecutive operation of the two $*$-involutions above, so that

$$\partial \to -q^2\bar{\partial}^* = q^4\partial - q^2(1+q^2)\rho^{-1}\bar{z}, \tag{3.33}$$

$$\bar{\partial} \to -q^{-2}\partial^* = q^{-4}\bar{\partial} - q^{-2}(1+q^{-2})z\rho^{-1}, \tag{3.34}$$

with $z, \bar{z}, dz, d\bar{z}$ unchanged. This replacement can be iterated n times and gives a symmetry which resembles that of a gauge transformation on a line bundle:

$$\begin{aligned} \partial \to \partial^{(n)} &\equiv q^{4n}\partial - q^2[2n]_q\rho^{-1}\bar{z} \\ &= q^{4n}\rho^{2n}\partial\rho^{-2n}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} \bar{\partial} \to \bar{\partial}^{(n)} &\equiv q^{-4n}\bar{\partial} - q^{-2}[2n]_{1/q}z\rho^{-1} \\ &= q^{-4n}\rho^{2n}\bar{\partial}\rho^{-2n}. \end{aligned} \tag{3.36}$$

For example, we have

$$\partial^{(n)}z = 1 + q^{-2}z\partial^{(n)}. \tag{3.37}$$

Making a particular choice of $\partial, \bar{\partial}$ is like fixing a gauge.

Many of the features of a calculus on a classical complex manifold are preserved. Define $\delta = dz\partial$ and $\bar{\delta} = d\bar{z}\bar{\partial}$ as the exterior derivatives on the holomorphic and anti-holomorphic functions on $\mathcal{C}^+$ respectively. We have:

$$[\delta, z] = dz, \quad [\delta, \bar{z}] = 0, \tag{3.38}$$

$$\left[\bar{\delta}, z\right] = 0, \quad \left[\bar{\delta}, \bar{z}\right] = d\bar{z}, \tag{3.39}$$

$$d = \delta + \bar{\delta}. \tag{3.40}$$

The action of δ and $\bar{\delta}$ can be extended consistently on forms as follows

$$\delta dz = dz\delta = 0, \quad \bar{\delta}d\bar{z} = d\bar{z}\bar{\delta} = 0, \tag{3.41}$$

$$\{\delta, d\bar{z}\} = 0, \quad \{\bar{\delta}, dz\} = 0, \tag{3.42}$$

$$\delta^2 = \bar{\delta}^2 = 0, \tag{3.43}$$

$$\{\delta, \bar{\delta}\} = 0, \tag{3.44}$$

where $\{\cdot,\cdot\}$, $[\cdot,\cdot]$ are the anticommutator and commutator respectively.

3.2 One-form Realization of the Exterior Differential Operator d

The calculus described in the previous section has a very interesting property. There exists a one-form Ξ having the property that

$$\Xi f \mp f\Xi = \lambda df, \tag{3.45}$$

where, as usual, the minus sign applies for functions or even forms and the plus sign for odd forms. Indeed, it is very easy to check that

$$\Xi = \xi - \xi^*, \tag{3.46}$$

where

$$\xi = qdz\rho^{-1}\bar{z}, \tag{3.47}$$

satisfies Eq.(3.45) and

$$\Xi^* = -\Xi. \tag{3.48}$$

It is also easy to check that

$$d\Xi = 2qd\bar{z}\rho^{-2}dz \tag{3.49}$$

and

$$\Xi^2 = q\lambda d\bar{z}\rho^{-2}dz. \tag{3.50}$$

Suitably normalized, $d\Xi$ is the natural area element on the quantum sphere. Notice that Ξ^2 commutes with all functions and forms, as required for consistency with the relation

$$d^2 = 0. \tag{3.51}$$

The existence of the form Ξ within the algebra of $z, \bar{z}, dz, d\bar{z}$ is especially interesting because no such form exists for the 3-D calculus on $SU_q(2)$ [16], from which we have derived the calculus on the quantum sphere (a one-form analogous to Ξ does exist for the two bicovariant calculi on $SU_q(2)$, but we have explained before why we didn't choose either of them). It is also interesting that $d\Xi$ and Ξ^2 do not vanish (as the corresponding expressions do in the bicovariant calculi on the quantum groups or in the calculus on quantum Euclidean space). The one-form Ξ is regular everywhere on the sphere, except at the point $z = \bar{z} = \infty$, which classically corresponds to the north pole.

3.3 Right Invariant Vector Fields on S_q^2

First let us recall some well-known facts about the vector fields on $SU_q(2)$ (see for example Ref.[19]). The enveloping algebra $\mathcal{U}$ of $SU_q(2)$ is usually said to be generated by the left-invariant vector fields H_L,$X_{L\pm}$ which are arranged in two matrices L^+ and L^-. The action of these vector fields corresponds to infinitesimal right transformation: $T \to TT'$. What we want now is the infinitesimal version of the left transformation given by Eq.(3.16), hence we shall use the right-invariant vector fields H_R,$X_{R\pm}$. Since only the right-invariant ones will be used, we will drop the subscript R hereafter.

The properties of the right-invariant vector fields are similar to those of the left-invariant ones. Note that if an $SU_q(2)$ matrix T is transformed from the right by another $SU_q(2)$ matrix T', then it is equivalent to say that the $SU_{1/q}(2)$ matrix T^{-1} is transformed from the left by another $SU_{1/q}(2)$ matrix T'^{-1}. Therefore one can simply write down all properties of the left-invariant vector fields and then make the replacements: $q \to 1/q$, $T \to T^{-1}$ and left-invariant fields$\to$right-invariant fields.

Consider the matrices of vector fields:

$$M^+ = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \quad M^- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_- & q^{-H/2} \end{pmatrix}. \tag{3.52}$$

The commutation relations between the vector fields are given by,

$$R_{12}M_2^+M_1^+ = M_1^+M_2^+R_{12}, \tag{3.53}$$
$$R_{12}M_2^-M_1^- = M_1^-M_2^-R_{12}, \tag{3.54}$$
$$R_{12}M_2^+M_1^- = M_1^-M_2^+R_{12}, \tag{3.55}$$

while the commutation relations between the vector fields and the elements of the quantum matrix in the smash product [19, 20, 21, 22] of $\mathcal{U}$ and $SU_q(2)$ are

$$T_1M_2^+ = M_2^+\mathcal{R}_{12}T_1, \tag{3.56}$$
$$T_1M_2^- = M_2^-\mathcal{R}_{21}^{-1}T_1, \tag{3.57}$$

where T is an $SU_q(2)$ matrix, $\mathcal{R} = q^{-1/2}R$ and R is the $GL_q(2)$ R-matrix. Clearly M^+, and M^- are the right-invariant counterparts of L^+ and L^-. The commutation relations between the M's and the T's tell us how the functions on $SU_q(2)$ are transformed by the vector fields H,X_+,X_-.

It is convenient to define a different basis for the vector fields,

$$\mathcal{Z}_+ = X_+q^{H/2}, \tag{3.58}$$
$$\mathcal{Z}_- = q^{H/2}X_- \tag{3.59}$$

and

$$\mathcal{H} = [H]_q = \tfrac{q^{2H}-1}{q^2-1}. \tag{3.60}$$

They satisfy the commutation relations

$$\mathcal{H}\mathcal{Z}_+ - q^4\mathcal{Z}_+\mathcal{H} = (1+q^2)\mathcal{Z}_+, \tag{3.61}$$
$$\mathcal{Z}_-\mathcal{H} - q^4\mathcal{H}\mathcal{Z}_- = (1+q^2)\mathcal{Z}_- \tag{3.62}$$

and

$$q\mathcal{Z}_+\mathcal{Z}_- - q^{-1}\mathcal{Z}_-\mathcal{Z}_+ = \mathcal{H}. \tag{3.63}$$

Using the expressions of $z, \bar{z}$ in terms of $\alpha, \beta, \gamma, \delta$, one can easily find the action of these vector fields on the variables $z, \bar{z}$:

$$\mathcal{Z}_+ z = q^2 z\mathcal{Z}_+ + q^{1/2}z^2, \tag{3.64}$$
$$\mathcal{Z}_+ \bar{z} = q^{-2}\bar{z}\mathcal{Z}_+ + q^{-3/2}, \tag{3.65}$$
$$\mathcal{H}z = q^4 z\mathcal{H} + (1+q^2)z, \tag{3.66}$$
$$\mathcal{H}\bar{z} = q^{-4}\bar{z}\mathcal{H} - q^{-4}(1+q^2)\bar{z}, \tag{3.67}$$
$$\mathcal{Z}_- z = q^2 z\mathcal{Z}_- - q^{1/2} \tag{3.68}$$

and

$$\mathcal{Z}_- \bar{z} = q^{-2}\bar{z}\mathcal{Z}_- - q^{-3/2}\bar{z}^2. \tag{3.69}$$

It is clear that a $*$-involution can be given:

$$\mathcal{Z}_+{}^* = \mathcal{Z}_-, \quad \mathcal{H}^* = \mathcal{H}. \tag{3.70}$$

Since all the relations listed above are closed in the vector fields and z, $\bar{z}$ (this would not be the case if we had used the left-invariant fields), we can now take these equations as the definition of the vector fields that generate the fractional transformation on S_q^2. We shall take our vector fields to commute with the exterior differentiation d. One can show that this is consistent for right-invariant vector fields in a left-covariant calculus and allows us to obtain the action of our vector fields on the differentials dz and $d\bar{z}$, as well as on the derivatives ∂ and $\bar{\partial}$. For instance (3.64) gives

$$\mathcal{Z}_+ dz = q^2 dz\mathcal{Z}_+ + q^{1/2}(dzz + zdz) \tag{3.71}$$

and

$$\partial\mathcal{Z}_+ = q^2\mathcal{Z}_+\partial + q^{-3/2}(1+q^2)z\partial. \tag{3.72}$$

It is interesting to see how Ξ and $d\Xi$ transform under the action of the right invariant vector fields or under the coaction of the fractional transformations (3.16). Using (3.64) to (3.69) one finds

$$\mathcal{Z}_+\Xi = \Xi\mathcal{Z}_+ + q^{-1/2}dz \tag{3.73}$$

and

$$\mathcal{H}\Xi = \Xi\mathcal{H}. \tag{3.74}$$

Eqs.(3.73) and (3.74) imply that $d\Xi$ commutes with $Z_\pm$ and $\mathcal{H}$, as expected for the invariant area element.

For the fractional transformation (3.16) one finds $\xi \to \xi'$ where

$$\xi' - \xi = -q(dz)cd^{-1}(1 + cd^{-1}z)^{-1} \tag{3.75}$$

and a similar formula for ξ^*. The right hand side of (3.75) is a closed one-form, since $(dz)^2 = 0$, so one could write

$$\xi' - \xi = -qd[\log_q(1 + cd^{-1}z)] \tag{3.76}$$

with a suitably defined quantum function $\log_q$. Because of Eq.(3.76), ξ can be interpreted as a connection. At any ráte

$$d\xi' = d\xi, \tag{3.77}$$

so that the area element two-form is invariant under finite transformations as well.

3.4 The Poisson Sphere

The commutation relations of the previous sections give us, in the limit $q \to 1$, a Poisson structure on the sphere. The Poisson Brackets (P.B.s) are obtained as usual as a limit

$$(f, g) = \lim_{h \to 0} \frac{fg \mp gf}{h}, \quad q^2 = e^h = 1 + h + [h^2], \tag{3.78}$$

where we use $+$ for f, g both odd and $-$ otherwise. For instance, the commutation relation (3.10) gives

$$z\bar{z} = (1 - h)\bar{z}z - h + [h^2] \tag{3.79}$$

and therefore [10] [7]

$$(\bar{z}, z) = \rho. \tag{3.80}$$

Similarly one finds

$$(dz, z) = zdz, \quad (d\bar{z}, z) = zd\bar{z}, \tag{3.81}$$

$$(dz, \bar{z}) = -\bar{z}dz, \quad (d\bar{z}, \bar{z}) = -\bar{z}d\bar{z} \tag{3.82}$$

and

$$(d\bar{z}, dz) = d\bar{z}dz. \tag{3.83}$$

In this classical limit functions and forms commute or anticommute according to their even or odd parity, as usual. The P.B. of two even quantities or of an even and an odd quantity is antisymmetric, that of two odd quantities is symmetric. It is

$$d(f, g) = (df, g) \pm (f, dg), \tag{3.84}$$

where the plus (minus) sign applies for even (odd) f. Notice that we have enlarged the concept of Poisson bracket to include differential forms. This is very natural when considering the classical limit of our commutation relations.

[7] Note that this Poisson structure is not the one usually considered on the sphere, $(z, \bar{z}) = \rho^2$, which is associated with the symplectic structure given by the Kähler (area) form on the sphere.

In the classical limit, Eq.(3.45) becomes

$$(\Xi, f) = df, \tag{3.85}$$

where

$$\Xi = \xi - \xi^* \tag{3.86}$$

and

$$\xi = dz\bar{z}\rho^{-1}, \quad \xi^* = d\bar{z}z\rho^{-1} \tag{3.87}$$

are ordinary classical differential forms. Now

$$d\Xi = 2d\bar{z}dz\rho^{-2} \tag{3.88}$$

and

$$\Xi^2 = 0. \tag{3.89}$$

Inspired by this example and by those of $CP_q(N)$ and the Grassmannians given in Secs.4 and 6, it is natural to consider the problem of constructing a Poisson structure on the algebra of differential forms so that (3.84), (3.85) and some other natural conditions are valid. This is attempted in [23] and interesting results are obtained.

As before, the variables z and $\bar{z}$ cover the sphere except for the north pole, while $w = 1/z$ and $\bar{w} = 1/\bar{z}$ miss the south pole. It is

$$(\bar{w}, w) = \bar{w}w(1 + \bar{w}w). \tag{3.90}$$

The Poisson structure is not symmetric between the north and south pole. All P.B.s of regular functions and forms vanish at the north pole $w = \bar{w} = 0$. Therefore, for Eq.(3.85) to be valid, the one-form Ξ must be singular at the north pole. Indeed one finds

$$\xi = \frac{dw\bar{w}}{1 + \bar{w}w} - \frac{dw}{w}, \quad \xi^* = \frac{d\bar{w}w}{1 + \bar{w}w} - \frac{d\bar{w}}{\bar{w}} \tag{3.91}$$

and

$$\Xi = \frac{wd\bar{w} - \bar{w}dw}{\bar{w}w(1 + \bar{w}w)}. \tag{3.92}$$

On the other hand the area two-form

$$d\Xi = 2\frac{d\bar{w}dw}{(1 + \bar{w}w)^2} \equiv \Omega \tag{3.93}$$

is regular everywhere on the sphere.

The singularity of Ξ at the north pole is not a real problem if we treat it in the sense of the theory of distributions. Consider a circle C of radius r encircling the origin of the w plane in a counter-clockwise direction and set

$$w = re^{i\theta}, \quad \bar{w} = re^{-i\theta}. \tag{3.94}$$

Using (3.91), we have

$$\int \Xi = \int \frac{\bar{w}dw - wd\bar{w}}{1+\bar{w}w} - 4\pi i. \tag{3.95}$$

As $r \to 0$ the integral in the right hand side tends to zero because the integrand is regular at the origin. Stokes' theorem can be satisfied even at the origin if we modify Eq.(3.93) to read

$$d\Xi = \Omega - 4\pi i\delta(w)\delta(\bar{w})d\bar{w}dw. \tag{3.96}$$

It is

$$\int_{S^2} \Omega = 4\pi i \tag{3.97}$$

so that

$$\int_{S^2} d\Xi = 0 \tag{3.98}$$

as it should be for a compact manifold without boundary. Notice that the additional delta function term in (3.96) also has zero P.B.s with all functions and forms as required by consistency.

3.5 Braided Quantum Spheres

We first review the general formulation [24] for obtaining the braiding of quantum spaces in terms of the universal R-matrix of the quantum group which coacts on the quantum space.

3.5.1 Braiding for Quantum Group Comodules

Let $\mathcal{A}$ be the algebra of functions on a quantum group and $\mathcal{V}$ an algebra on which $\mathcal{A}$ coacts from the left:

$$\begin{aligned} \Delta_L : \mathcal{V} &\to \mathcal{A} \otimes \mathcal{V} \\ v &\mapsto v^{(1')} \otimes v^{(2)}, \end{aligned} \tag{3.99}$$

where we have used the Sweedler-like notation for $\Delta_L(v)$.

Let $\mathcal{W}$ be another left $\mathcal{A}$-comodule algebra,

$$\begin{aligned} \Delta_L : \mathcal{W} &\to \mathcal{A} \otimes \mathcal{W} \\ w &\mapsto w^{(1')} \otimes w^{(2)}. \end{aligned} \tag{3.100}$$

It is known [24] that one can put $\mathcal{V}$ and $\mathcal{W}$ into a single left $\mathcal{A}$-comodule algebra with the multiplication between elements of $\mathcal{V}$ and $\mathcal{W}$ given by

$$vw = \mathcal{R}(w^{(1')}, v^{(1')})w^{(2)}v^{(2)}. \tag{3.101}$$

Here $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ is the universal R-matrix for the quantum enveloping algebra $\mathcal{U}$ dual to $\mathcal{A}$ (with respect to the pairing $\langle \cdot, \cdot \rangle$) and

$$\mathcal{R}(a,b) = \langle \mathcal{R}, a \otimes b \rangle. \tag{3.102}$$

It satisfies:

$$\mathcal{R}(f_{(1)}, g_{(1)}) f_{(2)} g_{(2)} = g_{(1)} f_{(1)} \mathcal{R}(f_{(2)}, g_{(2)}), \tag{3.103}$$

$$\mathcal{R}(fg, h) = \mathcal{R}(f, h_{(1)}) \mathcal{R}(g, h_{(2)}), \tag{3.104}$$

$$\mathcal{R}(f, gh) = \mathcal{R}(f_{(1)}, h) \mathcal{R}(f_{(2)}, g), \tag{3.105}$$

$$\mathcal{R}(1, f) = \mathcal{R}(f, 1) = \epsilon(f). \tag{3.106}$$

One can check that (3.101) is associative and is left-covariant [8].

$$\Delta_L(vw) = \Delta_L(v)\Delta_L(w). \tag{3.110}$$

For $\mathcal{A} = SU_q(2)$, it is

$$\mathcal{R}(T^i_j, T^k_l) = q^{-1/2} \hat{R}^{ki}_{jl}, \tag{3.111}$$

where $\hat{R}$ is the $GL_q(2)$ R-matrix.

The braiding formula (3.101) can be used for any number of ordered $\mathcal{A}$-comodules $\{\mathcal{V}_n\}_{n=1}^N$ so that it holds for $v \in \mathcal{V}_m$ and $w \in \mathcal{V}_n$ if $m < n$.

Since we know how z, z' and $\bar{z}'$ transform, we can use (3.101) to derive the braided commutation relations [8]. We will not repeat the derivation here but will only give the results

$$z\bar{z} = q^{-2}\bar{z}z - \lambda q^{-1}, \tag{3.112}$$

$$zz' = q^2 z'z - \lambda q z'^2, \tag{3.113}$$

$$z\bar{z}' = q^{-2}\bar{z}'z - \lambda q^{-1}. \tag{3.114}$$

For consistency with the $*$-involution of the braided algebra the braiding order of $z, \bar{z}, z'$ and $\bar{z}'$ has to be $z < z' < \bar{z}' < \bar{z}$ after we have fixed $z < z'$ and $z < \bar{z}$ as assumed in [7]. It is crucial that we braid separately $\mathcal{A} = \langle\{1, z\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$, and $\bar{\mathcal{A}} = \langle\{1, \bar{z}\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$ instead of simply braiding the whole algebra $\langle\{1, z, \bar{z}\}\rangle$ with $\langle\{1, z', \bar{z}'\}\rangle$. Otherwise we will not be able to have the usual properties of the $*$-involution (e.g. $(f(z)g(z'))^* = g(z')^* f(z)^*$) for the braiding relations.

An alternative derivation of the same braiding relations proceeds by first computing the braiding of two copies of the complex quantum plane on which $SU_q(2)$ coacts and then using the expressions of the stereographic variables z and $\bar{z}$ in terms of the coordinates x, y of the quantum plane

$$z = xy^{-1}, \quad \bar{z} = \bar{y}^{-1}\bar{x}. \tag{3.115}$$

[8]If on the other hand, one starts with two right $\mathcal{A}$-comodule algebras,

$$\begin{aligned} \Delta_R : \mathcal{V} &\rightarrow \mathcal{V} \otimes \mathcal{A} \\ v &\mapsto v^{(1)} \otimes v^{(2')}, \end{aligned} \tag{3.107}$$

$$\begin{aligned} \Delta_R : \mathcal{W} &\rightarrow \mathcal{W} \otimes \mathcal{A} \\ w &\mapsto w^{(1)} \otimes w^{(2')}, \end{aligned} \tag{3.108}$$

then the multiplication

$$vw = w^{(1)} v^{(1)} \mathcal{R}(v^{(2')}, w^{(2')}) \tag{3.109}$$

is associative (under the corresponding assumption), right covariant under Δ_R and makes $\mathcal{V}$ and $\mathcal{W}$ together a right $\mathcal{A}$-comodule algebra.

3.5.2 Anharmonic Ratios

Let us first review the classical case. The coordinates x, y on a plane transform as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.116}$$

by an $SL(2)$ matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (Since here we do not need the complex conjugates $\bar{x}$ and $\bar{y}$, T does not have to be an $SU(2)$ matrix.) The determinant-like object $xy' - yx'$ defined for x, y together with the coordinates of another point x', y' is invariant under the $SL(2)$ transformation. For each point we define $z = x/y$ so that

$$z - z' = y^{-1}(xy' - yx')y'^{-1}. \tag{3.117}$$

It now follows that with x_i, y_i for $i = 1, 2, 3, 4$ as coordinates of four points,

$$\begin{aligned} & (z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1} \\ = \ & (x_1 y_2 - y_1 x_2)(x_4 y_2 - y_4 x_2)^{-1}(x_4 y_3 - y_4 x_3)(x_1 y_3 - y_1 x_3)^{-1} \end{aligned} \tag{3.118}$$

is invariant because all the factors y_i^{-1} cancel and only the invariant parts $(x_i y_j - y_i x_j)$ survive. Therefore the anharmonic ratio is invariant under the $SL(2)$ transformation. (In fact it is invariant for T being a $GL(2)$ matrix.)

Permuting the indices in the above expression we may get other anharmonic ratios, but they are all functions of the one above. For example,

$$(z_2 - z_3)(z_2 - z_4)^{-1}(z_1 - z_4)(z_3 - z_1)^{-1} = (z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1} - 1. \tag{3.119}$$

The coordinates of the $SU_q(2)$ covariant quantum plane obey

$$xy = qyx, \tag{3.120}$$

an equation covariant under the transformation (3.116) with T now being an $SU_q(2)$ matrix. Braided quantum planes can be introduced by using (3.101). Let $\mathcal{V}$ be the i-th copy and $\mathcal{W}$ be the j-th one, then we have for $i < j$,

$$\begin{aligned} x_i y_j &= q y_j x_i + q\lambda x_j y_i, \\ x_i x_j &= q^2 x_j x_i, \\ y_i y_j &= q^2 y_j y_i, \\ y_i x_j &= q x_j y_i. \end{aligned} \tag{3.121}$$

In the deformed case we have to be more careful about the ordering. Let the deformed determinant-like object be

$$(ij) = x_i y_j - q y_i x_j, \tag{3.122}$$

which is invariant under the $SU_q(2)$ transformation, and let

$$[ij] = z_i - z_j = q^{-1} y_i^{-1} (ij) y_j^{-1}, \tag{3.123}$$

where $z_i = x_i y_i^{-1}$.

Using the relations

$$y_i(ij) = q(ij)y_i, \tag{3.124}$$

$$(ij)y_j = qy_j(ij) \tag{3.125}$$

for $i < j$ and

$$y_i(jk) = q^3(jk)y_i, \tag{3.126}$$

$$(ij)y_k = q^3 y_k(ij) \tag{3.127}$$

for $i < j < k$, we can see that, for example,

$$A = [12][24]^{-1}[34][13]^{-1} \tag{3.128}$$

is again invariant. Similarly, $B = [12][23]^{-1}[34][14]^{-1}$ as well as a number of others are invariant.

To find out whether these invariants are independent of one another, we now discuss the algebra of the $[ij]$'s.

Because $[ij] = [ik] + [kj]$ and $[ij] = -[ji]$ the algebra of $[ij]$ for $i, j = 1, 2, 3, 4$ is generated by only three elements $[12], [23], [34]$. It is easy to prove that

$$[ij][kl] = q^2[kl][ij] \tag{3.129}$$

if $i < j \leq k < l$.

It follows that we have

$$[ij][ik][jk] = q^4[jk][ik][ij] \tag{3.130}$$

for $i < j < k$, and

$$[12][34] + [14][23] = [12][24] + [24][23]. \tag{3.131}$$

Using these relations we can check the dependency between the different anharmonic ratios. For example, let $C = [13][23]^{-1}[24][14]^{-1}$, and $D = [14][13]^{-1}[23][24]^{-1}$, both invariant, then

$$\begin{aligned} B^{-1}AC &= 1 && (3.132)\\ q^2 B - D^{-1} &= -1. && (3.133) \end{aligned}$$

In this manner it can be checked that all products of four terms $[ij]$, $[kl]$, $[mn]^{-1}$, $[pr]^{-1}$ in arbitrary order, which are invariant, are functions of only one invariant, say, A. Namely, all invariants are related and just like in the classical case, there is only one independent anharmonic ratio. If one uses the $SU_q(2)$ covariant commutation relations (3.112) and (3.114), one can check that the anharmonic ratio commutes with all the $\bar{z}_i$'s and so commutes with its $*$-complex conjugate, which is also an invariant.

3.6 Integration

We want to determine the invariant integral $\langle f \rangle$ of a function $f(z, \bar{z})$ over the sphere.

3.6.1 Using the Definition

A left-invariant integral can be defined, up to a normalization constant, by requiring invariance under the action of the right-invariant vector fields

$$\langle \chi f(z,\bar{z}) \rangle = 0 \tag{3.134}$$

for $\chi = \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}$.

Using $\mathcal{H}$ and Eqs.(3.66) and (3.67) one finds that

$$\langle z^k \bar{z}^l g(\bar{z}z) \rangle = 0, \quad \text{unless } k = l. \tag{3.135}$$

(Here g is a convergence function such that $z^k\bar{z}^l g(\bar{z}z)$ belongs to the sphere.) Therefore we can restrict ourselves to integrals of the form $\langle f(\bar{z}z) \rangle$.

Eqs.(3.64) and (3.65) imply

$$\mathcal{Z}_+ \rho = \rho \mathcal{Z}_+ + q^{1/2} z \rho \tag{3.136}$$

and

$$\mathcal{Z}_+ \rho^{-l} = \rho^{-l} \mathcal{Z}_+ - q^{-3/2}[l]_{1/q} z \rho^{-l}. \tag{3.137}$$

From $\langle \mathcal{Z}_+(\bar{z}\rho^{-l}) \rangle = 0$, $l \geq 1$, one finds easily the recursion formula

$$[l+1]_q \langle \rho^{-l} \rangle = [l]_q \langle \rho^{-l+1} \rangle, \quad l \geq 1, \tag{3.138}$$

which gives

$$\langle \rho^{-l} \rangle = \frac{1}{[l+1]_q} \langle 1 \rangle, \quad l \geq 0. \tag{3.139}$$

Similarly

$$\langle \frac{\bar{z}z}{(1+\bar{z}z)^l} \rangle = (\frac{1}{[l]_q} - \frac{1}{[l+1]_q}) \langle 1 \rangle, \quad l \geq 1. \tag{3.140}$$

We leave it to the reader to find the expression for

$$\langle \frac{(\bar{z}z)^k}{(1+\bar{z}z)^l} \rangle, \quad l \geq k. \tag{3.141}$$

Notice that one can also compute the integral by using the "cyclic property" of the quantum integral [9]

$$\langle f(z,\bar{z}) g(z,\bar{z}) \rangle = \langle g(z,\bar{z}) f(q^{-2}z, q^2\bar{z}) \rangle, \tag{3.142}$$

which can be derived from the requirement of invariance under the action (3.134) of vector fields or from the requirement of invariance under finite fractional transformation.

[9] Similar cyclic properties have been found by H. Steinacker[25] for integrals over higher dimensional quantum spheres in quantum Euclidean space.

3.6.2 Using the Braiding

We can also compute the left-invariant integral by requiring its consistency with the braiding relations.

Since both z' and $\bar{z}'$ are always on the same side of either variable of their braided copy, z or $\bar{z}$, in the braiding order ($z < z' < \bar{z}' < \bar{z}$), the integration on $z', \bar{z}'$, has the following property:

if

$$f(z', \bar{z}')g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) f_i(z', \bar{z}'), \tag{3.143}$$

then

$$\langle f(z', \bar{z}')\rangle g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) \langle f_i(z', \bar{z}')\rangle, \tag{3.144}$$

where $\langle\cdot\rangle$ is the invariant integral on S_q^2. However,

$$f(z', \bar{z}')\langle g(z, \bar{z})\rangle \neq \sum_i \langle g_i(z, \bar{z})\rangle f_i(z', \bar{z}'). \tag{3.145}$$

The above property (3.144) can be used to derive explicit integral rules. For example, consider the case of $f(z', \bar{z}') = \bar{z}'\rho'^{-n}$, where $\rho' = 1 + \bar{z}'z'$ and $g(z, \bar{z}) = z$. Since

$$\bar{z}'\rho'^{-n}z = q^2 z\bar{z}'\rho'^{-n} + q^{1-2n}\lambda([n+1]_q - [n]_q\rho')\rho'^{-n}, \quad n \geq 0, \tag{3.146}$$

using (3.144) and $\langle \bar{z}'\rho'^{-n}\rangle = 0$ we get the recursion relation:

$$[n+1]_q \langle \rho'^{-n}\rangle = [n]_q \langle \rho'^{-(n-1)}\rangle, \quad n \geq 1. \tag{3.147}$$

This agree with the first method.

4 $CP_q(N)$ AS A COMPLEX MANIFOLD

4.1 $SU_q(N+1)$ Covariant Complex Quantum Space

For completeness, we list here the formulas we shall need to construct the complex projective space. Remember that the $SU_q(N+1)$ symmetry can be represented [26] on the complex quantum space C_q^{N+1} with coordinates $x_i, \bar{x}^i, i = 0, 1, ..., N$, which satisfy the relations

$$x_i x_j = q^{-1}\tilde{R}_{ij}^{kl} x_k x_l, \tag{4.1}$$

$$\bar{x}^i x_j = q(\tilde{R}^{-1})_{jl}^{ik} x_k \bar{x}^l \tag{4.2}$$

and

$$\bar{x}^i \bar{x}^j = q^{-1}\tilde{R}_{lk}^{ji} \bar{x}^k \bar{x}^l. \tag{4.3}$$

Here q is a real number, $\tilde{R}_{ij}^{kl}$ is the $GL_q(N+1)$ $\hat{R}$-matrix [1] with indices running from 0 to N, and $\bar{x}^i = x_i^*$ is the $*$-conjugate of x_i. The Hermitian length

$$L = x_i \bar{x}^i \tag{4.4}$$

is real and central. The $\tilde{R}$-matrix satisfies the characteristic equation

$$(\tilde{R}-q)(\tilde{R}+q^{-1})=0. \tag{4.5}$$

Derivatives $D^i, \bar{D}_i$ can be introduced (the usual symbols $\partial^a, \bar{\partial}_b$ are reserved below for the derivatives on $CP_q(N)$) which satisfy

$$D^i x_j = \delta^i_j + q\tilde{R}^{ik}_{jl} x_k D^l, \qquad D^i \bar{x}^j = q(\tilde{R}^{-1})^{ji}_{lk} \bar{x}^k D^l, \tag{4.6}$$

$$\bar{D}_i \bar{x}^j = \delta^i_j + q^{-1}(\tilde{R}^{-1})^{lj}_{ki} \bar{x}^k \bar{D}_l, \quad \bar{D}_i x_j = q^{-1}\tilde{\Phi}^{lk}_{ji} x_k \bar{D}_l \tag{4.7}$$

and

$$D^i D^j = q^{-1}\tilde{R}^{ji}_{lk} D^k D^l, \tag{4.8}$$

$$D^i \bar{D}_j = q^{-1}\tilde{\Phi}^{ki}_{lj} \bar{D}_k D^l, \tag{4.9}$$

$$\bar{D}_i \bar{D}_j = q^{-1}\tilde{R}^{kl}_{ij} \bar{D}_k \bar{D}_l. \tag{4.10}$$

Here we have defined

$$\tilde{\Phi}^{ij}_{kl} = \tilde{R}^{ji}_{lk} q^{2(i-l)} = \tilde{R}^{ji}_{lk} q^{2(k-j)}, \tag{4.11}$$

which satisfies

$$\tilde{\Phi}^{ri}_{sj}(\tilde{R}^{-1})^{jk}_{il} = (\tilde{R}^{-1})^{ri}_{sj}\tilde{\Phi}^{jk}_{il} = \delta^r_l \delta^k_s \tag{4.12}$$

and (summing over the index k)

$$\tilde{\Phi}^{ik}_{jk} = \delta^i_j q^{2i+1}, \quad \tilde{\Phi}^{ki}_{kj} = \delta^i_j q^{2(N-i)+1}. \tag{4.13}$$

Using

$$\tilde{R}^{ij}_{kl}(q^{-1}) = (\tilde{R}^{-1})^{ji}_{lk}(q) \tag{4.14}$$

and

$$\tilde{R}^{ij}_{kl} = \tilde{R}^{kl}_{ij}, \tag{4.15}$$

one can show that there is a symmetry of this algebra:

$$q \to q^{-1}, \tag{4.16}$$

$$x_i \to kq^{-2i}\bar{x}^i, \quad \bar{x}^i \to l x_i, \tag{4.17}$$

$$D^i \to k^{-1}q^{2i}\bar{D}_i, \quad \bar{D}_i \to l^{-1}D^i, \tag{4.18}$$

where k and l are arbitrary constants. Exchanging the barred and unbarred quantities in (4.16)-(4.18), we get another symmetry which is related to the inverse of this one.

Using the fact that L commutes with $x_i, \bar{x}^i$, a $*$-involution can be defined for D^i

$$(D^i)^* = -q^{-2i'} L^n \bar{D}_i L^{-n}, \tag{4.19}$$

where

$$i' = N - i + 1 \tag{4.20}$$

for any real number n. The $*$-involutions corresponding to different n's are related to one another by the symmetry of conjugation by L

$$a \longrightarrow L^m a L^{-m}, \tag{4.21}$$

where a can be any function or derivative and m is the difference in the n's.

The differentials $\xi_i = dx_i, \bar{\xi}^i = (\xi_i)^*$ satisfy

$$x_i \xi_j = q \tilde{R}^{kl}_{ij} \xi_k x_l, \tag{4.22}$$

$$\bar{x}^i \xi_j = q (\tilde{R}^{-1})^{ik}_{jl} \xi_k \bar{x}^l \tag{4.23}$$

and

$$\xi_i \xi_j = -q \tilde{R}^{kl}_{ij} \xi_k \xi_l, \tag{4.24}$$

$$\bar{\xi}^i \xi_j = -q (\tilde{R}^{-1})^{ik}_{jl} \xi_k \bar{\xi}^l. \tag{4.25}$$

All the above relations are covariant under the right $SU_q(N+1)$ transformation

$$x_i \longrightarrow x_j T^j_i, \qquad \bar{x}^i \longrightarrow (T^{-1})^i_j \bar{x}^j, \tag{4.26}$$

$$D^i \longrightarrow (T^{-1})^i_j D^j, \quad \bar{D}_i \longrightarrow \bar{D}_j q^{2i'} T^j_i q^{-2j'}, \tag{4.27}$$

$$\xi_i \longrightarrow \xi_j T^j_i, \qquad \bar{\xi}^i \longrightarrow (T^{-1})^i_j \bar{\xi}^j, \tag{4.28}$$

where $T^i_j \in SU_q(N+1)$. [10]

The holomorphic and antiholomorphic differentials $\delta, \bar{\delta}$ satisfy the undeformed Leibniz rule, $\delta^2 = \bar{\delta}^2 = 0$ and $\bar{\delta} x_j = x_j \bar{\delta}$ etc.

4.2 Algebra and Calculus on $CP_q(N)$

Define for $a = 1, ..., N$, [11]

$$z_a = x_0^{-1} x_a, \quad \bar{z}^a = \bar{x}^a (\bar{x}^0)^{-1}. \tag{4.29}$$

It follows from (4.1) and (4.2) that

$$z_a z_b = q^{-1} \hat{R}^{ce}_{ab} z_c z_e, \tag{4.30}$$

$$\bar{z}^a z_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} z_c \bar{z}^e - \lambda q^{-1} \delta^a_b, \tag{4.31}$$

where $\hat{R}^{ac}_{be}$ is the $GL_q(N)$ $\hat{R}$-matrix with indices running from 1 to N.

[10] Due to our conventions of using a right $SU_q(N+1)$ covariant quantum space here, (4.1)-(4.3) are different from the left-covariant ones (3.13) in the case of 2-dimensions. And as a consequence, the equations (4.30), (4.31) etc. below for the case of $N = 1$ are also different from what we obtained in the last section for the sphere S^2_q.

[11] The letters a, b, c, e etc. run from 1 to N, while i, j, k, l run from 0 to N.

It follows from (4.22) and (4.23) that

$$z_a dz_b = q\hat{R}^{ce}_{ab} dz_c z_e, \tag{4.32}$$

$$\bar{z}^a dz_b = q^{-1}(\hat{R}^{-1})^{ac}_{be} dz_c \bar{z}^e, \tag{4.33}$$

$$dz_a dz_b = -q\hat{R}^{ce}_{ab} dz_c dz_e \tag{4.34}$$

and

$$d\bar{z}^a dz_b = -q^{-1}(\hat{R}^{-1})^{ac}_{be} dz_c d\bar{z}^e. \tag{4.35}$$

The derivatives $\partial^a, \bar{\partial}_a$ are defined by requiring $\delta \equiv dz_a \partial^a$ and $\bar{\delta} \equiv d\bar{z}^a \bar{\partial}_a$ to be exterior differentials. It follows from (4.32) and (4.33) that

$$\partial^a z_b = \delta^a_b + q\hat{R}^{ac}_{be} z_c \partial^e, \tag{4.36}$$

$$\partial^a \bar{z}^b = q^{-1}(\hat{R}^{-1})^{ba}_{ec} \bar{z}^c \partial^e, \tag{4.37}$$

$$\bar{\partial}_a z_b = q\Phi^{ec}_{ba} z_c \bar{\partial}_e, \tag{4.38}$$

$$\bar{\partial}_a \bar{z}^b = \delta^b_a + q^{-1}(\hat{R}^{-1})^{eb}_{ca} \bar{z}^c \bar{\partial}_e, \tag{4.39}$$

$$\partial^b \partial^a = q^{-1}\hat{R}^{ab}_{ce} \partial^e \partial^c \tag{4.40}$$

and

$$\partial^a \bar{\partial}_b = q\Phi^{ca}_{eb} \bar{\partial}_c \partial^e, \tag{4.41}$$

where the Φ matrix is defined by

$$\Phi^{ca}_{db} = \hat{R}^{ac}_{bd} q^{2(c-b)} = \hat{R}^{ac}_{bd} q^{2(d-a)}. \tag{4.42}$$

Similarly as in the case of quantum spaces the algebra of the differential calculus on $CP_q(N)$ has the symmetry:

$$q \to q^{-1}, \tag{4.43}$$

$$z_a \to rq^{-2a}\bar{z}^a, \quad \bar{z}^a \to sz_a, \tag{4.44}$$

$$\partial^a \to r^{-1}q^{2a}\bar{\partial}_a, \quad \bar{\partial}_a \to s^{-1}\partial^a, \tag{4.45}$$

where $rs = q^2$. Again we also have another symmetry by exchanging the barred and unbarred quantities and $q \to 1/q$ in the above.

Also the $*$-involutions

$$z^*_a = \bar{z}^a, \tag{4.46}$$

$$dz^*_a = d\bar{z}^a \tag{4.47}$$

and

$$\partial^{a*} = -q^{2n-2a'} \rho^n \bar{\partial}_a \rho^{-n}, \tag{4.48}$$

where

$$a' = N - a + 1 \tag{4.49}$$

and

$$\rho = 1 + \sum_{a=1}^{N} z_a \bar{z}^a, \tag{4.50}$$

can be defined for any n. Corresponding to different n's they are related with one another by the symmetry of conjugation by ρ to some powers followed by a rescaling by appropriate powers of q.

In particular, the choice $n = N + 1$ gives the $*$-involution which has the correct classical limit of Hermitian conjugation with the standard measure $\rho^{-(N+1)}$ of $CP(N)$.

The transformation (4.26) induces a transformation on $CP_q(N)$

$$z_a \to (T_0^0 + z_b T_0^b)^{-1}(T_a^0 + z_c T_a^c). \tag{4.51}$$

One can then calculate how the differentials transform

$$dz_a \longrightarrow dz_b M_a^b, \quad d\bar{z}^a \longrightarrow (M^\dagger)_b^a d\bar{z}^b, \tag{4.52}$$

where M_a^b is a matrix of functions in z_a with coefficients in $SU_q(N+1)$ and $(M^\dagger)_b^a \equiv (M_a^b)^*$. Since $\delta, \bar{\delta}$ are invariant, the transformation on the derivatives follows

$$\partial^a \longrightarrow (M^{-1})_b^a \partial^b, \quad (\partial^a)^* \longrightarrow (\partial^b)^*((M^\dagger)^{-1})_a^b. \tag{4.53}$$

The covariance of the $CP_q(N)$ relations under the transformation (4.51), (4.52) and (4.53) follows directly from the covariance of C_q^{N+1}.

4.3 One-Form Realization of Exterior Differentials

Let us first recall that in Connes' non-commutative geometry [27], the calculus is quantized using the following operator representation for the differentials,

$$d\omega = F\omega - (-1)^k \omega F \tag{4.54}$$

where ω is a k-form and F is an operator such that $F^* = F$ and $F^2 = 1$. [12] In the bicovariant calculus on quantum groups [17], there exists a one-form η with the properties $\eta^* = -\eta$, $\eta^2 = 0$ and

$$df = [\eta, f]_\pm, \tag{4.55}$$

where $[a, b]_\pm = ab \pm ba$ is the graded commutator with plus sign only when both a and b are odd. It is interesting to ask when will such a realization of differentials exist? And what will be the properties of this special one-form? Instead of studying the operator aspect, we will first consider these questions in the simpler algebraic sense.

[12] The appropriate setting is a Fredholm module $(\mathcal{H}, F)$ where all these relations take place in the Hilbert space $\mathcal{H}$.

4.3.1 A Special One-Form

Let us first look at the example of the $SO_q(N)$ covariant quantum space [1, 2]. The quantum matrix T of $SO_q(N)$ satisfies in addition to

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}, \tag{4.56}$$

also the orthogonality relations [1]

$$T^t gT = g, \quad Tg^{-1}T^t = g^{-1}, \tag{4.57}$$

where the numerical quantum metric matrices $g = g_{ij}$ and $g^{-1} = g^{ij}$ can be chosen to be equal $g_{ij} = g^{ij}$. The coordinates x_i of the quantum Euclidean space satisfy the commutation relations

$$x_k x_l \hat{R}^{kl}_{ij} = qx_i x_j - \lambda\alpha L g_{ij}, \tag{4.58}$$

where $L = x_k x_l g^{kl} = x_k x^k$ and $\alpha = \frac{1}{1+q^{N-2}}$. The differentials of the coordinates $\xi_i = dx_i$ satisfy the commutation relations

$$x_i\xi_j = q\xi_k x_l \hat{R}^{kl}_{ij}. \tag{4.59}$$

It can be verified that

$$Lx_i = x_i L, \quad Ldx_i = q^2 dx_i L. \tag{4.60}$$

Hence $\eta = -q^{-1}dLL^{-1}$ satisfies

$$\lambda df = [\eta, f]_\pm. \tag{4.61}$$

Generalizing this idea, we have the following construction:

Construction 1 *Let A be an algebra generated by coordinates x_i and $(\Omega(A), d)$ be a differential calculus [13] over A. If there exists an element $a \in A$, unequal nonvanishing constants r, s such that*

$$ax_i = rx_i a, \quad adx_i = sdx_i a, \quad \forall i, \tag{4.62}$$

then

$$\lambda df = [\eta, f]_\pm \tag{4.63}$$

with

$$\eta = \frac{\lambda}{1 - s/r} daa^{-1}. \tag{4.64}$$

The normalization constant λ is introduced so that $\lambda/(1-s/r)$ is well defined as $r, s, q \to 1$.

[13] By this we mean an A-bimodule $\Omega(A)$ generated by x_i, dx_i with commutation relations specified such that $(d1) = 0$, graded Leibniz rule is satisfied and $d^2 = 0$.

It is not hard to prove that $\eta^2 = d\eta = 0$. As another example, in the $GL_q(N)$ quantum group [1, 20], the algebra is generated by the elements of the quantum matrix $T = (T^i_j)_{i,j=1,\ldots N}$ and the differentials dT^i_j. The quantum determinant $\Delta = det_q T$ satisfies

$$\Delta T^i_j = T^i_j \Delta, \quad \Delta dT^i_j = q^2 dT^i_j \Delta \tag{4.65}$$

and so in this case

$$\eta = -q^{-1} d\Delta\Delta^{-1}. \tag{4.66}$$

4.3.2 One-form Realization of the Exterior Differential for a $*$-Algebra

In the same manner as in the construction in section 1, we have the following:

Construction 2 ($*$-Algebra)
Let A be a $*$-involutive algebra with coordinates $z_i, \bar{z}_i$ and differentials $dz_i = \delta z_i, d\bar{z}_i = \bar{\delta}\bar{z}_i$ such that $\bar{z}_i = z_i^*, d\bar{z}_i = (dz_i)^*$. If there exists a real element $a \in A$ and real unequal nonvanishing constants r, s such that

$$az_i = rz_i a, \quad adz_i = sdz_i a, \quad \forall i, \tag{4.67}$$

then, as easily seen,

$$\lambda\delta f = [\eta, f]_\pm, \quad \eta = \frac{\lambda}{1 - s/r}\delta a a^{-1}, \tag{4.68}$$

$$\lambda\bar{\delta} f = [\bar{\eta}, f]_\pm, \quad \bar{\eta} = \frac{\lambda}{1 - r/s}\bar{\delta} a a^{-1} \tag{4.69}$$

and

$$\lambda df = [\Xi, f]_\pm, \quad \Xi = \eta + \bar{\eta}, \tag{4.70}$$

where $\pm$ applies for odd/even forms f.

Notice that (4.68) and (4.69), and therefore (4.67), imply that

$$ra\delta a = s\delta a a, \quad r\bar{\delta} a a = s a\bar{\delta} a. \tag{4.71}$$

It can be proved that $\eta^* = -\bar{\eta}$ and so $\Xi^* = -\Xi$. It holds that $\eta^2 = \bar{\eta}^2 = 0$. However $\Xi^2 = \eta\bar{\eta} + \bar{\eta}\eta = \lambda\delta\bar{\eta} = \lambda\bar{\delta}\eta$ will generally be nonzero. Note that

$$\lambda d\Xi = [\Xi, \Xi]_+ = 2\Xi^2. \tag{4.72}$$

Define

$$K = \delta\bar{\eta} = \bar{\delta}\eta \tag{4.73}$$

then

$$K = \frac{1}{2} d\Xi. \tag{4.74}$$

It follows that $dK = 0$ and $K^* = K$. Thus in the case $K \neq 0$, we will call it a Kähler form and K^n [14] will be non-zero and define a real volume element for an integral (invariant integral if K^n is invariant). K also has the very nice property of commuting with everything

$$Kz_a = z_a K, \quad K dz_a = dz_a K. \tag{4.75}$$

We see here an example of Connes' calculus [27] of the type $F^2 \neq 0$ rather than $F^2 = 0$.

We consider a few examples of this construction. In the case of the quantum sphere S_q^2, the element $\rho = 1 + \bar{z}z$ satisfies

$$\rho z = q^2 z \rho, \quad \rho dz = dz \rho. \tag{4.76}$$

Therefore, we obtain

$$\eta = q dz \rho^{-1} \bar{z}, \quad \bar{\eta} = -q d\bar{z} \rho^{-1} z \tag{4.77}$$

and K is just the area element

$$K = \bar{\delta}\eta = -q^3 dz d\bar{z} \rho^{-2}. \tag{4.78}$$

One can introduce the Kähler potential V defined by

$$K = \delta\bar{\delta}V. \tag{4.79}$$

It is

$$V = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{2k-1}}{[k]_q} \bar{z}^k z^k. \tag{4.80}$$

Such a one-form representation for the differential exists on both C_q^{N+1} and $CP_q(N)$. For C_q^{N+1}, we saw in the above that

$$Lx_i = x_i L, \quad L\xi_i = q^2 \xi_i L \tag{4.81}$$

and

$$\eta_0 = -q^{-1}\delta L L^{-1}, \quad \bar{\eta}_0 = q\bar{\delta} L L^{-1}. \tag{4.82}$$

In this case, K is not the Kähler form one usually assigns to C_q^{N+1}. Rather, it gives C_q^{N+1} the geometry of $CP_q(N)$ written in homogeneous coordinates.

Similar relations hold for $CP_q(N)$ in inhomogeneous coordinates. It is

$$\rho z_a = q^{-2} z_a \rho, \quad \rho dz_a = dz_a \rho \tag{4.83}$$

and therefore

$$\eta = -q^{-1}\delta\rho\rho^{-1}, \quad \bar{\eta} = q\bar{\delta}\rho\rho^{-1}. \tag{4.84}$$

[14] n = complex dimension of the algebra. We consider only deformations such that the Poincaré series of the deformed algebra and its classical counterpart match.

One can then compute

$$K = \bar{\delta}\eta = dz_a g^{a\bar{b}} d\bar{z}^b, \tag{4.85}$$

where the metric $g^{a\bar{b}}$ is

$$g^{a\bar{b}} = q^{-1}\rho^{-2}(\rho\delta_{ab} - q^2\bar{z}^a z_b) \tag{4.86}$$

with inverse $g_{\bar{b}c}$

$$g_{\bar{b}c}g^{c\bar{a}} = g^{a\bar{c}}g_{\bar{c}b} = \delta_{ab} \tag{4.87}$$

given by

$$g_{\bar{b}c} = q\rho(\delta_{bc} + \bar{z}^b z_c). \tag{4.88}$$

This metric is the quantum deformation of the standard Fubini-Study metric for $CP(N)$. It is $K = \delta\bar{\delta}V$, where the Kähler potential V is

$$V = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{2k-1}}{[k]_q}\sum_{1\leq a_1,a_2,\cdots,a_k\leq N} z_{a_k}z_{a_{k-1}}\cdots z_{a_1}\bar{z}^{a_1}\cdots\bar{z}^{a_{k-1}}\bar{z}^{a_k}. \tag{4.89}$$

Notice that under the transformation (4.51)

$$\eta \to \eta + qf^{-1}\delta f, \quad f = T_0^0 + z_b T_0^b \tag{4.90}$$

and so K is invariant. From (4.52) and (4.85), it follows that

$$g^{a\bar{b}} \to (M^{-1})^a_c g^{c\bar{d}}((M^\dagger)^{-1})^{\bar{d}}_{\bar{b}}, \tag{4.91}$$

$$g_{\bar{b}a} \to (M^\dagger)^{\bar{b}}_{\bar{d}} g_{\bar{d}c} M^c_a. \tag{4.92}$$

One can show that the following form dv_x in C_q^{N+1}

$$\begin{aligned} dv_x &\equiv \Pi_{j=0}^N(\bar{\xi}^j L^{-1/2})\Pi_{i=0}^N(L^{-1/2}\xi_i) && (4.93)\\ &= \rho^{-(N+1)}d\bar{z}^N\cdots d\bar{z}^1 dz_1\cdots dz_N\cdot\bar{\xi}^0(\bar{x}^0)^{-1}(x_0)^{-1}\xi_0 && (4.94)\end{aligned}$$

is invariant. Using this, one can prove that

$$dv_z \equiv \rho^{-(N+1)}d\bar{z}^N\cdots d\bar{z}^1 dz_1\cdots dz_N \tag{4.95}$$

is invariant also and is in fact equal to K^N (up to a numerical factor). The factor $\rho^{-(N+1)}$ justifies the choice $n = N+1$ for the involution (4.48).

4.4 Poisson Structures on $CP(N)$

The commutation relations in the previous sections give us, in the limit $q \to 1$, a Poisson structure on $CP(N)$. The Poisson Brackets (P.B.s) are obtained as the limit (this definition differs from (3.78) by a factor of two)

$$(f,g) = \lim_{h\to 0}\frac{fg\mp gf}{h}, \quad q = e^h = 1 + h + [h^2]. \tag{4.96}$$

It is straightforward to find

$$(z_a, z_b) = z_a z_b, \qquad a < b, \tag{4.97}$$

$$(z_a, \bar{z}^b) = \begin{cases} z_a \bar{z}^b, & a \neq b \\ 2(1 + \sum_{c=1}^{a} z_c \bar{z}^c), & a = b \end{cases}, \tag{4.98}$$

$$(z_a, dz_b) = \begin{cases} z_a dz_b + 2 z_b dz_a, & a < b \\ 2 z_a dz_a, & a = b \\ z_a dz_b, & a > b \end{cases}, \tag{4.99}$$

$$(\bar{z}^a, dz_b) = \begin{cases} -\bar{z}^a dz_b, & a \neq b \\ -2 \sum_{c=1}^{a} \bar{z}^c dz_c, & a = b \end{cases} \tag{4.100}$$

and those following from the $*$-involution, which satisfies

$$(f, g)^* = (g^*, f^*). \tag{4.101}$$

The P.B. of two differential forms f and g of degrees m and n respectively satisfies

$$(f, g) = (-1)^{mn+1} (g, f). \tag{4.102}$$

The exterior derivatives $\delta, \bar{\delta}, d$ act on the P.B.s distributively, for example

$$d(f, g) = (df, g) \pm (f, dg), \tag{4.103}$$

where the plus (minus) sign applies for even (odd) f. Notice that we have extended the concept of Poisson Bracket to include differential forms.

The Fubini-Study Kähler form

$$K = dz_a g^{a\bar{b}} d\bar{z}^b \tag{4.104}$$

has vanishing Poisson bracket with all functions and forms and, naturally, it is closed.

4.5 Integration

We now turn to the discussion of integration on $CP_q(N)$. We shall use the notation $\langle f(z, \bar{z}) \rangle$ for the right-invariant integral of a function $f(z, \bar{z})$ over $CP_q(N)$. It is defined, up to a normalization factor, by requiring

$$\langle \mathcal{O} f(z, \bar{z}) \rangle = 0 \tag{4.105}$$

for any left-invariant vector field $\mathcal{O}$ of $SU_q(N+1)$. We can work out the integral by looking at the explicit action of the vector fields on functions. This approach has been worked out for the case of the sphere but it gets rather complicated for the higher dimensional projective spaces. We shall follow a different and simpler approach here. First we notice that the identification

$$x_i / L^{1/2} = T_i^N, \quad \bar{x}^i / L^{1/2} = (T^{-1})^i_N, \quad i = 0, 1, ..., N, \tag{4.106}$$

where T is an $SU_q(N+1)$ matrix, reproduces (4.1)-(4.4). Thus if we define

$$\langle f(z,\bar{z})\rangle \equiv \langle f(z,\bar{z})|_{z_a=(T_0^N)^{-1}T_a^N, \bar{z}^a=(T^{-1})_N^a/(T^{-1})_N^0}\rangle_{SU_q(N+1)}, \tag{4.107}$$

where $\langle\cdot\rangle_{SU_q(N+1)}$ is the Haar measure [11] on $SU_q(N+1)$, then it follows immediately that (4.105) is satisfied. [15] Next we claim that

$$\langle (z_1)^{i_1}(\bar{z}^1)^{j_1}\cdots(z_N)^{i_N}(\bar{z}^N)^{j_N}\rangle = 0 \text{ unless } i_1 = j_1, ..., i_N = j_N. \tag{4.108}$$

This is because the integral is invariant under the finite transformation (4.51). For the particular choice $T^i_j = \delta^i_j\alpha_i$, with $|\alpha_i| = 1, \Pi_{i=0}^N\alpha_i = 1$, this gives

$$z_a \longrightarrow (\alpha_a/\alpha_0)z_a \tag{4.109}$$

and so (4.108) follows.

In [11], Woronowicz proved the following interesting property for the Haar measure

$$\langle f(T)g(T)\rangle_{SU_q(N+1)} = \langle g(T)f(DTD)\rangle_{SU_q(N+1)}, \tag{4.110}$$

where

$$(DTD)^i_j = D^i_k T^k_m D^m_j \tag{4.111}$$

and

$$D^i_j = q^{-N+2i}\delta^i_j \tag{4.112}$$

is the D-matrix for $SU_q(N+1)$. It follows from (4.110) that

$$\langle f(z,\bar{z})g(z,\bar{z})\rangle = \langle g(z,\bar{z})f(\mathcal{D}z, \mathcal{D}^{-1}\bar{z})\rangle, \tag{4.113}$$

where

$$\mathcal{D}^a_b = \delta^a_b q^{2a}, \quad a, b = 1, 2, ..., N. \tag{4.114}$$

Introducing

$$\rho_r = 1 + \textstyle\sum_{a=1}^r z_a\bar{z}^a, \tag{4.115}$$

one finds

$$\rho_r z_a = \begin{cases} z_a\rho_r & r < a \\ q^{-2}z_a\rho_r & r \geq a \end{cases}, \tag{4.116}$$

$$\rho_r\rho_s = \rho_s\rho_r \tag{4.117}$$

and

$$\bar{z}^a z_a = q^{-2}\rho_a - \rho_{a-1} \quad \text{(no sum)}. \tag{4.118}$$

Because of (4.108), it is sufficient to determine integrals of the form

$$\langle \rho_1{}^{-i_1}\cdots\rho_N{}^{-i_N}\rangle. \tag{4.119}$$

[15] A similar strategy of using the "angular" measure to define an integration has been employed by H. Steinacker [25] in constructing integration over quantum Euclidean space.

The values of the integers i_a for (4.119) to make sense will be determined later.

Consider

$$\begin{aligned}\langle \bar{z}_a \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} z_a \rangle &= \langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} z_a (q^{-2a} \bar{z}^a) \rangle \\ &= q^{-2a} \langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} (\rho_a - \rho_{a-1}) \rangle, \end{aligned} \tag{4.120}$$

where (4.113) is used. Applying (4.116)

$$\begin{aligned}\text{L.S.} &= q^{2(i_a+\cdots+i_N)} \langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \bar{z}^a z_a \rangle \\ &= q^{2I_a} \langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \bar{z}^a z_a \rangle, \end{aligned} \tag{4.121}$$

where we have denoted

$$I_a = i_a + \cdots + i_N. \tag{4.122}$$

Using (4.118) we get the recursion formula

$$\begin{aligned} &\langle \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}+1} \rho_a{}^{-i_a} \cdots \rho_N{}^{-i_N} \rangle [I_a + a]_q \\ &= \langle \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}} \rho_a{}^{-i_a+1} \cdots \rho_N{}^{-i_N} \rangle [I_a + a - 1]_q. \end{aligned} \tag{4.123}$$

It is obvious then that

$$\langle \rho_1{}^{-i_1} \cdots \rho_a{}^{-i_a} \rangle = \langle \rho_1{}^{-i_1} \cdots \rho_{a-1}{}^{-i_{a-1}-i_a} \rangle \frac{[a]_q}{[I_a + a]_q}. \tag{4.124}$$

By repeated use of the recursion formula, $\langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \rangle$ reduces finally to $\langle \rho_1{}^{-i_1-i_2\cdots-i_N} \rangle$ and

$$\langle \rho_1{}^{-I_1} \rangle = \frac{1}{[I_1 + 1]_q} \langle 1 \rangle. \tag{4.125}$$

Therefore

$$\langle \rho_1{}^{-i_1} \cdots \rho_N{}^{-i_N} \rangle = \langle 1 \rangle \Pi_{a=1}^N \frac{[a]_q}{[I_a + a]_q}. \tag{4.126}$$

For this to be positive definite, i_a should be restricted such that $I_a + a > 0$ for $a = 1, \cdots, N$.

4.6 Braided $CP_q(N)$

As described in [8] and also in section 3.5.1, it is sufficient to know the transformation property of the algebra to derive the braiding. But as demonstrated there, it is already quite complicated to obtain explicit formulas in the case of a one dimensional algebra. Therefore although we can derive the braiding for the $CP_q(N)$ using the general framework of 3.5.1, we will follow a different, easier path: first introduce the braiding for C_q^{N+1} quantum planes and then use it to derive a braiding for $CP_q(N)$ expressed in terms of inhomogeneous coordinates.

4.6.1 Braided C_q^{N+1}

Let the first copy of quantum plane be denoted by $x_i, \bar{x}^i$ and the second by $x'_i, \bar{x}'^i$ and let their commutation relations be:

$$x_i x'_j = \tau \tilde{R}^{kl}_{ij} x'_k x_l, \tag{4.127}$$

$$\bar{x}^i x'_j = \nu (\tilde{R}^{-1})^{ik}_{jl} x'_k \bar{x}^l \tag{4.128}$$

and their $*$-involutions for arbitrary numbers τ, ν. These are consistent and covariant, as one can easily check. One can choose $\tau = \nu^{-1}$ and the Hermitian length L will be central, $Lf' = f'L$, for any function f' of $x', \bar{x}'$. However, L' does not commute with $x, \bar{x}$. In the following, we don't need to assume that $\tau = \nu^{-1}$.

By assuming that the exterior derivatives of the two copies satisfy the Leibniz rule

$$\delta' f = \pm f \delta', \quad \bar{\delta}' f = \pm f \bar{\delta}', \tag{4.129}$$

$$\delta f' = \pm f' \delta, \quad \bar{\delta} f' = \pm f' \bar{\delta}, \tag{4.130}$$

where the plus (minus) signs apply for even (odd) f and f', and

$$\delta \delta' = -\delta' \delta, \quad \delta \bar{\delta}' = -\bar{\delta}' \delta, \tag{4.131}$$

$$\bar{\delta} \delta' = -\delta' \bar{\delta}, \quad \bar{\delta} \bar{\delta}' = -\bar{\delta}' \bar{\delta}, \tag{4.132}$$

one can derive the commutation relations between functions and forms. Identifying $\delta = dx_i D^i, \bar{\delta} = d\bar{x}^i \bar{D}_i$ for both copies, one can derive also the commutation relations between derivatives and functions of different copies. We will not write them down here.

4.6.2 Braided $CP_q(N)$

Using (4.127), (4.128), one can derive the braiding relations of two braided copies of $CP_q(N)$ in terms of the inhomogeneous coordinates

$$z_a z'_b = q \hat{R}^{ce}_{ab} (z'_c - q^{-1} \lambda z_c) z_e, \tag{4.133}$$

$$\bar{z}'^a z_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} z_c \bar{z}'^e - q^{-1} \lambda \delta^a_b \tag{4.134}$$

and their $*$-involutions. Notice that these are independent of the particular choice of τ and ν. Similarly, one can work out the commutation relations between functions and forms of different copies following the assumption that their exterior derivatives anticommute. We will not list them here.

5 QUANTUM PROJECTIVE GEOMETRY

We will show in this section that many concepts of projective geometry have an analogue in the deformed case. We shall study the collinearity conditions in Sec.5.1, the deformed anharmonic ratios (cross ratios) in Sec.5.2, the coplanarity conditions in Sec.5.3. In Sec.5.4 we will show that the anharmonic ratios are the building blocks of other invariants.

5.1 Collinearity Condition

Classically the collinearity conditions for m distinct points in $CP(N)$ can be given in terms of the inhomogeneous coordinates $\{z_a^A | A = 1, 2, \cdots, m; a = 1, 2, \cdots, N\}$ as

$$(z_a^A - z_a^B)(z_a^C - z_a^D)^{-1} = (z_b^A - z_b^B)(z_b^C - z_b^D)^{-1}, \tag{5.1}$$

where $A \neq B, C \neq D = 1, \cdots, m$ and $a, b = 1, \cdots, N$.

In the deformed case, the coordinates $\{z_a^A\}$ of m points must be braided for the commutation relations to be covariant, namely,

$$z_a^A z_b^B = q\hat{R}_{ab}^{ce}(z_c^B - q^{-1}\lambda z_c^A)z_e^A, \quad A \leq B, \tag{5.2}$$

as an extension of (4.133). Eq.(4.134) can also be generalized in the same way, but we shall not need it in this section. This braiding has the interesting property that the algebra of $CP_q(N)$ is *self-braided*, that is, (5.2) allows the choice $A = B$. This property makes it possible to talk about the coincidence of points. Actually, the whole differential calculus for braided $CP_q(N)$ described in Sec.4.6 has this property.

Another interesting fact about this braiding is that for a fixed index a the commutation relation is identical to that for braided S_q^2

$$z_a^A z_a^B = q^2 z_a^B z_a^A - q\lambda z_a^A z_a^A, \quad A \leq B. \tag{5.3}$$

Since there is no algebraic way to say that two "points" are distinct in the deformed case, the collinearity conditions should avoid using expressions like $(z_a^A - z_a^B)^{-1}$, which are ill defined. Denote

$$[AB]_a = z_a^A - z_a^B. \tag{5.4}$$

The collinearity conditions in the deformed case can be formulated as

$$[AB]_a[CD]_b = q^2[CD]_a[AB]_b, \quad \forall a, b \tag{5.5}$$

and $A < B \leq C < D$. By (5.2) this equation is formally equivalent to the quantum counterpart of (5.1):

$$[AB]_a[CD]_a^{-1} = [AB]_b[CD]_b^{-1}, \tag{5.6}$$

where the ordering of A, B, C, D is arbitrary. The advantage of this formulation is that (5.5) is a quadratic polynomial condition and polynomials are well defined in the braided algebra.

Therefore the algebra Q of functions of m collinear points is the quotient of the algebra $\mathcal{A}$ of m braided copies of $CP_q(N)$ over the ideal $I = \{f\alpha g : \forall f, g \in \mathcal{A}; \forall\alpha \in CC\}$ generated by α which stands for the collinearity conditions (5.5), i.e., $\alpha \in CC = \{[AB]_a[CD]_b - q^2[CD]_a[AB]_b : A < B \leq C < D\}$.

Two requirements have to be checked for this definition $Q = \mathcal{A}/I$ to make sense. The first one is that for any $f \in \mathcal{A}$ and $\alpha \in CC$,

$$f\alpha = \sum_i \alpha_i f_i, \quad \forall f \in \mathcal{A}, \tag{5.7}$$

for some $f_i \in \mathcal{A}$ and $\alpha_i \in CC$. This condition ensures that the ideal I generated by the collinearity conditions is not "larger" than what we want, as compared with the classical case.

The second requirement is the invariance of I under the fractional transformation (4.51). It can be checked that both requirements are satisfied.

5.2 Anharmonic Ratios

Classically the anharmonic ratio of four collinear points is an invariant of the projective mappings, which are the linear transformations of the homogeneous coordinates. In the deformed case, the homogeneous coordinates are the coordinates x_i of the $GL_q(N+1)$-covariant quantum space, and the linear transformations are the $GL_q(N+1)$ transformations which induce the fractional transformations (4.51) on the coordinates z_a of the projective space $CP_q(N)$.

We consider the following anharmonic ratio of $CP_q(N)$ for four collinear points $\{z_a^A | A = 1,2,3,4\}$

$$[A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1}, \tag{5.8}$$

where $A, B = 2,3$. We wish to show that it is invariant. After some calculations and denoting $\tau(A) = [1A]_a[14]_a^{-1}$, which is independent of the index a according to the collinearity condition, we get

$$[AB]_a \to U(B)^{-1}(\tau(A) - \tau(B))P_a(A)V(A)^{-1}, \tag{5.9}$$

where

$$\begin{aligned} U(B) &= T_0^0 + z_e^B T_0^e, & (5.10)\\ V(A) &= T_0^0 + q z_f^A T_0^f, & (5.11)\\ P_a(A) &= -[14]_b M_a^b(A) & (5.12)\end{aligned}$$

and

$$M_a^b(A) = (T_a^b T_0^0 - q^{-1} T_0^b T_a^0) + q z_c^A (T_a^b T_0^c - q^{-1} T_0^b T_a^c). \tag{5.13}$$

Then the anharmonic ratio (5.8) transforms as

$$\begin{aligned} [A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1} &\to U(1)^{-1}\tau(A)(1-\tau(A))^{-1}(1-\tau(B))\tau(B)^{-1}U(1) \\ &= \tau(A)(1-\tau(A))^{-1}(1-\tau(B))\tau(B)^{-1} \\ &= [A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1}, \end{aligned} \tag{5.14}$$

where we have used $z_a^1 \tau(A) = \tau(A) z_a^1$ for any $A \geq 1$, which is true because we can represent $\tau(A)$ as $[1A]_a[14]_a^{-1}$ with the same index a and then use $z_a^1[AB]_a = q^2[AB]_a z_a^1$.

Because of the nice property (5.3), we can use the results about the anharmonic ratios of S_q^2 (which is a special case of $CP_q(N)$ with $N = 1$ but no collinearity condition is needed there) in Sec.3.5.2. Note that all the invariants as functions of z_a^A for a fixed

a in $CP_q(N)$ are also invariants as functions of $z^A = z_a^A$ in S_q^2. The reason is the following. Consider the matrix T_b^a defined by

$$T_0^0 = \alpha, \quad T_a^0 = \beta, \tag{5.15}$$
$$T_0^a = \gamma, \quad T_a^a = \delta, \tag{5.16}$$

where $\alpha, \beta, \gamma, \delta$ are components of an $SU_q(2)$-matrix, $T_b^b = 1$ for all $b \neq 0, a$ and all other components vanishing. It is a $GL_q(N+1)$-matrix, but the transformation (4.51) of z_a^A by this matrix is the fractional transformation (3.16) on S_q^2 with coordinate $z^A = z_a^A$.

Therefore, by simply dropping the subscript a, the anharmonic ratio (5.8) becomes an anharmonic ratio of S_q^2. On the other hand, since all other anharmonic ratios of S_q^2 are functions of only one of them, their corresponding anharmonic ratios of $CP_q(N)$ (by putting in the subscript a) would be functions of (5.8) and hence are invariant. Therefore we have established the fact that all invariant anharmonic ratios of $CP_q(N)$ are functions of only one of them.

5.3 Coplanarity Condition

In this subsection we will get the coplanarity condition as a generalization of the collinearity condition (5.5).

For $r+1$ points spanning an r-dimensional hyperplane, we have

$$z^B = \sum_{A \in I} \sigma_A^B z^A, \tag{5.17}$$

where $\sigma_A^B = (x_0^B)^{-1} \nu_A^B x_0^A$ and $\sum_{A \in I} \sigma_A^B = 1$. By a change of variables for σ_A^B, and letting $I = \{1, 2, \cdots, r, r+1\}$, $B = 0$, it is

$$[01]_i = \sum_{j=1}^{r} \tau_j [j(j+1)]_i, \tag{5.18}$$

where $[AB]_i = z_i^A - z_i^B$ and the τ's are independent linear combinations of the σ's.

Choose a set K of r different integers from $1, 2, \cdots, N$. Consider the r equations (5.18) for $i \in K$. Let $K = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, $M_i^j = [j(j+1)]_{\alpha_i}$ and $M_i^0 = [01]_{\alpha_i}$. Then

$$\tau_j = M_i^0 (M^{-1})_j^i, \quad j = 1, 2, \cdots, r, \tag{5.19}$$

where M^{-1} is the inverse matrix of $(M_j^i)_{i,j=1}^r$.

Even though M is not a $GL_q(r)$-matrix we define

$$det_q(M) = \epsilon_{i_1 \cdots i_r} M_{i_1}^1 \cdots M_{i_r}^r, \tag{5.20}$$

where

$$\epsilon_{\sigma_1 \sigma_2 \cdots \sigma_r} = (-q)^{l(\sigma)} \tag{5.21}$$

for σ being a permutation of r objects with length $l(\sigma)$ and the ϵ tensor is 0 otherwise. M^{-1} is then found to be

$$(M^{-1})^i_j = (-1)^{j-1}\epsilon_{ii_2\cdots i_r} M^1_{i_2}\cdots M^{j-1}_{i_j} M^{j+1}_{i_{j+1}}\cdots M^r_{i_r}(det_q(M))^{-1}. \tag{5.22}$$

Hence by (5.19)

$$(-1)^{j-1}\tau_j = det_q(M(j))(det_q(M(0)))^{-1}, \tag{5.23}$$

where

$$det_q(M(j)) = \epsilon_{i_1\cdots i_r} M^0_{i_1} M^1_{i_2}\cdots M^{j-1}_{i_j} M^{j+1}_{i_{j+1}}\cdots M^r_{i_r} \tag{5.24}$$

(so that $det_q(M(0)) = det_q(M)$).

Since this solution of τ is independent of the choice of K, by choosing another set K' we have another matrix M' and $(-1)^{j-1}\tau_j = det_q(M'(j))(det_q(M'(0)))^{-1}$. Therefore we get the coplanarity condition

$$det_q(M(j))(det_q(M(0)))^{-1} = det_q(M'(j))(det_q(M'(0)))^{-1} \tag{5.25}$$

for all $j = 1, \cdots, r$ and any two sets of indices K and K'. This is obviously equivalent to

$$det_q(M(j))(det_q(M(k)))^{-1} = det_q(M'(j))(det_q(M'(k)))^{-1} \tag{5.26}$$

for all $j, k = 0, \cdots, r$.

If $N \geq 2r$ then one can choose $K < K'$, i.e., any element in K is smaller than any element in K', then one can show that

$$det_q(M(0))det_q(M'(0)) = q^r det_q(M'(0))det_q(M(0)) \tag{5.27}$$

and a polynomial type of coplanarity condition is available:

$$det_q(M(j))det_q(M'(k)) = q^r det_q(M'(j))det_q(M(k)). \tag{5.28}$$

The algebra of functions of $r+1$ coplanar points is then the quotient of the algebra generated by $\{z^A\}^r_{A=0}$ over the ideal generated by (5.28).

5.4 Other Invariants

The anharmonic ratios are important because they are the building blocks of invariants in classical projective geometry. For example, in the N-dimensional classical case for given $2(N+1)$ points with homogeneous coordinates $\{x^A_i\}$, inhomogeneous coordinates $\{z^A_a\}$ where $A = 1, \cdots, 2(N+1)$, $i = 0, 1, \cdots, N$ and $a = 1, \cdots, N$, we can construct an invariant

$$I = \frac{det(x^1, x^2, \cdots, x^N, x^{N+1})det(x^{N+2}, x^{N+3}, \cdots, x^{2(N+1)})}{det(x^1, x^2, \cdots, x^N, x^{N+2})det(x^{N+1}, x^{N+3}, \cdots, x^{2(N+1)})}, \tag{5.29}$$

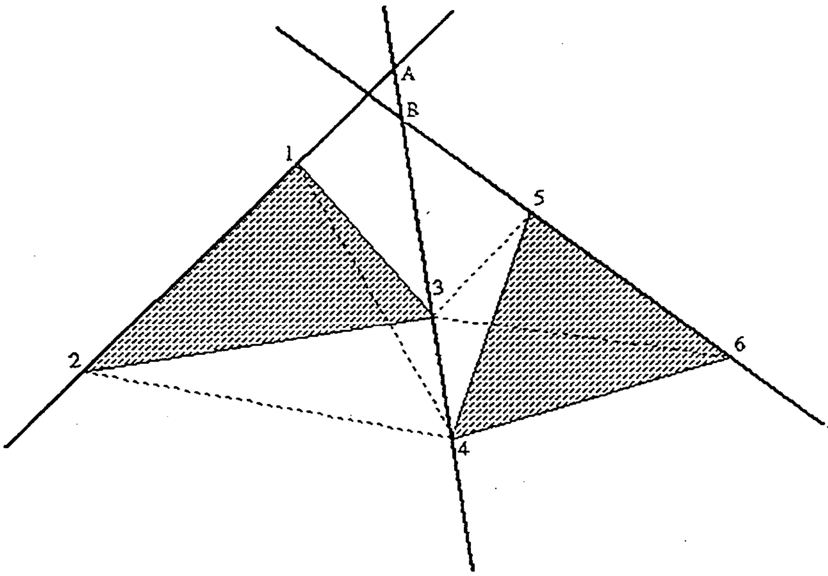

Figure 1 The invariant I as a cross ratio of $A, B, 3, 4$.

where $det(x^{A_0}, \cdots, x^{A_N})$ is the determinant of the matrix $M_j^i = x_j^{A_i}$, $i, j = 0, \cdots, N$, which equals the determinant of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_1^{A_0} & \cdots & z_1^{A_N} \\ \vdots & \ddots & \vdots \\ z_N^{A_0} & \cdots & z_N^{A_N} \end{pmatrix} \tag{5.30}$$

multiplied by the factor $x_0^{A_0} \cdots x_0^{A_N}$, which cancels between the numerator and denominator of I. It can be shown that this invariant I is in fact the anharmonic ratio of four points z, z', z^{N+1}, z^{N+2}, where z (z') is the intersection of the line fixed by z^{N+1}, z^{N+2} with the $(N-1)$-dimensional subspace fixed by $z^1, \cdots, z^N$ ($z^{N+3}, \cdots, z^{2(N+1)}$).

For the case of $N = 2$ (see Fig.4.1), I is the ratio of the areas of four triangles:

$$I = \frac{\triangle_{123} \triangle_{456}}{\triangle_{124} \triangle_{356}}, \tag{5.31}$$

which is easily found to be

$$I = \frac{\overline{A3}\,\overline{B4}}{\overline{A4}\,\overline{B3}}, \tag{5.32}$$

the anharmonic ratio of the four points $A, B, 3, 4$.

It is remarkable that all this can also be done in the quantum case. One can construct an invariant I_q using the quantum determinant and describe the intersection between subspaces of arbitrary dimension spanned by given points. It is shown that the invariant I_q is indeed an anharmonic ratio in the same sense as the classical case.

6 QUANTUM GRASSMANNIANS $G_q^{M,N}$

In this section, we study the quantum deformation of the Grassmannians.

6.1 The Algebra

Let C_a^i, $i = 1, 2, \cdots, M$, $a = 1, 2, \cdots, M+N$, be an $M \times (M+N)$ rectangular matrix satisfying the commutation relations

$$\hat{R}'^{ij}_{kl} C^k_c C^l_d = C^i_a C^j_b \hat{R}^{ab}_{cd}, \tag{6.1}$$

where $\hat{R}'^{ij}_{kl}$ is a $GL_q(M)$ $\hat{R}$-matrix, with indices i, j, k, l etc. going from 1 to M and $\hat{R}^{ab}_{cd}$ is a $GL_q(M+N)$ $\hat{R}$-matrix, with indices a, b, c, d etc. going from 1 to $M+N$. In compact notation, it is

$$\hat{R}'_{12} C_1 C_2 = C_1 C_2 \hat{R}_{12} \tag{6.2}$$

and (6.2) is right-covariant under the transformation

$$C \to CT, \tag{6.3}$$

where T^a_b is a $GL_q(M+N)$ quantum matrix and is also left-covariant under the transformation

$$C \to SC, \tag{6.4}$$

where S^i_j is a $GL_q(M)$ quantum matrix. Writing

$$C^i_a = (A^i_j, B^i_\alpha) \tag{6.5}$$

with $\alpha = 1, 2, \cdots, N$, we have

$$\begin{aligned} \hat{R}'_{12} A_1 A_2 &= A_1 A_2 \hat{R}'_{12}, \\ \hat{R}'_{12} B_1 B_2 &= B_1 B_2 \hat{R}''_{12}, \\ A_1 B_2 &= R'_{21} B_2 A_1, \end{aligned} \tag{6.6}$$

where $\hat{R}''^{\alpha\beta}_{\gamma\delta}$ is a $GL_q(N)$ $\hat{R}$-matrix, with indices $\alpha, \beta, \gamma, \delta$ etc. going from 1 to N.

Define the coordinates Z^i_α for the quantum Grassmannians $G_q^{M,N}$

$$Z = A^{-1} B. \tag{6.7}$$

Z is invariant under the transformation (6.4), while under (6.3), it transforms as

$$Z \to (\alpha + Z\gamma)^{-1}(\beta + Z\delta), \tag{6.8}$$

where $\alpha, \beta, \gamma, \delta$ are the sub-matrices of T

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{6.9}$$

It follows from (6.6) that Z satisfies

$$\hat{R}'_{21} Z_1 Z_2 = Z_1 Z_2 \hat{R}''_{12}. \tag{6.10}$$

6.1.1 $*-$structure

We consider q to be a real number. One can introduce the $*-$conjugate variables $(C_a^i)^*$ and impose the commutation relation

$$C_1^\dagger \hat{R}_{12}'^{-1} C_1 = C_2 \hat{R}_{12}^{-1} C_2^\dagger, \tag{6.11}$$

i.e.

$$\begin{aligned}(A^{-1})_2^\dagger \hat{R}_{12}'(A^{-1})_2 &= (A^{-1})_1 \hat{R}_{12}'(A^{-1})_1^\dagger,\\ B_1^\dagger A_2^{-1} &= A_2^{-1} B_1^\dagger R_{12}'^{-1},\\ B_1^\dagger \hat{R}_{12}'^{-1} B_1 &= B_2 \hat{R}_{12}''^{-1} B_2^\dagger - \lambda I_1 (AA^\dagger)_2,\end{aligned} \tag{6.12}$$

where $(I_1)_\beta^\alpha = \delta_\beta^\alpha$ is the identity matrix. These imply

$$Z_1^\dagger \hat{R}_{21}' Z_1 = Z_2 \hat{R}_{12}''^{-1} Z_2^\dagger - \lambda I_1 I_2. \tag{6.13}$$

Explicitly,

$$(Z^\dagger)_i^\alpha \hat{R}_{tj}'^{si} Z_\beta^j = Z_\gamma^s (\hat{R}''^{-1})_{\beta\delta}^{\alpha\gamma} (Z^\dagger)_t^\delta - \lambda \delta_\beta^\alpha \delta_t^s. \tag{6.14}$$

6.2 Calculus

One can introduce the following commutation relation for functions and one-forms

$$\hat{R}_{12}'^{-1} C_1 dC_2 = dC_1 C_2 \hat{R}_{12}, \tag{6.15}$$

i.e.

$$\begin{aligned}\hat{R}_{12}'^{-1} A_1 dA_2 &= dA_1 A_2 \hat{R}_{12}',\\ \hat{R}_{12}'^{-1} B_1 dB_2 &= dB_1 B_2 \hat{R}_{12}'',\\ dA_1 B_2 &= R_{12}'^{-1} B_2 dA_1,\\ A_1 dB_2 &= R_{21}'(dB_2 A_1 + \lambda dA_2 B_1 P_{12}),\end{aligned} \tag{6.16}$$

where $(P_{12})_{kl}^{ij} = \delta_l^i \delta_k^j$. Since $Z_\alpha^i = (A^{-1})_k^i B_\alpha^k$, it is easy to derive

$$dZ = A^{-1}(dB - dAZ) \tag{6.17}$$

and

$$\begin{aligned}Z_1 dA_2 &= dA_2 R_{12}'^{-1} Z_1,\\ dZ_1 A_2 &= A_2 R_{12}'^{-1} dZ_1,\\ Z_1 dB_2 &= (dB_2 Z_1 - \lambda Z_1 (dAZ)_2 P_{12}) R_{12}''.\end{aligned} \tag{6.18}$$

It follows

$$\hat{R}_{21}'^{-1} Z_1 dZ_2 = dZ_1 Z_2 \hat{R}_{12}''. \tag{6.19}$$

To introduce a $*-$structure for the calculus, it is consistent to take $(dZ_\alpha^i)^* = d(Z_\alpha^{i*})$. In addition we impose a complex structure on the calculus so that $d = \delta + \bar{\delta}$, where δ ($\bar{\delta}$) acts only on the holomorphic (antiholomorphic) part, satisfies Eqs.(3.43) and (3.44). This implies, after some calculation,

$$Z_1^\dagger \hat{R}_{21}' dZ_1 = dZ_2 \hat{R}_{12}''^{-1} Z_2^\dagger. \tag{6.20}$$

6.3 One-Form Realization

Introduce the matrix

$$E^i_j = C^i_a (C^\dagger)^a_j. \tag{6.21}$$

It is

$$E^\dagger = E \tag{6.22}$$

and

$$\hat{R}'^{-1}_{12} E_1 \hat{R}'^{-1}_{12} E_1 = E_1 \hat{R}'^{-1}_{12} E_1 \hat{R}'^{-1}_{12}. \tag{6.23}$$

One can show that

$$R'^{-1}_{I,J} E_J R'^{-1}_{J,I} E_I = E_I R'^{-1}_{I,J} E_J R'^{-1}_{J,I} \tag{6.24}$$

or equivalently,

$$R'_{I,J} E_I \bullet E_J = E_J \bullet E_I R'_{I,J}. \tag{6.25}$$

The bullet product is defined [28] inductively by

$$E_I \bullet E_J \equiv E_I R'^{-1}_{I,J} E_J R'_{I,J} \tag{6.26}$$

for any $I = (1'2' \cdots m')$, $J = (12 \cdots n)$, where

$$\begin{aligned} E_{(12\cdots M)} &\equiv E_1 \bullet E_2 \bullet \cdots \bullet E_M \\ &= E_1 (R'^{-1}_{12} E_2 R'_{12}) \cdots (R'^{-1}_{(M-1)M} \cdots R'^{-1}_{1M} E_M R'_{1M} R'_{2M} \cdots R'_{(M-1)M}) \end{aligned} \tag{6.27}$$

$$\begin{aligned} R_{I,II} &\equiv \langle \mathcal{R}, A_I \otimes A_{II} \rangle \\ &= \begin{array}{cccc} R_{1'n} & \cdot R_{1'(n-1)} & \cdot \cdots \cdot & R_{1'1} \\ \cdot R_{2'n} & \cdot R_{2'(n-1)} & \cdot \cdots \cdot & R_{2'1} \\ \vdots & \vdots & & \vdots \\ \cdot R_{m'n} & \cdot R_{m'(n-1)} & \cdot \cdots \cdot & R_{m'1}, \end{array} \end{aligned} \tag{6.28}$$

$\mathcal{R}$ is the universal R-matrix for $GL_q(M)$ and

$$A_I \equiv A_{(1'2'\cdots m')} \equiv A_{1'} A_{2'} \cdots A_{m'}, \quad A_{II} \equiv A_{(12\cdots n)} \equiv A_1 A_2 \cdots A_n. \tag{6.29}$$

Hence one can introduce the quantum determinant [28] for the generators E,

$$Det E \epsilon^{12\cdots M} = E_{(12\cdots M)} \epsilon^{12\cdots M}, \tag{6.30}$$

where $\epsilon^{12\cdots M}$ is the ϵ tensor for $GL_q(M)$.

Denote

$$L = Det E \tag{6.31}$$

and one can show that

$$\begin{aligned} L E^i_j &= E^i_j L, & (6.32) \\ C L &= L C & (6.33) \end{aligned}$$

and

$$dCL = q^{-2}LdC. \tag{6.34}$$

Using the general procedure stated in Sec.4.3 , we obtain the realization on the algebra generated by C_a^i, dC_a^i and their $*-$conjugates,

$$\eta = -qL^{-1}\delta L. \tag{6.35}$$

To find the one-form realization for the exterior differential operating on the complex Grassmannians Z, dZ, we introduce

$$\begin{aligned} X_l^k &= (A^{-1})_i^k E_j^i (A^{\dagger -1})_l^j \\ &= \delta_l^k + Z_\alpha^k (Z^\dagger)_l^\alpha. \end{aligned} \tag{6.36}$$

It is not hard to check that

$$\hat{R}'_{12} X_2 \hat{R}'_{12} X_2 = X_2 \hat{R}'_{12} X_2 \hat{R}'_{12}. \tag{6.37}$$

Since X commutes like the vector field Y, the quantum determinant

$$\begin{aligned} \rho &\equiv DetX \\ &= X_{(12\cdots M)}\epsilon^{12\cdots M} \end{aligned} \tag{6.38}$$

is central in the algebra of X. Here, the $\bullet$-product for X is

$$X_I \bullet X_J = R'^{-1}_{IJ} X_I R'_{IJ} X_J, \tag{6.39}$$

as for the vector field Y. In particular,

$$\begin{aligned} X_{(12\cdots M)} = &\ (R'^{-1}_{12} R'^{-1}_{23} \dots R'^{-1}_{1M} X_1 R_{1M} \dots R_{12}) \cdot (R'^{-1}_{23} R'^{-1}_{24} \dots R'^{-1}_{2M} X_2 R_{2M} \dots R_{23}) \\ &\cdots \cdot (R'^{-1}_{(M-1)M} X_{M-1} R_{(M-1)M}) X_M. \end{aligned} \tag{6.40}$$

Introducing the quantum determinants $det(A^{-1})$, $det(A^{\dagger -1})$

$$\begin{aligned} det(A^{-1})\epsilon^{12\cdots M} &= A_M^{-1} \cdots A_2^{-1} A_1^{-1} \epsilon^{12\cdots M}, \\ det(A^{\dagger -1})\epsilon^{12\cdots M} &= (A^{\dagger -1})_1 (A^{\dagger -1})_2 \cdots (A^{\dagger -1})_M \epsilon^{12\cdots M} \end{aligned} \tag{6.41}$$

for A^{-1} and $A^{\dagger -1}$ satisfying the "RTT"-like relations

$$\begin{aligned} \hat{R}'_{12}(q^{-1}) A_1^{-1} A_2^{-1} &= A_1^{-1} A_2^{-1} \hat{R}'_{12}(q^{-1}), \\ \hat{R}'_{12}(A^{\dagger -1})_1 (A^{\dagger -1})_2 &= (A^{\dagger -1})_1 (A^{\dagger -1})_2 \hat{R}'_{12}, \end{aligned} \tag{6.42}$$

it can be shown that

$$\rho = det(A^{-1})\, L\, det(A^{\dagger -1}), \tag{6.43}$$

$$Z\rho = q^2 \rho Z \tag{6.44}$$

and

$$dZ\rho = \rho dZ. \tag{6.45}$$

As a result, we have the one-form realization

$$\eta = -q^{-1}\rho^{-1}\delta\rho \tag{6.46}$$

for the exterior derivative acting on the algebra generated by Z^i_α, dZ^i_α and their $*$-conjugates. The Kähler form

$$K = \bar{\delta}\eta \tag{6.47}$$

is central as usual.

6.4 Braided $G_q^{M,N}$

Let Z, Z' be two copies of the quantum Grassmannians $G_q^{M,N}$ defined by

$$Z = A^{-1}B, \quad Z' = A'^{-1}B', \tag{6.48}$$

where $C^i_a = (A^i_j, B^i_\alpha)$, $C'^i_a = (A'^i_j, B'^i_\alpha)$ both satisfy the relations (6.2). Let the mixed commutation relations be

$$Q_{12}C_1C'_2 = C'_1C_2\hat{R}_{12}, \tag{6.49}$$

where Q is a numerical matrix. For (6.49) to be consistent with (6.2), we can take Q to be $\hat{R}'^{\pm 1}$. For either of these two choice, (6.49) is covariant under

$$C \to CT, \quad C' \to C'T, \tag{6.50}$$

where T^a_b is a $GL_q(M+N)$ quantum matrix and also under the transformation

$$C \to SC, \quad C' \to SC', \tag{6.51}$$

where S^i_j is a $GL_q(M)$ quantum matrix. We will pick $Q = \hat{R}'$ in the following

$$\hat{R}'_{12}C_1C'_2 = C'_1C_2\hat{R}_{12}. \tag{6.52}$$

Explicitly, it is

$$\begin{aligned} R'_{12}A_1A'_2 &= A'_2A_1R'_{12}, \\ R'_{12}B_1B'_2 &= B'_2B_1R''_{12}, \\ B_1A'_2 &= R'^{-1}_{12}A'_2B_1, \\ A_1B'_2 &= R'^{-1}_{12}B'_2A_1 + \lambda B_1A'_2P_{12}. \end{aligned} \tag{6.53}$$

It follows that [16]

$$Z_1Z'_2 = R'_{12}Z'_2Z_1R''_{12} - \lambda Z_1Z_2\hat{R}''_{12}. \tag{6.54}$$

[16]If we had made the other choice $Q = \hat{R}'^{-1}$ in the above, the relations (6.53) would be different, but (6.54) would remain the same.

One can introduce a $*$-structure to this braided algebra, the relation

$$C_1^\dagger \hat{R}'^{-1}_{12} C'_1 = C'_2 \hat{R}^{-1}_{12} C_2^\dagger \tag{6.55}$$

is consistent and is covariant under

$$C \to CT, \quad C' \to C'T, \tag{6.56}$$

and

$$C \to SC, \quad C' \to SC' \tag{6.57}$$

with the same T, S quantum matrices as explained before. It follows immediately

$$Z_1^\dagger \hat{R}'_{21} Z'_1 = Z'_2 \hat{R}''^{-1}_{12} Z_2^\dagger - \lambda I_1 I_2. \tag{6.58}$$

One can also show that the Kähler form K of the original copy (6.47) commutes also with the $Z', Z'^\dagger, dZ', dZ'^\dagger$.

This concludes our discussion for the quantum Grassmannians, with the case of complex projective spaces $CP_q(N) = G_q^{1,N}$ as a special case.[17]

7 Acknowledgement

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-9514797.

A RELATION TO CONNES' FORMULATION

Here we make a comment on the relation of our work to Connes' quantum Riemannian geometry [27]. We will try to re-formulate the differential and integral calculus on the quantum sphere in a way as close to his formulation as possible. We take $0 < q \leq 1$.

To do so we first give the spectral triple $(X, \mathcal{H}, \mathcal{D})$ for this case. X is the algebra of functions on S_q^2. $\mathcal{H}$ is the Hilbert space on which both functions and differential forms are realized as operators. It is chosen to be composed of two parts $\mathcal{H} = \mathcal{H}_0 \otimes V$. The first part $\mathcal{H}_0$ is any Hilbert space representing the algebra X. An example is [3]

$$\begin{aligned} \pi(z)|n\rangle &= (q^{-2n} - 1)^{1/2}|n-1\rangle, \\ \pi(\bar{z})|n\rangle &= (q^{-2(n+1)} - 1)^{1/2}|n+1\rangle, \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{A.1}$$

Another example is the Gel'fand-Naĭmark-Siegel construction using the integration $\langle \cdot \rangle$ introduced in Sec.3.6. The second part V is $\mathbf{C}^2$, as in the classical case. Operators on

[17]Notice that for $M = 1$, the numerical R-matrix becomes a number: $\hat{R}'_{12} = q$.

$\mathcal{H}$ are therefore 2×2 matrices with entries being operators on $\mathcal{H}_0$. Finally, the Dirac operator is an anti-self-adjoint operator [18] on $\mathcal{H}$:

$$\mathcal{D} = k \begin{pmatrix} i & \pi(\bar{z}) \\ -\pi(z) & -i \end{pmatrix}, \tag{A.2}$$

where k is a real number.

According to Connes we proceed as follows to find the differential calculus on S_q^2. The representation π of X on $\mathcal{H}_0$ is extended to be a representation on $\mathcal{H}$ for the universal differential calculus Ω_X by

$$\pi(a_0(da_1)\cdots(da_n)) = \pi(a_0)[\mathcal{D}, \pi(a_1)]\cdots[\mathcal{D}, \pi(a_n)]. \tag{A.3}$$

In particular, one finds

$$\pi(dz)|\psi\rangle = q^{-1}\lambda k\pi(\rho)\tau|\psi\rangle, \tag{A.4}$$

$$\pi(d\bar{z})|\psi\rangle = q^{-1}\lambda k\pi(\rho)\tau^\dagger|\psi\rangle, \quad |\psi\rangle \in \mathcal{H}, \tag{A.5}$$

where the γ-matrices $\tau = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ satisfy the deformed Clifford algebra $q\tau\tau^\dagger + q^{-1}\tau^\dagger\tau = \mathcal{I}$ for $\mathcal{I} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$.

It can be checked that the kernel of the map π for one-forms in Ω_X, is generated from

$$z(dz) - q^{-2}(dz)z, \quad z(d\bar{z}) - q^{-2}(d\bar{z})z, \tag{A.6}$$

and their $*$-involutions, by multiplying with functions from both sides. The kernel of π for two-forms is generated from

$$(dz)(dz) = 0, \quad (d\bar{z})(d\bar{z}) = 0 \tag{A.7}$$

by multiplying with functions and from the kernel of one-forms by multiplying with one-froms from both sides. The auxiliary fields form the ideal Aux defined to be the sum of the kernels of all degrees and the differential of them. So in our case Aux, in addition to the sum of kernels mentioned above, is generated by

$$d[z(d\bar{z}) - q^{-2}(d\bar{z})z] = (dz)(d\bar{z}) + q^{-2}(d\bar{z})(dz). \tag{A.8}$$

The other differentials are already contained in the sum. According to Connes, the differential calculus is obtained from the spectral triple by

$$\Omega(X) = \Omega_X/Aux. \tag{A.9}$$

This gives precisely the same covariant differential calculus described in Sec.3.1.

[18] It is a pure convention that we choose $\mathcal{D}$ to be anti-self-adjoint rather than self-adjoint like Connes usually does.

Next we consider the integration on S_q^2. Connes' formula for the integration is

$$\int \alpha = Tr_\omega(\gamma\pi(\alpha)|\mathcal{D}|^{-d}), \tag{A.10}$$

where Tr_ω is the Dixmier's trace [29] and γ is the $\mathbf{Z}_2$-grading operator. Here d is the dimension of the quantum space, which, according to Connes, is defined by the series of eigenvalues of $|\mathcal{D}|^{-1}$. In our case d determined that way is zero.

One should expect that Connes' prescription will not give the same invariant integration on S_q^2 (3.134) because while Connes' integration always has the cyclic property

$$\int \omega_1\omega_2 = \pm \int \omega_2\omega_1, \tag{A.11}$$

we know that the $SU_q(2)$-invariant integration does not. Remarkably, if we choose to use the classical dimension $d = 2$ of the two-sphere in the formula (A.10), we actually obtain the invariant integration. This is shown in the following.

Note that the calculus on S_q^2 is $\mathbf{Z}_2$-graded by

$$\gamma = k^{-2}\pi((dzd\bar{z} - d\bar{z}dz)\rho^{-2}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.12}$$

which satisfies

$$\gamma^2 = I, \quad \gamma^\dagger = \gamma, \tag{A.13}$$

$$\gamma\pi(a) = \pi(a)\gamma \quad \forall a \in X, \tag{A.14}$$

$$\gamma\mathcal{D} = -\mathcal{D}\gamma. \tag{A.15}$$

We define the integration on S_q^2 by the trace

$$\int \alpha = Tr(\gamma\pi(\alpha)|\mathcal{D}|^{-2}), \tag{A.16}$$

where Tr is the appropriate trace on the Hilbert space $\mathcal{H}$. (If the Hilbert space of (A.1) is used for $\mathcal{H}_0$, one should simply use the ordinary trace.) It can be directly checked that the integration is compatible with the differential calculus

$$\int Aux = 0, \tag{A.17}$$

by using the representations of the auxiliary fields

$$\pi(a)\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \tag{A.18}$$

for any $a \in X$ (including 0) and

$$|\mathcal{D}|^{-2} = qk^{-2}\pi(\rho^{-1})\begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}. \tag{A.19}$$

Using the cyclic property of the trace, it follows also that Stokes' theorem

$$\int d\alpha = 0 \tag{A.20}$$

is valid for any one-form α.

Since Stokes' theorem can be used to derive the recursion relations (3.138), the integration (A.16) coincides with the invariant integration on S_q^2 up to normalization.

Note that Eqs.(A.17) and (A.20) are valid for any choice of $\mathcal{H}_0$, hence the formula (A.16) gives the same invariant integration as long as an appropriate trace exists so that our selected integrable functions, e.g. ρ^{-n} for $n \geq 0$, multiplied by the area element (Kähler two-form) have finite integrals.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098