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Detectability and Invariance Properties for Set Dynamical Systems ⋆

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Abstract: Invariance properties and convergence of solutions of set dynamical systems are studied. Using a framework for systems with set-valued states, notions of stability and detectability, similar to the existing results for classical dynamical systems, are defined and used to obtain information about the convergence properties of solutions. In particular, it is shown that local stability, detectability, and boundedness can be combined to conclude convergence of set-valued solutions. Under the assumption of bounded solutions and outer semicontinuity of the set-valued maps that define the system's dynamics, invariance properties for set dynamical systems are also presented along with an invariance principle. The invariance principle involves the use of Lyapunov-like functions to locate invariant sets. Examples illustrate the results.

Keywords: Invariance principle, set dynamical systems, stability of nonlinear systems, Lyapunov methods, discrete event systems

1. INTRODUCTION

Complex technological systems available nowadays require a suitable representation for realistic behavior analysis. System dynamics often incorporate restrictions that the state variables must satisfy, as well as a model of the disturbances. Methods to check properties of the behavior of such systems, without the need of explicitly computing solutions are the main tools used in analysis and system design nowadays. For such purposes, formulations based on set-theoretic frameworks are well suited for design problems involving constraints, uncertainties, multiple operation points, among others.

Most popular methods to study the behavior of dynamical systems consider a classical state representation, namely a single-valued vector in some space. On the other hand, the generation of tools to study the properties of systems in which set-valued states represent variables or multivalued signals has experienced a more delayed development. Early approaches in this direction can be found in works such as Pelczar (1977), where a type of generalized systems is presented and basic stability properties are studied, with follow-up developments for the study of limit sets in Pelczar (1991). Notions of reachability for generalized pseudo dynamical systems are formally studied in Pelczar (1994). More recently, properties of systems with set-valued states in continuous time are studied in Artstein (1995) and a framework for the design of output feedback algorithms incorporating set-valued systems is developed

in Artstein and Raković (2011). Even though some early predecessors exists, a tool for set dynamical systems that resembles the widely used invariance principle is currently not available.

With the objective of developing an invariance principle so as to characterize the behavior of systems with set-valued solutions, in this paper we study convergence of solutions and invariance properties of set dynamical systems. Here, we consider systems with a set-valued state that evolves in discrete time according to a set-valued map and a constraint given in terms of a set, both in Euclidean space. Solutions associated to such systems are given by sequences of sets, rather than sequences of points as in the case of classical dynamical systems in discrete time. In particular, we extend the framework introduced in Sanfelice (2014) to set dynamical systems with outputs and formulate appropriate notions of detectability and stability. We follow similar ideas to those existing for classical dynamical systems as well as those in the literature of set-valued dynamical systems, such as Panasyuk (1986), Pelczar (1994), and Artstein (1995). The proposed concepts are then used to obtain information about the convergence properties of solutions and to state an invariance principle for set dynamical systems. In particular, our contributions include:

- Basic assumptions on the system data for key structural properties, namely, a semigroup property, a sequential compactness property, and closedness of the set-valued states and outputs.
- Relationships between local stability, detectability, and boundedness to conclude convergence of set-valued solutions.

- Results about convergence of set-valued solutions based on invariant sets involving the output map.
- An invariance principle, similar to the one available for continuous-time and discrete-time systems; see, e.g., LaSalle (1967). Under the said basic assumptions, our invariance principle uses a real-valued Lyapunov-like function to locate invariant sets of bounded and complete solutions.

Although results on convergence and stability properties for dynamical systems under a set-valued approach have been generated in the past, see Artstein and Raković (2011) for results concerning systems with output feedback, and results pertaining to invariance of sets are available (see Blanchini and Miani (2007), Artstein and Raković (2008)), our approach provides a tool to analyze convergence of solutions by using an invariance principle for set dynamical systems, which allows to use nonstrict Lyapunov functions for determining where set-valued solutions converge to. To the best of our knowledge, such tool is not available in the literature. [1]

The reminder of this paper is organized as follows. After basic notation is introduced, Section 2 presents concepts associated to properties that are relevant for set dynamical systems behavior. The framework for set dynamical systems with outputs is defined in Section 3. The main results are presented in Section 4. More precisely, results relating detectability, stability, and convergence of solutions are described in Section 4.1. Invariance properties for set dynamical systems are presented in Section 4.2, where a formulation of an invariance principle for set-valued systems is proposed. Examples throughout the paper illustrate the ideas.

2. PRELIMINARIES

Notation: The following notation is used throughout this paper. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers. $\mathbb{B}$ denotes the closed unit ball in Euclidean space. Given a set $\mathcal{A}$, its closure is denoted by $\overline{\mathcal{A}}$. Given a map V, $\mathrm{dom}\, V$ denotes its domain of definition. Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm and $|x|_1$ denotes the taxicab norm. For a closed set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the distance $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. Given a function $V : \mathrm{dom}\, V \to \mathbb{R}$ and a constant $r \in \mathbb{R}$, its r-sublevel set is given by $L_V(r) := \{x \in \mathrm{dom}\, V : V(x) \leq r\}$. For a map G, G^n denotes the n-th composition.

Some basic definitions and properties that are used to characterize set dynamical systems are given in this section. We consider the standard topology in $\mathbb{R}^n$. A set is closed if and only if it contains its boundary. A set is compact if bounded and closed.

Definition 2.1. (distance between sets). The Hausdorff distance between two closed sets $\mathcal{A}_1$, $\mathcal{A}_2 \subset \mathbb{R}^n$ is given by

$$d(\mathcal{A}_1, \mathcal{A}_2) = \max \left\{ \sup_{x \in \mathcal{A}_1} |x|_{\mathcal{A}_2}, \sup_{z \in \mathcal{A}_2} |z|_{\mathcal{A}_1} \right\}$$

---

[1] Note that Proposition 9.1 in Artstein and Raković (2011) is a convergence result that requires the Lyapunov function to strictly decrease while the convergence criterion of sets in Proposition 4.5 in Artstein and Raković (2008) requires a strict contraction property.

Definition 2.2. (inner and outer limit). For a sequence of sets $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$:

- The inner limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted by $\liminf_{i \to \infty} T_i$, is the set of all points $x \in \mathbb{R}^n$ for which there exist points $x_i \in T_i$, $i \in \mathbb{N}$, such that $\lim_{i \to \infty} x_i = x$;
- The outer limit of the sequence $\{T_i\}_{i=0}^{\infty}$, denoted by $\limsup_{i \to \infty} T_i$, is the set of all points $x \in \mathbb{R}^n$ for which there exist a subsequence $\{T_{i_k}\}_{k=0}^{\infty}$ of $\{T_i\}_{i=0}^{\infty}$ and points $x_k \in T_{i_k}$, for all $k \in \mathbb{N}$, such that $\lim_{k \to \infty} x_k = x$.

The *limit* of the sequence exists if the outer and the inner limit sets are equal; namely,

$$\lim_{i \to \infty} T_i = \liminf_{i \to \infty} T_i = \limsup_{i \to \infty} T_i$$

The inner and outer limits of a sequence always exist and are closed [2], although the limit itself might not exist. □

Definition 2.3. (union of a collection of sets). Let $\mathcal{C}$ be a collection of sets. The union of $\mathcal{C}$ is $\bigcup \mathcal{C} = \{x : x \in X \text{ for some } X \in \mathcal{C}\}$. □

Definition 2.4. (Rockafellar and Wets, 2009, convergence of a sequence of sets) When the limit of the sequence $\{T_i\}_{i=0}^{\infty}$ in $\mathbb{R}^n$ exists in the sense of Definition 2.2, and is equal to T, the sequence of sets $\{T_i\}_{i=0}^{\infty}$ is said to converge to the set T. □

3. SET DYNAMICAL SYSTEMS WITH OUTPUTS

We consider set dynamical systems defined by

$$\begin{aligned} \mathfrak{X}^+ &= G(\mathfrak{X}) \\ \mathcal{Y} &= H(\mathfrak{X}) \\ \mathfrak{X} &\subset D \end{aligned} \tag{1}$$

where $\mathfrak{X}$ is the set-valued state, $\mathcal{Y}$ is the system's output, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are set-valued maps defining the right-hand side and the output map, respectively, and $D \subset \mathbb{R}^n$ defines a constraint that solutions to the system must satisfy. We say that (D, G, H) is the data of (1). A solution to the system in (1) is defined as the sequence of nonempty sets $\{\mathfrak{X}_j\}_{j=0}^{J}$ and its associated output is defined by the sequence $\{\mathcal{Y}_j\}_{j=0}^{J}$, $J \in \mathbb{N} \cup \{\infty\}$, satisfying

$$\begin{aligned} \mathfrak{X}_{j+1} &= G(\mathfrak{X}_j) \\ \mathcal{Y}_j &= H(\mathfrak{X}_j) \\ \mathfrak{X}_j &\subset D \end{aligned}$$

over the domain of definition of the solution $\{\mathfrak{X}_j\}_{j=0}^{J}$, which is given by the collection $\{0, 1, 2, \ldots, J\} \cap \mathbb{N}$, and denoted by $\mathrm{dom}\, \mathfrak{X}$. The first entry of the solution, $\mathfrak{X}_0$, is the initial set. We assume $\mathfrak{X}_0$ to be compact. If a solution has $J = 0$ then we say that it is *trivial*, and if it has $J > 0$ we say that it is *nontrivial*. If it has $J = \infty$, we say that it is *complete*. A solution $\{\mathfrak{X}_j\}_{j=0}^{J}$ is said to be maximal if it cannot be further extended. Given an initial set $\mathfrak{X}_0 \subset \mathbb{R}^n$, $S(\mathfrak{X}_0)$ denotes the set of maximal solutions [3] to (1) from $\mathfrak{X}_0$.

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[2] For a sequence of empty sets, the inner and outer limit is the empty set.

[3] By Lemma 3.7, maximal solutions to (1) are unique.

To make notation easier to follow, at times, the sequence of sets $\{\mathfrak{X}_j\}_{j=0}^J$ is represented as $\mathfrak{X}_j$ (or even just $\mathfrak{X}$). We make the same notational simplification when referring to the output $\mathcal{Y}$. The term solution-output pair $\{\mathfrak{X}, \mathcal{Y}\}$ is used to represent a solution $\mathfrak{X}$ and its associated output $\mathcal{Y} = H(\mathfrak{X})$. Notation $\{\mathfrak{X}_j^i\}_{i=0}^\infty$ refers to the sequence of solutions $\mathfrak{X}_j^i$ indexed by i, where j is the associated discrete time.

3.1 Definitions and Assumptions

Definition 3.1. (stability of a set). The set $\mathcal{A}$ is stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution $\mathfrak{X}_j$ to (1) with $d(\mathfrak{X}_0, \mathcal{A}) \leq \delta$ satisfies $d(\mathfrak{X}_j, \mathcal{A}) \leq \epsilon$ for all $j \in \mathrm{dom}\ \mathfrak{X}_j$. □

For classical systems, detectability means that when the output remains at zero, the norm of solutions converges to zero as time goes to infinity. A related concept is defined for set dynamical systems next.

Definition 3.2. (detectability). Let the sets $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{N} \subset \mathbb{R}^m$, and $\mathcal{M} \subset \mathbb{R}^n$ be given, where $\mathcal{A}$ is closed. For the system in (1), the distance to $\mathcal{A}$ is detectable relative to $\mathcal{N}$ on $\mathcal{M}$ if for each complete solution $\mathfrak{X} \in S(\mathfrak{X}_0)$, with $\mathfrak{X}_j$ closed for each $j \in \mathrm{dom}\ \mathfrak{X}$, and associated output $\mathcal{Y}$

$$\left.\begin{array}{c} \mathfrak{X} \subset \mathcal{M} \\ \mathcal{Y} \subset \mathcal{N} \end{array}\right\} \implies \lim_{j \to \infty} d(\mathfrak{X}_j, \mathcal{A}) = 0$$

□

In the next sections, results characterizing key dynamical properties of a set dynamical system as in (1) are proposed. Those results require the data of (1) to satisfy certain mild conditions, which are stated in the following assumption.

Definition 3.3. (outer semicontinuity). The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=0}^\infty$ converging to a point $x \in \mathbb{R}^n$ and each sequence $\{y_i\}_{i=0}^\infty$ such that $y_i \in G(x_i)$ for each i, converging to a point y, it holds that $y \in G(x)$. It is *outer semicontinuous* if $G(x)$ is outer semicontinuous at each $x \in \mathbb{R}^n$. □

Assumption 3.4. The set dynamical system defined in (1), with data (D, G, H) satisfies the following properties:

(A0) The set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded [4], and, for each $x \in D$, $G(x)$ is a nonempty subset of $\mathbb{R}^n$.
(A1) The set $D \subset \mathbb{R}^n$ is closed.
(A2) The set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous, locally bounded, and, for each $x \in D$, $H(x)$ is a nonempty subset of $\mathbb{R}^n$.

3.2 Preliminary results

The following result establishes a structural property of the state and output trajectory of (1). A similar result appeared in (Sanfelice, 2014, Proposition 4.6).

Proposition 3.5. (basic properties of solutions) The following properties hold for system (1):

---

[4] The map G is locally bounded if for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \subset \mathbb{R}^n$ such that $G(K) \subset K'$.

(B1) For any solution $\mathfrak{X}$ to (1) and any $\hat{j} \in \mathrm{dom}\,\mathfrak{X}$, we have that $\hat{\mathfrak{X}}$ given by $\hat{\mathfrak{X}}_j = \mathfrak{X}_{j+\hat{j}}$ for each $j \in \mathrm{dom}\,\hat{\mathfrak{X}} = \left\{ j \ : \ j + \hat{j} \in \mathrm{dom}\,\mathfrak{X} \right\}$ is a solution to (1).
(B2) Suppose the data (D, G) of (1) satisfies (A0) and (A1) of Assumption 3.4. Let $\{\mathfrak{X}_0^i\}_{i=0}^\infty$ be an eventually bounded (with respect to $\mathbb{R}^n$) sequence of sets converging to a compact set $\mathfrak{X}_0$ and suppose $\{\mathfrak{X}^i\}_{i=0}^\infty$ is such that $\mathfrak{X}^i \in \mathcal{S}(\mathfrak{X}_0^i)$. Then, there exists a subsequence of $\{\mathfrak{X}^i\}_{i=0}^\infty$ converging to some $\mathfrak{X} \in \mathcal{S}(\mathfrak{X}_0)$.
(B3) Suppose the data (D, G, H) of (1) satisfies Assumption 3.4. Let $\{\mathfrak{X}_j^i\}_{i=0}^\infty$ and $\{\mathcal{Y}_j^i\}_{i=0}^\infty$ be a solution-output pair to (1), such that $\{\mathcal{Y}_j^i\}_{i=0}^\infty$ is bounded. Then, there exists a subsequence of $\{\mathcal{Y}_j^i\}_{i=0}^\infty$ that converges to a closed set.

Theorem 3.6. (Rockafellar and Wets, 2009, Theorem 4.18) Every sequence of nonempty sets $\{T_i\}_{i=0}^\infty$ in $\mathbb{R}^n$ either escapes to the horizon or has a subsequence converging to a nonempty set $T \subset \mathbb{R}^n$, i.e., there exists a subsequence $\{T_{i_k}\}_{k=0}^\infty$ of $\{T_i\}_{i=0}^\infty$ such that $\lim_{k \to \infty} T_{i_k} = T$.

Lemma 3.7. Given a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) and any subsequence $\{j_i\}_{i=0}^I$ of $\{j\}_{j=0}^J$, we have

$$\mathfrak{X}_{j_i} = G^{j_i}(\mathfrak{X}_0) \qquad \forall i \in \{0, 1, \dots, I\} \cap \mathbb{N} \qquad (2)$$

Moreover, every maximal solution to (1) is unique.

Now, we define the notion of ω-limit set. This notion is used later to establish convergence properties of solutions.

Definition 3.8. (ω-limit set). The ω-limit set of a solution $\{\mathfrak{X}_j\}_{j=0}^J$ to (1) is given by

$$\widetilde{\omega}(\mathfrak{X}_j) = \{Y \subset \mathbb{R}^n : \exists \{j_i\}_{i=0}^\infty, \lim_{i \to \infty} j_i = \infty, Y = \lim_{i \to \infty} \mathfrak{X}_{j_i}\}$$

$$(3)$$

Note that $\widetilde{\omega}(\mathfrak{X}_j)$ is a collection of sets. □

Theorem 3.9. (Sanfelice, 2014, Theorem 4.13) Suppose the data (D, G) of (1) satisfies Assumption 3.4 and that $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov-like function. Then, every solution $\{\mathfrak{X}_j\}_{j=0}^J$, $J \in \mathbb{N} \cup \{\infty\}$ to (1) from $\mathfrak{X}_0 \subset D$ satisfies

$$\left[0, \sup_{x \in \mathfrak{X}_{j+1}} V(x)\right] \subset \left[0, \sup_{x \in \mathfrak{X}_j} V(x)\right] \qquad (4)$$

for all $j \in \{0, 1, \dots, J-1\} \cap \mathbb{N}$. Moreover, if $J = \infty$ then

$$\lim_{j \to \infty} \left[0, \sup_{x \in \mathfrak{X}_j} V(x)\right] = \bigcap_{j \in \mathbb{N}} \left[0, \sup_{x \in \mathfrak{X}_j} V(x)\right] \qquad (5)$$

and if, furthermore, $\{\mathfrak{X}_j\}_{j=0}^\infty$ is bounded then there exists $r \in \mathbb{R}_{\geq 0}$ such that

$$V(\widetilde{\omega}(\mathfrak{X}_j)) \subset [0, r] \qquad (6)$$

4. DETECTABILITY AND AN INVARIANCE PRINCIPLE

This section pertains to the study of convergence and invariance properties of set dynamical systems. Notions and results on detectability and convergence are presented in Section 4.1. In Section 4.2, invariance properties of set

dynamical systems are characterized and then used to prove an invariance principle, which corresponds to the main result in this paper. The proofs of some of the results borrow ideas from variational analysis in Rockafellar and Wets (2009) and the results for discrete-time systems in LaSalle (1967) and Teel (2006).

4.1 Detectability and Convergence

Structural properties of the solutions to set dynamical systems are used in this section, along with detectability, to conclude convergence of solutions to compact sets.

Theorem 4.1. Let $\mathcal{A}$ and $\mathcal{M}$ be compact subsets of $\mathbb{R}^n$. Suppose system (1) satisfies Assumption 3.4, the set $\mathcal{A}$ is stable, and the distance to $\mathcal{A}$ is detectable relative to $\mathcal{N}$. Let $\mathfrak{X} \in S(\mathfrak{X}_0)$ be complete and $\mathcal{Y}$ be the associated output. If $\mathfrak{X} \subset \mathcal{M}$ and $\lim_{j \to \infty} \mathcal{Y}_j = \mathcal{N}$ then

$$\lim_{j \to \infty} d(\mathfrak{X}_j, \mathcal{A}) = 0$$

Example 4.2. (illustration of Theorem 4.1) Consider the set dynamical system in (1) with $G(\mathfrak{X}) = \{g(x) : x \in \mathfrak{X}\}$, and $H(\mathfrak{X}) = \{h(x) : x \in \mathfrak{X}\}$ where $g(x) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix}$, $h(x) = 0.5 - \dfrac{1}{1 + e^{-x_1}}$, and $D = \{x \in \mathbb{R}^2 : |x|_1 \le c_D\}$, with $c_D > 0$. This system satisfies Assumption 3.4 since both g and h are continuous and D is compact. Consider the compact sets $\mathcal{M} = \{x \in \mathbb{R}^2 : |x|_1 \le c_M\}$ with $c_M \ge 0$, $\mathcal{A} = \{(0,0)\}$ and $\mathcal{N} = \{0\}$. Let $\mathfrak{X}_j$ be a complete solution from $\mathfrak{X}_0 = \{x \in \mathbb{R}^2 : |x|_1 \le c_0\}$ with $c_0 < 1$ and let $\mathcal{Y}_j$ be its associated output. For $c_M < 1$ and $c_0 \le c_M$, $\mathfrak{X}_j \subset \mathcal{M}$ for all $j \in \mathrm{dom}\,\mathfrak{X}_j$ since g is contractive on $\mathcal{M}$ in the sense that $|g_1(x)| < |x_1| < 1$ and $|g_2(x)| < 1$ for each $x \in \mathcal{M}$. To establish stability of $\mathcal{A}$, let $\epsilon > 0$. For $\delta = \epsilon$, the solution $\mathfrak{X}_j$ from $\mathfrak{X}_0$ is such that $d(\mathfrak{X}_{j-1}, \mathcal{A}) \le d(\mathfrak{X}_j, \mathcal{A}) \le \delta < 1$, since g is contractive on $\mathfrak{X} \subset \mathcal{M}$, so $d(\mathfrak{X}_j, \mathcal{A}) \le \epsilon$ for all $j \in \mathrm{dom}\,\mathfrak{X}_j$. We have that $\mathcal{Y} = \mathcal{N}$ if and only if $\mathfrak{X} \subset \{x \in \mathbb{R}^2 : x_1 = 0\}$. So any solution with $\mathfrak{X} \subset \mathcal{M}$ and $\mathcal{Y} = \mathcal{N}$ is such that $\mathcal{Y}_j = \{0\}$ for all $j \in \mathrm{dom}\,\mathfrak{X}_j$, which implies that $\mathfrak{X}_0 \subset \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $\mathfrak{X}_1 = G(\mathfrak{X}_0) = \{(0,0)\}$. Then, $\lim_{j \to \infty} d(\mathfrak{X}_j, \mathcal{A}) = 0$. Thus, the distance to $\mathcal{A}$ is detectable relative to $\mathcal{N}$ on $\mathcal{M}$. The assumptions in Theorem 4.1 are satisfied, which implies that the set-valued solution converges to $\mathcal{A}$. A solution from the initial set $\mathfrak{X}_0$ with $c_0 = 0.6$, $c_M = 0.8$, and $c_D = 0.9$ is presented in Figure 1, where convergence to $\mathcal{A}$ can be observed.

Theorem 4.3. Let $\mathcal{A}$ and $\mathcal{M}$ be compact subsets of $\mathbb{R}^n$. Suppose system (1) satisfies Assumption 3.4, the set $\mathcal{A}$ is stable for system (1), and $\mathcal{A}$ is detectable relative to $\mathcal{N}$ on $\mathcal{M}$. Then, for each $\epsilon > 0$, there exists $j^* \in \mathbb{N}$ such that, for each solution-output pair $(\mathfrak{X}, \mathcal{Y})$ with $\mathfrak{X} \in S(\mathfrak{X}_0)$,

$$\left. \begin{aligned} \mathfrak{X} &\subset \mathcal{M} \\ \mathcal{Y} &= \mathcal{N} \end{aligned} \right\} \implies d(\mathfrak{X}_j, \mathcal{A}) \le \epsilon \quad \forall j \ge j^*, \; j \in \mathrm{dom}\,\mathfrak{X}$$

4.2 Invariance Properties

In this section we formulate a convergence result based on invariant sets involving the output map. We introduce the concept of Lyapunov-like functions in the sense of Sanfelice (2014), and use such functions to locate invariant sets for
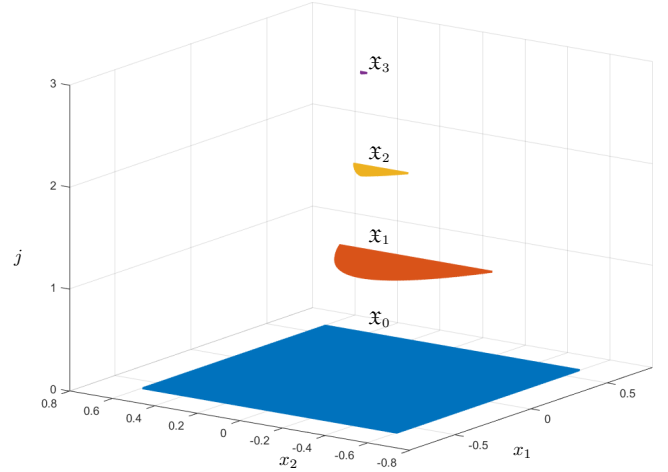

Fig. 1. *Solution to the set dynamical system in Example 4.2 from initial set $\mathfrak{X}_0 = \{x : |x|_1 \le 0.6\}$ (blue). As j increases, the solution converges to $\mathcal{A} = \{(0,0)\}$.*

solutions to set dynamical systems. We start by defining the concept of invariance for set dynamical systems.

Definition 4.4. (forward and backward invariance).
A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *forward invariant* for (1) if for every set $T \subset \mathcal{M} \cap D$ we have $G(T) \subset \mathcal{M} \cap D$. A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *backward invariant* for (1) if for every set $T' \subset \mathcal{M} \cap D$ for which there exists a set T with the property $T' = G(T)$, we have $T \subset \mathcal{M} \cap D$ for every such set T. A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be *invariant* if it is both forward and backward invariant. $\qquad\square$

Recall that, according to the definition of solution, we have that all maximal solutions to (1) are unique. The following result from Sanfelice (2014) characterizes properties of the omega limit sets of solutions to set dynamical systems.

Proposition 4.5. (Sanfelice, 2014, Proposition 4.9) Suppose the data (D,G) in (1) satisfies (A0) and (A1) in Assumption 3.4. Let $\mathfrak{X}_j$ be a bounded and complete solution to (1). Then, $\widetilde{\omega}(\mathfrak{X}_j)$ is nonempty, compact, and invariant.

Given a set dynamical system as in (1) and a set $\mathcal{N}$, we define

$$\mathcal{Y}^* := \{x \in D : H(x) \subset \mathcal{N}\}$$

which will be used in the formulation of the next results.

Theorem 4.6. Let $\mathcal{M}$ be compact and let $\mathfrak{X} \in S(\mathfrak{X}_0)$ be a complete solution to the system in (1) satisfying Assumption 3.4. If $\mathfrak{X} \subset \mathcal{M}$ and its associated output $\mathcal{Y}$ satisfies $\lim_{j \to \infty} \mathcal{Y}_j = \mathcal{N}$, then the solution $\mathfrak{X}$ converges to the largest invariant set contained in $\mathcal{M} \cap \mathcal{Y}^* \cap D$.

Proof Sketch: Since $\mathfrak{X}$ remains in $\mathcal{M}$ for all $j \in \mathrm{dom}\,\mathfrak{X}$, the solution is bounded and by Proposition 4.5, the omega limit set is nonempty, compact, and invariant. Since the system satisfies Assumption 3.4, from the definition of $\mathcal{Y}_j$ we have that $\lim_{j \to \infty} \mathcal{Y}_j = \lim_{j \to \infty} H(\mathfrak{X}_j) = \mathcal{N}$. If $\mathfrak{X}^*$ is an element of $\widetilde{\omega}(\mathfrak{X})$, there is a subsequence of $\mathfrak{X}_j$ indexed by j_i such that $\lim_{i \to \infty} H(\mathfrak{X}_{j_i}) = H(\mathfrak{X}^*) = \mathcal{N}$. Moreover, for each $\mathfrak{X}^* \in \widetilde{\omega}(\mathfrak{X})$, $\mathfrak{X}^* \subset \{x \in D : H(x) \subset \mathcal{N}\} = \mathcal{Y}^*$.

Theorem 4.6 is later used in proving our invariance principle, which is in Theorem 4.9. Before that, we make a link between detectability and stability.

Theorem 4.7. Let $\mathcal{A}$ and $\mathcal{M}$ be compact subsets of $\mathbb{R}^n$ and suppose, for the system described in (1), that the set $\mathcal{A}$ is stable. Then, the following statements are equivalent:

(1) The distance to $\mathcal{A}$ is detectable relative to $\mathcal{N}$ on $\mathcal{M}$.
(2) The largest invariant set contained in $\mathcal{M} \cap \mathcal{Y}^*$ is a subset of $\mathcal{A}$.

The following definition introduces the notion of Lyapunov-like functions for set dynamical systems.

Definition 4.8. (Lyapunov-like function). A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is called a Lyapunov-like function for system (1) if it satisfies

$$V(x) \geq 0 \qquad \forall x \in D \cup G(D)$$
$$V(\eta) - V(x) \leq 0 \qquad \forall x \in D, \eta \in G(x)$$

$\square$

Now we are ready to present our main result.

Theorem 4.9. (invariance principle) Suppose system (1) satisfies Assumption 3.4. Let V be a Lyapunov-like function for (1). Let $\mathfrak{X}_j$ be a bounded and complete solution to (1). Then, there is a number $r \in \mathbb{R}_{\geq 0}$ such that $\mathfrak{X}_j$ converges to the largest invariant set contained in

$$\bigcup E \cap L_V(r) \cap D$$

where

$$E := \left\{ X \subset D : \sup_{\eta \in G(X)} V(\eta) = \sup_{x \in X} V(x) \right\}$$

and $L_V(r) := \left\{ x \in D : V(x) \leq r \right\}$.

Proof Sketch: Since $\mathfrak{X}_j$ is bounded and complete, $\widetilde{\omega}(\mathfrak{X}_j)$ is nonempty, compact, and invariant. Using Theorem 3.9, since Assumption 3.4 holds for system (1), $V(\widetilde{\omega}(\mathfrak{X}_j)) \subset [0, r]$ for some $r \in \mathbb{R}_{\geq 0}$. Consider the set dynamical system in (1) with $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2) = H(\mathfrak{X}) = (H_1(\mathfrak{X}), H_2(\mathfrak{X}))$, where

$$H_1(\mathfrak{X}) = \left[0, \sup_{x \in \mathfrak{X}} V(x) - \sup_{\eta \in G(\mathfrak{X})} V(\eta) \right]$$
$$H_2(\mathfrak{X}) = \left[0, \sup_{x \in \mathfrak{X}} V(x) \right]$$

to which the given $\mathfrak{X}_j$ is a solution. We can apply Theorem 4.6, defining the sets $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$, $\mathcal{N}_1 = \{0\}$, $\mathcal{N}_2 = \{x \in D : V(x) \leq r\}$ and $\mathcal{M} = \{x \in D : V(\eta) \leq V(x)\}$. We have $\mathfrak{X}_j \subset \mathcal{M}$ for all $j \in \operatorname{dom} \mathfrak{X}$. Using Theorem 3.9 we have that $\sup_{x \in G(\mathfrak{X}_j)} V(x) \leq \sup_{x \in G(\mathfrak{X}_{j-1})} V(x)$ for each $j \in \operatorname{dom} \mathfrak{X}_j \setminus \{0\}$ and the sequence of points $\{\sup_{x \in \mathfrak{X}_j} V(x)\}_{j=0}^\infty$ converges to a real number $b \in [0, r]$, with $b = \lim_{j \to \infty} \sup_{x \in \mathfrak{X}_j} V(x)$. Furthermore, we have that $\lim_{j \to \infty} \left(\sup_{x \in G(\mathfrak{X}_j)} V(x) - \sup_{x \in G(\mathfrak{X}_{j-1})} V(x) \right) = 0$. Then, for the first component of the output, $\mathcal{Y}_1$, $\lim_{j \to \infty} \mathcal{Y}_{1_j} = \mathcal{N}_1$. The second output $\mathcal{Y}_2$ satisfies $\lim_{j \to \infty} \mathcal{Y}_{2_j} = \mathcal{N}_2$. Then, since $\lim_{j \to \infty} \mathcal{Y}_j = \mathcal{N}$, using Theorem 4.6 for this system, we have that the solution $\mathfrak{X}_j$ converges to the largest invariant set in $\mathcal{M} \cap \mathcal{Y}^* \cap D$, which corresponds to $\bigcup E \cap L_V(r) \cap D$. $\blacksquare$

Remark 4.10. If, for the set dynamical system in (1), we have that $G = g$, with $g : \mathbb{R}^n \to \mathbb{R}^n$, then Theorem 4.9 reduces to the original invariance principle for discrete-time systems given in LaSalle (1967), with a sublevel set of V instead of a level set as stated in the original principle. $\square$

Example 4.11. (illustration of Theorem 4.9) Consider the set dynamical as defined in (1) in $\mathbb{R}^2$ with $G(\mathfrak{X}) = \{g(x) : x \in \mathfrak{X}\}$ and $\quad g(x) = \begin{bmatrix} \dfrac{\alpha x_2}{1 + x_1^2} \\ \dfrac{\beta x_1}{1 + x_2^2} \end{bmatrix}$,

$D = [0, p] \times [0, q]$, with p, q real positive numbers and $\alpha, \beta \geq 0$. The set dynamical system data satisfies Assumption 3.4 since g is continuous and D is compact. Consider as the Lyapunov-like function $V(x) = x_1^2 + x_2^2$, which, for each $x \in D$ and $\eta = G(x)$, leads to

$$V(\eta) - V(x) = \left(\frac{\beta^2}{(1+x_2^2)^2} - 1 \right) x_1^2 + \left(\frac{\alpha^2}{(1+x_1^2)^2} - 1 \right) x_2^2$$
$$\leq (\beta^2 - 1)x_1^2 + (\alpha^2 - 1)x_2^2$$

For $\alpha^2 < 1$ and $\beta^2 < 1$, $V(\eta) - V(x) \leq 0$ for all $x \in \mathfrak{X}$, with $\mathfrak{X} \subset D$, $\eta \in G(\mathfrak{X})$, and V is a valid Lyapunov-like function for the set dynamical system. For each $x \in D \cup G(D)$, we have $V(x) \geq 0$. Moreover, let $\mathfrak{X}_j$ be a complete solution from $\mathfrak{X}_0 \subset D$. Since D is compact, $\mathfrak{X}$ is bounded. The assumptions in Theorem 4.9 are satisfied and the solution converges to the largest invariant set in $\bigcup E \cap L_V(r) \cap D$ for some $r \geq 0$. Since $\sup_{\eta \in G(X)} V(\eta) = \sup_{x \in X} V(x)$ only for $X = \{(0,0)\}$, the set E contains only the element $\{(0,0)\}$, and the largest invariant set is the origin. A solution from the initial set $\mathfrak{X}_0 = [0.5, 2] \times [0.5, 2]$, with $p = 5$, $q = 4$, $\alpha = \beta = 0.75$ is presented in Figure 2.

For $\alpha^2 = \beta^2 = 1$, condition $V(\eta) - V(x) \leq 0$ still holds, for all $x \in X$, with $X \subset D$, $\eta \in G(X)$, so V is a Lyapunov-like function for the set dynamical system. In this case, condition $\sup_{\eta \in G(X)} V(\eta) = \sup_{x \in X} V(x)$ is satisfied for X with elements x such that

$$\left(\frac{1}{(1+x_2^2)^2} - 1 \right) x_1^2 + \left(\frac{1}{(1+x_1^2)^2} - 1 \right) x_2^2 = 0$$

which holds for each x in $X = \{x \in D : x_1 = 0, x_2 \geq 0\} \cup \{x \in D : x_2 = 0, x_1 \geq 0\}$. Since this set of points belongs to the boundary of D and is invariant for G, we have that it defines the set E as well, namely, $E = \{x \in D : x_1 = 0, x_2 \geq 0\} \cup \{x \in D : x_2 = 0, x_1 \geq 0\}$. Since the assumptions in Theorem 4.9 are satisfied, the set-valued solution converges to the largest invariant set in $\bigcup E \cap L_V(r) \cap D$, in this case
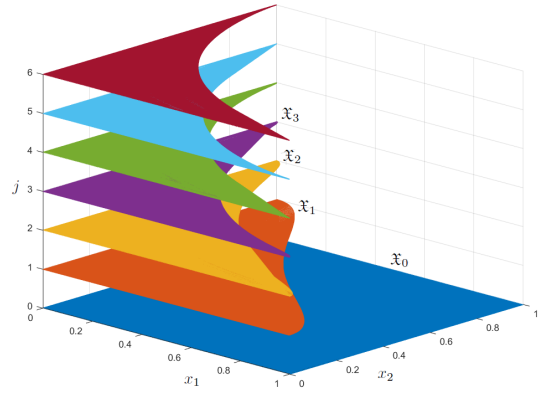
$$E \cap L_V(r) \qquad (7)$$

for some $r \geq 0$. For a solution from the initial set $\mathfrak{X}_0 = [0, 1] \times [0, 1]$, with $p = 5$, $q = 4$, the set-valued state converges to $\{x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 1\} \cup \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \leq 1\}$ which is the set in (7) with $r = \max_{x \in \mathfrak{X}_0} V(x) = 1$. Figure 3 shows the set-valued solution up to $j = 6$.
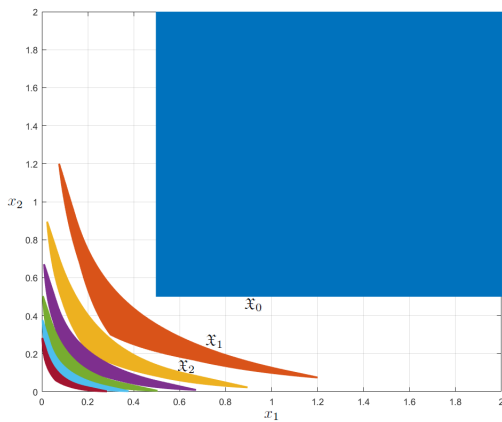
5. CONCLUSION

Convergence and invariance properties for set dynamical systems with outputs are studied in this paper. The
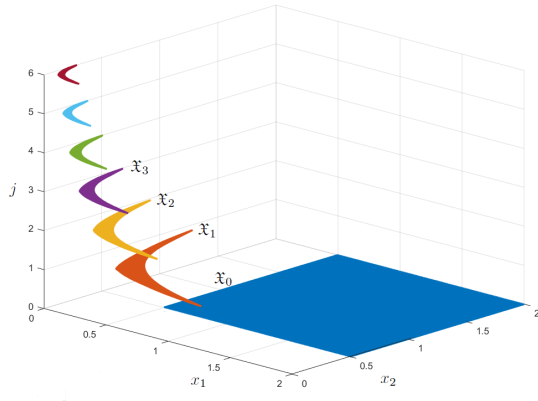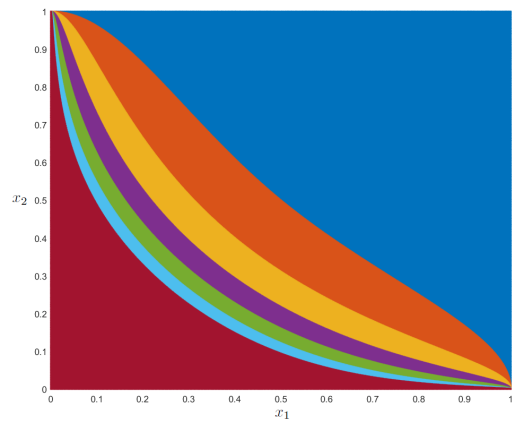
(a) Set-valued state evolution.

(b) Set-valued state convergence in the plane $x_1 - x_2$.

Fig. 2. *Solution to the set dynamical system in Example 4.11, with $\alpha = \beta = 0.75$ and $\mathfrak{X}_0 = [0.5, 2] \times [0.5, 2]$, for $j = 0$ to 6.*

mathematical framework in Sanfelice (2014) was extended to systems with outputs and notions of stability and detectability for these systems were defined. Then, under the assumption of bounded solutions and outer semicontinuity, results relating stability, detectability and state trajectory convergence were developed. Invariance properties for set dynamical systems were characterized and an invariance principle, involving the use of Lyapunov-like functions to locate invariant sets, under assumption of bounded solutions and outer semicontinuity of the set-valued map that defines the system dynamics, was presented.

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(a) Set-valued state evolution.

(b) Set-valued state convergence in the plane $x_1 - x_2$.

Fig. 3. *Solution to the set dynamical system in Example 4.11, with $\alpha = \beta = 1$ and $\mathfrak{X}_0 = [0, 1] \times [0, 1]$, for $j = 0$ to 6.*

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