

UC Santa Barbara

UC Santa Barbara Electronic Theses and Dissertations

Title

Categorical Resolutions of Singularities of Type A_n

Permalink

<https://escholarship.org/uc/item/2jb5d9mr>

ISBN

9798186381518

Author

Pion, Joel Ethan

Publication Date

2026-06-09

Peer reviewed|Thesis/dissertation

University of California
Santa Barbara

Categorical Resolutions of Singularities of Type A_n

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Joel E. Pion

Committee in charge:

Professor Xiaolei Zhao, Chair
Professor Zheng Liu
Professor David Morrison

June 2026

The Dissertation of Joel E. Pion is approved.

Professor Zheng Liu

Professor David Morrison

Professor Xiaolei Zhao, Committee Chair

June 2026

Categorical Resolutions of Singularities of Type A_n

Copyright © 2026

by

Joel E. Pion

Acknowledgements

I would like to thank my advisor, Xiaolei, for his immense patience, honesty and support throughout my PhD. I would like to thank my other committee members Dr. Zheng Liu and Dr. David Morrison for taking the time and energy to aid me through the PhD process. Thank you to my mother, Dr. Carla Pion, who has given me endless support, love, encouragement, and math suits at every step of life. Thank you to my father, Dr. Paul Pion, for showing his love and support (though he'd never admit it) in those ineffable ways that perfectly counter-balance my mom. Thank you to my brother, Dr. Luca Pion-Tonachini, who from the far reaches of Italy provided the perfect blend of intellectual stimulation and distraction. Thank you to my sister, Silke Pion, for always staying in touch and keeping me grounded. Also thank you Fara for being a cool sister-in-law. Thank you to my partner, (soon to be Dr.) Sarah Mantell, for your love and always having my back when I needed you. I would like to thank my many friends for their support and welcome distractions. Thank you Bloop Mantell for always being a gentleman and reminding me when dinner is. Thank you Wubba Wubba Pion for taking over Bloop's dinner reminder job and gracing the world with his presence. Thank you Charbydis Pion for always being cheerful, even when I've just done something you don't like for your own good. Thank you Bleem Pion for chasing my floss and battling my shadows. And thank you to Fiona, the best and hungriest of them all.

Curriculum Vitæ

Joel E. Pion

Education

2026	Ph.D. in Mathematics (Expected), University of California, Santa Barbara.
2022	M.A. in Mathematics, University of California, Santa Barbara.
2020	B.S. in Mathematics (Summa Cum Laude), California Polytechnic State University of San Luis Obispo

Publications

Pion, J.E., Negre, C.F.A. and Mniszewski, S.M. Quantum computing for a profusion of postman problem variants. *Quantum Mach. Intell.* 5, 24 (2023).
<https://doi.org/10.1007/s42484-023-00111-6>

Pion, J.E., Mniszewski, S.M. Toward computing bounds for Ramsey numbers using quantum annealing. *Quantum Inf Process* 24, 221 (2025).
<https://doi.org/10.1007/s11128-025-04839-x>

Abstract

Categorical Resolutions of Singularities of Type A_n

by

Joel E. Pion

Resolutions of singularities are a key idea in birational algebraic geometry. Resolutions are important as they allow one to find, for a singular scheme, a non-singular scheme in the same birational class. Resolutions have been studied extensively in classical algebraic geometry, but are not as well understood in the derived category setting. In this thesis we extend recent results for computing crepant categorical resolutions and their kernels in the A_1 and A_2 cases to the arbitrary A_n case.

Contents

Curriculum Vitae	v
Abstract	vi
1 Introduction	1
2 Singularities and Resolutions in Geometry	6
2.1 Resolutions and Rational Singularities	6
2.2 Canonical Bundle	13
2.3 Spinor Bundles and Quadrics	19
3 Categorical Resolutions of Singularities	25
3.1 Triangulated and Derived Categories	25
3.2 Categorical Resolutions of A_1 and A_2 Singularities	38
3.3 Categorical Resolutions of A_n Singularities	42
Bibliography	62

Chapter 1

Introduction

A major impetus behind algebraic geometry, both historically and in the modern day, is the goal of classifying and understanding the various classes of varieties and schemes. Classifying schemes up to isomorphism, like classifying up to isomorphism in many mathematical fields, can be too restrictive. A fruitful avenue of research is to weaken this condition to birationality. Many key properties are preserved under birationality and having the additional flexibility allows one to study “nicer” members of a birational class. This naturally leads to the study of resolutions of singularity. A common hindrance to using some powerful tools with schemes is the presence of singularities; a resolution provides a member of the birational class which is non-singular.

The existence of a resolution of a singularity in characteristic 0, under mild assumptions, was proven by Hironaka in 1964 [12]. With existence, a non-trivial problem, resolved one then turns their attention to finding a “best” resolution. What best means can depend on context, but generally one wants

to find the smallest resolution one can. Removing the chaff from your stalks in a manner of speaking. A special version of these minimal resolutions is called a crepant resolution. Unfortunately we do not in general have crepant resolutions, especially when dealing with higher dimensional schemes. Just as isomorphisms were too restrictive, it may behoove us to weaken our notion of crepancy.

In the last century derived categories have become a central tool in the field of algebraic geometry. They have granted powerful homological tools to further our understanding. This has driven work to translate and study classical geometric ideas in the derived category setting. Centrally relevant to this thesis is that of translating the notion of resolutions of singularities to categorical resolutions. Versions of the notion of categorical resolutions were introduced in papers like [3, 5, 20]. The version we are taking in this paper is closest in statement to the one introduced by Kuznetsov in [17].

Definition 1.1: A **categorical resolution** of a triangulated category $\mathcal{D}$ is defined by a triple $(\tilde{\mathcal{D}}, \sigma_*, \sigma^*)$. The triangulated category $\tilde{\mathcal{D}}$ is an admissible subcategory of $D^b(X)$ where X is a non-singular scheme. The functors σ_*, σ^* must be defined so that $\sigma_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$, $\sigma^* : D^{\text{perf}} \rightarrow \tilde{\mathcal{D}}$ where $\sigma^* \dashv \sigma_*$ and the natural morphism induces $\text{id}_{D^{\text{perf}}} \cong \sigma_* \sigma^*$.

Alongside categorical resolutions, categorical notions of crepancy were introduced. We use a version that we will call weak crepancy.

Definition 1.2: Let $(\tilde{\mathcal{D}}, \sigma_*, \sigma^*)$ be a categorical resolution of a triangulated

category $\mathcal{D}$. We say the resolution is **weakly crepant** if $\sigma_* \dashv \sigma^*$.

In this thesis the author studies schemes, X , containing singularities of type A_n with the goal of understanding $D^b(X)$. A good way to understand $D^b(X)$ is to relate it to some $D^b(\tilde{X})$ where $\tilde{X}$ is non-singular. A strong way to relate them is through a Verdier localization.

Definition 1.3: Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories and let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. Then F is a **Verdier localization** (or just localization) if $\overline{F} : \mathcal{D}/\ker(F) \rightarrow \mathcal{D}'$, $\overline{F}$ defined in Lemma 3.22, is an equivalence.

To understand $D^b(X)$ one would like to find a weakly crepant categorical resolution, which is a localization, as well as compute the kernel of this resolution. The resolution gives us a scheme, $\tilde{X}$, birational to X which is smooth. The crepancy gives the resolution a notion of minimality. The localization, along with the computation of the kernel of the resolution gives a “First Isomorphism Theorem” relation between X and $\tilde{X}$. We do this with the main result of the thesis:

Theorem 1.4: *Let X be a projective variety with an isolated A_n singularity. Then $D^b(X)$ has a weakly crepant categorical resolution of a singularity which is a localization. The kernel of the resolution is generated by $\langle \mathcal{A}_{E_1}, \dots, \mathcal{A}_{E_{\lceil \frac{n}{2} \rceil}} \rangle$.*

where we use the notation $\mathcal{A}_{E_i}$ to mean

$$\mathcal{A}_{E_i} = \begin{cases} \langle \tilde{j}_{i*} c_i^* S \rangle & \text{if } m \text{ is odd and } (n \text{ is even or } i \leq \lfloor \frac{n}{2} \rfloor) \\ \langle \text{cone}(\tilde{j}_{i*} S' \rightarrow \tilde{j}_{i*} S''[2]) \rangle & \text{if } m \text{ is odd and } n \text{ is odd and } i = \lceil \frac{n}{2} \rceil \\ \langle \tilde{j}_{i*} c_i^* S', \tilde{j}_{i*} c_i^* S'' \rangle & \text{if } m \text{ is even and } (n \text{ is even or } i \leq \lfloor \frac{n}{2} \rfloor) \\ \langle \tilde{j}_{i*} S \rangle & \text{if } m \text{ is even and } n \text{ is odd and } i = \lceil \frac{n}{2} \rceil \end{cases}$$

The terms S, S', S'' are spinor bundles defined over the quadric exceptional divisors which arise when resolving the A_n singularity by repeatedly blowing-up the singularity. The construction will be done via induction. The A_1 case was proven by Cattani, Giovenzana, Liu, Magni, Martinelli, Pertusi, and Song in [4] and by Kuznetsov and Shinder in [19]. The A_2 case was proven by Fietz in [8]. These two cases act as the base for our induction. Moreover the ideas developed in these papers were invaluable to the work in this thesis. Many of the techniques developed in them were used or adapted to be used in the A_n situation.

The work in this thesis is outlined as follows. Chapter 2 covers the non-derived geometric results which are necessary for the main result on A_n singularities. Section 2.1 introduces resolutions and singularities in the classical geometric setting. Section 2.2 reviews the notion of Gorenstein. Here we also go over how one can define the canonical bundle when in singular setting. At the end of chapter 2 in section 2.3 we define spinor bundles on both smooth and nodal quadrics. Chapter 3 covers the derived category portion of the thesis. In section 3.1 we develop the necessary theory in the derived category that

we need to prove our result. Section 3.2 summarizes the key results of the A_1 singularity and A_2 singularity cases in [4, 19, 8] necessary for the base case of our induction. Finally in section 3.3 we prove the main result of the thesis. Here we construct a crepant categorical resolution of an A_n singularity which is also a localization. We also compute the kernel of this categorical resolution.

Chapter 2

Singularities and Resolutions in Geometry

2.1 Resolutions and Rational Singularities

In this section we will introduce the idea of singularities and their resolutions in a geometric setting. We will then take a closer look at singularities of type A_n .

Definition 2.1: Let X be a normal scheme of finite type over a field of characteristic zero. Then X has **rational singularities** if there exists a proper birational map $\sigma : \tilde{X} \rightarrow X$ from a regular scheme $\tilde{X}$ such that the natural map in the derived category $\mathcal{O}_X \rightarrow \sigma_* \mathcal{O}_{\tilde{X}}$ is an isomorphism.

Definition 2.2: The **surface singularities of type ADE** are rational sin-

gularities in $\mathbb{A}^3$ given by the equations [22]:

$$A_n : x_0^{n+1} + x_1^2 + x_2^2 = 0 \quad (n \geq 1)$$

$$D_n : x_0^{n-1} + x_1^2 x_0 + x_2^2 = 0 \quad (n \geq 4)$$

$$E_6 : x_0^4 + x_1^3 + x_2^2 = 0$$

$$E_7 : x_0^3 x_1 + x_1^3 + x_2^2 = 0$$

$$E_8 : x_0^5 + x_1^3 + x_2^2 = 0$$

The singularities share the name of Dynkin diagrams as the diagrams show up in their resolutions. Each of these equations has a resolution where the exceptional divisor is built of copies of $\mathbb{P}^1$ configured in the following way. If one considers each $\mathbb{P}^1$ as a vertex, and connects two vertices if the $\mathbb{P}^1$'s intersect, the corresponding Dynkin Diagram is formed.

Definition 2.3: Let X be a scheme of dimension $m \geq 2$, let $n \in \mathbb{N}$, and let $p \in X$. We say X has an A_n **singularity at** p if

$$\widehat{\mathcal{O}_{X,p}} \cong k[[x_0, x_1, \dots, x_m]] / (x_0^{n+1} + x_1^2 + \dots + x_m^2)$$

Theorem 1.11 from [25] can be specialized to say that m -dimensional singularities defined locally by $x_0^{r_0} + \dots + x_m^{r_m}$ are rational if and only if $\frac{1}{r_0} + \dots + \frac{1}{r_m} > 1$. So singularities of type A_n are rational.

The definition we use for higher dimensional singularities of type A_n is a natural extension of the surface singularity. The exceptional divisor of the resolution can be viewed, in a similar manner to the surface case, as a Dynkin diagram of type A_n . This is exhibited in example 2.

Definition 2.4: A **regular local ring**, S , is a Noetherian local ring whose maximal ideal can be generated by $\dim(S)$ elements.

Definition 2.5: A point x on a scheme X is **non-singular** if the local ring $\mathcal{O}_{X,x}$ is regular. Otherwise x on X is **singular**. The scheme X is **non-singular** if every point on the scheme is non-singular, otherwise X is **singular**.

Definition 2.6: A **resolution of singularity** of a scheme X is a non-singular scheme $\tilde{X}$ and a proper birational morphism $\sigma : \tilde{X} \rightarrow X$ that is an isomorphism over X_{sm} (the smooth locus of X).

Definition 2.7: Let X be a scheme and let Z be a closed subvariety of X . Let $\mathcal{I}$ be the ideal sheaf corresponding to Z . Then the **blow-up of X along Z** is defined as

$$b : \text{Proj} \left(\bigoplus_{n \geq 0} \mathcal{I}^n \right) \rightarrow X$$

and the **exceptional divisor** is defined as $b^{-1}(Z)$.

Remark 1. Our focus will be on blow-ups where Z is a point. The following diagram illustrates the blow-up of a point more concretely.

$$\begin{array}{ccccc} \pi^{-1}(X \setminus \{0\}) & \xrightarrow{\text{Closure}} & \tilde{X} & \hookrightarrow & \{(\vec{x}, [\vec{y}]) \mid x_i y_j = x_j y_i\} & \hookrightarrow & \mathbb{A}^{m+1} \times \mathbb{P}^m \\ \pi \downarrow & & b \downarrow & & \pi \downarrow & & \\ X \setminus \{0\} & \hookrightarrow & X & \hookrightarrow & \mathbb{A}^{m+1} & & \end{array}$$

Theorem 2.8 (Hironaka Main Theorem 1 [12]): *Let k be a field of characteristic 0 and X a scheme over k , reduced and irreducible, then there exists a*

subscheme Z of X such that Z is supported on the singular locus of X and if $\sigma : \tilde{X} \rightarrow X$ is the blow-up of X with center Z , then $\tilde{X}$ is non-singular.

This landmark theorem published in 1964 by Hironaka guarantees the existence of resolutions of singularities for schemes in characteristic 0 under very mild assumptions. This non-trivial theorem is proven by constructing a finite sequence of blow-ups for a given scheme. One can walk through how to produce a resolution through a sequence of blow-ups in examples 1 and 2.

Example 1: Let $n \in \mathbb{N}$. Let X be defined by the zero locus of

$$x_0^{n+1} + x_1^2 + \dots + x_m^2 \text{ in } \mathbb{A}^{m+1}.$$

Note X has an A_n singularity at the origin. We shall blow-up X at the origin. Thus we seek the smallest closed subset of $\mathbb{A}^{m+1} \times \mathbb{P}^m$ containing the zero locus of:

1. $x_0^{n+1} + x_1^2 + \dots + x_m^2$
2. $x_i y_j - x_j y_i$ for $0 \leq i < j \leq m$.

On each affine open subset $\{y_i = 1\}$ we get an equation

$$x_0^{n+1} + x_0^2 y_1^2 + \dots + x_0^2 y_m^2 = 0$$

when $i = 0$ or

$$x_0^{n-1} x_i^2 y_0^2 + x_i^2 y_1^2 + \dots + x_i^2 + \dots + x_i^2 y_m^2 = 0$$

when $i \neq 0$. These equations then reduces by factoring out an unnecessary component to obtain the globally defined

$$x_0^{n-1}y_0^2 + y_1^2 + \dots + y_m^2 = 0.$$

With the addition of this equation we have our blow-up of the A_n singularity. Note when $n > 2$ that the blow-up has a single A_{n-2} singularity at the point $((0, \dots, 0), [1 : 0 : \dots : 0])$ and that the exceptional divisor, E , is defined by

$$y_1^2 + \dots + y_m^2 = 0.$$

We have $E \cong \tilde{X} \times_X \{0\}$ where $\tilde{X}$ is the blow-up of X . This can be verified affine-locally by using the map $x_i \mapsto 0$. Letting Z be our origin this means we get a cartesian diagram:

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} \\ \rho \downarrow & & \downarrow \sigma \\ Z & \xrightarrow{i} & X \end{array}$$

In example 1 we walked through the blow-up of an A_n singularity. Later we will want to understand the geometry of what happens when we repeat the blow-up procedure to eventually resolve the singularity. We will walk through the computation now:

Example 2: Let us begin as before with a variety, X , defined by

$$x_0^{n+1} + x_1^2 + \dots + x_m^2 = 0.$$

Assume $n \geq 3$. From example 1 we know the singularity of X lies at the origin and that the blow-up of X , $\tilde{X}$, is defined by the following equations in $\mathbb{A}^{m+1} \times \mathbb{P}^m$:

1. $x_0^{n+1} + x_1^2 + \dots + x_m^2 = 0$
2. $x_i y_j = x_j y_i$ for $i, j \in \{0, \dots, m\}$
3. $x_0^{n-1} y_0^2 + y_1^2 + \dots + y_m^2 = 0$

The exceptional divisor E is then defined by the equation

$$y_1^2 + \dots + y_m^2 = 0 \text{ in } \mathbb{P}^m$$

and $\tilde{X}$ has a singularity at $((0, \dots, 0), [1 : 0 : \dots : 0])$. Now we can take the blow-up of $\tilde{X}$ at its singularity to get $\tilde{\tilde{X}}$ which will be defined in $\mathbb{A}^{m+1} \times \mathbb{P}^m \times \mathbb{P}^m$ by the equations:

1. $x_0^{n+1} + x_1^2 + \dots + x_m^2 = 0$
2. $x_i y_j = x_j y_i$ for $i, j \in \{0, \dots, m\}$
3. $x_0^{n-1} y_0^2 + y_1^2 + \dots + y_m^2 = 0$
4. $a_i x_j = a_j x_i$ for $i, j \in \{0, \dots, m\}$
5. $x_0^{n-3} z_0^2 + z_1^2 + \dots + z_m^2 = 0$

where $a_i = z_i$ when $i \neq 0$ and $a_0 = x_0$. The exceptional divisor of this second blow-up, E' , would then be defined by

$$z_1^2 + \dots + z_m^2 = 0 \text{ in } \mathbb{P}^m.$$

The singularity in $\tilde{\tilde{X}}$, if it still exists, would lie at the coordinate

$$((0, \dots, 0), [1 : 0 : \dots : 0], [1 : 0 : \dots : 0]).$$

The strict transform of E in $\tilde{\tilde{X}}$ under the blow-up is just the blow-up of E , $\tilde{E}$, at the point $((0, \dots, 0), [1 : 0 : \dots : 0])$ by Lemma 080E in [1]. Thus we can see that $\tilde{E}$ in $\tilde{\tilde{X}}$ is just defined by the equations:

1. $x_i = 0$ for $i \in \{0, \dots, m\}$
2. $y_1^2 + \dots + y_m^2 = 0$
3. $y_i z_j = z_i y_j$ for $i, j \in \{0, \dots, m\}$
4. $z_0 = 0$
5. $z_1^2 + \dots + z_m^2 = 0$

Notably we can see that if a singularity remains in $\tilde{\tilde{X}}$ then it would not lie on the strict transform of E . Thus the strict transform of E , in the resolution of the singularity of $\tilde{\tilde{X}}$ defined by blowing up the singular point at each step, is isomorphic to $\tilde{E}$.

2.2 Canonical Bundle

In this section we discuss the definition and properties of the canonical bundle. We do this as the canonical bundle is often introduced in a smooth. As we care about singular schemes, we introduce how one can define the canonical bundle in some non-smooth settings. We also review the notion of Gorenstein.

Definition 2.9: Let R be a commutative ring. The **Krull dimension** of R is $\sup\{n \mid P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n \subsetneq R \text{ and } P_i \text{ a prime ideal } \forall i\}$.

Remark 2. When referring to the dimension of a ring, unless otherwise stated, we will mean the Krull dimension.

Definition 2.10: Let R be a commutative ring. Then $a_1, \dots, a_d$ is a **regular sequence** of R if:

1. $R \neq (a_1, \dots, a_d)$
2. a_r is not a zero divisor in $R/(a_1, \dots, a_{r-1})$ for $r \in \{1, \dots, d\}$

and if R is a local ring then the **depth** of R is the maximum d across regular sequences in R .

Definition 2.11: A local ring R is **Cohen-Macaulay** if $\text{depth}(R) = \dim(R)$.

A ring or scheme is **Cohen-Macaulay** if locally Cohen-Macaulay.

Lemma 2.12 (Thm II.8.21A [11]): *A regular local ring is Cohen-Macaulay*

Definition 2.13: A **local complete intersection ring** is the quotient of a regular local ring by an ideal generated by regular sequence.

Definition 2.14: Let R be the quotient of a regular local ring S . Then the **canonical module**, also known as the **dualizing module**, is

$$\omega_R \equiv \text{Ext}_S^{\dim(S)-\dim(R)}(R, S).$$

Let $X \subset \mathbb{P}^m$ be a projective scheme of codimension d , then the **dualizing sheaf** or **canonical sheaf** of X is

$$\omega_X \equiv \mathcal{E}xt_{\mathbb{P}^m}^d(\mathcal{O}_X, \omega_{\mathbb{P}^m}).$$

Note that the standard definition for canonical bundle for non-singular varieties, as defined in [11], aligns with our definition when mutually defined by Corollary III.7.12 in [11]. Also $\omega_{\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^m}(-m-1)$, see II.8.20.1 in [11]. The next lemma connects the local definition of canonical module with the global definition of canonical sheaf.

Lemma 2.15: *Let $X \subset \mathbb{P}^m$ be a projective scheme and p a point in X . Then we have that $\omega_{X,p} \cong \omega_{\mathcal{O}_{X,p}}$.*

Proof. Note that viewed affine locally we have $\omega_{\mathcal{O}_{\mathbb{P}^m,p}} \cong k[y_1, \dots, y_m]_p$ and we have $\mathcal{O}_{X,p} \cong (k[y_1, \dots, y_m]/I)_p$. So we have,

$$\omega_{X,p} \cong \mathcal{E}xt_{\mathbb{P}^m}^d(\mathcal{O}_X, \omega_{\mathbb{P}^m})_p \cong \text{Ext}_{\mathcal{O}_{\mathbb{P}^m,p}}^d(\mathcal{O}_{X,p}, \omega_{\mathcal{O}_{\mathbb{P}^m,p}}) \cong \omega_{\mathcal{O}_{X,p}}.$$

The outer isomorphisms follow from definition and the interior isomorphism follows directly from Theorem III.6.8 in [11] as X Noetherian and $\mathcal{O}_X$ coherent.

□

Definition 2.16: Let R be the quotient of a regular local ring S . Then R is **Gorenstein** if R is Cohen-Macaulay and $\text{Ext}_S^{\dim(S)-\dim(R)}(R, S)$ is a free R -module of rank 1. A scheme or ring is **Gorenstein** if locally Gorenstein.

Remark 3. There are other and more general ways of defining Gorenstein/notions of dualizing, which are equivalent when mutually defined. See [10] or chapter 21 in [7] or chapter 08XG/0DWE in [1].

The following lemma justifies the term dualizing module.

Lemma 2.17: Let S be a regular local ring of dimension n , I an ideal generated by a regular sequence $(a_1, \dots, a_d)$, and $R = S/I$. Then $\text{Ext}_S^d(\text{Ext}_S^d(R, S), S) \cong R$ and $\text{Ext}_S^j(R, S) = 0$ when $j \neq d$.

Proof. Note $\omega_S = S$, as $\text{Ext}_S^0(S, S) = \text{Hom}_S(S, S) \cong S$, and so with Theorem 21.15 in [7] our definition of dualizing module is compatible for the following results. As a_i annihilates R we have by Proposition 21.10 in [7] that

$$\text{Ext}_{S/(a_1, \dots, a_{i+1})}^s(R, S/(a_1, \dots, a_{i+1})) \cong \text{Ext}_{S/(a_1, \dots, a_i)}^{s+1}(R, S/(a_1, \dots, a_i))$$

for all s . We know $\text{Hom}_R(\text{Hom}_R(R, R), R) \cong R$. Thus using our Ext formula above we deduce that $\text{Ext}_S^d(\text{Ext}_S^d(R, S), S) \cong R$ by iteratively replacing $S/(a_1, \dots, a_{i+1})$ with $S/(a_1, \dots, a_i)$. We also conclude $\text{Ext}_S^j(R, S) = 0$ when $j \neq d$

as $\text{Ext}_R^a(\text{Ext}_R^a(R, R), R) \cong 0$ for $a \neq 0$ because $\text{Hom}_R(R, -) \cong \text{Id}(-)$ as functors. $\square$

Proposition 2.18 (Huneke Prop 3.2/Cor 3.5 [13]): *Let S be a regular local ring of dimension n , I an ideal generated by a regular sequence $(a_1, \dots, a_d)$, and $R = S/I$ a local complete intersection ring. Then R is Gorenstein.*

Proof. Consider the Koszul complex defined as:

$$0 \rightarrow \bigwedge^d S^d \rightarrow \dots \rightarrow \bigwedge^{d-j} S^d \rightarrow \dots \rightarrow S^d \rightarrow S \rightarrow 0$$

with morphism (using $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with the one in position i)

$$\begin{aligned} \delta_{d-j} : \bigwedge^{d-j} S^d &\longrightarrow \bigwedge^{d-j-1} S^d \\ e_1 \wedge \dots \wedge e_{d-j} &\longmapsto \sum_{i=1}^{d-j} (-1)^{i+1} a_i \cdot e_1 \wedge \dots \wedge e_{i-1} \wedge e_{i+1} \wedge \dots \wedge e_{d-j} \end{aligned}$$

By Lemma 062F in [1], as $(a_1, \dots, a_d)$ is regular, we get that the following sequence is a free S -resolution, $K^\bullet \rightarrow R$:

$$0 \rightarrow \bigwedge^d S^d \rightarrow \dots \rightarrow \bigwedge^{d-j} S^d \rightarrow \dots \rightarrow S^d \rightarrow S \rightarrow R \rightarrow 0$$

Now we will prove using $I \cdot R = 0$ that $I \cdot \text{Ext}_S^d(R, S) = 0$. As K_i is free for each i we can lift maps as in the following diagram by mapping the basis in the necessary way.

$$\begin{array}{ccccc} & & K_i & & \\ & \swarrow \tilde{f} & \downarrow f & & \\ K_{i+1} & \xrightarrow{\delta_{i+1}} & \text{im}(\delta_{i+1}) & \xrightarrow{\text{inclusion}} & K_i \end{array}$$

Thus we can construct the chain homotopy in the following diagram with $s_0 = \tilde{f}$ given $f = (\cdot a)$ and $s_i = \tilde{f}$ given $f = (\cdot a) - s_{i-1}\delta_i$ for $i \geq 1$.

$$\begin{array}{ccccccccccccccccccc}
0 & \longrightarrow & K_d & \xrightarrow{\delta_d} & \cdots & \xrightarrow{\delta_3} & K_2 & \xrightarrow{\delta_2} & K_1 & \xrightarrow{\delta_1} & K_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
& & \cdot a \downarrow & \swarrow s_{d-1} & & \cdot a \downarrow & \swarrow s_2 & & \cdot a \downarrow & \swarrow s_1 & & \cdot a \downarrow & \swarrow s_0 & & 0 \downarrow \swarrow \cdot a \\
0 & \longrightarrow & K_d & \xrightarrow{\delta_d} & \cdots & \xrightarrow{\delta_3} & K_2 & \xrightarrow{\delta_2} & K_1 & \xrightarrow{\delta_1} & K_0 & \xrightarrow{\pi} & R & \longrightarrow & 0
\end{array}$$

Thus we find that $(\cdot a)$ is a nullhomotopy on our resolution $K^\bullet \rightarrow R$. Then after applying $\text{Hom}(-, S)$ to the diagram we still have $(\cdot a)$ is a nullhomotopy proving $I \cdot \text{Ext}_S^d(R, S) = 0$. So as K_d is one-dimensional over S we know $\text{Ext}_S^d(R, S) \cong S/J$ for some ideal $J \supset I$.

Now using the fact $\text{Ext}_S^j(R, S) = 0$ when $j \neq d$ from Lemma 2.17, we get $\text{Hom}(K^\bullet, S) \rightarrow \text{Ext}_S^d(R, S)$ is a free resolution of $\text{Ext}_S^d(R, S)$. Referring to Lemma 2.17 again we have $\text{Ext}_S^d(\text{Ext}_S^d(R, S), S) \cong R$, so we can interchange the roles of R and $\text{Ext}_S^d(R, S)$ from the above argument to get $R \cong S/I$ with $I \supset J$. Thus $\omega_R \cong R$ and so R is Gorenstein. $\square$

Lemma 2.19 (special case of 16.4 in [16]): *Let $j : E \rightarrow X$ be a Cartier divisor with X a Gorenstein scheme. Then $\omega_E \cong j^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))$.*

Proof. Consider the Cartier short exact sequence,

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0.$$

Now we can apply $\mathcal{H}om_{\mathcal{O}_X}(\cdot, \omega_X)$ to our Cartier short exact sequence to get the long exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{H}om(\mathcal{O}_E, \omega_X) & \longrightarrow & \mathcal{H}om(\mathcal{O}_X, \omega_X) & \longrightarrow & \mathcal{H}om(\mathcal{O}_X(-E), \omega_X) \\
& & & & & & \searrow \\
& & & & & & \mathcal{E}xt^1(\mathcal{O}_E, \omega_X) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_X, \omega_X)
\end{array}$$

We have $\mathcal{E}xt^1(\mathcal{O}_X, \omega_X) = 0$ by Proposition III.6.3 in [11]. Locally $\mathcal{H}om(\mathcal{O}_E, \omega_X)$ is $\text{Hom}(R/f, R) = 0$ for some ring R as E Cartier and X Gorenstein. After noting $\mathcal{H}om(\mathcal{O}_X, \omega_X) \cong \omega_X$, $\mathcal{H}om(\mathcal{O}_X(-E), \omega_X) \cong \omega_X(E)$, and also that $\mathcal{E}xt^1(\mathcal{O}_E, \omega_X) = \omega_E$, we get the short exact sequence

$$0 \rightarrow \omega_X \rightarrow \omega_X(E) \rightarrow \omega_E \rightarrow 0.$$

As $\omega_X(E)$ is locally free as X Gorenstein, hence flat by Proposition III.9.2 in [11], we get the following commuting rows of short exact sequences from the sequence we just constructed. Tensoring $\omega_X(E)$ with our Cartier divisor sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \omega_X(E) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E) & \xrightarrow{\frac{a}{q} \otimes q \mapsto \frac{a}{q}} & \omega_X(E) & \xrightarrow{a \mapsto a \otimes 1} & \omega_X(E) \otimes_{\mathcal{O}_X} \mathcal{O}_E \longrightarrow 0 \\
& & \downarrow \frac{a}{q} \otimes q \mapsto a & & \downarrow x \mapsto x & & \downarrow a \otimes b \mapsto \beta(a) \cdot b \\
0 & \longrightarrow & \omega_X & \xrightarrow{a \mapsto \frac{a}{q}} & \omega_X(E) & \xrightarrow{\beta} & \omega_E \longrightarrow 0
\end{array}$$

As the first two down arrows are isomorphisms and β is surjective, by snake lemma we can see that the the last down arrow is also an isomorphism. Thus $\omega_E \cong j^*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))$. $\square$

Definition 2.20: Let $\sigma : \tilde{X} \rightarrow X$ be a resolution of a singularity. We call our resolution σ **crepant** if $\sigma^* \omega_X \cong \omega_{\tilde{X}}$.

All ADE-surface singularities have a crepant resolution, see section 1.1 in [22]. Crepant resolutions are rarer to find when working in higher dimensions.

As an example, even in the nice case of Gorenstein quotient singularities we are only guaranteed the existence of crepant resolutions up to dimension 3, see [3].

2.3 Spinor Bundles and Quadrics

The exceptional divisor of our blow-up of the A_n singularity is a quadric as seen in example 1. We will provide some properties of quadrics arising in this way. Then we will discuss specific bundles, called spinor bundles, and how they are defined on quadrics. These spinor bundles will be salient to our computations of the categorical resolutions discussed later.

Definition 2.21: A **quadric** of dimension m over a field k is a subspace of $\mathbb{P}^{m+1}$ defined as the zero locus of a single homogeneous polynomial of degree two.

Remark 4. When k is an algebraically closed field with $\text{char } k \neq 2$, we can write any quadric in $k[x_0, \dots, x_m]$ in the form of $x_0^2 + \dots + x_i^2$ using a linear change of coordinates. Let $x = [x_0, \dots, x_m]$. Then one can represent our quadric as xAx^T where A is a symmetric matrix. We can guarantee A is symmetric by just replacing A with $(A + A^T)/2$. As A symmetric, we can diagonalize A as $PAP^T = D$ for some matrices P, D . This is done by a process similar to standard Gauss-Jordan elimination and works here as A is symmetric. As k algebraically closed, allowing scaling by $\frac{1}{\sqrt{d_i}}$ for nonzero diagonal entry d_i , we

can choose P so that D is a diagonal matrix with 1's along the diagonal followed by 0's. Thus giving a linear change of coordinates to rewrite our quadric in the desired form.

This means we can refer to the unique (up to isomorphism) smooth quadric and nodal quadric of dimensions m : $x_0^2 + \dots + x_m^2$ and $x_1^2 + \dots + x_m^2$ respectively.

Proposition 2.22: *The canonical bundle of a quadric, E , defined by $(q = 0)$ in $\mathbb{P}^m$ is $\mathcal{O}_E(1 - m)$.*

Proof. As E is a closed subscheme and local complete intersection in $\mathbb{P}^m$, by Theorem III.7.11 in [11] we have $\omega_E \cong \omega_{\mathbb{P}^m} \otimes_{\mathcal{O}_{\mathbb{P}^m}} \mathcal{H}om_E(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_E)$ where $\mathcal{I}$ is the ideal sheaf of E . Using the short exact sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^m}(-2) \xrightarrow{\cdot q} \mathcal{O}_{\mathbb{P}^m} \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

we see $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^m}(-2)$. Thus we get $\mathcal{H}om_E(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_E) \cong \mathcal{O}_E(2)$ and so we find that $\omega_E = \mathcal{O}_E(1 - m)$ as desired. $\square$

Proposition 2.23: *Let $\sigma : \tilde{X} \rightarrow X$ be the blow-up of a variety with exceptional divisor $j : E \rightarrow \tilde{X}$ a quadric in $\mathbb{P}^m$. Then $\omega_{\tilde{X}} = \sigma^* \omega_X \otimes \mathcal{O}_{\tilde{X}}((m - 2)E)$*

Proof. As σ is an isomorphism outside the exceptional divisor we have that $\omega_{\tilde{X}} = \sigma^* \omega_X \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}(aE)$ for some integer a , as $K_{\tilde{X}} - \sigma^* K_X$ may only differ by copies of E . Now let us observe,

$$\mathcal{O}_E(1 - m) \cong \omega_E \cong j^*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E)) \cong j^*(\sigma^* \omega_X \otimes \mathcal{O}_{\tilde{X}}((a + 1)E)) \cong \mathcal{O}_E(-a - 1)$$

Where the first isomorphism is Lemma 2.22, the second is Lemma 2.19 and the third isomorphism is from having an isomorphism outside the exceptional divisor. Note the last isomorphism comes from $\mathcal{O}_{\tilde{X}}(E) = \mathcal{O}_{\tilde{X}}(-1)$ (02OS [1]) and the fact $\sigma^*\omega_X$ is trivial on E . Since $\text{Pic}(E)$ is torsion-free as noted in II.6.5 in [11] we may conclude that $-a - 1 = -m + 1$ or rather that have $\omega_{\tilde{X}} \cong \sigma^*\omega_X \otimes \mathcal{O}_{\tilde{X}}(m - 2)$. $\square$

We shall define the Spinor Bundles on a quadric now. To do so we must first define a Grassmannian variety.

Definition 2.24: A **Grassmannian variety**, $Gr(a, b)$, is the space of a -dimensional subspaces of a b -dimensional vector space.

Remark 5. Let V be a b -dimensional vector space with basis $e_1, \dots, e_b$. We have the following commutative diagram:

$$\begin{array}{ccc} & \mathbb{P}(\bigwedge^a(V)) & \\ f \nearrow & & \searrow g \\ Gr(a, b) & \xrightarrow{\quad\quad\quad} & \mathbb{P}^{\binom{b}{a}} \end{array}$$

The map f takes an a -dimensional subspace, A , of V with basis $v_1, \dots, v_a$ to $v_1 \wedge \dots \wedge v_a$. The partial function g linearly (where defined) takes nontrivial $v_1 \wedge \dots \wedge v_a$ to the projective coordinate [determinant of i^{th} $a \times a$ minor of $[v_1 \dots v_a]_i$. Thus by dimension, away from zero, g is a bijection. We have f is injective as A is reconstructable as the set of vectors w such that $w \wedge v_1 \wedge \dots \wedge v_a = 0$. Now let us see that gf is well-defined. Given any other basis of A , $v'_1, \dots, v'_a$,

there will exist an $a \times a$ matrix C such that $[v_1 \dots v_a]C = [v'_1 \dots v'_a]$. For any $a \times a$ minor we have the determinants only differing by the determinant of C . This embedding gives what are called Plücker coordinates and gives the Grassmannian as a projective variety, see example 6.6 of [9] for more details.

Let $J \subset \{1, \dots, b\}$ be of size a . We can define U_J to be the open set of $Gr(a, b)$ of subspaces whose basis matrix $[v_1 \dots v_a]$ has an invertible $a \times a$ minor made of the rows in J . Given $A \in U_J$, similar to before, we can choose a basis of A so its basis in V is written as M_A where row $j_i \in J$ of M_A is row i in id_a . Then over U_J we can define sections $s_1, \dots, s_a$ so that $s_i(A) = i^{\text{th}}$ column of M_A . This sheaf is locally free of rank a .

Definition 2.25: The **universal bundle**, also known as **tautological bundle**, of the Grassmannian is the sheaf constructed at the end of remark 5.

Let E be a smooth quadric of dimension m . Then a paper by Ottaviani, see Corollary 1.2 in [21], provides an embedding of E into a Grassmannian. This breaks down into two cases based on dimension. If E is of odd dimension we get an embedding $s : E \rightarrow Gr(2^{\frac{m-1}{2}}, 2^{\frac{m-1}{2}+1})$. If E is even dimensional we get two embeddings $s', s'' : E \rightarrow Gr(2^{\frac{m}{2}-1}, 2^{\frac{m}{2}})$. Here we are thinking of these being Grassmannians of projective space and so the projective space dimensions are actually one less for each entry. This now allows us to define the spinor bundle.

Definition 2.26: Let E be a smooth quadric of dimension m . Let U be the universal bundle of the Grassmannian. Then the **spinor bundle** on E is $s^*(U)$

when m is odd and are $s'^*(U)$, $s''^*(U)$ when m is even.

Now we shall define spinor bundles for nodal quadrics. Let E_s be our smooth quadric and let E_n be our nodal quadric which is the cone of E_s . Then we have a rational map $c_0 : E_n \rightarrow E_s$ which is the projection sending

$$[x_0 : x_1 : \dots : x_m] \mapsto [x_1 : \dots : x_m].$$

This extends to a morphism, c , from the blow-up of E_n at its singularity to E_s sending

$$([x_0 : x_1 : \dots : x_m], [y_1 : \dots : y_m]) \mapsto [y_1 : \dots : y_m].$$

This relationship is illustrated in the following diagram:

$$\begin{array}{ccc} & Bl(E_n) & \\ \psi \swarrow & & \searrow c \\ E_n & \xrightarrow{c_0} & E_s \end{array}$$

Definition 2.27: Let E be a nodal quadric of dimension m and the cone of a smooth quadric E_s . We define the **spinor bundle** on E based on its dimension. When m is even, there is a spinor bundle S on E_s . We then define the spinor bundle on E as $\psi_* c^* S$. When m is odd, we get spinor bundles S', S'' on E_s . We then define the spinor bundles on E as $\psi_* c^* S', \psi_* c^* S''$.

One can refer to [21] to learn more properties of spinor bundles over a smooth quadric. Likewise one can look to [8],[19] for more information on spinor bundles over nodal quadrics.

Remark 6. Later we will be interested in what happens to our spinor bundles under subsequent blow-ups of our ambient space when resolving our A_n singularity. Using example 2 we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & Bl(E_n) & \xrightarrow{\text{inc.}} & \tilde{\tilde{X}} \\
 & & \downarrow \text{id} & & \downarrow \tilde{\sigma} \\
 E_s & \xleftarrow{c} & Bl(E_n) & \xrightarrow{\text{inc.}} & \tilde{X} \\
 & & \downarrow \psi & & \downarrow \sigma \\
 & & E_n & \xrightarrow{j} & X
 \end{array}$$

where $\tilde{\tilde{X}}$ is the resolution of our A_n singularity. Thus we can see the pushforward of the inclusion of our spinor bundles lift as described in the diagram.

Chapter 3

Categorical Resolutions of Singularities

3.1 Triangulated and Derived Categories

In this section we develop the derived category tools we will need later when discussing categorical resolutions. We will develop some properties for semiorthogonal decompositions and Verdier quotients, among other objects. Also key lemmas for our main result will be established. Let us start by introducing perfect complexes.

Definition 3.1: Let R be a ring. A **perfect complex** in $D(R)$ is a complex quasi-isomorphic to a bounded complex of finite projective R -modules.

Definition 3.2: Let X be a scheme. A **strictly perfect complex** is one that is non-zero at a finite number of indices and a direct summand of a finite free $\mathcal{O}_X$ -module at all indices.

Definition 3.3: Let X be a scheme. A **perfect complex**, P , is one where there exists an open cover of X , $\{U_i\}_i$, such that for each U_i we have $P|_{U_i}$ is quasi-isomorphic to a strictly perfect complex of $\mathcal{O}_{U_i}$ -modules..

Proposition 3.4: Let X be a scheme and $j : E \rightarrow X$ a Cartier divisor. Then $j_*(\mathcal{O}_E(i))$ is perfect in $D^b(X)$ for all i .

Proof. As E a Cartier divisor of X we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow j_*(\mathcal{O}_E) \rightarrow 0.$$

We know $\mathcal{O}_X(i)$, for any i , is flat by III.9.2 in [11] as $\mathcal{O}_X(i)$ is coherent and locally free. So we get a short exact sequence,

$$0 \rightarrow \mathcal{O}_X(-E)(i) \rightarrow \mathcal{O}_X(i) \rightarrow j_*(\mathcal{O}_E(i)) \rightarrow 0.$$

Note the commuting of j_* and twisting by i follows from the projection formula.

Our short exact sequence gives rise to a distinguished triangle in $D^b(X)$,

$$\mathcal{O}_{\tilde{X}}(-E)(-i) \rightarrow \mathcal{O}_{\tilde{X}}(-i) \rightarrow j_*(\mathcal{O}_E(-i)) \rightarrow \mathcal{O}_{\tilde{X}}(-E)(-i)[1].$$

So as the first two terms are perfect, as they are bounded complexes of locally free sheaves, we conclude by Lemma 08CR in [1] that $j_*(\mathcal{O}_E(-i))$ is also perfect.

□

Definition 3.5: Let $\mathcal{D}$ be a k -linear triangulated category (so $\text{Hom}(A, B)$ is always a k -vector space) where $\dim_k \text{Hom}(A, B)$ is finite. A **Serre functor** on $\mathcal{D}$ is a k -linear equivalence $S : \mathcal{D} \rightarrow \mathcal{D}$ such that we have isomorphisms $\text{Hom}(A, B) \cong \text{Hom}(B, S(A))^\vee$ functorial in A, B .

Definition 3.6: Let $f : X \rightarrow Y$ with X, Y schemes. Then the **relative canonical bundle**, denoted $\omega_{X/Y}$, is $\omega_X \otimes f^*(\omega_Y)^{-1}$.

Proposition 3.7: Let $f : X \rightarrow Y$ be a proper morphism between proper Gorenstein schemes of dimensions m', m over a field k . Let $K \in D^{\text{perf}}(Y)$. Then $f^!(K) \cong f^*(K) \otimes \omega_{X/Y}[m' - m]$, where $f^!$ is the right adjoint of f_* in the derived category.

Proof. By a special case of Lemma 0A9T in [1] as f, X, Y proper and K perfect, $f^!(K) \cong f^*(K) \otimes f^!(\mathcal{O}_Y)$. Thus we just need to understand $f^!(\mathcal{O}_Y)$. To do that will first compute $i^!(\mathcal{O}_{\text{spec}(k)})$. Let $i : Y \rightarrow k, i' : X \rightarrow k$ be the structure morphisms. Let $F \in D^{\text{perf}}(Y)$ and $K \in D^b(\text{spec}(k))$.

$$\begin{aligned} \text{Hom}(i_*(F), K) &\cong \text{Hom}(K, i_*(F) \otimes \omega_{\text{spec}(k)})^\vee \\ &\cong \text{Hom}(K \otimes \omega_{\text{spec}(k)}^{-1}, i_*F)^\vee \\ &\cong \text{Hom}(i^*(K) \otimes i^*(\omega_{\text{spec}(k)}^{-1}), F)^\vee \\ &\cong \text{Hom}(F, i^*(K) \otimes i^*(\omega_{\text{spec}(k)}^{-1}) \otimes \omega_Y[m]) \end{aligned}$$

The first isomorphism follows from an application of the Serre functor in $D^b(\text{spec}(k))$ using Lemma 0FY8 in [1] which works as $D^{\text{perf}}(\text{spec}(k)) = D^b(\text{spec}(k))$ as $\text{spec}(k)$ smooth. The second isomorphism follows from the flatness of $\omega_{\text{spec}(k)}$ since then we can treat the derived tensor as a nonderived tensor. The third isomorphism comes from the adjunction $i^* \dashv i_*$ plus the fact pullbacks commute with tensors. The last isomorphism is again an application of the Serre functor using Lemma 0FY8 in [1] which works as Y Gorenstein and F perfect. Note

the shift by m comes from the fact Y is Gorenstein and using Lemma 0FVV in [1]. After observing $\omega_{\text{spec}(k)} \cong \mathcal{O}_{\text{spec}(k)}$ we find that $i^!(\mathcal{O}_{\text{spec}(k)}) \cong \omega_Y[m]$ by just doing the same computation as above with $K = \mathcal{O}_{\text{spec}(k)}$. Using the same argument we get that $i^!(\mathcal{O}_{\text{spec}(k)}) \cong \omega_X[m']$. A consequence of the Yoneda lemma is that $i \circ f = i'$ implies $f^! \circ i^! = i'^!$. Thus,

$$\omega_X[m'] \cong i'^!(\mathcal{O}_{\text{spec}(k)}) \cong f^! \circ i^!(\mathcal{O}_{\text{spec}(k)}) \cong f^!(\omega_Y[m]) \cong f^*(\omega_Y[m]) \otimes f^!(\mathcal{O}_Y).$$

So we may conclude the isomorphism $f^!(\mathcal{O}_Y) \cong \omega_{X/Y}[m' - m]$ and thus also that $f^!(\cdot) \cong f^*(\cdot) \otimes \omega_{X/Y}[m' - m]$ holds. $\square$

Definition 3.8: Let $\mathcal{D}$ be a triangulated category and let $\mathcal{T}$ be a triangulated subcategory. We say $\mathcal{T}$ is **left admissible** if the inclusion $i : \mathcal{T} \rightarrow \mathcal{D}$ has a left adjoint denoted i^* . Similarly we say $\mathcal{T}$ is **right admissible** if the inclusion $i : \mathcal{T} \rightarrow \mathcal{D}$ has a right adjoint denoted $i^!$. We call $\mathcal{T}$ **admissible** if it is both left and right admissible.

Proposition 3.9: Let $\mathcal{D}$ be a triangulated category with a right admissible subcategory L . Let $X \in \mathcal{D}$. Then defining $i : L \rightarrow \mathcal{D}$ inclusion we get that $ii^!X \rightarrow X \rightarrow \text{cone}(ii^!X \rightarrow X)$ is the unique (up to isomorphism) distinguished triangle of the form $A \rightarrow X \rightarrow B$ with $A \in L$ and $B \in L^\perp$.

Proof. First observe that the adjunction $\text{Hom}_L(i^!X, i^!X) \cong \text{Hom}_{\mathcal{D}}(ii^!X, X)$ gives a distinguished triangle induced by identity,

$$ii^!X \rightarrow X \rightarrow \text{cone}(ii^!X \rightarrow X) \rightarrow ii^!X[1].$$

Let $A \in L$. Then observe $\mathrm{Hom}_{\mathcal{D}}(A, ii^!X) \cong \mathrm{Hom}_L(A, i^!X) \cong \mathrm{Hom}_{\mathcal{D}}(A, X)$ which follows as L a full subcategory and from the adjunction $i \dashv i^!$. Then by Lemma OCQQ in [1] we find that $\mathrm{cone}(ii^!X \rightarrow X) \in L^\perp$. Thus we have our existence.

Now we show uniqueness. Consider another distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow A[1]$$

with $A \in L$ and $B \in L^\perp$. By Proposition 1.34 in [14] we know a distinguished triangle induces long exact sequences of Hom's in either position. Note by definition for any $G \in L$ and using the closure of triangulated subcategories under shift,

$$\mathrm{Hom}_{\mathcal{D}}(G, B[-1]) = 0 = \mathrm{Hom}_{\mathcal{D}}(G, B).$$

Using the long exact sequence induced by the Hom's this implies

$$\mathrm{Hom}_{\mathcal{D}}(G, A) \cong \mathrm{Hom}_{\mathcal{D}}(G, X) \cong \mathrm{Hom}_{\mathcal{D}}(G, ii^!X).$$

As L a full subcategory and since we know

$$\mathrm{Hom}_L(G, A) \cong \mathrm{Hom}_L(G, i^!X)$$

then by the Yoneda lemma we conclude that

$$A \cong ii^!X.$$

This in turn implies that $B \cong \mathrm{cone}(ii^!X \rightarrow X)$. Thus we've shown uniqueness. □

Note that a symmetric proof of Lemma 3.9 is true for left admissible subcategories and $\text{cone}(X \rightarrow ii^*X)[-1] \rightarrow X \rightarrow ii^*X$.

Definition 3.10: Let $\mathcal{D}$ be a triangulated category with triangulated subcategory $\mathcal{T}$. The **right orthogonal** of $\mathcal{T}$ in $\mathcal{D}$ is $\mathcal{T}^\perp := \{A \in \mathcal{D} \mid \text{Hom}(\mathcal{T}, A) = 0\}$. Similarly, the **left orthogonal** of $\mathcal{T}$ in $\mathcal{D}$ is ${}^\perp\mathcal{T} := \{A \in \mathcal{D} \mid \text{Hom}(A, \mathcal{T}) = 0\}$.

Definition 3.11: Let $\mathcal{D}$ be a triangulated category with $D_n, \dots, D_1$ triangulated subcategories. We call $(D_n, \dots, D_1)$ a **semiorthogonal collection** if $D_i \subset D_j^\perp$ for $j < i \in \{1, \dots, n\}$. We call this collection **full** if $\mathcal{D}$ can be generated by $D_n, \dots, D_1$ by shifts and cones.

Definition 3.12: Let $\mathcal{D}$ be a triangulated category with full semiorthogonal collection $(D_n, \dots, D_1)$. We call $\langle D_n, \dots, D_1 \rangle$ a semiorthogonal decomposition of $\mathcal{D}$ if for any $A \in \mathcal{D}$ there is a filtration of A by $\langle D_n, \dots, D_1 \rangle$. By this we mean there exists $A_i \in D_i$ and $T_i \in \mathcal{D}$ such that

$$0 \rightarrow A_1 = T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n = A$$

and $\text{cone}(T_{i-1} \rightarrow T_i) \cong A_i$.

Note that any such filtration is unique and functorial, see Lemma 2.4 in [18]. A semiorthogonal decomposition of a derived category $D^b(X)$ is the best, in this sense, as one can hope for. For any connected noetherian scheme there does not exist a fully orthogonal decomposition of $D^b(X)$, see Proposition 3.10 in [14].

Proposition 3.13: *Let $\mathcal{D}$ be a triangulated category and let $(D_n, \dots, D_1)$ be a full semiorthogonal collection of admissible subcategories of $\mathcal{D}$. Let $A \in \mathcal{D}$. Then $\langle D_n, \dots, D_1 \rangle$ is a semiorthogonal decomposition of $\mathcal{D}$.*

Proof. Let $i_n : D_n \rightarrow \mathcal{D}$ be the inclusions. Then by Lemma 3.9 we have a distinguished triangle

$$\text{cone}(A \rightarrow ii^*A)[-1] \rightarrow A \rightarrow ii^*A \rightarrow \text{cone}(A \rightarrow ii^*A)$$

with $ii^*A \in D_n$ and $\text{cone}(A \rightarrow ii^*A)[-1] \in {}^\perp D_n \supset \langle D_{n-1}, \dots, D_1 \rangle$. Now we quickly show that in fact $\text{cone}(A \rightarrow ii^*A)[-1]$ is isomorphic to something in $\langle D_{n-1}, \dots, D_1 \rangle$. Let us relabel $\text{cone}(A \rightarrow ii^*A)[-1]$ for now as L . By Lemma 3.9 we have a distinguished triangle

$$L_2 \rightarrow L \rightarrow L_1 \rightarrow L_2[1]$$

with $L_1 \in \langle D_{n-1}, \dots, D_1 \rangle^\perp$ and $L_2 \in \langle D_{n-1}, \dots, D_1 \rangle$. Then we may use Lemma 3.9 again to get a distinguished triangle

$$S_2 \rightarrow L_1 \rightarrow S_1 \rightarrow S_2[1]$$

with $S_1 \in D_n^\perp$ and $S_2 \in D_n$. As $L_1 \in \langle D_{n-1}, \dots, D_1 \rangle^\perp \supset ({}^\perp D_n)^\perp \supset D_n$ and $S_2 \in D_n$ we know $S_1 \in \langle D_{n-1}, \dots, D_1 \rangle^\perp, D_n^\perp$ and so by Lemma 1.61 in [14] we get that $S_1 = 0$. Thus $S_2 \cong L_1$. As $L_2 \in \langle D_{n-1}, \dots, D_1 \rangle \subset {}^\perp D_n$ we can find $L_1 \in {}^\perp D_n$ too. As $L_1 \in {}^\perp D_n$ and $L_1 \cong S_2 \in D_n$ we can conclude that $L_1 = 0$. Thus $L \cong L_2 \in \langle D_{n-1}, \dots, D_1 \rangle$ as we were hoping.

By repeating this process with $\text{cone}(A \rightarrow ii^*A[-1])$ replacing A we get the required filtration. Note the process terminates as there are finite terms. $\square$

Proposition 3.14: *Let $\tilde{\mathcal{D}}$ be a triangulated category with an admissible semiorthogonal decomposition $\langle D_n, \dots, D_1 \rangle$. Let $A \in \mathcal{D}$. Let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n$ be the filtration for A with respect to our semiorthogonal decomposition. Let $i_1 : D_1 \rightarrow \mathcal{D}$ be inclusion. Then $i_1 i_1^! A \cong A_1$.*

Proof. Let $r_j : \langle D_{j-1}, \dots, D_1 \rangle \rightarrow \langle D_j, \dots, D_1 \rangle$ be inclusions. We will now show that r_j has a right adjoint. Let $X \in \langle D_j, \dots, D_1 \rangle$. Then we have a semiorthogonal filtration of X which we can write as

$$0 \rightarrow X_2 \rightarrow \dots \rightarrow X_j \rightarrow X.$$

As $0, \text{cone}(0 \rightarrow X_2) \in D_1$ we have that $X_2 \in D_1$. As we have the relationship $X_2, \text{cone}(X_2 \rightarrow X_3) \in \langle D_2, D_1 \rangle$ we also know $X_3 \in \langle D_2, D_1 \rangle$. Continuing this process we find that $X_j \in \langle D_{j-1}, \dots, D_1 \rangle$. Also $D_j \subset \langle D_{j-1}, \dots, D_1 \rangle^\perp$ and

$$X_j \rightarrow X \rightarrow \text{cone}(X_j \rightarrow X) \rightarrow X_j[1]$$

is a distinguished triangle and so by Lemma OCQS in [1] we have that each r_j is right admissible. Now we can apply Lemma 3.9 to see that our filtration for A can be rewritten as

$$0 \rightarrow r_2^! \circ \dots \circ r_n^! A \rightarrow \dots \rightarrow r_n^! A \rightarrow A.$$

Clearly $r_n \circ \dots \circ r_2 = i_1$ and so $i_1 i_1^! (A) = r_2^! \dots r_n^! A = A_1$. □

Lemma 3.15: *Let $\langle \tilde{\mathcal{D}}^\perp, \tilde{\mathcal{D}} \rangle$ be a semiorthogonal decomposition with $\mathcal{D} = {}^\perp \sigma^*(\tilde{\mathcal{D}}^\perp)$ and $\sigma_* \dashv \sigma^*$. The exact functor σ^* need only be defined on $\tilde{\mathcal{D}}^\perp$. Then $\sigma_*(\mathcal{D}) \subset \tilde{\mathcal{D}}$.*

Proof. First observe

$$\mathrm{Hom}(\sigma_* B, C) \cong \mathrm{Hom}(B, \sigma^* C) = 0$$

for $B \in \mathcal{D}$, $C \in \tilde{\mathcal{D}}^\perp$ and so $\sigma_* \mathcal{D} \subset {}^\perp(\tilde{\mathcal{D}}^\perp)$. Also $\tilde{\mathcal{D}} \subset {}^\perp(\tilde{\mathcal{D}}^\perp)$. Let $A \in \sigma_* \mathcal{D}$. Using a filtration we get a distinguished triangle

$$A_1 \rightarrow A \rightarrow A_2 \rightarrow A_1[1]$$

with $A_1 \in \tilde{\mathcal{D}}$ and $A_2 \in \tilde{\mathcal{D}}^\perp$. Let $B \in {}^\perp(\tilde{\mathcal{D}}^\perp)$. Using a long exact sequence of $\mathrm{Hom}(B, -)$ we see that $\mathrm{Hom}(B, A) \cong \mathrm{Hom}(B, A_1)$ as we have that $\mathrm{Hom}(B, A_2) \cong 0$. Thus by applying Yoneda lemma we conclude that $A \cong A_1$, so $\sigma_*(\mathcal{D}) \subset \tilde{\mathcal{D}}$. $\square$

Lemma 3.16: *Let $\tilde{\mathcal{D}}$ be a triangulated category subcategory with $\mathcal{D} = {}^\perp \sigma^*(\tilde{\mathcal{D}}^\perp)$. Let $\sigma^* \dashv \sigma_*$ and $\sigma_* \circ \sigma^* \cong \mathrm{id}$, where σ^* need only be defined on $\tilde{\mathcal{D}}^{\mathrm{perf}}$. Then $\sigma^*(\tilde{\mathcal{D}}^{\mathrm{perf}}) \subset \mathcal{D}$.*

Proof. Let $P \in \tilde{\mathcal{D}}^{\mathrm{perf}}$ and let $A \in \sigma^* \tilde{\mathcal{D}}^\perp$. Then

$$\mathrm{Hom}(\sigma^* P, A) \cong \mathrm{Hom}(P, \sigma_* A) = 0$$

as $\sigma_* A \in \sigma_* \sigma^* \tilde{\mathcal{D}}^\perp = \tilde{\mathcal{D}}^\perp$. Therefore $\sigma^* P \in \mathcal{D}$. Hence $\sigma^* \tilde{\mathcal{D}}^{\mathrm{perf}} \subset \mathcal{D}$. $\square$

Lemma 3.17: *Let $\mathcal{D}$ be a triangulated category with semiorthogonal decomposition $\langle A, B \rangle$. Let S be a triangulated subcategory of $\mathcal{D}$ such that $A \subset S$. then S has semiorthogonal decomposition $\langle A, B \cap S \rangle$.*

Proof. Let $X \in S$. Then using the filtration provided by the semiorthogonal decomposition $\langle A, B \rangle$ of $\mathcal{D}$ we have the filtration:

$$\begin{array}{ccccc} 0 & \longrightarrow & X_B & \longrightarrow & X \\ & & & & \downarrow \\ & & & & X_A \end{array}$$

with $X_B \in B$ and $X_A \in A$. By hypothesis we have $X_A \in S$. So as $X, X_A \in S$, $X_B \rightarrow X \rightarrow X_A \rightarrow X_A[1]$ distinguished, and S a triangulated subcategory we see that $X_B \in S$ by definition. Thus we can conclude that $\langle A, B \cap S \rangle$ is a semiorthogonal decomposition of S . $\square$

Lemma 3.18: *Let $\langle A, B \rangle = \langle A, C \rangle$ be semiorthogonal decompositions. Then $B = C$.*

Proof. Let $X_C \in C$. Then using the filtration provided by the semiorthogonal decomposition $\langle A, B \rangle$ of $\mathcal{D}$ we have the filtration:

$$\begin{array}{ccccc} 0 & \longrightarrow & X_B & \longrightarrow & X_C \\ & & & & \downarrow \\ & & & & X_A \end{array}$$

Using the distinguished triangle $X_B \rightarrow X_C \rightarrow X_A \rightarrow X_B[1]$ we get a long exact sequence induced by $\text{Hom}(\cdot, X_A)$:

$$\cdots \rightarrow \text{Hom}(X_B, X_A) \rightarrow \text{Hom}(X_C, X_A) \rightarrow \text{Hom}(X_A, X_A) \rightarrow \text{Hom}(X_B[1], X_A) \rightarrow \cdots$$

Note that $\text{Hom}(X_B[1], X_A) = \text{Hom}(X_C, X_A) = 0$ by assumption and so by exactness $\text{Hom}(X_A, X_A) = 0$ which implies that $X_A \cong 0$. And so this implies

that $X_B \cong X_C$. Thus as X_C was arbitrary in C we see that $C \subset B$. By symmetry we get that $B = C$ as desired. $\square$

Lemma 3.19: *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{D}$ be a right admissible subcategory with inclusion $i : \mathcal{D} \rightarrow \mathcal{T}$. Let $A \in \mathcal{D}$. Then $i^!A \cong A$.*

Proof. Let $B \in \mathcal{D}$. Observe $\text{Hom}(B, i^!A) \cong \text{Hom}(B, A)$. As B was arbitrary, by the Yoneda lemma we have $ii^!A \cong A$. $\square$

Definition 3.20: Let $\mathcal{D}$ be a triangulated category with triangulated subcategory $\mathcal{T}$. The **Verdier quotient** (or just quotient) of $\mathcal{D}$ by $\mathcal{T}$ is the triangulated category $\mathcal{D}/\mathcal{T}$. The objects of $\mathcal{D}/\mathcal{T}$ are the same as $\mathcal{D}$. The morphisms of $\mathcal{D}/\mathcal{T}$, $A \rightarrow B$, are defined as pairs of morphisms in $\mathcal{D}$, $(s : N \rightarrow A, f : N \rightarrow B)$ where $\text{cone}(s : N \rightarrow A) \in \mathcal{T}$. Two morphisms $(s_1 : N_1 \rightarrow A, f_1 : N_1 \rightarrow B)$ and $(s_2 : N_2 \rightarrow A, f_2 : N_2 \rightarrow B)$ in $\mathcal{D}/\mathcal{T}$ are equivalent if there exists a morphism in $\mathcal{D}/\mathcal{T}$ between N_1 and N_2 , $(s_3 : N_3 \rightarrow N_1, f_3 : N_3 \rightarrow N_2)$, making the diagram commute in $\mathcal{D}$.

The Verdier quotient comes with an exact functor $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{T}$ which is identity on objects and maps $f : A \rightarrow B$ to $(\text{id} : A \rightarrow A, f : A \rightarrow B)$. For a more detailed understanding of the construction and its properties refer to sections 04VB, 05RA in [1].

Lemma 3.21 (05RK in [1]): *For Q defined above the kernel, $\ker(Q)$, is the strictly fully subcategory of $\mathcal{D}$, $\{A \in \mathcal{D} \mid \exists B \in \mathcal{D}, A \oplus B \in \mathcal{T}\}$.*

Lemma 3.22 (05RJ in [1]): *Let $\mathcal{D}$ be a triangulated category and $\mathcal{T}$ a full triangulated subcategory. Let $\mathcal{D}'$ be a triangulated category and $F : \mathcal{D} \rightarrow \mathcal{D}'$ an exact functor with $\mathcal{T} \subset \ker(F)$. Then there exists a unique exact functor $\overline{F} : \mathcal{D}/\mathcal{T} \rightarrow \mathcal{D}'$ such that $F = \overline{F} \circ Q$ with $Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{T}$ defined above.*

Lemma 3.22 is often referred to as the universal property of Verdier quotients. It is useful for factoring morphisms through the quotient.

Definition 3.23: Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories and let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. Then F is a **Verdier localization** (or just localization) if $\overline{F} : \mathcal{D}/\ker(F) \rightarrow \mathcal{D}'$, $\overline{F}$ defined in Lemma 3.22, is an equivalence.

Definition 3.24: Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor between triangulated categories. We call F a **categorical contraction** if $\overline{F}$ is fully faithful and every object of $\mathcal{D}'$ can be written as a direct summand of objects in $\mathcal{D}/\ker(F)$.

A Verdier localization essentially captures what it means for a “first isomorphism theorem” to hold. A categorical contraction is a slight weakening of this where we only require the isomorphism to hold on what can be thought of as a dense subset of the codomain. The following lemma can be considered as a “second isomorphism theorem.”

Lemma 3.25: *Let $\mathcal{T}$ be a triangulated category with semiorthogonal decomposition $\langle \mathcal{D}^\perp, \mathcal{D} \rangle$. Let $\mathcal{K}$ be a triangulated subcategory of $\mathcal{T}$ with semiorthogonal decomposition $\langle \mathcal{D}^\perp, \mathcal{N} \rangle$. Then the canonical map from $\mathcal{D}/(\mathcal{D} \cap \mathcal{K})$ to $\mathcal{T}/\mathcal{K}$ is fully faithful.*

Proof. First we note by the universal property of the Verdier quotient, see Lemma 05RJ in [1], that there exists a canonical map $\mathcal{D}/(\mathcal{D} \cap \mathcal{K}) \rightarrow \mathcal{T}/\mathcal{K}$ induced by the inclusion $\mathcal{D} \rightarrow \mathcal{T}$.

Now let $A \in \mathcal{K}$ and $B \in \mathcal{D}$ and $\phi : B \rightarrow A$ a morphism. The following diagram commutes using the adjunction of $i, i^!$ (see definition 0037 in [1] for reference) and the fact $ii^!B = B$ by Lemma 3.19:

$$\begin{array}{ccc} A & \xleftarrow{\phi} & iB \\ & \swarrow & \downarrow i^!(\phi) \\ & & ii^!A \end{array}$$

By definition $i^!A \in \mathcal{D}$. By Lemma 0CQS in [1] we have a distinguished triangle $S \rightarrow A \rightarrow R \rightarrow S[1]$ where $S \in \mathcal{N} \subset \mathcal{D}$ and $R \in \mathcal{D}^\perp$. Thus using Lemma 3.19 we get a distinguished triangle: $S \rightarrow i^!A \rightarrow i^!R \rightarrow S[1]$. Observe $\text{Hom}(i^!R, i^!R) \cong \text{Hom}(ii^!R, R) = 0$ as $ii^!R \in \mathcal{D}$ and $R \in \mathcal{D}^\perp$. Thus $i^!R = 0$. So we know $S \cong i^!A$. This means that $i^!A \in \mathcal{D}$. Thus $i^!A \in \mathcal{D} \cap \mathcal{K}$. So we may apply the dual of Proposition 2.3.5 from [24] to conclude that indeed the canonical map from $\mathcal{D}/(\mathcal{D} \cap \mathcal{K})$ to $\mathcal{T}/\mathcal{K}$ is fully faithful.

□

Lemma 3.26: *Let $\sigma : \mathcal{T} \rightarrow \mathcal{S}$ be an essentially surjective morphism of triangulated categories where $\mathcal{T}$ has a semiorthogonal decomposition $< \mathcal{D}^\perp, \mathcal{D} >$, $\mathcal{D}$ is right admissible, and $\mathcal{D}^\perp \subset \ker \sigma$. Then $\sigma|_{\mathcal{D}}$ is essentially surjective.*

Proof. Let $A \in \mathcal{S}$. By hypothesis there exists a $B \in \mathcal{T}$ such that $\sigma(B) \cong A$.

By Lemma 0CQS in [1] we have a distinguished triangle

$$S \rightarrow B \rightarrow R \rightarrow S[1]$$

where $S \in \mathcal{D}$ and $R \in \mathcal{D}^\perp$. Thus we have a distinguished triangle

$$\sigma S \rightarrow \sigma B \rightarrow \sigma R \rightarrow \sigma S[1].$$

As $\mathcal{D}^\perp \subset \ker \sigma$ we have that $\sigma R = 0$. Thus $\sigma S \cong \sigma B \cong A$ as required. $\square$

3.2 Categorical Resolutions of A_1 and A_2 Singularities

In this section we introduce the definition of a categorical resolution. We also go over some key results in the literature about categorical resolutions for singularities of type A_1 and A_2 .

Definition 3.27: A **categorical resolution** of a triangulated category $\mathcal{D}$ is defined by a triple $(\tilde{\mathcal{D}}, \sigma_*, \sigma^*)$. The triangulated category $\tilde{\mathcal{D}}$ is an admissible subcategory of $D^b(X)$ where X is a non-singular scheme. The functors σ_*, σ^* must be defined so that $\sigma_* : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$, $\sigma^* : D^{\text{perf}} \rightarrow \tilde{\mathcal{D}}$ where $\sigma^* \dashv \sigma_*$ and the natural morphism induces $\text{id}_{D^{\text{perf}}} \cong \sigma_* \sigma^*$.

Proposition 3.28: Let $\sigma : \tilde{X} \rightarrow X$ be a resolution of singularity where X has a single rational singularity. Then $\sigma_* \sigma^* \cong \text{id}_{D^{\text{perf}}(X)}$.

Proof. Observe that

$$\sigma_*\sigma^*(P) \cong \sigma_*(\sigma^*(P) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}) \cong \sigma_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} P \cong P$$

for $P \in D^{\text{perf}}(X)$ using the projection formula from Lemma 0B54 in [1] and the fact $\sigma_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$ since we have a rational singularity. $\square$

Proposition 3.28 provides the justification for the name categorical resolution as we get that the classic resolution of singularity induces a categorical resolution when the singularity is rational. This definition has the weakness that we are restricting ourselves to rational singularities when considering geometric realizations of a categorical resolution.

Proposition 3.29: *Let $\tilde{\sigma} : \tilde{\tilde{X}} \rightarrow \tilde{X}$, $\sigma : \tilde{X} \rightarrow X$, $\sigma\tilde{\sigma} : \tilde{\tilde{X}} \rightarrow X$ be morphisms, with $\tilde{\sigma}$, $\sigma\tilde{\sigma}$ each a resolutions of a single rational singularity. Then $\sigma_*\sigma^* \cong \text{id}_{D^{\text{perf}}(X)}$.*

Proof. By Proposition 3.28 we know that $\sigma_*\tilde{\sigma}_*\tilde{\sigma}^*\sigma^* \cong (\sigma\tilde{\sigma})_*(\sigma\tilde{\sigma})^* \cong \text{id}_{D^{\text{perf}}(X)}$ and $\tilde{\sigma}_*\tilde{\sigma}^* \cong \text{id}_{D^{\text{perf}}(\tilde{X})}$. Thus indeed $\sigma_*\sigma^* \cong \text{id}_{D^{\text{perf}}(X)}$. $\square$

Definition 3.30: Let $(\tilde{\mathcal{D}}, \sigma_*, \sigma^*)$ be a categorical resolution of a triangulated category $\mathcal{D}$. We say the resolution is **weakly crepant** if $\sigma_* \dashv \sigma^*$.

Observe that Proposition 3.7 implies that a crepant resolution of a singularity induces a weakly crepant categorical resolution justifying the name.

Definition 3.31: Let X be a variety and L a line bundle on X . A **dual Lefschetz decomposition** is a semiorthogonal decomposition of $D^b(X)$ of the form $D^b(X) = \langle A_{m-1} \otimes L^{1-m}, \dots, A_1 \otimes L^{-1}, A_0 \rangle$ where $A_{m-1} \subset \dots \subset A_1 \subset A_0$.

The dual Lefschetz decomposition is an important tool for computing categorical resolutions. For example consider a cartier divisor, E , in X . Let's say E is the exceptional divisor of a blow-up and let L be the conormal bundle of E in X . Then a dual Lefschetz decomposition of E with respect to L ends up giving a semiorthogonal decomposition of X which often includes the “critical components” of X , see Proposition 4.1 in [17]. The following two theorems give a categorical resolution and its kernel for an A_1 singularity. The third theorem does the same for an A_2 singularity.

Theorem 3.32 (Cattani, Giovenzana, Liu, Magni, Martinelli, Pertusi, Song, Thm1.1/Proposition 3.6 [4]): *Let X be a quasiprojective variety with an isolated nodal singularity, and assume $\dim(X) \geq 2$. Then there exists a weakly crepant categorical resolution $\sigma_* : \tilde{\mathcal{D}} \rightarrow D^b(X)$ such that σ_* is a categorical contraction. Let $\tilde{X}$ be the resolution of X by blowing up the singular point. We have $\tilde{\mathcal{D}}$ is an admissible subcategory of $D^b(\tilde{X})$. Let S be the spinor bundle on the exceptional divisor E when $\dim(X)$ is even and let S', S'' be the spinor bundles on E when $\dim(X)$ is odd. Notating j as the inclusion of E into $\tilde{X}$, the kernel of σ_* is generated by $\langle j_* S \rangle$ when $\dim(X)$ is even and is generated by $\langle \text{cone}(j_* S' \rightarrow j_* S''[2]) \rangle$ when $\dim(X)$ is odd.*

Theorem 3.33 (Kuznetsov, Shinder, Thm 5.8/Lemma 5.10 [19]): *Let X be a variety of dimension $m \geq 2$ over an algebraically closed field k such that $\text{char}(k) \neq 2$. Let X have isolated A_1 singularities $x_1, \dots, x_r$ and no other*

singularities. Let

$$\sigma : \tilde{X} \rightarrow X$$

be the blow up of all the singular points and $j_i : E_i \rightarrow \tilde{X}$ the embedding of the exceptional divisor over x_i . Let S_i be a spinor bundle on E_i . Then the subcategory

$$\mathcal{D} := \{F \in D^b(\tilde{X}) \mid j^*F \in \langle S_i, \mathcal{O}_{E_i} \rangle \forall i \in \{1, \dots, r\}\}$$

is admissible in $D^b(\tilde{X})$. Moreover $\sigma_*|_{\mathcal{D}} : \mathcal{D} \rightarrow D^b(X)$ is a weakly crepant localization with kernel generated by $\langle j_*(S_i) \rangle_{i=1}^r$ if $\dim X$ is even or generated by $\langle \text{cone}(j_*(S'_i) \rightarrow j_*(S''_i)[2]) \rangle_{i=1}^r$ if $\dim X$ is odd.

Theorem 3.34 (Fietz, Theorem 4.3 [8]): *Let X be a variety with an isolated A_2 singularity at a point $Z \in X$ and let $\dim(X) \geq 2$. Then there is a weakly crepant categorical resolution $\sigma_* : \tilde{\mathcal{D}} \rightarrow D^b(X)$ which is a localization. The kernel of σ_* is $\langle j_*S \rangle$ when $\dim(X)$ is odd and $\langle j_*S_1, j_*S_2 \rangle$ when $\dim(X)$ is even where $j : E \rightarrow \tilde{X}$ is the inclusion of the exceptions divisor and S, S_1, S_2 are the corresponding spinor bundles on E .*

The techniques used in proving these three results establish much of the ground work necessary for us to extend the result to A_n . Moreover, these results are used as the base case for the induction process used to prove the main result of this thesis.

3.3 Categorical Resolutions of A_n Singularities

In this section we will give a construction of a weakly crepant categorical resolution for an A_n singularity. In addition we will compute the kernel of this resolution. We will do this by showing the work on the A_1 singularity [4, 19] and A_2 singularity [8] cases extend to A_n .

Definition 3.35: Let k be a field and let $\mathcal{D}$ be a k -linear triangulated category. An object $E \in \mathcal{D}$ is called **exceptional** if $\mathrm{Hom}(E, E) = k$ and for any non-trivial shift, $n \neq 0$, $\mathrm{Hom}(E, E[n]) = 0$. A sequence of exceptional objects $E_m, \dots, E_1$ is called an **exceptional collection** if $(E_m, \dots, E_1)$ is a semiorthogonal collection.

Proposition 3.36: *Let E be a projective quadric of dimension $m - 1$. Then $D^b(E)$ has a semiorthogonal decomposition*

$$\langle \mathcal{A}_E, \mathcal{O}_E, \mathcal{O}_E(1), \dots, \mathcal{O}_E(m-2) \rangle .$$

Proof. We shall prove this by first showing that $(\mathcal{O}_E, \mathcal{O}_E(1), \dots, \mathcal{O}_E(m-2))$ is an exceptional collection and then applying a theorem of Bondal and a theorem of Kuznetsov.

We shall consider $\mathrm{Hom}_{D^b(E)}(\mathcal{O}_E(a), \mathcal{O}_E(b)[s])$ for arbitrary a, b, s .

$$\begin{aligned}
\mathrm{Hom}_{D^b(E)}(\mathcal{O}_E(a), \mathcal{O}_E(b)[s]) &\cong \mathrm{Hom}_{D^+(Q\mathrm{Coh}(E))}(\mathcal{O}_E(a), \mathcal{O}_E(b)[s]) \\
&\cong \mathrm{Ext}^s(\mathcal{O}_E(a), \mathcal{O}_E(b)) \\
&\cong \mathrm{Ext}^s(\mathcal{O}_E, \mathcal{O}_E(b-a)) \\
&\cong H^s(\mathcal{O}_E(b-a))
\end{aligned}$$

As $\mathcal{O}_E(a), \mathcal{O}_E(b)[s]$ bounded and coherent, the first two isomorphism follow by Proposition 3.5 and Proposition 2.56 in [14] respectively. The third isomorphism follows from Proposition III.6.7 in [11] and the fourth isomorphism follows from the fact $\mathrm{Hom}(\mathcal{O}_E, \cdot) = \Gamma(E, \cdot)$ as functors.

Now we shall look at the short exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^m}(a-2) \xrightarrow{\cdot Q} \mathcal{O}_{\mathbb{P}^m}(a) \xrightarrow{\text{projection}} i_*\mathcal{O}_E(a) \rightarrow 0$$

Where Q is the homogeneous degree 2 defining equation of E . From this short exact sequence we get a long exact sequence of cohomology, namely,

$$\dots \rightarrow H^s(\mathcal{O}_{\mathbb{P}^m}(a-2)) \rightarrow H^s(\mathcal{O}_{\mathbb{P}^m}(a)) \rightarrow H^s(i_*\mathcal{O}_E(a)) \rightarrow H^{s+1}(\mathcal{O}_{\mathbb{P}^m}(a-2)) \rightarrow \dots$$

We know that

$$H^s(\mathcal{O}_{\mathbb{P}^m}(r)) = \begin{cases} k & \text{if } s = 0, r = 0 \\ 0 & \text{if } s = 0, r < 0 \\ 0 & \text{if } s \neq 0, r \geq -m \end{cases}$$

by Lemma 01XT from [1]. Thus we can deduce from our long exact sequence, after applying Lemma III.2.10 in [11], that

$$H^s(\mathcal{O}_E(r)) = \begin{cases} k & \text{if } s = 0, r = 0 \\ 0 & \text{if } s = 0, r \geq 2 - m \\ 0 & \text{if } s \neq 0, r \geq 2 - m \end{cases}$$

which gives us

$$\text{Hom}(\mathcal{O}_E(a), \mathcal{O}_E(b)[s]) = \begin{cases} k & \text{if } s = 0, b = a \\ 0 & \text{if } s = 0, b - a \geq 2 - m \\ 0 & \text{if } s \neq 0, b - a \geq 2 - m \end{cases}$$

implying $(\mathcal{O}_E, \mathcal{O}_E(1), \dots, \mathcal{O}_E(m-2))$ is indeed an exceptional collection.

Thus by Theorem 3.2 in [2] we have that $\langle \mathcal{O}_E, \mathcal{O}_E(1), \dots, \mathcal{O}_E(m-2) \rangle$ is an admissible subcategory of $D^b(E)$. So by Lemma 2.6 from [18] we get

$$D^b(E) = \langle \mathcal{A}_E, \mathcal{O}_E, \dots, \mathcal{O}_E(m-2) \rangle$$

is a semi-orthogonal decomposition where

$$\mathcal{A}_E = \langle \mathcal{O}_E, \mathcal{O}_E(1), \dots, \mathcal{O}_E(m-2) \rangle^\perp.$$

□

The following theorem of Efimov is a key theorem in computing the kernel of our categorical resolution. We will use it similarly to how it was applied in the A_1 and A_2 case. However, this version of the theorem is phrased in the way of Theorem 5.2 in [19].

Theorem 3.37 (Thm 8.22 in [6]): *Let $\sigma : \tilde{X} \rightarrow X$ be a proper birational morphism. Let $i : Z \rightarrow X$ be a closed subscheme and let $\sigma^{-1}(Z)$, we'll call E , be a cartier divisor. Assume σ is an isomorphism away from E, Z . Let $\mathcal{I}$ be the ideal sheaf corresponding to Z . Suppose $\sigma_*\mathcal{O}_{\tilde{X}}(-mE) \cong \mathcal{I}^m$ for all $m \geq 0$. Let the following be a cartesian square:*

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} \\ \rho \downarrow & & \sigma \downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

If $\rho_ : D^b(E) \rightarrow D^b(Z)$ is a Verdier localization then we get that the derived functor $\sigma_* : D^b(\tilde{X}) \rightarrow D^b(X)$ is a Verdier localization whose kernel is generated by $j_*(\ker(\rho_*))$.*

Definition 3.38: Let $Z \in X$ be a normal isolated singularity. Let m be the maximal ideal in the local ring $\mathcal{O}_{X,Z}$. Then we call Z an **acyclic projectively normal singularity** if the canonical map $m^j/m^{j+1} \rightarrow H^0(E, \mathcal{O}_E(j))$ is an isomorphism and $H^i(E, \mathcal{O}_E(j)) = 0$ for all $j \geq 0, i > 0$.

The idea of an acyclic projectively normal singularity is to capture a sufficient condition for what kinds of singularities we can apply Efimov's theorem. The properties are sufficient to give the “meat” of thereom's condition which is: $\sigma_*\mathcal{O}_{\tilde{X}}(-mE) \cong \mathcal{I}^m$ for all $m \geq 0$. Lemma 5.5 and Corollary 5.6 in [19] show this connection.

The next lemma is proved similarly to the A_2 case in [8].

Lemma 3.39: *Let X be a projective m -dimensional variety with an isolated A_n singularity, Z , with $n \geq 2$. Then (X, Z) is an acyclic projectively normal singularity.*

Proof. Note we have shown earlier (see proof of Lemma 3.36) that we know $H^i(E, \mathcal{O}_E(l)) = 0$ for $i > 0$, $l \geq 0$. Using the beginning of the long exact sequence,

$$0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^m}(l-2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^m}(l)) \rightarrow H^0(i_*\mathcal{O}_E(l)) \rightarrow 0,$$

also from the proof Lemma 3.36 we can see that

$$H^0(E, \mathcal{O}_E(l)) \cong \frac{k[y_0, \dots, y_m]_l}{(y_1^2 + \dots + y_m^2)k[y_0, \dots, y_m]_{l-2}}$$

using Lemma 01XT from [1]. Let us note that $\mathfrak{m}_{X,Z} \cong (x_0, x_1, \dots, x_m)$ in $\frac{k[x_0, x_1, \dots, x_m]}{x_0^{n+1} + x_1^2 + \dots + x_m^2}(x_0, x_1, \dots, x_m)$. So we can see $\frac{\mathfrak{m}_{X,Z}^l}{\mathfrak{m}_{X,Z}^{l+1}} \cong \frac{(x_0, x_1, \dots, x_m)^l}{(x_0, x_1, \dots, x_m)^{l+1}}$. Let us consider the map

$$\phi : \frac{k[y_0, \dots, y_m]_l}{(y_1^2 + \dots + y_m^2)k[y_0, \dots, y_m]_{l-2}} \xrightarrow{y_i \rightarrow x_i} \frac{(x_0, \dots, x_m)^l}{(x_0, \dots, x_m)^{l+1}}.$$

If we show this map is well-defined and an isomorphism that will be enough to show its inverse, the canonical map,

$$\frac{(x_0, \dots, x_m)^l}{(x_0, \dots, x_m)^{l+1}} \xrightarrow{x_i \rightarrow y_i} \frac{k[y_0, \dots, y_m]_l}{(y_1^2 + \dots + y_m^2)k[y_0, \dots, y_m]_{l-2}}$$

is an isomorphism like we want. Let $g \in k[y_0, \dots, y_m]_{l-2}$, then in $\frac{(x_0, \dots, x_m)^l}{(x_0, \dots, x_m)^{l+1}}$,

$$\begin{aligned} \phi((y_1^2 + \dots + y_m^2)g(y_0, y_1, \dots, y_m)) &= (x_1^2 + \dots + x_m^2)g(x_0, x_1, \dots, x_m) \\ &= (x_0^{n+1} + x_1^2 + \dots + x_m^2)g(x_0, x_1, \dots, x_m) \\ &= 0 \end{aligned}$$

with the second to last equality following as degree of $x_0^{n+1}g \geq l+1$. Thus ϕ is well-defined.

Now we show ϕ surjective. Generically an element of $\frac{(x_0, \dots, x_m)^l}{(x_0, \dots, x_m)^{l+1}}$ looks like $\frac{\sum a_i \prod x_i}{(\sum b_j \prod x_j) + c}$ with monomials in the numerator of degree at least l , monomials in the denominator degree at least 1, and c in $k - \{0\}$. Observe

$$\frac{\sum a_i \prod x_i}{(\sum b_j \prod x_j) + c} = \frac{\sum a_i \prod x_i}{c}$$

as

$$c(\sum a_i \prod x_i) - ((\sum b_j \prod x_j) + c) * (\sum a_i \prod x_i) = 0$$

as

$$(\sum a_i \prod x_i) * ((\sum b_j \prod x_j)) = 0$$

as terms will have degree at least $l+1$. Hence all elements of $\frac{(x_0, \dots, x_m)^l}{(x_0, \dots, x_m)^{l+1}}$ are writeable as $\sum a_i \prod x_i$ with each monomial of degree l . Thus ϕ is surjective.

Now we show ϕ is injective. Let $p(y_0, \dots, y_m) \in k[y_0, \dots, y_m]_l$ such that we have $\phi(p) = 0$. The only homogeneous degree l terms in the ideal generated as $(\geq \text{degree } l+1, x_0^{n+1} + x_1^2 + \dots + x_m^2)$ are those of the form $(x_1^2 + \dots + x_m^2)f(x)$ with $f(x)$ degree $l-2$. Thus $p(y_0, \dots, y_m) = 0$ as required. $\square$

Lemma 3.40: *Let D_0 be a triangulated category and let $(\sigma_*, \sigma^*, \tilde{\mathcal{D}})$ be a weakly crepant categorical resolution of D_0 with the resolution an admissible subcategory of the bounded derived category of coherent sheaves for a smooth projective variety Y . Let $\tilde{\mathcal{D}} \subset D_0$ be a right admissible triangulated subcategory such that $\tilde{\mathcal{D}}^\perp \subset D_0^{\text{perf}}$. Let $\mathcal{D} = {}^\perp \sigma^*(\tilde{\mathcal{D}}^\perp)$, with the last perp taken in $\tilde{\mathcal{D}}$. Then $(\sigma_*|_{\mathcal{D}}, \sigma^*|_{\tilde{\mathcal{D}}^{\text{perf}}}, \mathcal{D})$ is a weakly crepant categorical resolution of $\tilde{\mathcal{D}}$. Also if σ_* is a localization, then $\sigma_*|_{\mathcal{D}}$ is a localization with $\ker(\sigma_*) = \ker(\sigma_*|_{\mathcal{D}})$.*

Proof. First we note that the definition of $\mathcal{D}$ makes sense as $\tilde{\mathcal{D}}^\perp \subset D_0^{\text{perf}}$ so that σ^* is well defined on $\tilde{\mathcal{D}}^\perp$.

Let $i : \sigma^*(\tilde{\mathcal{D}}^\perp) \rightarrow D^b(Y)$ and $i' : \tilde{\mathcal{D}}^\perp \rightarrow D_0$ be inclusion. As a consequence of Lemma 2.6 in [18] as $\tilde{\mathcal{D}}$ right admissible we have that $\tilde{\mathcal{D}}^\perp$ is left admissible. Observe that by weak crepancy we have that σ_*, σ^* have left adjoints, by left admissibility i' has a left adjoint, and as $\tilde{\mathcal{D}}^\perp \subset D_0^{\text{perf}}$ we see that $i = (\sigma^* i' \sigma_*)|_{\sigma^*(\tilde{\mathcal{D}}^\perp)}$ also has a left adjoint. So by Lemma 2.6 and 2.7 in [18], as Y smooth and projective, we deduce $\mathcal{D}$ is admissible in $D^b(Y)$.

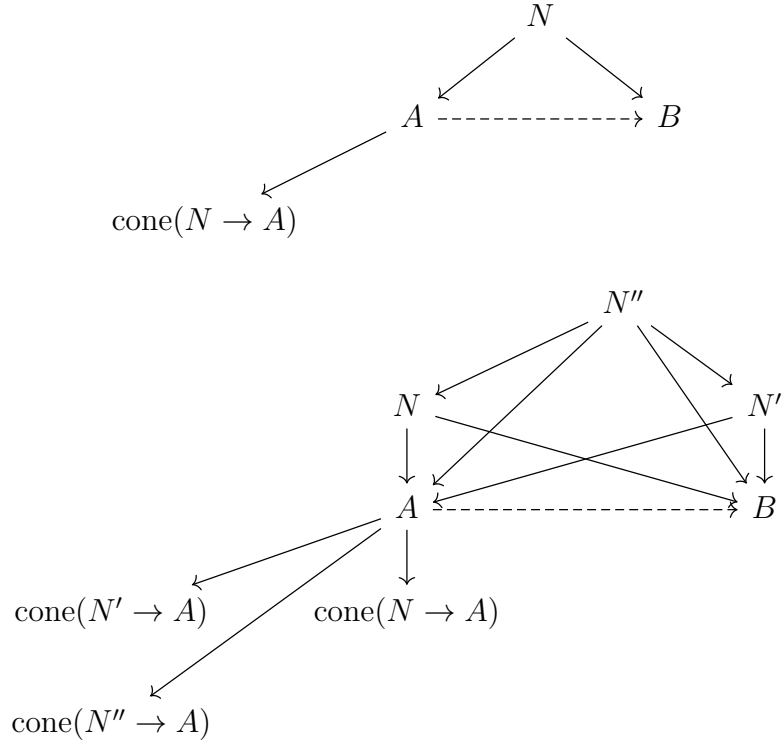
$$\begin{array}{ccccc}
 \tilde{\mathcal{D}} & & \xleftarrow{i} & \sigma^*(\tilde{\mathcal{D}}^\perp) & \\
 \downarrow \sigma_* & \nearrow \sigma^* & \searrow \sigma_*|_{\tilde{\mathcal{D}}^\perp} & \searrow \sigma^*|_{\tilde{\mathcal{D}}^\perp} & \\
 D_0 & \xleftarrow{i'_2} & D_0^{\text{perf}} & \xleftarrow{i'_1} & \tilde{\mathcal{D}}^\perp \\
 & \nwarrow i' & & &
 \end{array}$$

By Lemma 3.15 we get that $\tilde{\sigma}_*|_{\mathcal{D}} : \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ is well-defined and by Lemma 3.16 we get that $\tilde{\sigma}^*|_{\tilde{\mathcal{D}}^{\text{perf}}} : \tilde{\mathcal{D}}^{\text{perf}} \rightarrow \mathcal{D}$ is well-defined. Observe that $\sigma_* \dashv \sigma^* \dashv \sigma_*$

implies $\sigma_*|_{\mathcal{D}} \dashv \sigma^*|_{\tilde{\mathcal{D}}^{\text{perf}}} \dashv \sigma_*|_{\mathcal{D}}$. Also $\sigma_*\sigma^* \cong \text{id}_{D_0^{\text{perf}}}$ implies $\sigma_*|_{\mathcal{D}}\sigma^*|_{\tilde{\mathcal{D}}^{\text{perf}}} \cong \text{id}_{\tilde{\mathcal{D}}^{\text{perf}}}$. Thus $(\sigma_*|_{\mathcal{D}}, \sigma^*|_{\tilde{\mathcal{D}}^{\text{perf}}}, D)$ is a weakly crepant categorical resolution of $\tilde{\mathcal{D}}$.

Now we will assume σ_* is a localization. Let $A \in \ker(\sigma_*)$ and $B \in \tilde{\mathcal{D}}^\perp$. We have that $\text{Hom}_{\tilde{\mathcal{D}}}(A, \sigma^*B) \cong \text{Hom}_{\tilde{\mathcal{D}}}(\sigma_*A, B) = 0$ as $\sigma_*A = 0$ by definition. Thus $A \in \mathcal{D}$. So $\ker \sigma_* \subset \mathcal{D}$ which means $\ker \sigma_* = \ker \sigma_*|_{\mathcal{D}}$.

Note then that $\mathcal{D}/\ker(\sigma_*)$ is a full subcategory of $\tilde{\mathcal{D}}/\ker(\sigma_*)$. This can be seen from the following two diagrams,



Let $A, B \in \mathcal{D}$ and assume

$$\text{cone}(N \rightarrow A), \text{cone}(N' \rightarrow A), \text{cone}(N'' \rightarrow A) \in \ker(\sigma_*) \subset \mathcal{D}.$$

As triangulated categories are closed under cone we get that $N, N', N'' \in \mathcal{D}$ as

well. Thus the first diagram gives us that each morphism between objects in $\mathcal{D}$ in $\tilde{\mathcal{D}}/\ker(\sigma_*)$ is also a morphism in $\mathcal{D}/\ker(\sigma_*)$. The second diagram gives us that two morphisms between objects in $\mathcal{D}$ are equal in $\tilde{\mathcal{D}}/\ker(\sigma_*)$ if and only if they are equal in $\mathcal{D}/\ker(\sigma_*)$. Thus $\overline{\sigma_*|_{\mathcal{D}}}$ is fully faithful as $\overline{\sigma_*}$ is fully faithful. Now let $A \in \tilde{\mathcal{D}}$. As σ_* essentially surjective there exists $\tilde{A} \in \tilde{\mathcal{D}}$ such that $\sigma_*(\tilde{A}) \cong A$. Observe $\text{Hom}(\tilde{A}, \sigma^*(\tilde{\mathcal{D}}^\perp)) \cong \text{Hom}(\sigma_*\tilde{A}, \tilde{\mathcal{D}}^\perp) \cong \text{Hom}(A, \tilde{\mathcal{D}}^\perp) = 0$ by weak crepancy and definition. Thus $\tilde{A} \in \mathcal{D}$. Hence $\sigma_*|_{\mathcal{D}}$ is essentially surjective. Therefore we may conclude that $\overline{\sigma_*|_{\mathcal{D}}}$ is an equivalence and that $\sigma_*|_{\mathcal{D}}$ is a localization. $\square$

Lemma 3.41: *Let $\phi : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{D}}$ and $\psi : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ be a Verdier localizations with kernels $\langle A_1, \dots, A_m \rangle$ and $\langle B_1, \dots, B_n \rangle$ respectively. Then there are $B'_i \in \tilde{\mathcal{D}}$ such that $\phi(B'_i) = B_i$. The composition $\psi\phi$ is a Verdier localization with kernel $\langle A_1, \dots, A_m, B'_1, \dots, B'_n \rangle$ for any such B'_i .*

Proof. By essential surjectivity of ϕ there exists B'_i such that $\phi(B'_i) = B_i$. Of course $\langle A_1, \dots, A_m, B'_1, \dots, B'_n \rangle \subset \ker(\psi\phi)$. Let $L \in \ker(\psi\phi)$. Then we know that $\phi(L) \in \langle B_1, \dots, B_n \rangle$. As shifts and cones commute with ϕ and $\phi(L)$ can be written as a combinations of shifts and cones of $B_1, \dots, B_n$, using the same combination of shifts and cones on $B'_1, \dots, B'_n$ we get a term B'_L such that $\phi(B'_L) = \phi(L)$. Thus $B'_L \cong L$ in the localization as under the localization ϕ is fully faithful. This means we have a distinguished triangle,

$$L \rightarrow B'_L \rightarrow \text{cone}(L \rightarrow B'_L),$$

with $B'_L \in \langle B'_1, \dots, B'_n \rangle$ and $\text{cone}(L \rightarrow B'_L) \in \ker(\phi) = \langle A_1, \dots, A_m \rangle$. Thus by two out of three property we get $L \in \langle A_1, \dots, A_m, B'_1, \dots, B'_n \rangle$. So $\ker(\psi\phi) = \langle A_1, \dots, A_m, B'_1, \dots, B'_n \rangle$.

Now we need to show that the $\psi\phi$ is a localization. By the universal property of Verdier quotients (Lemma 05RJ in [1]) we have that $\overline{\psi\phi}$ is a triangulated functor and $\overline{\psi} \circ Q_2 \circ \overline{\phi} \circ Q_1 = \overline{\psi\phi} \circ Q_3$ where each Q_i is the relevant quotient functor.

$$\begin{array}{ccccc}
 & & Q_3 & & \\
 & \nearrow & & \searrow & \\
 \tilde{\mathcal{D}} & \xrightarrow{Q_1} & \tilde{\mathcal{D}}/\ker \phi & \xrightarrow{\overline{Q}_1} & \tilde{\mathcal{D}}/\ker \psi\phi \\
 & \searrow \phi & \downarrow \overline{\phi} & & \uparrow \overline{\psi\phi} \\
 & & \tilde{\mathcal{D}} & \xrightarrow{Q_2} & \tilde{\mathcal{D}}/\ker \psi \\
 & & & \searrow \psi & \downarrow \overline{\psi} \\
 & & & & \mathcal{D}
 \end{array}$$

As $\overline{\psi}$, $\overline{\phi}$ and each Q_i essentially surjective we get that $\overline{\psi\phi}$ is essentially surjective.

Next we will show fullness. Now consider a morphism $\overline{\psi\phi}(R) \rightarrow \overline{\psi\phi}(S)$. Since Q_i 's are surjective on objects via identity, we can write this as a morphism

$$\overline{\psi} \circ Q_2 \circ \overline{\phi} \circ Q_1(R) \rightarrow \overline{\psi} \circ Q_2 \circ \overline{\phi} \circ Q_1(S).$$

As $\overline{\psi}$ full we may lift to a morphism

$$Q_2 \circ \overline{\phi} \circ Q_1(R) \rightarrow Q_2 \circ \overline{\phi} \circ Q_1(S).$$

Thus we have a diagram in $\tilde{\mathcal{D}}$:

$$\begin{array}{ccc}
 & N & \\
 \swarrow & & \searrow \\
 \overline{\phi} \circ Q_1(R) & \dashrightarrow & \overline{\phi} \circ Q_1(S) \\
 \swarrow & & \\
 \text{cone}(N \rightarrow \overline{\phi} \circ Q(R)) & &
 \end{array}$$

Where $\text{cone}(N \rightarrow \overline{\phi} \circ Q(R)) \in \ker(\psi)$. As $\overline{\phi}$ is an equivalence we can lift to a diagram in $\tilde{\mathcal{D}}/\ker(\phi)$:

$$\begin{array}{ccc}
 & \tilde{N} & \\
 \swarrow & & \searrow \\
 Q_1(R) & \dashrightarrow & Q_1(S) \\
 \swarrow & & \\
 \text{cone}(\tilde{N} \rightarrow Q_1(R)) & &
 \end{array}$$

where $\overline{\phi}(\tilde{N}) \cong N$ and $\overline{\phi}(\text{cone}(\tilde{N} \rightarrow Q_1(R))) = \text{cone}(N \rightarrow \overline{\phi} \circ Q_1(R))$. Note that Q_3 factors as $\overline{Q_1} \circ Q_1$ by the universal property of Verdier quotients. Now observe that $\text{cone}(\tilde{N} \rightarrow Q_1(R)) \in \ker(\psi\phi)$ as $\text{cone}(N \rightarrow \overline{\phi} \circ Q_1(R)) \in \ker(\psi)$. Thus we have lifted to a morphism $R \rightarrow S$ in $\tilde{\mathcal{D}}/\ker(\psi\phi)$ by construction. Therefore $\overline{\psi\phi}$ is full.

Finally we shall show faithfulness. Suppose $f_1, f_2 \in \text{Hom}(A, B)$ in $\tilde{\mathcal{D}}/\ker(\psi\phi)$ such that $\overline{\psi\phi}(f_1) = \overline{\psi\phi}(f_2)$. Similar to before we have that $Q_2 \circ \overline{\phi} \circ Q_1(f_1) = Q_2 \circ \overline{\phi} \circ Q_1(f_2)$ as ψ fully faithful. Then we get a morphism equality diagram in $\tilde{\mathcal{D}}$ which we can lift to $\tilde{\mathcal{D}}/\ker(\phi)$ as $\overline{\phi}$ an equivalence. Then as the cones are still in $\ker(\psi\phi)$ we have an equality diagram between f_1 and f_2 . Thus $\psi\phi$ is

indeed a localization. $\square$

Let X be a projective scheme of dimension $m \geq 2$ over a field k with an isolated A_n singularity. Then we may take the blow-up of X at the singularity, which we will call Z , to produce $\tilde{X}$ which has an A_{n-2} singularity, refer to example 1. Then we repeat this procedure of blowing-up until we've fully resolved the singularity, see example 2. We shall call this non-singular variety $\tilde{\tilde{X}}$. At each blow-up step we get an exceptional divisor E_i which is a nodal quadric, except possibly in the last step it could be a smooth quadric. On each nodal quadric, except possibly the last one, consider the resolution of E_i via blowing up the singularity notated $\psi_i : Bl(E_i) \rightarrow E_i$. We also have for each step a map c_i from $Bl(E_i)$ to a smooth quadric, refer to definition 2.27. The spinor bundle on E_i is $\psi_{i*}c_i^*S$ for a spinor bundle S on the smooth quadric. We also have an inclusion map $\tilde{j}_i : Bl(E_i) \rightarrow \tilde{\tilde{X}}$, see remark 6. Let us use the notation, $\mathcal{A}_{E_i}$, to mean

$$\mathcal{A}_{E_i} = \begin{cases} \langle \tilde{j}_{i*}c_i^*S \rangle & \text{if } m \text{ is odd and } (n \text{ is even or } i \leq \lfloor \frac{n}{2} \rfloor) \\ \langle \text{cone}(\tilde{j}_{i*}S' \rightarrow \tilde{j}_{i*}S''[2]) \rangle & \text{if } m \text{ is odd and } n \text{ is odd and } i = \lceil \frac{n}{2} \rceil \\ \langle \tilde{j}_{i*}c_i^*S', \tilde{j}_{i*}c_i^*S'' \rangle & \text{if } m \text{ is even and } (n \text{ is even or } i \leq \lfloor \frac{n}{2} \rfloor) \\ \langle \tilde{j}_{i*}S \rangle & \text{if } m \text{ is even and } n \text{ is odd and } i = \lceil \frac{n}{2} \rceil \end{cases}$$

To clarify this definition, we essentially always split into two cases based on the parity of the dimension of X . The only exception occurs at the last blow-up to resolve the singularity of X . Here the cases split based on whether the last

singularity being resolved is an A_1 or an A_2 singularity.

Theorem 3.42: *Let X be a projective variety with an isolated A_n singularity. Then $D^b(X)$ has a weakly crepant categorical resolution of a singularity which is a localization. The kernel of the resolution is generated by $\langle \mathcal{A}_{E_1}, \dots, \mathcal{A}_{E_{\lceil \frac{n}{2} \rceil}} \rangle$.*

Proof. **Step 1:** Setup the problem.

We shall proceed by induction. The base cases of A_1 and A_2 are true by Theorem 3.33 and Theorem 3.34 respectively. Suppose for a projective variety X of $\dim m \geq 2$ with an isolated A_{n-2} singularity, $n \geq 3$, there exists a weakly crepant categorical resolution, $(\tilde{\mathcal{D}}, \tilde{\sigma}_*, \tilde{\sigma}^*)$, resulting from repeatedly blowing up at the singularity, with kernel generated by $\langle \mathcal{A}_{E_2}, \dots, \mathcal{A}_{E_{\lceil \frac{n}{2} \rceil}} \rangle$.

$$\begin{array}{ccc}
 \tilde{E} & \xrightarrow{\tilde{j}} & \tilde{X} \\
 \tilde{\rho} \downarrow & \tilde{i} \nearrow & \tilde{\sigma} \downarrow \\
 \tilde{Z} & & \tilde{X} \\
 & E \xrightarrow{j} & \\
 & \rho \downarrow & \sigma \downarrow \\
 & Z \xrightarrow{i} & X
 \end{array}$$

Note that $E, \tilde{E}$ are exceptional divisors and $\tilde{Z}$ is the A_{n-2} singularity in $\tilde{X}$. The current set-up is portrayed in the diagram above. For now we will focus on the lower half of the diagram in the derived category setting:

$$\begin{array}{ccc}
 D^b(E) & \xrightarrow{j_*} & D^b(\tilde{X}) \\
 \rho_* \downarrow & & \sigma_* \downarrow \\
 D^b(Z) & \xrightarrow{i_*} & D^b(X)
 \end{array}$$

From Proposition 3.36, as E is a nodal quadric, we have a semiorthogonal decomposition:

$$D^b(E) = \langle \mathcal{A}_E, \mathcal{O}_E, \dots, \mathcal{O}_E(m-2) \rangle.$$

We will justify the notational compatibility of $\mathcal{A}_E$ later.

Step 2: Convert the semiorthogonal decomposition of $D^b(E)$ into a dual Lefschetz decomposition.

Since E is a complete intersection, as $E \subset \mathbb{P}^m$ is a hypersurface in a smooth variety, we know E is Gorenstein by Lemma 2.18. By Lemma 2.22 we get $\omega_E \cong \mathcal{O}_E(1-m)$. As for any i we know $\mathcal{O}_E(i)$, when viewed as a complex, is a line bundle concentrated in degree 0 we have by Lemma 0BCJ in [1] that $\mathcal{O}_E(i)$ is a perfect complex. So we know

$$\langle \mathcal{O}_E, \dots, \mathcal{O}_E(m-2) \rangle \subset D^{\text{perf}}(E).$$

Thus as E a Gorenstein projective variety we produce via repeated applications of Lemma 2.15 from [15] the semiorthogonal decomposition:

$$D^b(E) = \langle \mathcal{O}_E(2-m), \mathcal{O}_E(3-m), \dots, \mathcal{O}_E(-1), \mathcal{A}_E, \mathcal{O}_E \rangle.$$

As $\langle \mathcal{O}_E \rangle \subset \langle \mathcal{A}_E, \mathcal{O}_E \rangle$ we have constructed our dual Lefschetz decomposition.

Step 3: Push the dual Lefschetz decomposition of $D^b(E)$ to $D^b(\tilde{X})$.

As E is an effective Cartier divisor of $\tilde{X}$, by Lemma 02OS in [1], we can compute the conormal bundle as

$$\mathcal{N}_{E/\tilde{X}}^\vee \cong \mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_{\tilde{X}}(1)|_E \cong \mathcal{O}_E(1)$$

where the first isomorphism is from 21.2.H in [23], the second is from 02OS in [1], and the third isomorphism follows as the pullback of the structure sheaf is the structure sheaf. As E is a Cartier divisor of X , by Proposition 4.1 from [17] we know that

$$D^b(\tilde{X}) = \langle j_*\mathcal{O}_E(2-m), \dots, j_*\mathcal{O}_E(-1), \tilde{\mathcal{D}} \rangle$$

is a semiorthogonal decomposition where

$$\tilde{\mathcal{D}} = \{F \in D^b(\tilde{X}) \mid j^*F \in \langle \mathcal{A}_E, \mathcal{O}_E \rangle\}.$$

Step 4: Show $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ is almost a weakly crepant categorical resolution.

We shall prove that $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ has all the properties of a weakly crepant categorical resolution, except potentially being an admissible subcategory of a bounded derived category of a non-singular variety. We must first show well-definedness.

Step 4.1: Prove $\sigma^*(D^{\text{perf}}(X)) \subset \tilde{\mathcal{D}}$.

Let $G \in D^{\text{perf}}(X)$. As $\tilde{\mathcal{D}} = \{F \in D^b(\tilde{X}) \mid j^*F \in \langle \mathcal{A}_E, \mathcal{O}_E \rangle\}$ we win if we can show $j^*\sigma^*G \in \langle \mathcal{A}_E, \mathcal{O}_E \rangle$. Note $j^*\sigma^* = \rho^*i^*$, see Lemma 0D5S in [1], so we have reduced the problem to showing $\rho^*i^*G \in \langle \mathcal{A}_E, \mathcal{O}_E \rangle$. As $i^*G \in D^{\text{perf}}(Z)$ by Lemma 09UA in [1] and $D^{\text{perf}}(Z)$ is just complexes of finite-dimensional vector spaces over k , we only need to understand $\rho^*(k)$. As k flat over k we may treat $\rho^*(k)$ as non-derived and compute $\rho^*(k) = \rho^*(\mathcal{O}_Z) = \mathcal{O}_E \in \langle \mathcal{O}_E \rangle$. Thus indeed $j^*\sigma^*(G) = \rho^*i^*G \in \langle \mathcal{A}_E, \mathcal{O}_E \rangle$ and so $\sigma^*(G) \in \tilde{\mathcal{D}}$.

Step 4.2: Show $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ is almost a categorical resolution.

Certainly $\sigma^* \vdash \sigma_*$ as it follows from the classical adjunction. And we know $\sigma_*\sigma^* \cong \text{id}_{D^{\text{perf}}(X)}$ from Lemma 3.29 and the well-definedness of the derived functor $\sigma^* : D^{\text{perf}}(X) \rightarrow \tilde{\mathcal{D}}$.

Thus our final step for $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ is to show weak crepancy. To do this we will follow closely the ideas of Lemma 4.6 in [17] and Lemma 1.9 in [8]. We go through it ourselves as our setting is mildly different since we do not have a resolution, but merely a blow-up at this stage.

Step 4.3: Show $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ is weakly crepant.

By Lemma 3.4 we see $j_*(\mathcal{O}_E(-i))$ is perfect for each i and so $\tilde{\mathcal{D}}^\perp \subset D^{\text{perf}}(\tilde{X})$. As $\tilde{X}$ is a projective variety, see II.7.16 in [11], Gorenstein, by Proposition 2.23, and since we just showed $D^b(\tilde{X}) = \langle \tilde{\mathcal{D}}^\perp, \tilde{\mathcal{D}} \rangle$ with $\tilde{\mathcal{D}}^\perp \subset D^{\text{perf}}(\tilde{X})$ we can use Lemma 2.15 from [15] to determine that $\tilde{\mathcal{D}}, \tilde{\mathcal{D}}^\perp$ are admissible subcategories of $D^b(\tilde{X})$. Let $\delta : \tilde{\mathcal{D}} \rightarrow D^b(\tilde{X})$ be our embedding. Thus δ has left and right adjoints $\delta^*, \delta^!$ as $\tilde{\mathcal{D}}$ admissible. So we can write the left and right adjoints of $\sigma_*|_{\tilde{\mathcal{D}}}$ as $\delta^*\sigma^*, \delta^!\sigma^!$.

$$\begin{array}{ccc}
 D^b(\tilde{X}) & \xrightleftharpoons[\delta]{\delta^*, \delta^!} & \tilde{\mathcal{D}} \\
 \downarrow \sigma_* & \swarrow \sigma^* & \nearrow \sigma_*|_{\tilde{\mathcal{D}}} \\
 D^b(X) & \xleftarrow[\text{incl.}]{\quad} D^{\text{perf}}(X) & \xrightarrow{\sigma^*} \tilde{\mathcal{D}}
 \end{array}$$

The goal is to show that $\delta^*\sigma^*$ and $\delta^!\sigma^!$ are equal over $D^{\text{perf}}(X)$. Let's suppose $F \in D^{\text{perf}}(X)$. Note that δ^* is defined as $\delta^*(R) = B$ for $A \rightarrow R \rightarrow B \rightarrow A[1]$ a distinguished triangle with $A \in \tilde{\mathcal{D}}^\perp$ and $B \in \tilde{\mathcal{D}}$, see Lemma 0CQT in [1]. Recall

we showed for $G \in D^{\text{perf}}(X)$ that $\sigma^*G \in \tilde{\mathcal{D}}$. So as $0 \rightarrow \sigma^*F \rightarrow \sigma^*F \rightarrow 0[1]$ is a distinguished triangle in $\tilde{\mathcal{D}}$ we can compute $\delta\delta^*\sigma^*F = \sigma^*F$.

Combining Lemma 2.23 and Lemma 3.7 gives $\sigma^!(F) \cong \sigma^*(F) \otimes \mathcal{O}_{\tilde{X}}((m-2)E)$. Again let us consider the short exact sequence for effective Cartier divisors,

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-E) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow i_*\mathcal{O}_E \rightarrow 0.$$

The short exact sequence then induces a distinguished triangle,

$$\mathcal{O}_{\tilde{X}}(-E) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow i_*\mathcal{O}_E \rightarrow \mathcal{O}_{\tilde{X}}(-E)[1].$$

Note that $\sigma^*F \otimes \mathcal{O}_{\tilde{X}}(lE)$ for $l \in \{1, \dots, m-2\}$ is perfect as the pullback of perfect complexes is perfect, since the pullback of locally free sheaves is locally free, and the tensor product of perfect complexes is perfect by Lemma 068T in [1]. As being perfect means one has a projective resolution, by tensoring we can get a collection of distinguished triangles in $D^b(\tilde{X})$ of the form

$$\sigma^*F \otimes \mathcal{O}_{\tilde{X}}((l-1)E) \rightarrow \sigma^*F \otimes \mathcal{O}_{\tilde{X}}(lE) \rightarrow \sigma^*F \otimes \mathcal{O}_{\tilde{X}}(lE) \otimes i_*\mathcal{O}_E$$

for $l \in \{1, \dots, m-2\}$. Now we compute:

$$\begin{aligned} \sigma^*F \otimes \mathcal{O}_{\tilde{X}}(lE) \otimes j_*\mathcal{O}_E &\cong j_*(j^*(\sigma^*F \otimes \mathcal{O}_{\tilde{X}}(lE))) \\ &\cong j_*(j^*\sigma^*F \otimes j^*\mathcal{O}_{\tilde{X}}(lE)) \\ &\cong j_*(\rho^*i^*F \otimes \mathcal{O}_E(-l)) \\ &\in \langle j_*\mathcal{O}_E(-l) \rangle. \end{aligned}$$

The first line follows from projection formula, Lemma 0B54 in [1], the second follows as tensor and pull back commute by Lemma 079U in [1], the third follows from the fact $\sigma j = i\rho$, and the fourth line follows from $\rho^*(D^b(Z)) \in \langle \mathcal{O}_E \rangle$ which we showed earlier and the fact that $\mathcal{O}_E(-l)$ perfect which makes our tensor product work as if underived. As $\sigma^*F \in \tilde{\mathcal{D}}$ and considering the family of distinguished triangles we just constructed, we have a filtration

$$0 \rightarrow \sigma^*F \rightarrow \sigma^*F \otimes \mathcal{O}_{\tilde{X}}(E) \rightarrow \dots \rightarrow \sigma^*F \otimes \mathcal{O}_{\tilde{X}}((m-2)E)$$

with respect to our semiorthogonal decomposition

$$D^b(\tilde{X}) = \langle j_*\mathcal{O}_E(2-m), \dots, j_*\mathcal{O}_E(-1), \tilde{\mathcal{D}} \rangle.$$

Thus by Lemma 3.14 we get that

$$\delta\delta^!\sigma^!F \cong \delta\delta^!(\sigma^*F \otimes \mathcal{O}_{\tilde{X}}((m-2)E)) \cong \sigma^*F \cong \delta\delta^*\sigma^*F.$$

So $\delta^*\sigma^*$ and $\delta^!\sigma^!$ are equal over $D^{\text{perf}}(X)$. Thus we conclude $\sigma^* \dashv \sigma_*|_{\tilde{\mathcal{D}}} \dashv \sigma^*$ and our resolution is weakly crepant. We now have our weakly crepant “almost” categorical resolution $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$.

Step 5: Produce our weakly crepant categorical resolution of $D^b(X)$.

Now we want to combine our induction hypothesis of the existence of a categorical resolution of singularities for $\tilde{X}$, $(\tilde{\sigma}_*, \tilde{\sigma}^*, \tilde{\mathcal{D}} \subset D^b(\tilde{X}))$, with $(\sigma_*|_{\tilde{\mathcal{D}}}, \sigma^*, \tilde{\mathcal{D}})$ to produce a categorical resolution, $\mathcal{D}$, which fits into the following diagram.

$$\begin{array}{ccccc}
D^b(\tilde{X}) & \xleftarrow{\quad} & \tilde{\mathcal{D}} & \xleftarrow{\quad} & \mathcal{D} \\
\downarrow \tilde{\sigma}_* & \swarrow \tilde{\sigma}_*| & & \swarrow \tilde{\sigma}_*| & \\
D^b(\tilde{X}) & \xleftarrow{\quad} & \tilde{\mathcal{D}} & & \\
\downarrow \sigma_* & \swarrow \sigma_*| & & & \\
D^b(X) & & & &
\end{array}$$

First recall that $\tilde{\mathcal{D}}^\perp = \langle j_*\mathcal{O}_E(2-m), \dots, j_*\mathcal{O}_E(-1) \rangle \subset D^{\text{perf}}(\tilde{X})$. Thus we can define

$$\mathcal{D} = {}^\perp\tilde{\sigma}^*(\tilde{\mathcal{D}}^\perp) \subset D^b(\tilde{X})$$

which we will now show has the desired properties and fits into the diagram as drawn. Observe that $\tilde{\mathcal{D}}^\perp$ is left admissible by Lemma 2.6 in [18] and as $i : \tilde{\sigma}^*(\tilde{\mathcal{D}}^\perp) \rightarrow D^b(\tilde{X}) \cong \tilde{\sigma}^*i\tilde{\sigma}_*$ which each have left adjoints we get $\tilde{\sigma}^*(\tilde{\mathcal{D}}^\perp)$ is left admissible. By Lemma 2.6 and 2.7 of [18] as $\tilde{X}$ is smooth and projective we get that $\mathcal{D}$ is admissible. By Lemma 3.40 we get that $(\tilde{\sigma}_*, \tilde{\sigma}^*, \mathcal{D})$ is a weakly crepant resolution of $\tilde{\mathcal{D}}$. Thus $(\sigma_*\tilde{\sigma}_*, \tilde{\sigma}^*\sigma^*, \mathcal{D})$ is a weakly crepant resolution of $D^b(X)$ as adjunction holds under composition and $\sigma_*\tilde{\sigma}_*\tilde{\sigma}^*\sigma^* \cong \sigma_*\sigma^* \cong \text{id}$.

Step 6: Show our weakly crepant categorical resolution of $D^b(X)$ is a Verdier localization and compute the kernel of the resolution.

As our singularity (X, Z) is acyclic projectively normal by Lemma 3.39 we can apply Corollary 5.6 from [19] to get that $\sigma_* : D^b(\tilde{X}) \rightarrow D^b(X)$ is a Verdier localization with kernel $\langle j_*(\mathcal{O}_E^\perp) \rangle$. By combining Proposition 2.8, 3.27, and 3.31 from [8] we get that $\mathcal{A}_E \cong \langle S_1, S_2 \rangle$ when E is odd dimensional and

$\mathcal{A}_E \cong \langle S_0 \rangle$ when E is even dimensional, where S_0, S_1, S_2 are the spinor sheaves on their respective nodal quadrics. Thus we have

$$\ker \sigma_* = \langle j_* \mathcal{O}_E(2-m), j_* \mathcal{O}_E(3-m), \dots, j_* \mathcal{O}_E(-1), j_* \mathcal{A}_E \rangle.$$

Using the semiorthogonal decomposition we computed earlier,

$$D^b(\tilde{X}) = \langle j_* \mathcal{O}_E(2-m), \dots, j_* \mathcal{O}_E(-1), \tilde{\mathcal{D}} \rangle,$$

we can see that $\ker(\sigma_*) \supset \tilde{\mathcal{D}}^\perp$. By lemma 3.17 we have a semiorthogonal decomposition $\ker \sigma_* = \langle \tilde{\mathcal{D}}^\perp, \ker \sigma_* \cap \tilde{\mathcal{D}} \rangle$. By Lemma 3.18 we then get that $\ker \sigma_* \cap \tilde{\mathcal{D}} = \langle j_* \mathcal{A}_E \rangle$. Thus the kernel of $\sigma_* : \tilde{\mathcal{D}} \rightarrow D^b(X)$ is $\langle j_* \mathcal{A}_E \rangle$.

Also by Lemma 3.25 and Lemma 3.26 we have that $\sigma_* : \tilde{\mathcal{D}} \rightarrow D^b(X)$ is a Verdier localization and by Lemma 3.40 we get that $\tilde{\sigma}_* : \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ is a Verdier localization with kernel $\langle j_* \mathcal{A}_E \rangle$. So by Lemma 3.41 we get that $\tilde{\sigma}_* \sigma_* : \mathcal{D} \rightarrow D^b(X)$ is also a Verdier localization with kernel generated by $\langle \ker \tilde{\sigma}_*, \text{lift of } j_* \mathcal{A}_E \rangle$. Using remark 6 we can see that $\tilde{\sigma}_* \tilde{j}_{1*} = j_* \psi_{1*}$. In the respective cases $S_0 = \psi_{1*} c_1^* S$, $S_1 = \psi_{1*} c_1^* S'$, $S_2 = \psi_{1*} c_1^* S''$ we see a lift of $j_* S_0, j_* S_1, j_* S_2$ respectively are $\tilde{j}_{1*} c_1^* S, \tilde{j}_{1*} c_1^* S', \tilde{j}_{1*} c_1^* S''$. Also we have that for $A \in \{S, S_1, S_2\}$ that

$$\begin{aligned} \text{Hom}(\tilde{j}_{1*} c_1^* A, \tilde{\sigma}^*(\tilde{\mathcal{D}}^\perp)) &\cong \text{Hom}(\tilde{\sigma}_* \tilde{j}_{1*} c_1^* A, \tilde{\mathcal{D}}^\perp) \\ &\cong \text{Hom}(j_* \psi_{1*} c^* A, \tilde{\mathcal{D}}^\perp) \\ &\cong 0 \end{aligned}$$

as $j_* \psi_{1*} c^* A \in \ker \sigma_* \subset \tilde{\mathcal{D}}$. So indeed $\tilde{j}_{1*} c_1^* S, \tilde{j}_{1*} c_1^* S', \tilde{j}_{1*} c_1^* S''$ are in $\mathcal{D}$. Thus the kernel of our resolution is generated by $\langle \mathcal{A}_{E_1}, \dots, \mathcal{A}_{E_{\lceil \frac{n}{2} \rceil}} \rangle$. $\square$

Bibliography

- [1] T. S. P. AUTHORS-STACKS, *The stacks project*, 2024.
- [2] A. I. BONDAL, *Representation of associative algebras and coherent sheaves*, Mathematics of the USSR-Izvestiya, 34 (1990), pp. 23–42.
- [3] T. BRIDGELAND, A. KING, AND M. REID, *The mckay correspondence as an equivalence of derived categories*, Journal of the American Mathematical Society, 14 (2001), pp. 535–554.
- [4] W. CATTANI, F. GIOVENZANA, S. LIU, P. MAGNI, L. MARTINELLI, L. PERTUSI, AND J. SONG, *Kernels of categorical resolutions of nodal singularities*, Rendiconti del Circolo Matematico di Palermo Series 2, 72 (2023), pp. 3077–3105.
- [5] M. V. DEN BERGH, *Non-commutative crepant resolutions*, 2009.
- [6] A. I. EFIMOV, *Homotopy finiteness of some dg categories from algebraic geometry*, 2018.
- [7] D. EISENBUD, *Commutative Algebra : With a View Toward Algebraic Geometry.*, Graduate Texts in Mathematics Ser. ; v. 150, Springer, New York, NY, 2013.
- [8] C. FIETZ, *Categorical resolutions of cuspidal singularities*, arXiv preprint arXiv:2411.19380, (2024).
- [9] J. HARRIS, *Grassmannians and Related Varieties*, Springer New York, New York, NY, 1992, pp. 63–71.

- [10] R. HARTSHORNE, *Residues and duality*, Springer, 1966.
- [11] —, *Algebraic Geometry*, vol. 52 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1977.
- [12] H. HIRONAKA, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Annals of Mathematics, 79 (1964), pp. 109–203.
- [13] C. HUNEKE, *Hyman bass and ubiquity: Gorenstein rings*, 2002.
- [14] D. HUYBRECHTS, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2006.
- [15] M. KALCK, N. PAVIC, AND E. SHINDER, *Obstructions to semiorthogonal decompositions for singular threefolds i*, Moscow Mathematical Journal, (2019).
- [16] J. KOLLÁR, ed., *Flips and Abundance for Algebraic Threefolds*, vol. 211 of Astérisque, Société Mathématique de France, Paris, 1992.
- [17] A. KUZNETSOV, *Lefschetz decompositions and categorical resolutions of singularities*, 2006.
- [18] —, *Hochschild homology and semiorthogonal decompositions*, Moscow Mathematical Journal, 9 (2009), pp. 847–868.
- [19] A. KUZNETSOV AND E. SHINDER, *Categorical absorptions of singularities and degenerations*, Épijournal de Géométrie Algébrique, Special volume in honour of... (2024).
- [20] V. A. LUNTS, *Categorical resolution of singularities*, 2009.
- [21] G. OTTAVIANI, *Spinor bundles on quadrics*, Transactions of the American Mathematical Society, 307 (1988), pp. 301–316.
- [22] M. REID, *Young person’s guide to canonical singularities*, Proceedings of Symposia in Pure Mathematics, 46 (1987), pp. 345–414.

- [23] R. VAKIL, *The rising sea: Foundations of algebraic geometry*, draft (november 18, 2017 version), Stanford University, 2017. Early out-of-date version. Available online: <http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf>.
- [24] J.-L. VERDIER AND M. GEORGES, *Descatégories dérivées des catégories abéliennes*, Société mathématique de France, 1996.
- [25] K. WATANABE, *On plurigenera of normal isolated singularities. i*, Mathematische Annalen, 250 (1980), pp. 65–94.