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Regular and rigid curves on some Calabi–Yau and general-type complete intersections

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Let X be either a general hypersurface of degree $n+1$ in $\mathbb{P}^n$ or a general $(2, n)$ complete intersection in $\mathbb{P}^{n+1}, n \geq 4$. We construct balanced rational curves on X of all high enough degrees. If $n = 4$ or $g = 1$, we construct rigid curves of genus g on X of all high enough degrees. As an application we construct some rigid bundles on Calabi–Yau threefolds. In addition, we construct some low-degree balanced rational curves on hypersurfaces of degree $n+2$ in $\mathbb{P}^n$.

Keywords: Calabi–Yau complete intersection; rigid curve; isolated curve; degeneration methods.

Mathematics Subject Classification 2020: 14j30, 14j32, 14j60, 14j70

1. Introduction

1.1. *Setup*

Let C be a curve of genus g on a variety X of dimension $m \geq 3$ whose canonical bundle K_X is ample or trivial. Deformations of C in X (and sometimes with X) are controlled by the normal bundle $N = N_{C/X}$. In particular, C is said to be *rigid* (or sometimes *isolated*) if $H^0(N) = 0$, i.e. C does not move on X, even infinitesimally. C is *regular* or *strongly unobstructed* if $H^1(N) = 0$ which means that deformations of C in X are unobstructed and not oversize. Now in general one has

$$\chi(N) = C \cdot (-K_X) + (m-3)(1-g).$$

This number is always < 0 if K_X is ample and $g > 0$. On the other hand, when X is a Calabi–Yau (K_X trivial) and either $m = 3$ or $g = 1$, one always has $\chi(N) = 0$. So in that case rigidity ($h^0 = 0$) is equivalent to regularity ($h^1 = 0$). One "expects" any curve on such X to be rigid: e.g. Clemens has famously conjectured that on a general quintic threefold in $\mathbb{P}^4$, all rational curves are rigid. However, the most obvious curves one can construct are often not rigid, so it is a nontrivial problem, partially motivated by Physics [7, 15], to construct rigid, or more generally regular curves (and relatedly, vector bundles) on Calabi–Yau manifolds, in particular those that are complete intersections in projective space, the so-called CICY manifolds, which anyhow contain a lot of non-rigid curves.

In the case K_X ample, one has $C > (-K_X) < 0$ in the above formula for $\chi(N)$, hence a regular curve $C \subset X$ (more generally, a curve with $\chi(N) \geq 0$) must anyhow have genus 0 and canonical degree $C \cdot K_X \leq m - 3$. which is equivalent to $\chi(N) \geq 0$. And whenever $0 \leq \chi(N) \leq \mathrm{rk}(N)$ and $g = 0$, regularity is equivalent to balancedness of N (no $\mathcal{O}(-2)$ summands).

1.2. *Known results*

Results to date have largely focused on rigid curves on CY 3-folds. They go back to Clemens [1], who first constructed infinitely many rigid rational curves on the general quintic in $\mathbb{P}^4$. This was extended by Katz [4], then by Ekedahl *et al.* [3], to rigid rational curves on all CICY threefolds (CICY3fs). Subsequently, based on Clemens's method, Kley [5] constructed on any CICY3f X curves of positive but sufficiently low genus g (depending on X, e.g. $g < 35$ for the quintic) and sufficiently high degree (depending on the genus). For the quintic threefold, any $g \geq 0$ and large d, Zahariuc [18] constructs a map of degree d to its image from a smooth curve of genus g to X which is set-theoretically isolated (it is not proved the map is an embedding or infinitesimally rigid). Other existence and nonexistence results for curves of small degree (relative to the genus or otherwise) were obtained by Knutsen [6], Clemens and Kley [2] and Yu [16, 17]. I am not aware of results in the literature for curves on higher-dimensional CICYs. Some results on vector bundles are in [7, 13] and references therein. On the other hand results on balanced rational and irrational curves on *Fano* hypersurfaces were obtained in [9, 11].

1.3. *New results*

The purpose of this paper is to enlarge the known collection of "good" curves and bundles on CICY and general-type manifolds by constructing, on some m-dimensional CICYs, curves C of all large enough degrees with the following properties:

- C rigid of any genus, $m = 3$; or
- C rigid of genus 1, $m \geq 3$; or
- C rational and balanced, $m \geq 3$;

additionally, we will construct some balanced rational curves of degree at most $n-4$ on a general canonical (degree-$(n+2)$) hypersurface in $\mathbb{P}^n$.

Precisely, we will prove the following.

Theorem 1. *Let X be either a general hypersurface of degree $n+1$ in $\mathbb{P}^n$ or a general $(2,n)$ complete intersection in $\mathbb{P}^{n+1}, n \geq 4$, and let e be an integer. Then*

(i) *if $e \geq 2n-1$, there exist smooth rational curves of degree e on X with normal bundle $(n-4)\mathcal{O} \oplus 2\mathcal{O}(-1)$;*
(ii) *let $g \geq 1$ be an integer and assume*

$$e \geq 2n(n-1)(g+1)+1. \tag{1}$$

If either $g=1$ or $n=4$, then there exists a smooth rigid curve of genus g and degree e on X.

As noted above, this yields, e.g. the first construction of rigid curves of any genus on the quintic 3-fold, as well as the first construction of rigid or balanced curves on higher-dimensional CYs. Note the assumption $g=1$ or $n=4$ implies $\chi(N)=0$, so in that case rigidity is equivalent to regularity. For $g=0$ we have $\chi(N)=n-4$ so the curve cannot be rigid if $n>4$. Anyhow we have $\chi(N) \geq 0$ in all cases.

The case $g=1, n=4$ (already done by Kley [5]) has the following application to rigid vector bundles. See Sec. 5 for the proof.

Corollary 2. *If X is either a general quintic threefold in $\mathbb{P}^4$ or a general $(2,4)$ complete intersection in $\mathbb{P}^5$, then for every integer $e \geq 49$ X carries a rigid, indecomposable, semistable rank-2 vector bundle E with $c_1(E)=0, c_2(E)=e\lambda$ where λ is the class of a line.*

On the $(2,4)$ complete intersection, two other rigid rank-2 bundles (with odd c_1) were constructed by Thomas [13], who also constructs numerous other rigid examples on K3 fibrations. He has also constructed in [12] an example of a curve and a bundle on a CY3f of special moduli that are set-theoretically isolated but not (infinitesimally) rigid.

Finally, we will undertake a (necessarily limited) incursion into the forbidding territory of general type by proving the following.

Theorem 3. *Let X be a general hypersurface of degree $n+2$ in $\mathbb{P}^n, n \geq 5$. Then X contains rational curves of degree $e \leq n-4$ with normal bundle*

$$N = (n-4-e)\mathcal{O} \oplus (e+2)\mathcal{O}(-1).$$

For $n=5,6$ the 7. So, Theorem 3 is only interesting for $n \geq 7$. Of course here again $\chi(N) \geq 0$.

Actually the proof of Theorem 3 extends to hypersurfaces of degree $d \geq n+2$ to yield balanced rational curves of degree $\leq (n-4)/(d-n-1)$.

1.4. *Methods*

The idea of the proof of Theorem 1 is to use a suitable degeneration of X to a reducible normal-crossing variety $X_1 \cup X_2$. In the case, where X is a hypersurface we use a so-called quasi-cone degeneration where X_1 is "resolved quasi-cone" i.e. the blowup at a point q of a quintic with multiplicity n at q, and X_2 is a degree-n hypersurface in $\mathbb{P}^n$. In the case, where X is a $(2, n)$ complete intersection, X_0 is a the complete intersection of a degree-n hypersurface with a reducible quadric, so that X_1, X_2 are both degree-n hypersurfaces in hyperplanes $H_1, H_2 \simeq \mathbb{P}^{n+1}$ such that $X_1 \cap H_1 \cap H_2 = X_2 \cap H_1 \cap H_2$. We use results from [9, 11] about existence of curves on a degree-n hypersurface with "good" normal bundle, together with a new notion, introduced in Sec. 1, of relative regularity of a curve or bundle, which is analogous to a special case of the notion of "ultra-balance" introduced in [11] but where the "test points" are not general but lie on a divisor in a specified system.

Theorem 3 is proved similarly.

The question of existence of good curves of genus $g > 0$ and high degree on other CICY types of any dimension, or of rigid curves of genus > 1 on any CICYs of dimension > 3 (where $\chi(N) < 0$) remains open.

As for general type we mention here the following.

Conjecture. A general hypersurface of degree $\geq n+2$ in $\mathbb{P}^n, n \geq 4$ does not contain any irreducible rational curves of degree $\geq n-3$.

The conjecture is true for $n = 4$ (no rational curves at all) by [14]. As for $n = 5$, an easy dimension count shows that a general septic $X \subset \mathbb{P}^5$ contains no irreducible conics. The case of higher-degree rational curves is not clear.

I would like to thank Sheldon Katz and Richard Thomas for helpful comments and references.

1.5. *Notation and conventions*

In this paper, we work over $\mathbb{C}$.

On a curve C isomorphic to $\mathbb{P}^1$ we denote by $\mathcal{O}(k)$ the line bundle of degree k. A bundle on C is *balanced* if it has the form $a\mathcal{O}(k) \oplus b\mathcal{O}(k-1)$. In this case, the subbundle $a\mathcal{O}(k)$ is uniquely determined, called the *upper subbundle*.

A *quasi-cone* is a hypersurface X of degree d in $\mathbb{P}^n$ with a point of multiplicity $d-1$, called the *quasi-vertex*. Via projection, the blowup of X in the quasi-vertex is realized as the blowup of $\mathbb{P}^n$ in a $(d, d-1)$ complete intersection Y, where the exceptional divisor is the birational transform of the unique hypersurface of degree $d-1$ containing Y. The blow-up of a variety X in a subvariety Y is denoted by $B_Y X$.

2. Fan, Fang and Quasi-cone Degenerations

See [9] for details. We recall that a *fan* (also called a 2-fan) is a reducible normal-crossing variety of the form

$$P_0 = P_1 \cup_E P_2,$$

where

$$P_1 = B_p\mathbb{P}^n, \quad P_2 = \mathbb{P}^n$$

and $E \subset P_1$ is the exceptional divisor and $E \subset P_2$ is a hyperplane. The family

$$\mathcal{P} = B_{(p,0)}(\mathbb{P}^n \times \mathbb{A}^1)$$

is called a standard fan degeneration and realizes P_0 as the special fiber in a family with general fiber $\mathbb{P}^n$.

A *hypersurface of type* (d_1, d_2) in P_0 has the form

$$X_0 = X_1 \cup_Z X_2,$$

where

$$X_1 \in |d_1H_1 - d_2E|_{P_1}, \quad X_2 \in |d_2H_2|_{P_2}, \quad X_1 \cap Z = X_2 \cap Z$$

$(H_1, H_2$ are the respective hyperplanes). If $d_2 = d_1 - 1$, X_0 is said to be of *quasi-cone type*. Given a family $\bar{\mathcal{X}} \subset \mathbb{P}^n \times \mathbb{A}^1$ of hypersurfaces of degree d_1 whose special fiber has multiplicity d_2 at p, its birational transform $\mathcal{X} \subset \mathcal{P}$ is a family of hypersurfaces in $\mathbb{P}^n$ specializing to one of type (d_1, d_2).

More generally, a *fang of type* ℓ is a variety of the form

$$P_0 = P_1 \cup_E P_2,$$

where

$$P_1 = B_{\mathbb{P}^\ell}\mathbb{P}^n, \quad P_2 = B_{\mathbb{P}^{n-1-\ell}}\mathbb{P}^n, \quad E = \mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell}.$$

This is the special fiber of the degeneration

$$\mathcal{P} = B_{\mathbb{P}^\ell \times 0}\mathbb{P}^n \times \mathbb{A}^1 \to \mathbb{A}^1.$$

3. Relatively Regular Bundles and Curves

The purpose of this section is to study a property of vector bundles which, when applied to normal bundles, is helpful in studying the normal bundle to a union of curves.

Let C be a smooth curve, L a line bundle on C, $0 \neq V \subset H^0(L)$ a linear system, and $D \in V$ a reduced member.

Definition 4. (i) A vector bundle E on C is said to be *regular relative to* D if it is regular, i.e. $H^1(E) = 0$ and, for any subset $D_1 \subset D$ and a general quotient U of $E|_{D_1}$ locally of rank 1, the composite map

$$\rho_U : H^0(E) \to E|_{D_1} \to U$$

has maximal rank. E is said to be regular relative to V is it is regular relative to some divisor (or equivalently, a general divisor) $D \in V$.

(ii) A curve C on a variety X endowed with a linear system V is said to be regular relative to V if the corresponding property holds for its normal bundle $N = N_{C/X}$ and the restricted system $V|_C$.

More explicitly, condition (i) means that for any distinct $p_1, \ldots, p_k \in D$ and respective general 1-dimensional quotients $U_1, \ldots, U_k$ of $E_{p_1}, \ldots, E_{p_k}$, the natural map $H^0(E) \to \bigoplus_{i=1}^k U_i$ has maximal rank. Or equivalently, the kernel E' of the natural map $E \to \bigoplus U_i$ has either $H^1(E') = 0$ or $H^0(E') = 0$.

This notion is essentially meaningless in genus 0:

Lemma 5. *If D is any reduced divisor on $\mathbb{P}^1$ then any regular vector bundle is regular relative to D.*

Proof. We use the above notation. The proof is by induction on $h^0(E)$ which may be assumed > 0. We may also assume $D_1 = D$ is nontrivial and write $D = D' + p$. Let $L \subset E$ get a line subbundle of maximal degree which we may assume subjects to $U|_p$ which is 1-dimensional. Then letting $E' \subset E$ denote the kernel of $E \to U_p$, we get an exact diagram

$$\begin{array}{ccccccc} 0 \to H^0(E') & \to & H^0(E) & \to & \mathbb{C}_p & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \to \quad U' & \to & U & \to & U|_p & \to 0 \end{array}$$

As the right vertical arrow is an isomorphism and the left vertical arrow has maximal rank by induction, it is easy to see that the middle vertical arrow likewise has maximal rank. □

In the higher-genus case relative regularity seems related to the property of ultra-balancedness studied in [11], but there is no implication either way. Relative regularity is stronger in that the support of the quotient is a general divisor in V which may not be a general divisor on C; it is weaker in that the quotient must have local rank 1.

The main result of this section is

Proposition 6. *For $X = \mathbb{P}^n, n \geq 3$ (respectively, X a general hypersurface of degree n in $\mathbb{P}^n, n \geq 4$) and $g \geq 0$, there exists a curve of genus g and degree $e \geq 2(g+1)n$ (respectively, $e \geq 2(g+1)n(n-1)$) in X that is regular relative to $|-K_X|$.*

Proof.

Case 1. $X = \mathbb{P}^n$.

We follow closely in [11, Proof of Theorem 29], using induction on the genus g. Thus, we will be constructing curves on a fang and smoothing them as the fang smooths to $\mathbb{P}^n$. As the case $g = 0$ is automatic we begin with the case $g = 1$. Thus let ℓ be an integer in $[0, n-1]$ and consider a fang of type ℓ (see Sec. 2)

$$P_0 = P_1 \cup_Q P_2, \quad Q := \mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell},$$

where P_1, respectively, P_2 is the blowup of $\mathbb{P}^n$ in $\mathbb{P}^\ell$, respectively, $\mathbb{P}^{n-1-\ell}$, with common exceptional divisor $Y = \mathbb{P}^\ell \times \mathbb{P}^{n-1-\ell}$. As $\mathbb{P}^n$ degenerates to P_0, hyperplanes have different types of limits depending on the dimension of the limiting intersection with $\mathbb{P}^\ell$. At one extreme, if the limiting intersection is transverse, the limit in P_0 will have the form $H_1' \cup H_2'$ where H_1' is the birational transform of a hyperplane in $\mathbb{P}^n$ transverse to $\mathbb{P}^\ell$, while H_2' is the birational transform of a hyperplane containing $\mathbb{P}^{n-1-\ell}$. This is the type we will use below.

Thus let

$$C_1 \subset P_1, \quad C_2 \subset P_2$$

be rational curves with the property that

$$C_1 \cap Q = C_2 \cap Q = \{p, q\}.$$

Let

$$C_0 = C_1 \cup_{p,q} C_2 \subset P_0$$

be a the resulting curve of arithmetic genus 1 as in loc. cit. and set $e_i = \deg(C_i), i = 1, 2$ so the degree of the smoothing of C_0 in $\mathbb{P}^n$ is $e = e_1 + e_2 - 2$. As C_0 is a locally complete intersection in P_0, it has a locally free normal bundle denoted $N_0 = N_{C_0/P_0}$. Also set $N_i = N_{C_i/\mathbb{P}_i}, i = 1, 2$. In [11, Lemma 31], it is proven that we may assume N_1, N_2 are balanced. Then

$$\chi(N_0) = (e_1 + e_2 - 2)(n + 1),$$
$$\chi(N_1) = e_1(n + 1) + (n - 3) - 2(n - \ell - 1),$$
$$\chi(N_2) = e_2(n + 1) + (n - 3) - 2\ell.$$

Now let $\ell = [(n - 1)/2]$ which makes $\chi(N_1) \leq e_1(n + 1)$. Let

$$H'_{1,1} \cup H'_{2,1}, \ldots, H'_{1,n+1} \cup H_{2,n+1}$$

be $n + 1$ general hyperplane limits as above, and let D consist of $s = \chi(N_1)$ many points on $C_1 \cap (H'_{1,1} \cup \cdots \cup H'_{1,n+1})$ plus $\chi(N) - s$ many points on $C_2 \cap (H'_{2,1} \cup \cdots \cup H'_{2,n+1})$, and let U be a general, locally rank ≤ 1 quotient of N_D and let N' be the kernel of $N \to U$. To prove $H^0(N) \to U$ has maximal rank we may assume U has length $\chi(N)$, i.e. has rank exactly 1 at each point of D. Then we must prove $H^0(N') = 0$. Now because N_1, N_2 are balanced we have $H^0(N'|_{C_1}) = 0$ and $H^0(N'_{C_2}(-p - q)) = 0$, hence $H^0(N') = 0$. Thus proves that C_0 is regular relative to the limit of $|(n+1)H|$ hence its smoothing in $\mathbb{P}^n$ is regular relative to $|(n+1)H|$. This proves our assertion in genus 1.

In the general case, we use induction on the genus based on a fan (i.e. fang of type $\ell = 0$) degeneration

$$P_0 = P_1 \cup_E P_2, \quad P_1 = B_q\mathbb{P}^n, \quad P_2 = \mathbb{P}^n$$

with $E \subset \mathbb{P}_1$ the exceptional divisor and $E \subset \mathbb{P}_2$ a hyperplane. We use a limit anticanonical divisor that is a union of limit hyperplanes of the form

$$D = \bigcup_{i=1}^{n+1} H'_i, \quad H'_i = H'_{1,i} \cup H'_{2,i}$$

with each $H'_{1,i} \subset P_1$ the birational transform of a hyperplane through q and $H'_{2,i} \subset P_2$ a hyperplane with $H'_{1,i} \cdot E = H'_{2,i} \cdot E$. This is the opposite extreme of hyperplane limit types from the one used above. We consider a lci curve

$$C_0 = C_1 \cup C_2,$$

where $C_2 \subset P_2$ a $|-K_{P_2}|$-regular curve of genus $g-1$ and degree $e-1$ and

$$C_1 = C_{1,1} \cup \bigcup_{i=1}^{e-3} L_i$$

consists of the birational transforms of a plane cubic nodal at q (so that $C_{1,1} \cdot E = 2$) plus lines through q. Then an argument similar to the above but simpler with $H'_{1,i} \cdot L_j = 0, \forall i, j$ shows that, for a locally-rank-1 quotient U of $N_0|_D$ the map $H^0(N_0) \to U$ is an isomorphism as required. This concludes the proof in the case $X = \mathbb{P}^n$.

Case 2. $X \subset \mathbb{P}^n$ of degree n.

We use the usual "quasi-cone" degeneration as in [11], with the same notation:

$$X_0 = X_1 \cup_F X_2 \subset P_1 \cup_Q P_2.$$

Thus $P_1 = B_q\mathbb{P}^n, P_2 = \mathbb{P}^n$ and $Q = \mathbb{P}^{n-1}$ is embedded in P_1 as the exceptional divisor and in P_2 as a hyperplane; $X_1 \subset P_1$ is the blow-up in q of a general hypersurface of degree n in $\mathbb{P}^n$ multiplicity $n-1$ at q, X_2 is a hypersurface of degree $n-1$ and $F = X_1 \cap Q = X_2 \cap Q$. Projection from q realizes X_1 as the blow-up of $\mathbb{P}^{n-1}$ in a $(n-1, n)$ complete intersection Y. As in loc. cit. we consider a curve

$$C_0 = C_1 \cup C_2$$

with $C_1 \subset X_1$ the birational transform of a curve C'_1 of degree e_1 and genus g in $\mathbb{P}^{n-1}$ that is regular with respect to the anticanonical $|-K_{\mathbb{P}^{n-1}}|$, and C_2 a disjoint union of lines with trivial normal bundle. Here as in loc. cit. $e = e_1 n - a, a \leq n$. Now a limit anticanonical divisor on X_0 has the form F'_n which is the birational transform of a hypersurface of degree n in $\mathbb{P}^{n-1}$ containing Y while N_{C_1/X_1} is a general corank-1 modification of $N_{C'_1/\mathbb{P}^{n-1}}$ at the a points $\{p_1, \ldots, p_a\} = C'_1 \cap Y$. From this it is easy to see that C_1 is regular relative to the divisor $F'_n \cdot C_1$ and hence that C_0 is regular relative to $F'_n \cdot C_0$. Therefore as C_0 smooths to a curve C on a general degree-n hypersurface X, C is regular relative to $|-K_X|$. □

4. Lines and Conics on Some Hypersurfaces

In this section, we will study some rational curves that will serve as "tails" in the construction of good curves as in Theorem 1. First, an easy remark about their normal bundle:

Lemma 7. *Let X be a general hypersurface of degree d in $\mathbb{P}^n$.*

(i) *If $d \leq 2n-3$, X contains a line with balanced normal bundle.*
(ii) *If $d \leq (3n-2)/2$, X contains a conic with balanced normal bundle.*

Proof. Line case: Let L be the line $V(x_2,\dots,x_n)$ in $\mathbb{P}^n$. A hypersurface X of degree d containing L has equation $\sum_{i=2}^n x_i f_i(x_0,x_1), \deg(f_i)=d-1$. The normal sequence reads

$$0 \to N_{L/X} \to (n-1)\mathcal{O}_L(1) \to \mathcal{O}_L(d) \to 0,$$

where the right map is $(f_2,\dots,f_n)$ which is a general map. On the other hand, let $E=(n-1-d)\mathcal{O}_L(1)\oplus(d-1)\mathcal{O}_L$ be the unique balanced bundle of rank $n-2$ and degree $n-1-d$ on L. Then there clearly exists a fiberwise injection $E \to (n-1)\mathcal{O}_L(1)$ whose cokernel, for degree reasons, must be $\mathcal{O}_L(d)$. By openness of the balancedness property (i.e. the fact that a balanced bundle is rigid), it follows that $N \simeq E$.

The conic case is similar (note that in that case the middle term in the normal sequence is $\mathcal{O}(4)\oplus(n-2)\mathcal{O}(2)$ which is not balanced, but this doesn't matter). □

Remark 8. In case $n-1 \le d \le 2n-3$ above, the normal bundle $N=N_{L/X}$ is $N=(2n-d-3)\mathcal{O}\oplus(d-n+1)\mathcal{O}(-1)$ for the line. In the case $n \le d \le (3n-2)/2$ we have $N=(3n-2d-2)\mathcal{O}\oplus(2d-2n)\mathcal{O}(-1)$ for the conic. The argument doesn't extend to higher-degree rational curves because they are not complete intersections.

Let P denote the blow-up of $\mathbb{P}^n$ at a point q. By a *resolved quasi-cone* in P we mean the birational transform $X \subset P$ of a quasi-cone, i.e. a hypersurface $\bar{X}$ of degree d in $\mathbb{P}^n$ having multiplicity $d-1$ at q, the quasi-vertex. Projection from q realizes X as the blow-up of $\mathbb{P}^n$ in a $(d,d-1)$ complete intersection curve Y so that the exceptional divisor corresponds to the unique degree-$(d-1)$ hypersurface containing Y, while hyperplane sections from $\mathbb{P}^n$ correspond to hypersurfaces of degree d containing Y. In particular, taking $d=n+1$, $(n-1)$-secant lines to Y in $\mathbb{P}^{n-1}$ correspond to conics in $\bar{X}$ through the quasi-vertex q.

Before stating the next result we recall that for a balanced bundle $a\mathcal{O}(m+1)\oplus b\mathcal{O}(m), a>0$ on $\mathbb{P}^1$ the upper subbundle is by definition $a\mathcal{O}(m+1)$ and the upper subspace at a point p is the fiber of the upper subbundle at p.

Lemma 9.
Let X be either

Case (a). *A general, degree-n hypersurface in $\mathbb{P}^n, n \ge 4$, or*
Case (b). *A general resolved quasi-cone of degree $(n+1)$ in P.*

Let $L \subset X$ be either

Case (a). *A general line, or*
Case (b). *The transform of a general conic through the quasi-vertex.*

Let Z be either

Case (a). *$Z = X \cap H$ a general hyperplane section from $\mathbb{P}^n$, or*
Case (b). *the exceptional divisor; and let $p = Z \cdot L$.*

Then:

(i) *We have* $N := N_{L/X} = (n-3)\mathcal{O} \oplus \mathcal{O}(-1)$.
(ii) *Varying* X *with fixed* L *and* Z, *and identifying* $N|_p \simeq T_pZ$, *the upper subspace* $(n-3)\mathcal{O}|_p \subset T_pZ$ *becomes a general hyperplane.*
(iii) *There is a deformation* $\{(X_t, L_t) : t \in T\}$, *fixing* Z, *such that the image of the map*

$$T \ni t \mapsto L_t \cap Z$$

contains a neighborhood of p.

Remark 10. From the proof below it will follow that X actually contains a 1-parameter family of such curves.

Proof of Lemma. Note that all our assertions are open in L Z and X, so it suffices to prove them for some special case.

Case (a). (i) This is just Lemma 7 above. Note that it implies that L moves on X in a smooth $(n-3)$-dimensional family and, because the restriction map $H^0(N) \to N|_p$ for general $p \in L$ has $(n-3)$-dimensional image, this family fills up an $(n-2)$-dimensional scroll that is a divisor on X.

(ii) Let x_0 be the equation of H and $x_1, \ldots, x_n$ be general linear forms, and consider the case of a simplex

$$X_0 = H_1 \cup \cdots \cup H_n, \quad H_i = V(x_i)$$

and let $A_1 \cup \cdots \cup A_n = X_0 \cap H$ be its H-section. X_0 has singular locus

$$S = \bigcup S_{ij}, S_{ij} := H_i \cap H_j, \quad 1 \leq i < j \leq n.$$

Let $L \subset H_1$ be a general line and

$$p_j = L \cdot H_j = L \cdot S_{1j}, \quad j = 2, \ldots, n; \quad p = L \cdot H = L \cdot A_1.$$

Set $f_0 = x_1 \cdots x_n$, let g be a general degree-$(n-1)$ form vanishing at $p_2, \ldots, p_n$, and let $X_1 \subset \mathbb{P}^n$ be the hypersurface with equation $f_0 + x_0 g$. Thus, X_1 contains $p_2, \ldots, p_n$ and because x_0 is the equation of H in $\mathbb{P}^n$, one has $X_1 \cap H = X_0 \cap H = Z$ and this has equation f_0 in H. Set $Q_j = V(g) \cap S_{1j}, j = 2, \ldots, n$, which is a degree-$(n-1)$ subvariety in S_{1j} containing p_j, and $X_1 \cap S_{1j}$ consists of the hyperplane $V(x_0)$ plus Q_j. Let t be a coordinate on $\mathbb{A}^1$ and consider the linear family, depending on g

$$\pi = \pi(g) : \mathcal{X}(g) = V(f_0 + t x_0 g) \subset \mathbb{P}^n \times \mathbb{A}^1 \to \mathbb{A}^1, \quad X_t = \pi^{-1}(t).$$

This is a pencil of hypersurfaces with fixed H-section Z. Then $\mathcal{X}(g)$ is singular at $S \cap X_1$ and, away from $S_0 = S \cap V(x_0, g)$, $\mathcal{X}_g$ has singularity of type (3-fold ordinary double point)$\times \mathbb{A}^{n-3}$ and so admits a small resolution $\mathcal{X}' \to \mathbb{A}^1$ with fiber X'_0. Note that S_0 also coincides with the singular locus of a general fiber X_t.

The normal bundle N_{L/X'_0} is a corank-1 down modification (i.e. subsheaf of colength 1) of N_{L/H_1} at $p_2, \ldots, p_n$. Identifying $N_{L/H_1}|_{p_j} \simeq T_{p_j} S_{1j}$, this is the down modification corresponding to the subspace $T_{p_j}Q_j, j = 2, \dots, n$. Since these subspaces may be chosen generally it follows first that the modification is general, so that $N_{L/X'_0} = (n-3)\mathcal{O} \oplus \mathcal{O}(-1)$. Moreover clearly, and as one can check by a coordinate computation, as the hyperplanes $T_{p_j}Q_j, j = 2, \dots, n$ vary, so does the $(n-3)\mathcal{O}$ subsheaf and its fiber at p. Explicitly, write $N_{L/H_1} = L_2 \oplus \cdots \oplus L_{n-1}$ where $L_i \simeq \mathcal{O}(1)$ and the fiber of L_i at p_i corresponds to $T_{p_i}Q_i, i = 2, \dots, n-1$. Then the down modification corresponding to $T_{p_i}Q_i, i = 2, \dots, n-1$ replaces each L_i by $\mathcal{O}_L$ so it is just $(n-2)\mathcal{O}_L$ with basis $e_2, \dots, e_{n-1}$. Then if the hyperplane $T_{p_n}Q_n$ is represented by $(\alpha_2, \dots, \alpha_{n-1})$ in this basis, then the $(n-3)\mathcal{O}$ subsheaf is generated by $\alpha_3 e_2 - \alpha_2 e_3, \dots, \alpha_{n-1}e_{n-2} - \alpha_{n-2}e_{n-1}$ and this clearly moves with $T_{p_n}Q_n$. is generated by $\alpha_2\phi_2(p_n) + ... + \alpha_{n-1}\phi_{n-1}(p_n)$, the $\mathcal{O}$ subsheaf is generated by $\alpha_2\phi_2 + ... + \alpha_{n-1}\phi_{n-1}$ and by varying te α_i only we van vary the $(n-3)\mathcal{O}$ subsheaf and its fibre at p.

(iii) What (ii) shows is that the set of limit lines in the family $X(g)$ is a smooth $(n-3)$-parameter family which traces out on A_1 a smooth divisor $D(g)$ whose tangent hyperplane at p corresponds to the aforementioned $(n-3)\mathcal{O}$ subsheaf. This depends on g through the Q_i curves. As g varies, the latter computation shows that $D(g)$ will vary and with it the tangent hyperplane $T_pD(g)$. Therefore, the divisors $D(g)$ will sweep out A_1, filling up an open set.

Case (b). It is convenient to start with a line $L \subset \mathbb{P}^{n-1}$ with $n-1$ distinct points $p_1, \dots, p_{n-1}$, then let F_n, F_{n+1} be general hypersurfaces of the indicated degrees containing $p_1, \dots, p_{n-1}$, then let $Y = F_n \cap F_{n+1}$ and $P = B_Y\mathbb{P}^{n-1}$. Then $N_{L/P}$ is the down modification of $N_{L/\mathbb{P}^{n-1}} = (n-2)\mathcal{O}(1)$ in the rank-1 quotients corresponding to $T_{p_i}Y, i = 1, \dots, n-1$. Because these tangent spaces are general, we have $N_{L/P} = (n-3)\mathcal{O} \oplus \mathcal{O}(-1)$. Then because we can vary Y while fixing $Z = F_{n-1}$, assertions (ii) and (iii) are clear. □

5. Conclusions

Proof of Theorem 1.

$(2, n)$ *complete intersection case:* Assume first $e = 2e_1$ even. We first prove assertion (ii), so we assume $g = 1$ or $n = 4$. Let

$$X_0 = X_1 \cup X_2 \subset \mathbb{P}^{n+1},$$

where X_q, X_2 are general degree-n hypersurfaces in respective hyperplanes $P_1, P_2 \subset \mathbb{P}^{n+1}$ with the property that

$$X_1 \cap P_1 \cap P_2 = X_2 \cap P_1 \cap P_2 =: Z.$$

We may assume Z is a general hyperplane section of X_1 and X_2. A smoothing of X_0 is given by a smoothing of the reducible quadric $P_1 \cup P_2$, and has total space that is singular along a divisor $Z_q \subset Z$.

Let $C_1 \subset X_1$ be a curve of degree $e_1 = e/2$ and genus g, regular with respect to $|\mathcal{O}_{X_1}(1)| = |-K_{X_1}|$ (cf. Proposition 6), and meeting Z transversely in $p_1, \ldots, p_{e_1}$ and disjoint from Z_q. As $(g-1)(n-3) = 0$ and N_{C_1/X_1} is balanced, C_1 moves in a smooth family of dimension $e_1 + n - 5 = \chi(N_{C_1/X_1})$ on X_1. Because $e_1 > n(n-1)(g+1)$ by hypothesis (1), clearly Now because

$$e_1 - 2 = e_1 + n - 4 - (n-2) = \chi(N_{C_1/X_1}(-p_i)) > 0,$$

hence $H^1(N_{C_1/X_1}(-p_i) = 0$, so each p_i moves on Z filling up an analytic open set U_i. By restricting, we may assume the U_i pairwise disjoint. On the other hand, consider a balanced line $L \subset X_2$ with normal bundle $(n-3)\mathcal{O} \oplus \mathcal{O}(-1)$. As we saw in Lemma 9, as the pair (X_2, L) moves while fixing Z, the point $L \cap Z$ moves, filling up a Zariski open set V (NB it is obviously necessary here that the hypersurface move together with the line). As V must intersect each U_i, we may assume that we have a balanced line L_i in X_2 through p_i for $i = 1, \ldots, e_1$. Moreover by Lemma 9, Case (a), the may assume the upper subspace M_i of $N_{L_i/X_2}|_{p_i}$ is general as subspace of $T_{p_i}Z$. Let $N_1 \subset N_{C_1/X_1}$ be the down modification corresponding to $M_1, \ldots, M_{e_1}$ and $N_0 = N_{C_0/X_0}$.

Now recall we are assuming either $g = 1$ or $n = 4$. Then because N_{C_1/X_1} is regular relative to $|\mathcal{O}(1)|$ and the modification is general, we have $H^0(N_1) = 0$. let

$$C_2 = \bigcup_{i=1}^{e_1} L_i, \quad C_0 = C_1 \cup C_2.$$

Now note that $H^0(N_{C_0/X_0}) = H^0(N_1)$ and as we have seen this vanishes. Thanks to our assumption that either $n = 4$ or $g = 1$, we have $\chi(N_{C_0/X_0}) = 0$ so it follows that $H^1(N_{C_0/X_0}) = 0$ as well. Thus, C_0 is rigid on X_0 and deforms with it to a rigid curve of degree $2e_1 = e$ on a general $(2, n)$ complete intersection in $\mathbb{P}^n$. This completes the proof of assertion (ii) e is even.

As for assertion (i), i.e. the case $g = 0$, still assuming e is even, the argument is similar but simpler. Using the same notation, note that it suffices to prove $h^0(N_0) = n - 4$. We have an exact sequence

$$0 \to H^0(N_0) \to H^0(N_{C_1/X_1}) \oplus H^0(N_{C_2/X_2}) \xrightarrow{\rho} \bigoplus_{i=1}^{e_1} M_i.$$

By Proposition 6, we may assume $C_1 \subset X_1$ is regular with respect to $\mathcal{O}(1)$. Then because the M_i are general hyperplanes in $N_{C_1/X_1}|_{p_i}$ it follows that

$$h^0(N_0) = h^0(N_{C_1/X_1}) - e_1 = n - 4.$$

Now consider the case $e = 2e_1 + 1$ odd. The idea is to use the same C_1 of degree e_1 and to replace one of the lines, say L_1, by a suitable conic M. Now recall that the smoothing of X_0 corresponds to smoothing the reducible quadric with equation x_1x_2 to one with equation q. The total space of the family has local equation x_1x_2+tq. As such it is singular in $x_1 = x_2 = t = q = 0$, i.e. the intersection Z_q of Z, the double locus of the special fiber, with the quadric $q = 0$. There the total space admits a small resolution $\tilde{\mathcal{X}}$ by blowing up $x_1 = q_0$ (this makes sense globally) and the special fiber in $\tilde{\mathcal{X}}$ replaces the component X_2 by its blowup in Z_q. Choosing the quadratic q suitably, we can arrange that M meets Z_q in exactly 1 point and there transversely. Then the birational transform of M meets that of Z in exactly 1 point and has normal bundle $(n-3)\mathcal{O} \oplus \mathcal{O}(-1)$ by Lemma 7, so we can proceed as before, for both assertion (i) and (ii).

Hypersurface case:
Suppose e is even. Here, we use a standard quasi-cone degeneration to

$$X_0 = X_1 \cup_Z X_2$$

with X_1 a resolved quasi-cone of degree $n+1$ and X_2 is a hypersurface of degree n in $\mathbb{P}^n$. The degeneration has smooth total space. As in the $(2,n)$ case, we have a lci curve

$$C_0 = C_1 \cup C_2$$

with C_1 a disjoint union of $e_1 = e/2$ many "conics" with normal bundle $(n-3)\mathcal{O} \oplus \mathcal{O}(-1)$ as in Lemma 9, Case (b), and C_2 is a curve of genus g and degree $e_1 = e/2$ on X_2 that is regular relative to $|-K_{X_2}| = |\mathcal{O}_{X_2}(1)|$, and with

$$C_1 \cap Z = C_2 \cap Z.$$

Then an argument as above shows that $H^0(N_{C_0/X_0}) = 0$ so we can conclude as above.

Finally in case e is odd we replace one of the "conics" by a "twisted cubic". This is obtained by starting with a conic $M \subset \mathbb{P}^{n-1}$ with $2n-1$ points $p_1, \ldots, p_{2n-1}$, choosing general hypersurfaces F_n, F_{n+1} through $p_1, \ldots, p_{2n-1}$ and blowing up $Y = F_n \cap F_{n+1}$. The birational transform $\tilde{M}$ of M meets $Z = \tilde{F}_n$ in 1 point and contributes 2 to the total degree of C_0 and its smoothing. Even though the M has normal bundle $N = \mathcal{O}(4) \oplus (n-3)\mathcal{O}(2)$ which is unbalanced, the down modification of N in > 0 points is balanced and so $\tilde{M}$ has balanced normal bundle i.e. $(n-2)\mathcal{O} \oplus \mathcal{O}(-1)$. □

Remark 11. It is proved in [14] that the general sextic in $\mathbb{P}^4$ contains no rational or elliptic curves, thus showing that Theorem 3 is sharp.

Remark 12. Let C be a rational curve on a CICY X as in Theorem 1 with normal bundle $N = (n-4)\mathcal{O}_C \oplus 2\mathcal{O}_C(-1)$. Then for $p \in C$ the image of restriction $H^0(N) \to N|_p$ is $(n-4)$-dimensional. It follows that C moves in X filling up an $(n-3)$-dimensional ruled subvariety birational to a fibration with fiber $\mathbb{P}^1$, i.e. either a $\mathbb{P}^1$-bundle (e odd) or a conic bundle (e even).

For example, for $n = 5$ and $X \subset \mathbb{P}^5$ a sextic, such a rational curve fills up a surface. When C is a line, the degree of the surface is easily computed to be $d = c_7(\mathrm{Sym}^6(Q)) \cdot c_1(Q)$ where Q is the tautological rank-2 bundle on the Grassmannian $\mathbb{G}(1,5)$. Evaluating this number is a routine if tedious calculation (which we have undertaken at the referee's behest). If r_1, r_2 are the Chern roots of Q then those of $\mathrm{Sym}^6(Q)$ are $6r_1, 5r_1 + r_1, \ldots, 6r_2$, hence

$$c_7(Q) = 4320r_1r_2(r_1^5 + r_2^5) + 37584r_1^2r_2^2(r_1^3 + r_2^3) + 98064r_1^3r_2^3(r_1 + r_2)$$

hence, where $c_i = c_i(Q)$,

$$d = 4320c_1^6c_2 + 15984c_1^4 + 6912c_1^2c_2^3.$$

The Chern numbers is question are, respectively, $8!144 = 5806080, 2, 1$ hence

$$d = 25082304480.$$

Remark 13. If C is as in Remark 12 then clearly $T_X|_C = \mathcal{O}(2) \oplus (n-2)\mathcal{O} \oplus \mathcal{O}(-1)$ which is not balanced. Thus in the terminology of [11], C is never ambient-balanced.

Remark 14. The above method of constructing curves yields rigid nodal lci curves of any genus on X_0 for any n. However for $n > 4, g > 1$ these curves have $H^1(N) \neq 0$, so it's not clear these curves smooth out with X_0.

Proof of Corollary 2. The proof is based on the Serre construction (see [8, §I.5.1], which, though formulated for projective spaces is actually mostly valid for arbitrary smooth varieties, as already noted in [8, §I.5.3]). If C is an elliptic curve as in Theorem 1, part (ii), then the computations in [8, §I.5.1], show that $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X) = \mathcal{O}_C$ and there is an exact sequence

$$H^1(\mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X) \to H^0(\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X)) \to H^2(\mathcal{O}_X).$$

Since the extreme groups clearly vanish, the element $1 \in H^0(\mathcal{O}_C)$ yields a uniquely determined sheaf E as an extension of $\mathcal{I}_C$ by $\mathcal{O}_X$, i.e. one has exact

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C \to 0. \tag{2}$$

Then the computations in loc. cit. show that E is locally free with $c_1(E) = 0, c_2(E) = [C] = e\lambda$ (the inequality $e \geq 49$ comes from (1)).

Now an easy diagram chase around the exact sequence

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C \to 0$$

shows that $h^0(E) = 1, h^1(E) = 0$, so the unique section up to scalars of E extends to (infinitesimal) deformations, therefore C deforms with deformations of E. Conversely the functoriality of the Serre construction shows that deformations of C induce deformations of E. Thus, there is an isomorphism between the deformation functors of C and E. Therefore since C is rigid, so is E. Finally, since $\mathrm{Pic}(X)$ is generated by the hyperplane class, in decomposability follows from the Chern classes while (proper) semistability follows from the above exact sequence. □

Proof of Theorem 3. For the proof we use a standard quasi-cone degeneration where $C \subset X \subset \mathbb{P}^n$ degenerates to

$$C_0 = C_1 \subset X_1 \subset X_0 = X_1 \cup_Z X_2 \subset P_0 = P_1 \cup P_2,$$

where $X_1 \subset P_1$ a resolved quasi-cone of degree $n+2$, i.e. the birational transform of a hypersurface of degree $n+2$ with a point of multiplicity $n+1$, also realizable as $B_Y\mathbb{P}^{n-1}$, the blowup of $\mathbb{P}^{n-1}$ in Y which is a codimension-2 complete intersection of general hypersurfaces F_{n+1}, F_{n+2} of the indicated degrees; $X_2 \subset P_2 = \mathbb{P}^n$ a degree-$(n+1)$ hypersurface; $Z = X_1 \cap X_2 \simeq F_{n+1}$ coincides with the birational transform of F_{n+1}. For $C_1 \subset X_1$ we take a curve constructed as follows. Start with a general rational curve C_1' of degree e in $\mathbb{P}^{n-1}$, i.e. a rational normal curve in a general $\mathbb{P}^e \subset \mathbb{P}^{n-1}$, let F_{n+1} be a general hypersurface and then let F_{n+2} be a general hypersurface through $C_1' \cap F_{n+1}$. Then take for C_1 the birational transform of C_1' in the blowup of $Y = F_{n+1} \cap F_{n+2}$. Because $C_1' \cap Y = C_1' \cap F_{n+1}$, we have $C_1 \cap Z = \emptyset$. Because $N_{C_1'/\mathbb{P}^e}$ is balanced, i.e. $N_{C_1'/\mathbb{P}^e} = (e-1)\mathcal{O}(e+2)$, it follows that

$$N_{C_1'/\mathbb{P}^{n-1}} = (e-1)\mathcal{O}(e+2) \oplus (n-e)\mathcal{O}(e).$$

Then because the tangent spaces to Y at $Y \cap C_1'$ can be specified arbitrarily (compare [9, Proof of Theorem 20]), it follows that N_{C_1/X_1} is a general, locally corank-1 down modification of $N_{C_1'/\mathbb{P}^{n-1}}$ at $e(n+1)$ points, it follows that

$$N_{C_1/X_1} = (n-e-4)\mathcal{O} \oplus (e+2)\mathcal{O}(-1).$$

Because $H^1(N_{C_1/X_1}) = H^1(N_{C_1/X_0}) = 0$ as C_1 is disjoint from Z, it follows that C_1 deforms along as X_0 smooths to a general degree-$(n+2)$ hypersurface in $\mathbb{P}^n$. □

Remark 15. If $X_d \subset \mathbb{P}^n$ is a hypersurface of degree d and $C \to X_d$ is a curve of degree e and genus g with normal bundle N, then $\chi(N) = (n+1-d)e + n - 4 - g(n-2)$. Thus if $d > n+1$ and $e > \frac{n-4}{d-n-1}$ (any $g \geq 0$), then $\chi(N) < 0$ and in particular $H^1(N) \neq 0$. Indeed one does not expect such a curve (e.g. rational curve) to exist on a *general* X_d. In [10], however one can find some constructions for "well-behaved" families of *special* hypersurfaces X_d, codimension $h^1(N) = -\chi(N)$ in the family of all hypersurfaces, endowed with "good" curves.

Notations as above, the above argument in proof of Theorem 3, applied to a rational curve C_0 of degree $e > n-4$, yields a curve C_1 whose normal bundle N_{C_1/X_1} has $\mathcal{O}(-2)$ summands, hence has $H^1 \neq 0$, so it's not clear if C_1 deforms along as X_0 deforms to a general X_d.

On the other hand, whenever $d \geq n+2$ the same argument, using a degree-d quasi-cone degeneration, does produce balanced rational curves of degree $e \leq (n-4)/(d-n-1)$. In view of Lemma 7, this is interesting when we can take $e \geq 3$ which means $3 \leq (n-4)/(d-n-1)$ i.e. $n+2 \leq d \leq (4n-1)/3$, $n \geq 7$. For example, taking $n = 10, d = 13$, we get a balanced twisted cubic on a general degree-13 hypersurface $X_{13} \subset \mathbb{P}^{10}$.

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