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1. INTRODUCTION

We will describe the method of phase contours and its application to some problems in the study of strong interactions. Phase contours are curves along which the phase of a scattering amplitude is constant. They are sections of complex surfaces in the space of the invariants s, t and u. We will give a brief discussion of the following topics:

(i) Properties of phase contours.

(ii) Phase contours for pion-nucleon scattering.

(iii) Phase contours in a Regge model for pion-nucleon scattering.

(iv) Resonance poles, and zeros, in a crossing symmetric model.

(v) Fixed angle scattering at high energy.

A more detailed discussion of these topics will be given in forthcoming papers[1),2),3)].

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† To read the paper.

2. PROPERTIES OF PHASE CONTOURS

We limit our discussion to the scattering of spinless bosons of mass m, except when considering pion-nucleon scattering.[1] The phase $\phi(s,t)$ of a scattering amplitude $F(s,t)$ is defined by

$$\phi(s,t) = \operatorname{Im}[\log \{F(s,t)\}]. \qquad (2.1)$$

It is also necessary to define the phase at an initial point (s_0, t_0). When the amplitude has zeros or poles, the phase depends on the route chosen from the initial point to the point (s,t), so we must always specify the route taken.

A phase contour is defined by

$$\phi(s,t) = C, \qquad (2.2)$$

where C is a real constant. It is useful to study their properties both for real s and t, and for complex s and fixed t. These properties include:

a) Phase contours, for different values of C, do not meet each other, except at zeros or poles of the scattering amplitude, and at certain other singularities.

b) The phase change clockwise round a zero is -2π, and round a pole is 2π.

c) For fixed t and complex s, the phase is an harmonic function, and the phase contours are orthogonal to the modulus contours.

d) The asymptotic phase in a Regge model for a symmetric amplitude, for $t < 4m^2$, assuming a non-zero residue is given by

$$\phi(s,t) \sim \pi[1 - \tfrac{1}{2}\alpha(t)] , \qquad (2.3)$$

as $s \to +\infty$ along $s + i0$ (above the real branch cut). In complex directions, as $|s| \to \infty$

$$\phi(|s|e^{i\theta},t) \sim \pi[1 - \tfrac{1}{2}\alpha(t)] + \alpha\theta . \qquad (2.4)$$

The simple relation of the phase to the power $s^{\alpha(t)}$ should be noted.

e) Phase contours contain information about the oscillations of $\mathrm{Im}\, F(s,t)$ as s moves along the real axis. These oscillations, together with information about poles and zeros, can be used to set bounds on the high energy behavior of F.

f) From the optical theorem, we have

$$\mathrm{Im}\, F(s,0) > 0 , \qquad \text{for} \qquad s > 4m^2. \qquad (2.5)$$

We define the phase $\phi(s,t)$ at an initial point along $t = 0$, namely along $s + i0$ (above the branch cut),

$$\phi(s \to +\infty, 0) = \tfrac{1}{2}\pi . \qquad (2.6)$$

g) The phase contours

$$\phi(s,t) = 0 , \qquad \text{or} \qquad \pi, \qquad (2.7)$$

cannot enter the region

$$0 \leqslant t < 4m^2, \qquad s > 4m^2. \qquad (2.8)$$

For a symmetric amplitude, they also cannot enter the region,

$$0 \leqslant u < 4m^2, \qquad s > 4m^2. \qquad (2.9)$$

Except when considering low and medium energy pion-nucleon phase contours, we work with a Regge model in which there is dominance by a single, continuously rising, Regge trajectory, as $s \to \infty$ for fixed t, and there is symmetry between all three channels, s, t and u. The method of phase contours can equally well be applied to other models, or it can be used to generalize existing models.

3. PHASE CONTOURS FOR PION-NUCLEON SCATTERING AMPLITUDES

We have used the results of the phase shift analysis of pion-nucleon scattering to obtain phase contours for invariant amplitudes in the energy range

$$0 < T_\pi < 1.4 \text{ GeV}, \qquad (3.1)$$

where T_π denotes the pion kinetic energy. The amplitudes that we have studied,[1] are the symmetric and antisymmetric combinations of π^+p and π^-p, namely,

$$A'^{(+)},\quad B^{(+)},\quad A'^{(-)},\quad B^{(-)} . \qquad (3.2)$$

We have used the 1966 phase shifts of Lovelace[4] in the range (3.1), and some preliminary results of Johnson[5] in the range $0 < T_\pi < 1.6$ GeV. Our purpose is to show that phase contours provide:

(a) A useful visual aid for comparing different phase shift solutions and for indicating regions where the resulting scattering amplitudes are complicated. In these regions one might conclude that the continuation of the phase shift solutions is doubtful and more experiments are desirable.

(b) They determine the location of real zeros of the amplitudes from the crossover points of phase contours and indicate complex zeros.

(c) They may help to give an indication of how to match high energy models for scattering, onto low energy models. For this purpose we have obtained phase contours for the extrapolated Regge model described in Section 4.

Phase contours for $A'^{(+)}$ derived from the Lovelace (1966) phase shifts[4)] are shown in Fig. 1. There are several points of special interest.

(i) The phase, in an energy range from 0.2 to 1.2 GeV at angles from 0° to 90°, lies in the range

$$90^\circ < \phi(s,t) < 150^\circ. \tag{3.3}$$

Above 1.2 GeV, there is a wider range of phases which suggests the onset of high energy effects, possibly of Regge type.

(ii) There are two real zeros indicated by the crossing of phase contours. There are probably also some nearby complex zeros indicated by the bunching of phase contours, especially along a scattering angle of about 120°.

The corresponding modulus contours are shown in Fig. 2. This shows the minima in the modulus near the crossover points of the phase contours, and also in regions where there was a bunching of phase contours in Fig. 1.

The phase contours for $A'^{(+)}$, based on the Johnson phase shifts,[5] are shown in Fig. 3. They resemble those of Fig. 1 in their general features. However the zero at 1.27 GeV in Fig. 1 has disappeared. It appears to be replaced by a complex zero near the physical region, which would be one of a pair of complex zeros.

All the other amplitudes indicate considerable complexity around $T_\pi = 0.85$ GeV, which is just below the 1688 resonance. As an example the phase contours for $B^{(+)}$ are shown in Fig. 4. Rapid phase changes like those near 0.85 GeV indicate rapid changes in polarization. They also suggest that more experimental information would be valuable.

4. PHASE CONTOURS IN A REGGE MODEL FOR PION-NUCLEON SCATTERING

We have calculated the phase contours for the pion-nucleon amplitudes (3.2) using extrapolation from a Regge model for high energy scattering.[1] Our extrapolation is not meant to be realistic at this stage, but it is hoped that it gives some orientation on what might be attributed to Regge effects from the few poles that dominate near-forward and near-backward scattering at high energies. We have done this simply by adding the Regge solutions based on P, P' and ρ, exchange in the t channel,[6] and N, N^*, exchange in the u channel.[7] We have also assumed that the trajectories fall linearly as t decreases (or u).

The results are illustrated in Fig. 5 for the invariant amplitude $A'^{(+)}$. It is interesting to find that real zeros arise from the interference of different Regge terms. The contours in

$0 < (-t) < 1\ (\mathrm{GeV})^2$ illustrate the behavior of the phase near the dip and peak of the differential cross section near the forward direction, which is attributed to the zero in the residue of the Regge term from ρ exchange. The shape of the contours at large angles is characteristic of our Regge model with rising Regge trajectories.

5. RESONANCE POLES AND ZEROS IN A CROSSING SYMMETRIC MODEL

Phase contours provide a method for studying part of the consistency problem in strong interactions, and may help towards formulating approximations in which the bootstrap problem is meaningful. However at this stage we do not introduce unitarity, but consider only the consistency that is required by crossing symmetry for a given high energy behavior, taking into account the associated resonance poles (Regge poles).

We consider a symmetric scattering amplitude for equal mass spinless bosons.[2)] Our aim is to construct a model, or a class of models, that satisfies the crossing conditions and has given high energy behavior. The latter is based on Regge terms of the type,

$$\frac{b(t)\, s^{\alpha(t)} \exp[i\pi\{1 - \frac{1}{2}\alpha(t)\}]}{\sin[\frac{1}{2}\pi\alpha(t)]\,\Gamma(\alpha)}\,, \qquad (5.1)$$

where $\mathrm{Re}\{\alpha(t)\}$ is a rising function of t.

We assume that there are no real poles on the physical sheet, and we neglect the local distortions of phase contours that are introduced at low energies in the physical regions. This leads to the Regge dominance model for phase contours in the physical regions that is illustrated for the s channel in Fig. 6.

The phases in regions of crossed branch cuts are obtained from (5.1). In these regions it is important to specify whether the real values of s, t and u, lie above or below their respective branch cuts, on the physical sheet. The formula (5.1) applies along $s + i0$, and $\alpha(t)$ is real for $t < 4m^2$. For $t > 4m^2$, $\text{Im}\,\alpha > 0$ for $t + i0$ (t above the cut), and $\text{Im}\,\alpha < 0$ for $t - i0$ (below the cut but still on the physical sheet).

The form of the solution depends on how many real zeros of the scattering amplitude lie on the boundary of the physical sheet. The location of the first of these zeros (the one nearest the symmetry point $(4m^2/3,\ 4m^2/3,\ 4m^2/3)$, is directly related to the scattering length. There is an infinite sequence of other zeros, but not all of them need be on the physical sheet. They are associated with interference between resonance poles, or between a resonance pole and a background term.

One of our solutions for the phase contours is illustrated in Fig. 7. This solution corresponds to an infinite sequence of zeros on the physical sheet $(s + i0,\ t - i0,\ u - i0)$. A complex section is given in Fig. 8, with t real and $t > 4m^2$. This shows how these zeros are associated with phase contours and resonance poles in the complex s plane including part of the unphysical sheet.

6. FIXED ANGLE SCATTERING AT HIGH ENERGY

We have considered[3] the problem of fixed angle scattering at high energy within the general framework of the crossing symmetric amplitude described in Section 5. The method of phase contours is used to discuss the relation between high energy behavior at fixed momentum transfer and at fixed angle. The former is given by our assumption of dominance by Regge poles with rising trajectories in each channel.

As an example we will consider the phase contours shown in Fig. 7. These correspond to the limit appropriate for fixed angle scattering with $\text{Im } s > 0$. We reduce the scattering amplitude to a Herglotz function by factoring out zeros and oscillations. In this example there are no zeros of F in $\text{Im } s > 0$, and no zeros of $\text{Im } F$ along real $s < 0$.

With certain simplifying assumptions we can write,

$$F(s, \cos \theta) = \frac{H(s, \cos \theta)}{R(s, \cos \theta)} , \qquad (6.1)$$

where H is a Herglotz function and R is an entire function in the variable s. The order $p(\theta)$ of the entire function is related to the phase by means of Jensen's theorem, which gives

$$\phi(s, \cos \theta) \sim C\, s^{p(\theta)} . \qquad (6.2)$$

Using Polya's inequality, one can obtain bounds on the scattering amplitude in the upper half s plane,

$$\exp[-A\, s^{p(\theta)}] < |F(s, \cos\theta)| < \exp[-A(\cos \pi p - \epsilon) s^{p(\theta)}]. \quad (6.3)$$

If we generalize (6.1) to allow a sequence of zeros in $\text{Im } s > 0$ for fixed $\cos\theta$, we obtain

$$F(s, \cos\theta) = \frac{E(s, \cos\theta)\, H(s, \cos\theta)}{R(s, \cos\theta)}, \quad (6.4)$$

where E is an entire function of order $q(\theta)$. Then, if $q > p$, we obtain the bounds for large $|s|$,

$$\exp\left[B(\cos q\pi - \epsilon) s^{q}\right] < F(s, \cos\theta) < \exp[B s^{q}], \quad (6.5)$$

except near the zeros of E. This gives

$$q(\theta) \geqslant \frac{1}{2}, \qquad \text{for all } \theta. \quad (6.6)$$

No such condition is required if $p(\theta) > q(\theta)$ for all θ.

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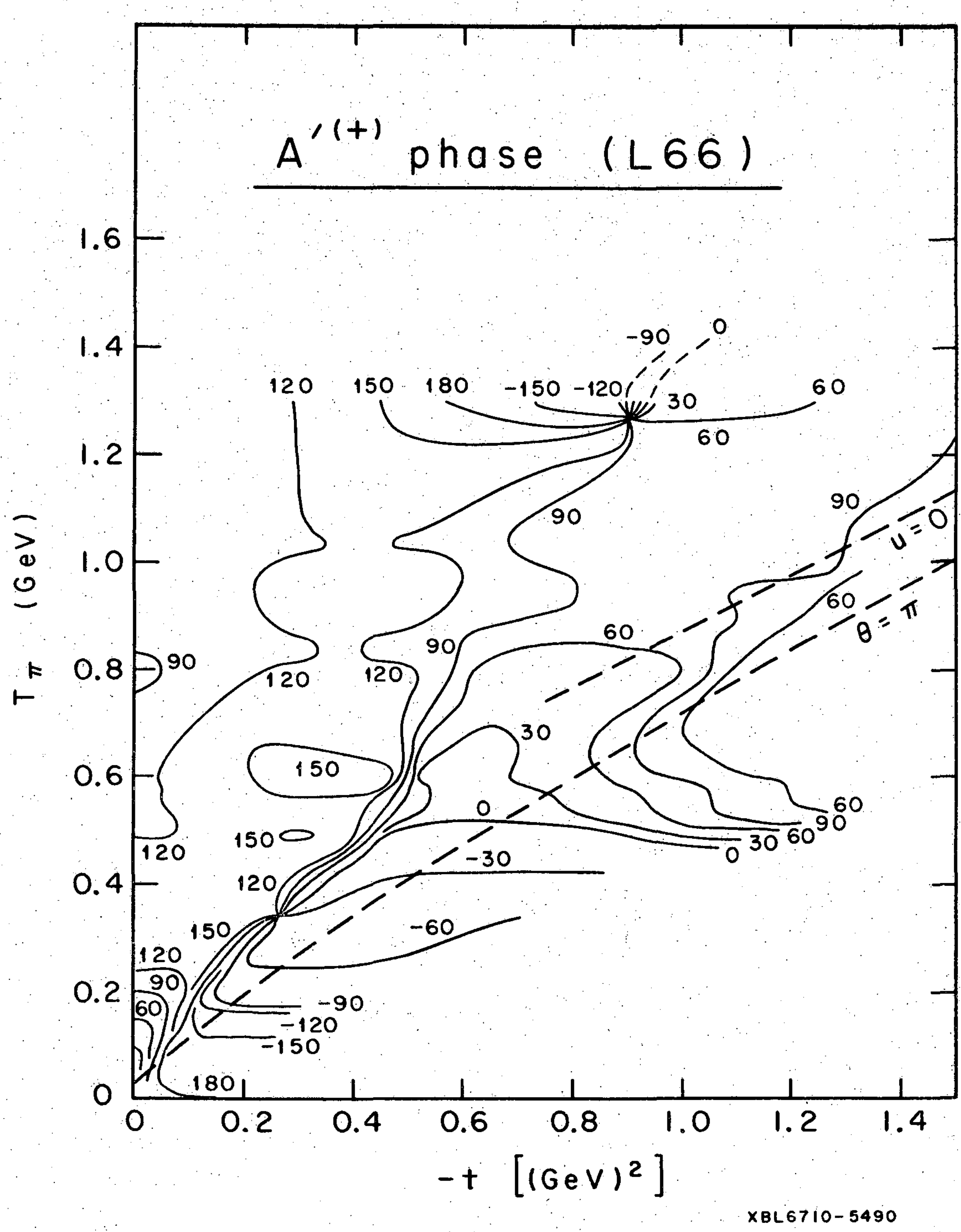

Fig. 1. Phase contours for $A'^{(+)}$ as functions of the pion kinetic energy T_π and the momentum transfer variable (-t). Derived from the Lovelace (1966) phase shifts for πN scattering.

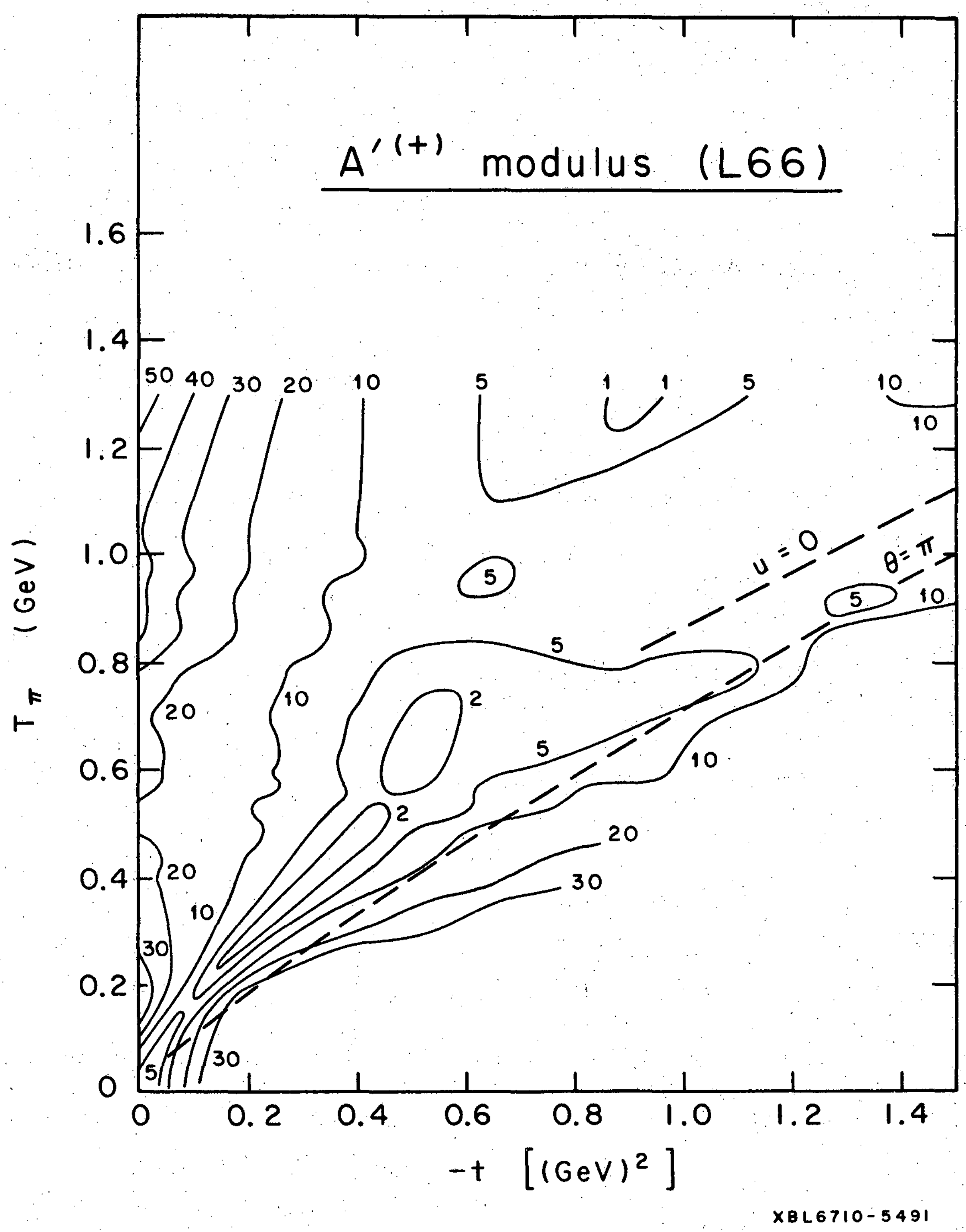

Fig. 2. Modulus contours for $A'^{(+)}$, derived from the Lovelace (1966) phase shifts for πN scattering.

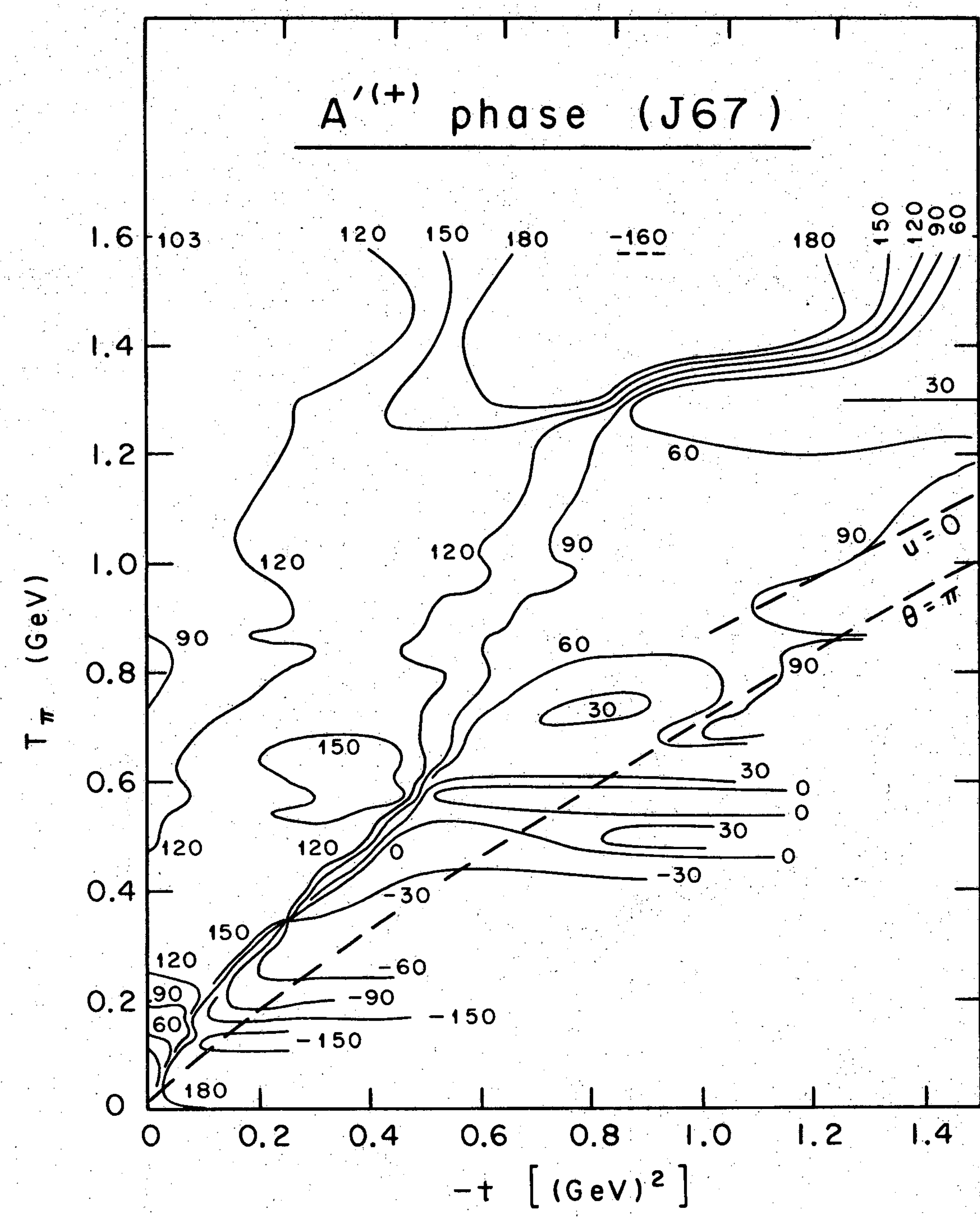

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Fig. 3. Phase contours for $A'^{(+)}$ derived from the Johnson (1967) phase shifts for πN scattering.

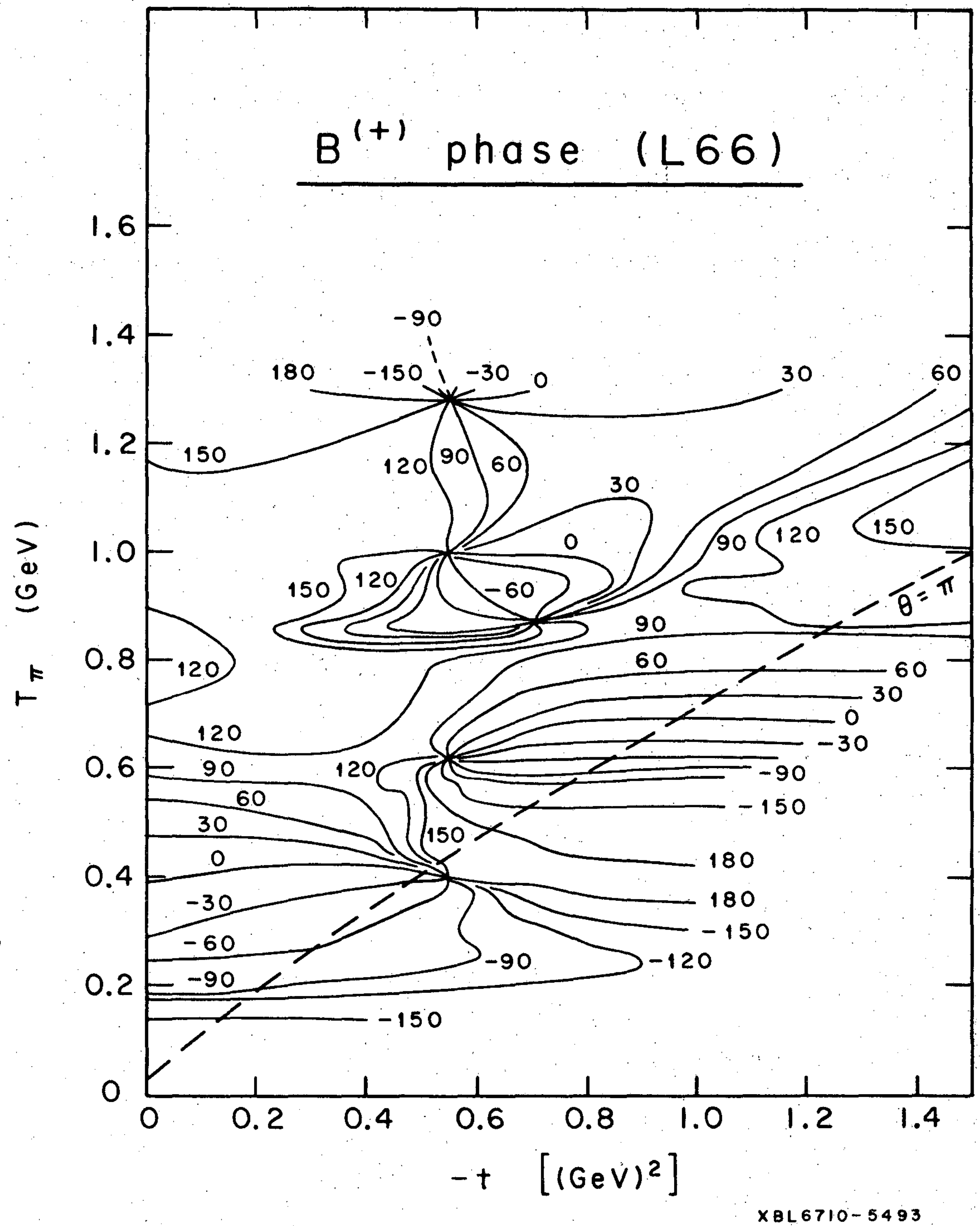

Fig. 4. Phase contours for $B^{(+)}$, derived from the Lovelace (1966) phase shifts for πN scattering.

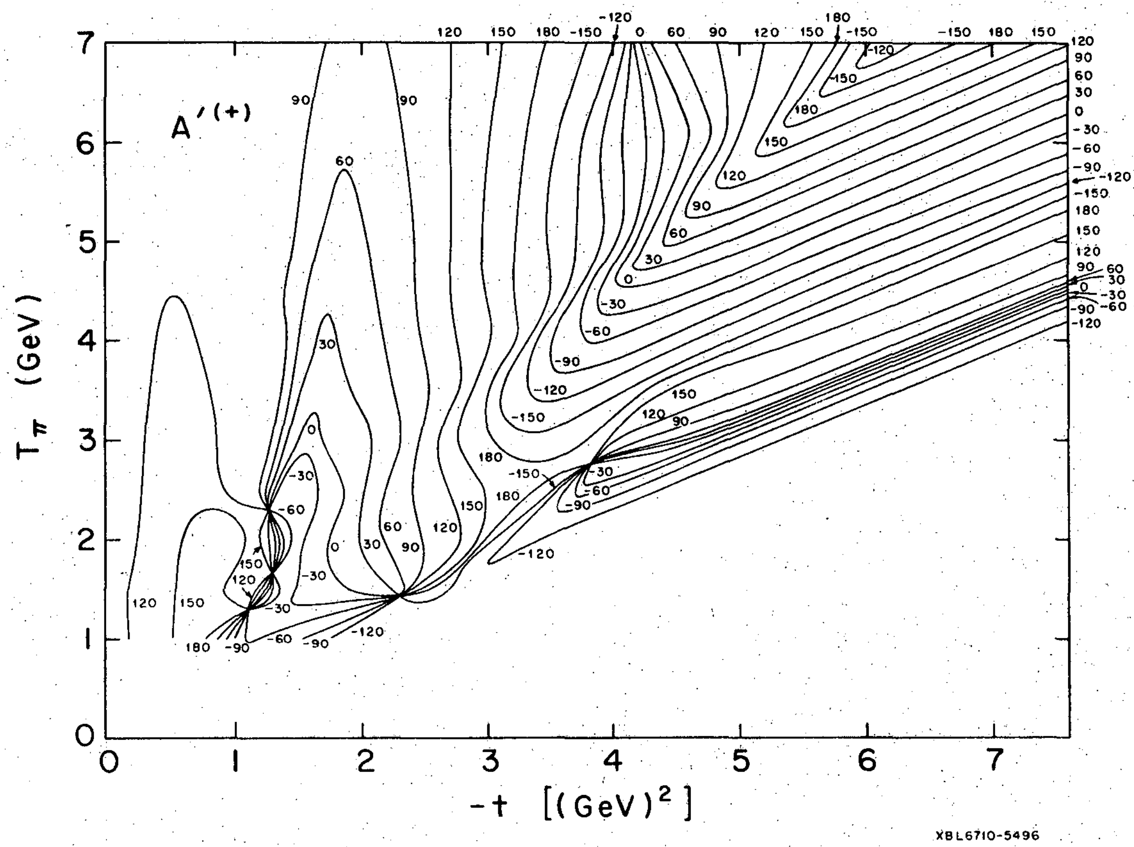

Fig. 5. Phase contours for $A'^{(+)}$, derived by extrapolating from the high energy Regge solutions for πN scattering.

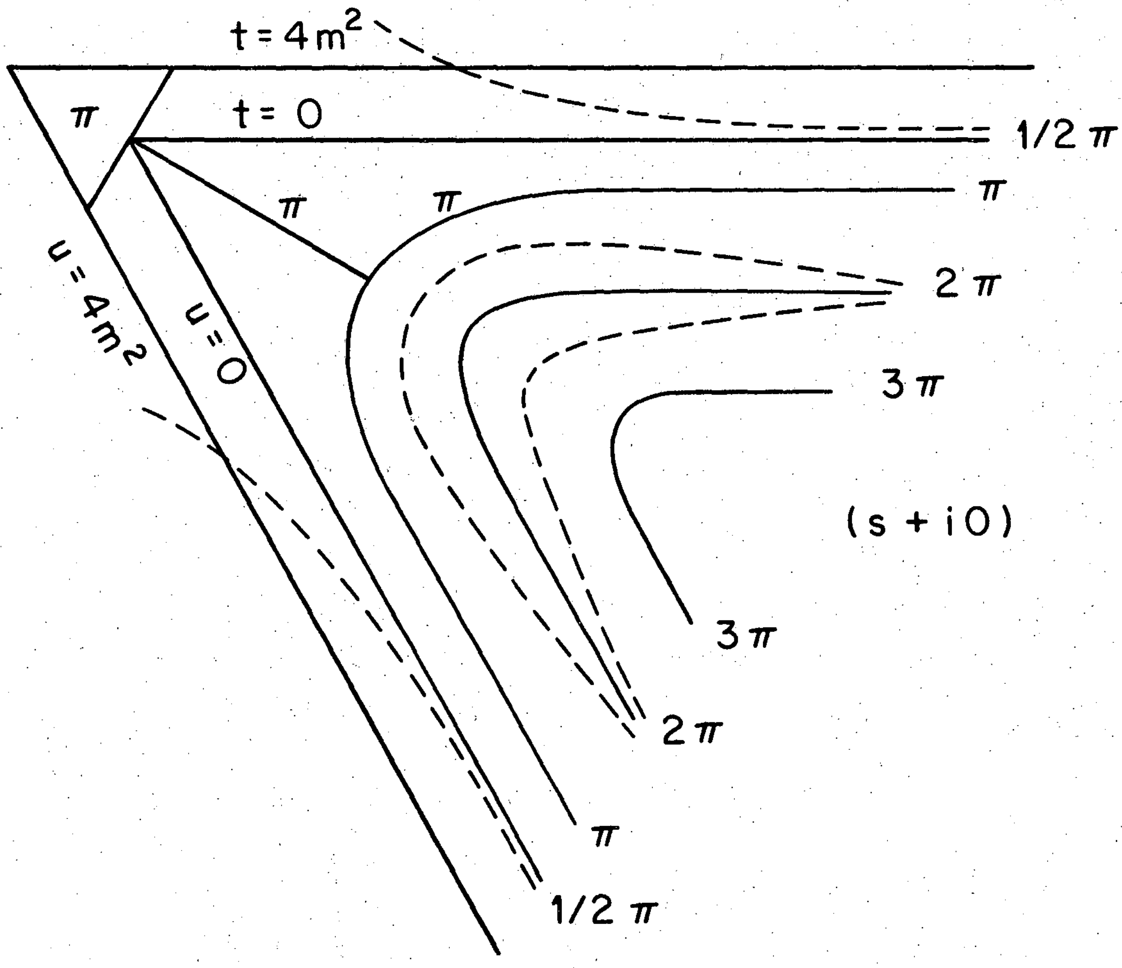

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Fig. 6. Phase contours in the s channel for a Regge model. Broken lines correspond to half integer multiples of π.

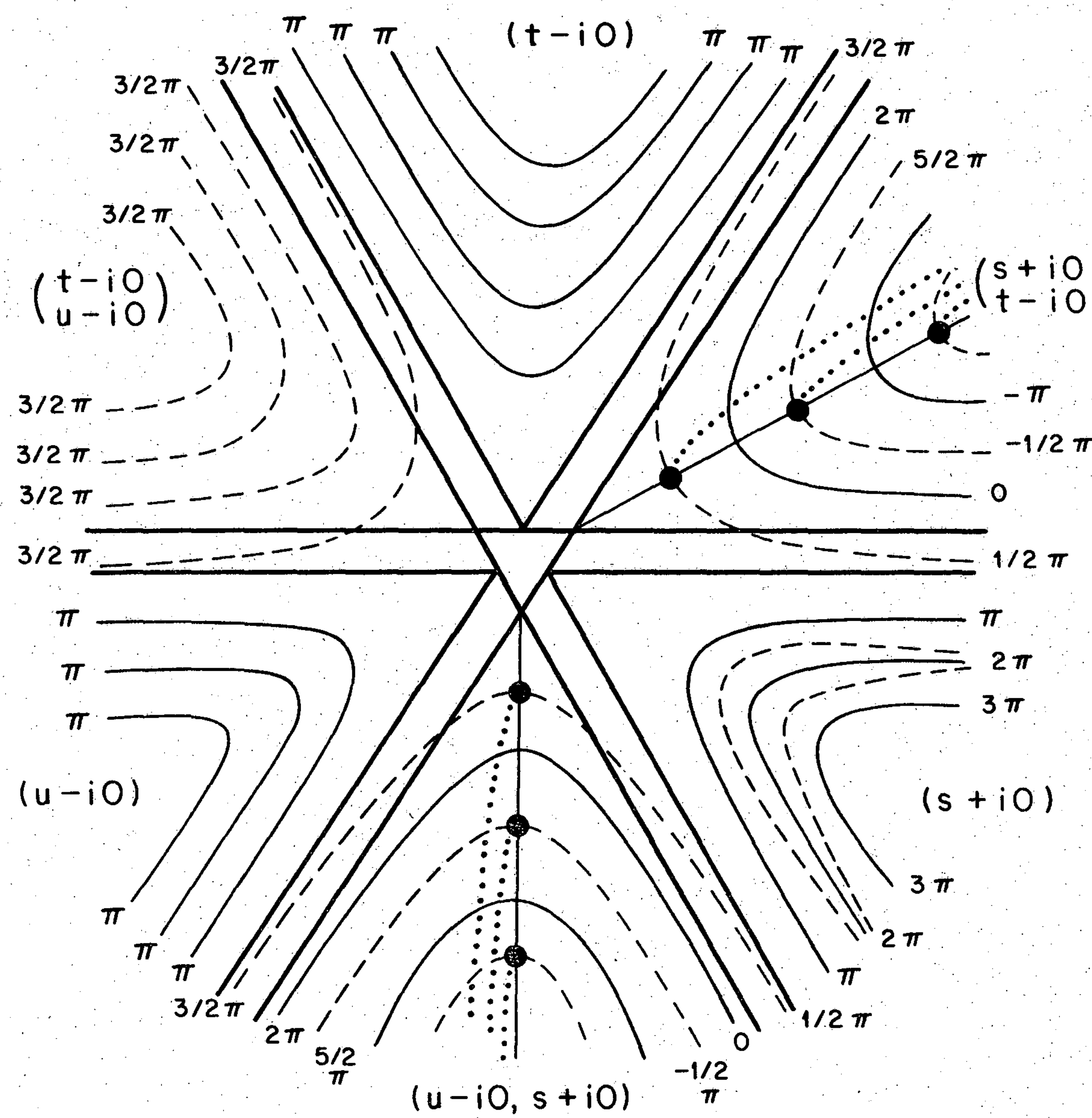

Fig. 7. Phase contours for a crossing symmetric amplitude in the limit (s + i0, t - i0, u - i0). The apparent lack of symmetry is due to the existence of complex zeros, some of which are indicated by the dotted lines. The black circles indicate real zeros.

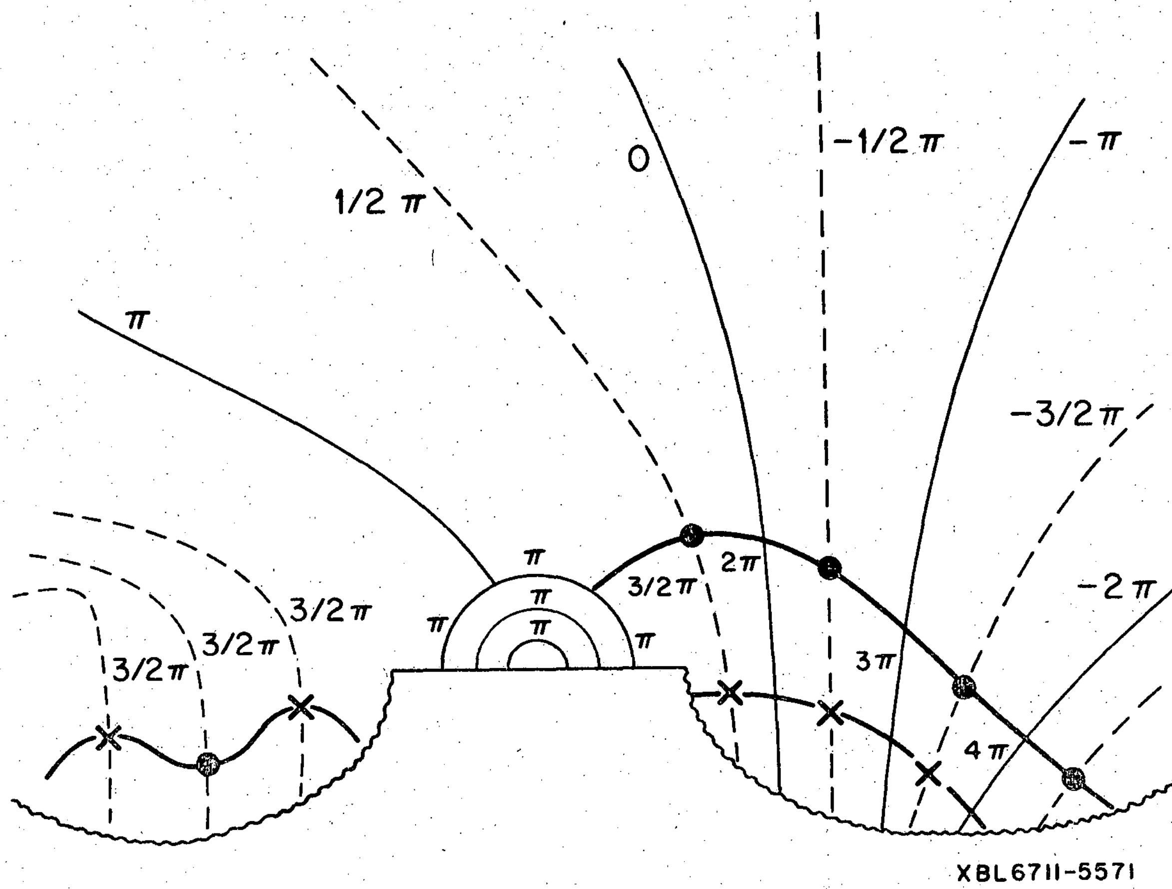

Fig. 8. Phase contours in the complex s plane, for fixed $t > 4m^2$, that correspond to the real section shown in Fig. 7. Poles are denoted by crosses and zeros by circles.