

UC Santa Cruz

UC Santa Cruz Electronic Theses and Dissertations

Title

Asymptotics of determinants for finite sections of operators with almost periodic diagonals

Permalink

<https://escholarship.org/uc/item/2fq9h2v6>

Author

Zhou, Zheng

Publication Date

2019

Copyright Information

This work is made available under the terms of a Creative Commons Attribution License, available at <https://creativecommons.org/licenses/by/4.0/>

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
SANTA CRUZ

**ASYMPTOTICS OF DETERMINANTS FOR FINITE SECTIONS
OF OPERATORS WITH ALMOST PERIODIC DIAGONALS**

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Zheng Zhou

June 2019

The Dissertation of Zheng Zhou
is approved:

Prof. Torsten Ehrhardt, Chair

Prof. Estelle Basor

Prof. Jie Qing

Lori Kletzer
Vice Provost and Dean of Graduate Studies

Copyright © by

Zheng Zhou

2019

Contents

Abstract	iv
1 Introduction	1
1.1 Background	1
1.2 About the paper	9
2 Preliminaries	15
2.1 Banach algebras of almost periodic sequences	15
2.2 Banach algebras of operators on $\ell^2(\mathbb{Z})$	21
2.3 Banach algebras associated with suitable Banach algebras	27
2.3.1 Banach algebras of operators on $\ell^2(\mathbb{Z}^+)$	28
2.3.2 Banach algebras of sequences of finite sections	30
3 Main results	39
3.1 First versions of the limit theorem	39
3.2 A uniform version of the limit theorem	46
4 Further Remarks	54
4.1 Special cases of the limit theorem	54
4.1.1 The structure of finitely generated subgroups Ξ of \mathbb{R}/\mathbb{Z}	57
4.1.2 The limit theorem for finitely generated subgroups	62
4.1.3 The case of block Laurent operators revisited	65
4.2 Future problems and remarks	67
4.2.1 The inverse closedness problem	67
4.2.2 In connection with Random Matrix Theory	69
Bibliography	71

Abstract

Asymptotics of determinants for finite sections of operators with almost periodic
diagonals

by

Zheng Zhou

Let $A = (a_{j,k})_{j,k=-\infty}^{\infty}$ be a bounded linear operator on $l^2(\mathbb{Z})$ whose diagonals $D_n(A) = (a_{j,j+n})_{j=-\infty}^{\infty} \in l^\infty(\mathbb{Z})$ are almost periodic sequences. For certain classes of such operators and under certain conditions, we are going to determine the asymptotics of the determinants $\det A_{n_1, n_2}$ of the finite sections $A_{n_1, n_2} = (a_{j,k})_{j,k=n_1}^{n_2-1}$ as their size $n_2 - n_1$ tends to infinity. Examples of such operators include block Toeplitz operators and the almost Mathieu operator.

Keywords: Szegő-Widom limit theorem, Toeplitz operator, almost Mathieu operator, determinants of finite sections

Chapter 1

Introduction

1.1 Background

Asymptotics of determinants of finite sections. For a symbol $a \in L^\infty(\mathbb{T})$, $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ being the unit circle, the $n \times n$ Toeplitz matrices are defined by

$$T_n(a) = (a_{j-k}), \quad 0 \leq j, k \leq n-1, \quad (1.1.1)$$

where a_k stands for the k -th Fourier coefficients of a ,

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{ix}) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

Under certain assumption on the symbol a , the strong Szegő-Widom limit theorem [30, 31] states that

$$\lim_{n \rightarrow \infty} \frac{\det T_n(a)}{G[a]^n} = E[a],$$

where $G[a]$ and $E[a]$ are explicitly defined non-zero constants.

The Toeplitz matrices $T_n(a)$ can be viewed as the finite sections of Laurent

operators $L(a)$, i.e., as the compressions

$$T_n(a) = P_n L(a) P_n|_{\text{im} P_n},$$

where the Laurent operators

$$L(a) = (a_{j-k}), \quad j, k \in \mathbb{Z}, \quad (1.1.2)$$

are acting on $l^2(\mathbb{Z})$, and P_n are the finite section projections

$$P_n : \{x_k\}_{k \in \mathbb{Z}} \mapsto \{y_k\}_{k \in \mathbb{Z}}, \quad y_k = \begin{cases} x_k & \text{if } 0 \leq k < n \\ 0 & \text{if } k < 0 \text{ or } k \geq n. \end{cases}$$

Note that Laurent operators are constant on each diagonal, that means they are shift-invariant in the sense that $U_{-n} L(a) U_n = L(a)$ for all $n \in \mathbb{Z}$, where $U_n = L(t^n)$ are the shift operators

$$U_n : \{x_k\}_{k \in \mathbb{Z}} \mapsto \{x_{k-n}\}_{k \in \mathbb{Z}} \quad (1.1.3)$$

acting on $l^2(\mathbb{Z})$.

For $n_1, n_2 \in \mathbb{Z}$ such that $n_2 > n_1$ let us define more general finite section projections

$$P_{n_1, n_2} : \{x_k\}_{k \in \mathbb{Z}} \mapsto \{y_k\}_{k \in \mathbb{Z}}, \quad y_k = \begin{cases} x_k & \text{if } n_1 \leq k < n_2 \\ 0 & \text{if } k < n_1 \text{ or } k \geq n_2 \end{cases} \quad (1.1.4)$$

thought of as operators acting on $l^2(\mathbb{Z})$. The goal of the paper is to describe the asymptotic behavior of the determinants

$$\det P_{n_1, n_2} A P_{n_1, n_2}$$

as the size $n_2 - n_1$ of the finite sections tends to ∞ , where A is a bounded linear operator on $l^2(\mathbb{Z})$ with almost periodic diagonals which has to satisfy various conditions. As usual, the finite sections $P_{n_1, n_2} A P_{n_1, n_2}$ therein are identified with matrices of order $n_2 - n_1$. Notice, however, that even in the case of band operators with periodic diagonals (see the comments on Theorem 1.1.1 below) one *cannot* expect an asymptotic behavior of the kind

$$\lim_{n_2 - n_1 \rightarrow \infty} \frac{\det P_{n_1, n_2} A P_{n_1, n_2}}{G^{n_2 - n_1}} = E. \quad (1.1.5)$$

In fact, in order for a limit to exist one has to consider appropriate sequences of integers $h_1 = \{h_1(n)\}_{n \geq 0}$ and $h_2 = \{h_2(n)\}_{n \geq 0}$ (which will be called *fractal sequences* in this paper) such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, under the appropriate assumptions on A , one of our main results (referred to as the *fractal version of the limit theorem*) states that

$$\lim_{n \rightarrow \infty} \frac{\det P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)}}{G^{h(n)}} = E_{h_1, h_2}, \quad (1.1.6)$$

where G is a nonzero constant only depending on A and E_{h_1, h_2} is a constant depending on A and the sequences h_1 and h_2 . This generalizes the results of [14], where the case of $h_1(n) = 0$ and very particular fractal sequences $h(n) = h_2(n)$ (therein referred to as *distinguished sequences*) was considered.

The notion of fractal sequences is general enough to allow a “complete” understanding of the determinant asymptotics in the following sense. Namely, as another main result (referred to as the *uniform version of the limit theorem*) we will establish that

$$\lim_{n_2 - n_1 \rightarrow \infty} \left(\frac{\det P_{n_1, n_2} A P_{n_1, n_2}}{G^{n_2 - n_1}} - E[n_1, n_2] \right) = 0 \quad (1.1.7)$$

under appropriate conditions on A . To make this a non-trivial statement, it is also established that $E[n_1, n_2] = \Theta_{A,1}(\tau_{n_1})\Theta_{A,2}(\tau_{n_2})$ with explicitly defined elements τ_n of a compact Hausdorff space and *continuous* complex-valued functions $\Theta_{A,1}$ and $\Theta_{A,2}$. The limit (1.1.7), being of the form $\lim_{n_2-n_1 \rightarrow \infty} \sigma_{n_1, n_2} = 0$, is understood in the usual way, i.e., meaning that for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $|\sigma_{n_1, n_2}| < \epsilon$ whenever $n_1, n_2 \in \mathbb{Z}$ and $n_2 - n_1 > k$.

Let us now introduce the notions of almost periodic sequences and band dominated operators with almost periodic diagonals, the latter being the class of operators A for which the limits (1.1.6) and (1.1.7) will be established.

Almost periodic sequences. Denote by $AP(\mathbb{Z})$ the set of *almost periodic sequences* consisting of all $a = \{a(k)\}_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$ for which the set

$$\{U_n a : n \in \mathbb{Z}\}$$

is relatively compact in the norm topology of $l^\infty(\mathbb{Z})$. Here

$$U_n : a \in l^\infty(\mathbb{Z}) \mapsto b \in l^\infty(\mathbb{Z}), \quad b(k) := a(k - n) \quad (1.1.8)$$

is the shift operator acting (isometrically) on $l^\infty(\mathbb{Z})$. Despite the difference of the underlying spaces in (1.1.3) and (1.1.8), we will use the same symbol for brevity. There is an equivalent definition of $AP(\mathbb{Z})$ as the closure in $l^\infty(\mathbb{Z})$ of the set of all finite linear combinations of sequences $e_\xi \in l^\infty(\mathbb{Z})$, where

$$e_\xi(k) = e^{2\pi i k \xi}, \quad k \in \mathbb{Z}, \quad \xi \in \mathbb{R}. \quad (1.1.9)$$

Note that e_ξ depends on $\xi \in \mathbb{R}$ only modulo \mathbb{Z} . Therefore, it is more appropriate to think of ξ as an element in \mathbb{R}/\mathbb{Z} , the additive group arising from \mathbb{R} by identifying

two numbers whose difference is an integer.

For each $a = \{a(k)\}_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$, its mean $M(a)$ is well-defined by the limit

$$M(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a(k). \quad (1.1.10)$$

The Fourier coefficients of a sequence $a \in AP(\mathbb{Z})$ are defined by

$$a_\xi = M(ae_{-\xi}), \quad \xi \in \mathbb{R}/\mathbb{Z}. \quad (1.1.11)$$

It is known that the set of all ξ for which $a_\xi \neq 0$ is at most countable for each $a \in AP(\mathbb{Z})$. This set is called the Fourier spectrum of a . The theory of almost periodic sequences is similar to the theory of almost periodic functions on \mathbb{R} . For details and basic information we refer to [8].

Band-dominated operators with almost periodic diagonals. We define the class \mathcal{OAP} of *operators with almost periodic diagonals* as the set of all bounded linear operators A on $l^2(\mathbb{Z})$ whose n -th diagonal $a^{(n)}$ belongs to $AP(\mathbb{Z})$ for each $n \in \mathbb{Z}$. In other words,

$$a^{(n)} = D_n(A) := D(AU_{-n}) \in AP(\mathbb{Z}), \quad (1.1.12)$$

where $D(A) \in l^\infty(\mathbb{Z})$ stands for the main diagonal of a bounded linear operator A on $l^2(\mathbb{Z})$. While a Laurent operator can be written as

$$L(a) = \sum_{n \in \mathbb{Z}} a^{(n)} U_n$$

where $a^{(n)} \in \mathbb{C}$ are constants, an operator $A \in \mathcal{OAP}$ can be formally written as

$$A = \sum_{n \in \mathbb{Z}} (a^{(n)} I) U_n \quad (1.1.13)$$

where $a^{(n)} = D_n(A) \in AP(\mathbb{Z})$ is the sequence representing the n -th diagonal of A and aI stands for the multiplication operator generated by $a = \{a(k)\}_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$, i.e.,

$$aI : \{x_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \mapsto \{a(k)x_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}). \quad (1.1.14)$$

A subclass of \mathcal{OAP} is the collection of *band-dominated operators with almost periodic diagonals*, the notion of which is used rather loosely. One can define it as the closure of the set of all band operators with almost periodic diagonals with respect to some appropriate norm. Band operators are operators of the form (1.1.13) with the sum being finite. For more on the theory of band-dominated operators, see [19, 20, 21, 24].

The strong Szegő-Widom limit theorem. The class $AP(\mathbb{Z})$ includes periodic sequences as a special case. Indeed, the sequence e_ξ is periodic if and only if $\xi \in \mathbb{Q}/\mathbb{Z}$. An operator acting on $l^2(\mathbb{Z})$ whose diagonals are periodic sequences with period N can be identified with a block Laurent operator, which is defined by (1.1.2) with a being an $N \times N$ matrix-valued function whose Fourier coefficients a_n are $N \times N$ matrices.

The classical strong Szegő-Widom limit theorem [31] describes the asymptotics of the determinants of the finite sections $P_{n_1, n_2} L(a) P_{n_1, n_2}$ of block Laurent operators in the case when $n_1 = 0$ and $n_2 = nN$ as $n \rightarrow \infty$. Let us recall this result for generating functions in $B^{N \times N}$, where $B = W \cap F\ell_{1/2, 1/2}^{2, 2}$ is the Banach algebra of

all $a \in L^\infty(\mathbb{T})$ for which

$$\|a\|_B := \sum_{n=-\infty}^{\infty} |a_n| + \left(\sum_{n=-\infty}^{\infty} |n| \cdot |a_n|^2 \right)^{1/2} < \infty.$$

Further information and different versions of this limit theorem for various classes of functions can be found in [5, 6, 11, 24, 29] and in the references given there.

Theorem 1.1.1 (Szegő-Widom limit theorem). *For $a \in B^{N \times N}$ assume that $\det a(t) \neq 0$ for all $t \in \mathbb{T}$ and that $\det a(t)$ has winding number zero. Then*

$$\lim_{n \rightarrow \infty} \frac{\det P_{0,nN} L(a) P_{0,nN}}{G[a]^n} = E[a], \quad (1.1.15)$$

where $G[a] = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det a(e^{ix}) dx \right)$ and $E[a] = \det T(a) T(a^{-1})$.

Therein $T(a) = (a_{j-k})_{j,k=0}^\infty$ is the block Toeplitz operator thought of as acting on $l^2(\mathbb{Z}^+)$. The constant $E[a]$ is given in terms of a well-defined operator determinant. In the scalar case ($N = 1$) the constant $E[a]$ admits a more explicit expression and is always nonzero. In general ($N \geq 2$), the constant $E[a]$ can happen to be zero and explicit expressions are known only in very special cases (e.g., if a or a^{-1} are trigonometric matrix polynomials).

The classical Szegő-Widom limit theorem can be used to determine the asymptotics of $\det P_{n_1, n_2} L(a) P_{n_1, n_2}$ when the block size fits the size of the finite sections, i.e., when N divides $n_2 - n_1$. Indeed, for fixed $k \in \{0, \dots, N - 1\}$ the identity

$$\det P_{k, k+nN} L(a) P_{k, k+nN} = \det P_{0, nN} L(a^{[k]}) P_{0, nN}$$

holds, where $a^{[k]}$ are slightly modified symbols determined by $U_{-k} L(a) U_k = L(a^{[k]})$. These symbols yield the same constant G but in general different values for E . This observation corroborates the statement made above that one cannot expect

an asymptotics of the kind (1.1.5), and gives an indication of the notion of fractal sequences in this case.

In Subsection 4.1.3, as a consequence to our main results, we will obtain a limit theorem for the determinants $\det P_{n_1, n_2} L(a) P_{n_1, n_2}$ where the condition that N divides $n_2 - n_1$ is removed.

The almost Mathieu operators. The other most prominent example of operators to which the main results can be applied (under certain conditions) is the almost Mathieu operator defined on $l^2(\mathbb{Z})$,

$$M_a = U_1 + aI + U_{-1}, \tag{1.1.16}$$

where $a \in AP(\mathbb{Z})$ is given by $a(n) = \beta \cos 2\pi(\xi n + \delta)$, and β , ξ , and δ are certain (real) constants. The interesting case is when ξ is irrational. Then the spectrum of the almost Mathieu operator is a Cantor-like set. This was an open conjecture attracting quite some attention before it was finally proved by Avila and Jitomirskaya [1]. For more on the history and on general information about the almost Mathieu operators see [1, 3] and the references therein.

One can ask the question for which values of ξ , β , δ , and λ , it is possible to apply our results to $A = M_a - \lambda I$ and obtain a corresponding asymptotics (1.1.6) and (1.1.7). One restriction is that ξ is not a Liouville number, another one is that A is invertible on $l^2(\mathbb{Z})$. These restrictions can probably not be relaxed very much. If a conjecture raised in Subsection 4.2.1 turns out to be true, then these are the only restrictions (assuming M_a to be self-adjoint).

Regardless of whether the conjecture is true, we can still say that for non-Liouville numbers ξ , our results can be applied to $A = M_a - \lambda I$ whenever $|\lambda|$ is sufficiently large. For details see again Subsection 4.2.1.

1.2 About the paper

The notion of fractal sequences. In Sections 2.1 and 2.2 we will introduce the notion of a fractal sequence. It is an integer sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$, i.e., $h = \{h(n)\}_{n \geq 0}$, characterized by a property that guarantees the existence of certain limits in various settings. More precisely, we will encounter

- (i) fractal sequences for an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} ,
- (ii) fractal sequences for a Banach subalgebra \mathcal{A} of $l^\infty(\mathbb{Z})$,
- (iii) fractal sequences for a Banach subalgebra \mathcal{R} of $\mathcal{L}(l^2(\mathbb{Z}))$.

The name “fractal” is chosen because (in the appropriate setting, see Lemma 3.2.3) each integer sequence has subsequence which is fractal. Furthermore, there seems to be a connection with the work of Roch and Silbermann on fractal algebras of approximation sequences (see [23, 22] and the references therein). Although their notion of fractality is used in the setting of C^* -algebras, it can be given for Banach algebras as well. It is not too hard to see that if h_1 and h_2 are fractal sequences (in our sense), then the Banach algebra \mathcal{S}_{h_1, h_2} to be defined in Theorem 2.3.4 is fractal in the sense of Roch/Silbermann. Note that this Banach algebra plays the crucial role in the proof of our main results in Section 3.1.

On the assumptions encountered in the limit theorems. In our main results, the fractal and the uniform version of the limit theorem (Theorems 3.1.4 and 3.2.4), several assumptions have to be imposed. One may wonder about the form of these assumptions and whether they are really necessary. We want to argue here that they are at least close to be necessary. We will encounter three kinds of assumptions on the “symbol” A that occurs in the limits (1.1.6) and (1.1.7), which could be referred to as

- (i) a smoothness assumption,
- (ii) a regularity assumption,
- (iii) a diophantine assumption.

The first two already occur naturally in the Szegő-Widom limit theorem, where $A = L(a)$. Therein first condition comes down to assume the function a to be sufficiently smooth. The second one is the requirement that $\det a(t)$ does not vanish and has winding number zero. Equivalently, this means that $\det a$ has a continuous logarithm or that the matrix function a is the finite product of exponentials. (To be precise, the equivalence holds for various classes of smooth functions. It breaks down for the rather exotic classes of Krein algebras. For details see Proposition 6.4 and the concluding remarks in [11].)

In our limit theorems the first assumption corresponds to assuming a sufficiently fast decay of the diagonals of A (therefore the restriction to band-dominated operators) and in addition a sufficiently fast decay of the Fourier coefficients of the almost periodic sequences that occur as the diagonals of A . More specifically, we will consider weighted Wiener-type algebras of such operators, which involve an “admissible” weight β on an additive subgroup $\Xi \subseteq \mathbb{R}/\mathbb{Z}$ containing the Fourier spectra of the diagonals of A .

The regularity assumption in our limit theorems comes down to assuming that A is a finite product of exponentials, i.e., $A = e^{A_1} \cdots e^{A_r}$, where A_1, \dots, A_r have to belong to the afore-mentioned Wiener-type algebras. This is stronger than invertibility, but seems natural in view of what is assumed in the classical Szegő-Widom limit theorem. For another interpretation of this condition and its relation to inverse closedness, see Section 4.2.1.

Finally, the diophantine assumption is peculiar to the almost periodic case. It occurs in the form that the weight β has to be “compatible”. The *compatibility*

condition involves the underlying group Ξ of the Fourier spectra, and it is conceivable that not every Ξ has an admissible and compatible weight. However, if the elements of Ξ have “nice” diophantine properties then the existence of such weights is guaranteed by non-trivial results from diophantine approximation (see the beginning of Section 4.1 and Subsection 4.1.2 in particular for details). As already demonstrated in [14] and as we will indicate next, without the compatibility condition the asymptotics of the determinants may not take the desired form, the reason being that the trace computations break down.

Method of proof: reduction of determinants to traces, and the compatibility condition. Let us make one important point pertaining to the method of proof and explaining how the compatibility condition arises from the asymptotics of the traces. First of all, if one wants to study the asymptotics of the determinants $\det P_{h_1(n),h_2(n)}AP_{h_1(n),h_2(n)}$ for band-dominated operators A one has to be able to describe the asymptotics of the traces of the finite sections of such operators in the first place. Indeed, this can be seen by considering the special case of a diagonal operator $A = e^a I$ with $a \in AP(\mathbb{Z})$ and notice that

$$\det \left(P_{h_1(n),h_2(n)}(e^a I)P_{h_1(n),h_2(n)} \right) = \exp \left(\text{trace } P_{h_1(n),h_2(n)}(aI)P_{h_1(n),h_2(n)} \right).$$

Conversely, our method of establishing the determinant asymptotics consists in assuming that $A = e^{A_1} \cdots e^{A_r}$, where A_k are operators taken from certain subclasses of \mathcal{OAP} , and then reducing asymptotics of $\det P_{h_1(n),h_2(n)}AP_{h_1(n),h_2(n)}$ to the asymptotics of the traces

$$\text{trace} \left(P_{h_1(n),h_2(n)}(A_1 + \cdots + A_r)P_{h_1(n),h_2(n)} \right).$$

This idea is realized in the first of our main results, the *abstract version of the*

limit theorem (Theorem 3.1.3). Notice that the above trace equals

$$\text{trace}(P_{h_1(n),h_2(n)}(aI)P_{h_1(n),h_2(n)}) = \sum_{k=h_1(n)}^{h_2(n)-1} a(k), \quad (1.2.1)$$

where $a = D(A_1 + \cdots + A_r) \in AP(\mathbb{Z})$. Thus, to summarize, we are led to the problem of describing the asymptotics of traces (1.2.1) for $a = \{a(k)\}_{k \in \mathbb{Z}} \in AP(\mathbb{Z})$.

Since we assume that $h(n) := h_2(n) - h_1(n) \rightarrow \infty$ as $n \rightarrow \infty$, the theory of almost periodic sequences implies that (1.2.1) equals

$$h(n) \cdot M(a) + o(h(n)), \quad n \rightarrow \infty,$$

where $M(a)$ is the mean of $a \in AP(\mathbb{Z})$ defined in (1.1.10). However, what we would like to have (in view of the desired asymptotics (1.1.6)) is an asymptotics of the kind

$$h(n) \cdot M(a) + C_{h_1, h_2} + o(1), \quad n \rightarrow \infty, \quad (1.2.2)$$

with some constant C_{h_1, h_2} possibly depending on the underlying sequences h_1 and h_2 . In order to see when we can expect such a behavior, consider the case where $a \in AP(\mathbb{Z})$ is a finite sum of the form

$$a = \sum_{\xi} a_{\xi} e_{\xi}, \quad a_{\xi} \in \mathbb{C}. \quad (1.2.3)$$

Then we have (for details see the proof of Theorem 2.1.3 below)

$$\sum_{k=h_1(n)}^{h_2(n)-1} a(k) = h(n) \cdot M(a) + \sum_{\xi \neq 0} a_{\xi} \frac{e^{2\pi i h_1(n)\xi} - e^{2\pi i h_2(n)\xi}}{1 - e^{2\pi i \xi}}.$$

In order for the last term to converge as $n \rightarrow \infty$ one should assume that the

sequences h_1 and h_2 are such that the limits

$$\lim_{n \rightarrow \infty} e^{2\pi i h_j(n)\xi} =: \tau_j(\xi), \quad j = 1, 2,$$

exist for each ξ over which the summation is taken. This condition will later be conceptualized by saying that h_1 and h_2 are *fractal sequences*. If this holds then the trace equals

$$h(n) \cdot M(a) + \sum_{\xi \neq 0} a_\xi \frac{\tau_1(\xi) - \tau_2(\xi)}{1 - e^{2\pi i \xi}} + o(1), \quad n \rightarrow \infty, \quad (1.2.4)$$

where the second term is a well-defined constant C_{h_1, h_2} depending on h_1 and h_2 .

Things get more complicated if we consider $a \in AP(\mathbb{Z})$ for which the sum (1.2.3) involves infinitely many terms. Then we encounter a “small denominator problem” in the expression for the traces and in its expected asymptotics (1.2.4). In some cases, when the Fourier spectrum of a is “non-Liouville” and the decay of the Fourier coefficients a_ξ is sufficiently fast we can still guarantee the asymptotics (1.2.4). A corresponding result is established in Theorem 2.1.3, which involves the *compatibility condition*. The relationship with diophantine properties will be discussed in Section 4.1 (see Subsection 4.1.1 in particular).

On the other hand, there exists $a \in AP(\mathbb{Z})$ for which an asymptotics of the form (1.2.2) does not hold, even if we assume h_1 and h_2 to be fractal sequences. This was shown by explicit (though complicated) examples in Section 6 of [14], and will be briefly mentioned at the beginning of Section 4.1. Therefore, one cannot completely eliminate the diophantine assumption in the formulation of our limit theorems.

Outline of the paper. The paper is organized as follows. Section 2.1 deals with Banach algebras of almost periodic sequences. The notions of fractality

and of admissible and compatible weights on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} are introduced as well.

In Section 2.2 we consider Banach algebras of operators on $l^2(\mathbb{Z})$ which are characterized by properties called suitability, shift-invariance and rigidity. Based on Section 2.1, concrete examples of such Banach algebras are described.

Sections 2.3 and 3.1 are devoted to the abstract version of the limit theorem and its proof. Note that the abstract version applies to operators A taken from any suitable, shift-invariant, rigid, and unital Banach algebra of operators on $l^2(\mathbb{Z})$. The proof is based on a Banach algebra method introduced by one of the authors in [11] for the classical Szegő-Widom limit theorem. This approach was also employed in [14].

At the end of Section 3.1 we establish the fractal version of the limit theorem using the previous results. In Section 3.2 we obtain the uniform version, and we show that certain quantities appearing in the limit are continuous functions.

Section 4.1 discusses a variety of issues that naturally arise when trying to apply the limit theorems to concrete operators. In particular we will discuss the case of finitely generated groups and explain the relationship between the compatibility condition and diophantine properties. We will also establish a generalization of the block Szegő-Widom limit theorem dealing with “non-standard” finite sections of Laurent operators. Finally a conjecture concerning an inverse closedness problem is raised in Section 4.2, and the applicability of our results to almost Mathieu operators is discussed.

Chapter 2

Preliminaries

2.1 Banach algebras of almost periodic sequences

A Banach subalgebra \mathcal{A} of $l^\infty(\mathbb{Z})$ is called *shift-invariant* if for each $a \in \mathcal{A}$ and $n \in \mathbb{Z}$ we have $U_n a \in \mathcal{A}$ and

$$\|U_n a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}.$$

Let \mathcal{A} be a shift-invariant Banach subalgebra of $l^\infty(\mathbb{Z})$. A sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is called *fractal for \mathcal{A}* if for each $a \in \mathcal{A}$ there exists an element $Ua \in \mathcal{A}$ such that

$$\lim_{n \rightarrow \infty} \|U_{-h(n)} a - Ua\|_{\mathcal{A}} = 0. \quad (2.1.1)$$

It is easy to see that then the map $U : \mathcal{A} \rightarrow \mathcal{A}$ is an isometric Banach algebra homomorphism. Indeed, note that U is multiplicative because each U_n acts multiplicatively on \mathcal{A} . If \mathcal{A} contains the unit element of $l^\infty(\mathbb{Z})$, then U is unital. As the sequence h determines uniquely the map U by (2.1.1), we will also say that h is *fractal for \mathcal{A} with associated U* .

In what follows we are going to describe shift-invariant Banach subalgebras \mathcal{A} of $AP(\mathbb{Z}) \subseteq l^\infty(\mathbb{Z})$ which are characterized by an additive subgroup of \mathbb{R}/\mathbb{Z} . For this purpose we extend our conceptual framework with the following definitions.

For an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} denote by $\text{Hom}(\Xi, \mathbb{T})$ the set of all group homomorphisms $\tau : \Xi \rightarrow \mathbb{T}$. Recall that $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ is the (multiplicative) circle group. Then $\text{Hom}(\Xi, \mathbb{T})$ is a compact abelian group with group multiplication defined by $(\tau_1\tau_2)(\xi) = \tau_1(\xi)\tau_2(\xi)$. The topology of $\text{Hom}(\Xi, \mathbb{T})$ arises from the local bases at τ given by the collection of all neighborhoods

$$U_{\xi_1, \dots, \xi_N; \varepsilon}[\tau] = \left\{ \tau' \in \text{Hom}(\Xi, \mathbb{T}) : |\tau'(\xi_k) - \tau(\xi_k)| < \varepsilon \text{ for all } 1 \leq k \leq N \right\} \quad (2.1.2)$$

with $\varepsilon > 0$, $N \in \mathbb{N}$, $\xi_1, \dots, \xi_N \in \Xi$. Note that if we consider Ξ with discrete topology, then $\text{Hom}(\Xi, \mathbb{T})$ is the dual group of Ξ .

Given an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} , we say that a sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is *fractal for Ξ* if for each $\xi \in \Xi$ the limit

$$\tau(\xi) := \lim_{n \rightarrow \infty} e^{2\pi i h(n)\xi} \quad (2.1.3)$$

exists. Obviously, in this case, $\tau \in \text{Hom}(\Xi, \mathbb{T})$. Therefore we will say that the sequence h is *fractal for Ξ with associated $\tau \in \text{Hom}(\Xi, \mathbb{T})$* .

We can give another interpretation of (2.1.3). To each $n \in \mathbb{Z}$ there exists a naturally associated $\tau_n \in \text{Hom}(\Xi, \mathbb{T})$ defined by $\tau_n(\xi) = e^{2\pi i n \xi}$. Using this notation, h being fractal for Ξ is equivalent to saying that $\tau_{h(n)} \rightarrow \tau$ in the topology of $\text{Hom}(\Xi, \mathbb{T})$ as $n \rightarrow \infty$.

Proposition 2.1.1. *Let Ξ be an additive subgroup of \mathbb{R}/\mathbb{Z} , and let \mathcal{A} be a shift-*

invariant Banach subalgebra of $AP(\mathbb{Z})$ such that the linear span of

$$\{e_\xi : \xi \in \Xi\}$$

is contained and dense in \mathcal{A} . Let $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$.

- (a) The sequence h is fractal for \mathcal{A} if and only if it is fractal for Ξ .
- (b) If this is true, and if h is fractal for Ξ with associated $\tau \in \text{Hom}(\Xi, \mathbb{T})$, then h is fractal for \mathcal{A} with the associated U given by

$$U : \mathcal{A} \rightarrow \mathcal{A}, \quad \sum_{\xi} a_{\xi} e_{\xi} \mapsto \sum_{\xi} a_{\xi} \tau(\xi) e_{\xi}. \quad (2.1.4)$$

Therein, the operation is defined for finite linear combinations and extends by continuity to all of \mathcal{A} .

Proof. Let us first show that if h is fractal for \mathcal{A} with associated U , then it is fractal for Ξ . Indeed, for each $\xi \in \Xi$,

$$\|U_{-h(n)}e_{\xi} - Ue_{\xi}\|_{\mathcal{A}} = \|e^{2\pi ih(n)\xi}e_{\xi} - Ue_{\xi}\|_{\mathcal{A}} \rightarrow 0$$

as $n \rightarrow \infty$. This implies that $e^{2\pi ih(n)\xi}e_{\xi}$ is a Cauchy sequence in \mathcal{A} , and therefore so is the scalar sequence $e^{2\pi ih(n)\xi}$. Hence the limit (2.1.3) exists. In fact, we have $Ue_{\xi} = \tau(\xi)e_{\xi}$.

Now let us assume that h is fractal for Ξ with associated $\tau \in \text{Hom}(\Xi, \mathbb{T})$. We are going to show that h is fractal for \mathcal{A} with associated U given above. First consider those $a \in \mathcal{A}$ which can be written as a finite linear combination $a = \sum_{\xi} a_{\xi} e_{\xi}$. Since $U_{-h(n)}e_{\xi} = e^{2\pi ih(n)\xi}e_{\xi}$ it follows that

$$U_{-h(n)}a = \sum_{\xi} a_{\xi} e^{2\pi ih(n)\xi} e_{\xi} \rightarrow \sum_{\xi} a_{\xi} \tau(\xi) e_{\xi} =: Ua,$$

as $n \rightarrow \infty$, where the convergence is in the norm of \mathcal{A} . Considering $U_{-h(n)}$ as a bounded linear operator on the Banach space \mathcal{A} we thus have strong convergence on the dense subset of \mathcal{A} . Because of shift-invariance, i.e., $\|U_{-h(n)}a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$, the operator norm of $U_{-h(n)}$ equals one. A standard approximation argument implies that we have strong convergence of $U_{-h(n)}$ on all of \mathcal{A} , and that the operator U (already defined on a dense subset) extends by continuity to all of \mathcal{A} . In fact, we have $\|Ua\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$. \square

Let us now proceed with describing a concrete class of shift-invariant Banach algebras \mathcal{A} to which the previous proposition can be applied. They arise from any additive subgroup Ξ of \mathbb{R}/\mathbb{Z} and can be considered as a weighted Wiener-type algebra of almost periodic sequences. We call a mapping $\beta : \Xi \rightarrow \mathbb{R}^+$ an *admissible weight on Ξ* if

$$1 \leq \beta(\xi_1 + \xi_2) \leq \beta(\xi_1)\beta(\xi_2) \tag{2.1.5}$$

for each $\xi_1, \xi_2 \in \Xi$. For such Ξ and β let $APW(\mathbb{Z}, \Xi, \beta)$ be the set of all sequences $a \in l^\infty(\mathbb{Z})$ of the form

$$a = \sum_{\xi \in \Xi} a_\xi e_\xi \tag{2.1.6}$$

for which

$$\|a\|_{\Xi, \beta} := \sum_{\xi \in \Xi} \beta(\xi) |a_\xi| < \infty. \tag{2.1.7}$$

As usual, in the case when Ξ is uncountable it is agreed that at most countably many of the a_ξ 's are nonzero, and only over those the sum is taken.

The following result is almost obvious and was proved in [14, Thm. 2.6]. It implies that Proposition 2.1.1 can be applied to $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$. In addition, it is easy to see that the map U is given by formula (2.1.4) for all $a \in APW(\mathbb{Z}, \Xi, \beta)$.

Proposition 2.1.2. *Let β be an admissible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . Then $APW(\mathbb{Z}, \Xi, \beta)$ is a shift-invariant, continuously embedded Banach subalgebra of $AP(\mathbb{Z})$, and the linear span of $\{e_\xi : \xi \in \Xi\}$ is a dense subset.*

We need one more property about weights. A weight β is said to be *compatible* on Ξ if

$$C_\beta := \inf_{\xi \in \Xi, \xi \neq 0} \beta(\xi) \cdot \|\xi\|_{\mathbb{R}/\mathbb{Z}} > 0. \quad (2.1.8)$$

Therein, we use the natural metric on \mathbb{R}/\mathbb{Z} given by

$$\|\xi\|_{\mathbb{R}/\mathbb{Z}} = \inf\{|\xi - n| : n \in \mathbb{Z}\}. \quad (2.1.9)$$

The next theorem gives us information about the asymptotics of the trace of the finite sections $P_{h_1(n), h_2(n)}(aI)P_{h_1(n), h_2(n)}$ under certain conditions (see also (1.2.1)).

Theorem 2.1.3. *Let β be an admissible and compatible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . If $h_1, h_2 : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ are two fractal sequences for Ξ with associated $\tau_1, \tau_2 \in \text{Hom}(\Xi, \mathbb{T})$, respectively, then for each $a \in APW(\mathbb{Z}, \Xi, \beta)$ we have*

$$\sum_{k=h_1(n)}^{h_2(n)-1} a(k) = (h_2(n) - h_1(n)) \cdot M(a) + F_a(\tau_1) - F_a(\tau_2) + o(1) \quad (2.1.10)$$

as $n \rightarrow \infty$, where

$$F_a(\tau) := \sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{\tau(\xi)}{1 - e^{2\pi i \xi}} \quad (2.1.11)$$

is a well defined constant depending on a and $\tau \in \text{Hom}(\Xi, \mathbb{T})$.

Proof. First note that $F_a(\tau)$ is well-defined for each $\tau \in \text{Hom}(\Xi, \mathbb{T})$ with a given by (2.1.6). Indeed, the corresponding series (2.1.11) is absolutely convergent since

β is compatible, since (2.1.7) is finite, and since

$$\frac{1}{|1 - e^{2\pi i \xi}|} = \frac{1}{2|\sin(\pi \xi)|} \leq \frac{1}{4\|\xi\|_{\mathbb{R}/\mathbb{Z}}} \leq \frac{\beta(\xi)}{4C_\beta}, \quad \xi \in \Xi, \xi \neq 0. \quad (2.1.12)$$

Recall that we identify $\xi \in \mathbb{R}/\mathbb{Z}$ with any real number representing it.

Now let $a \in APW(\mathbb{Z}, \Xi, \beta)$ be of the form (2.1.6). Since

$$\sum_{k=h_1(n)}^{h_2(n)-1} a(k) = \sum_{\xi \in \Xi} a_\xi \left(\sum_{k=h_1(n)}^{h_2(n)-1} e_\xi(k) \right),$$

$M(a) = a_0$ and $e_\xi(k) = e^{2\pi i k \xi}$, we obtain that

$$\begin{aligned} \sum_{k=h_1(n)}^{h_2(n)-1} a(k) - (h_2(n) - h_1(n)) \cdot M(a) &= \sum_{\xi \in \Xi, \xi \neq 0} a_\xi \left(\sum_{k=h_1(n)}^{h_2(n)-1} e_\xi(k) \right) \\ &= \sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{e^{2\pi i h_1(n)\xi} - e^{2\pi i h_2(n)\xi}}{1 - e^{2\pi i \xi}}. \end{aligned}$$

Since h_1 and h_2 are fractal for Ξ with associated τ_1 and τ_2 we have that for each fixed $\xi \in \Xi$, $\xi \neq 0$,

$$\frac{e^{2\pi i h_1(n)\xi} - e^{2\pi i h_2(n)\xi}}{1 - e^{2\pi i \xi}} \rightarrow \frac{\tau_1(\xi) - \tau_2(\xi)}{1 - e^{2\pi i \xi}}$$

as $n \rightarrow \infty$ by (2.1.3). Now use the estimate (2.1.12) together with a dominated convergence argument and the finiteness of (2.1.7) to see that

$$\sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{e^{2\pi i h_1(n)\xi} - e^{2\pi i h_2(n)\xi}}{1 - e^{2\pi i \xi}} \rightarrow \sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{\tau_1(\xi) - \tau_2(\xi)}{1 - e^{2\pi i \xi}} = F_a(\tau_1) - F_a(\tau_2)$$

as $n \rightarrow \infty$. This concludes the proof of (2.1.10). \square

One can consider F_a as a function in $\tau \in \text{Hom}(\Xi, \mathbb{T})$. In fact, we will show later in Proposition 3.2.1 that F_a is continuous on $\text{Hom}(\Xi, \mathbb{T})$ for each fixed a .

2.2 Banach algebras of operators on $l^2(\mathbb{Z})$

In this section we are going to characterize certain classes of Banach algebras of operators on $l^2(\mathbb{Z})$ for which we will prove an abstract version of a Szegő-Widom type limit theorem in Section 3.1. The proof of this limit theorem is based on a “Banach algebra approach”, which was introduced in [11] and used also in [14]. The main goal of the approach is to reduce the asymptotics of determinants to the asymptotics of traces of certain operators.

The operators A that belong to these classes of Banach algebras can be considered as “symbols” for the corresponding finite sections $A_{n_1, n_2} = P_{n_1, n_2} A P_{n_1, n_2}$. Moreover, these operators also serve as “symbols” for corresponding compression operators on $l^2(\mathbb{Z}^+)$, quite analogous to the classical case of symbols of Toeplitz operators.

The main property that such Banach algebras have to possess is *suitability* (in the sense of [11] or [14]). In addition, the notions of *shift-invariance* and *rigidity* are needed, and the notion of *fractal sequences* will occur again.

Let $\mathcal{L}(H)$ denote the Banach algebra of all bounded linear operators on a Hilbert space H . Furthermore, let P, J stand for the following operators on $l^2(\mathbb{Z})$,

$$P : (x_n)_{n \in \mathbb{Z}} \mapsto (y_n)_{n \in \mathbb{Z}} \text{ with } y_n = \begin{cases} x_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases}$$

$$J : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{-n-1})_{n \in \mathbb{Z}}.$$

For each operator $A \in \mathcal{L}(l^2(\mathbb{Z}))$, define

$$T(A) := PAP, \quad H(A) := PAJP, \quad \tilde{A} := JAJ.$$

Identifying the image of P with $l^2(\mathbb{Z}^+)$, we will consider $T(A)$ and $H(A)$ as operators acting on $l^2(\mathbb{Z}^+)$. The notation above is inspired by the classical notation for Toeplitz and Hankel operators. In fact, the following simple, but important identities hold for any $A, B \in \mathcal{L}(l^2(\mathbb{Z}))$,

$$T(AB) = T(A)T(B) + H(A)H(\tilde{B}), \quad (2.2.1)$$

$$H(AB) = T(A)H(B) + H(A)T(\tilde{B}), \quad (2.2.2)$$

generalizing the classical identities for Toeplitz and Hankel operators.

Rigidity and suitability. A set \mathcal{R} of bounded linear operators on $l^2(\mathbb{Z})$ is called *rigid* if for each $A \in \mathcal{R}$ the following statement holds:

If $T(A)$ or $T(\tilde{A})$ is compact, then $A = 0$.

The notion is modelled after the corresponding property for Toeplitz operators. It was proved in [14, Thm. 3.1] that the class \mathcal{OAP} is rigid.

A Banach algebra \mathcal{R} of bounded linear operators on $l^2(\mathbb{Z})$ will be called *suitable* if the following conditions hold:

- (a) \mathcal{R} is continuously embedded into $\mathcal{L}(l^2(\mathbb{Z}))$ and

$$\|A\|_{\mathcal{L}(l^2(\mathbb{Z}))} \leq \|A\|_{\mathcal{R}} \text{ for all } A \in \mathcal{R}.$$

- (b) For all $A, B \in \mathcal{R}$, the operators $H(A)H(\tilde{B})$ and $H(\tilde{A})H(B)$ are trace class, and there exists $M > 0$ such that

$$\max\{\|H(A)H(\tilde{B})\|_{c_1}, \|H(\tilde{A})H(B)\|_{c_1}\} \leq M\|A\|_{\mathcal{R}}\|B\|_{\mathcal{R}} \text{ for all } A, B \in \mathcal{R}. \quad (2.2.3)$$

Therein, for $1 \leq p < \infty$, let $\mathcal{C}_p(H)$ stand for the Schatten-von Neumann class of operators on a Hilbert space H , i.e., the set of all (compact) operators $A \in \mathcal{L}(H)$ for which

$$\|A\|_{\mathcal{C}_p} := \left(\sum_{n \geq 0} s_n(A)^p \right)^{1/p} < \infty,$$

where $s_n(A)$ refers to the n -th singular value of A . The operators belonging to \mathcal{C}_1 are called trace class operators. We refer to [16] for more information about these concepts.

For sake of illustration, let us present an example of a class of suitable Banach algebras (see also [14, Ex. 3.2 and Prop. 3.3]).

Example 2.2.1. For $p, q \geq 1$, define

$$\mathcal{R}_{p,q} := \left\{ A \in \mathcal{L}(l^2(\mathbb{Z})) : H(A) \in \mathcal{C}_p(l^2(\mathbb{Z}^+)), H(\tilde{A}) \in \mathcal{C}_q(l^2(\mathbb{Z}^+)) \right\}$$

along with a norm

$$\|A\|_{\mathcal{R}_{p,q}} := \|A\|_{\mathcal{L}(l^2(\mathbb{Z}))} + \|H(A)\|_{\mathcal{C}_p} + \|H(\tilde{A})\|_{\mathcal{C}_q}.$$

With the above norm, $\mathcal{R}_{p,q}$ is a Banach algebra, and it is suitable if, in addition, $1/p + 1/q = 1$.

Shift-invariance and fractal sequences. Finally, let us define the notions of shift-invariance and fractal sequences for Banach algebras of operators on $l^2(\mathbb{Z})$. A Banach algebra \mathcal{R} of bounded linear operators on $l^2(\mathbb{Z})$ is said to be *shift-invariant* if

$$U_{-n}AU_n \in \mathcal{R} \quad \text{and} \quad \|U_{-n}AU_n\|_{\mathcal{R}} = \|A\|_{\mathcal{R}} \quad (2.2.4)$$

for each $A \in \mathcal{R}$ and $n \in \mathbb{Z}$. Occasionally, we will also use the notation

$$\mathcal{U}^n : A \in \mathcal{R} \mapsto U_{-n}AU_n \in \mathcal{R}, \quad (2.2.5)$$

noting that \mathcal{U}^n is an isometric Banach algebra isomorphism on \mathcal{R} .

Assuming \mathcal{R} to be shift-invariant, we call a sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ to be *fractal* for \mathcal{R} if for each $A \in \mathcal{R}$ there exists an element $\mathcal{U}A \in \mathcal{R}$ such that

$$\lim_{n \rightarrow \infty} \|U_{-h(n)}AU_{h(n)} - \mathcal{U}A\|_{\mathcal{R}} = 0. \quad (2.2.6)$$

It is easy to see that in this case $\mathcal{U} : \mathcal{R} \rightarrow \mathcal{R}$ is an isometric Banach algebras homomorphism, which is unital if \mathcal{R} is unital. We will also say that the sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is *fractal with associated Banach algebra homomorphism \mathcal{U}* on \mathcal{R} .

The Banach algebras $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$. Let \mathcal{A} be a shift-invariant and continuously embedded Banach subalgebra of $l^\infty(\mathbb{Z})$. We are going to introduce corresponding Banach algebras $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ of operators on $l^2(\mathbb{Z})$ as follows. For given $\alpha_1, \alpha_2 \geq 0$, define the weight function α on \mathbb{Z} by

$$\alpha(k) = \begin{cases} (1+k)^{\alpha_1} & \text{if } k \geq 0, \\ (1+|k|)^{\alpha_2} & \text{if } k < 0. \end{cases} \quad (2.2.7)$$

Now let $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ refer to the set of all operators $A \in \mathcal{L}(l^2(\mathbb{Z}))$ for which $D_k(A) \in \mathcal{A}$ for each $k \in \mathbb{Z}$ and

$$\|A\|_{\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})} := \sum_{k \in \mathbb{Z}} \alpha(k) \|D_k(A)\|_{\mathcal{A}} < \infty.$$

Recalling (1.1.12) note that $D_k(A)$ stands for the k -th diagonal of A , a sequence

in $l^\infty(\mathbb{Z})$. It is easy to see that $A \in \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ if and only if it can be written as

$$A = \sum_{k \in \mathbb{Z}} (a^{(k)} I) U_k \quad (2.2.8)$$

with $a^{(k)} \in \mathcal{A}$ ($k \in \mathbb{Z}$) such that

$$\|A\|_{\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})} = \sum_{k \in \mathbb{Z}} \alpha(k) \|a^{(k)}\|_{\mathcal{A}} < \infty.$$

Note that (2.2.8) converges absolutely both in the norm of $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ and in the operator norm. Clearly, $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ is a Banach space. In fact, we have the following results.

Proposition 2.2.2. *Let $\alpha_1, \alpha_2 \geq 0$, and let \mathcal{A} be a shift-invariant and continuously embedded Banach subalgebra of $l^\infty(\mathbb{Z})$. Then $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ is a shift-invariant and continuously embedded Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$. Moreover,*

(a) *if $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and if $\|a\|_{l^\infty(\mathbb{Z})} \leq \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$, then*

$\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ is suitable,

(b) *if $\mathcal{A} \subseteq AP(\mathbb{Z})$, then $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ is rigid.*

For a detailed proof we refer to [14, Thm. 3.5]. The cases of $\alpha_1 = 0$ or $\alpha_2 = 0$ in (a) were not explicitly stated there, but the proof proceeds along the same lines. To elaborate a little bit more, let us remark that the proof of (a) is based on the fact that $\mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ is continuously embedded in $\mathcal{R}_{p, q}$ as defined in Example 2.2.1 with $p = 1/\alpha_2$ and $q = 1/\alpha_1$. Part (b) follows from the fact that the class \mathcal{OAP} is rigid. This was proved in [14, Thm. 3.1] and uses the property that for each $a \in AP(\mathbb{Z})$ there exists a strictly increasing integer sequence $k : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ such that $U_{-k(n)}a \rightarrow a$ in $l^\infty(\mathbb{Z})$ as $n \rightarrow \infty$.

Concerning the notion of fractal sequences the following result holds.

Proposition 2.2.3. *Let $\alpha_1, \alpha_2 \geq 0$, and let \mathcal{A} be a shift-invariant Banach subalgebra of $l^\infty(\mathbb{Z})$. Then*

(a) *a sequence $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is fractal for $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ if and only if h is fractal for \mathcal{A} .*

(b) *If this is true, and if h is fractal for \mathcal{A} with associated $U : \mathcal{A} \rightarrow \mathcal{A}$, then h is fractal for \mathcal{R} with the associated \mathcal{U} given by*

$$\mathcal{U} : \sum_{k \in \mathbb{Z}} (a^{(k)} I) U_k \mapsto \sum_{k \in \mathbb{Z}} (U(a^{(k)}) I) U_k. \quad (2.2.9)$$

Therein, the operator is defined for finite linear combinations and extends by continuity to all of \mathcal{R} .

Proof. To show the ‘only if’ part of (a), assume that h is fractal for \mathcal{R} with associated \mathcal{U} . Considering the special case $A = aI$ with $a \in \mathcal{A}$ we have

$$\|U_{-h(n)} A U_{h(n)} - \mathcal{U} A\|_{\mathcal{R}} = \|(U_{-h(n)} a) I - \mathcal{U}(aI)\|_{\mathcal{R}} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, $\mathcal{U}(aI) = U(a)I$ for some $U(a) \in \mathcal{A}$, and $\|U_{-h(n)} a - U(a)\|_{\mathcal{A}} \rightarrow 0$ as $n \rightarrow \infty$. Hence, if we identify \mathcal{A} with $\{aI : a \in \mathcal{A}\}$, then $U : \mathcal{A} \rightarrow \mathcal{A}$ is the restriction map of \mathcal{U} onto \mathcal{A} and (2.1.1) is satisfied. Therefore, h is fractal for \mathcal{A} .

Let us now show (b) and the ‘if’ part of (a). Assume that h is fractal for \mathcal{A} with associated U . We are going to show that h is fractal for \mathcal{R} with the associated \mathcal{U} given above. Firstly consider those $A \in \mathcal{R}$ which can be written as a finite linear combination $A = \sum_k (a^{(k)} I) U_k$, $a^{(k)} \in \mathcal{A}$. Observing that

$$U_{-h(n)} A U_{h(n)} = \sum_k ((U_{-h(n)} a^{(k)}) I) U_k \rightarrow \sum_k (U(a^{(k)}) I) U_k = \mathcal{U} A$$

as $n \rightarrow \infty$, where the convergence is in the norm of \mathcal{R} . In other words, we have strong convergence of the bounded linear operators $\mathcal{U}^{h(n)}$ defined by $A \in \mathcal{R} \mapsto U_{-h(n)}AU_{h(n)} \in \mathcal{R}$ on the dense subset of \mathcal{R} . In addition, the norm of $U^{h(n)}$ equals one because of shift-invariance, and we have strong convergence of $\mathcal{U}^{h(n)}$ to \mathcal{U} on all of \mathcal{R} followed by a standard approximation argument. It follows that the operator \mathcal{U} extends by continuity to all of \mathcal{R} and $\|\mathcal{U}A\|_{\mathcal{R}} = \|A\|_{\mathcal{R}}$ for all $A \in \mathcal{R}$. \square

Besides establishing an abstract version of a Szegő-Widom limit theorem in Section 3.1, we will specialize it to the Banach algebras $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$, where Ξ and β have to satisfy certain conditions. Apart from Theorem 2.1.3, we will only need the following summary of statements resulting from Propositions 2.1.1, 2.1.2, 2.2.2, and 2.2.3.

Corollary 2.2.4. *Let β be an admissible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} , and let $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$.*

(i) $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$ is a rigid, suitable, shift-invariant and unital Banach algebra of bounded linear operators on $l^2(\mathbb{Z})$.

(ii) If $h : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ is a fractal sequence for Ξ with associated $\tau \in \text{Hom}(\Xi, \mathbb{T})$, then h is fractal for \mathcal{R} with associated \mathcal{U} given by

$$\mathcal{U} : \sum_{n \in \mathbb{Z}} \sum_{\xi \in \Xi} a_{\xi}^{(n)}(e_{\xi}I)U_n \mapsto \sum_{n \in \mathbb{Z}} \sum_{\xi \in \Xi} a_{\xi}^{(n)}\tau(\xi)(e_{\xi}I)U_n. \quad (2.2.10)$$

2.3 Banach algebras associated with suitable Banach algebras

In this section, we will continue to prepare the proof of the abstract version of the limit theorem in the next section. We construct three types of Banach

algebras which are naturally associated with any suitable, rigid, shift-invariant and unital Banach algebra \mathcal{R} .

2.3.1 Banach algebras of operators on $l^2(\mathbb{Z}^+)$

The first two are the Banach algebras $\mathcal{O}(\mathcal{R})$ and $\mathcal{O}(\tilde{\mathcal{R}})$. They were already considered in [14, Sect. 4.1], and it suffices to cite the corresponding result from there. Recall that $\tilde{A} = JAJ$.

Proposition 2.3.1. *Let \mathcal{R} be a rigid, suitable and unital Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$.*

(a) *The set*

$$\mathcal{O}(\mathcal{R}) := \{T(A) + K : A \in \mathcal{R}, K \in \mathcal{C}_1(l^2(\mathbb{Z}^+))\} \quad (2.3.1)$$

is a unital Banach algebra with the norm

$$\|T(A) + K\|_{\mathcal{O}(\mathcal{R})} := \|A\|_{\mathcal{R}} + \|K\|_{\mathcal{C}_1}. \quad (2.3.2)$$

(b) *The set*

$$\mathcal{O}(\tilde{\mathcal{R}}) := \{T(\tilde{A}) + K : A \in \mathcal{R}, K \in \mathcal{C}_1(l^2(\mathbb{Z}^+))\} \quad (2.3.3)$$

is a unital Banach algebra with the norm

$$\|T(\tilde{A}) + K\|_{\mathcal{O}(\tilde{\mathcal{R}})} := \|A\|_{\mathcal{R}} + \|K\|_{\mathcal{C}_1}. \quad (2.3.4)$$

We also need the following two results, which were established in [14, Sect. 3.3].

Proposition 2.3.2. *Let \mathcal{R} be a suitable and unital Banach algebra, and let*

$A_1, \dots, A_r \in \mathcal{R}$. Then the functions

$$F_0(\lambda_1, \dots, \lambda_r) := T(e^{\lambda_1 A_1} \dots e^{\lambda_r A_r}) e^{-\lambda_r T(A_r)} \dots e^{-\lambda_1 T(A_1)} - P,$$

$$F_1(\lambda_1, \dots, \lambda_r) := T(e^{\lambda_1 \widetilde{A}_1} \dots e^{\lambda_r \widetilde{A}_r}) e^{-\lambda_r T(\widetilde{A}_r)} \dots e^{-\lambda_1 T(\widetilde{A}_1)} - P$$

are analytic with respect to each variable $\lambda_k \in \mathbb{C}$ and take values in $\mathcal{C}_1(l^2(\mathbb{Z}^+))$.

As a consequence, the operator determinants

$$\det T(e^{\lambda_1 A_1} \dots e^{\lambda_r A_r}) e^{-\lambda_r T(A_r)} \dots e^{-\lambda_1 T(A_1)},$$

and

$$\det T(e^{\lambda_1 \widetilde{A}_1} \dots e^{\lambda_r \widetilde{A}_r}) e^{-\lambda_r T(\widetilde{A}_r)} \dots e^{-\lambda_1 T(\widetilde{A}_1)}$$

are well-defined and depend analytically on each of the complex variables λ_k . For the definition and basic properties of operator determinants, see, e.g., [16]. As it turns out, these two kinds of operator determinants are related to each other.

Proposition 2.3.3. *Let \mathcal{R} be a suitable and unital Banach algebra, and let $A_1, \dots, A_r \in \mathcal{R}$. Then for each $\lambda_1, \dots, \lambda_r \in \mathbb{C}$, the operator determinant*

$$f(\lambda_1, \dots, \lambda_r) := \det T(e^{\lambda_1 \widetilde{A}_1} \dots e^{\lambda_r \widetilde{A}_r}) e^{-\lambda_r T(\widetilde{A}_r)} \dots e^{-\lambda_1 T(\widetilde{A}_1)}$$

is equal to the operator determinant

$$g(\lambda_1, \dots, \lambda_r) := \det e^{\lambda_1 T(A_1)} \dots e^{\lambda_r T(A_r)} T(e^{-\lambda_r A_r} \dots e^{-\lambda_1 A_1}).$$

2.3.2 Banach algebras of sequences of finite sections

The third kind of Banach algebras associated with \mathcal{R} are Banach algebras $\mathcal{S}_{h_1, h_2}(\mathcal{R})$ of sequences of matrices. In contrast to [14, Sect. 4.2] we need to consider a more general situation, which involves fractal sequences h_1 and h_2 for \mathcal{R} (instead of distinguished sequences) and is motivated by the following.

Let $h_1, h_2 : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ be two fractal sequences for \mathcal{R} such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, for $A \in \mathcal{R} \subseteq \mathcal{L}(l^2(\mathbb{Z}))$,

$$P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)} = U_{h_1(n)} P_{h(n)} P(U_{-h_1(n)} A U_{h_1(n)}) P P_{h(n)} U_{-h_1(n)},$$

i.e., the sequence of finite sections obtained through $P_{h_1(n), h_2(n)}$ can be identified with a sequence of $h(n) \times h(n)$ matrices. Using the notation defined in (2.2.5) we obtain

$$\det P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)} = \det P_{h(n)} T(\mathcal{U}^{h_1(n)} A) P_{h(n)}, \quad (2.3.5)$$

$$\text{trace } P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)} = \text{trace } P_{h(n)} T(\mathcal{U}^{h_1(n)} A) P_{h(n)}. \quad (2.3.6)$$

Therefore, as far as traces and determinants are concerned, instead of the finite sections $P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)}$ it suffices to consider $P_{h(n)} T(\mathcal{U}^{h_1(n)} A) P_{h(n)}$. In the following theorem we define the Banach algebra $\mathcal{S}_{h_1, h_2}(\mathcal{R})$, the elements of which include such sequences.

We also need the reflection operator $W_n : l^2(\mathbb{Z}^+) \rightarrow l^2(\mathbb{Z}^+)$, $n \in \mathbb{Z}^+$, defined by

$$W_n : (x_k)_{k \in \mathbb{Z}^+} \mapsto (y_k)_{k \in \mathbb{Z}^+}, \quad y_k = \begin{cases} x_{n-1-k} & \text{if } 0 \leq k < n \\ 0 & \text{if } k \geq n. \end{cases} \quad (2.3.7)$$

Note that $W_n^2 = P_n$ and $\text{im } W_n = \text{im } P_n$.

Theorem 2.3.4. *Let \mathcal{R} be a rigid, suitable, shift-invariant and unital Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$, and let h_1 and h_2 be fractal sequences for \mathcal{R} with associated \mathcal{U}_1 and \mathcal{U}_2 , respectively, such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

Then the set $\mathcal{S}_{h_1, h_2}(\mathcal{R})$ consisting of all sequences $(A_n)_{n \geq 1}$ of operators $A_n : \text{im}P_{h(n)} \rightarrow \text{im}P_{h(n)}$ of the form

$$A_n = P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + G_n \quad (2.3.8)$$

with $A \in \mathcal{R}$, $K, L \in \mathcal{C}_1(l^2(\mathbb{Z}^+))$, $G_n \in \mathcal{C}_1(\text{im}P_{h(n)})$ and $\|G_n\|_{\mathcal{C}_1} \rightarrow 0$ forms a unital Banach algebra with respect to the operations

$$(A_n) + (B_n) = (A_n + B_n), \quad (A_n) \cdot (B_n) = (A_n B_n), \quad \lambda(A_n) = (\lambda A_n)$$

and with the norm

$$\|(A_n)_{n \geq 1}\|_{\mathcal{S}_{h_1, h_2}(\mathcal{R})} := \|A\|_{\mathcal{R}} + \|K\|_{\mathcal{C}_1} + \|L\|_{\mathcal{C}_1} + \sup_{n \geq 1} \|G_n\|_{\mathcal{C}_1}. \quad (2.3.9)$$

Moreover, the set $\mathcal{J}_h(\mathcal{R})$ of all sequences (J_n) of the form

$$J_n = P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + G_n$$

with $K, L \in \mathcal{C}_1(l^2(\mathbb{Z}^+))$, $G_n \in \mathcal{C}_1(\text{im}P_{h(n)})$ and $\|G_n\|_{\mathcal{C}_1} \rightarrow 0$ forms a closed two-sided ideal of $\mathcal{S}_{h_1, h_2}(\mathcal{R})$.

Proof. Since h_1 and h_2 are fractal sequences for \mathcal{R} with associated isometric Banach algebra homomorphisms \mathcal{U}_1 and \mathcal{U}_2 on \mathcal{R} , we have for any $A \in \mathcal{R}$ that

$$\lim_{n \rightarrow \infty} \|\mathcal{U}^{h_1(n)}A - \mathcal{U}_1A\|_{\mathcal{R}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\mathcal{U}^{h_2(n)}A - \mathcal{U}_2A\|_{\mathcal{R}} = 0. \quad (2.3.10)$$

Note that this implies, in particular, convergence in the operator norm.

Let us first show that the norm (2.3.9) is well-defined. For this we need to show that the various terms on the right hand side of (2.3.8) are uniquely determined by the sequence (A_n) . Therefore, consider a sequence

$$P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + G_n = 0.$$

Since $W_{h(n)} \rightarrow 0$ weakly, taking the strong limit as $n \rightarrow \infty$ yields that $T(\mathcal{U}_1A) + K = 0$. Because of rigidity we have $K = 0$ and $\mathcal{U}_1A = 0$, which implies that $A = 0$ as well. It follows that

$$W_{h(n)}LW_{h(n)} + G_n = 0.$$

Multiplying with $W_{h(n)}$ from both sides and noting that $W_{h(n)}^2 = P_{h(n)}$, we obtain $L = 0$ by taking again strong limits. Thus $G_n = 0$ as well, and this shows that the norm is well-defined.

It is obvious that $\mathcal{S}_{h_1, h_2}(\mathcal{R})$ is a linear space, which is complete with respect to its norm. It remains to show that it is indeed a Banach algebra. Consider

$$A_n = P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} + P_{h(n)}K_1P_{h(n)} + W_{h(n)}L_1W_{h(n)} + G_n^{(1)}, \quad (2.3.11)$$

$$B_n = P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} + P_{h(n)}K_2P_{h(n)} + W_{h(n)}L_2W_{h(n)} + G_n^{(2)}. \quad (2.3.12)$$

We have to show that $(A_nB_n) \in \mathcal{S}_{h_1, h_2}(\mathcal{R})$ and

$$\|(A_nB_n)\|_{\mathcal{S}_{h_1, h_2}(\mathcal{R})} \leq C \|(A_n)\|_{\mathcal{S}_{h_1, h_2}(\mathcal{R})} \|(B_n)\|_{\mathcal{S}_{h_1, h_2}(\mathcal{R})}$$

for some constant C . For that purpose, we multiply each term in the first sum with each term in the second sum and are led to consider several cases. Note that if one of the factors is $G_n^{(1)}$ or $G_n^{(2)}$, then the product can easily be taken care of.

The remaining products are dealt with as follows. Firstly,

$$\begin{aligned}
P_{h(n)}K_1P_{h(n)} \cdot P_{h(n)}K_2P_{h(n)} &= P_{h(n)}K_1K_2P_{h(n)} - P_{h(n)}K_1Q_{h(n)}K_2P_{h(n)}, \\
W_{h(n)}L_1W_{h(n)} \cdot W_{h(n)}L_2W_{h(n)} &= W_{h(n)}L_1L_2W_{h(n)} - W_{h(n)}L_1Q_{h(n)}L_2W_{h(n)}, \\
P_{h(n)}K_1P_{h(n)} \cdot W_{h(n)}L_2W_{h(n)} &= P_{h(n)}K_1W_{h(n)}L_2W_{h(n)}, \\
W_{h(n)}L_1W_{h(n)} \cdot P_{h(n)}K_2P_{h(n)} &= W_{h(n)}L_1W_{h(n)}K_2P_{h(n)}.
\end{aligned}$$

Since $Q_{h(n)} := P - P_{h(n)}$ converges to zero strongly, and $W_{h(n)}$ converges to zero weakly on $l^2(\mathbb{Z}^+)$, the last term in the first two equations as well as the terms in the third and fourth equation converge to zero in the trace norm.

Now consider the following product and write it as

$$\begin{aligned}
&P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} \cdot P_{h(n)}K_2P_{h(n)} \\
&= P_{h(n)}T(\mathcal{U}^{h_1(n)}A)K_2P_{h(n)} - P_{h(n)}T(\mathcal{U}^{h_1(n)}A)Q_{h(n)}K_2P_{h(n)} \\
&= P_{h(n)}T(\mathcal{U}_1A)K_2P_{h(n)} + P_{h(n)}T(\mathcal{U}^{h_1(n)}A - \mathcal{U}_1A)K_2P_{h(n)} - P_{h(n)}T(\mathcal{U}^{h_1(n)}A)Q_{h(n)}K_2P_{h(n)} \\
&= P_{h(n)}T(\mathcal{U}_1A)K_2P_{h(n)} + C'_n,
\end{aligned}$$

where C'_n consist of the last two terms, both of which converge to zero in the trace norm due to (2.3.10) and since $Q_n = Q_n^* \rightarrow 0$ strongly. Similarly,

$$\begin{aligned}
&P_{h(n)}K_1P_{h(n)} \cdot P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} \\
&= P_{h(n)}K_1T(\mathcal{U}_1B)P_{h(n)} + P_{h(n)}K_1T(\mathcal{U}^{h_1(n)}B - \mathcal{U}_1B)P_{h(n)} - P_{h(n)}K_1Q_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} \\
&= P_{h(n)}K_1T(\mathcal{U}_1B)P_{h(n)} + C''_n,
\end{aligned}$$

where C''_n converges to zero in the trace norm for the same reasons. Moreover,

observe that we have the estimates

$$\|C'_n\|_{\mathcal{C}_1} \leq 3\|A\|_{\mathcal{R}}\|K_2\|_{\mathcal{C}_1}, \quad \|C''_n\|_{\mathcal{C}_1} \leq 3\|K_1\|_{\mathcal{C}_1}\|B\|_{\mathcal{R}}$$

for all n .

Before proceeding to the next cases, remark that

$$W_{h(n)}P = P_{h(n)}PJU_{-h(n)} \quad \text{and} \quad PW_{h(n)} = U_{h(n)}JPP_{h(n)}, \quad (2.3.13)$$

and therefore

$$\begin{aligned} W_{h(n)}T(\mathcal{U}^{h_1(n)}A)W_{h(n)} &= P_{h(n)}PJU_{-h(n)}(\mathcal{U}^{h_1(n)}A)U_{h(n)}JPP_{h(n)} \\ &= P_{h(n)}T(\widetilde{\mathcal{U}^{h_2(n)}A})P_{h(n)}. \end{aligned}$$

Here recall that $\tilde{A} = JAJ$, that $\mathcal{U}^{h_i(n)}$ are defined in (2.2.5), and note that $h(n) = h_2(n) - h_1(n)$. Using this identity, the following two products can be written as

$$\begin{aligned} P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} \cdot W_{h(n)}L_2W_{h(n)} &= W_{h(n)} \left(P_{h(n)}T(\widetilde{\mathcal{U}^{h_2(n)}A})P_{h(n)}L_2P_{h(n)} \right) W_{h(n)}, \\ W_{h(n)}L_1W_{h(n)} \cdot P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} &= W_{h(n)} \left(P_{h(n)}L_1P_{h(n)}T(\widetilde{\mathcal{U}^{h_2(n)}B})P_{h(n)} \right) W_{h(n)}. \end{aligned}$$

The resulting expressions are of the same form as those in the last two cases except for the $W_{h(n)}$'s on each side. One can deal with them in the same manner to obtain

$$\begin{aligned} P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} \cdot W_{h(n)}L_2W_{h(n)} &= W_{h(n)}T(\widetilde{\mathcal{U}_2A})L_2W_{h(n)} + D'_n, \\ W_{h(n)}L_1W_{h(n)} \cdot P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} &= W_{h(n)}L_1T(\widetilde{\mathcal{U}_2B})W_{h(n)} + D''_n, \end{aligned}$$

where D'_n and D''_n are sequences tending to zero in the trace norm, while in addition

their trace norms are bounded.

Finally, the last case to consider is that of the product

$$\begin{aligned} & P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} \cdot P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} \\ &= P_{h(n)}T(\mathcal{U}^{h_1(n)}(AB))P_{h(n)} - P_{h(n)}H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B})P_{h(n)} \\ &\quad - P_{h(n)}T(\mathcal{U}^{h_1(n)}A)Q_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)}. \end{aligned}$$

Here we used (2.2.1) and that $\mathcal{U}^{h_1(n)}$ is multiplicative. The second term $H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B})$ is trace class and we have

$$\|H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B})\|_{\mathcal{C}_1} \leq M\|\mathcal{U}^{h_1(n)}A\|_{\mathcal{R}}\|\mathcal{U}^{h_1(n)}B\|_{\mathcal{R}} = M\|A\|_{\mathcal{R}}\|B\|_{\mathcal{R}}$$

by (2.2.3) and since \mathcal{R} is shift-invariant. In fact, we can write

$$\begin{aligned} & H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B}) \\ &= H(\mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B}) + H(\mathcal{U}^{h_1(n)}A - \mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B}) + H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B - \mathcal{U}_1B}). \end{aligned}$$

Using (2.3.10) and the estimate of the kind we just employed, it is easily seen that the last two terms converge to zero in the trace norm as $n \rightarrow \infty$. Their trace norm can be estimated uniformly as well. Therefore, the second term becomes

$$P_{h(n)}H(\mathcal{U}^{h_1(n)}A)H(\widetilde{\mathcal{U}^{h_1(n)}B})P_{h(n)} = P_{h(n)}H(\mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B})P_{h(n)} + E'_n$$

with $H(\mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B})$ being trace class, and $E'_n \rightarrow 0$ in trace norm, and $\|E'_n\| \leq 2M\|A\|_{\mathcal{R}}\|B\|_{\mathcal{R}}$. Regarding the third term we note that

$$P_{h(n)}T(\mathcal{U}^{h_1(n)}A)Q_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} = W_{h(n)}H(\widetilde{\mathcal{U}^{h_2(n)}A})H(\mathcal{U}^{h_2(n)}B)W_{h(n)}.$$

Indeed, this identity can be verified by using (2.3.13) and the identity $PQ_{h(n)}P = U_{h(n)}PU_{-h(n)}$. With the same kind of arguments as just employed it follows that this equals

$$W_{h(n)}H(\widetilde{\mathcal{U}_2A})H(\mathcal{U}_2B)W_{h(n)} + E_n''$$

with $H(\widetilde{\mathcal{U}_2A})H(\mathcal{U}_2B)$ being trace class, and $E_n'' \rightarrow 0$ in trace norm, and $\|E_n''\| \leq 2M\|A\|_{\mathcal{R}}\|B\|_{\mathcal{R}}$. We now arrive at

$$\begin{aligned} & P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)} \cdot P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)} \\ &= P_{h(n)}T(\mathcal{U}^{h_1(n)}(AB))P_{h(n)} - P_{h(n)}H(\mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B})P_{h(n)} \\ &\quad - P_{h(n)}H(\widetilde{\mathcal{U}_2A})H(\mathcal{U}_2B)W_{h(n)} - E_n' - E_n''. \end{aligned}$$

To summarize what we have done so far, the product of the above (A_n) and (B_n) can be written as

$$A_nB_n = P_{h(n)}T(\mathcal{U}^{h_1(n)}(AB))P_{h(n)} + P_{h(n)}KP_{h(n)} + W_{h(n)}LW_{h(n)} + G_n, \quad (2.3.14)$$

where

$$K = K_1K_2 + T(\mathcal{U}_1A)K_2 + K_1T(\mathcal{U}_1B) - H(\mathcal{U}_1A)H(\widetilde{\mathcal{U}_1B}), \quad (2.3.15)$$

$$L = L_1L_2 + T(\widetilde{\mathcal{U}_2A})L_2 + L_1T(\widetilde{\mathcal{U}_2B}) - H(\widetilde{\mathcal{U}_2A})H(\mathcal{U}_2B) \quad (2.3.16)$$

and where G_n is a sequence of trace class operators converging to zero in the trace norm. Expressing G_n explicitly using the above computations, it is easily seen that there exists a constant C such that

$$\|(A_nB_n)\|_{\mathcal{S}_{h_1,h_2}(\mathcal{R})} \leq C \|(A_n)\|_{\mathcal{S}_{h_1,h_2}(\mathcal{R})} \|(B_n)\|_{\mathcal{S}_{h_1,h_2}(\mathcal{R})}.$$

Therefore, we can conclude that $\mathcal{S}_{h_1, h_2}(\mathcal{R})$ is a Banach algebra.

Finally, formulas (2.3.14)–(2.3.16) imply that $\mathcal{J}_h(\mathcal{R})$ is an ideal of $\mathcal{S}_{h_1, h_2}(\mathcal{R})$, and it is indeed closed by the definition of the norm. \square

Theorem 2.3.5. *Under the assumptions of the preceding theorem, the mappings \mathcal{W}_{h_1, h_2} and $\widetilde{\mathcal{W}}_{h_1, h_2}$ defined by*

$$\mathcal{W}_{h_1, h_2} : (A_n)_{n \geq 1} \in \mathcal{S}_{h_1, h_2}(\mathcal{R}) \mapsto T(\mathcal{U}_1 A) + K \in \mathcal{O}(\mathcal{R}), \quad (2.3.17)$$

$$\widetilde{\mathcal{W}}_{h_1, h_2} : (A_n)_{n \geq 1} \in \mathcal{S}_{h_1, h_2}(\mathcal{R}) \mapsto T(\widetilde{\mathcal{U}}_2 A) + L \in \mathcal{O}(\widetilde{\mathcal{R}}), \quad (2.3.18)$$

where (A_n) is of the form (2.3.8), are well defined unital Banach algebra homomorphisms.

Here, as before, \mathcal{U}_1 and \mathcal{U}_2 are the (isometric, unital) Banach algebra homomorphisms on \mathcal{R} associated with the fractal sequences h_1 and h_2 .

Proof. The norms of these mappings applied to a sequence (A_n) of the form (2.3.8) are given by

$$\begin{aligned} \|\mathcal{W}_{h_1, h_2}((A_n))\|_{\mathcal{O}(\mathcal{R})} &= \|\mathcal{U}_1 A\|_{\mathcal{R}} + \|K\|_{\mathcal{C}_1} = \|A\|_{\mathcal{R}} + \|K\|_{\mathcal{C}_1}, \\ \|\widetilde{\mathcal{W}}_{h_1, h_2}((A_n))\|_{\mathcal{O}(\widetilde{\mathcal{R}})} &= \|\widetilde{\mathcal{U}}_2 A\|_{\widetilde{\mathcal{R}}} + \|L\|_{\mathcal{C}_1} = \|A\|_{\mathcal{R}} + \|L\|_{\mathcal{C}_1}. \end{aligned}$$

In view of (2.3.9) this implies that these mappings are well defined and continuous. Linearity is obvious. Multiplicativity follows from formulas (2.3.14)–(2.3.16). For instance, if (A_n) and (B_n) are given by (2.3.11) and (2.3.12), then

$$\mathcal{W}_{h_1, h_2}((A_n)) = T(\mathcal{U}_1 A) + K_1, \quad \mathcal{W}_{h_1, h_2}((B_n)) = T(\mathcal{U}_1 B) + K_2,$$

while

$$\mathcal{W}_{h_1, h_2}((A_n)(B_n)) = T(\mathcal{U}_1(AB)) + K_1 K_2 + T(\mathcal{U}_1 A) K_2 + K_1 T(\mathcal{U}_1 B) - H(\mathcal{U}_1 A) H(\widetilde{\mathcal{U}_1 B}).$$

Now take (2.2.1) into account and note that \mathcal{U}_1 is multiplicative as well. \square

Chapter 3

Main results

3.1 First versions of the limit theorem

In this section we establish the abstract version and a more concrete version of the limit theorems. Both deal with the asymptotics of the determinants of $P_{h_1(n),h_2(n)}AP_{h_1(n),h_2(n)}$ where h_1 and h_2 are fractal sequences.

The abstract version reduces the determinant asymptotics to the asymptotics of traces. It applies to operators A taken from any suitable, rigid, shift-invariant and unital Banach algebra \mathcal{R} .

In the more concrete version (referred to as the fractal version), we specialize to the case of the Banach algebras $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$. Here the trace computation is carried out by help of Theorem 2.1.3.

We need two more auxiliary results for the abstract version. The first one is a direct consequence of the results of the previous section.

Proposition 3.1.1. *Let \mathcal{R} be a rigid, suitable, shift-invariant and unital Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$, and let h_1 and h_2 be fractal sequences for \mathcal{R} with associated \mathcal{U}_1 and \mathcal{U}_2 , respectively, such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as*

$n \rightarrow \infty$.

Then for any $A_1, \dots, A_r \in \mathcal{R}$, the sequence (B_n) defined by

$$B_n := P_{h(n)}T(\mathcal{U}^{h_1(n)}(e^{A_1} \dots e^{A_r}))P_{h(n)} \cdot e^{-P_{h(n)}T(\mathcal{U}^{h_1(n)}A_r)P_{h(n)}} \dots e^{-P_{h(n)}T(\mathcal{U}^{h_1(n)}A_1)P_{h(n)}} \quad (3.1.1)$$

belongs to $\mathcal{S}_{h_1, h_2}(\mathcal{R})$. Furthermore, there exist $K, L \in \mathcal{C}_1(l^2(\mathbb{Z}^+))$, and $G_n \in \mathcal{C}_1(\text{im}P_{h(n)})$ with $\|G_n\|_{\mathcal{C}_1} \rightarrow 0$ such that

$$B_n = P_{h(n)} + P_{h(n)}K P_{h(n)} + W_{h(n)}L W_{h(n)} + G_n. \quad (3.1.2)$$

The operators K and L are determined by

$$P + K = T(\mathcal{U}_1(e^{A_1} \dots e^{A_r}))e^{-T(\mathcal{U}_1 A_r)} \dots e^{-T(\mathcal{U}_1 A_1)}, \quad (3.1.3)$$

$$P + L = T(\mathcal{U}_2(\widetilde{e^{A_1} \dots e^{A_r}}))e^{-T(\widetilde{\mathcal{U}_2 A_r})} \dots e^{-T(\widetilde{\mathcal{U}_2 A_1})}. \quad (3.1.4)$$

Proof. Note that for any $(A_n) \in \mathcal{S}_{h_1, h_2}(\mathcal{R})$, $(e^{A_n}) = e^{(A_n)}$, and hence (B_n) defined above is in $\mathcal{S}_{h_1, h_2}(\mathcal{R})$. Define the bounded linear map

$$\Lambda : A \in \mathcal{R} \mapsto (P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)}) \in \mathcal{S}_{h_1, h_2}(\mathcal{R}). \quad (3.1.5)$$

We rewrite (B_n) as

$$(B_n) = \Lambda(e^{A_1} \dots e^{A_r})e^{-\Lambda(A_r)} \dots e^{-\Lambda(A_1)}.$$

Furthermore, denote by Φ the natural homomorphism

$$\Phi : \mathcal{S}_{h_1, h_2}(\mathcal{R}) \rightarrow \mathcal{S}_{h_1, h_2}(\mathcal{R})/\mathcal{J}_h(\mathcal{R}), \quad (A_n) \mapsto (A_n) + \mathcal{J}_h(\mathcal{R}).$$

Then $\Phi \circ \Lambda : \mathcal{R} \rightarrow \mathcal{S}_{h_1, h_2}(\mathcal{R})/\mathcal{J}_h(\mathcal{R})$ is a continuous Banach algebra homomorphism. Indeed, in view of (2.3.11), (2.3.12), and (2.3.14) we have

$$\begin{aligned} (\Phi \circ \Lambda)(A)(\Phi \circ \Lambda)(B) &= (P_{h(n)}T(\mathcal{U}^{h_1(n)}A)P_{h(n)}) \cdot (P_{h(n)}T(\mathcal{U}^{h_1(n)}B)P_{h(n)}) + \mathcal{J}_h(\mathcal{R}) \\ &= (P_{h(n)}T(\mathcal{U}^{h_1(n)}(AB))P_{h(n)}) + \mathcal{J}_h(\mathcal{R}) = (\Phi \circ \Lambda)(AB). \end{aligned}$$

Applying the homomorphism Φ to (B_n) yields

$$\begin{aligned} \Phi((B_n)) &= (\Phi \circ \Lambda)(e^{A_1} \dots e^{A_r} e^{-(\Phi \circ \Lambda)(A_r)} \dots e^{-(\Phi \circ \Lambda)(A_1)}) \\ &= (\Phi \circ \Lambda)(e^{A_1} \dots e^{A_r} e^{-A_r} \dots e^{-A_1}) \\ &= (\Phi \circ \Lambda)(I) \\ &= (P_{h(n)}) + \mathcal{J}_h(\mathcal{R}), \end{aligned}$$

which shows (B_n) is of the form (3.1.2). To determine the operators K and L , we apply the Banach algebra homomorphisms \mathcal{W}_{h_1, h_2} and $\widetilde{\mathcal{W}_{h_1, h_2}}$ to both sides of (3.1.2). \square

The second auxiliary result is the following. Its (simple) proof can be found, e.g., in Lemmas 9.1 and 9.3 of [11].

Lemma 3.1.2. *Let $A_n = P_n + P_n K P_n + W_n L W_n + C_n$ with*

$$K, L \in \mathcal{C}_1(l^2(\mathbb{Z}^+)) , C_n \in \mathcal{C}_1(\text{im}P_n) \text{ and } \|C_n\|_{\mathcal{C}_1} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then

$$\lim_{n \rightarrow \infty} \det A_n = \det(P + K) \det(P + L). \quad (3.1.6)$$

Now we are able to state the first of our main results.

Theorem 3.1.3 (Abstract version of the limit theorem). *Let \mathcal{R} be a rigid, suitable, shift-invariant and unital Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$, and let h_1 and h_2 be fractal sequences for \mathcal{R} with associated \mathcal{U}_1 and \mathcal{U}_2 , respectively, such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

If $A_1, \dots, A_r \in \mathcal{R}$ and $A = e^{A_1} \dots e^{A_r}$, then

$$\lim_{n \rightarrow \infty} \frac{\det(P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)})}{\exp(\text{trace}(P_{h_1(n), h_2(n)}(A_1 + \dots + A_r) P_{h_1(n), h_2(n)}))} = \det(\mathcal{B}_1) \det(\mathcal{B}_2), \quad (3.1.7)$$

where

$$\mathcal{B}_1 = T(\mathcal{U}_1 A) \cdot e^{-T(\mathcal{U}_1 A_r)} \dots e^{-T(\mathcal{U}_1 A_1)}, \quad (3.1.8)$$

$$\mathcal{B}_2 = e^{T(\mathcal{U}_2 A_1)} \dots e^{T(\mathcal{U}_2 A_r)} \cdot T(\mathcal{U}_2(A^{-1})). \quad (3.1.9)$$

Proof. As pointed out in (2.3.5),

$$\det(P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)}) = \det(P_{h(n)} T(\mathcal{U}^{h_1(n)} A) P_{h(n)}),$$

and therefore it suffices to consider the latter. Let (B_n) be the sequence defined by (3.1.1). Taking the determinant, we have

$$\begin{aligned} \det B_n &= \det \left(P_{h(n)} T(\mathcal{U}^{h_1(n)}(A)) P_{h(n)} \cdot e^{-P_{h(n)} T(\mathcal{U}^{h_1(n)} A_r) P_{h(n)}} \dots e^{-P_{h(n)} T(\mathcal{U}^{h_1(n)} A_1) P_{h(n)}} \right) \\ &= \det \left(P_{h(n)} T(\mathcal{U}^{h_1(n)}(A)) P_{h(n)} \right) \cdot e^{-\text{trace}(P_{h(n)} T(\mathcal{U}^{h_1(n)} A_r) P_{h(n)})} \dots e^{-\text{trace}(P_{h(n)} T(\mathcal{U}^{h_1(n)} A_1) P_{h(n)})} \\ &= \det \left(P_{h(n)} T(\mathcal{U}^{h_1(n)}(A)) P_{h(n)} \right) \cdot e^{-\text{trace}(P_{h(n)} T(\mathcal{U}^{h_1(n)}(A_1 + \dots + A_r)) P_{h(n)})} \\ &= \det \left(P_{h(n)} T(\mathcal{U}^{h_1(n)}(A)) P_{h(n)} \right) \cdot e^{-\text{trace}(P_{h_1(n), h_2(n)}(A_1 + \dots + A_r) P_{h_1(n), h_2(n)})}. \end{aligned}$$

In the last step we used (2.3.6). By Proposition 3.1.1, B_n is of the form (3.1.2),

where

$$\begin{aligned}\det(P + K) &= \det \left(T(\mathcal{U}_1 A) \cdot e^{-T(\mathcal{U}_1 A_r)} \dots e^{-T(\mathcal{U}_1 A_1)} \right), \\ \det(P + L) &= \det \left(T(\widetilde{\mathcal{U}}_2 A) \cdot e^{-T(\widetilde{\mathcal{U}}_2 A_r)} \dots e^{-T(\widetilde{\mathcal{U}}_2 A_1)} \right).\end{aligned}$$

We apply Lemma 3.1.2 to obtain the limit (3.1.7). Using Proposition 2.3.3 the second operator determinant can be rewritten as

$$\det \left(e^{T(\mathcal{U}_2 A_1)} \dots e^{T(\mathcal{U}_2 A_r)} \cdot T \left(\mathcal{U}_2(A^{-1}) \right) \right),$$

and both determinants together yield the well-defined constant $\det(\mathcal{B}_1) \det(\mathcal{B}_2)$.

□

In what follows we will apply the previous theorem to the Banach algebra $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$, where β is an admissible and compatible weight on the additive subgroup Ξ of \mathbb{R}/\mathbb{Z} , and $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$. Note that Corollary 2.2.4(i) implies that \mathcal{R} is a rigid, suitable, shift-invariant, and unital Banach algebra. Moreover, the compatibility condition will facilitate the computation of the trace by Theorem 2.1.3.

Let us recall the notation that is used in the following theorem. First, $M(a)$ stands for the mean of an almost periodic sequence a as defined in (1.1.10), and $D(A) \in l^\infty(\mathbb{Z})$ denotes the main diagonal of an operator $A \in \mathcal{L}(l^2(\mathbb{Z}))$.

Furthermore, Corollary 2.2.4(ii) implies that if h is fractal for Ξ with associated $\tau \in \text{Hom}(\Xi, \mathbb{T})$, then h is fractal for \mathcal{R} with associated \mathcal{U}^τ given by

$$\mathcal{U}^\tau : \mathcal{R} \rightarrow \mathcal{R}, \quad \sum_{n \in \mathbb{Z}} \sum_{\xi \in \Xi} a_\xi^{(n)}(e_\xi I) U_n \mapsto \sum_{n \in \mathbb{Z}} \sum_{\xi \in \Xi} a_\xi^{(n)} \tau(\xi)(e_\xi I) U_n. \quad (3.1.10)$$

In fact, it is easy to see that \mathcal{U}^τ is well-defined for all $\tau \in \text{Hom}(\Xi, \mathbb{T})$ and is a

unital isometric Banach algebra isomorphism on \mathcal{R} (see also Proposition 3.2.2(a) below).

Therefore, for any $\tau \in \text{Hom}(\Xi, \mathbb{T})$ and any $A \in \mathcal{R}$ of the form

$$A = e^{A_1} \dots e^{A_r}$$

with $A_1, \dots, A_r \in \mathcal{R}$, it is possible to define the constants

$$\Theta_{A,1}(\tau) = \exp \left(\sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{\tau(\xi)}{1 - e^{2\pi i \xi}} \right) \cdot \det \left(T(\mathcal{U}^\tau A) e^{-T(\mathcal{U}^\tau A_r)} \dots e^{-T(\mathcal{U}^\tau A_1)} \right), \quad (3.1.11)$$

$$\Theta_{A,2}(\tau) = \exp \left(\sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{-\tau(\xi)}{1 - e^{2\pi i \xi}} \right) \cdot \det \left(e^{T(\mathcal{U}^\tau A_1)} \dots e^{T(\mathcal{U}^\tau A_r)} T(\mathcal{U}^\tau A^{-1}) \right), \quad (3.1.12)$$

where

$$a = \sum_{\xi \in \Xi} a_\xi e_\xi = D(A_1 + \dots + A_r) \in \mathcal{A}. \quad (3.1.13)$$

Observe that the operator determinants are well-defined by Proposition 2.3.2. Since β is compatible on Ξ , the sums are finite as shown in Theorem 2.1.3. Note that the definition of the constants depends also on the choice of A_1, \dots, A_r , which is omitted in the notation.

Theorem 3.1.4 (Fractal version of the limit theorem). *Let β be an admissible and compatible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . Let $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$. Suppose that h_1 and h_2 are fractal sequences for Ξ with associated τ_1 and τ_2 in $\text{Hom}(\Xi, \mathbb{T})$, respectively, such that $h(n) := h_2(n) - h_1(n) > 0$ and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$.*

If $A_1, \dots, A_r \in \mathcal{R}$ and $A = e^{A_1} \cdots e^{A_r}$, then

$$\lim_{n \rightarrow \infty} \frac{\det(P_{h_1(n), h_2(n)} A P_{h_1(n), h_2(n)})}{G^{h_2(n) - h_1(n)}} = \Theta_{A,1}(\tau_1) \Theta_{A,2}(\tau_2), \quad (3.1.14)$$

where

$$G = \exp(M(a)), \quad a = D(A_1 + \cdots + A_r). \quad (3.1.15)$$

Proof. Because of Corollary 2.2.4(i), Theorem 3.1.3 can be applied and we are left with the evaluation of the asymptotics for

$$\begin{aligned} \text{trace}(P_{h_1(n), h_2(n)}(A_1 + \cdots + A_r)P_{h_1(n), h_2(n)}) &= \sum_{k=h_1(n)}^{h_2(n)-1} a(k) \\ &= \text{trace}(P_{h_1(n), h_2(n)}(aI)P_{h_1(n), h_2(n)}) \end{aligned}$$

with a given by (3.1.15). Theorem 2.1.3 can be applied and yields

$$\sum_{k=h_1(n)}^{h_2(n)-1} a(k) = (h_2(n) - h_1(n)) \cdot M(a) + F_a(\tau_1) - F_a(\tau_2) + o(1), \quad n \rightarrow \infty,$$

where $F_a(\tau)$ is given by (2.1.11). Combining the exponentials of the constants $F_a(\tau_k)$ with the operator determinants $\det(\mathcal{B}_k)$ gives the constants $\Theta_{A,k}(\tau_k)$. \square

Remark 3.1.5. *The constant G defined in the above theorem is only defined implicitly in terms of A . Namely, different choices of A_1, \dots, A_r may yield the same A , but different constants G . It was conjectured in [14] that this can always happen in the almost periodic setting.*

3.2 A uniform version of the limit theorem

In this section let us assume that $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(APW(\mathbb{Z}, \Xi, \beta))$ with the same assumptions as before: $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, Ξ is an additive subgroup of \mathbb{R}/\mathbb{Z} , and β is an admissible and compatible weight on Ξ .

In this setting the constants $\Theta_{A,1}(\tau)$ and $\Theta_{A,2}(\tau)$ are well-defined for all $\tau \in \text{Hom}(\Xi, \mathbb{T})$. For fixed $A = e^{A_1} \cdots e^{A_r}$ with $A_1, \dots, A_r \in \mathcal{R}$ these quantities can be considered as functions on $\text{Hom}(\Xi, \mathbb{T})$. The goal of the following two propositions is to show that these functions are continuous on $\text{Hom}(\Xi, \mathbb{T})$. After this we will derive the uniform version of the limit theorem.

Recall that $\text{Hom}(\Xi, \mathbb{T})$ is a compact topological space with the topology determined by the local bases of the form

$$U_{\xi_1, \dots, \xi_N; \varepsilon}[\tau] = \left\{ \tau' \in \text{Hom}(\Xi, \mathbb{T}) : |\tau'(\xi_k) - \tau(\xi_k)| < \varepsilon \text{ for all } 1 \leq k \leq N \right\}, \quad (3.2.1)$$

where $\varepsilon > 0$, $N \in \mathbb{N}$, $\xi_1, \dots, \xi_N \in \Xi$.

Proposition 3.2.1. *Let β be an admissible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . For each fixed $a = \sum_{\xi \in \Xi} a_\xi e_\xi \in \mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$,*

(a) *the function $\psi_a : \text{Hom}(\Xi, \mathbb{T}) \rightarrow \mathcal{A}$ defined by*

$$\psi_a(\tau) = U^\tau a := \sum_{\xi \in \Xi} a_\xi \tau(\xi) e_\xi \quad (3.2.2)$$

is well-defined and continuous;

(b) *the function $F_a : \text{Hom}(\Xi, \mathbb{T}) \rightarrow \mathbb{C}$ defined by*

$$F_a(\tau) = \sum_{\xi \in \Xi, \xi \neq 0} a_\xi \frac{\tau(\xi)}{1 - e^{2\pi i \xi}}$$

is well-defined and continuous provided the weight β is compatible.

Proof. (a): Since the spectrum of a is at most countable and $\sum_{\xi \in \Xi} |a_\xi| \beta(\xi) < \infty$, for any given $\epsilon > 0$, there is a finite subset $S = \{\xi_1, \dots, \xi_n\}$ of Ξ , such that

$$\sum_{\xi \in \Xi \setminus S} |a_\xi| \cdot \beta(\xi) < \frac{\epsilon}{4}.$$

For each fixed $\tau \in \text{Hom}(\Xi, \mathbb{T})$, consider $U = U_{\xi_1, \dots, \xi_n, \epsilon_0}[\tau]$ defined by (3.2.1) with $\epsilon_0 = \frac{\epsilon}{2\|a\|_{\mathcal{A}}}$, and note that

$$\begin{aligned} \|U^\tau a - U^{\tau'} a\|_{\mathcal{A}} &= \sum_{\xi \in S} |a_\xi| \cdot |\tau(\xi) - \tau'(\xi)| \beta(\xi) + \sum_{\xi \in \Xi \setminus S} |a_\xi| \cdot |\tau(\xi) - \tau'(\xi)| \beta(\xi) \\ &< \epsilon_0 \sum_{\xi \in S} |a_\xi| \beta(\xi) + 2 \sum_{\xi \in \Xi \setminus S} |a_\xi| \beta(\xi) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for any $\tau' \in U$. It implies that ψ_a is a continuous function on $\text{Hom}(\Xi, \mathbb{T})$ for each fixed a .

(b): Similarly, for any given $\epsilon > 0$, there exists a finite subset $S = \{\xi_1, \dots, \xi_n\}$ of Ξ , such that

$$\sum_{\xi \in \Xi \setminus S} |a_\xi| \cdot \beta(\xi) < C_\beta \epsilon,$$

where C_β is the constant given in (2.1.8). For each fixed $\tau \in \text{Hom}(\Xi, \mathbb{T})$, consider the open neighborhood $U = U_{\xi_1, \dots, \xi_n, \epsilon_0}[\tau]$ with $\epsilon_0 = \frac{C_\beta \epsilon}{\|a\|_{\mathcal{A}}}$. Then, for any $\tau' \in U$,

using estimate (2.1.12)

$$\begin{aligned}
|F_a(\tau) - F_a(\tau')| &\leq \sum_{\xi \in S, \xi \neq 0} |a_\xi| \frac{|\tau(\xi) - \tau'(\xi)|}{|1 - e^{2\pi i \xi}|} + \sum_{\xi \in \Xi \setminus S, \xi \neq 0} |a_\xi| \frac{|\tau(\xi) - \tau'(\xi)|}{|1 - e^{2\pi i \xi}|} \\
&\leq \frac{1}{4C_\beta} \sum_{\xi \in S, \xi \neq 0} |a_\xi| \beta(\xi) \cdot |\tau(\xi) - \tau'(\xi)| + \frac{1}{2C_\beta} \sum_{\xi \in \Xi \setminus S} |a_\xi| \beta(\xi) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since the choice of ϵ and τ is arbitrary, F_a is a continuous function on $\text{Hom}(\Xi, \mathbb{T})$.

□

Proposition 3.2.2. *Let β be an admissible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . Let $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\mathcal{A} = \text{APW}(\mathbb{Z}, \Xi, \beta)$.*

(a) *For each fixed $A \in \mathcal{R}$, the function $\Psi_A(\tau)$ defined by*

$$\Psi_A : \text{Hom}(\Xi, \mathbb{T}) \rightarrow \mathcal{R}, \quad \tau \mapsto \mathcal{U}^\tau A \tag{3.2.3}$$

is well-defined and continuous.

(b) *For each fixed $A = e^{A_1} \cdots e^{A_r}$ with $A_1, \dots, A_r \in \mathcal{R}$, the functions*

$$\begin{aligned}
\Delta_{A,1}(\tau) &= \det \left(T(\mathcal{U}^\tau A) e^{-T(\mathcal{U}^\tau A_r)} \cdots e^{-T(\mathcal{U}^\tau A_1)} \right), \\
\Delta_{A,2}(\tau) &= \det \left(e^{T(\mathcal{U}^\tau A_1)} \cdots e^{T(\mathcal{U}^\tau A_r)} T(\mathcal{U}^\tau A^{-1}) \right)
\end{aligned}$$

are well-defined and continuous on $\text{Hom}(\Xi, \mathbb{T})$.

(c) *For each fixed $A = e^{A_1} \cdots e^{A_r}$ with $A_1, \dots, A_r \in \mathcal{R}$, the functions $\Theta_{A,1}(\tau)$ and $\Theta_{A,2}(\tau)$ given by (3.1.11) and (3.1.12) are well-defined and continuous on $\text{Hom}(\Xi, \mathbb{T})$ provided that the weight β is compatible.*

Proof. (a): Recall that

$$\|A\|_{\mathcal{R}} = \sum_{k \in \mathbb{Z}} \alpha(k) \|a^{(k)}\|_{\mathcal{A}} = \sum_{k \in \mathbb{Z}} \alpha(k) \sum_{\xi \in \Xi} |a_{\xi}^{(k)}| \beta(\xi),$$

where

$$A =: \sum_{k \in \mathbb{Z}} (a^{(k)} I) U_k \quad \text{and} \quad a^{(k)} =: \sum_{\xi \in \Xi} a_{\xi}^{(k)} e_{\xi}.$$

For any given $\epsilon > 0$, there exist $K \in \mathbb{N}$ and a finite subset $S = \{\xi_1, \dots, \xi_n\}$ of Ξ , such that

$$\sum_{|k| \geq K} \alpha(k) \|a^{(k)}\|_{\mathcal{A}} < \frac{\epsilon}{6},$$

and

$$\sum_{\xi \in \Xi \setminus S} |a_{\xi}^{(k)}| \beta(\xi) < \frac{\epsilon}{12KM_K}$$

whenever $|k| < K$, where $M_K = \max\{\alpha(k) : |k| < K\}$. For a fixed $\tau \in \text{Hom}(\Xi, \mathbb{T})$, consider the open neighborhood $U = U_{\xi_1, \dots, \xi_n, \epsilon_0}[\tau]$ with $\epsilon_0 = \frac{\epsilon}{3\|A\|_{\mathcal{R}}}$. We have

$$\begin{aligned} \|\mathcal{U}^{\tau} A - \mathcal{U}^{\tau'} A\|_{\mathcal{A}} &= \sum_{|k| < K} \alpha(k) \|\mathcal{U}^{\tau} a^{(k)} - \mathcal{U}^{\tau'} a^{(k)}\|_{\mathcal{A}} + \sum_{|k| \geq K} \alpha(k) \|\mathcal{U}^{\tau} a^{(k)} - \mathcal{U}^{\tau'} a^{(k)}\|_{\mathcal{A}} \\ &< \sum_{|k| < K} \alpha(k) \left(\sum_{\xi \in S} |a_{\xi}^{(k)}| \cdot |\tau(\xi) - \tau'(\xi)| \beta(\xi) + 2 \sum_{\xi \in \Xi \setminus S} |a_{\xi}^{(k)}| \beta(\xi) \right) + \frac{\epsilon}{3} \\ &< \sum_{|k| < K} \alpha(k) \epsilon_0 \sum_{\xi \in S} |a_{\xi}^{(k)}| \beta(\xi) + \frac{\epsilon}{6KM} \sum_{|k| < K} \alpha(k) + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for any $\tau' \in U$. Since the choices of ϵ and τ are arbitrary, Ψ_A is a continuous function on $\text{Hom}(\Xi, \mathbb{T})$ for each fixed A .

(b): It suffices to show that the two operators

$$\begin{aligned}\mathcal{B}_1(\tau) &:= T(\mathcal{U}^\tau A)e^{-T(\mathcal{U}^\tau A_r)} \dots e^{-T(\mathcal{U}^\tau A_1)}, \\ \mathcal{B}_2(\tau) &:= e^{T(\mathcal{U}^\tau A_1)} \dots e^{T(\mathcal{U}^\tau A_r)} T(\mathcal{U}^\tau A^{-1})\end{aligned}$$

are both continuous functions from $\text{Hom}(\Xi, \mathbb{T})$ to $\mathcal{O}(\mathcal{R})$. Note that,

$$\|T(\mathcal{U}^\tau A) - T(\mathcal{U}^{\tau'} A)\|_{\mathcal{O}(\mathcal{R})} = \|\mathcal{U}^\tau A - \mathcal{U}^{\tau'} A\|_{\mathcal{R}},$$

and thus

$$\chi_A : \text{Hom}(\Xi, \mathbb{T}) \rightarrow \mathcal{O}(\mathcal{R}), \quad \tau \mapsto T(\mathcal{U}^\tau A) \tag{3.2.4}$$

is continuous for each fixed $A \in \mathcal{R}$ by part (a).

Next, note that the exponential function is continuous in any Banach algebra, and thus

$$\eta_A : \text{Hom}(\Xi, \mathbb{T}) \rightarrow \mathcal{O}(\mathcal{R}), \quad \tau \mapsto e^{-T(\mathcal{U}^\tau A)}$$

is continuous from $\text{Hom}(\Xi, \mathbb{T})$ to $\mathcal{O}(\mathcal{R})$ for each fixed $A \in \mathcal{R}$. Finally, for fixed $A_1, \dots, A_r \in \mathcal{R}$ and $A = e^{A_1} \dots e^{A_r}$, we have that

$$\begin{aligned}\mathcal{B}_1(\tau) &= \chi_A(\tau)\eta_{A_r}(\tau) \dots \eta_{A_1}(\tau), \\ \mathcal{B}_2(\tau) &= \eta_{-A_1}(\tau) \dots \eta_{-A_r}(\tau)\chi_{A^{-1}}(\tau)\end{aligned}$$

are continuous functions from $\text{Hom}(\Xi, \mathbb{T})$ to $\mathcal{O}(\mathcal{R})$, and in addition, $\mathcal{B}_1(\tau) - P$ and $\mathcal{B}_2(\tau) - P$ are both trace class by Proposition 2.3.2. It follows that $\Delta_{A,1}(\tau)$ and $\Delta_{A,2}(\tau)$ are well-defined continuous on $\text{Hom}(\Xi, \mathbb{T})$.

Part (c) follows directly from part (b) above and from Proposition 3.2.1(b). \square

We also need the following simple lemma. Therein the additional assumption

on Ξ being at most countable is imposed.

Lemma 3.2.3. *Let Ξ be an at most countable additive subgroup of \mathbb{R}/\mathbb{Z} . For any sequence $n = \{n(k)\}_{k=1}^{\infty}$ of integers there exists a subsequence $h = \{h(k)\}_{k=1}^{\infty}$ of n which is fractal for Ξ .*

Proof. The proof is based on a standard diagonal argument. Since Ξ is countable we can assume that $\Xi = \{\xi_t : t \in \mathbb{N}\}$. Starting with $n_0 := n$ one can recursively construct a collection of integer sequences $n_t = \{n_t(k)\}_{k=1}^{\infty}$, $t \in \mathbb{N}$, such that n_t is a subsequence of n_{t-1} and such that $\{e^{2\pi i n_t(k) \xi_t}\}_{k=1}^{\infty}$ converges for every fixed $t \in \mathbb{N}$. Indeed, suppose we are given n_{t-1} . Then we can consider $\{e^{2\pi i n_{t-1}(k) \xi_t}\}_{k=1}^{\infty}$ and select a convergent subsequence $\{e^{2\pi i n_t(k) \xi_t}\}_{k=1}^{\infty}$, thereby defining a subsequence n_t of n_{t-1} . Moreover, we can keep the first $t-1$ terms of the sequence n_{t-1} unchanged when passing to n_t , i.e., $n_t(k) = n_{t-1}(k)$, $k = 1, \dots, t-1$.

Having defined sequences $n_t = \{n_t(k)\}_{k=1}^{\infty}$ for all $t \in \mathbb{N}$, we define

$$h(k) =: n_k(k).$$

It is straightforward to verify that $h = \{h(k)\}_{k=1}^{\infty}$ is a subsequence of each sequence n_t , $t = 0, 1, \dots$, in particular of the sequence $n = n_0$. Now h being a subsequence of n_t implies that $\{e^{2\pi i h(k) \xi_t}\}_{k=1}^{\infty}$ converges for every fixed $t \in \mathbb{N}$. But this implies that $h = \{h(k)\}_{k=1}^{\infty}$ is fractal for Ξ by the definition of fractality. \square

For each $n \in \mathbb{Z}$ we define a corresponding $\tau_n \in \text{Hom}(\Xi, \mathbb{T})$ by

$$\tau_n(\xi) := e^{2\pi i n \xi}, \quad \xi \in \Xi. \tag{3.2.5}$$

As a consequence of Theorem 3.1.4 we obtain the following version of the limit theorem.

Theorem 3.2.4 (Uniform version of the limit theorem). *Let β be an admissible and compatible weight on an at most countable additive subgroup Ξ of \mathbb{R}/\mathbb{Z} . Let $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$.*

If $A_1, \dots, A_r \in \mathcal{R}$ and $A = e^{A_1} \dots e^{A_r}$, then

$$\lim_{n_2 - n_1 \rightarrow \infty} \left(\frac{\det(P_{n_1, n_2} A P_{n_1, n_2})}{G^{n_2 - n_1}} - \Theta_{A,1}(\tau_{n_1}) \Theta_{A,2}(\tau_{n_2}) \right) = 0, \quad (3.2.6)$$

where

$$G = \exp(M(a)), \quad a = D(A_1 + \dots + A_r),$$

and with $\Theta_{A,1}$ and $\Theta_{A,2}$ given by (3.1.11) and (3.1.12).

Proof. For each pair of integers (n_1, n_2) such that $n_2 - n_1 > 0$ define the quantity

$$F[n_1, n_2] := \frac{\det(P_{n_1, n_2} A P_{n_1, n_2})}{G^{n_2 - n_1}} - \Theta_{A,1}(\tau_{n_1}) \Theta_{A,2}(\tau_{n_2}).$$

We will prove the theorem by contradiction. Assume that (3.2.6) does not hold.

Then there exists an $\epsilon > 0$ such that for each $k \in \mathbb{N}$ there exists two integers $n_1(k)$ and $n_2(k)$ with $n_2(k) - n_1(k) > k$ such that

$$|F[n_1(k), n_2(k)]| \geq \epsilon.$$

Thus we obtain two integer sequences $\{n_1(k)\}_{k=1}^{\infty}$ and $\{n_2(k)\}_{k=1}^{\infty}$ such that $n_2(k) - n_1(k) \rightarrow \infty$ as $k \rightarrow \infty$. By applying Lemma 3.2.3 twice, there exists a strictly increasing sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers such that $h_1 = \{h_1(j)\}_{j=1}^{\infty} := \{n_1(k_j)\}_{j=1}^{\infty}$ and $h_2 = \{h_2(j)\}_{j=1}^{\infty} := \{n_2(k_j)\}_{j=1}^{\infty}$ are both fractal sequences for Ξ . Assume that these fractal sequences h_1 and h_2 have associated $\tau_{h_1}, \tau_{h_2} \in \text{Hom}(\Xi, \mathbb{T})$. Then $\tau_{h_i(j)}$ converges to τ_{h_i} in the natural topology of $\text{Hom}(\Xi, \mathbb{T})$ as

$j \rightarrow \infty$ (for $i = 1, 2$), and therefore

$$\Theta_{A,1}(\tau_{h_1(j)})\Theta_{A,2}(\tau_{h_2(j)}) \rightarrow \Theta_{A,1}(\tau_{h_1})\Theta_{A,2}(\tau_{h_2})$$

as $j \rightarrow \infty$ by Proposition 3.2.2. On the other hand,

$$\frac{\det(P_{h_1(j),h_2(j)}AP_{h_1(j),h_2(j)})}{G^{h_2(j)-h_1(j)}} \rightarrow \Theta_{A,1}(\tau_{h_1})\Theta_{A,2}(\tau_{h_2})$$

as $j \rightarrow \infty$ by Theorem 3.1.4, which contradicts the fact that $|F[h_1(j), h_2(j)]| \geq \epsilon$ for each $j \in \mathbb{N}$ since h_1 and h_2 are subsequences of $\{n_1(k)\}_{k=1}^{\infty}$ and $\{n_2(k)\}_{k=1}^{\infty}$, respectively. \square

Note that the additional assumption that Ξ is at most countable is not a serious restriction. Indeed, for a given operator A with almost periodic diagonals, one can take for Ξ the additive subgroup of \mathbb{R}/\mathbb{Z} generated by the union of the Fourier spectra of all the diagonals $D_k(A) \in AP(\mathbb{Z})$ of A . Recall that the Fourier spectrum of any sequence in $AP(\mathbb{Z})$ is at most countable.

Chapter 4

Further Remarks

4.1 Special cases of the limit theorem

If one wants to verify whether the limit theorems (Theorem 3.1.4 or Theorem 3.2.4) can be applied to a concrete operator $A \in \mathcal{OAP} \subseteq \mathcal{L}(l^2(\mathbb{Z}))$ one faces the following problems:

- (i) Does there exist an admissible and compatible weight β on Ξ , where Ξ is a subgroup of \mathbb{R}/\mathbb{Z} containing the Fourier spectra of all the diagonals $D_k(A) \in AP(\mathbb{Z})$ of A ?
- (ii) Does the operator A belong to $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(APW(\mathbb{Z}, \Xi, \beta))$?
- (iii) Moreover, can A be written as a product of exponentials of operators in \mathcal{R} ?

As we will see below, the first problem is related to diophantine approximation, and the last problem naturally leads to the question whether \mathcal{R} is inverse closed in $\mathcal{L}(l^2(\mathbb{Z}))$.

The questions are also related to each other in the following sense. On the one hand, the compatibility condition requires the weight β to grow sufficiently fast

(depending on Ξ). A fast growing weight, on the other hand, severely restricts the class \mathcal{R} . In addition, it may prevent the inverse closedness of \mathcal{R} in $\mathcal{L}(l^2(\mathbb{Z}))$. For this reason, it would be desirable to consider (admissible and compatible) weights β satisfying the *Gelfand-Raikov-Shilov condition* (or, *GRS-condition*), namely

$$\lim_{k \rightarrow \infty} \beta(k\xi)^{1/k} = 1 \quad \text{for all } \xi \in \Xi. \quad (4.1.1)$$

(Therein k is a positive integer and $k\xi \in \Xi$ is well-defined.) In fact, this conditions is necessary for the inverse closedness of \mathcal{R} in $\mathcal{L}(l^2(\mathbb{Z}))$.

Consequently, we may pose the following question: For which (countable) subgroups Ξ of \mathbb{R}/\mathbb{Z} do there exist admissible and compatible weights β satisfying in addition the GRS-condition?

This question is not easy to answer in general, and we will discuss it to some extent for finitely generated groups Ξ . However, let us first present a simple positive example (mentioned already in [14, Ex. 2.9]) as well as a (not so simple) counter-example.

Example 4.1.1. *Let $\Xi = \mathbb{Q}/\mathbb{Z}$. This is a countable subgroup of \mathbb{R}/\mathbb{Z} which is not finitely generated. Yet, we can define an admissible and compatible weight on Ξ simply by $\beta(\xi) = q$ whenever $\xi = [p/q]$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ being coprime. This weight also satisfies the GRS-condition.*

Here and in what follows, we will notationally distinguish between an equivalence class $[x] \in \mathbb{R}/\mathbb{Z}$ and its representative $x \in \mathbb{R}$. Furthermore, for a given subset $S \subseteq \mathbb{R}/\mathbb{Z}$, let $\text{gr } S$ stand for the subgroup of \mathbb{R}/\mathbb{Z} generated by the set S . In other words, $\text{gr } S$ consists of all finite integer linear combinations of elements from S .

Example 4.1.2. *Let $\Xi_\xi = \text{gr}\{[\xi]\}$ denote the subgroup generated by an irrational*

number ξ . In Section 6 of [14] (see Example 6.7 in particular), the following class of examples has been exhibited.

Let $1 < b < c$ and α be real numbers such that $0 < \alpha < 1 - \frac{\log b}{\log c} < 1$. Then one can construct a Liouville number ξ and a (strictly increasing) sequence $h = \{h(n)\}_{n=1}^{\infty}$ of integers such that

(i) the sequence h is distinguished for Ξ_{ξ} in sense of [14] (hence fractal for Ξ_{ξ} in our sense), and

(ii)

$$\sum_{k=0}^{h(n)-1} a(k) = h(n)^{\alpha}(1 + o(1)), \quad n \rightarrow \infty,$$

where $a = \sum_{k=1}^{\infty} b^{-k} e_{k\xi} \in AP(\mathbb{Z})$.

This class of examples has the following consequences. Suppose there exists an admissible and compatible weight β on Ξ_{ξ} satisfying in addition

$$\lim_{k \rightarrow \infty} \beta(k\xi)^{1/k} < b,$$

(the latter being the case if β satisfies the GRS-condition). Then $a \in APW(\mathbb{Z}, \Xi, \beta)$, and it would be possible to apply Theorem 2.1.3 with $h_1(n) = 0$ and $h_2(n) = h(n)$ and obtain the asymptotics (2.1.10). But this contradicts the asymptotics given in (ii).

Therefore we can conclude that for the Liouville numbers ξ constructed above, there exists no weight β on Ξ_{ξ} which is admissible, compatible, and satisfies the GRS-condition.

It is an open question whether there exist Liouville numbers ξ for which there exists no admissible and compatible weight β on Ξ_{ξ} (regardless of the GRS-condition).

4.1.1 The structure of finitely generated subgroups Ξ of \mathbb{R}/\mathbb{Z}

Let us now consider finitely generated subgroups Ξ of \mathbb{R}/\mathbb{Z} . Such subgroups are necessarily countable. The following two results characterize the structure of such subgroups. Therein, let $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})$ denote the set of congruence classes modulo N . The proof of the first proposition is straightforward. The second one could certainly be derived from the general structure theorem for finitely generated abelian groups, but we provide a proof which is constructive.

Proposition 4.1.3. *For $n \geq 0$, let ξ_1, \dots, ξ_n be real numbers such that $\{\xi_1, \dots, \xi_n, 1\}$ is linearly independent over \mathbb{Q} , and let $N \in \mathbb{N}$. Then the map*

$$(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z}_N \mapsto \left[\alpha_1 \xi_1 + \dots + \alpha_n \xi_n + \alpha_{n+1} \frac{1}{N} \right] \in \mathbb{R}/\mathbb{Z} \quad (4.1.2)$$

is a well-defined group isomorphism between the additive group $\mathbb{Z}^n \times \mathbb{Z}_N$ and the finitely generated subgroup $\Xi = \text{gr} \left\{ [\xi_1], \dots, [\xi_n], \left[\frac{1}{N} \right] \right\}$ of \mathbb{R}/\mathbb{Z} .

Proposition 4.1.4. *If Ξ is a finitely generated subgroup of \mathbb{R}/\mathbb{Z} , then there exists $n \geq 0$ and $N \geq 1$ such that Ξ is group-isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_N$ via an isomorphism of the form (4.1.2).*

Proof. The group Ξ being finitely generated means that there exist generators $[\xi_1], \dots, [\xi_m] \in \mathbb{R}/\mathbb{Z}$ such that

$$\Xi = \text{gr} \left\{ [\xi_1], \dots, [\xi_m] \right\} := \left\{ \sum_{k=1}^m \alpha_k [\xi_k] : \alpha_k \in \mathbb{Z} \right\}.$$

The numbers ξ_k therein are either irrational or rational. Among all tuples $([\xi_1], \dots, [\xi_m])$ of elements in \mathbb{R}/\mathbb{Z} generating the group Ξ consider one in which the number of irrational generators is as small as possible, say equal to n . Therefore, without

loss of generality, we can assume that

$$\Xi = \text{gr} \{ [\xi_1], \dots, [\xi_n], [\xi_{n+1}], \dots, [\xi_{n+k}] \} \quad (4.1.3)$$

where $\xi_1, \dots, \xi_n \in \mathbb{R} \setminus \mathbb{Q}$, $\xi_{n+1}, \dots, \xi_{n+k} \in \mathbb{Q}$, and $n, k \geq 0$.

Concerning the number k of rational generators we made no assumption at this point. However, it is possible to modify the rational generators in (4.1.3) and keeping the irrational generators unchanged such that the resulting representation has precisely one rational generator and in addition that this rational generator equals $[\frac{1}{N}]$ for some integer $N \geq 1$. In other words, we can turn (4.1.3) into

$$\Xi = \text{gr} \{ [\xi_1], \dots, [\xi_n], [\frac{1}{N}] \}. \quad (4.1.4)$$

The argument is as follows. First of all, if $k = 0$, we can add the ‘dummy’ element $[1] = [0]$. If $k \geq 1$, then each rational generator $[\frac{p}{q}]$ in which $p \in \mathbb{Z}$ and $q \geq 1$ are coprime can be replaced by $[\frac{1}{q}]$. This is because $\text{gr}\{[\frac{p}{q}]\} = \text{gr}\{[\frac{1}{q}]\}$. If we have more than one rational generator ($k \geq 2$) then we can replace two generators $[\frac{1}{q_1}]$ and $[\frac{1}{q_2}]$ by a single generator $[\frac{1}{q}]$ with q being the least common multiple of q_1 and q_2 . Here the reason is that $\text{gr}\{[\frac{1}{q_1}], [\frac{1}{q_2}]\} = \text{gr}\{[\frac{1}{q}]\}$.

We can now assume that Ξ is given by (4.1.4) and that the number n of irrational generators cannot be reduced. We claim that $\{\xi_1, \dots, \xi_n, 1\}$ is rationally independent over \mathbb{Q} . Suppose this is not the case. Then $n \geq 1$ and there are $a_1, \dots, a_n \in \mathbb{Z}$ such that

$$a_1 \xi_1 + \dots + a_n \xi_n = \frac{p}{q} \in \mathbb{Q}$$

while not all a_1, \dots, a_n are zero. In fact, one can assume that $\text{gcd}(a_1, \dots, a_n) = 1$.

It is well-known and straightforward to show (e.g., by induction on n) that there exists a unimodular $n \times n$ matrix M with entries in \mathbb{Z} whose last row is (a_1, \dots, a_n) . Define the elements $[\xi'_1], \dots, [\xi'_n]$ by

$$\begin{bmatrix} [\xi'_1] \\ \vdots \\ [\xi'_n] \end{bmatrix} := M \begin{bmatrix} [\xi_1] \\ \vdots \\ [\xi_n] \end{bmatrix}$$

and note that $[\xi'_n] = [\frac{p}{q}]$. Because M^{-1} has entries in \mathbb{Z} as well, it is easily seen that

$$\text{gr} \{[\xi_1], \dots, [\xi_n]\} = \text{gr} \{[\xi'_1], \dots, [\xi'_n]\}.$$

Therefore,

$$\Xi = \text{gr} \left\{ [\xi'_1], \dots, [\xi'_{n-1}], [\frac{p}{q}], [\frac{1}{N}] \right\}.$$

But this means that the number of irrational generators of Ξ can be reduced, contrary to our assumption.

We therefore conclude that $\{\xi_1, \dots, \xi_n, 1\}$ is linearly independent over \mathbb{Q} , which makes it possible to apply Proposition 4.1.3. Since Ξ is given by (4.1.4), it follows that Ξ is group-isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_N$ via the map (4.1.2). \square

After having seen that every finitely generated subgroup of \mathbb{R}/\mathbb{Z} is group-isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_N$, the question now is for which of them does there exist an admissible and compatible weight satisfying the GRS-condition. This depends, of course, on the irrational generators ξ_1, \dots, ξ_n featured in the description of the group Ξ .

Before going into this, let us mention the trivial case of $n = 0$, i.e., $\Xi = \text{gr}\{[\frac{1}{N}]\} \cong \mathbb{Z}_N$. Since this group is finite the choice of the weight β is irrelevant (one can take $\beta \equiv 1$). The corresponding Banach algebra $APW(\mathbb{Z}, \Xi, \beta)$ consists

of all sequences in $l^\infty(\mathbb{Z})$ that have period N . Even though this is the trivial case, Theorem 3.1.4 or Theorem 3.2.4 implies a generalization of the classical strong Szegő-Widom limit theorem (see Subsection 4.1.3 below).

Let us now consider the case $n \geq 1$. We denote by $\mathcal{S}_n \subseteq (\mathbb{R}/\mathbb{Z})^n$ the set of all (ξ_1, \dots, ξ_n) for which there exist $\omega > 0$ and $C > 0$ such that

$$\|\alpha_1 \xi_1 + \dots + \alpha_n \xi_n\|_{\mathbb{R}/\mathbb{Z}} \geq C \left(\max_{1 \leq i \leq n} |\alpha_i| \right)^{-\omega} \quad (4.1.5)$$

for all $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \setminus \{0\}$.

As the following result shows, absence from \mathcal{S}_n is rather exceptional. For a proof see, e.g., Section 3.5.3 ($n = 1$) and Section 4.3.2 ($n \geq 2$) of [4]. Note that the set \mathcal{S}_1 corresponds to irrational numbers which are not Liouville numbers.

Theorem 4.1.5. *The complement of \mathcal{S}_n in $(\mathbb{R}/\mathbb{Z})^n$ has Hausdorff dimension $n-1$, hence Lebesgue measure zero in $(\mathbb{R}/\mathbb{Z})^n$.*

It is easy to see that condition (4.1.5) implies that $\{\xi_1, \dots, \xi_n, 1\}$ is linearly independent over \mathbb{Q} . Hence for each $N \geq 1$, the group

$$\Xi = \text{gr} \left\{ [\xi_1], \dots, [\xi_n], \left[\frac{1}{N} \right] \right\} \quad (4.1.6)$$

can be identified with $\mathbb{Z}^n \times \mathbb{Z}_N$ as described in Proposition 4.1.3.

Proposition 4.1.6. *For $n \geq 1$, let $(\xi_1, \dots, \xi_n) \in \mathcal{S}_n$, and let ω be the constant in (4.1.5). Then for each $N \geq 1$ one can define an admissible and compatible weight on*

$$\Xi = \left\{ \xi = \left[\alpha_1 \xi_1 + \dots + \alpha_n \xi_n + \alpha_{n+1} \frac{1}{N} \right] : \alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z}_N \right\},$$

satisfying the GRS-condition, by

$$\beta(\xi) := (1 + |\alpha|)^\omega, \quad |\alpha| = \max_{1 \leq i \leq n} |\alpha_i|. \quad (4.1.7)$$

Proof. Obviously, the weight is admissible and satisfies the GRS-condition. Note that $\|\frac{x}{N}\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{N}\|x\|_{\mathbb{R}/\mathbb{Z}}$. Therefore, whenever $|\alpha| \neq 0$,

$$\begin{aligned} \|\xi\| &= \left\| \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n + \frac{\alpha_{n+1}}{N} \right\| \geq \frac{1}{N} \|N\alpha_1 \xi_1 + \cdots + N\alpha_n \xi_n + \alpha_{n+1}\| \\ &= \frac{1}{N} \|N\alpha_1 \xi_1 + \cdots + N\alpha_n \xi_n\| \geq \frac{C}{N^{1+\omega}} (1 + |\alpha|)^{-\omega} = \frac{C}{N^{1+\omega} \beta(\xi)}. \end{aligned}$$

Taking also into account the trivial case of $|\alpha| = 0$, $\alpha_{n+1} \neq 0$, this implies that β is compatible by the definition (2.1.8). \square

In order to practically apply the previous proposition one would have to know when $(\xi_1, \dots, \xi_n) \in \mathcal{S}_n$. We confine ourselves to briefly state two examples, both of which rely on deep results about diophantine approximation (see, e.g., [7, 27]). These examples were discussed thoroughly in [14, Section 2.3], where also further references can be found.

Example 4.1.7. (Roth-Schmidt, [25, 28]) *Let ξ_1, \dots, ξ_n be (real) algebraic numbers such that $\{\xi_1, \dots, \xi_n, 1\}$ is linearly independent over \mathbb{Q} . Then for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\|\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n\|_{\mathbb{R}/\mathbb{Z}} \geq C_\varepsilon \left(\max_{1 \leq i \leq N} |\alpha_i| \right)^{-n-\varepsilon}$$

for every $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \setminus \{0\}$.

The second example concerns logarithms of algebraic numbers.

Example 4.1.8. (Baker-Feldman, [2, 15]) Let $\Lambda = \{\lambda \in \mathbb{C} : \exp(\lambda) \in \overline{\mathbb{Q}}\}$, where $\overline{\mathbb{Q}}$ denotes the set of all (possibly complex) algebraic numbers. Consider

$$\xi_1, \dots, \xi_m \in \Lambda \cap \mathbb{R}, \quad \xi_{m+1}, \dots, \xi_n \in i\Lambda \cap \mathbb{R},$$

such that $\{\xi_1, \dots, \xi_m\}$ and $\{\xi_{m+1}, \dots, \xi_n\}$ are linearly independent over \mathbb{Q} . Then there exists (effectively computable) constants $C > 0$ and $\omega > 0$ such that (4.1.5) holds.

4.1.2 The limit theorem for finitely generated subgroups

Let Ξ be a finitely generated subgroup of \mathbb{R}/\mathbb{Z} which is group-isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_N$. Then, as we will see shortly, $\text{Hom}(\Xi, \mathbb{T})$ is naturally isomorphic as a compact group to $\mathbb{T}^n \times \mathbb{T}_N$, where $\mathbb{T}_N := \{\omega \in \mathbb{C} : \omega^N = 1\}$.

We will use this observation in order to replace in the uniform version of the limit theorem (Theorem 3.2.4) the continuous functions $\Theta_{A,1}$ and $\Theta_{A,2}$ defined on $\text{Hom}(\Xi, \mathbb{T})$ by continuous functions defined on $\mathbb{T}^n \times \mathbb{T}_N$.

Assume that Ξ is group-isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_N$ via the group-isomorphism (4.1.2). Then $\text{Hom}(\Xi, \mathbb{T})$ can be identified with $\mathbb{T}^n \times \mathbb{T}_N$ as follows. Each $(t_1, \dots, t_n, t_{n+1}) \in \mathbb{T}^n \times \mathbb{T}_N$ gives rise to a $\tau \in \text{Hom}(\Xi, \mathbb{T})$ defined by

$$\tau(\xi) = t_1^{\alpha_1} \dots t_n^{\alpha_n} t_{n+1}^{\alpha_{n+1}}$$

with $\xi = [\alpha_1 \xi_1 + \dots + \alpha_n \xi_n + \alpha_{n+1} \frac{1}{N}]$, $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z}_N$. Conversely, it is easy to see, that each $\tau \in \text{Hom}(\Xi, \mathbb{T})$ arises in this way. In fact, the underlying map

$$\Lambda : (t_1, \dots, t_n, t_{n+1}) \mapsto \tau$$

is both a group-isomorphism and a homeomorphism between $\mathbb{T}^n \times \mathbb{T}_N$ and $\text{Hom}(\Xi, \mathbb{T})$.

The following theorem also includes the case $n = 0$, in which we stipulate $\Xi = \text{gr}\{\frac{1}{N}\}$ and $\beta \equiv 1$. For $n \geq 1$, notice that the weight β depends on the parameter ω given by (4.1.5), which in turn depends on (ξ_1, \dots, ξ_n) .

Theorem 4.1.9 (Limit theorem for finitely generated groups). *Let $n \geq 0$, $(\xi_1, \dots, \xi_n) \in \mathcal{S}_n$, and $N \geq 1$, and consider the group Ξ with the weight β defined by (4.1.6) and (4.1.7). Let $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(\mathcal{A})$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\mathcal{A} = \text{APW}(\mathbb{Z}, \Xi, \beta)$.*

If $A_1, \dots, A_r \in \mathcal{R}$ and $A = e^{A_1} \cdots e^{A_r}$, then there exist continuous functions

$$\hat{\Theta}_{A,k} : \mathbb{T}^n \times \mathbb{T}_N \rightarrow \mathbb{C}, \quad k = 1, 2,$$

such that

$$\lim_{n_2 - n_1 \rightarrow \infty} \left(\frac{\det(P_{n_1, n_2} A P_{n_1, n_2})}{G^{n_2 - n_1}} - E_{A,1}(n_1) E_{A,2}(n_2) \right) = 0 \quad (4.1.8)$$

with

$$E_{A,k}(m) = \hat{\Theta}_{A,k}(e^{2\pi i \xi_1 m}, \dots, e^{2\pi i \xi_n m}, e^{2\pi i \frac{1}{N} m}), \quad k = 1, 2, \quad m \in \mathbb{Z},$$

$$G = \exp(M(a)), \quad a = D(A_1 + \cdots + A_r).$$

Proof. In Theorem 3.2.4 we encountered the functions $\Theta_{A,1}$ and $\Theta_{A,2}$ defined by (3.1.11) and (3.1.12), and we know from Proposition 3.2.2(c) that these functions are continuous on $\text{Hom}(\Xi, \mathbb{T})$. Considering the composition of these functions with Λ ,

$$\hat{\Theta}_{A,k}(t_1, \dots, t_n, t_{n+1}) = \Theta_{A,k}(\Lambda(t_1, \dots, t_n, t_{n+1})),$$

we obtain continuous function $\hat{\Theta}_{A,k} : \mathbb{T}^n \times \mathbb{T}_N \rightarrow \mathbb{C}$. The limit (3.2.6) contains

the quantities $\Theta_{A,k}(\tau_m)$ where $\tau_m \in \text{Hom}(\Xi, \mathbb{T})$ is given by $\tau_m(\xi) = e^{2\pi i m \xi}$, $\xi \in \Xi$.

It is easy to see that

$$\Lambda(e^{2\pi i \xi_1 m}, \dots, e^{2\pi i \xi_n m}, e^{2\pi i \frac{1}{N} m}) = \tau_m.$$

Hence

$$\Theta_{A,k}(\tau_m) = \widehat{\Theta}_{A,k}(e^{2\pi i \xi_1 m}, \dots, e^{2\pi i \xi_n m}, e^{2\pi i \frac{1}{N} m}).$$

We now obtain formula (4.1.8) from formula (3.2.6). □

Note that we did not write down formulas for $\widehat{\Theta}_{A,k}(t_1, \dots, t_n, t_{n+1})$ in terms of operator determinants showing the dependence on $(t_1, \dots, t_{n+1}) \in \mathbb{T}^n \times \mathbb{T}_N$ explicitly since the concrete evaluation of such operator determinants seems to be illusive in general. Still one could ask whether such an evaluation is possible in such cases where A is an almost Mathieu operator.

While $\widehat{\Theta}_{A,k}(t_1, \dots, t_n, t_{n+1})$ are continuous, one could further ask if (or under what conditions) these functions are differentiable in $t_1, \dots, t_n \in \mathbb{T}$. We leave this as an open question.

Another issue is the vanishing of these operator determinants. Numerical evidence suggests that for certain (exceptional) A it is possible that the functions $\Theta_{A,1}(\tau)$ and $\Theta_{A,2}(\tau)$ can vanish at particular values of τ . However, an even more “undesirable” situation would occur if one of the functions vanishes identically on $\text{Hom}(\Xi, \mathbb{T})$. For the case $n \geq 1$ we do not know whether this can be ruled out. In the case where $n = 0$ and $N \geq 2$ it can happen, see Subsection 4.1.3 below.

4.1.3 The case of block Laurent operators revisited

Let us consider the case of block Laurent operators again and establish a generalization of the strong Szegő-Widom limit theorem. Such a generalization seems to be stated here for the first time. The classical result (in the form of Theorem 1.1.1) is recovered from the theorem below with $k_1 = k_2$, although for a slightly different class of symbols.

To obtain the generalization we apply Theorem 3.2.4 (or Theorem 4.1.9) to the case of $\Xi \cong \mathbb{Z}_N$. Then the operators under consideration are block Laurent operators where the symbol is a smooth $N \times N$ matrix valued function. To be more specific, define for $\alpha_1, \alpha_2 \geq 0$ the Wiener class W_{α_1, α_2} consisting of all $a \in L^1(\mathbb{T})$ such that

$$\|a\|_{W_{\alpha_1, \alpha_2}} = \sum_{k \in \mathbb{Z}} \alpha(k) |a_k| < \infty,$$

where $\alpha(k)$ is defined by (2.2.7).

Theorem 4.1.10 (Generalized block Szegő-Widom limit theorem). *For $N \geq 1$ and for $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$, let $a \in W_{\alpha_1, \alpha_2}^{N \times N}$ be such that $\det a(t) \neq 0$ for all $t \in \mathbb{T}$ and $\det a(t)$ has winding number zero. Then for each $k_1, k_2 \in \{0, \dots, N-1\}$ we have that*

$$\lim_{m \rightarrow \infty} \frac{\det P_{k_1, k_2 + mN} L(a) P_{k_1, k_2 + mN}}{G^{k_2 - k_1 + mN}} = E_{a,1}[k_1] E_{a,2}[k_2],$$

where $G = \exp\left(\frac{1}{2\pi N} \int_0^{2\pi} \log \det a(e^{ix}) dx\right)$ and $E_{a,1}[k_1]$ and $E_{a,2}[k_2]$ are certain constants.

Proof. We first have to use a result from [11], more precisely, the equivalence of the statements (i) and (ii) in [11, Prop. 6.4] in the setting $\mathcal{S} = W_{\alpha_1, \alpha_2}$. Note that statement (ii) amounts to the above assumptions on the symbol a , whereas statement (i) asserts that we can write $a = e^{a_1} \dots e^{a_r}$ for certain $a_1, \dots, a_r \in$

$W_{\alpha_1, \alpha_2}^{N \times N}$. Therefore, it follows that

$$L(a) = e^{L(a_1)} \dots e^{L(a_r)}$$

with $L(a_1), \dots, L(a_r) \in \mathcal{R}$, $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(APW(\mathbb{Z}, \Xi, \beta))$, $\Xi = \text{gr}\{\frac{1}{N}\}$, $\beta \equiv 1$.

We can now apply Theorem 4.1.9 with $n_1 = k_1$, $n_2 = k_2 + mN$ and $m \rightarrow \infty$. Note that in the terms of the fractal version (Theorem 3.1.4) we would consider the fractal sequences $h_1(m) = k_1$ and $h_2(m) = k_2 + mN$. The expressions for the constants in (4.1.8) are given by

$$E_{A,j}(n_j) = \widehat{\Theta}_{A,j}(e^{2\pi i n_j \frac{1}{N}}) = \widehat{\Theta}_{A,j}(e^{2\pi i k_j \frac{1}{N}}) =: E_{a,j}[k_j], \quad j = 1, 2.$$

They do not depend on m , but only on k_1 and k_2 . The computation of the constant G can be done straightforwardly. \square

In the classical case and when $N \geq 2$ it is possible that the constant $E[a]$ in (1.1.15) is zero (even though this is considered ‘exceptional’). In our situation the same can happen when $N \geq 2$. For instance, in the case of the symbol

$$a(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

one finds that both constants $E_{a,1}[k_1]$ and $E_{a,2}[k_2]$ are zero for all k_1, k_2 , while with little effort it can be seen that a is a product of exponentials of trigonometric matrix functions. We leave the details to the reader.

4.2 Future problems and remarks

4.2.1 The inverse closedness problem

Problem (iii) mentioned at the beginning of Section 4.1 asks the question how to decide whether a given operator $A \in \mathcal{R} \subseteq \mathcal{L}(l^2(\mathbb{Z}))$ is the finite product of exponentials of operators in \mathcal{R} . It would be desirable to have at least some (non-trivial) sufficient conditions available. We will make a connection between this problem and the question whether \mathcal{R} is inverse closed in $\mathcal{L}(l^2(\mathbb{Z}))$.

Recall that a unital Banach subalgebra \mathcal{R} of $\mathcal{L}(l^2(\mathbb{Z}))$ is inverse closed in $\mathcal{L}(l^2(\mathbb{Z}))$ if

$$\mathcal{G}(\mathcal{R}) = \mathcal{R} \cap \mathcal{G}(\mathcal{L}(l^2(\mathbb{Z}))).$$

Here $\mathcal{G}(\mathcal{B})$ stands for the group of invertible elements in a unital Banach algebra \mathcal{B} .

The group $\mathcal{G}(\mathcal{R})$ may consist of several connected components. It is well known (see, e.g. [26, Thm. 10.34]) that A is a finite product of exponentials of elements in \mathcal{R} if and only if A belongs to the connected component of $\mathcal{G}(\mathcal{R})$ containing the identity operator. With this equivalence, the following proposition is obvious.

Proposition 4.2.1. *Let \mathcal{R} be unital, inverse closed Banach subalgebra of $\mathcal{L}(l^2(\mathbb{Z}))$. For $A \in \mathcal{R}$ assume that there exists a continuous function $\sigma : [0, 1] \rightarrow \mathcal{R}$ such that*

$$\sigma(0) = I, \quad \sigma(1) = A, \quad \text{and} \quad \sigma(t) \text{ is invertible in } \mathcal{L}(l^2(\mathbb{Z})) \text{ for all } t \in [0, 1].$$

Then $A = e^{A_1} \cdots e^{A_r}$ for certain $A_1, \dots, A_r \in \mathcal{R}$.

Under the assumption that \mathcal{R} is inverse closed, this proposition provides useful sufficient criteria for $A \in \mathcal{R}$ to be a product of exponentials. For instance, a function σ with the required properties exists if the unbounded component of the

spectrum of A in $\mathcal{L}(l^2(\mathbb{Z}))$ contains zero. The latter is the case, for instance, if A is invertible in $\mathcal{L}(l^2(\mathbb{Z}))$ and is self-adjoint.

It would be interesting to know under which conditions \mathcal{R} has the inverse closedness property. We raise the following conjecture.

Conjecture 4.2.2. *Let β be an admissible weight on an additive subgroup Ξ of \mathbb{R}/\mathbb{Z} satisfying the GRS-condition, and let $\alpha_1, \alpha_2 \geq 0$. Then the Banach algebra $\mathcal{R} = \mathcal{W}_{\alpha_1, \alpha_2}(APW(\mathbb{Z}, \Xi, \beta))$ is inverse closed in $\mathcal{L}(l^2(\mathbb{Z}))$.*

Assume that β is an admissible weight on Ξ . Then the GRS-condition is necessary for the inverse closedness of \mathcal{R} in $\mathcal{L}(l^2(\mathbb{Z}))$. In fact, it is easy to see (e.g., by using Gelfand theory) that the GRS-condition is necessary and sufficient for the inverse closedness of $\mathcal{A} = APW(\mathbb{Z}, \Xi, \beta)$ in $l^\infty(\mathbb{Z})$. On the other hand, proving that the GRS-condition implies the inverse closedness of \mathcal{R} in $\mathcal{L}(l^2(\mathbb{Z}))$ seems difficult.

In the case of finitely generated groups $\Xi \cong \mathbb{Z}^n$ the conjecture is motivated by the work of Gröchenig and Leinert [17], where a similar inverse closedness property is proved. They consider (non-commutative) weighted Wiener-type algebras with twisted convolutions as the product and symmetric weights on \mathbb{Z}^d . Their algebras are isomorphic as Banach algebras to our algebras \mathcal{R} with $d = n + 1$ in certain cases. Unfortunately, the difference is that their algebras are represented on $l^2(\mathbb{Z}^d)$, whereas our \mathcal{R} is represented on $l^2(\mathbb{Z})$.

The almost Mathieu operators revisited. Recall that the almost Mathieu operator is given by

$$M_a = U_1 + aI + U_{-1}, \tag{4.2.1}$$

where $a \in AP(\mathbb{Z})$ is $a(n) = \beta \cos 2\pi(\xi n + \delta)$. We assume that β, ξ , and δ are real numbers in which case M_a is a selfadjoint operator on $l^2(\mathbb{Z})$.

Suppose in addition that ξ is not a Liouville number. This means that ξ is either rational or belongs to the set \mathcal{S}_1 defined in Subsection 4.1.1. In both cases we can conclude that there exists an admissible and compatible weight β on the group $\Xi_\xi = \text{gr}\{[\xi]\}$ (see Proposition 4.1.6). In addition, this weight satisfies the GRS-condition. It is also clear that

$$M_a \in \mathcal{R} := \mathcal{W}_{1/2,1/2}(APW(\mathbb{Z}, \Xi_\xi, \beta)).$$

Now assume that the above conjecture is true. Then we can conclude that if $\lambda \in \mathbb{C}$ is such that $A = M_a - \lambda I$ is an invertible operator on $l^2(\mathbb{Z})$, then A is a finite product of exponentials of elements in \mathcal{R} . Hence our limit theorems (e.g., Theorem 4.1.9) are applicable to A .

Without having the conjecture available we can still say something, namely, that the limit theorems apply to $A = M_a - \lambda I$ provided that $\lambda \in \mathbb{C}$ is such that $|\lambda| > \|M_a\|_{\mathcal{R}}$. Indeed, in this case A possesses a logarithm in \mathcal{R} . Notice that the norm of $\|M_a\|_{\mathcal{R}}$ depends on β and thus on the diophantine properties of ξ .

4.2.2 In connection with Random Matrix Theory

In Random Matrix Theory, the bulk scaling limit (the probability that an interval contains no eigenvalues) and edge scaling limit (which describes the distribution of the extreme smallest or largest eigenvalue) can often be expressed as a Fredholm determinant with certain kernel function K ,

$$\det(I - K)|_{L^2(I)} := 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_I \cdots \int_I \det_{i,j=1}^k K(x_i, x_j) \prod_{i=1}^k dx_i. \quad (4.2.2)$$

A Toeplitz determinant can be represented as a Fredholm determinant with certain kernel function on $L^2(\mathbb{T})$, and the asymptotics of determinants of Toeplitz matrices and other Toeplitz-like matrices has been put into great use to evaluate

the asymptotics of Fredholm determinants, e.g., with the sine kernel [10, 12, 18, 32] and with the Bessel kernel [9, 13].

In the setting of band-dominated operators with almost periodic diagonals, one can deal with a larger scale of Toeplitz-like operators, and the limit theorems presented in this paper can provide a great tool to evaluate the corresponding Fredholm determinants under certain assumptions. It will be interesting to directly look at random matrix ensembles representing matrices with almost periodic diagonals as well.

Bibliography

- [1] A. Avila, S. Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170**, no. 1 (2009), 303-342.
- [2] A. Baker, Linear forms in the logarithms of algebraic numbers III, *Mathematica* **14** (1967), 220-228.
- [3] F.P. Boca, *Rotation C^* -Algebras and Almost Mathieu Operators*, Theta Ser. Adv. Math., vol. 1, The Theta Foundation, Bucharest, 2001.
- [4] V.I. Bernik and M.M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, **137**, Cambridge University Press, Cambridge, 1999.
- [5] A. Böttcher, B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices*, Springer-Verlag, Berlin, 1999.
- [6] A. Böttcher, B. Silbermann, *Analysis of Toeplitz operators*, 2nd ed., Springer-Verlag, Berlin, 2006, Prepared jointly with A. Karlovich.
- [7] J.W.S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Math. Physics, **45**, Cambridge University Press, New York, 1957.

- [8] C. Corduneanu, *Almost Periodic Functions*, Chelsea Publ. Co., New York, 1989.
- [9] P. Deift, A. Its, J. Vasilevska, *Asymptotics for a determinant with a confluent hypergeometric kernel*, Int. Math. Res. Not., 2010.
- [10] P. Deift, A. Its, I. Krasovsky, X. Zhou, *The Widom-Dyson constant and related questions of the asymptotic analysis of Toeplitz determinants*, Proceedings of the AMS meeting, Atlanta 2005, J. Comput. Appl. Math., **202**(2007), 26-47.
- [11] T. Ehrhardt, *A new algebraic approach to the Szegő-Widom limit theorem*, Acta Math. Hungar., **99** (2003), no. 3, 233-261.
- [12] T. Ehrhardt, *Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel*, Comm. Math. Phys. **262**(2006), no. 2, 317-341.
- [13] T. Ehrhardt, *The asymptotics of a Bessel-kernel determinant which arises in random matrix theory*, Adv. Math., **225**(2010), no. 6, 3088-3133.
- [14] T. Ehrhardt, S. Roch, B. Silbermann, *A strong Szegő-Widom limit theorem for operators with almost periodic diagonal*, J. Funct. Anal. **260** (2011), no. 1, 30-75.
- [15] N.I. Feldman, *An improvement of the estimate of a linear form in the logarithms of algebraic numbers*, Mat. Sb. (N.S.) **77 (119)** (1968), 423-436 (in Russian).
- [16] I. Gohberg and S. Goldberg and M.A. Kaashoek, *Classes of linear operators, Vol. 2*, Birkhäuser, **63**(2012).

- [17] K. Gröchenig and M. Leinert, *Inverse-closed Banach subalgebras of higher-dimensional non-commutative tori*, *Studia Math.*, **234** (2016), no. 1, 49-58.
- [18] I. Krasovsky, *Gap probability in the spectrum of random matrices and asymptotics of polynomial orthogonal on an arc of the unit circle*, *Int. Math. Res. Not.*, **2004**(2004), 1249-1272.
- [19] V.S. Rabinovich, S. Roch, B. Silbermann, *Finite sections of band-dominated operators with almost periodic coefficients*, *Oper. Theory Adv. Appl.*, **170** (2007), 205-228.
- [20] V.S. Rabinovich, S. Roch, *Finite sections of band-dominated operators on discrete groups*, *Oper. Theory Adv. Appl.*, **220** (2012), 239-253.
- [21] S. Roch, *Finite section of band-dominated operators*, *Mem. Amer. Math. Soc.* **191** (895) (2008).
- [22] S. Roch, *Beyond fractality: piecewise fractal and quasifractal algebra*, *Operator Theory: Advances and Applications*, **268**, Birkhäuser/Springer, 2018, 413–428.
- [23] S. Roch, B. Silbermann, *C^* -algebra techniques in numerical analysis*, *J. Operator Theory* **35**, no. 2 (1996), 241–280.
- [24] S. Roch, B. Silbermann, *Szegö limit theorems for operators with almost periodic diagonals*, *Oper. Matrices* **1** (2007), no. 1, 1–29.
- [25] K.F. Roth, *Rational approximations to algebraic numbers*, *Mathematika*, **2**(1955), 1-20.
- [26] W. Rudin, *Functional Analysis*, *Internat. Ser. Pure Appl. Math*, 1991.

- [27] M.G. Kreĭn, *Certain new Banach algebras and theorems of the type of the Wiener-Lévy theorems for series and Fourier integrals*, Mat. Issled., **1**(1996), no. 1, 82-108.
- [28] W.M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Math., **125**(1970), 189-201.
- [29] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory*, Amer. Math. Soc. Colloq. Publ., vol. 54, Amer. Math. Soc., Providence, RI, 2005.
- [30] G. Szegő, *On certain Hermitian forms with associated the Fourier series of a positive function*, Comm. Sémin. Math. Univ. Lund, (1952), 228-238.
- [31] H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants. II*, Advances in Math., **21**(1976), no. 1, 1-29.
- [32] H. Widom, *The strong Szegő limit theorem for circular arcs*, Indiana Univ. Math. J., **21**(1971), 277-283.