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Surface Bundles Over Low-Dimensional Manifolds

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Jonathan Alcaraz

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Dissertation Committee:

Dr. Stefano Vidussi, Chairperson
Dr. Matthew Durham
Dr. Peter Samuelson

The Dissertation of Jonathan Alcaraz is approved:

Committee Chairperson

University of California, Riverside

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To my parents, for their love and support.

ABSTRACT OF THE DISSERTATION

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by

Jonathan Alcaraz

Doctor of Philosophy, Graduate Program in Mathematics
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Surface bundles over circles and surfaces have been and continue to be well studied. The main focus of this thesis is manifolds which admit multiple surface bundle structures over low-dimensional manifolds. In [Sal15], Salter proved certain cohomological properties of 4-manifolds with multiple surface bundle structures over surfaces. In this thesis, we will analyze the extent to which analogous results are true in other dimensions as well as construct examples of spaces with multiple surface bundle structures with punctured surfaces as fibers.

List of Figures

1.1 The sphere ($S^2 = \Sigma_0$), the torus ($T^2 = \Sigma_1$), the surface of genus 2 (Σ_2) . 3
1.2 A punctured surface deformation retracts to a compact surface with one boundary component 5

Contents

1 Preliminaries & Background **1**
1.1 A Brief History of Fiber Bundles 1
1.2 Low-Dimensional Manifolds and Their Fundamental Groups 2
1.2.1 Surfaces and Surface Groups 3
1.2.2 Punctured Surfaces and Free Groups 4
1.2.3 3-manifolds and 3-manifold Groups 5
1.2.4 The Finitely Generated Normal Property of Groups 7
1.3 Surface Bundles 8
1.3.1 Fibered 3-manifolds and Seifert-Fibered Spaces 14
1.3.2 Surface Bundles Over Surfaces 19

2 Surface-Bundles Over 3-manifolds **25**
2.1 Almost Unique Surface-Fibered 5-manifolds 26
2.1.1 A Group Theoretic Generalization of Lemma 1.3 26
2.1.2 Applying Lemma 2.2 to Surface Bundles Over 3-manifolds . . 29
2.2 Examples of 5-manifolds With Multiple Surface-Fiberings 31
2.2.1 Trivial Bundles With Multiple Fiberings 31
2.2.2 Pullback Bundles 32
2.3 Further Questions 36

3 Free-by-Surface Groups **38**
3.1 Multisections 39
3.2 Groups With Multiple Free-by-Surface Structures 41
3.3 Further Questions 43

Chapter 1

Preliminaries & Background

1.1 A Brief History of Fiber Bundles

The study of fiber bundles dates back to 1935 when Whitney wrote in [Whi35] about spaces "in which the points themselves are spaces of some simple sort." He was particularly interested in the case when the "points themselves" were spheres and hence termed these spaces *sphere spaces*. Along with other topologists and differential geometers, Whitney continued to study these objects into the 1940s and eventually they coined the term *sphere bundles*. Study of these objects continued, leading to the general study of fiber bundles, and in 1951, Steenrod wrote the first expository text in the subject, *The Topology of Fibre Bundles* [Ste51].

During the mid 20th century, physicists began to observe certain symmetries, called "gauge" symmetries, of field theories. In the early 1950s, physicists Yang and

Mills began their study of non-abelian gauge theories in order to understand the so-called strong interactions. This eventually lead to the standard model of particle physics, which unifies three fundamental forces of physics in the language of gauge theory. Though developed independently from the theory of fiber bundles, the mathematical formalization of gauge theory can be understood in terms of *principal bundles*, that is a fiber bundle with a fiber-preserving, regular action by diffeomorphisms from a Lie group called the *structure group* of the bundle.

Since then, much of the study has focused on *surface bundles*, that is fiber bundles whose fibers are 2-dimensional manifolds, particularly those whose bases are themselves circles or surfaces. In this thesis, we will explore certain properties of such surface bundles and generalize them to higher dimensions.

1.2 Low-Dimensional Manifolds and Their Fundamental Groups

As mentioned above, we will be discussing surface bundles over low-dimensional manifolds, that is, surface bundles whose base space is a circle, surface, or 3-manifold. As such, we need to understand certain properties of these spaces and their fundamental groups. Since the fundamental group of the circle is simply the free cyclic group, we will focus on the latter two.

1.2.1 Surfaces and Surface Groups

In general, when discussing surfaces, we refer to manifolds of dimension 2 which are connected, orientable, and closed, that is, they are compact and without boundary. Such surfaces are completely classified by their genus, so we denote the surface of genus g by Σ_g. See Figure 1.1 for cartoon examples of these surfaces.

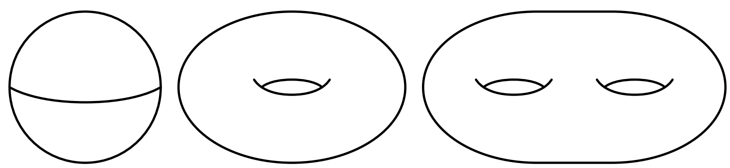


Figure 1.1: The sphere ($S^2 = \Sigma_0$), the torus ($T^2 = \Sigma_1$), the surface of genus 2 (Σ_2)

Typically, we focus on surfaces whose genera are at least two. We often refer to such surfaces as *hyperbolic surfaces* due to the Gauss-Bonnet theorem. That is, when Σ_g is endowed with a Riemannian metric of constant curvature κ, then

$$\kappa = \frac{2\pi\chi(\Sigma_g)}{\mathrm{Area}(\Sigma_g)} \tag{1.1}$$

where $\chi(\Sigma_g)$ is the Euler characteristic of Σ_g. For surfaces, $\chi(\Sigma_g) = 2-2g$ and thus κ is negative—and hence Σ_g is hyperbolic—precisely when $g \geqslant 2$. A more detailed explanation of this can be seen in an introductory text in differential geometry such as [Kob95].

The fundamental group of a surface of genus g can be computed by considering Σ_g as the quotient space obtained by taking a regular $4g$-gon and identifying suitable sides. With this model, we can use the Seifert-Van Kampen Theorem to compute the fundamental group of a surface as:

$$\pi_1\Sigma_g = \left\langle a_1, \dots, a_g, b_1, \dots, b_g \;\middle|\; \prod_{i=1}^{g}[a_i, b_i] \right\rangle . \tag{1.2}$$

Any group with such a presentation is referred to as the *surface group of genus* g and is often denoted Π_g. Moreover, any surface group can be realized as the fundamental group of a surface and in fact, Σ_g is an Eilenberg-MacLane space $K(\Pi_g, 1)$, namely its homotopy groups are

$$\pi_i\Sigma_g = \begin{cases} \Pi_g & \text{if } i = 1 \\ \{1\} & \text{otherwise.} \end{cases} \tag{1.3}$$

1.2.2 Punctured Surfaces and Free Groups

So far, we've been discussing closed surfaces, but there may be instances where we'd like to consider surfaces with finitely many points removed. Such surfaces are called *punctured surfaces*, denoted Σ_g^b where b is the number of points removed, called *punctures*. A punctured surface admits a deformation retraction to a compact subsurface with as many boundary components as the punctures in the original surface.

Further, the punctured surface is homotopy equivalent to a wedge of circles. The number of circles is equal to $2g + b - 1$ where $g \geqslant 0$ is the genus of the surface and $b \geqslant 1$ is the number of punctures or boundary components. To see this, for a fixed

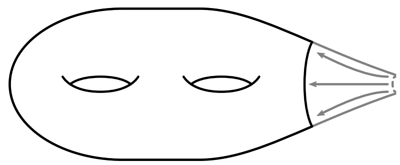


Figure 1.2: A punctured surface deformation retracts to a compact surface with one boundary component

g, one can induct on b. When $b = 1$, Σ_g^1 can deformation retract onto the 1-skeleton of Σ_g—a wedge of $2g$ circles. With each additional puncture, the 1-skeleton gains an extra circle.

So the fundamental group of a punctured surface is the free group of rank $2g+b-1$,

$$\pi_1 \Sigma_g^b \cong F_{2g+b-1} \,. \tag{1.4}$$

Moreover, any free group of finite rank can be exhibited as the fundamental group of a punctured surface.

1.2.3 3-manifolds and 3-manifold Groups

Unless otherwise stated, when we refer to the 3-manifolds, we mean manifolds of dimension 3 which are connected, orientable, and closed.

Definition 1.1. The *connected sum* of two 3-manifolds N_1 and N_2 is a 3-manifold N, denoted

$$N = N_1 \# N_2, \tag{1.5}$$

obtained by removing the interior of closed balls B_1 and B_2 from each of N_1 and N_2 respectively and gluing them together along their boundaries via an orientation-reversing diffeomorphism of $\varphi\colon \partial B_1 \to \partial B_2$, namely

$$N_1 \# N_2 = (N_1 - \mathrm{Int}(B_1)) \cup_{\varphi} (N_2 - \mathrm{Int}(B_2)). \tag{1.6}$$

Note that any two orientation-reversing self-diffeomorphisms of S^2 are isotopic and hence $N_1 \# N_2$ does not depend on φ.

A 3-manifold N is said to be *prime* if for any decomposition of the form (1.5), either $N_1 = S^3$ or $N_2 = S^3$. A 3-manifold is *irreducible* if every embedded 2-sphere bounds a 3-ball.

Consider an irreducible 3-manifold N expressed as a connected sum as in (1.5). Let S denote the embedded 2-sphere in N given by the boundaries of $N_i - \mathrm{Int}(B_i)$. Since N is irreducible, S bounds a 3-ball, say $B \subseteq N$. Since the components of $N - S$ are $N_i - B_i$, then either $B \subseteq N_1 - \mathrm{Int}(B_1)$ or $B \subseteq N_2 - \mathrm{Int}(B_2)$. Without loss of generality, suppose the former. Since B is a compact submanifold of codimension 0 in the compact manifold $N_1 - \mathrm{Int}(B_1)$, they must be equal. Therefore, $N_1 = B \cup B_1 = S^3$ and N is prime. From this, it follows that S^3 itself is a prime 3-manifold. Moreover, the only prime 3-manifold which is not irreducible is $S^2 \times S^1$. More information regarding prime 3-manifolds can be found in [Hat07] including proof of the following theorem:

Theorem 1.1 (PRIME DECOMPOSITION). Any 3-manifold N can be decomposed as

$$N = N_1 \# N_2 \# \cdots \# N_n \tag{1.7}$$

where N_i are prime. Moreover, this decomposition is unique up to permutation and inserting or removing S^3.

Any non-prime 3-manifold N therefore admits a prime decomposition as in 1.7 where all $N_i \not\cong S^3$ and $n > 1$. In particular, $\pi_1 N_i$ are nontrivial by the Poincaré conjecture and thus

$$\pi_1 N = \pi_1 N_1 * \cdots * \pi_1 N_n \tag{1.8}$$

is a *proper free product*. These decompositions of 3-manifolds and their fundamental groups will be helpful for working with these objects. In the next section we will see a useful property of the fundamental groups of non-prime 3-manifolds.

1.2.4 The Finitely Generated Normal Property of Groups

Definition 1.2. A group has the *finitely generated normal (f.g.n.)* property if each of its non-trivial finitely generated normal subgroups has finite index.

This notation is taken from [KS73]. In the literature, such as [Cat03], this property is often referred to as the *NINF* property since *normal* nontrivial subgroups of *infinite* index are *not finitely-generated*. Here let's see some examples.

Example 1.1. Surface groups $\Pi_g = \pi_1 \Sigma_g$ have the f.g.n. property for $g \geqslant 2$.

A thorough proof of this classical result can be found in [Cat03].

Example 1.2. Free groups F_n have the f.g.n. property.

Proof. In the case where $n = 1$, follows since any nontrivial subgroup of $\mathbb{Z}$ has finite index.

In the case where $n \geqslant 2$, consider F_n as the fundamental group $\pi_1 Y$ where Y is a wedge of n circles. If K is a non trivial finitely generated normal subgroup of $\pi_1 Y$, then let Z be the cover of Y corresponding to K, namely $K = \pi_1 Z$. Thus, K is a free group and Z is a wedge of circles. Since K is finitely generated, Z is compact and since standard covering theory shows that compact covers of a compact space are of finite degree, K has finite index in $\pi_1 Y$. □

Example 1.3. *Proper free products*, i.e. free products of non-trivial groups, have the f.g.n. property.

The proof of this can be found in the last section of [Bau66].

1.3 Surface Bundles

Definition 1.3. A *surface bundle structure* on a space X, or *surface fibering* pf X, is a locally trivial fiber bundle $p : X \to B$ such that all fibers are diffeomorphic to a surface of genus at least 2, that is, every point $b \in B$ admits a *trivialization* *neighborhood* U with a diffeomorphism $\varphi\colon p^{-1}(U) \to \Sigma \times U$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \varphi\ } & \Sigma \times U \\ & \searrow^{p} & \downarrow{\scriptstyle \pi_U} \\ & & U \end{array} \tag{1.9}$$

commutes. The surface $p^{-1}(b) \cong \Sigma$ is called the *fiber* over b and we use the phrase *surface bundle over* B, or simply a *surface bundle*, to describe the structure (X, p). We refer to B as the *base space* and to X as the *total space*.

We will specifically study surface bundles whose base spaces are low-dimensional manifolds such as circles, surfaces, or 3-manifolds, in which case the total space X is also a manifold with dimension

$$\dim(X) = \dim(\Sigma) + \dim(B) = 2 + \dim(B)\,. \tag{1.10}$$

Example 1.4. Given a manifold B and a hyperbolic surface Σ the space $\Sigma \times B$ is a surface bundle with the projection map

$$p\colon \Sigma \times B \to B : (x, b) \mapsto b\,. \tag{1.11}$$

We call such an example a *trivial bundle* since the trivialization neighborhood of any point can be taken to be the entire base space B.

Example 1.5. Let Σ be a hyperbolic surface and $\varphi\colon \Sigma \to \Sigma$ be an orientation-preserving diffeomorphism. The *mapping torus* of Σ with respect to φ,

$$N_\varphi := \frac{\Sigma \times [0,1]}{(x,0) \sim (\varphi(x),1)} \tag{1.12}$$

admits a surface bundle structure

$$p\colon N_\varphi \to S^1 : [x,t] \mapsto [t] \tag{1.13}$$

with fiber Σ. The diffeomorphism type of N_φ is determined by isotopy type of φ, that is, if φ_0 is isotopic to φ_1, then $N_{\varphi_0} \cong N_{\varphi_1}$. The isotopy class of φ is often referred to as the *monodromy* of the bundle N_φ.

Example 1.6. Consider two hyperbolic surfaces Σ_1 and Σ_2. The trivial bundle in this case admits two fiberings:

$$p_i\colon \Sigma_1 \times \Sigma_2 \to \Sigma_i\,, \tag{1.14}$$

one for each projection.

Given a surface fibering $X \to B$ over a connected manifold with connected fibers, we get a long exact sequence of homotopy groups:

$$\cdots \to \pi_2 B \to \pi_1 \Sigma \to \pi_1 X \to \pi_1 B \to 1 \tag{1.15}$$

where Σ is the fiber and the map $\pi_1\Sigma \to \pi_1 X$ is the map induced by the inclusion of $\Sigma \hookrightarrow B$. The proof of this statement can be found in introductory texts to algebraic topology such as [Hat02]. In many cases of our interest, the map $\pi_2 B \to \pi_1\Sigma$ is trivial. For example, if B has a contractible universal cover, as in the case where $B = S^1$ or B is a hyperbolic surface, then the long exact sequence becomes a short exact sequence

$$1 \to \pi_1\Sigma \to \pi_1 X \to \pi_1 B \to 1\,. \tag{1.16}$$

For a given space X, we may wonder whether it admits more than one surface bundle structure. We can see from Example 1.6 that it is possible for a space to have multiple surface fiberings. In order to study this further, we should first clarify what it means for surface bundles structures to be distinct.

Definition 1.4. Two surface fiberings $p_1\colon X \to B_1$ and $p_2\colon X \to B_2$ are *fiberwise diffeomorphic* if there exist diffeomorphisms $\varphi\colon X \to X$ and $\alpha\colon B_1 \to B_2$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\alpha} & B_2 \end{array} \tag{1.17}$$

This definition is an equivalence of surface fiberings, but one may find it too coarse for some purposes. For example, the fiberings in Example 1.6 would be reasonably expected to be distinct, which is the case in most instances. However, in the case where $\Sigma_1 = \Sigma_2$, we can use the diffeomorphisms $\varphi\colon \Sigma_1 \times \Sigma_2 \to \Sigma_1 \times \Sigma_2$ defined by $\varphi(x, y) = (y, x)$ and let α be the identity map $\Sigma_1 \to \Sigma_2$. This satisfies the definition above.

Definition 1.5. Two surface fiberings $X \to B_1$ and $X \to B_2$ are *π_1-fiberwise diffeomorphic* if they are fiberwise diffeomorphic and $\varphi_*(\pi_1\Sigma_1) = \pi_1\Sigma_1$ where Σ_1 is the fiber of $X \to B_1$ and we identify $\pi_1\Sigma_1$ with the kernel of $\pi_1 X \to \pi_1 B_1$.

In Example 1.6, if $\Sigma_1 = \Sigma_2$, then $\varphi_*(\pi_1\Sigma_1 \oplus 1) = 1 \oplus \pi_1\Sigma_2$. Therefore, the projections p_1 and p_2 are fiberwise diffeomorphic but not π_1-fiberwise diffeomorphic.

In general, if $p_1\colon X \to B_1$ and $p_2\colon X \to B_2$ are fiberwise diffeomorphic, it follows that $\varphi_*(\pi_1\Sigma_1) = \pi_1\Sigma_2$. So we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1\Sigma_1 & \longrightarrow & \pi_1 X & \longrightarrow & \pi_1 B_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \alpha_*} & & \\ 1 & \longrightarrow & \pi_1\Sigma_2 & \longrightarrow & \pi_1 X & \longrightarrow & \pi_1 B_2 & \longrightarrow & 1 \end{array} \tag{1.18}$$

In particular, if p_1 and p_2 are π_1-fiberwise diffeomorphic, then φ_* restricts to an automorphism of $\pi_1\Sigma_1$ and so $\pi_1\Sigma_1 = \pi_1\Sigma_2$. In fact, the converse is true as shown in Proposition 2.1 of [Sal16]. Therefore, we introduce the following definition:

Definition 1.6. We say two surface bundle structures on X are *equivalent* if their induced maps on fundamental groups have the same kernel.

We can call a surface fibering $p\colon X \to B$ unique if all other surface fiberings of X are equivalent to p. However, in many cases, a fibering is not unique. In fact, we can ask how many ways a surface bundle fibers. As such, we introduce the following definition.

Definition 1.7. Given a surface bundle X, the *surface fibering number* is given by

$$\mathrm{sFib}(X) = \#\big\{\text{surface fiberings } p\colon X \to B\big\}/\sim \;\in \mathbb{N} \cup \{\infty\} \tag{1.19}$$

where $p_1 \sim p_2$ if and only if they are equivalent.

Given a surface bundle $p\colon X \to B$ over a manifold B, consider the induced map on cohomology $p^*\colon H^1(B;\mathbb{Z}) \to H^1(X;\mathbb{Z})$. By identifying $H^1(X;\mathbb{Z}) = \mathrm{Hom}(\pi_1 X, \mathbb{Z})$ and $H^1(B;\mathbb{Z}) = \mathrm{Hom}(\pi_1 B, \mathbb{Z})$, we can realize p^* by defining $p^*(\alpha) = \alpha \circ p_*$ where p_* is the map induced on fundamental groups.

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{p_*} & \pi_1 B \\ & \searrow^{p^*\alpha} & \downarrow \alpha \\ & & \mathbb{Z} \end{array} \tag{1.20}$$

Moreover, since the map p_* is surjective, if $p^*\alpha = \alpha \circ p_* = 0$, then $\alpha = 0$. Thus p^* is injective and in particular

$$b_1 B \leqslant b_1 X. \tag{1.21}$$

The injectivity of p^* allows us to think of $H^1(B;\mathbb{Z})$ as a subgroup of $H^1(M;\mathbb{Z})$ or by considering rational coefficients, $H^1(B;\mathbb{Q})$ as a vector subspace of $H^1(M;\mathbb{Q})$.

1.3.1 Fibered 3-manifolds and Seifert-Fibered Spaces

A 3-manifold may admit bundle structures as either a surface bundle over a circle or as a circle bundle over a surface; in fact, the trivial bundle shows that a 3-manifold can admit both structures simultaneously. In the study of 3-manifold topolgy, the phrase *fibered 3-manifold* usually refers to the former, but not the latter. For the sake of this thesis, we will study surface bundles over circles as a starting point for studying surface bundles in general and we will look at circle bundles over surfaces as groundwork for exploring surface bundles over 3-manifolds.

	circle bundle	surface bundle
base	Σ_g	S^1
fiber	S^1	Σ_g

Surface Bundles Over Circles

As discussed in Example 1.5, a mapping torus is an example of a surface bundle over the circle. In fact, this is the only example of a surface bundle over a circle. To see this, we use the following bijective correspondence, valid for any dimension of base manifold B:

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{oriented }\Sigma\text{- bundles over }B\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conjugacy classes of}\\ \text{representations}\\ \rho\colon \pi_1 B \to \mathrm{MCG}(\Sigma)\end{array}\right\} \tag{1.22}$$

where $\mathrm{MCG}(\Sigma)$ is the *mapping class group* of Σ, that is the group of isotopy classes of orientation-preserving diffeomorphisms $\Sigma \to \Sigma$. A more thorough discussion of this correspondence can be found in [FM12] in its full generality, however in the case where $B = S^1$ and hence $\pi_1 B = \mathbb{Z}$, the image of ρ is the cyclic subgroup generated by some mapping class of Σ. In particular, a mapping torus N_φ will correspond with the representation $\rho\colon \mathbb{Z} \to \mathrm{MCG}(\Sigma)$ given by $1 \mapsto [\varphi]$. So any Σ-bundle over S^1 is equivalent to the mapping torus with monodromy given by the generator of $\rho(\mathbb{Z})$.

Since $\pi_2 S^1$ is trivial, then a surface fibering $N \to S^1$ of a 3-manifold N determines the short exact sequence

$$1 \to \pi_1\Sigma \to \pi_1 N \to \mathbb{Z} \to 1. \tag{1.23}$$

Since $\pi_1\Sigma$ is a normal finitely-generated subgroup of infinite index in $\pi_1 N$, the fundamental group of a surface bundle over a circle does not have the f.g.n. property. In [Thu86], Thurston describes a norm on $H_2(N;\mathbb{R})$ for any 3-manifold N with $b_1 N > 0$. The unit ball with this norm is a polyhedron with vertices at points in $H_2(N;\mathbb{Q}) \subseteq H_2(N;\mathbb{R})$. Moreover, if N admits a fibering $N \to S^1$ with fiber Σ, then $[\Sigma]$, called a *fibered class*, determines a ray in $H_2(N;\mathbb{R})$ which passes through a top-dimensional face of the polyhedron. Moreover, Thurston determines that any other ray of rational slope through this face is determined by the fiber of some other non-equivalent surface fibering of N.

In particular, if $b_1 N > 1$, then $\mathrm{sFib}(N) = \infty$. In the case of a fibered 3-manifold with $b_1 N = 1$, we claim $\mathrm{sFib}(N) = 1$. The techniques used to prove this claim are generalized and used to prove Lemma 2.2.

Circle Bundles Over Surfaces and Seifert-fibered Spaces

Another class of 3-manifolds which admit fibrations are circle bundles over surfaces, that is, locally trivial fiber bundles over surfaces whose fibers are circles. Assuming the base surface Σ is hyperbolic, then a circle bundle N induces a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1 N \to \pi_1 \Sigma \to 1 \tag{1.24}$$

and since Z is a normal finitely-generated normal subgroup of infinite index, the fundamental group of N does not have the f.g.n. property.

A generalization of circle bundles over surfaces are *Seifert-fibered spaces*, which are 3-manifolds which can be decomposed as a disjoint union of copies of embedded circles called *fibers* so that each fiber has a tubular neighborhood which is a *standard fibered torus*, that is, the mapping torus of an open disc with respect to rotation by a rational multiple of 2π. We refer to fibers where this is rotation by an integer multiple of 2π as *ordinary* fibers and otherwise refer to the fiber as *exceptional*, of which there are only finitely many. The quotient given by identifying the fibers is a surface with certain marked points, called *orbifold points*, corresponding to the exceptional fibers. We call this quotient an *orbisurface*. The fundamental group of a

Seifert-fibered space admits the exact sequence

$$\mathbb{Z} \to \pi_1 N \to \pi_1^{orb}\Sigma \to 1 \tag{1.25}$$

where $\pi_1^{orb}\Sigma$ is the *orbifold fundamental group* of Σ. Except in the case where N is a spherical 3-manifold—that is a 3-manifold which is finitely covered by the 3-sphere—the map $\mathbb{Z} \to \pi_1 N$ will be injective and $\pi_1^{orb}\Sigma$ is infinite, hence $\pi_1 N$ does not have the f.g.n. property.

3-manifold Groups and the Finitely Generated Normal Property

We've seen in this section three classes of 3-manifolds whose fundamental groups lack the f.g.n. property, namely

- surface bundles over circles,
- circle bundles over surfaces of genus at least 1, and more generally,
- aspherical Seifert-fibered spaces.

All of which are prime since any non-prime 3-manifold can be decomposed as $N = N_1 \# N_2$ with $N_1 \ncong S^3 \ncong N_2$ and thus

$$\pi_1 N = \pi_1(N_1 \# N_2) = \pi_1 N_1 * \pi_1 N_2 \tag{1.26}$$

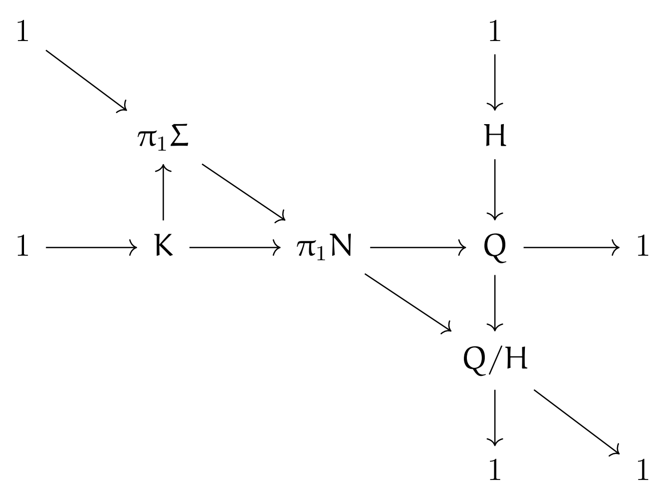
would be a free product of nontrivial groups and by Example 1.3 would satisfy the f.g.n. property.

One might wonder what other types of 3-manifolds have fundamental groups which fail to satisfy the f.g.n. property. The following theorem from [Hem76] elaborates on this question.

Theorem 1.2. Let N be a 3-manifold and K a finitely generated normal subgroup of $\pi_1 N$ with infinite quotient Q. Then one of the following happens:

1. $K \cong \mathbb{Z}$, in which case N is Seifert-fibered over a surface Σ.

2. K is a surface group and Q contains a finite normal subgroup H so that the following diagram commutes:



(1.27)

where either

(a) $Q/H \cong \mathbb{Z}$, in which case N is a Σ-bundle over S^1, or

(b) $Q/H \cong \mathbb{Z}_2 * \mathbb{Z}_2$, in which case N has a two-sheeted regular cover which is a Σ-bundle over S^1 arising from the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1\Sigma & \longrightarrow & \pi_1\tilde{N} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1\Sigma & \longrightarrow & \pi_1 N & \longrightarrow & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}_2 & = & \mathbb{Z}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{1.28}$$

We can summarize this theorem by saying that the only 3-manifolds whose fundamental groups do not have the f.g.n. property are finitely covered by a surface bundle over a circle or are Seifert-fibered space. Moreover the only nontrivial finitely generated normal subgroups of infinite index in 3-manifold groups are surface groups or $\mathbb{Z}$.

1.3.2 Surface Bundles Over Surfaces

In this section, we will see how surface bundles over surfaces are similar to and differ from surface bundles over circles as well as look at examples of surface bundles over surfaces.

Given a surface bundle $p \colon M \to \Sigma$ over a surface, we've seen that we can consider $H^1(\Sigma; \mathbb{Q})$ as a vector subspace of $H^1(M; \mathbb{Q})$ since $p^* \colon H^1(\Sigma; \mathbb{Q}) \to H^1(M; \mathbb{Q})$ is injective. The following lemma shows how distinct fiberings interact with each other in cohomology.

Lemma 1.3. Consider a 4-manifold M with non-equivalent surface-fiberings $p_1 \colon M \to \Sigma_1$ and $p_2 \colon M \to \Sigma_2$ over hyperbolic surfaces Σ_1 and Σ_2. Then

$$p_1^*(H^1(\Sigma_1;\mathbb{Q})) \cap p_2^*(H^1(\Sigma_2;\mathbb{Q})) = \{0\} \subseteq H^1(M;\mathbb{Q}). \tag{1.29}$$

This result can be found as Lemma 3.3 in [Sal16] and a proof of a more general result can be found in the following chapter.

In particular, if M is a surface bundle over a hyperbolic surface Σ where $b_1\Sigma = b_1M$, then this surface bundle structure is unique. This is analogous to the case of a surface bundle N over S^1 and $b_1N = 1$. In fact, we can take this idea a bit further with the following corollary:

Corollary 1.4. Suppose $p \colon M \to \Sigma$ is a surface bundle structure on a 4-manifold M. If $b_1M < b_1\Sigma + 4$, then p is unique.

Proof. Suppose M admits distinct surface bundle structures $p_1 \colon M \to \Sigma_1$ and $p_2 \colon M \to \Sigma_2$ where Σ_2 has genus g. By Lemma 1.3,

$$b_1M \geqslant b_1\Sigma_1 + b_1\Sigma_2 \tag{1.30}$$

Since $b_1\Sigma_2 = 2g$ and $g \geqslant 2$, then $b_1M \geqslant b_1\Sigma_1 + 4$. ☐

This result also shows that the case where $b_1\Sigma < b_1M$ does not necessarily follow analogously to the lower dimensional case, since a surface bundle over S^1 admits infinitely many surface bundle structures whenever $1 < b_1N$. Moreover, it follows from [Joh94] that any 4-manifold admits at most finitely many surface bundle structures over hyperbolic surfaces. In fact, we see in [Sal15] that given any positive integer n, there is a 4-manifold which admits at least n distinct surface bundle structures. Salter's proof of this is constructive; it builds 4-manifolds with n distinct surface fiberings, but it is not clear whether the constructed manifolds admit any other surface bundle structures.

Constructing the Atiyah-Kodaira Fibration

In [Kod67] and [Ati69], Kodaira and Atiyah independently developed constuctions of a 4-manifold which admits two distinct nontrivial fiberings. Here, we will see the construction of this useful example of a nontrivial surface bundle. We provide the full details here since we will discuss a modification to this construction in chapter 3.

Start with the compact oriented surface Σ_2 of genus 2. Consider a double cover of Σ_2 by a surface Σ_3 of genus 3. The deck transformation of this cover gives a free $\mathbb{Z}_2$-action on Σ_3. Denote by ι the involution on Σ_3 given by the nontrivial deck transformation.

We define the map

$$\rho : \pi_1(\Sigma_3) \xrightarrow{\text{Ab}} H_1(\Sigma_3; \mathbb{Z}) \xrightarrow{\text{mod } 2} H_1(\Sigma_3; \mathbb{Z}_2) \cong \mathbb{Z}_2^{\oplus 6} \tag{1.31}$$

There exists a regular cover $f : \Sigma_h \to \Sigma_3$ such that $\ker(\rho) \cong \pi_1(\Sigma_h)$. Moreover, f is cover of degree

$$[\pi_1(\Sigma_3) : \pi_1(\Sigma_h)] = \left|\mathbb{Z}_2^{\oplus 6}\right| = 64. \tag{1.32}$$

Covering space theory says tells us that

$$\chi(\Sigma_h) = 64\chi(\Sigma_3) \tag{1.33}$$

$$2 - 2h = 64(2 - 2 \cdot 3) \tag{1.34}$$

$$h = 129 \tag{1.35}$$

Consider the graphs $\Gamma_f = \{x, f(x)\} \subseteq \Sigma_{129} \times \Sigma_3$ and $\Gamma_{\iota f} = \{x, \iota f(x)\} \subseteq \Sigma_{129} \times \Sigma_3$ of f and $\iota \circ f$ respectively. Since ι has no fixed points, then $\Gamma_f \cap \Gamma_{\iota f} = \emptyset$. Moreover, these graphs represent cohomology classes in $H^2(\Sigma_{129} \times \Sigma_3; \mathbb{Z})$ and it can be shown that $[\Gamma_f] + [\Gamma_{\iota f}]$ is even.

As a result [BPVdV84], we can construct a degree 2 branch cover branched along $\Gamma_f \sqcup \Gamma_{\iota f}$, call this surface M_{AK}. So we have a 2-fold branch cover

$$\varphi : M_{AK} \to \Sigma_{129} \times \Sigma_3 \tag{1.36}$$

which we post compose with projection onto Σ_{129} to get

$$M_{AK} \to \Sigma_{129}, \tag{1.37}$$

a surface bundle over Σ_{129}. To determine the fiber, pick a point $x \in \Sigma_{129}$. The preimage of this point with respect to the projection is $\{x\} \times \Sigma_3$ and the preimage of this with respect to φ is a double branch cover of Σ_3 branched at 2 points, $f(x)$ and $\iota f(x)$. The Hurwitz formula tells us that the genus of this branch cover is given by

$$2g - 2 = 2(2 \cdot 3 - 2) + 2 \tag{1.38}$$

$$g = 6. \tag{1.39}$$

So M_{AK} fibers as

$$\Sigma_6 \hookrightarrow Z \twoheadrightarrow \Sigma_{129} \tag{1.40}$$

Alternatively, we could post compose with the other projection giving a fibering

$$M_{AK} \to \Sigma_3 \tag{1.41}$$

Pick $x \in \Sigma_3$. The preimage of this point with respect to the projection is $\Sigma_{129} \times \{x\}$. Note this intersects the ramification locus at $f^{-1}(x) \cup f^{-1}(\iota x)$. Since f is a 64-sheeted cover, this is a discrete set of 128 points. The preimage of Σ_{129} with respect to φ is a double branch cover of Σ_{129} branched at these points. Again applying the Hurwitz formula, the fiber is a surface with genus given by

$$2g - 2 = 2(2 \cdot 129 - 2) - 128 \tag{1.42}$$

$$g = 321. \tag{1.43}$$

So M_{AK} also fibers as

$$\Sigma_{321} \hookrightarrow M_{AK} \twoheadrightarrow \Sigma_3 \tag{1.44}$$

While both fiberings were known since their initial discovery in the 1960s, it wasn't until [Che18] when Chen proved that indeed $\mathrm{sFib}(M_{AK}) = 2$.

Chapter 2

Surface-Bundles Over 3-manifolds

Given a surface bundle $p\colon X \to B$ with base either S^1 or a surface Σ, the following table summarizes the relationship between the Betti numbers of the base and total spaces with the number of distinct fibering of the total space:

	$b_1B = b_1X$	$b_1B < b_1X$
$B = S^1$	$\mathrm{sFib}(X) = 1$	$\mathrm{sFib}(X) = \infty$
$B = \Sigma$	$\mathrm{sFib}(X) = 1$	$1 \leqslant \mathrm{sFib}(X) < \infty$

In this chapter, we will explore the case when B is a 3-manifold and answer analogous questions to those answered by this table.

2.1 Almost Unique Surface-Fibered 5-manifolds

In this section, we will explore a generalization of Lemma 1.3 in purely in terms of group theory and see how it applies to surface bundles over 3-manifolds.

2.1.1 A Group Theoretic Generalization of Lemma 1.3

The following two lemma approach the proof of Lemma 1.3 in terms of only the abstract group theoretic properties.

Lemma 2.1. Let G be a group which admits epimorphisms $p_1 \colon G \to Q_1$ and $p_2 \colon G \to Q_2$ with distinct finitely generated non-trivial kernels K_1 and K_2 respectively.

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
& \searrow & & & & & & & \\
& & K_2 & & & & & & \\
& & & \searrow & & & & & \\
1 & \longrightarrow & K_1 & \longrightarrow & G & \xrightarrow{p_1} & Q_1 & \longrightarrow & 1 \\
& & & & & \searrow^{p_2} & & & \\
& & & & & & Q_2 & & \\
& & & & & & & \searrow & \\
& & & & & & & & 1
\end{array} \tag{2.1}$$

If $[Q_1 : p_1(K_2)] < \infty$, then

$$p_1^*(\mathrm{Hom}(Q_1, \mathbb{Q})) \cap p_2^*(\mathrm{Hom}(Q_2, \mathbb{Q})) = \{0\}\,. \tag{2.2}$$

Proof. Firstly, we notice that if a character $\alpha \in \mathrm{Hom}(G, \mathbb{Q})$ vanishes on a finite index subgroup of G, then $\alpha = 0$. To see why, consider such a character α. Since it vanishes on a finite index subgroup of G, $\ker(\alpha)$ has finite index in G. Therefore, $G/\ker(\alpha)$ is a finite subgroup of $\mathbb{Q}$. However, the only finite subgroup of $\mathbb{Q}$ is the trivial group. Thus $\alpha = 0$.

Take $\alpha \in p_1^*(\mathrm{Hom}(Q_1, \mathbb{Q})) \cap p_2^*(\mathrm{Hom}(Q_2, \mathbb{Q})) \subseteq \mathrm{Hom}(G, \mathbb{Q})$. Since $\alpha \in p_1^*(\mathrm{Hom}(Q_1, \mathbb{Q}))$, then $\alpha = p_1^*(\beta)$, for some $\beta \in \mathrm{Hom}(Q_1, \mathbb{Q})$. In particular, if $k \in K_1$, then $\alpha(k) = \beta(p_1(k)) = \beta(0) = 0$, so α vanishes on K_1. Similarly, α also vanishes on K_2, and thus vanishes on the product of the subgroups K_1 and K_2,

$$K_1K_2 = \{k_1k_2 \, : \, k_i \in K_i\}. \tag{2.3}$$

Since K_i are normal subgroups, then so is K_1K_2. It suffices to show that K_1K_2 is finite index.

Indeed, the composition

$$G \to Q_1 \to Q_1/p_1(K_2) \tag{2.4}$$

has kernel exactly K_1K_2. Since $[Q_1 : p_1(K_2)] < \infty$, then K_1K_2 has finite index in G. □

Lemma 2.2. Let G be a group which admits epimorphisms $p_1 \colon G \to Q_1$ and $p_2 \colon G \to Q_2$ with distinct finitely generated non-trivial kernels K_1 and K_2 respectively such that either of the following are true:

1. K_1 and Q_1 have the f.g.n. property and Q_2 contains no finite normal subgroups

2. Q_1 and Q_2 have the f.g.n. property.

Then

$$p_1^*(\mathrm{Hom}(Q_1, \mathbb{Q})) \cap p_2^*(\mathrm{Hom}(Q_2, \mathbb{Q})) = \{0\}. \tag{2.5}$$

Proof. Firstly notice that if either Q_i is finite, the equality 2.5 is satisfied trivially since $\mathrm{Hom}(Q_i, \mathbb{Q}) = \{0\}$ when Q_i is finite. For the remainder of this proof, we will assume both Q_i are infinite.

Let $\Gamma = p_1(K_2)$, a finitely-generated normal subgroup of Q_1. In either case, Q_1 has the f.g.n. property, so either

(i) Γ is trivial, or

(ii) $[Q_1 : \Gamma]$ is finite.

If $\Gamma = 1$, then $K_2 \trianglelefteq K_1$. We consider the short exact sequence

$$1 \to K_1/K_2 \to Q_2 \to Q_1 \to 1\,. \tag{2.6}$$

Given case (1), since K_1 has the f.g.n. property and K_2 is finitely generated and nontrivial, then $[K_1 : K_2]$ is finite. We see that $K_1 = K_2$ since Q_2 contains no normal finite subgroups. However, this contradicts the assumption that K_1 and K_2 are distinct.

Given case (2), since Q_2 has the f.g.n. property and Q_1 is infinite, the short exact sequence 2.6 shows that K_1/K_2 is a finitely generated normal subgroup of Q_2 with infinite index and hence must be trivial. Therefore, $K_1 = K_2$, contradicting the assumption that they are distinct.

In either case, we get that $[Q_1 : \Gamma]$ is finite, so the result follows from Lemma 2.1. □

We can take this result and apply it to surface bundles over certain 3-manifolds.

2.1.2 Applying Lemma 2.2 to Surface Bundles Over 3-manifolds

Theorem 2.3. If $p_1 \colon X \to N_1$ and $p_2 \colon X \to N_2$ are distinct surface bundles over 3-manifolds N_1 and N_2 and $\pi_1 N_1$ has the f.g.n. property, then

$$p_1^*(H^1(N_1;\mathbb{Q})) \cap p_2^*(H^1(N_2;\mathbb{Q})) = \{0\}. \tag{2.7}$$

Proof. If $\pi_1 N_2$ is finite, then the equality 2.7 is satisfied trivially as in the proof of Lemma 2.2. For the remainder of this proof, we shall assume that $\pi_1 N_2$ is infinite. First, identify $H^1(X;\mathbb{Q}) = \mathrm{Hom}(\pi_1 X, \mathbb{Q})$. Since p_i are distinct surface bundles, then the kernels of p_{i*} are distinct surface subgroups of $\pi_1 X$; in particular, the kernels are finitely generated and satisfy the f.g.n. property.

If N_2 is not prime, then in particular, $\pi_1 N_2$ has the f.g.n. property, so the result follows from Lemma 2.2 part 2.

If N_2 is prime, then we claim that $\pi_1 N_2$ contains no normal finite subgroups. To see this, suppose $H \leqslant \pi_1 N_2$ is such a subgroup. In particular, H would have infinite index in $\pi_1 N_2$. By Theorem 1.2, H would be either be a copy of $\mathbb{Z}$ or a surface group, neither of which is finite. Thus $\pi_1 N_2$ contains no non-trivial normal finite subgroups and therefore the result follows from Lemma 2.2 part 1. □

Theorem 2.4. Let $p \colon X \to N$ be a surface bundle over a 3-manifold such that $\pi_1 N$ has the f.g.n. property and $b_1 X = b_1 N$, then any other surface fibering of X has a rational homology sphere as a base space.

Proof. Suppose $\pi \colon X \to N'$ is another surface fibering of X. Then by Theorem 2.3,

$$p^*(H^1(N;\mathbb{Q})) \cap \pi^*(H^1(N';\mathbb{Q})) = \{0\} \subseteq H^1(X;\mathbb{Q}) \,. \tag{2.8}$$

Since $b_1 N = b_1 X$, $p^*(H^1(N;\mathbb{Q})) = H^1(X;\mathbb{Q})$ and thus $H^1(N';\mathbb{Q}) = \{0\}$. By Poincaré duality, $H^2(N';\mathbb{Q}) = \{0\}$ and so N' is a rational homology sphere. □

An important note regarding the results in this section is that the fundamental groups of the base 3-manifolds have the f.g.n. property. We now explore examples where this fails to be true.

2.2 Examples of 5-manifolds With Multiple Surface-Fiberings

In this section we will discuss some general methods for constructing 5-manifolds with multiple surface bundle structures. All of these examples will have a base space which is a surface bundle over a circle, and hence lack the f.g.n. property.

2.2.1 Trivial Bundles With Multiple Fiberings

Example 2.1. Consider a 3-manifold N with a surface fibering $p\colon N \to S^1$ with fiber F. Given any hyperbolic surface Σ, we can take the trivial Σ-bundle over N, $q\colon \Sigma \times N \to N$. We can construct another fibering of the total space by

$$\mathrm{id} \times p\colon \Sigma \times N \to \Sigma \times S^1\colon (x, y) \mapsto (x, p(y)). \tag{2.9}$$

These fiberings are not equivalent since the kernel of $q_*\colon \pi_1(\Sigma \times N) \to \pi_1 N$ is $\pi_1\Sigma \times \{1\}$ while the kernel of $(\mathrm{id} \times p)_*\colon \pi_1(\Sigma \times N) \to \pi_1(\Sigma \times S^1)$ is $\{1\} \times \pi_1 F$.

We can use this construction to exhibit two examples of interest.

Theorem 2.5. There is a surface bundle over a 3-manifold $p\colon X \to B$ with $b_1 X = b_1 B$ and $\mathrm{sFib}(X) > 1$.

Proof. Let N be a surface-fibered 3-manifold with $b_1 N = 1$. We could construct such a 3-manifold using 0-surgery on a fibered knot of genus at least one. A more detailed explanation of such a construction can be found in [Rol76]. Further, let Σ be some hyperbolic surface. By the construction described above, $\Sigma \times N$ fibers trivially over N and also fibers over $\Sigma \times S^1$, so

$$\mathrm{sFib}(\Sigma \times N) \geqslant 2. \tag{2.10}$$

Moreover, $b_1(\Sigma \times N) = b_1(\Sigma \times S^1)$. □

Theorem 2.6. There is a 5-manifold X with $\mathrm{sFib}(X) = \infty$.

Proof. Let N be a surface-fibered 3-manifold with $b_1 N > 1$. In particular, N admits infinitely many surface bundle structures. For each fibering, $p_\alpha \colon N \to S^1$, $\Sigma \times N$ admits a distinct surface bundle structure over $\Sigma \times S^1$, namely $\mathrm{id} \times p_\alpha$. □

2.2.2 Pullback Bundles

We can generalize the construction above to a non-trivial bundle with multiple surface bundle structures.

Example 2.2. Suppose $p_1 \colon N_1 \to S^1$ and $p_2 \colon N_2 \to S^1$ are distinct surface bundles with fibers F_1 and F_2 respectively. Let X be the pullback $p_1^* N_1 = p_2^* N_2$ and $q_1 \colon X \to N_1$ and $q_2 \colon X \to N_2$ be the projection maps so that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{q_1} & N_1 \\ \big\downarrow{\scriptstyle q_2} & & \big\downarrow{\scriptstyle p_1} \\ N_2 & \xrightarrow{p_2} & S^1 \end{array} \tag{2.11}$$

More explicitly,

$$X = \{(x_1, x_2) \in N_1 \times N_2 \, : \, p_1(x_1) = p_2(x_2)\} \tag{2.12}$$

and the maps q_i are the projections onto the first and second coordinate respectively. With this construction, it is easy to see that q_1 is a fiber bundle with fiber

$$q_1^{-1}(x) = (\{x\} \times N_2) \cap X \tag{2.13}$$

$$= \{(x, x_2) \, : \, p_2(x_2) = p_1(x)\} \tag{2.14}$$

$$= \{x\} \times p_2^{-1}(p_1(x)) \tag{2.15}$$

$$= \{x\} \times F_2 \tag{2.16}$$

$$\cong F_2 \tag{2.17}$$

and similarly, q_2 is a fiber bundle with fiber $q_2^{-1}(x) \cong F_1$. Moreover, if we take a fixed $\theta \in S^1$ then

$$q_1^{-1}(p_1^{-1}(\theta)) = q_2^{-1}(p_2^{-1}(\theta)) = F_1 \times F_2 \subseteq X \subseteq N_1 \times N_2. \tag{2.18}$$

In fact, X is the total space of a fiber bundle over S^1 with fiber $F_1 \times F_2$ where the bundle structure is given by the map $p := p_1 \circ q_1 = p_2 \circ q_2 \colon X \to S^1$. If N_1 is the mapping torus of $\varphi_1 \colon F_1 \to F_1$ and N_2 is the mapping torus of $\varphi_2 \colon F_2 \to F_2$, then X is the mapping torus of

$$\varphi \colon F_1 \times F_2 \to F_1 \times F_2 : (x_1, x_2) \mapsto (\varphi_1(x_1), \varphi_2(x_2)) \tag{2.19}$$

To see how this generalized the previous example, notice that if N_1 is any surface bundle over a circle and N_2 is a trivial bundle over a circle, then X is equivalent to a trivial bundle over N_1.

Lemma 2.7. If X is constructed as in Example 2.2, then the following diagram commutes:

$$\begin{array}{ccc} \pi_1 N_1 & \xrightarrow{\rho_{q_1}} & \mathrm{MCG}(F_2) \\ {\scriptstyle p_{1*}}\downarrow & \nearrow{\scriptstyle \rho_{p_2}} & \\ \pi_1 S^1 & & \end{array} \tag{2.20}$$

where ρ_{q_1} and ρ_{p_2} are the monodromy representations of q_1 and p_2 respectively.

Proof. Since N_1 is the total space of an F_1-bundle over S^1, it determines a short exact sequence

$$1 \to \pi_1 F_1 \to \pi_1 N_1 \to \pi_1 S^1 \to 1. \tag{2.21}$$

We claim that $\rho_{q_1}(\pi_1 F_1) = [\mathrm{id}] \in \mathrm{MCG}(F_2)$. Indeed, we see that the bundle structures $q_i \colon X \to N_i$ restrict to the trivial bundle structures $F_1 \times F_2 \to F_i$ and thus

$\rho_{q_1}(\pi_1 F_1)$ acts trivially on F_2. Hence ρ_{q_1} factors through $\pi_1 S^1$, namely

$$\begin{array}{ccc} \pi_1 N_1 & \xrightarrow{\rho_{\pi_1}} & MCG(F_2) \\ {\scriptstyle p_{1*}}\downarrow & \nearrow_{g} & \\ \pi_1 S^1 & & \end{array} \tag{2.22}$$

It remains to show $g = \rho_{p_2}$. Since p_{1*} is induced by p_1 this follows from the commuting square 2.11.

□

Theorem 2.8. There is a 5-manifold X with surface fiberings $p_1 \colon X \to N_1$ and $p_2 \colon X \to N_2$ such that

$$b_1 X = b_1 N_1 = b_1 N_2 = 1. \tag{2.23}$$

Proof. Let $p_1 \colon N_1 \to S^1$ and $p_2 \colon N_2 \to S^1$ be surface bundles such that $b_1 N_1 = b_1 N_2 = 1$ and let X be the pullback bundle as described above. It follows from [Mor87] that there are splittings

$$H_1(X;\mathbb{Q}) \cong H_1(N_1;\mathbb{Q}) \oplus H_1(F_2;\mathbb{Q})_{\pi_1 N_1} \tag{2.24}$$

and

$$H_1(N_2;\mathbb{Q}) \cong H_1(S^1;\mathbb{Q}) \oplus H_1(F_2;\mathbb{Q})_{\pi_1 S^1} \tag{2.25}$$

where $H_1(F_2;\mathbb{Q})_{\pi_1 N_1}$ and $H_1(F_2;\mathbb{Q})_{\pi_1 S^1}$ are the *coinvariant homology* of the bundles π_1 and p_2 respectively, namely

$$H_1(F;\mathbb{Q})_{\pi_1 B} = H_1(F;\mathbb{Q})/\{v - gv \ : \ g \in \pi_1 B\} \tag{2.26}$$

where the action of $\pi_1 B$ on $H_1(F;\mathbb{Q})$ is induced by the monodromy representation; that is, if $\rho\colon \pi_1 B \to \mathrm{MCG}(F)$ is the monodromy representation of a surface bundle over B, then $gv = \rho(g)_*(v)$ for $v \in H_1(F;\mathbb{Q})$. Further, it follows from Lemma 2.7 that $\rho_{\pi_1}(\pi_1 N_1) = \rho_2(\pi_1 S^1)$ and thus

$$H_1(F_2;\mathbb{Q})_{\pi_1 N_1} \cong H_1(F_2;\mathbb{Q})_{\pi_1 S^1}. \tag{2.27}$$

Since $b_1 N_2 = b_1 S^1 = 1$, $H_1(F_2;\mathbb{Q})_{\pi_1 S^1} = 0$ by 2.25 and hence $H_1(F_2;\mathbb{Q})_{\pi_1 N_1} = 0$. Therefore $H_1(X;\mathbb{Q}) \cong H_1(N_1;\mathbb{Q})$ by 2.24.

□

2.3 Further Questions

Question 2.1. The examples of surface bundles over 3-manifolds we've constructed in this chapter have at least 2 surface bundle structures and we've seen that one can construct a 5-manifold with infinitely many surface bundle structures. Does there exist a 5-manifold X with $\mathrm{sFib}(X) < \infty$?

Question 2.2. Examples 2.1 and 2.2 construct 5-manifolds which admit multiple surface bundle structures, but in both cases, all of the base spaces are surface fibered 3-manifolds. Does there exist a 5-manifold with multiple surface bundle structures, at least one of which has a base space whose fundamental group satisfies the f.g.n. property?

Chapter 3

Free-by-Surface Groups

Following the notation of [Joh94], we will refer to the class of groups which can be realized as the fundamental group of a surface of finite type by $\mathcal{D}$. In the language introduced in Chapter 1, this is the union of the class of surface groups with the class of free groups. A group extension of a $\mathcal{D}$-group by another $\mathcal{D}$-group is referred to as a $\mathcal{D}^2$-group. These $\mathcal{D}^2$-groups come in four types:

- *Surface-by-surface group*: A group extension of a surface group by another surface group.
- *Free-by-free group*: A group extension of a free group by another free group.
- *Surface-by-free group*: A group extension of a surface group by a free group.
- *Free-by-surface group*: A group extension of a free group by a surface group.

Note that the group cohomological dimension of a group of the first two types on this list would be 4 and 2 respectively, whereas that of the latter two types would be 3.

In Chapter 1, we discussed surface bundles over surfaces and the fundamental groups of the total spaces of such bundles are $\mathcal{D}^2$-groups of the first type on the list above. Moreover, for each surface bundle structure a 4-manifold admits, its fundamental group admits a $\mathcal{D}^2$-structure of the first type. In this chapter, we will discuss a topological approach at constructing groups with multiple $\mathcal{D}^2$-structures of the last type.

3.1 Multisections

Definition 3.1. A *multisection*, or a k-*section* of a surface bundle $p\colon X \to B$ is a submanifold $S \subseteq X$ such that the restriction $p|_S\colon S \to B$ is a degree k cover.

In [BH16], the authors use a similar definition in the broader context of Lefschetz fibrations.

Lemma 3.1. If S is a k-section of surface bundle $p\colon X \to B$ over a hyperbolic surface B, with fiber Σ_g, then the restriction $\hat{p} = p|_{X-S}$ is a fiber bundle over B with fiber Σ_g^k, that is Σ_g with k points removed.

Proof. Fix a point $b \in B$ and let U be a trivialization neighborhood of p around b which is evenly covered by $p|_S$. We claim U is a trivialization neighborhood of $\hat{p}$.

We can see that

$$\hat{p}^{-1}(U) = p^{-1}(U) - S \tag{3.1}$$

Since U is a trivialization neighborhood of p, then there is a diffeomorphism

$$\varphi\colon p^{-1}(U) \xrightarrow{\sim} U \times \Sigma_g \tag{3.2}$$

Since U is evenly covered by $p|_S$, then $p^{-1}(U) \cap S$ is made up of components $\{U_1, \ldots, U_k\}$, each diffeomorphic to U. Since S is transverse to the fibers of p, $\varphi(U_i) = U \times \{x_i\}$ for some $x_i \in \Sigma_g$. So $\varphi(\hat{p}^{-1}(U)) = U \times (\Sigma_g - \{x_1, \ldots, x_k\})$. □

This strategy allows us to take existing surface bundles and turn them into punctured surface bundles over surfaces, that is, surface bundles over surfaces where the fibers are punctured surfaces. In particular, the fundamental groups of such bundles are groups G which admit group extensions of the form

$$1 \to F_n \to G \to \Pi_h \to 1 \tag{3.3}$$

where h is the genus of the base surface and $n = 2g + k - 1$, where n is the genus of the fiber and k is the number of punctures in the fiber respectively. Such an extension is called a *free-by-surface structure* of G.

3.2 Groups With Multiple Free-by-Surface Structures

We start with the simplest example.

Example 3.1. Given a surface Σ, the space $X = \Sigma \times \Sigma - \Delta$ has fundamental group with two free-by-surface structures.

The product $\Sigma \times \Sigma$ admits two trivial fiberings

$$p_1 \colon \Sigma \times \Sigma \to \Sigma \colon (x, y) \mapsto x \tag{3.4}$$

and

$$p_2 \colon \Sigma \times \Sigma \to \Sigma \colon (x, y) \mapsto y\,. \tag{3.5}$$

The diagonal $\Delta = \{(x, x)\} \subseteq \Sigma \times \Sigma$ is a 1-section of both p_i. Thus by Lemma 3.1, the restrictions $\hat{p}_i = p_i|_{\Sigma \times \Sigma - \Delta}$ are surface fiberings with base Σ and fiber Σ^*, that is Σ with one point removed. Note that $\hat{p}_i$ are distinct since p_i are. If g is the genus of the surface Σ, then the fundamental group of X admits two inequivalent $\mathcal{D}^2$-structures as

$$1 \to F_{2g} \to \pi_1 X \xrightarrow{\hat{p}_{i*}} \Pi_g \to 1. \tag{3.6}$$

Note, though the kernels of $\hat{p}_{i*}$ are distinct, they are both isomorphic to F_{2g}. Next, we will generalize this example to one where that is not the case.

Example 3.2. Consider a covering map $f \colon \Sigma_g \to \Sigma_h$. The complement C of the graph Γ_f of f in $\Sigma_g \times \Sigma_h$ has fundamental group with two free-by-surface structures.

Similarly to Example 3.1, the graph Γ_f is a multisection of the projections

$$p_1 \colon \Sigma_g \times \Sigma_h \to \Sigma_g \colon (x, y) \mapsto x \tag{3.7}$$

and

$$p_2 \colon \Sigma_g \times \Sigma_h \to \Sigma_h \colon (x, y) \mapsto y\,. \tag{3.8}$$

The graph Γ_f is simultaneously a 1-section of p_1 and an n-section of p_2 where n is the degree of f. Therefore, the fundamental group of C admits $\mathcal{D}^2$-structures as

$$1 \to F_{2h} \to \pi_1 C \xrightarrow{p_{1*}} \Pi_g \to 1 \tag{3.9}$$

and

$$1 \to F_{2g+n-1} \to \pi_1 C \xrightarrow{p_{2*}} \Pi_h \to 1. \tag{3.10}$$

Example 3.3. The Atiyah-Kodaira fibration contains a submanifold which is a multisection of the two fiberings described in Chapter 1.

Let M_{AK} denote the Atiyah-Kodaira bundle as constructed in Chapter 1 with surface fiberings $p_1 \colon M_{AK} \to \Sigma_{129}$ and $p_2 \colon M_{AK} \to \Sigma_3$. Recall that M_{AK} is a branch cover of $\Sigma_{129} \times \Sigma_3$ along the $\Gamma_f \cup \Gamma_{\iota f}$. Let $\Gamma \subseteq M_{AK}$ be the inverse image of the branch locus.

Firstly, we notice that Γ is a 2-section of p_1. Since p_1 has fiber Σ_6, then the restriction $\hat{p}_1 = p_1|_{M_{AK}-\Gamma}$ has fiber Σ_6^2. Thus the fundamental group of $M_{AK} - \Gamma$ admits an extension as

$$1 \to \pi_1 \Sigma_6^2 \to \pi_1(M_{AK} - \Gamma) \to \pi_1 \Sigma_{129} \to 1 \tag{3.11}$$

or equivalently,

$$1 \to F_{13} \to \pi_1(M_{AK} - \Gamma) \to \Pi_{129} \to 1 \tag{3.12}$$

since the fundamental group of Σ_6^2 is a free group of rank $2(6) + 2 - 1 = 13$.

Similarly, we notice that Γ is 128-section of p_2. Since p_2 has fiber Σ_{321}, then the restriction $\hat{p}_2$ has fiber Σ_{321}^{128} and thus $M_{AK} - \Gamma$ also admits a short exact sequence

$$1 \to F_{769} \to \pi_1(M_{AK} - \Gamma) \to \Pi_3 \to 1 \tag{3.13}$$

since the fundamental group of Σ_{321}^{128} is a free group of rank $2(321) + 128 - 1 = 769$.

3.3 Further Questions

Question 3.1. Do there exist any non-product groups which simultaneously extend a free group by a surface group and a surface group by a free group?

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