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Parameter estimation of multiple transient signals

Yingbo Hua
Department of Electrical and Electronics Engineering, The University of Melbourne, Parkville, Victoria 3052, Australia

Tapan K. Sarkar
Department of Electrical and Computer Engineering, Syracuse University, Syracuse, NY 13244-1240, USA

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Abstract. The parameter estimation of multiple transient signals (waves) impinging on a uniform linear array is addressed. Each transient signal is assumed to be a sum of complex exponentials, arriving at the array from a different direction. The method we use to solve this problem is a novel generalization of the subspace decomposition and matrix pencil approach which has been previously used for processing one-dimensional transient signals. Simulation results show that this method can be near efficient statistically.

Zusammenfassung. Es wird die Parameterschätzung transienter Signale (Wellen) untersucht, die auf eine gleichförmige, lineare Antennenanordnung einfallen. Jedes transiente Signal wird als eine Summe komplexer Exponentialfunktionen angenommen, die an der Antennenanordnung aus verschiedenen Richtungen eintreffen. Die von uns gebrauchte Methode zur Lösung des Problems ist eine neuartige Verallgemeinerung der Unterraumzerlegung und des Bandmatrixansatzes, der für die Verarbeitung eindimensionaler transienter Signale gebraucht wurde. Simulationsergebnisse zeigen, daß diese Methode beinahe statistisch wirksam (efficient) sein kann.

Résumé. Nous nous intéressons dans cet article à l'estimation des paramètres de signaux transitoires multiples arrivant sur un réseau uniforme linéaire. Chaque signal transitoire est suppose être la somme d'exponentielles complexes arrivant sur le réseau d'une direction différente. La méthode que nous utilisons pour résoudre ce problème est une généralisation originale de la décomposition sur sous-espaces et de l'approche par matrices 'pencil' qui a déjà été utilisée auparavant pour le traitement de signaux transitoires mono-dimensionnels. Des résultats de simulation montrent que cette méthode peut être proche de l'efficacité au sens statistique.

Keywords. Multiple transients, complex exponentials, parameter estimation, subspace decomposition, matrix pencil, multiple-target identification, wideband wave direction finding.

1. Introduction

In many applications, the study of transient signals modeled by the sum of complex exponentials is important. Generally, the transient signals can be processed for characterization of a system (such as a radar target [1]). To our knowledge, however, all studies of transient signals have only focused on single-transient signals, i.e., one-dimensional transient signals [2, 4]. In this paper, we will address a more general and more difficult problem: estimating the parameters of multiple transient signals arriving at a uniform linear array from different

Correspondence to: Yingbo Hua, Department of Electrical and Electronics Engineering, The University of Melbourne, Parkville, Victoria 3052, Australia.

directions. The multiple-transient signals can be used for (simultaneous) characterization of multiple systems such as multiple radar targets. Study of the multiple-transient problem appears not available in our research literature, and no method is available for solving this problem except a few general parameter estimation approaches (like the least square error approach) which are computationally too expensive. We will develop a method directly based on the multiple-transient signals. The method to be developed is a novel generalization of the subspace decomposition and matrix pencil (SDMP) approach which has been previously used for the single-transient problem. In the noiseless case, the SDMP method yields the exact values of the desired parameters. In the noisy case, the SDMP method can be near efficient statistically, as shown in simulation.

The following sections will be organized as follows. In Section 2, the multiple-transient problem is formulated and analyzed. In Section 3, the SDMP method is developed. In Section 4, simulation results are presented.

2. The problem formulation

We assume a uniform linear array of identical omnidirectional sensors on which several plane waves arrive from different directions. Then, the array output can be modelled by

$$y_{k,h}=x_{k,h}+n_{k,h}, \tag{2.1}$$

with

$$x_{k,h}=\sum_{i=1}^{I} s_i(k+\tau_i h), \tag{2.2}$$

$$\tau_i=\frac{\mathrm{d}\sin\theta_i}{v\delta t}, \tag{2.3}$$

where $x_{k,h}$ and $n_{k,h}$ are the signal and noise components, respectively; k $(=0,1,\ldots,K-1)$ the index of temporal samples (snapshots); h $(=0,1,\ldots,H-1)$ the index of spatial samples (sensors); $s_i(k)$ the ith signal received at the sensor zero; I the total number of impinging waves during the K snapshots; d the distance between adjacent sensors; θ_i the arrival angle of the ith signal against the broadside; v the velocity of all impinging waves; δt the sampling (snapshot) time interval.

We assume further that, during the K snapshots, each transient signal is a sum of complex exponentials, i.e.,

$$s_i(k)=\sum_{j=1}^{J(i)} b_{ij}z_{ij}^{k}, \tag{2.4}$$

where z_{ij} is the jth (signal) pole of the ith signal, b_{ij} the associated amplitude, and $J(i)$ the number of poles associated with the ith signal. $\{z_{ij};j=1,\ldots,J(i)\}$ are assumed to be distinct. Then, using (2.4) in (2.2) leads to

$$x_{k,h}=\sum_{i=1}^{I}\sum_{j=1}^{J(i)} b_{ij}z_{ij}^{k+\tau_i h}. \tag{2.5}$$

The objective now is to estimate the spatial parameters $\{\tau_i;i=1,\ldots,I\}$, which are one-to-one functions of the angular parameters $\{\theta_i;i=1,\ldots,I\}$ via (2.3), and the temporal parameters $\{b_{ij},z_{ij};j=1,\ldots,J(i),i=1,\ldots,I\}$, given the observed data $\{y_{k,h};k=0,1,\ldots,K-1,\ h=0,1,\ldots,H-1\}$. The order parameters, I and $J(i)$, are assumed to be known (or chosen).

Since $x_{k,h}$ is linearly dependent on the amplitudes b_{ij}, the amplitudes can easily be found by solving a set of linear equations once the other parameters are available. Hence, the important task is to find z_{ij} and τ_i.

We now write $x_{k,h}$ in the following form:

$$x_{k,h}=\sum_{k=0}^{K-1}\sum_{h=0}^{H-1} b_{ij}z_{ij}^{k}g_{ij}^{h}, \tag{2.6}$$

where

$$g_{ij}=z_{ij}^{\tau_i}. \tag{2.7}$$

Clearly, $x_{k,h}$ is a two-dimensional exponential signal as opposed to the one-dimensional exponential signals treated in [1, 2, 4]. This form suggests that the methods developed for one-dimensional exponential signals can be used to find z_{ij} and g_{ij} if both $\{z_{ij};\ j=1,\ldots,J(I),\ i=1,\ldots,I\}$ and $\{g_{ij};j=1,\ldots,J(I),\ i=1,\ldots,I\}$ are distinct. Specifically,

z_{ij} (if distinct) can be found (using methods in [1, 2, 4]) from the multiple sequences (or measurements) $\{x_{k,h}; k=0, 1, \ldots, K-1\}$ for $h=0, 1, \ldots, H-1$, and g_{ij} (if distinct) from the multiple sequences $\{x_{k,h}; h=0, 1, \ldots, H-1\}$ for $k=0, 1, \ldots, K-1$. However, there are two important remaining issues about the two-dimensional problem which cannot be answered from the knowledge that has been developed for the one-dimensional problem. The first issue is how a better method can be developed directly based on the two-dimensional structure of $x_{k,h}$ to find z_{ij} and g_{ij} regardless whether they are distinct or not. Note that the assumption of distinct θ_i and distinct $\{z_{ij}; j=1, \ldots, J(I)\}$ does not imply the distinction of $\{z_{ij}; j=1, \ldots, J(I),\ i=1, \ldots, I\}$ and $\{g_{ij}; j=1, \ldots, J(I),\ i=1, \ldots, I\}$. The second issue is how we pair the estimated z_{ij} and g_{ij} so that (2.7) can be used to find τ_i. These two issues will be answered in the next section.

3. The SDMP method

Our approach starts from the following generalized data matrix:

$$\boldsymbol{X}=\begin{bmatrix} \boldsymbol{X}_0 & \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_N \\ \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_{N+1} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{X}_{H-N-1} & \boldsymbol{X}_{H-N} & \cdots & \boldsymbol{X}_{H-1} \end{bmatrix}, \tag{3.1}$$

where the submatrix $\boldsymbol{X}_h$ $(h=0, 1, \ldots, H-1)$ is defined as

$$\boldsymbol{X}_h=\begin{bmatrix} x(0;h) & x(1;h) & \cdots & x(M;h) \\ x(1;h) & x(2;h) & \cdots & x(M+1;h) \\ \vdots & \vdots & & \vdots \\ x(K-M-1;h) & x(K-M;h) & \cdots & x(K-1;h) \end{bmatrix}, \tag{3.2}$$

in which $x(k;h)=x_{k,h}$ (for simple text processing purpose). M and N are changeable integers. Note that if $M=K-1$ and $N=0$ (or $M=0$ and $N=H-1$) then $\boldsymbol{X}$ becomes the 'original' data matrix (i.e., the two-dimensional array of $x_{k,h}$). However, the choice of different values of M and N affects significantly the estimation accuracy as will be shown in Section 4.

3.1. The structure of the noiseless generalized data matrix

For convenience, we will let

$$\{b_{ij}; j=1, \ldots, J(i),\ i=1, \ldots, I\} = \{b_{ij}\} = \{b_p; p=1, \ldots, P\} = \{b_p\},$$

$$\{z_{ij}; j=1, \ldots, J(i),\ i=1, \ldots, I\} = \{z_{ij}\} = \{z_p; p=1, \ldots, P\} = \{z_p\}$$

and

$$\{g_{ij}; j=1, \ldots, J(i),\ i=1, \ldots, I\} = \{g_{ij}\} = \{g_p; p=1, \ldots, P\} = \{g_p\},$$

where

$$P=\sum_{i=1}^{I} J(i).$$

Using (2.6) in (3.2), we get the decomposition

$$\boldsymbol{X}_h=\boldsymbol{Z}_{K-M}\boldsymbol{B}\boldsymbol{G}_d^h\boldsymbol{Z}_{M+1}^{\mathrm{T}}, \tag{3.3}$$

where

$$\boldsymbol{Z}_m=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_P \\ \vdots & \vdots & & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_P^{m-1} \end{bmatrix}, \tag{3.4}$$

$$\boldsymbol{B}=\mathrm{diag}[b_1, b_2, \ldots, b_P], \tag{3.5}$$

$$\boldsymbol{G}_d=\mathrm{diag}[g_1, g_2, \ldots, g_P]. \tag{3.6}$$

The superscript T denotes the matrix transpose, and the superscript h the exponential power h. $\boldsymbol{Z}_m$ has the dimension $m \times P$. Note that $\boldsymbol{Z}_{K-M}$ is $\boldsymbol{Z}_m$ with m replaced by $K-M$, and $\boldsymbol{Z}_{M+1}$ is $\boldsymbol{Z}_m$ with m replaced by $M+1$. Substituting (3.3) into (3.1) yields

$$\boldsymbol{X}=\boldsymbol{Z}_{K-M,H-N}\boldsymbol{B}\boldsymbol{Z}_{M+1,N+1}^{\mathrm{T}}, \tag{3.7}$$

where $Z_{K-M,H-N}$ and $Z_{M+1,N+1}$ are defined by

$$Z_{m,n}=\begin{bmatrix} Z_m G_d^0 \\ Z_m G_d^1 \\ \vdots \\ Z_m G_d^{n-1} \end{bmatrix}, \tag{3.8}$$

which has the dimension $mn \times P$.

We now construct four matrices: X_a, X_b, X_c and X_d from X as follows.

RULE 1.
X_a is X with its $(M+1)$th, $2(M+1)$th, ..., $(N+1)(M+1)$th columns deleted;
X_b is X with its 1st, $(M+1+1)$th, ..., $(N(M+1)+1)$th columns deleted;
X_c is X with its last $M+1$ columns deleted; and
X_d is X with its first $M+1$ columns deleted.

Following the approach leading to (3.3) and (3.7), we can similarly show the following decompositions:

$$X_a = Z_{K-M,H-N} B Z_{M,N+1}^T, \tag{3.9}$$

$$X_b = Z_{K-M,H-N} B Z_d Z_{M,N+1}^T, \tag{3.10}$$

$$X_c = Z_{K-M,H-N} B Z_{M+1,N}^T, \tag{3.11}$$

$$X_d = Z_{K-M,H-N} B G_d Z_{M,N+1}^T, \tag{3.12}$$

where G_d is defined in (3.6); $Z_{K-M,H-N}$, $Z_{M,N+1}$ and $Z_{M+1,N}$ are defined by (3.8); and

$$Z_d = \text{diag}[z_1, z_2, \ldots, z_P]. \tag{3.13}$$

Now, the two matrix pencils $X_b - zX_a$ and $X_d - gX_c$ can be expressed as

$$X_b - zX_a = Z_{K-M,H-N} B (Z_d - zI_P) Z_{M,N+1}^T, \tag{3.14}$$

$$X_d - gX_c = Z_{K-M,H-N} B (G_d - gI_P) Z_{M+1,N}^T. \tag{3.15}$$

where I_P is the $P \times P$ identity matrix, and z and g are complex variables. Based on (3.14) and (3.15), it can be shown [2] that if the P-column matrices $Z_{K-M,H-N}$, $Z_{M,N+1}$ and $Z_{M+1,N}$ in (3.14) and (3.15) are all of the full rank P, then $\{z_p\}$ are the P rank reducing numbers (generalized eigenvalues) of $X_b - zX_a$, and $\{g_p\}$ are the P rank reducing numbers of $X_d - gX_c$.

The two integers M and N can be adjusted so that $Z_{K-M,H-N}$, $Z_{M,N+1}$ and $Z_{M+1,N}$ in (3.14) and (3.15) are all of the full rank P. Obviously, each of the above three P-column matrices must have no less than P rows, which means that M and N must satisfy

$$(K-M)(H-N) \geq P, \tag{3.16}$$

$$M(N+1) \geq P, \tag{3.17}$$

$$(M+1)N \geq P. \tag{3.18}$$

Depending on the signal parameters, the above conditions may not be sufficient for the three P-column matrices to be of the full rank P. However, we conjecture that the following (more restrictive) conditions:

$$K-P \geq M \geq P, \tag{3.19}$$

$$H-P \geq N \geq P, \tag{3.20}$$

are sufficient (provided that each signal contains distinct poles and arrives at the array from a different direction). Although a general proof of the sufficiency appears difficult and has not been obtained, it can be shown (see Appendix A) that (3.19) and (3.20) are sufficient when all transients have an identical set of (signal) poles, i.e., when $\{z_{ij}; j=1, \ldots, J(i)\}$ are the same for all i.

Since the P-column matrices $Z_{K-M,H-N}$, $Z_{M,N+1}$ and $Z_{M+1,N}$ in (3.14) and (3.15) are submarices of $Z_{K-M,H-N}$ or $Z_{M+1,N+1}$ in (3.7), the 'necessary' conditions (3.16)–(3.18) or the 'sufficient' conditions (3.19)–(3.20) are also the corresponding conditions for the noiseless generalized data matrix X to be of the rank P.

3.2. Estimating the poles

Based on the structure analysis of the generalized data matrix X, the poles can be estimated as follows. Let Y be defined as X with $x_{k,h}$ replaced by the noisy measurement $y_{k,h}$. Since X has the rank P, we need a rank-P matrix Y_T such that the Frobenious matrix norm $\|Y - Y_T\|_F$ is minimized. Such Y_T is known to be given by the rank-P singular

value decomposition (SVD) truncation:

$$\boldsymbol{Y}_{\mathrm{T}} = \sum_{i=1}^{P} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{H}} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{H}}, \tag{3.21}$$

where

$$\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_P], \tag{3.22}$$

$$\boldsymbol{S} = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_P], \tag{3.23}$$

$$\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_P]. \tag{3.24}$$

$\sigma_1 > \sigma_2 > \cdots > \sigma_P$ are the P largest singular values of $\boldsymbol{Y}$. (If P is unknown, it can be chosen to be the number of dominant singular values of $\boldsymbol{Y}$.) $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the corresponding principal left and right singular vectors of $\boldsymbol{Y}$. The superscript H denotes the conjugate transpose.

Note that the columns of $\boldsymbol{U}$ span the principal column subspace of $\boldsymbol{Y}$, and the columns of $\boldsymbol{V}$ span the principal column subspace of $\boldsymbol{Y}^{\mathrm{H}}$. Also note that $\boldsymbol{Y}$ for $M = M_1$ and $N = N_1$ is equal to $\boldsymbol{Y}^{\mathrm{T}}$ for $M = K - M_1$ and $N = H - N_1$, and hence $\boldsymbol{U}$ for $M = M_1$ and $N = N_1$ is equal to $\boldsymbol{V}^*$ (i.e., $\boldsymbol{V}$ conjugate) for $M = K - M_1$ and $N = H - N_1$. Therefore, $\boldsymbol{U}$ and $\boldsymbol{V}$ are similarly structured, and any information obtained from $\boldsymbol{V}$ for $M = M_1$ and $N = N_1$ can be similarly obtained from $\boldsymbol{U}$ for $M = K - M_1$ and $N = H - N_1$. In the sequel, only $\boldsymbol{V}$ will be used.

Now, $\boldsymbol{Y}_a$, $\boldsymbol{Y}_b$, $\boldsymbol{Y}_c$ and $\boldsymbol{Y}_d$ (the noisy version of $\boldsymbol{X}_a$, $\boldsymbol{X}_b$, $\boldsymbol{X}_c$ and $\boldsymbol{X}_d$) should be constructed from $\boldsymbol{Y}_T$ according to Rule 1. Then, it follows from (3.21) that

$$\boldsymbol{Y}_a = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}_a^{\mathrm{H}}, \tag{3.25}$$

$$\boldsymbol{Y}_b = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}_b^{\mathrm{H}}, \tag{3.26}$$

$$\boldsymbol{Y}_c = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}_c^{\mathrm{H}}, \tag{3.27}$$

$$\boldsymbol{Y}_d = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}_d^{\mathrm{H}}, \tag{3.28}$$

where $\boldsymbol{V}_a$, $\boldsymbol{V}_b$, $\boldsymbol{V}_c$ and $\boldsymbol{V}_d$ are defined as follows.

RULE 2.

$\boldsymbol{V}_a$ is $\boldsymbol{V}$ with its $(M+1)$th, $2(M+1)$th, ..., $(N+1)(M+1)$th rows deleted;

$\boldsymbol{V}_b$ is $\boldsymbol{V}$ with its 1st, $(M+1+1)$th, ..., $(N(M+1)+1)$th rows deleted;

$\boldsymbol{V}_c$ is $\boldsymbol{V}$ with its last $M+1$ rows deleted; and

$\boldsymbol{V}_d$ is $\boldsymbol{V}$ with its first $M+1$ rows deleted.

Then, we have

$$\boldsymbol{Y}_b - z\boldsymbol{Y}_a = \boldsymbol{U}\boldsymbol{S}(\boldsymbol{V}_b^{\mathrm{H}} - z\boldsymbol{V}_a^{\mathrm{H}}), \tag{3.29}$$

$$\boldsymbol{Y}_d - g\boldsymbol{Y}_c = \boldsymbol{U}\boldsymbol{S}(\boldsymbol{V}_d^{\mathrm{H}} - g\boldsymbol{V}_c^{\mathrm{H}}). \tag{3.30}$$

The above equations imply that in the noiseless case, the generalized eigenvalues of $\boldsymbol{Y}_b - z\boldsymbol{Y}_a$ are those of $\boldsymbol{V}_b - z\boldsymbol{V}_a$, the generalized eigenvalues of $\boldsymbol{Y}_d - g\boldsymbol{Y}_c$ are those of $\boldsymbol{V}_d - g\boldsymbol{V}_c$. Hence the poles, $\{z_p\}$ and $\{g_p\}$, can be retrieved from the smaller matrix pencils: $\boldsymbol{V}_b - z\boldsymbol{V}_a$ and $\boldsymbol{V}_d - g\boldsymbol{V}_c$, respectively.

The above technique for constructing matrix pencils is a generalization of the state-space method used in [5]. The resulting matrix pencils $\boldsymbol{V}_b - z\boldsymbol{V}_a$ and $\boldsymbol{V}_d - g\boldsymbol{V}_c$ can be further compressed and filtered before the generalized eigenvalues are computed. Several algorithms for solving the generalized eigenvalue problem are recently reviewed in [3] and available in the references thereof.

3.3. Pairing the poles

Again based on the structure analysis of $\boldsymbol{X}$, we know from (3.7) and (3.21) that in the noiseless case, $\mathrm{range}(\boldsymbol{Z}_{M+1,N+1}) = \mathrm{range}(\boldsymbol{V}^*)$, provided that the sufficient conditions (3.19) and (3.20) are satisfied. This means that in the noiseless case,

$$\begin{aligned} &\mathrm{span}[\boldsymbol{z}_1 \otimes \boldsymbol{g}_1, \boldsymbol{z}_2 \otimes \boldsymbol{g}_2, \ldots, \boldsymbol{z}_P \otimes \boldsymbol{g}_P] \\ &\quad = \mathrm{span}[\boldsymbol{v}_1^*, \boldsymbol{v}_2^*, \ldots, \boldsymbol{v}_P^*] \\ &\quad \perp \mathrm{span}[\boldsymbol{v}_{P+1}^*, \boldsymbol{v}_{P+2}^*, \ldots, \boldsymbol{v}_{(M+1)(N+1)}^*], \end{aligned} \tag{3.31}$$

where $\boldsymbol{v}_p$ for $p > P$ are the nonprincipal right singular vectors of $\boldsymbol{Y}$, $\perp$ denotes the orthogonality, $\otimes$ the Kronecker product, $\boldsymbol{z}_p = [1, z_p, z_p^2, \ldots, z_p^M]^{\mathrm{T}}$, $\boldsymbol{g}_p = [1, g_p, g_p^2, \ldots, g_p^N]^{\mathrm{T}}$ and $\boldsymbol{z}_p \otimes \boldsymbol{g}_p$ is the pth column of $\boldsymbol{Z}_{M+1,N+1}$. It suggests that we can find the correct pairing between the estimated $\{z_{p'}\}$ and $\{g_{p''}\}$ by minimizing the following cost function:

$$E = \sum_{p>P} |\boldsymbol{v}_p^{\mathrm{T}}(\boldsymbol{z}_{p'} \otimes \boldsymbol{g}_{p''})|^2, \tag{3.32}$$

with respect to p'' for each p'.

4. Simulation results

Assume that the array output is given by

$$y_{k,h}=s_1(k+\tau_1 h)+s_2(k+\tau_2 h)+\sigma n_{k,h}, \quad (4.1)$$

where

$$s_1(k)=a_1\exp(\alpha_1 k)\cos(\omega_1 k+\phi_1), \quad (4.2)$$

$$s_2(k)=a_2\exp(\alpha_2 k)\cos(\omega_2 k+\phi_2). \quad (4.3)$$

Here $k=0, 1, \ldots, 24$; $h=0, 1, \ldots, 9$; $a_1=a_2=1$; $\alpha_1=-0.05$; $\alpha_2=-0.1$; $\omega_1=0.5$; $\omega_2=1.0$; $\tau_1=0.1$; $\tau_2=0.2$; $\phi_1=0$ and $\phi_2=90°$. The above data set indicates that there are two signals arriving at the uniform linear array of $H=10$ sensors from the two directions that correspond to τ_1 and τ_2, and each signal consists of a complex conjugate pair of poles (i.e., $z_{11}=\exp(\alpha_1+j\omega_1)$, $z_{12}=\exp(\alpha_1-j\omega_1)$, $z_{21}=\exp(\alpha_2+j\omega_2)$ and $z_{22}=\exp(\alpha_2-j\omega_2)$). The number of temporal samples (snapshots) is $K=25$. $\sigma n_{k,h}$ is the noise with deviation equal to σ, where $n_{k,h}$ was generated by using IMSL routine GGNML (which generates pseudo-white Gaussian noise of deviation one).

In Tables 1–5, statistical results are given for the spatial parameters (τ_1 and τ_2) and the temporal parameters (angular frequencies ω_1 and ω_2). After the estimates of $\{z_p\}$ and $\{g_p\}$ were obtained by using the SDMP method, they were correctly paired and then used to yield the estimates of $\{\tau_i\}$ according to (2.7). $\{\omega_i\}$ were obtained by using $\omega_i=|\mathrm{Im}(\log(z_{ij}))|$. (Note that j can be either 1 or 2 in the above computation without affecting the results since z_{i1} is always the complex conjugate of z_{i2}.)

For $\sigma=0.001$ (SNR $=-20\log_{10}\sigma=60$ dB), Tables 1–4 show 20-run sample deviations of the four parameters: τ_1, τ_2, ω_1 and ω_2, divided by the corresponding (square rooted) Cramér–Rao bounds. The Cramér–Rao bounds were computed with the ten unknowns: a_1, a_2, α_1, α_2, ω_1, ω_2, ϕ_1, ϕ_2, τ_1 and τ_2, assuming that the noise is white Gaussian. Note that the Cramér–Rao bounds are the diagonal elements of the inverted Fisher's information matrix [6]. From these tables, it can be

Table 1
Deviations of τ_1 divided by $\mathrm{CRB}^{1/2}=0.1041\times10^{-3}$

	$M=2$	3	4	5	6	7
$N=1$		2.56	1.82	1.89	1.91	1.96
2	2.52	2.97	1.23	1 27	1.32	1.35
3	9.84	2.10	1.12	1.15	1.17	1.22
4	10.78	3.47	1.33	1.20	1.19	1.23
5	10.38	4.10	1.49	1.23	1.24	1.28

Table 2
Deviations of τ_2 divided by $\mathrm{CRB}^{1/2}=0.7903\times10^{-4}$

	$M=2$	3	4	5	6	7
$N=1$		2.36	2.08	1.53	1.34	1.36
2	12.72	3.07	1.37	1.06	1.05	1.20
3	8.97	3.38	1.52	1.11	1.17	1.30
4	6.77	3.64	1.99	1.40	1.27	1.38
5	3.42	2.26	1.52	1.16	1.09	1.25

Table 3
Deviations of ω_1 divided by $\mathrm{CRB}^{1/2}=0.2601\times10^{-4}$

	$M=2$	3	4	5	6	7
$N=1$		9.66	6.10	1.60	1.29	1.27
2	13.02	7.42	4.98	1.57	1.34	1.30
3	13.07	6.75	3.76	1.65	1.44	1.37
4	11.43	6.06	2.78	1.73	1.49	1.41
5	10.57	5.01	2.48	1.75	1.47	1.40

Table 4
Deviations of ω_2 divided by $\mathrm{CRB}^{1/2}=0.4927\times10^{-4}$

	$M=2$	3	4	5	6	7
$N=1$		6.89	7.57	2.58	1.38	1.34
2	7.00	5.84	6.90	2.43	1.35	1.32
3	5.36	5.04	5.76	2.21	1.33	1.23
4	5.60	5.44	4.95	1.89	1.28	1.16
5	6.26	5.86	4.65	1.74	1.22	1.17

Table 5
Square rooted efficiencies as functions of σ

σ	τ_1	τ_2	ω_1	ω_2
0.001	1.32	1.05	1.34	1.35
0.01	1.32	1.04	1.35	1.36
0.02	1.32	1.03	1.35	1.36
0.05	1.32	1.02	1.36	1.39
0.07	2.81	12.03	97.14	51.75

seen that the efficiencies (i.e., ratio of variance over Cramér–Rao bound) are close to one (the optimal value) when $M \geqslant 6$ and $N \geqslant 2$.

For $M=6$ and $N=2$, Table 5 shows the square rooted (20-run) efficiencies of the estimated τ_1, τ_2, ω_1 and ω_2 as functions of the noise level σ. It can be seen that the efficiencies stay constant (nearly optimal) until σ reaches threshold 0.07 (SNR = 23 dB).

Note that our experience in simulation has suggested that good choices of M and N satisfy $\frac{2}{3}K \geqslant M \geqslant \frac{1}{3}K$ and $\frac{2}{3}H \geqslant N \geqslant \frac{1}{3}H$ as supported by Tables 1–4. This result is similar to that for the single-transient problem studied in [2].

5. Conclusions

The multiple-transient problem has been studied in detail. The SDMP method has been developed for solving this problem. The simulation results have shown that this method can be near efficient statistically. This method is also the most efficient in computation among all near or exact optimum methods known so far.

Appendix A

In the following, we will show that the conditions (3.19) and (3.20) are sufficient for $\boldsymbol{Z}_{K-M,H-N}$, $\boldsymbol{Z}_{M,N+1}$ and $\boldsymbol{Z}_{M+1,N}$ to be of the full rank P when $\{z_{ij}; j=1, \ldots, J(i)\}$ (distinct poles of the ith signal) are the same for all i.

It is easy to verify that '(3.19) and (3.20) are sufficient for $\boldsymbol{Z}_{K-M,H-N}$, $\boldsymbol{Z}_{M,N+1}$ and $\boldsymbol{Z}_{M+1,N}$ to be of the full rank P' is equivalent to 'the conditions $m \geqslant P$ and $n \geqslant P$ are sufficient for $\boldsymbol{Z}_{m,n}$ to be of the full rank P'.

Then, it can be seen that although $\{z_{ij}; j=1, \ldots, J(i); i=1, \ldots, I\}$ or $\{z_p; p=1, \ldots, P\}$ are not distinct, $\{g_{ij}; j=1, \ldots, J(i); i=1, \ldots, I\}$ or $\{g_p; p=1, \ldots, P\}$ are distinct because $g_{ij}=z_{ij}^{\tau_i}$ where $\{\tau_i; i=1, \ldots, I\}$ are distinct.

We can rearrange the order of rows of $\boldsymbol{Z}_{m,n}$ so that the resulting matrix becomes

$$\boldsymbol{Z}'_{m,n}=\begin{bmatrix} \boldsymbol{G}_n\boldsymbol{Z}_d^0 \\ \boldsymbol{G}_n\boldsymbol{Z}_d^1 \\ \vdots \\ \boldsymbol{G}_n\boldsymbol{Z}_d^{m-1} \end{bmatrix}, \tag{A.1}$$

where $\boldsymbol{Z}_d$ is defined by (3.13), and

$$\boldsymbol{G}_n=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ g_1 & g_2 & \cdots & g_P \\ \vdots & \vdots & & \vdots \\ g_1^{n-1} & g_2^{n-1} & \cdots & g_P^{n-1} \end{bmatrix}. \tag{A.2}$$

Note that $\operatorname{rank}(\boldsymbol{Z}'_{m,n}) = \operatorname{rank}(\boldsymbol{Z}_{m,n})$. Since $n \geqslant P$ and $\{g_p\}$ are distinct, $\boldsymbol{G}_n$ is of the full rank P. Furthermore, since $\boldsymbol{G}_n$ is the top n-row submatrix of $\boldsymbol{Z}'_{m,n}$, $\boldsymbol{Z}'_{m,n}$ is of the full rank P, and hence so is $\boldsymbol{Z}_{m,n}$.

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