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PHENOMENON OCCURS WHEN THERE ARE LARGE RELATIVE CHANGES IN BOUNDARY CURVATURE

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by

Paul Concus and Robert Finn

The first recorded observations of capillarity phenomena seem due to Leonardo da Vinci, who observed the rise of water in a small tube. These phenomena have since attracted the attention of leading natural philosophers in every generation, to the extent that one can trace the development of many methods of modern analysis and computation by observing the techniques applied in attempts to determine the shape and height of capillary free-surface interfaces.

Geometrical formulation

By the beginning of the nineteenth century it was recognized that the problem can be cast in geometrical language. Consider a surface separating air and liquid in an equilibrium configuration (e.g., a capillary free surface in a soda straw). Then this surface can be determined by the condition that its mean curvature H at a point be proportional to the height u of that point above a certain reference level -- $2H = \kappa u(x,y)$, where κ is a physical constant -- and by the condition that the surface meets the bounding walls in an angle γ, called the contact angle, which depends only on the materials.

This last condition has been and continues to be a source of some uncertainty; in the authors' view, the results discussed below lend strong support to its validity.

H can be expressed analytically in terms of the derivatives of u; thus, the surface is described by the solution of a (nonlinear) partial differential equation, with nonlinear boundary condition.

Given κ and γ, it is easy to find the average rise height $\overline{u}$ over a section. For the case of a tube with sectional area A and sectional boundary length L, placed vertically in an infinite reservoir, there holds $\overline{u} = (L/\kappa A)\cos\gamma$. This classical formula of Laplace yields, as a special case, the well-known result for a circular tube of radius r, $\overline{u} = (2/\kappa r)\cos\gamma$.

The wedge phenomenon

The formula for the average height can be deceptive if taken as an indication of the distribution of heights over the section. To see what can happen, we consider a water-air interface in a tube with the section shown in fig. 1, with ℓ at least 0.8 cm and with $0 \leq \gamma < \pi/2$. It turns out: (i) *whenever* $\alpha + \gamma \geq \pi/2$, *the maximum fluid height is less than* 0.8 cm; (ii) *if* $\alpha + \gamma < \pi/2$ *the fluid rises to an infinite height in the corner.*

Thus the fluid surface near the vertex depends discontinuously on the angle α.

Fig. 2 shows an experiment made by T. Coburn at the Stanford University Medical School. Here a drop of water has been placed between two acrylic plastic plates meeting at angles $\alpha \approx 12°$ in fig. 2a, and $\alpha \approx 9°$ in fig. 2b. In the latter case the measured rise height is more than ten times the maximum possible for any configuration with $\alpha + \gamma \geq \pi/2$.

In gravity-free environments, such as those encountered in space vehicles, the discontinuity in behavior becomes startling. It can be shown that *in the absence of gravity, if* $\alpha + \gamma < \pi/2$ *the equations defining the surface possess no solution!* This result holds at any corner, and depends in no other way on the shape of the container.

Our prediction of this phenomenon was tested by W. Masica at the NASA zero-gravity facility in Cleveland. Fig. 3 shows observations during a five-second interval of free fall, for a regular hexagonal cylinder of acrylic plastic, partly filled with alcohol solutions whose concentrations were varied to change γ. In fig. 3a, $\gamma \approx 48^\circ$; the free surface is a lower spherical cap (as can be derived exactly for any regular polygonal container whenever $\alpha + \gamma \geq \pi/2$).

Fig. 3b shows the case $\gamma \approx 25^\circ$; here the fluid flows up into the corner and covers part of the top of the container. Were the cylinder infinitely long, the fluid presumably would flow out along the corners to infinity.

The above mathematical results cannot be attributed to the failure of the contact-angle condition at a sharp corner. First, the mathematical solutions, when they exist, are *uniquely* determined by the data at the smooth boundary points, among all possible such solutions, with *no* growth conditions imposed. (Such a strong uniqueness theorem does not hold, for example, for the Laplace equation.)

Second, all results can be obtained as a limiting case for a sequence of smooth boundaries having rounded corners. For the gravity-free case, one finds the result: *Suppose there is a point on the boundary at which the boundary curvature exceeds* L/A. *Then there is a critical angle* γ_0, $0 < \gamma_0 \leq \pi/2$, *such that there is no solution whenever* $0 \leq \gamma < \gamma_0$. Thus, in the absence of gravity, there may be no solution even when the boundary is smooth.

The reasoning that shows this behavior shows also that solutions are *always* unstable with respect to boundary perturbation. This follows because one could always alter the boundary in a smooth way (e.g., by a groove scored into the cylinder wall) so as to obtain a new boundary surface for which there is no solution.

Another corollary of the method is that in a gravitational field all solutions are bounded, depending only on distance to the boundary (and depending in no other way on the shape of the boundary).

We note that in the case of "negative" gravity, as occurs for a drop of liquid pendent from a ceiling, such a universal bound no longer seems to hold. There exists, in this case, a "singular" solution in the form of an infinite spike, tending to minus infinity. In the positive- or zero-gravity cases, which correspond to the situations described above, no such solution can exist.

The trapezoid phenomenon

Much progress has been made recently in clarifying the effects of boundary geometry on the solution surface in positive gravitational fields; however, for the negative- and zero-gravity situations a satisfactory theory is not yet available. For example, it can be shown that in the gravity-free case *if the section is a rectangle of any side ratio, a solution exists if* $\gamma > \pi/4$ *and no solution exists if* $\gamma < \pi/4$. Thus, the criterion for a regular polygon still holds in this case. (It should be remarked that the solution, when it exists, is no longer known explicitly if the sides are unequal.)

Now pick a $\gamma > \pi/4$, say $\gamma = \pi/3$. Then *there exists a trapezoid whose angles are as close to* $\pi/2$ *as desired, and whose opposite sides are as nearly equal as desired, such that there is no solution in the trapezoid for boundary angle* γ. Thus again a discontinuity in behavior is encountered. In this case no satisfactory general criteria for deciding when a solution will exist have yet been found.

The pendent drop

The situation becomes in some ways still more striking when one studies the configuration of a pendent drop (negative-gravity case). If one attempts in practice to increase the size of the drop, it soon becomes unstable and breaks apart; however, formal solutions of the equations continue to exist for any vertex height, and exhibit remarkable analytical properties. Fig. 4 shows the result of a lengthy calculation made by W. Thomson in 1886.

In fig. 5 are shown the results of numerical calculations of vertical sections for vertex heights $u_0 = -4, -8$, and -16 (dotted curves); these are compared with the singular "spike" solution described above (solid curve). The analytical properties of these solutions are discussed in some detail in a paper by these authors to appear shortly, and it is conjectured -- but not completely proved -- that the solutions converge to the "spike" as $u_0 \to -\infty$.

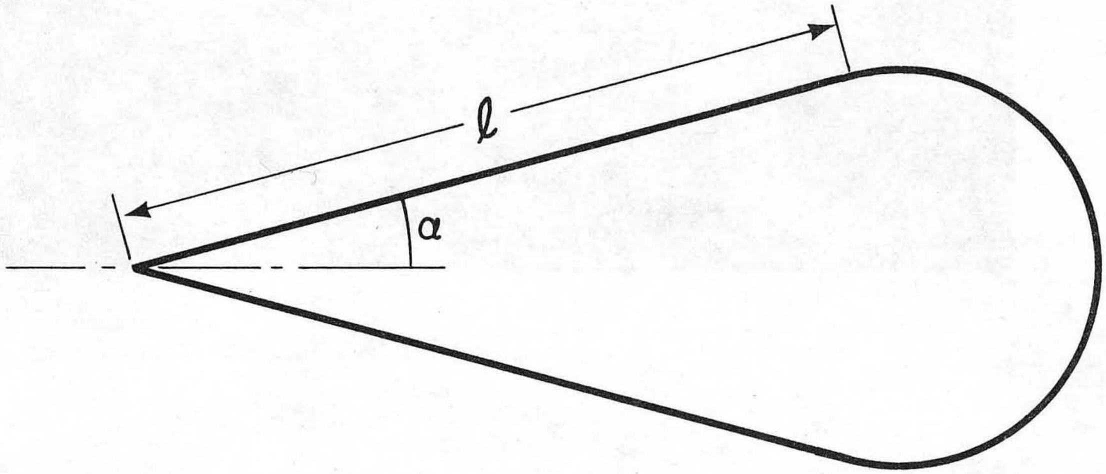
Extremal properties

Extremal problems for capillary surfaces have apparently been neglected in the literature since Laplace's time. We mention a few:

(i) Among all sections with given boundary length, which raises the largest (or smallest) volume of liquid? Answer: the classical formula of Laplace yields the result that all sections with given boundary length raise the same volume, independent of the shape!

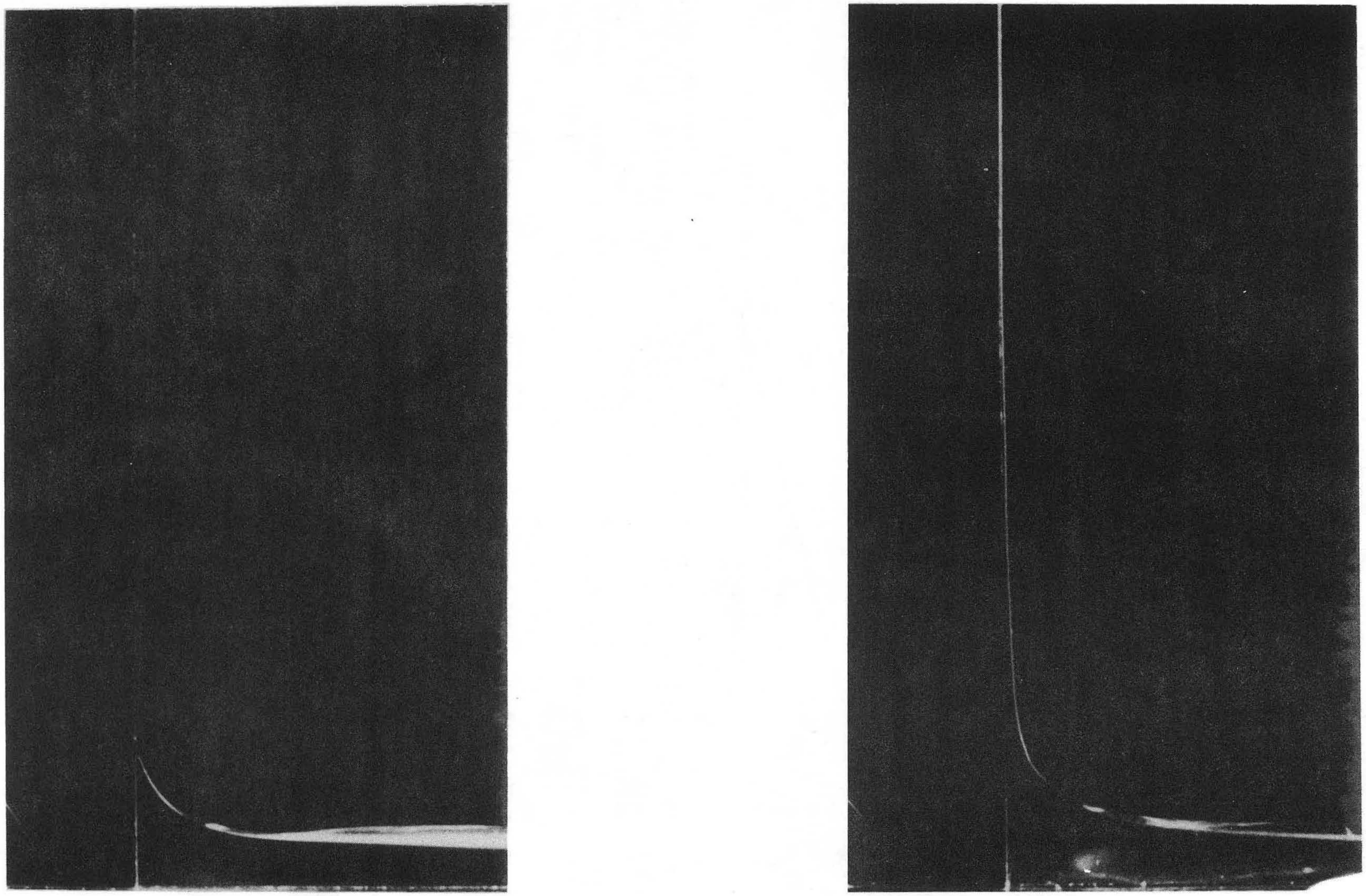
(ii) Among all sections of given area, which raises the largest (or smallest) volume? Answer: a section of given area can be constructed so as to raise an arbitrarily large volume; however, a circular disk raises the smallest possible volume, as follows from (i) and the isoperimetric inequality.

(iii) Among all sections of given area, for which is the maximum rise height a minimum? We conjecture the answer is a circular disk, but as yet we have no proof.



XBL 783-360

Fig. 1

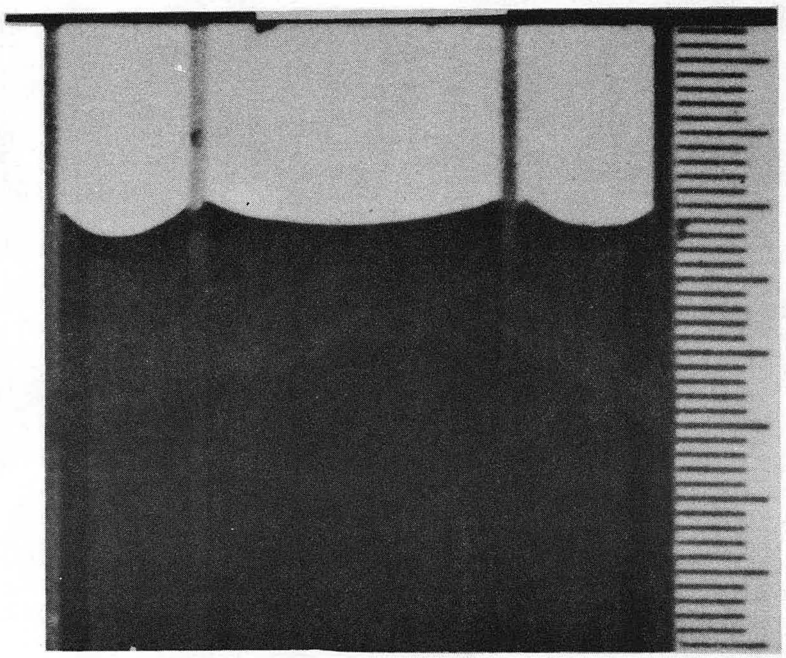


(a)

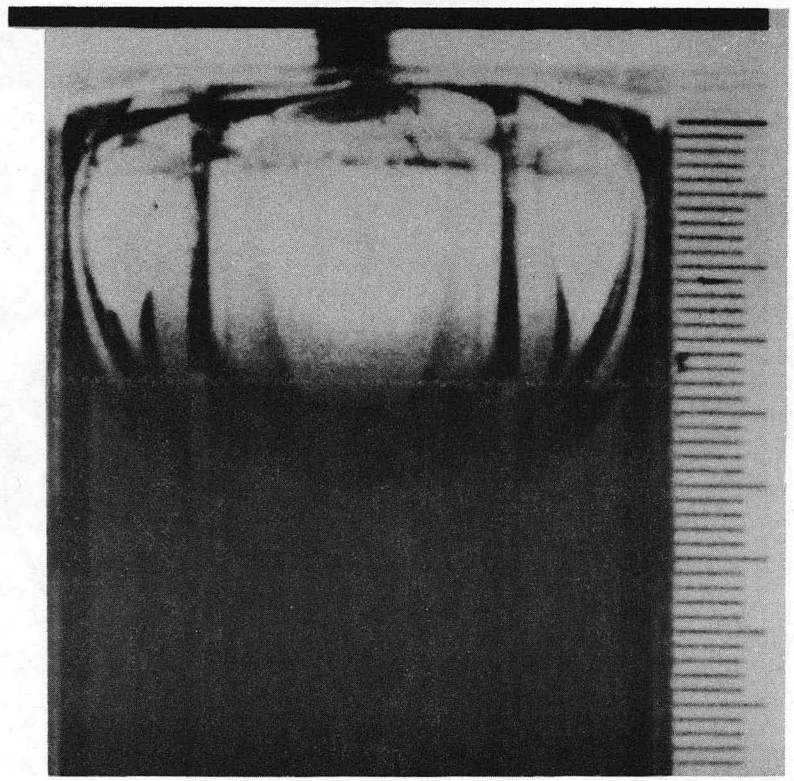
(b)

Fig. 2

XBB 741-331



(a)



(b)

Fig. 3

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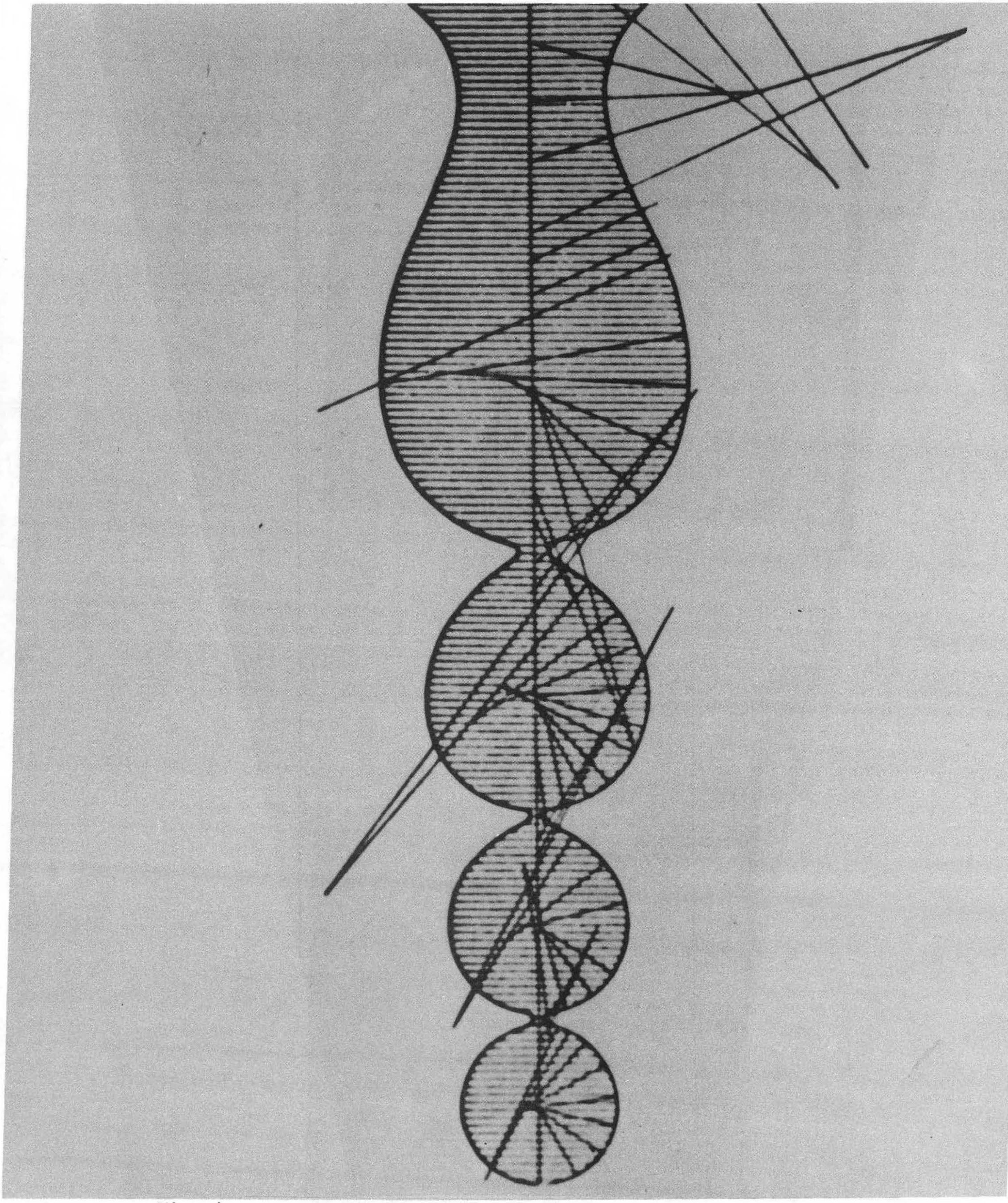
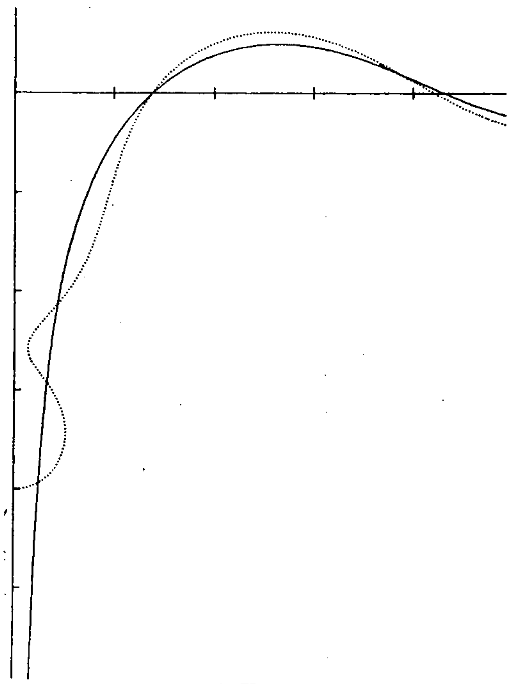
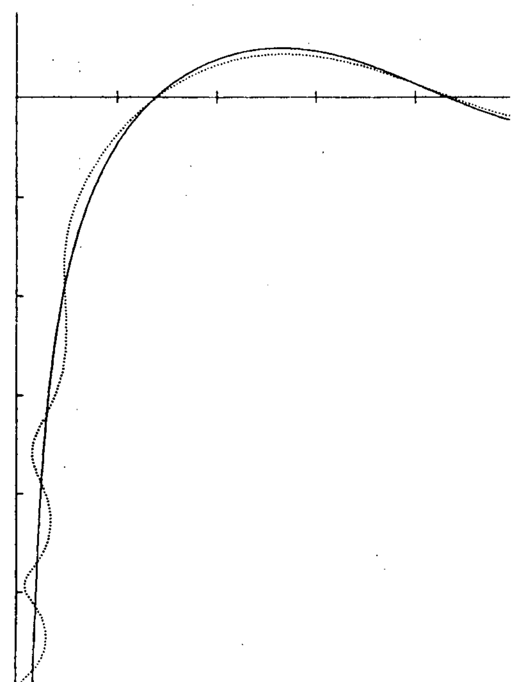


Fig. 4



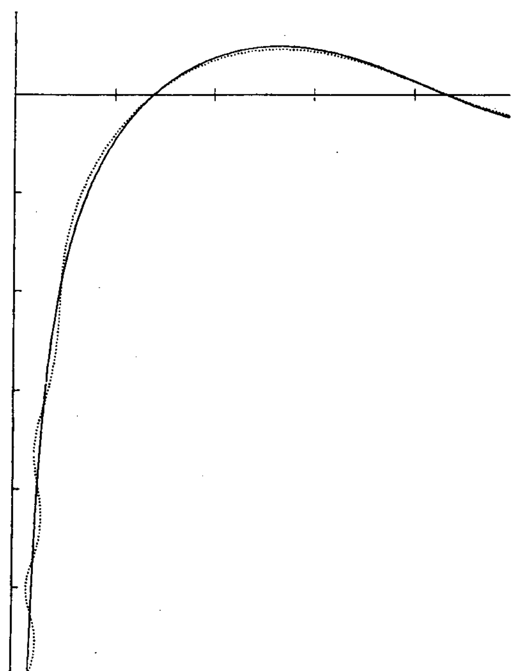
$u_0 = -4,\ U(r)$

Fig. 5*a*



$u_0 = -8\ \ U(r)$

Fig. 5*b*



$u_0 = -16,\ U(r)$

Fig. 5*c*

Fig. 5

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