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Los Angeles

Extreme Calabi–Yau Varieties and the Topology of the Hilbert Scheme of Points

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Jas Singh

2026

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2026

ABSTRACT OF THE DISSERTATION

Extreme Calabi–Yau Varieties and the Topology of the Hilbert Scheme of Points

by

Jas Singh
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2026
Professor Burt Totaro, Chair

In Chapter 1, I will survey the foundational theory of K3 surfaces and showcase a proof of the unboundedness of elliptic K3 surfaces through the unboundedness of a discrete invariant known as the minimal multisection degree. This invariant was used crucially in previous boundedness results for elliptic Calabi–Yau varieties.

In Chapter 2, I reproduce my article [59][1] wherein I constructed a family of smooth, projective Calabi–Yau varieties in each dimension with conjecturally extremal (and nevertheless extreme) discrete invariants. Namely, their index and the sum of their Betti numbers grows doubly-exponentially in dimension.

In Chapter 3, I present my approach to a work-in-progress study of the topology of Hilbert schemes of points in affine space, and compute the integral homology of the Hilbert scheme of 5 points in affine space.

[1]First published in *Mathematische Zetischrift*.

The dissertation of Jas Singh is approved.

Romyar Thomas Sharifi

Raphaël Alexis Rouquier

Joaquín Moraga Saez

Burt Totaro, Committee Chair

University of California, Los Angeles

2026

To my mother.

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VITA

2020 B.S. (Mathematics), University of California, Los Angeles.

2022 M.A. (Mathematics), University of California, Los Angeles.

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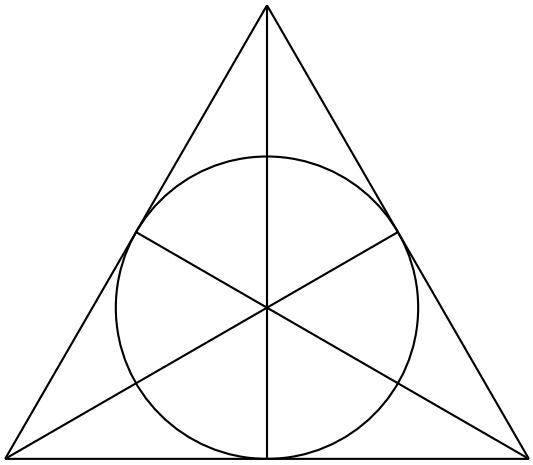
...Yet there have been and still are mathematicians and philosophers who doubt whether the whole universe, or to speak more widely the whole of being, was only created in Euclid's geometry. They even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity.

Ivan Karamazov in *The Brothers Karamozov*

Fyodor Dostoyevsky



CHAPTER 1

Elliptic K3 surfaces

1.1 Introduction

In this chapter, we discuss the unboundedness of elliptic, projective K3 surfaces. Our interest in this study is as a means to understand the major open questions of boundedness of Calabi–Yau varieties. Higher dimensional Calabi–Yau varieties, where some boundedness is expected, will be studied in Chapter 2.

For our purposes, we use the following definition of a *Calabi–Yau* variety. Also, for simplicity, we work over the complex numbers.

Definition 1.1.1. *A normal, projective variety X is* Calabi–Yau *if its canonical divisor satisfies $K_X \sim_{\mathbb{Q}} 0$, i.e. is torsion in the divisor class group of X.*

We also recall the definition of a *K3 surface*.

Definition 1.1.2. *A* K3 surface *is a compact, connected, Kähler manifold X of complex dimension 2 satisfying*

(i) $K_X \sim 0$

(ii) $H^1(X; \mathcal{O}_X) = 0$.

Note here that we are not assuming K3 surfaces to be projective.

In birational geometry and the minimal model program, we study algebraic varieties in a trichotomy based on "positivity" of the canonical divisor. Not all varieties fit into one these three pure types, but it is expected by the minimal model program that varieties with reasonable singularities are all built out of them. See [57] for a more in depth survey of this notion.

For example, let C be a smooth, projective curve of genus g. As the degree of the canonical

divisor of C equals $2g-2$, the degree is:

$$\begin{aligned} \text{negative} &\iff g = 0 \\ \text{zero} &\iff g = 1 \\ \text{positive} &\iff g \geq 2. \end{aligned}$$

This mimics the trichotomy of compact Riemann surfaces due to the uniformization theorem. The universal cover of a compact Riemann surface of genus g is:

$$\begin{aligned} \mathbb{CP}^1 &\iff g = 0 \\ \mathbb{C} &\iff g = 1 \\ \mathbb{D} &\iff g \geq 2. \end{aligned}$$

Similarly, let X be a hypersurface in $\mathbb{P}^n$ of degree d. By the adjunction formula, the canonical divisor of X is:

$$\begin{aligned} \text{anti-ample} &\iff d < n+1 \\ \text{trivial} &\iff d = n+1 \\ \text{ample} &\iff d > n+1. \end{aligned}$$

Calabi–Yau varieties are the boundary case in this essential trichotomy. A major open question is the *(birational) boundedness* of (*strict*) Calabi–Yau varieties in fixed dimension (and with some restriction on the singularities). (A Calabi–Yau variety X, say smooth, is *strict* if it is simply connected and $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim(X)$. We will omit this adjective for brevity.) That is, is there a finite type scheme $\mathcal{Y}$ and a projective morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ so that all Calabi–Yau varieties of dimension n are isomorphic (resp. birational) to a fiber?

This is known for $n = 1$ via the j-invariant. For $n = 2$, this is false, as the degree of a projective K3 surface can be arbitrarily large. For $n \geq 3$, this question is very deep and is an active area of current research. For instance, in [17], Engel, Filipazzi, Greer, Mauri, and Svaldi have recently proven birational boundedness of Calabi–Yau varieties which admit certain fibration structures.

The simplest type of fibration is an *elliptic fibration*.

Definition 1.1.3. *An* elliptic *Calabi–Yau variety is a projective morphism $X \longrightarrow Y$ with connected fibers so that the generic fiber is a smooth genus 1 curve. Such a morphism is called an* elliptic fibration.

A few remarks are in order. For one, a Calabi–Yau variety can admit many (or no) inequivalent elliptic fibrations. Secondly, we are not insisting that the generic fiber, which is a curve over the function field $\mathbb{C}(Y)$, has a $\mathbb{C}(Y)$-rational point. That is to say, the fibration need not admit a rational section. A looser notion we study is the existence of a (rational) *multisection*.

Definition 1.1.4. *A* multisection *of an elliptic fibration $X \longrightarrow Y$ is an irreducible, reduced divisor $D \subseteq X$ which dominates the base Y. The degree of the multisection D is the intersection number $\langle D, F \rangle$ for F a fiber.*

For instance, a multisection of degree 1 is precisely a rational section. An elliptic Calabi–Yau variety need not admit any section, so it's natural to ask what the smallest possible degree a multisection can have is.

Definition 1.1.5. *Let $X \longrightarrow Y$ be an elliptic Calabi–Yau variety with fiber F. The* minimal multisection degree*, denoted* $\mathrm{mmd}(X/Y)$*, is the minimum of the multisection degrees $\langle D, F \rangle$ among all multisections D.*

We note here that there is a potential ambiguity in how our definitions interpret the term *elliptic K3 surface*. We defined elliptic fibrations in the context of Calabi–Yau varieties,

which we defined to be projective. However, we also explicitly stated that K3 surfaces need not be projective. And indeed, non-projective K3 surfaces will play an important role in the coming argument. There are non-projective K3 surfaces which admit elliptic fibrations, but the existence of a multisection forces projectivity. Indeed, if D is a multisection and F is a fiber, we compute

$$(D + kF)^2 = D^2 + 2k\langle D, F\rangle.$$

If k is sufficiently large, this is positive, as $\langle D, F\rangle \geq 1$. As stated in Lemma 1.3.1, the existence of a divisor with positive self-intersection implies projectivity. In general, we will not insist that elliptic K3 surfaces are projective, and will state projectivity when we need it.

The minimal multisection degree has an important relationship to boundedness of elliptic Calabi–Yau varieties. One of the first results on boundedness of Calabi–Yau varieties was Gross' proof of the birational boundedness of elliptic Calabi–Yau threefolds in [29], which uses boundedness of the minimal multisection degree for elliptic Calabi–Yau 3-folds as a key input. In all dimensions $n \geq 3$, Birkar, di Cerbo, and Svaldi have shown birational boundedness of elliptic Calabi–Yau varieties that admit a rational section in [6]. That is, the elliptic Calabi–Yau n-folds $X \longrightarrow Y$ with $\mathrm{mmd}(X/Y) = 1$ form a birationally bounded family.

Recently, in [17], Engel, Filipazzi, Greer, Mauri, and Svaldi have dropped the hypothesis of the existence of a rational section and have shown birational boundedness for fibered Calabi–Yau varieties in every dimension $n \geq 3$. As a result, the minimal multisection degree is uniformly bounded for each $n \geq 3$. Our main result for this section is that this is false in dimension 2.

Theorem 1.1.6. *There is a sequence of elliptic, projective K3 surfaces $X_n \longrightarrow \mathbb{P}^1$ so that $\{\mathrm{mmd}(X_n/\mathbb{P}^1)\}_n$ is unbounded.*

Corollary 1.1.7. *Elliptic K3 surfaces form an unbounded family.*

Birkar, di Cerbo, and Svaldi state the unboundedness of elliptic, projective K3 surfaces

in [6], so this result is not new. Rather, our aim is to provide a full proof of this fact using the minimal multisection degree.

The minimal multisection degree of an elliptic, projective K3 surface is equal to the torsion order of its class in the Tate–Shafarevich group of its Jacobian. Definitions will be given at the end of this chapter, but we note here the resulting corollary.

Corollary 1.1.8. *There is no upper bound to the torsion orders of elements of* $Ш(Y/\mathbb{P}^1)$ *ranging over elliptic, projective K3 surfaces* $Y \longrightarrow \mathbb{P}^1$ *with a section.*

It is this formulation of the minimal multisection degree in terms of the Tate–Shafarevich group that Gross uses in proving birational boundedness of elliptic Calabi–Yau 3-folds. Developing a theory of Tate–Shafarevich groups, and showing their finiteness, is also core to the argument of boundedness of fibered Calabi–Yau varieties in [17].

A stronger result than Corollary 1.1.8 is known.

Theorem 1.1.9. *Let* $X \longrightarrow \mathbb{P}^1$ *be an elliptic, projective K3 surface with a section. Then*

$$Ш(X/\mathbb{P}^1) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}.$$

See [37, Ch. 11, Rmk. 5.13]. For a projective K3 surface X, $1 \leq \rho(X) \leq 20$, so in fact $Ш(X/\mathbb{P}^1)$ has unbounded torsion orders for any individual elliptic, projective K3 surface X with a section.

1.2 Preliminaries

We introduce here the preliminary results on K3 surfaces we use to prove Theorem 1.1.6. Our fundamental reference is Huybrechts' wonderful book [37]. The essence of our construction of the unbounded sequence in Theorem 1.1.6 is the principle that K3 surfaces are determined and controlled by their Hodge theory, formalized for instance, in the surjectivity

of the period map (Theorem 1.2.12). The main results of this section (on the deformation theory of K3 surfaces and the period mapping) will require us to consider analytic K3 surfaces, so we remark here that we are not supposing K3 surfaces to be projective, and will make this explicit when we do.

1.2.1 Examples

We first introduce the easiest families of K3 surfaces to describe.

1. A smooth quartic surface X_4 in $\mathbb{P}^3$ is a K3 surface. This is easy to see, using the adjunction formula and the ideal sheaf exact sequence. We refer to these as *quartic K3 surfaces*. Not all projective K3 surfaces are quartic, but even non-projective K3 surfaces are arbitrarily close to a quartic K3 surface, as discussed after Theorem 1.2.4.

   The quartic surface

   $$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

   is known as the *Fermat* quartic. To make visualization over the real numbers possible, we replace x_i with ζx_i for $i = 1, 2$, where ζ is a primitive 8th root of unity. The equation then becomes

   $$X = \{x_0^4 - x_1^4 = x_2^4 - x_3^4\},$$

   which has real points.

   The equation factors as

   $$(x_0^2 - x_1^2)(x_0^2 + x_1^2) = (x_2^2 - x_3^2)(x_2^2 + x_3^2),$$

   so it follows that the following ratios are equal on X:

   $$\frac{x_0^2 - x_1^2}{x_2^2 + x_3^2} = \frac{x_2^2 - x_3^2}{x_0^2 + x_1^2}.$$

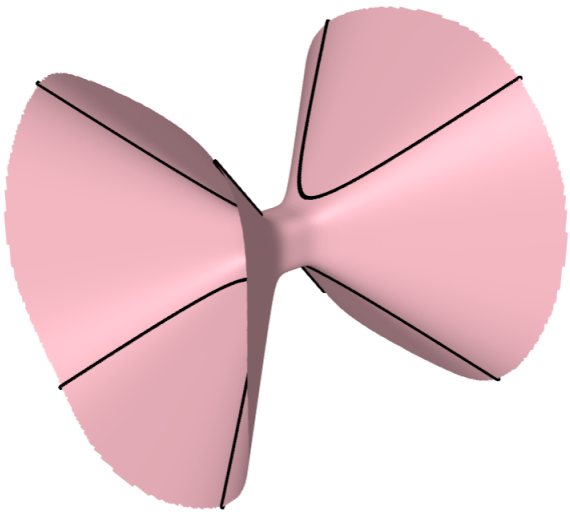


Figure 1.1: The real Fermat quartic with a marked elliptic curve.

We therefore define a morphism

$$X \longrightarrow \mathbb{P}^1$$

$$[x_0 : x_1 : x_2 : x_3] \longmapsto [x_0^2 - x_1^2 : x_2^2 + x_3^2]$$

$$= [x_2^2 - x_3^2 : x_0^2 + x_1^2].$$

The fiber over a point $[a : b]$ is the intersection of two quadrics

$$\{a(x_2^2 + x_3^2) = b(x_0^2 - x_1^2)\} \cap \{a(x_0^2 + x_1^2) = b(x_2^2 - x_3^2)\},$$

which is wholly contained in X. A complete intersection of two quadrics in $\mathbb{P}^3$ is a classic normal form for an elliptic curve, and we have thus defined an elliptic pencil on the Fermat quartic. See Figure 1.1.

2. Let A be an abelian surface and let μ_2 act on A via $x \mapsto -x$. The fixed points of this action are the 2-torsion points, the set of which we denote as $A[2]$. There are 16 of

these, as A is isomorphic to $(S^1)^4$ as a real Lie group. The action of μ_2 on A extends to the blowup at the 2-torsion points $\mathrm{Bl}_{A[2]}A$. The quotient

$$X = \frac{\mathrm{Bl}_{A[2]}A}{\mu_2}$$

is a projective K3 surface, referred to as a *Kummer surface*.

$$\begin{array}{ccc} \mathrm{Bl}_{A[2]}A & \xrightarrow{b} & A \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

Smoothness of X can be computed locally. To show that ω_X is trivial, we take a holomorphic 2-form ω on A which vanishes nowhere and is invariant under the group law. Then $b^*\omega$ is nowhere vanishing and invariant under the μ_2 action. It is therefore pulled back from a nowhere vanishing 2-form on X, which can be shown to be holomorphic. As for the vanishing of $H^1(X;\mathcal{O}_X)$, we use that $H^0(A;\Omega^1)^{\mu_2} = 0$. See [3, Ch. VIII, Prop. 11] for details.

Principally polarized abelian surfaces are either the Jacobian of a genus 2 curve, or the product of two elliptic curves. (A proof is given in [7, Ch. 11, Cor 8.2].) If A is the Jacobian of a genus 2 curve, then A/μ_2 can be embedded in $\mathbb{P}^4$ as a quartic whose singular set is precisely 16 nodes. This is classically known as *the* Kummer surface, and its minimal resolution is the Kummer surface associated to A as just defined. See Figure 1.2. On the other hand, if $A = E_1 \times E_2$, then the associated Kummer surface X comes with two projection maps $X \longrightarrow E_i/\mu_2 \cong \mathbb{P}^1$. So X is an elliptic K3 surface.

3. Consider a pencil of plane cubics $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with smooth generic fiber. The base locus B of this pencil consists of 9 points, being the intersection of two cubics. Let $b : \mathrm{Bl}_B\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at these 9 points. There is a morphism

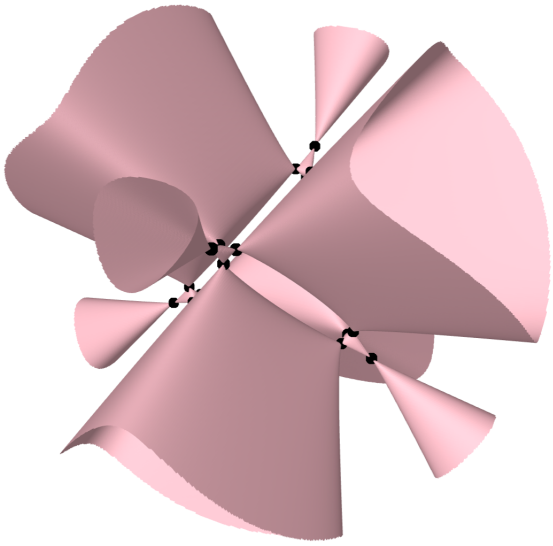


Figure 1.2: The Kummer surface with its 16 marked nodes.

$p : \mathrm{Bl}_B\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ resolving the indeterminacy of ψ.

$$\begin{array}{ccccc} & & \mathrm{Bl}_B\mathbb{P}^2 & & \\ & \overset{b}{\swarrow} & & \overset{p}{\searrow} & \\ \mathbb{P}^2 & & \overset{\psi}{\dashrightarrow} & & \mathbb{P}^1 \end{array}$$

Take two distinct points $a, b \in \mathbb{P}^1$ and let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the double cover branched over $\{a, b\}$. Let $\pi : X \longrightarrow \mathbb{P}^1$ be the pullback of p along f.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathrm{Bl}_B\mathbb{P}^2 \\ {\scriptstyle\pi}\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow[f]{} & \mathbb{P}^1 \end{array}$$

We claim $\pi : X \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface. That π has elliptic fibers is clear, as $\pi^{-1}(t) = p^{-1}(f(t))$, which is a plane cubic.

As for the canonical divisor, ϕ is a double cover branched over $F_a \coprod F_b$, the disjoint

union of two fibers. We have therefore that

$$K_X \sim \phi^* \left(K_{\mathrm{Bl}_B P^2} + \frac{1}{2}(F_a + F_b) \right).$$

Let $E_1, \ldots, E_9$ be the exceptional divisors of $b : \mathrm{Bl}_B \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ and $E = \sum E_i$. Then

$$\begin{aligned} K_{\mathrm{Bl}_B \mathbb{P}^2} &\sim b^*(K_{\mathbb{P}^2}) + E \\ &\sim b^*(\mathcal{O}(-3)) + E. \end{aligned}$$

Furthermore, $F_a \sim F_b \sim b^*(\mathcal{O}(3)) - E$. It follows that

$$\begin{aligned} K_X &\sim \phi^* \left(K_{\mathrm{Bl}_B \mathbb{P}^2} + \frac{1}{2}(F_a + F_b) \right) \\ &\sim \phi^*(K_{\mathrm{Bl}_B \mathbb{P}^2} + (b^*(\mathcal{O}(3)) - E)) \\ &\sim \phi^*((b^*(\mathcal{O}(-3)) + E) + (b^*(\mathcal{O}(3)) - E)) \\ &\sim \phi^*(0) \\ &\sim 0. \end{aligned}$$

We are left to show $H^1(X; \mathcal{O}_X) = 0$. The Leray spectral sequence for $\pi : X \longrightarrow \mathbb{P}^1$ converges as follows:

$$E_2^{ij}(X) = H^i(\mathbb{P}^1; R^j \pi_* \mathcal{O}_X) \implies H^{i+j}(X; \mathcal{O}_X).$$

As π has connected fibers, $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$. Hence, $E_2^{1,0} = 0$. We now seek to show $E_2^{0,1} = 0$ as well. Indeed, $E_2^{0,1} = H^0(\mathbb{P}^1; R^1 \pi_* \mathcal{O}_X)$. At $t \in \mathbb{P}^1$, the fiber of $R^1 \pi_* \mathcal{O}_X$ is $H^1(\pi^{-1}(t); \mathcal{O}_{\pi^{-1}(t)})$. The dimension of this is the arithmetic genus of the fiber, which is 1, as the fibers are plane cubics. $R^1 \pi_* \mathcal{O}_X$ is therefore a line bundle on $\mathbb{P}^1$. To show that $H^0(\mathbb{P}^1; R^1 \pi_* \mathcal{O}_X) = 0$, it suffices to show that the degree of $R^1 \pi_* \mathcal{O}_X$ is negative.

For all $t \in \mathbb{P}^1$, $\pi^{-1}(t) = p^{-1}(f(t))$, so $R^1\pi_*\mathcal{O}_X = f^*(R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2})$. Hence,

$$\begin{aligned}\deg(R^1\pi_*\mathcal{O}_X) &= \deg(f^*(R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2})) \\ &= \deg(f)\deg(R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) \\ &= 2\deg(R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}).\end{aligned}$$

To show that $\deg(R^1\pi_*\mathcal{O}_X) < 0$, it suffices to show that $\deg(R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) < 0$. Equivalently, that $H^0(\mathbb{P}^1; R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) = 0$.

Indeed, the Leray spectral sequence applied to $p : \mathrm{Bl}_B\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ yields the convergence

$$E_2^{ij}(\mathrm{Bl}_B\mathbb{P}^2) = H^i(\mathbb{P}^1; R^jp_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) \implies H^{i+j}(\mathrm{Bl}_B\mathbb{P}^2; \mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}),$$

so we are trying to show $E_2^{0,1}(\mathrm{Bl}_B\mathbb{P}^2) = 0$. In this case we understand the E_∞ page. $h^{1,0}$ is a birational invariant, so $H^1(\mathrm{Bl}_B\mathbb{P}^2; \mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) = H^1(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}) = 0$. Similarly to above, $E_2^{1,0}(\mathrm{Bl}_B\mathbb{P}^2) = 0$ as p has connected fibers. We claim furthermore that the differential $d_2^{0,1}$ coming out of $E_2^{0,1}(\mathrm{Bl}_B\mathbb{P}^2)$ is 0. Indeed, as $\mathbb{P}^1$ is 1-dimensional, no sheaf in the Zariski topology has cohomology above degree 1 by Grothendieck's vanishing theorem [30, Thm. 3.6.5]. Hence, $E_2^{ij} = 0$ for $i \geq 2$. The codomain of $d_2^{0,1}$ is $E_2^{2,0} = 0$, so $d_2^{0,1} = 0$. Also, the differential going into $E_2^{0,1}$ is $E_2^{-2,2} \longrightarrow E_2^{0,1}$, and the domain is 0, so $d_2^{-2,2} = 0$.

We have shown that $E_2^{0,1}(\mathrm{Bl}_B\mathbb{P}^2)$ has trivial differentials going into and out of it. Additionally, $E_2^{1,0}(\mathrm{Bl}_B\mathbb{P}^2) = 0$. These two terms are hence the associated graded pieces to a filtration of $H^1(\mathrm{Bl}_B\mathbb{P}^2; \mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) = 0$. It can only be, then, that $H^0(\mathbb{P}^1; R^1p_*\mathcal{O}_{\mathrm{Bl}_B\mathbb{P}^2}) = E_2^{0,1}(\mathrm{Bl}_B\mathbb{P}^2) = 0$.

It follows that $H^0(\mathbb{P}^1; R^1\pi_*\mathcal{O}_X) = 0$. As we already showed $E_2^{1,0}(X) = 0$, we conclude that $H^1(X; \mathcal{O}_X) = 0$.

1.2.2 Lattices

Definition 1.2.1. *A* lattice *is a finitely-generated free abelian group* Λ *with a nondegenerate, symmetric bilinear form*

$$\langle -, - \rangle : \Lambda \otimes \Lambda \longrightarrow \mathbb{Z}.$$

We denote

$$\lambda^2 = \langle \lambda, \lambda \rangle$$

for convenience.

A K3 surface X is a compact, connected, complex manifold of dimension 2. As complex manifolds admit canonical orientations, there is a canonical identification $H^4(X;\mathbb{Z}) = \mathbb{Z}$. The cup product therefore induces a symmetric bilinear form

$$H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \xrightarrow{\cup} H^4(X;\mathbb{Z}) = \mathbb{Z}.$$

We claim that $H^2(X;\mathbb{Z})$ is a lattice. Compactness ensures it is finitely generated. As for torsion, consider a prime ℓ and the Kummer exact sequence

$$1 \longrightarrow \mu_\ell \longrightarrow \mathcal{O}^* \xrightarrow{(-)^\ell} \mathcal{O}^* \longrightarrow 1,$$

which induces the exact sequence on cohomology

$$H^1(X;\mu_\ell) \longrightarrow H^1(X;\mathcal{O}^*) \xrightarrow{\cdot \ell} H^1(X;\mathcal{O}^*).$$

We will show $H^1(X;\mu_\ell) \cong H^1(X;\mathbb{Z}/\ell\mathbb{Z}) = 0$. Indeed, the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 1$$

induces the exact sequence

$$H^0(X;\mathcal{O}) \longrightarrow H^0(X;\mathcal{O}^*) \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow H^1(X;\mathcal{O}).$$

The map $H^0(X;\mathcal{O}) \longrightarrow H^0(X;\mathcal{O}^*)$ is identified with $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$, which is surjective. Hence, the map $H^1(X;\mathbb{Z}) \longrightarrow H^1(X;\mathcal{O})$ is injective. By definition of a K3 surface, $H^1(X;\mathcal{O}) = 0$, so $H^1(X;\mathbb{Z}) = 0$. Then, $H^1(X;\mu_\ell) \cong H^1(X;\mathbb{Z}/\ell\mathbb{Z}) = 0$, and it follows that multiplication by ℓ on $H^1(X;\mathcal{O}^*)$ is injective for all primes ℓ. So $H^1(X;\mathcal{O}^*)$ is torsion free. Looking again at the exponential exact sequence, we have

$$H^1(X;\mathcal{O}^*) \longrightarrow H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathcal{O}).$$

We have just shown that $H^1(X;\mathcal{O}^*)$ is torsion-free. $H^2(X;\mathcal{O})$ is a complex vector space, and is hence torsion free. It follows that $H^2(X;\mathbb{Z})$ is also torsion free, and is hence a lattice when equipped with the cup product pairing. In fact, this lattice is independent of the choice of K3 surface X, and we refer to this as the *K3 lattice* Λ_{K3}.

The structure of the K3 lattice is fully known. Let U be the hyperbolic plane, i.e. the group $\mathbb{Z}^2$ with bilinear form given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The E_8 lattice is $\mathbb{Z}^8$ with bilinear form given by

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

Given a lattice Λ and a nonzero integer n, we let $\Lambda(n)$ be the lattice given by the same underlying group and with bilinear form

$$\langle -, - \rangle_{\Lambda(n)} = n \langle -, - \rangle_{\Lambda}.$$

Proposition 1.2.2. *The K3 lattice Λ_{K3} is isomorphic to*

$$E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

See [37, Ch. 1, Prop. 3.5] for the proof, which uses abstract lattice theory. Note that some authors define E_8 the way we defined $E_8(-1)$. We use this convention to align with Huybrechts.

In fact, a much stronger result is known.

Definition 1.2.3. *Two compact, complex manifolds X_0, X_1 are* deformation equivalent *if there is a smooth, proper family $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ over a connected base $\mathcal{B}$ with points b_0, b_1 so that $\pi^{-1}(b_0) \cong X_0$ and $\pi^{-1}(b_1) \cong X_1$.*

Ehresmann's fibration theorem states that a C^∞, proper family over a smooth base is

locally *diffeomorphically* trivial. Hence, if two compact, complex manifolds X_0 and X_1 are deformation equivalent, we can pick a C^∞ curve connecting the two points b_0 and b_1 in $\mathcal{B}$ as above, and conclude via Ehresmann's theorem that X_0 and X_1 are diffeomorphic.

Theorem 1.2.4 (Kodaira)**.** *Any two K3 surfaces are deformation equivalent.*

This is due to Kodaira in [47, Thm. 13]. A proof is also given in [37, Ch. 7, Thm 1.1]. The arguments differ, but both use the Local Torelli Theorem (Theorem 1.2.8) to show that any given K3 surface is arbitrarily close (in the sense of a deformation over a small disk) to a smooth quartic surface in $\mathbb{P}^3$. Kodaira argues first that all K3 surfaces are deformation equivalent to an elliptic K3 surface with controlled singular fibers (type I_1 or II), whereas Huybrechts characterizes quartic surfaces in the period domain and appeals to their density therein. As all smooth quartic surfaces are deformation equivalent, the theorem is proven.

Hence, all K3 surfaces are diffeomorphic, and a K3 surface is determined by a choice of holomorphic structure on the same underlying real 4-manifold. This extra structure yields further lattices within the K3 lattice itself, such as the Picard group $\mathrm{Pic}(X)$. In general the Picard group can be very large, with its connected component at the identity $\mathrm{Pic}^0(X)$ being an abelian variety. Positive dimensional abelian varieties are not finite rank abelian groups. However, the exponential exact sequence yields

$$H^1(X;\mathcal{O}) \longrightarrow H^1(X;\mathcal{O}^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}),$$

and by definition of a K3 surface, $H^1(X,\mathcal{O}) = 0$. Hence, $\mathrm{Pic}(X)$ lives inside of $H^2(X;\mathbb{Z}) \cong \Lambda_{\mathrm{K3}}$ and is therefore a lattice. This shows that the Picard and Néron–Severi groups are equal. [37, Ch. 1, Prop. 2.4] shows further that linear and numerical equivalence agree for projective K3 surfaces.

1.2.3 Hodge theory

The Hodge numbers of every K3 surface can be computed as follows.

Proposition 1.2.5. *The Hodge diamond of every K3 surface is the following.*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof. By symmetry, we need only compute $h^{p,q}(X)$ for

(i) $(0,0)$

(ii) $(1,0)$

(iii) $(0,2)$

(iv) $(1,1)$

Indeed,

(i) $h^{0,0}(X) = h^0(X, \mathcal{O}) = 1$.

(ii) $h^{1,0}(X) = h^1(X, \mathcal{O}) = 0$, by definition of a K3 surface.

(iii) $h^{0,2}(X) = h^0(X, \Omega_X^2) = h^0(X, \mathcal{O}) = 1$, as $\Omega_X^2 = K_X$ is trivial.

So far, we have the following Hodge diamond.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The sums of the rows of the Hodge diamond are the Betti numbers. It follows that $h^{1,1} = b_2 - 2$. Furthermore, $b_0 = 1$, $b_4 = 1$, and the odd Betti numbers vanish. Hence, we have

$$\chi_{\text{top}}(X) = 4 + h^{1,1}.$$

We proceed by Noether's formula:

$$\chi(X, \mathcal{O}) = \frac{K_X^2 + \chi_{\text{top}}(X)}{12}.$$

The Hodge numbers we computed above show that $\chi(X, \mathcal{O}) = 1 - 0 + 1 = 2$. And by definition, $K_X \sim 0$. Hence, $\chi_{\text{top}}(X) = 24$ and $h^{1,1} = 20$. □

A K3 surface X has a pure Hodge structure of weight 2 on $H^2(X; \mathbb{Z})$, which can thus be carried to a Hodge structure on Λ_{K3}. As with any Hodge structure, $H^{2,0}(X) = \overline{H^{0,2}(X)}$. Moreover, $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X)$ and $H^{0,2}(X)$ under the cup product pairing $\langle -, - \rangle$. This Hodge structure is thus determined by the line $H^{2,0}(X) \subseteq H^2(X; \mathbb{C}) \cong \Lambda_{\text{K3}} \otimes \mathbb{C}$. These Hodge structures are parametrized by the relevant *period domain*.

Definition 1.2.6. *The* period domain *of the K3 lattice is*

$$\mathcal{D} = \{\ell \in \mathbb{P}(\Lambda_{\text{K3}} \otimes \mathbb{C}) \mid \textit{ for all nonzero } x \in \ell, x^2 = 0 \textit{ and } \langle x, \overline{x} \rangle > 0\}.$$

The condition $x^2 = 0$ cuts out a quadric hypersurface in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \cong \mathbb{P}^{21}$. This quadric is smooth, as $\langle -, - \rangle$ on the K3 lattice is nondegenerate. The condition $\langle x, \overline{x} \rangle > 0$ yields an open subset (in the analytic topology) of this quadric. $\mathcal{D}$ is therefore a 20-dimensional complex manifold.

Given a K3 surface X and a fixed isometry $\phi : H^2(X; \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$, we have an associated point $\phi(H^{2,0}(X))$ in the period domain $\mathcal{D}$. If we perturb X slightly, we may study how its image in the period domain varies.

Definition 1.2.7. *A* deformation *of a K3 surface X is a smooth, proper family $\pi : \mathcal{X} \longrightarrow \mathbb{D}$, where $\mathbb{D}$ is a complex disk of some dimension centered at the origin, such that $\mathcal{X}_0 = X$ and all other fibers are also K3 surfaces.*

(For such a family, we denote $\mathcal{X}_t = \pi^{-1}(t)$.) The condition that all other fibers are K3 surfaces is automatic for sufficiently small disks $\mathbb{D}$, as in the proof of [37, Ch. 5, Prop. 2.1].

We can assemble the cohomology of all the fibers $H^2(\mathcal{X}_t; \mathbb{Z})$ of a deformation $\pi : \mathcal{X} \longrightarrow \mathbb{D}$ together into $R^2\pi_*\mathbb{Z}$ on $\mathbb{D}$, which is a locally constant sheaf (by Ehresmann's theorem) on $\mathbb{D}$ with stalk at $t \in \mathbb{D}$ equal to $H^2(\mathcal{X}_t; \mathbb{Z})$. As $\mathbb{D}$ is connected, each stalk is isomorphic to $H^2(\mathcal{X}_0; \mathbb{Z}) = H^2(X; \mathbb{Z})$. As $\mathbb{D}$ is simply connected, this isomorphism is, in fact, canonical, and the local system $R^2\pi_*\mathbb{Z}$ is constant over $\mathbb{D}$ with value $H^2(X; \mathbb{Z})$. We therefore identify

$$H^2(\mathcal{X}_t; \mathbb{Z}) = H^2(X; \mathbb{Z}) \xrightarrow[\sim]{\phi} \Lambda_{\mathrm{K3}}.$$

In other words, ϕ extends to an isomorphism of local systems

$$R^2\pi_*\mathbb{Z} \xrightarrow[\sim]{\phi} \underline{\Lambda_{\mathrm{K3}}},$$

where $\underline{\Lambda_{\mathrm{K3}}}$ is the constant local system over $\mathbb{D}$ with value Λ_{K3}.

What is not constant, however, is the Hodge structure on this lattice. Indeed, we form $(R^2\pi_*\mathbb{Z}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{D}}$, which is a holomorphic vector bundle with fiber over $t \in \mathbb{D}$ equal to $H^2(\mathcal{X}_t; \mathbb{C})$.

This comes with a holomorphic subbundle $\pi_*\Omega^2_{\mathcal{X}/\mathbb{D}} \subseteq (R^2\pi_*\mathbb{Z}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{D}}$, which fiberwise yields the inclusion $H^{2,0}(\mathcal{X}_t) \subseteq H^2(\mathcal{X}_t; \mathbb{C})$. This is one piece of a *variation of Hodge structures*, and we therefore receive a *holomorphic* map

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\mathcal{P}} & \mathcal{D} \\ t & \longmapsto & \phi(H^{2,0}(\mathcal{X}_t)), \end{array}$$

which we refer to as the (local) *period map*. The theory of period maps was developed by Griffiths in his seminal papers [25, 26, 27].

The condition that $K_X \sim 0$ says that any K3 surface X has a nowhere vanishing holomorphic 2-form ω. As it is nowhere vanishing, the pairing it defines on holomorphic vector fields is nondegenerate, so we have an isomorphism $T_X \xrightarrow{\sim} \Omega_X$ via $v \mapsto \omega(v, -)$. Using the Hodge diamond for a K3 surface, as computed in Proposition 1.2.5, this implies that $H^0(X; T_X) = H^2(X; T_X) = 0$, and $h^1(X; T_X) = 20$. It follows from the general principles of deformation theory (see, for instance, [37, Ch. 6, §2.3]) that there is a *universal* deformation $\pi_{\text{univ}} : \mathcal{X}_{\text{univ}} \longrightarrow \mathbb{D}_{\text{univ}}$ over a small complex ball in $\mathbb{C}^{20}$. (A deformation is *universal* if for every other deformation $\pi : \mathcal{X} \longrightarrow \mathbb{D}$, there is a unique holomorphic map $f : \mathbb{D} \longrightarrow \mathbb{D}_{\text{univ}}$ sending $0 \mapsto 0$ so that $\mathcal{X}$ is the pullback of π_{univ} along f.)

$$\begin{array}{ccc} \mathcal{X} & \dashrightarrow & \mathcal{X}_{\text{univ}} \\ {\scriptstyle\pi}\downarrow & \ulcorner & \downarrow{\scriptstyle\pi_{\text{univ}}} \\ \mathbb{D} & \xrightarrow[\exists! f]{} & \mathbb{D}_{\text{univ}} \end{array}$$

There is therefore a period map associated to the universal deformation $\mathcal{P} : \mathbb{D}_{\text{univ}} \longrightarrow \mathcal{D}$.

Theorem 1.2.8 (Local Torelli Theorem)**.** *The period map $\mathcal{P} : \mathbb{D}_{\text{univ}} \longrightarrow \mathcal{D}$ is a local biholomorphism.*

This is proven in [37, Ch. 6, Prop. 2.8]. It suffices to show that the derivative is nonsingular, which follows from the general theory of period maps due to Griffiths (namely,

the so-called *Griffiths transversality*).

The discussion above can be globalized to the *moduli space of marked K3 surfaces*.

Definition 1.2.9. *A* marked *K3 surface is a pair* (X, ϕ) *where* X *is a K3 surface and* $\phi : H^2(X;\mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ *is an isometry.*

It's necessary for the global theory to consider a relative form of this definition.

Definition 1.2.10. *Let* $\mathcal{B}$ *be a complex analytic space. A* marked K3 surface over $\mathcal{B}$ *is a pair*

$$\left(\pi : \mathcal{X} \longrightarrow \mathcal{B}, \phi : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda_{\mathrm{K3}}}\right)$$

where $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ *is a smooth, proper family of K3 surfaces and* $\phi : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda_{\mathrm{K3}}}$ *is an isometry.*

With this, we can define the relevant moduli functor.

Definition 1.2.11. *Let* $\mathsf{ComplexAnalyticSpaces}$ *be the category of complex analytic spaces. We define the* moduli functor of marked K3 surfaces *as* $\mathcal{M}_{\mathrm{K3}} : \mathsf{ComplexAnalyticSpaces}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$ *via*

$$\mathcal{B} \mapsto \{\textit{marked K3 surfaces over } \mathcal{B}\}/\cong.$$

By gluing the universal deformations $\mathcal{X}_{\mathrm{univ}} \longrightarrow \mathbb{D}_{\mathrm{univ}}$, along with their markings, we form the moduli space of marked K3 surfaces M_{K3} with its universal family $\pi_{\mathrm{univ}} : U \longrightarrow M_{\mathrm{K3}}$ and marking $\phi_{\mathrm{univ}} : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \underline{\Lambda_{\mathrm{K3}}}$. The pair $(\pi_{\mathrm{univ}}, \phi_{\mathrm{univ}})$ represents the moduli functor $\mathcal{M}_{\mathrm{K3}}$. The universal deformations provide charts proving that M_{K3} is a 20-dimensional complex manifold, but crucially it is *not* Hausdorff. See [37, Ch. 6, §3]. A brief example of the principle of moduli functors and representability is given in Section 3.1.

We have therefore the *global* period map:

$$M_{\mathrm{K3}} \xrightarrow{\mathcal{P}} \mathcal{D}$$

$$(X, \phi) \longmapsto \phi(H^{2,0}(X)).$$

An incredible result due to Todorov in [62, Thm. 1] is that the global period map is surjective.

Theorem 1.2.12 (Surjectivity of the period map)**.** *$\mathcal{P} : M_{\mathrm{K3}} \longrightarrow \mathcal{D}$ is surjective.*

See [37, Ch. 6, Thm. 3.1] for further discussion. By the Local Torelli Theorem (Theorem 1.2.8), this map is a local biholomorphism.

This provides a powerful tool to define K3 surfaces by simply defining an element of $\mathcal{D}$, which is a matter purely of the Hodge theory of Λ_{K3}. We will define the elliptic, projective K3 surfaces of Theorem 1.1.6 in exactly this fashion.

1.3 Construction of elliptic K3 surfaces

We now determine in lattice-theoretic terms when a K3 surface is projective, elliptic, and how to determine the minimal multisection degree. Namely, we will interpret these properties in terms of the intersection theory of the K3 surface, and thereby in terms of how the Picard group sits in the K3 lattice.

First of all, we can determine when a K3 surface is projective from its Picard lattice.

Lemma 1.3.1. *A K3 surface is projective if and only if there is some $L \in \mathrm{Pic}(X)$ so that $L^2 > 0$.*

This holds for any compact, complex surface; see [2, Ch. IV, Thm. 6.2]. Note that such an L need not itself be ample, even if it is effective, in case the surface contains negative curves. If X is an elliptic K3 surface with a section, let F be the fiber and C be the section. Then

$F^2 = 0$, $\langle F, C\rangle = 1$, and $C^2 = -2$. We can compute $(C+2F)^2 = 2 > 0$, but $\langle C + 2F, C\rangle = 0$, so $C + 2F$ is not ample, even though it is effective with positive square.

We now determine when a projective K3 surface X admits an elliptic fibration. Suppose there is an elliptic fibration $\pi : X \longrightarrow \mathbb{P}^1$. For the class of a fiber F, we have that $F^2 = 0$ and $F \neq 0$. The existence of such an $F \in \mathrm{Pic}(X)$ is thus necessary for the existence of an elliptic fibration on X. Huybrechts proves the converse in [37, Ch. 11, Prop 1.3(i)].

Lemma 1.3.2. *A projective K3 surface X admits an elliptic fibration if and only if there is a nontrivial $L \in \mathrm{Pic}(X)$ so that $L^2 = 0$.*

Note that L may not itself be the fiber of an elliptic fibration, as indeed the complete linear system $|L|$ need not necessarily define a morphism, or even be nef. For example, consider again an elliptic K3 surface with a section with fiber F and section C. Let $L = F + C$. Then $L^2 = 0$ and L is effective. However, L is not nef, as $\langle L, C\rangle = -1$, so it cannot be the fiber of an elliptic pencil.

If we began with an effective L so that $L^2 = 0$ but L is not nef, then there must be some (-2)-curve so that $\langle L, C\rangle < 0$. We may reflect L about this curve

$$L' = L + \langle L, C\rangle C$$

so that $\langle L', C\rangle = -\langle L, C\rangle > 0$ and $(L')^2 = 0$. The proof of Lemma 1.3.2 goes by continuing to reflect about (-2)-curves until we reach a nef, effective divisor L with $L^2 = 0$ to form our elliptic pencil.

Consider now the lattice $U(n)$ for $n \geq 1$. That is, the group $\mathbb{Z}^2$ with bilinear form induced by the matrix

$$\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$$

We have that $(1,1)^2 = 2n > 0$ and $(1,0)^2 = 0$, so a K3 surface with $\mathrm{Pic}(X) \cong U(n)$ is projective and elliptic.

Proposition 1.3.3. *There exists a K3 surface X_n with* $\mathrm{Pic}(X_n) \cong U(n)$.

In [37, Ch. 14, Cor. 3.1], Huybrechts uses abstract lattice theory to show, in fact, that for any even lattice Λ with signature $(1, \rho - 1)$ for $\rho \leq 10$, there exists a projective K3 surface with $\mathrm{Pic}(X) \cong \Lambda$, and that the primitive embedding $\Lambda \longrightarrow \Lambda_{\mathrm{K3}}$ is unique up to $O(\Lambda_{\mathrm{K3}})$. We can avoid the lattice theory by directly computing a primitive embedding $U(n) \longrightarrow \Lambda_{\mathrm{K3}}$, and then following along with the remainder of the argument.

Proof of Proposition 1.3.3. $U(n)$ has a basis e_1, e_2 satisfying

$$\begin{pmatrix} e_1^2 & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & e_2^2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}.$$

We must therefore find two primitive vectors $v_1, v_2 \in \Lambda_{\mathrm{K3}}$ with the same numbers as above. There are no nontrivial solutions to $v^2 = 0$ in $E_8(-1)$, so we need only work in the $U^{\oplus 3}$ part of $\Lambda_{\mathrm{K3}} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. We define the primitive vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ n-1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

from which it is easy to compute

$$\begin{pmatrix} v_1^2 & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & v_2^2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}.$$

We have thus formed primitive embeddings

$$U(n) \hookrightarrow U^{\oplus 3} \hookrightarrow \Lambda_{\mathrm{K3}}.$$

We now seek to use surjectivity of the period map (Theorem 1.2.12) to show the existence of our desired K3 surface X_n. To do so, we must define a Hodge structure on Λ_{K3}. Recall that for any Kähler manifold X, $c_1(\mathrm{Pic}(X)) \subseteq H^{1,1}(X)$, so we certainly must place $U(n)$ in bidegree $(1,1)$. We let $T = U(n)^\perp$ in Λ_{K3} and pick any weight 2 Hodge structure on T with Hodge numbers $T^{0,2} = T^{2,0} = 1$, $T^{1,1} = 18$, and all others 0. Then $\Lambda_{\mathrm{K3}} = U(n) \oplus T$ inherits a Hodge structure.

From surjectivity of the period map, it follows that there exists a K3 surface X_n with $\phi : H^2(X;\mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ a *Hodge* isometry, for the Hodge structure on Λ_{K3} we just defined. We have by definition that $U(n) \subseteq \Lambda^{1,1}_{\mathrm{K3}}$. Hence, $\phi^{-1}(U(n)) \subseteq H^{1,1}(X;\mathbb{Z})$. By the Lefschetz $(1,1)$ theorem, $c_1(\mathrm{Pic}(X_n)) = H^{1,1}(X;\mathbb{Z})$. We have therefore formed a primitive embedding

$$U(n) \hookrightarrow \mathrm{Pic}(X_n).$$

To assure that this inclusion is an equality, it suffices to have $T^{1,1} \cap \Lambda_{\mathrm{K3}} = 0$. This can be assured for a very general Hodge structure on T, which is therefore exactly equal to the transcendental lattice $T(X) = \mathrm{Pic}(X)^\perp$. □

Proof of Theorem 1.1.6. Take the K3 surface X_n from Proposition 1.3.3 with $\mathrm{Pic}(X_n) \cong U(n)$. By Lemma 1.3.1 and Lemma 1.3.2, X_n is projective and elliptic. We will show that for any elliptic fibration $X_n \longrightarrow \mathbb{P}^1$ with fiber F and any multisection D that $\langle F, D\rangle \geq n$.

Via the isomorphism $\mathrm{Pic}(X_n) \cong U(n)$, we may write F as $(x,y) \in \mathbb{Z}^2$. As F is a fiber, it satisfies $F^2 = 0$. Thus, we have

$$0 = F^2 = 2nxy,$$

so that either $x = 0$ or $y = 0$. Suppose now that D is a multisection of $X_n \longrightarrow \mathbb{P}^1$. We may write it as $(a, b) \in \mathbb{Z}^2$ and compute

$$\langle F, D \rangle = n(ay + bx)$$

As $\langle F, D \rangle \geq 1$ (being the intersection of two distinct effective curves) and $ay+bx$ is an integer, it must be that $\langle F, D \rangle \geq n$. Hence, $\mathrm{mmd}(X_n/\mathbb{P}^1) \geq n$ for any elliptic fibration $X_n \longrightarrow \mathbb{P}^1$. This construction was valid for any $n \geq 1$, so $\{\mathrm{mmd}(X_n/\mathbb{P}^1)\}_n$ is unbounded. □

Any elliptic, projective K3 surface $X \longrightarrow \mathbb{P}^1$ defines an associated Jacobian $J(X) \longrightarrow \mathbb{P}^1$, which is an elliptic, projective K3 surface with a section. $J(X)$ is defined to be the unique relatively minimal smooth model of the function field of $\mathrm{Jac}(E/\mathbb{C}(\mathbb{P}^1))$, where E is the generic fiber of $X \longrightarrow \mathbb{P}^1$.

Given any elliptic, projective K3 surface $X \longrightarrow \mathbb{P}^1$ with a section, let U be the open subset of X which is smooth over $\mathbb{P}^1$. That is, remove from each fiber X_t the points which are singular in the curve X_t. Then $U \longrightarrow \mathbb{P}^1$ is a group scheme. The *Tate–Shafarevich group* of $X/\mathbb{P}^1$ is the set of torsors of this group scheme, denote $Ш(X/\mathbb{P}^1)$. Equivalently, it is the set of pairs (X', ϕ) with X' an elliptic, projective K3 surface and $\phi : X \xrightarrow{\sim} J(X')$ an isomorphism over $\mathbb{P}^1$.

As discussed in [37, Ch. 11, Rmk. 5.9], the minimal multisection degree of $X'/\mathbb{P}^1$ (referred to as the *index* therein) is exactly the torsion order of $[(X', \phi)]$ in $Ш(X/\mathbb{P}^1)$. Unboundedness of minimal multisection degrees thus implies unboundedness of the set of torsion orders of Tate–Shafarevich groups of elliptic K3 surfaces with a section, as stated in Corollary 1.1.8. As discussed in Theorem 1.1.9, the Tate–Shafarevich group of any one elliptic, projective K3 surface already has unbounded torsion orders.

CHAPTER 2

Smooth Calabi–Yau varieties with large index and Betti numbers

We reproduce the article [59], first published in *Mathematische Zeitschrift*, with minor changes in formatting and some corrected typos.

2.1 Introduction

We work over the complex numbers. A normal, projective variety X is called *Calabi–Yau* if its canonical divisor satisfies $K_X \sim_{\mathbb{Q}} 0$. In other words, K_X is a torsion element in the divisor class group of X. Note that other authors insist that Calabi–Yau varieties are simply connected, or that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. We do not insist on these additional constraints.

The index of a Calabi–Yau variety X is the torsion order of K_X, i.e. the smallest positive m so that $mK_X \sim 0$. We are interested in how large the index of smooth, projective Calabi–Yau n-folds can be.

Let s_n be Sylvester's sequence, defined by $s_0 = 2$, $s_{n+1} = s_0 \cdots s_n + 1$, which grows doubly exponentially. Our first main result is the following.

Theorem 2.1.1. *For $n \geq 1$ there exists a smooth, projective Calabi–Yau n-fold $V^{(n)}$ with index $(s_{n-1} - 1)(2s_{n-1} - 3)$.*

n	index($V^{(n)}$)
1	1
2	6
3	66
4	3486
5	6521466
6	21300104111286
7	226847426110811738551148466

Table 2.1: Extremely large indices of smooth, projective Calabi–Yau varieties

In small dimensions, these examples are known to have the largest possible index, as proven for 3-folds in [49, Corollary 5]. For instance, in dimension 1, any elliptic curve will do. In dimension 2, $V^{(2)} = (E_1 \times E_2)/\mu_6$, where E_1 is the elliptic curve whose automorphism group is the cyclic group μ_6 (which is the largest possible), and where E_2 is any elliptic curve acted on by μ_6 via translation by a 6-torsion point. In dimension 3, $V^{(3)} = (S \times E)/\mu_{66}$, where S is the K3 surface with a non-symplectic automorphism of order 66 (which is the largest possible), and E is an elliptic curve acted on via translation by a 66-torsion point.

A major conjecture due to Shokurov is that the indices of Calabi–Yau varieties are bounded in each dimension. More precisely, Xu formulated the following conjecture in [66, Conjecture 1.8.3].

Conjecture 2.1.2 (Index Conjecture)**.** *For a positive integer n and a subset $I \subseteq [0,1]$ satisfying the descending chain condition, there is a positive integer $c(n,I)$ so that for any klt Calabi–Yau pair (X,D) of dimension n with the coefficients of D lying in I, we have $c(n,I)(K_X + D) \sim 0$.*

A pair (X,D) is Calabi–Yau if $K_X + D \sim_{\mathbb{Q}} 0$, where X is a normal, projective variety and D is an effective $\mathbb{Q}$-divisor on X. The index of the pair is defined to be the torsion order of $K_X + D$. The index conjecture thereby asserts a uniform upper bound to the indices of klt Calabi–Yau pairs with fixed dimension n and (dcc) set of allowable coefficients I for the boundary divisor.

The conjecture was proven for surfaces in [56, Corollary 1.11]. In dimension 3, the index conjecture has been proven for terminal 3-folds in [45, Theorem 3.2] and [51, Corollary], klt 3-folds in [43, Corollary 1.7], lc pairs [66, Theorem 1.8.7], and slc pairs in [44, Corollary 1.6]. In dimension 4, [44, Corollary 1.7] shows the index conjecture for lc pairs which are not klt and [66, Theorem 1.8.7] shows it for lc pairs with nonzero boundary which are klt. Most recently, the index conjecture has been proven for rationally connected varieties by Birkar in [5, Theorem 1.6]. Note that the Calabi–Yau pairs with large index defined by Esser, Totaro,

and Wang in [18] are rationally connected, but the $V^{(n)}$ of Theorem 2.1.1 are not.

This conjecture remains open, so it remains unclear if there is any uniform upper bound to the indices of Calabi–Yau varieties of a fixed dimension, even with a fixed set of allowable coefficients. Nevertheless, Esser, Totaro, and Wang conjectured in [18] that $(s_{n-1}-1)(2s_{n-1}-3)$ is precisely that upper bound for terminal Calabi–Yau n-folds, and they found explicit examples achieving that value. We formulate therefore a similar conjecture in the smooth case, which follows from theirs.

Conjecture 2.1.3. *The largest possible index of any smooth, projective Calabi–Yau n-fold is* $(s_{n-1}-1)(2s_{n-1}-3)$.

Let $I_{\mathrm{sm}}(n)$, $I_{\mathrm{term}}(n)$ be the sets of indices of smooth, terminal Calabi–Yau n-folds, respectively. Additionally, let $I_{\text{klt st}}(n)$ be the set of indices of klt Calabi–Yau pairs with *standard coefficients* of dimension n. (We say that a pair (X, D) has standard coefficients if the coefficients of D are all of the form $1-\frac{1}{d}$ for d a positive integer. So, in the notation of the index conjecture, we are taking $I = \{1-\frac{1}{d} \mid d \in \mathbb{Z}^{>0}\}$.)

In [18], Esser, Totaro, and Wang constructed a klt Calabi–Yau pair with standard coefficients of dimension $n-1$ with large index, and then used it to construct a terminal Calabi–Yau variety of dimension n with the same index. In fact, it's true that $I_{\text{klt st}}(n-1) \subseteq I_{\mathrm{term}}(n)$, and Esser, Totaro, and Wang's conjecture can be rephrased as saying that both sets have the same maximum: $(s_{n-1}-1)(2s_{n-1}-3)$. Masamura conjectured further in [50] that $I_{\text{klt st}}(n-1) = I_{\mathrm{term}}(n)$ and proved that $I_{\mathrm{sm}}(n) \subseteq I_{\text{klt st}}(n-1) \subseteq I_{\mathrm{term}}(n)$. Given Esser, Totaro, and Wang's conjectural upper bound on the indices of terminal Calabi–Yau n-folds, our Theorem 2.1.1 would show that $I_{\mathrm{sm}}(n)$ and $I_{\mathrm{term}}(n)$ share the same maximum, which is consistent with Masamura's conjecture.

The construction of the $V^{(n)}$ in Theorem 2.1.1 will also lead to the definition of smooth Calabi–Yau varieties with extreme topological invariants.

n	$\sum_{i=0}^{2n} b_i(W^{(n)})$	$\chi(W^{(n)})$
1	4	0
2	24	24
3	1008	-960
4	1820448	1820448
5	5940926462016	-5940922821120
6	63271205161020798539584896	63271205161020798539584896

Table 2.2: Extremely large Betti numbers and Euler characteristics of smooth, projective Calabi–Yau varieties

Theorem 2.1.4. *For every $n \geq 1$, there exists a smooth, projective Calabi–Yau n-fold $W^{(n)}$ with the following properties:*

- *The sum of the Betti numbers of $W^{(n)}$ is $2(s_0 - 1) \cdots (s_n - 1)$.*
- *When n is even, the Euler characteristic of $W^{(n)}$ is $2(s_0 - 1) \cdots (s_n - 1)$.*
- *When n is odd, the Euler characteristic of $W^{(n)}$ is $-(s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 6)$.*

We now elaborate on our strategy to construct the varieties $V^{(n)}$ and $W^{(n)}$ of our main theorems.

Let's start with $V^{(n)}$. Let $m = (s_{n-1}-1)(2s_{n-1}-3)$. Esser, Totaro, and Wang constructed their klt Calabi–Yau pair of large index by defining a hypersurface X in an n-dimensional weighted projective space $\mathbb{P}$ which is stabilized by a cyclic subgroup μ_m of toric automorphisms of $\mathbb{P}$, and taking the quotient pair X/μ_m. To construct their terminal example from this, take $Z \longrightarrow X$ a μ_m-equivariant terminalization, and let E be an elliptic curve with a fixed m-torsion point. Then μ_m acts on E via translation by this point and $(Z \times E)/\mu_m$ is a terminal Calabi–Yau n-fold with index m. If we can instead find a μ_m-equivariant, crepant resolution of singularities $\widetilde{X} \longrightarrow X$, the same procedure will yield a smooth Calabi–Yau

n-fold of index m:

$$V^{(n)} = \frac{\widetilde{X} \times E}{\mu_m}.$$

As for $W^{(n)}$, Esser, Totaro, and Wang prove that the hypersurface $X \subseteq \mathbb{P}$ mentioned in the construction of $V^{(n)}$ has the values in Theorem 2.1.4 as its *orbifold* Betti numbers, as defined by Chen and Ruan in [11]. In [67], Yasuda proved that for a projective variety with Gorenstein quotient singularities (such as X), its orbifold cohomology agrees as a Hodge structure with the cohomology of any crepant resolution, should one exist. As mentioned above, our strategy to construct $V^{(n)}$ hinges on finding a crepant resolution $\widetilde{X}$ of the hypersurface X. This crepant resolution will be the $W^{(n-1)}$ in Theorem 2.1.4. So the V and W in Theorems 2.1.1 and 2.1.4 respectively are related by

$$V^{(n)} = \frac{W^{(n-1)} \times E}{\mu_m}.$$

We propose the following conjecture, which follows from the analogous conjecture of Esser, Totaro, and Wang due to the equality between orbifold Betti numbers and Betti numbers of a crepant resolution proven by Yasuda.

Conjecture 2.1.5. *$W^{(n)}$ has the largest sum of Betti numbers of any smooth, projective Calabi–Yau n-fold. When n is odd, it also has the smallest negative Euler characteristic among such varieties.*

To prove our two main theorems, we are reduced to finding a μ_m-equivariant, crepant, projective resolution of the weighted projective hypersurface X we discussed in the construction of $V^{(n)}$. As X is *quasi-smooth* in $\mathbb{P}$ (i.e. its affine cone is smooth away from the origin), and as the μ_m action is by toric automorphisms, we will show that to find an equivariant, crepant, projective resolution of X, it suffices to find a toric, crepant, projective resolution of $\mathbb{P}$. Finding such a crepant resolution of $\mathbb{P}$ is the main technical work of this paper.

There is no guarantee that any crepant resolution of $\mathbb{P}$ exists. Indeed, the singularities of a weighted projective space are of the form $\mathbb{C}^d/\mu_k$ where μ_k acts diagonally. These do

not always admit crepant resolutions. For example, as discussed in [48], $\mathbb{C}^d/\mu_2$ is resolved by blowing up the origin, which leads to a single exceptional divisor with discrepancy $\frac{d}{2}-1$. This is therefore Gorenstein for d even and terminal for $d > 2$. Varieties which are $\mathbb{Q}$-factorial and terminal do not admit (non-trivial) crepant resolutions, so for instance $\mathbb{C}^4/\mu_2$ is a Gorenstein quotient singularity with no crepant resolution.

The strongest general result we are aware of is by Dais, Haase, and Ziegler in [14], stating that toric local complete intersection (lci) singularities admit projective, crepant resolutions. This is inapplicable in our setting, as the singularities we resolve are in general not lci.

The paper is structured as follows:

- Section 2.2 describes some of the relevant notation and background on Sylvester's sequence and lattice polytopes.
- Section 2.3 provides background information on toric varieties, and describes in combinatorial terms what is needed to provide a toric, crepant, projective resolution of a weighted projective space.
- Section 2.4 describes a solution to this combinatorial problem for the weighted projective spaces in question, and thereby comes to toric, crepant, projective resolutions of them.
- Section 2.5 uses these resolutions to resolve the hypersurfaces in question and provide examples of smooth, projective Calabi–Yau varieties of large index and Betti numbers, proving our main theorems.

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2.2 Notation and background

2.2.1 Sylvester's sequence

Definition 2.2.1 (A000058 on [54]). *Sylvester's sequence s_n is defined recursively via*

$$s_0 = 2$$
$$s_{n+1} = s_0 s_1 \cdots s_n + 1.$$

The recurrence relationship can be rewritten as

$$\sum_{i=0}^{n} \frac{1}{s_i} = 1 - \frac{1}{s_{n+1} - 1},$$

so it follows that

$$\sum_{i=0}^{\infty} \frac{1}{s_i} = 1,$$

as $s_n \to \infty$. This convergence is extremely fast, as Sylvester's sequence is doubly exponential. In fact, $s_n > 2^{2^{n-1}}$. This is the fastest a sum of unit fractions can converge to 1. That is, any sum of n unit fractions which is less than 1 is at most $\sum_{i=0}^{n-1} \frac{1}{s_i}$, as proven in [13].

These properties have been used in a number of recent results by Esser, Totaro, and Wang in finding conjectural extremal values for various numerical invariants of varieties in all dimensions, such as the index and sum of Betti numbers described in Section 2.1.

2.2.2 Lattice polytopes

The combinatorial problem we must solve to find crepant resolutions of weighted projective spaces comes down to triangulating lattice polytopes (a *lattice point* in $\mathbb{R}^n$ is a point in $\mathbb{Z}^n$). We refer to [32] by Haase, Paffenholz, Piechnik, and Santos for more details on lattice polytopes. By convention, all of our polytopes are convex.

Definition 2.2.2. *A* (lattice) polytope *in* $\mathbb{R}^n$ *is the convex hull of a finite set of (lattice) points in* $\mathbb{R}^n$*. A* (lattice) simplex *in* $\mathbb{R}^n$ *is the convex hull of a nonempty affinely independent set of (lattice) points in* $\mathbb{R}^n$*.*

A hyperplane H in $\mathbb{R}^n$ splits $\mathbb{R}^n$ into two closed half-spaces. If a polytope P is wholly contained in one of these half-spaces, we say $H \cap P$ is a *face* of P, which is itself a polytope. A *facet* of P is a face of codimension 1. By definition, the empty set and P are both faces of P.

A polytope can be defined either as a convex hull of finitely many vertices or as an intersection of finitely many closed half-spaces. For example,

$$\mathrm{Conv}(e_1, \ldots, e_n, 0) = \left\{ x \in \mathbb{R}^n \;\middle|\; 1 - \sum_{i=1}^{n} x_i \geq 0 \right\} \cap \bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^n \;\middle|\; x_i \geq 0 \right\},$$

where e_i denotes the ith standard basis vector in $\mathbb{R}^n$.

Moreover, an n-dimensional simplex in $\mathbb{R}^n$ with the origin in its interior can be described as the convex hull of an affine basis $\{v_0, \ldots, v_n\}$ of $\mathbb{R}^n$, or as the intersection

$$\bigcap_{i=0}^{n} \{x \in \mathbb{R}^n \mid \langle u_i, x \rangle + 1 \geq 0\},$$

for an affine basis $\{u_0, \ldots, u_n\}$ of $\mathbb{R}^n$ whose convex hull contains the origin.

We can swap between these representations using *polar duality*. Our reference for this fact is [68, Theorem 2.11].

Definition 2.2.3. *Let* P *be a polytope in* $\mathbb{R}^n$*. The* polar dual *of* P *is*

$$\breve{P} = \{y \in \mathbb{R}^n \mid \langle x, y \rangle + 1 \geq 0 \text{ for all } x \in P\}.$$

Note that in [68], Ziegler uses a slightly different definition of (and notation for) the polar dual of a polytope. We choose the definition written here as it is the one used in SageMath

[58], which has been an invaluable computational tool.

Proposition 2.2.4 (Polar duality)**.** *Let σ be an n-dimensional simplex in $\mathbb{R}^n$ containing 0 in its interior.*

- *$\check{\sigma}$ is an n-dimensional simplex in $\mathbb{R}^n$ containing 0 in its interior.*
- *If*

$$\sigma = \mathrm{Conv}(v_0, \ldots, v_n) = \bigcap_{i=0}^{n} \{x \in \mathbb{R}^n \mid \langle u_i, x\rangle + 1 \geq 0\},$$

then

$$\check{\sigma} = \mathrm{Conv}(u_0, \ldots, u_n) = \bigcap_{i=0}^{n} \{x \in \mathbb{R}^n \mid \langle v_i, x\rangle + 1 \geq 0\}.$$

Definition 2.2.5. *Let P be a (lattice) polytope. A* (lattice) subdivision *$\mathcal{S}$ of P is a finite set of (lattice) polytopes with the following properties:*

- *Every face of every member of $\mathcal{S}$ is a member of $\mathcal{S}$.*
- *The intersection of any two elements of $\mathcal{S}$ is a common face of both of them.*
- *The union of the polytopes in $\mathcal{S}$ is P.*

A polytope $Q \in \mathcal{S}$ is called a *cell* if $\dim(Q) = \dim(P)$. If every nonempty element of $\mathcal{S}$ is a simplex, we say $\mathcal{S}$ is a *triangulation*.

For a k-dimensional polytope $P \subseteq \mathbb{R}^n$ we define its *normalized volume* to be

$$\text{n-vol}(P) = k! \,\mathrm{vol}(P),$$

where $\mathrm{vol}(P)$ is the volume of P in the k-dimensional subspace $\mathrm{span}(P)$. This allows for convenient comparison of the volume of simplices in different dimensions. We compute the

normalized volume of a simplex $\sigma = \text{Conv}(v_0, \ldots, v_n)$ in $\mathbb{R}^n$ as

$$\text{n-vol}(\sigma) = \left| \det \begin{pmatrix} v_0 & \ldots & v_n \\ -1 & \ldots & -1 \end{pmatrix} \right|,$$

where we view the v_i as column vectors.

Convention. Polytope *means lattice polytope and* subdivision *means lattice subdivision.*

Definition 2.2.6. *We say a simplex σ is* unimodular *if* n-vol$(\sigma) = 1$. *Furthermore, a triangulation $\mathcal{T}$ is called* unimodular *if its cells are unimodular.*

If a simplex σ admits a unimodular triangulation $\mathcal{T}$ (which is not guaranteed past dimension 2), n-vol(σ) is the number of cells in $\mathcal{T}$.

Definition 2.2.7. *A subdivision $\mathcal{S}$ of an n-dimensional polytope P in $\mathbb{R}^n$ is called* regular *if there is a continuous, piecewise linear function $\psi : P \longrightarrow \mathbb{R}$ so that $\psi(P \cap \mathbb{Z}^n) \subseteq \mathbb{Q}$ and is* strictly convex *with respect to $\mathcal{S}$. That is, ψ is convex and the domains of linearity of ψ are precisely the cells of $\mathcal{S}$. (The graph of ψ can be viewed as a folding of the subdivision $\mathcal{S}$ into a strictly convex shape.)*

Note that some authors' definition of regularity imposes $\psi(P \cap \mathbb{Z}^n) \subseteq \mathbb{Z}$ rather than $\mathbb{Q}$. This is equivalent to the definition we give. Indeed, the polytopes we are working with are compact and hence $P \cap \mathbb{Z}^n$ is finite, so such a ψ with $\psi(P \cap \mathbb{Z}^n) \subseteq \mathbb{Q}$ actually has image in $\frac{1}{N}\mathbb{Z}$ for a positive integer N. We may then multiply ψ by N to land in $\mathbb{Z}$.

Definition 2.2.8. *A subdivision $\mathcal{S}$ is called* star-shaped *(at the origin) if every cell contains the origin as a vertex.*

2.3 Background on resolving toric varieties

We describe in this section our strategy to find toric, crepant, projective resolutions of weighted projective spaces. Our main reference for facts about toric varieties is the short

text [22], whose notation we borrow. We recall some of the basic facts here.

A toric variety is defined by a lattice N and a strongly convex rational polyhedral fan Δ in $N \otimes \mathbb{R}$. We denote this variety by $X(\Delta)$, which comes equipped with a natural action and equivariant open immersion of the algebraic torus $T_{X(\Delta)} = T_N = N \otimes \mathbb{C}^\times$.

Key geometric properties of $X(\Delta)$ correspond to combinatorial properties of the fan Δ, which encodes the stratification of $X(\Delta)$ into orbit closures. For example, we will crucially use the following characterization of smoothness.

Proposition 2.3.1. *[22, §2.1] A toric variety $X(\Delta)$ is smooth if and only if every maximal cone $\sigma \in \Delta$ is generated by part of an integral basis of the lattice N.*

Similarly, torus invariant divisors can be fully encoded combinatorially. Let $\Delta(1)$ be the set of one dimensional cones, i.e. rays, in Δ. Then for every $\rho \in \Delta(1)$, there is a corresponding orbit closure $D_\rho = V(\rho)$ which is a torus invariant, irreducible Weil divisor. Every torus invariant Weil divisor is of the form $\sum_{\rho \in \Delta(1)} a_\rho D_\rho$ for $a_\rho \in \mathbb{Z}$. For example, the canonical divisor of any toric variety $X(\Delta)$ is given by

$$K_{X(\Delta)} = -\sum_{\rho \in \Delta(1)} D_\rho.$$

Torus invariant ($\mathbb{Q}$-)Cartier divisors correspond to continuous functions $\psi : |\Delta| \longrightarrow \mathbb{R}$ which restrict to integral (rational) linear functions on every cone in Δ. Here, $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ is the support of the fan Δ. If a torus invariant Weil divisor $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ is Cartier, the associated piecewise linear continuous function $\psi_D : |\Delta| \longrightarrow \mathbb{R}$ is called the support function of D, and is uniquely determined by the condition $\psi_D(n_\rho) = -a_\rho$, where n_ρ is the minimal nonzero lattice point on the ray ρ.

Important properties of a torus invariant Cartier divisor can be determined from its support function. For our purposes, we will primarily need the following characterization of ampleness.

Proposition 2.3.2. *[22, §3.4] Let $X(\Delta)$ be a toric variety so that $|\Delta| = N \otimes \mathbb{R}$. A torus invariant Cartier divisor D on $X(\Delta)$ is ample if and only if its support function ψ is strictly convex, i.e. it is convex and for distinct maximal cones $\sigma, \sigma' \in \Delta$, the linear functions extending $\psi|_\sigma$ and $\psi|_{\sigma'}$ are distinct.*

The construction of toric varieties is functorial. Let (N, Δ) and (N', Δ') be lattices equipped with fans. Given a $\mathbb{Z}$-linear morphism $f : N \longrightarrow N'$ so that for every $\sigma \in \Delta$ there is a $\sigma' \in \Delta'$ satisfying $f(\sigma) \subseteq \sigma'$, there is a morphism $\phi : X(\Delta) \longrightarrow X(\Delta')$ which is equivariant with respect to the torus actions. On tori, ϕ is given by $f \otimes \mathrm{id}_{\mathbb{C}^\times} : N \otimes \mathbb{C}^\times \longrightarrow N' \otimes \mathbb{C}^\times$.

Proposition 2.3.3. *$f : N \longrightarrow N'$ is an isomorphism if and only if $\phi : X(\Delta) \longrightarrow X(\Delta')$ is birational.*

Proof. If f is an isomorphism then $\phi|_{T_N} = f \otimes \mathrm{id} : N \otimes \mathbb{C}^\times \longrightarrow N' \otimes \mathbb{C}^\times$ is an isomorphism, so ϕ is birational.

Conversely, suppose ϕ is birational. Then $\phi|_{T_N} : T_N \longrightarrow T_{N'}$ is a birational homomorphism of algebraic tori. As such, it is an isomorphism. As $N \mapsto N \otimes \mathbb{C}^\times$ induces an equivalence of categories between lattices and algebraic tori, it follows that f is an isomorphism. □

Proposition 2.3.4. *Given $\phi : X(\Delta) \longrightarrow X(\Delta')$ arising from $f : N \longrightarrow N'$ and a torus invariant Cartier divisor D' on $X(\Delta')$ with support function $\psi_{D'}$, we have $\psi_{\phi^* D'} = \psi_{D'} \circ f$.*

Proof. Let M' be the dual of N'. For $\sigma' \in \Delta'$, let $-u \in M'$ be the linear function whose restriction to σ' agrees with $\psi_{D'}$, so that $D'|_{U_{\sigma'}} = \mathrm{div}(\chi^u)$. Let $\sigma \in \Delta$ be so that $f(\sigma) \subseteq \sigma'$. Then ϕ restricts to a morphism of affine toric varieties $U_\sigma \longrightarrow U_{\sigma'}$ defined on coordinate rings $A_{\sigma'} \longrightarrow A_\sigma$ by taking a monomial χ^m to $\chi^{m \circ f}$. Hence, $\phi^*(D')|_{U_\sigma} = \mathrm{div}(\chi^{u \circ f})$. As such, we have shown that $\psi_{\phi^* D'}$ agrees with $\psi_{D'} \circ f$ on any cone $\sigma' \in \Delta'$. Hence, they agree on the whole support of Δ'. □

A common way to construct toric varieties is to take a polytope P with the origin in its relative interior and to form the fan Δ_P consisting of cones generated by all the proper faces

of P. For example, in the lattice $\mathbb{Z}^{n+1}/(1,\ldots,1)$, the polytope $\mathrm{Conv}(\overline{e_0},\ldots,\overline{e_n})$ (where $\overline{e_i}$ is the image of the ith standard basis vector in the quotient $\mathbb{Z}^{n+1}/(1,\ldots,1)$) determines a very familiar toric variety, namely $\mathbb{P}^n$. More generally, given positive integer weights $a_0,\ldots,a_n$, the polytope $\mathrm{Conv}(\overline{e_0},\ldots,\overline{e_n})$ in the lattice $\mathbb{Z}^{n+1}/(a_0,\ldots,a_n)$ defines the weighted projective space $\mathbb{P}(a_0,\ldots,a_n)$. (If $a_n = 1$, we can project to the first n coordinates and work in the lattice $\mathbb{Z}^n$ with the polytope $P = \mathrm{Conv}(e_0,\ldots,e_{n-1},-\sum a_i e_i)$.) More generally still, if we have a star-shaped subdivision $\mathcal{S}$ of a polytope P, we can form the fan $\Delta_{\mathcal{S}}$ consisting of cones generated by all the elements of $\mathcal{S}$. For the remainder of this section, $\mathcal{S}$ is a star-shaped subdivision of P.

The identity morphism $\mathrm{id} : N \longrightarrow N$ sends cones in $\Delta_{\mathcal{S}}$ to Δ_P and hence defines a morphism of toric varieties $\pi : X(\Delta_{\mathcal{S}}) \longrightarrow X(\Delta_P)$. It is automatic that π is equivariant, and by Proposition 2.3.3 it's also birational. We will show below that π is crepant under some hypotheses, and determine conditions for when π is a projective resolution of singularities. As our goal is to resolve weighted projective spaces, we will focus on the case of interest where P is the simplex $\mathrm{Conv}(\overline{e_0},\ldots,\overline{e_n})$ in $\mathbb{Z}^{n+1}/(a_0,\ldots,a_n)$. Let $\mathbb{P} = \mathbb{P}(a_0,\ldots,a_n) = X(\Delta_P)$.

First, we show that $\pi : X(\Delta_{\mathcal{S}}) \longrightarrow \mathbb{P}$ is crepant. Let ψ be the support function of $K_{\mathbb{P}}$, which is uniquely determined by $\psi(\overline{e_i}) = 1$ for all i. Let's suppose ψ is piecewise $\mathbb{Z}$-linear, i.e. that $K_{\mathbb{P}}$ is Cartier. This is not a substantial assumption to make, as any normal variety with a $\mathbb{Q}$-Cartier canonical divisor admitting a crepant resolution must actually have a Cartier canonical divisor, and this assumption is satisfied in the case of the weighted projective spaces we study in Section 2.4. As π is defined by $\mathrm{id} : N \longrightarrow N$, we have by Proposition 2.3.4 that $\pi^*(K_{\mathbb{P}})$ has support function ψ. The support function of $K_{X(\Delta_{\mathcal{S}})}$ is defined by $\psi_{K_{X(\Delta_{\mathcal{S}})}}(n_\rho) = 1$ for every ray ρ of $\Delta_{\mathcal{S}}$, where n_ρ is the minimal lattice point on ρ. Any such ray ρ is generated by some lattice point $m \in \partial P$. As such, $\psi(m) \leq 1$. Because ψ is piecewise $\mathbb{Z}$-linear, $\psi(n_\rho) \in \mathbb{Z}$. Furthermore, $0 < \psi(n_\rho) \leq \psi(m) \leq 1$. Hence, $\psi(n_\rho) = 1$ for every ray ρ of $\Delta_{\mathcal{S}}$. As such, ψ equals the support function of the canonical divisor $\psi_{K_{X(\Delta_{\mathcal{S}})}}$ which proves that $\pi^*(K_{X(\Delta_P)}) = K_{X(\Delta_{\mathcal{S}})}$. That is, π is crepant.

$X(\Delta_{\mathcal{S}})$ is projective if and only if it has an ample $\mathbb{Q}$-Cartier divisor. By Proposition 2.3.2 above, it suffices to find a piecewise $\mathbb{Q}$-linear continuous function $|\Delta_{\mathcal{S}}| \longrightarrow \mathbb{R}$ which is strictly convex with respect to $\mathcal{S}$. That is, it suffices to show that $\mathcal{S}$ is a regular subdivision of P as defined in Section 2.2.2.

Finally, $X(\Delta_{\mathcal{S}})$ is smooth if and only if the nonzero vertices of every cell $\sigma \in \mathcal{S}$ form an integral basis of the lattice, i.e. are unimodular simplices, as defined in Section 2.2.2.

Altogether, we have determined that to find a toric, crepant, projective resolution of a weighted projective space $\mathbb{P}$, we must find a regular, unimodular, star-shaped triangulation $\mathcal{T}$ of the simplex P corresponding to $\mathbb{P}$ described above.

2.4 Triangles

Let us proceed to find such a triangulation, as described in Section 2.3. We define

$$\begin{aligned} d_1^{(n)} &= 2s_{n-1} - 2 \\ d_2^{(n)} &= s_n - 1 \end{aligned}$$

and

$$\begin{aligned} w_1^{(n)} &= \left(-\frac{d_1^{(n)}}{s_0}, \ldots, -\frac{d_1^{(n)}}{s_{n-2}}, -1 \right) \\ w_2^{(n)} &= \left(-\frac{d_2^{(n)}}{s_0}, \ldots, -\frac{d_2^{(n)}}{s_{n-1}} \right). \end{aligned}$$

With this, we define

$$\begin{aligned} P_1^{(n)} &= \mathrm{Conv}\left(e_0^{(n)}, \ldots, e_{n-1}^{(n)}, w_1^{(n)} \right) \\ P_2^{(n)} &= \mathrm{Conv}\left(e_0^{(n)}, \ldots, e_{n-1}^{(n)}, w_2^{(n)} \right), \end{aligned}$$

where $e_i^{(n)}$ is the ith standard basis vector in $\mathbb{R}^n$ (indexed beginning with 0) and Conv denotes the convex hull. When it is clear from context, the superscript (n) will be omitted for brevity.

These are both simplices in $\mathbb{R}^n$, and correspond as in Section 2.3 to the weighted projective spaces

$$\mathbb{P}_1^{(n)} = \mathbb{P}\left(d_1^{(n)}/s_0, \ldots, d_1^{(n)}/s_{n-2}, 1, 1\right)$$
$$\mathbb{P}_2^{(n)} = \mathbb{P}\left(d_2^{(n)}/s_0, \ldots, d_2^{(n)}/s_{n-1}, 1\right).$$

To use the strategy from Section 2.3 to crepantly resolve these weighted projective spaces, we must first verify that the canonical divisors $K_{\mathbb{P}_i^{(n)}}$ are Cartier for $i = 1, 2$ and any $n \geq 1$. By [4, Corollary 4A.5], a torus invariant divisor $D = \sum c_i D_i$ in $\mathbb{P}(a_0, \ldots, a_n)$ is Cartier if $\text{lcm}(a_0, \ldots, a_n) | \sum c_i a_i$, where $D_i = \{x_i = 0\}$ is the torus invariant divisor corresponding to the ray generated by $\overline{e_i}$. The converse is true supposing $\mathbb{P}(a_0, \ldots, a_n)$ is *well-formed*, as will be defined in Section 2.5. For $D = K_{\mathbb{P}}$, this means that $K_{\mathbb{P}}$ is Cartier if $a_i \mid \sum_j a_j$ for all i. This holds for both weighted projective spaces in question. We will use the relation:

$$\sum_{j=0}^{k} \frac{1}{s_j} = 1 - \frac{1}{s_{k+1} - 1}.$$

For $i = 1$, we compute the sum of the weights to be

$$\begin{aligned}\sum_{j=0}^{n-2} \frac{d_1^{(n)}}{s_j} + 1 + 1 &= (2s_{n-1} - 2)\left(1 - \frac{1}{s_{n-1} - 1}\right) + 2 \\ &= 2s_{n-1} - 2 \\ &= d_1^{(n)}.\end{aligned}$$

All of the weights divide $d_1^{(n)}$, so we deduce that $K_{\mathbb{P}_1^{(n)}}$ is Cartier for all n.

For $i = 2$, the sum of the weights is

$$\begin{aligned}\sum_{j=0}^{n-1} \frac{d_2^{(n)}}{s_j} + 1 &= (s_n - 1)\left(1 - \frac{1}{s_n - 1}\right) + 1 \\ &= s_n - 1 \\ &= d_2^{(n)}.\end{aligned}$$

All of the weights divide $d_2^{(n)}$, so $K_{\mathbb{P}_2^{(n)}}$ is Cartier for all n.

The sources of the large index examples in [18] are hypersurfaces in $\mathbb{P}_1$. We seek to find a toric, crepant, projective resolution of $\mathbb{P}_1$. To do so, as discussed in Section 2.3, is equivalent to finding a regular, unimodular, star-shaped triangulation of P_1.

Proposition 2.4.1. *For all n, there is a regular, unimodular, star-shaped triangulation of $P_1^{(n)}$.*

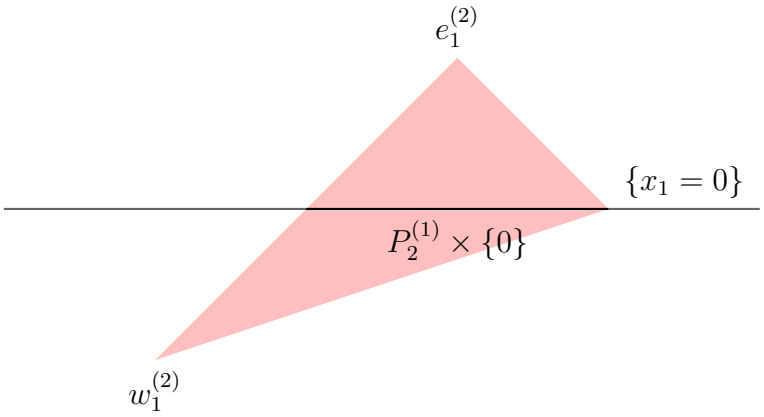
In the following section, we reduce this to finding a regular, unimodular, star-shaped triangulation of P_2.

2.4.1 Reducing to P_2

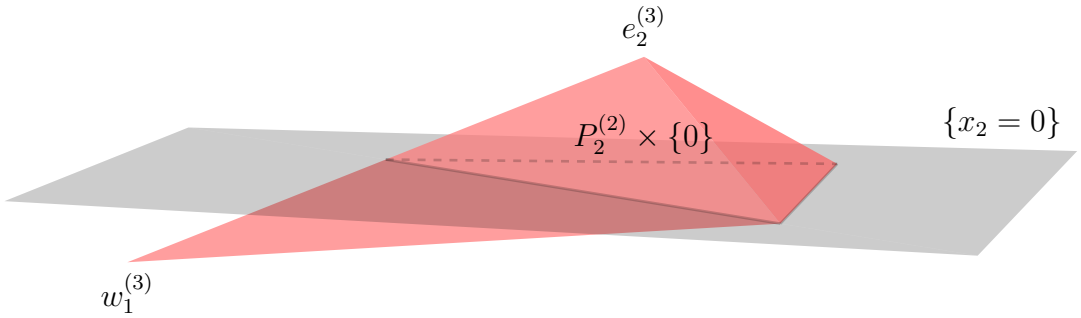
Notice that $w_1^{(n+1)} = (2w_2^{(n)}, -1)$. From this, we can show that the family $P_2^{(n)}$ sits inside the family $P_1^{(n+1)}$ as the hyperplane section setting $x_n = 0$. See Figure 2.1. More precisely, we can show the following.

Lemma 2.4.2. $P_1^{(n+1)} = \mathrm{Conv}\left(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\right)$.

Proof. We first show $\mathrm{Conv}(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}) \subseteq P_1^{(n+1)}$. Indeed, $e_n^{(n+1)}$ and $w_1^{(n+1)}$ are in $P_1^{(n+1)}$ by definition. Recall that $P_2^{(n)} \times \{0\}$ is the convex hull of the first n standard basis vectors in $\mathbb{R}^{n+1}$ and $(w_2^{(n)}, 0)$. The standard basis vectors are all, by definition, in $P_1^{(n+1)}$. As



(a) $P_1^{(2)}$ with its cross section $P_2^{(1)} \times \{0\} = P_1^{(2)} \cap \{x_1 = 0\}$.



(b) $P_1^{(3)}$ with its cross section $P_2^{(2)} \times \{0\} = P_1^{(3)} \cap \{x_2 = 0\}$.

Figure 2.1: $P_2^{(n)}$ as a hyperplane section of $P_1^{(n+1)}$

for $(w_2^{(n)}, 0)$, we may realize it as a convex combination of the vertices of $P_1^{(n+1)}$ via

$$\left(w_2^{(n)}, 0\right) = \frac{1}{2} w_1^{(n+1)} + \frac{1}{2} e_n^{(n+1)},$$

showing that it is also in $P_1^{(n+1)}$. As $P_1^{(n+1)}$ is convex, the containment

$$\mathrm{Conv}\left(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\right) \subseteq P_1^{(n+1)}$$

is proven.

As we have this containment, to show that these polytopes are equal to each other it suffices to show that they have the same volume. Indeed, they have the same dimension so it suffices to compare the normalized volume. The normalized volume of $P_1^{(n+1)}$ is

$$\text{n-vol}\left(P_1^{(n+1)}\right) = \left| \det \begin{pmatrix} & & I_{n+1} & & w_1^{(n+1)} \\ \hline -1 & \dots & -1 & & -1 \end{pmatrix} \right|.$$

Recall that $w_1^{(n+1)} = (2w_2^{(n)}, -1)$. We may then rewrite the above as

$$\left| \det \begin{pmatrix} & & I_{n+1} & & 2w_2^{(n)} \\ & & & & -1 \\ \hline -1 & \dots & -1 & & -1 \end{pmatrix} \right|,$$

which we compute as

$$\left|\det\left(\begin{array}{ccc|c} & I_{n+1} & & \begin{array}{c}2w_2^{(n)}\\ \vdots \\ -1\end{array} \\ \hline -1 & \dots & -1 & -1\end{array}\right)\right| = \left|\det\left(\begin{array}{ccc|c} & I_{n+1} & & \begin{array}{c}2w_2^{(n)}\\ \vdots \\ 0\end{array} \\ \hline -1 & \dots & -1 & -2\end{array}\right)\right|$$

$$= 2\left|\det\left(\begin{array}{ccc|c} & I_{n+1} & & \begin{array}{c}w_2^{(n)}\\ \vdots \\ 0\end{array} \\ \hline -1 & \dots & -1 & -1\end{array}\right)\right|$$

$$= 2\Big(\text{n-vol}\Big(P_2^{(n)}\Big)\Big).$$

Observe that $\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\Big)$ contains

$$\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}\Big) \cup \text{Conv}\Big(P_2^{(n)} \times \{0\}, w_1^{(n+1)}\Big).$$

To compute the normalized volume of this union, we need only sum the normalized volumes of $\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}\Big)$ and $\text{Conv}\Big(P_2^{(n)} \times \{0\}, w_1^{(n+1)}\Big)$, as their intersection is $P_2^{(n)} \times \{0\}$ which has 0 volume in $\mathbb{R}^{n+1}$. To compute these two volumes, we appeal to the following lemma.

Lemma 2.4.3. *Let $P = \text{Conv}(v_0, \dots, v_n)$ be a simplex in $\mathbb{R}^n$. Let $w \in \mathbb{R}^{n+1}$ have last component ± 1. Then* $\text{n-vol}(\text{Conv}(P \times \{0\}, w)) = \text{n-vol}(P)$.

Proof. Write $w = (w', \pm 1)$. Then n-vol(Conv$(P \times \{0\}, w)$) is given by the determinant

$$\left| \det \begin{pmatrix} v_0 & \dots & v_n & w' \\ 0 & \dots & 0 & \pm 1 \\ -1 & \dots & -1 & -1 \end{pmatrix} \right|.$$

By minor expansion, this determinant equals

$$\left| \det \begin{pmatrix} v_0 & \dots & v_n \\ -1 & \dots & -1 \end{pmatrix} \right|,$$

which is precisely n-vol(P). □

Using this, we compute

$$\begin{aligned} \text{n-vol}\Big(P_2^{(n)}\Big) &= \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}\Big)\Big) \\ &= \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, w_1^{(n+1)}\Big)\Big). \end{aligned}$$

We therefore know

$$\begin{aligned} \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\Big)\Big) &\geq \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}\Big)\Big) \\ &\quad + \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, w_1^{(n+1)}\Big)\Big) \\ &= 2\Big(\text{n-vol}\Big(P_2^{(n)}\Big)\Big) \\ &= \text{n-vol}\Big(P_1^{(n+1)}\Big) \\ &\geq \text{n-vol}\Big(\text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\Big)\Big). \end{aligned}$$

So we have shown

$$P_1^{(n+1)} = \text{Conv}\Big(P_2^{(n)} \times \{0\}, e_n^{(n+1)}, w_1^{(n+1)}\Big).$$

□

From this we can extend subdivisions of $P_2^{(n)}$ to $P_1^{(n+1)}$ by taking cones.

Definition 2.4.4. *Let H be a hyperplane in $\mathbb{R}^n$ and let P be a polytope in H with a subdivision $\mathcal{S}$. For $z \in \mathbb{Z}^n - H$ we define a subdivision $\mathrm{Cone}(z, \mathcal{S})$ of $\mathrm{Conv}(z, P)$ via*

$$\mathrm{Cone}(z, \mathcal{S}) = \{\mathrm{Conv}(z, \sigma) \mid \sigma \in \mathcal{S}\}.$$

By Lemma 2.4.3, we have the following.

Lemma 2.4.5. *Let P be a simplex in $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ with a unimodular triangulation $\mathcal{T}$. Let $z \in \mathbb{R}^n \times \{\pm 1\}$. Then $\mathrm{Cone}(z, \mathcal{T})$ is a unimodular triangulation of $\mathrm{Conv}(z, P)$.*

Given a triangulation $\mathcal{T}$ of $P_2^{(n)}$, we form triangulations $\mathrm{Cone}\left(e_n^{(n+1)}, \mathcal{T}\right)$ and $\mathrm{Cone}\left(w_1^{(n+1)}, \mathcal{T}\right)$ of $\mathrm{Conv}\left(e_n^{(n+1)}, P_2^{(n)} \times \{0\}\right)$ and $\mathrm{Conv}\left(w_1^{(n+1)}, P_2^{(n)} \times \{0\}\right)$. By Lemma 2.4.5, these are both unimodular. Using Lemma 2.4.2, we may glue these along their intersection $P_2^{(n)} \times \{0\}$ to get a unimodular triangulation $\mathcal{T}'$ of $P_1^{(n+1)}$.

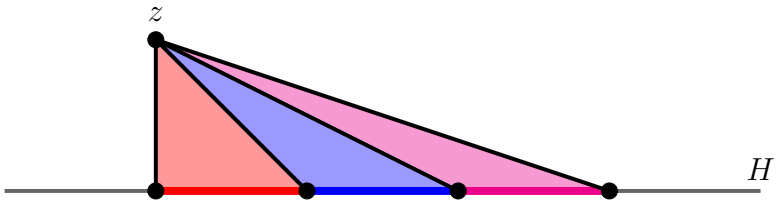
We now seek to show that if $\mathcal{T}$ is regular, then $\mathcal{T}'$ is regular as well. To do so, we use the following two lemmas.

Lemma 2.4.6. *Let H be a hyperplane of $\mathbb{R}^n$ and $P \subseteq H$ a polytope with a regular subdivision $\mathcal{S}$. Let z be a lattice point not in H. Then $\mathrm{Cone}(z, \mathcal{S})$ is a regular subdivision of $\mathrm{Conv}(z, P)$. See Figure 2.2.*

Proof. Let $\psi : P \longrightarrow \mathbb{R}$ be a continuous, piecewise $\mathbb{Q}$-linear function which is strictly convex with respect to $\mathcal{S}$. That is, ψ witnesses the regularity of $\mathcal{S}$. We extend ψ to $\Psi : \mathrm{Conv}(z, P) \longrightarrow \mathbb{R}$ by defining $\Psi|_P = \psi$, setting $\Psi(z)$ to be any rational number ω, and extending ψ linearly to z. That is, for any $y \in \mathrm{Conv}(z, P)$ (other than z itself) we consider the line connecting z to y and let x be its intersection with P. Then $y = tx + (1-t)z$



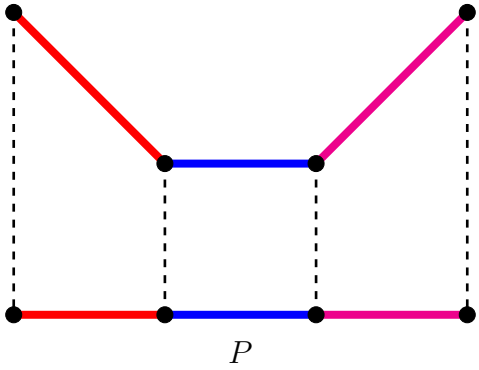
(a) A polytope $P \subseteq H$ with a regular subdivision $\mathcal{S}$.



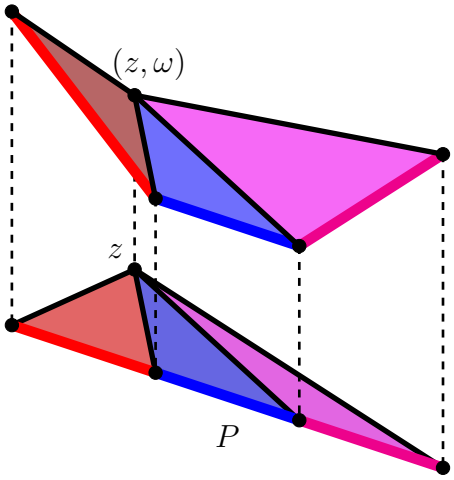
(b) $\mathrm{Conv}(z, P)$ with the subdivision $\mathrm{Cone}(z, \mathcal{S})$.

Figure 2.2: The cone of a subdivision.

for a unique $t \in [0, 1]$, and we define $\Psi(y) = t\psi(x) + (1 - t)\omega$. See Figure 2.3 for a visual of Ψ.



(a) The graph of ψ lying over P.



(b) The graph of Ψ lying over $\mathrm{Conv}(z, P)$.

Figure 2.3: Extending a witness of regularity of a subdivision to its cone.

Ψ is a continuous function $\mathrm{Conv}(z, P) \longrightarrow \mathbb{R}$. As $\psi(\mathbb{Z}^n \cap P) \subseteq \mathbb{Q}$ and $\omega \in \mathbb{Q}$ it follows that $\Psi(\mathbb{Z}^n \cap \mathrm{Conv}(z, P)) \subseteq \mathbb{Q}$. We are left to show the strict convexity of Ψ with respect to $\mathrm{Cone}(z, \mathcal{S})$.

We first show its convexity. For a cell $\sigma \in \mathcal{S}$, let ψ_σ be the linear function $H \longrightarrow \mathbb{R}$ extending $\psi|_\sigma$. By convexity of ψ, we have

$$\psi = \max_\sigma \psi_\sigma, \qquad (*)$$

where σ ranges over the cells of $\mathcal{S}$. Define Ψ_σ to be the linear extension of ψ_σ with $\Psi_\sigma(z) = \omega$. Then we have

$$\Psi|_{\mathrm{Conv}(z,\sigma)} = \Psi_\sigma|_{\mathrm{Conv}(z,\sigma)}.$$

If we can show that

$$\Psi = \max_\sigma \Psi_\sigma, \tag{$**$}$$

where σ ranges over the cells of $\mathcal{S}$, then we will have shown Ψ is the max of a set of linear functions (which are convex), and hence that Ψ is convex.

It suffices to show that for $\sigma \in \mathcal{S}$ a cell and $y \in \mathrm{Conv}(z, \sigma)$ we have $\Psi(y) = \Psi_\sigma(y) \geq \Psi_\tau(y)$ for any other cell τ of $\mathcal{S}$. Indeed, write $y = tx + (1-t)z$ for $t \in [0,1]$ and $x \in \sigma$. As $x \in \sigma$, $\psi(x) = \psi_\sigma(x)$. By $(*)$ we have $\psi_\sigma(x) \geq \psi_\tau(x)$. Then we compute

$$\begin{aligned}
\Psi(y) &= \Psi_\sigma(y) \\
&= t\psi_\sigma(x) + (1-t)\omega \\
&\geq t\psi_\tau(x) + (1-t)\omega \\
&= \Psi_\tau(y),
\end{aligned}$$

and we have proven $(**)$ so convexity of Ψ follows.

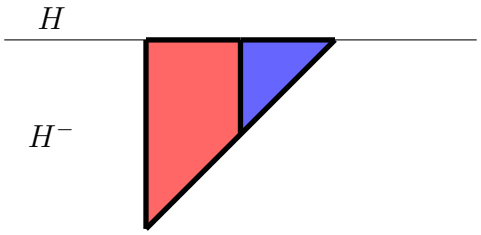
We are left to show strict convexity with respect to $\mathrm{Cone}(z, \mathcal{S})$, i.e. that the domains of linearity of Ψ are precisely the cells of $\mathrm{Cone}(z, \mathcal{S})$. These cells are of the form $\mathrm{Conv}(z, \sigma)$ for σ a cell of $\mathcal{S}$. Ψ is by definition linear on these cells. Suppose then that there exist two cells $\mathrm{Conv}(z, \sigma_1), \mathrm{Conv}(z, \sigma_2)$ of $\mathrm{Cone}(z, \mathcal{S})$ so that $\Psi_{\sigma_1} = \Psi_{\sigma_2}$. As $\psi_{\sigma_i} = \Psi_{\sigma_i}|_H$, we have $\psi_{\sigma_1} = \psi_{\sigma_2}$. Therefore, because ψ is strictly convex with respect to $\mathcal{S}$, we have that $\sigma_1 = \sigma_2$. Hence, $\mathrm{Conv}(z, \sigma_1) = \mathrm{Conv}(z, \sigma_2)$, so we have shown that Ψ is strictly convex with respect to $\mathrm{Cone}(z, \mathcal{S})$, witnessing therefore the regularity of this subdivision. □

Lemma 2.4.7. *Let H be a hyperplane in $\mathbb{R}^n$, which splits $\mathbb{R}^n$ into two closed half-spaces H^-*

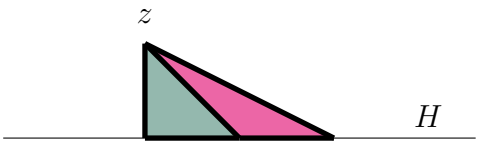
and H^+. *Let* P *be a polytope in* $\mathbb{R}^n$ *and define*

$$P^- = P \cap H^-$$
$$P^+ = P \cap H^+$$
$$P^0 = P \cap H.$$

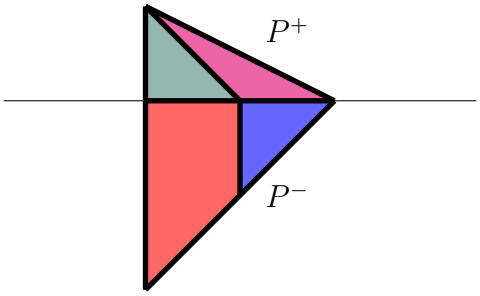
Suppose that P^0 *is a facet of* $P^\pm$ *and that* $P^+ = \mathrm{Conv}(z, P^0)$ *for some lattice point* z *not in* H^-. *Suppose additionally that* P^- *has a regular subdivision* $\mathcal{S}^-$ *which restricts to a regular subdivision* $\mathcal{S}^0$ *of* P^0. *Let* $\mathcal{S}^+ = \mathrm{Cone}(z, \mathcal{S}^0)$. *Form the subdivision* $\mathcal{S}$ *of* P *by gluing* $\mathcal{S}^+$ *and* $\mathcal{S}^-$ *along* $\mathcal{S}^0$. *We claim that* $\mathcal{S}$ *is regular. See Figure 2.4.*



(a) $P^- \subseteq H^-$ with regular subdivision $\mathcal{S}^-$.



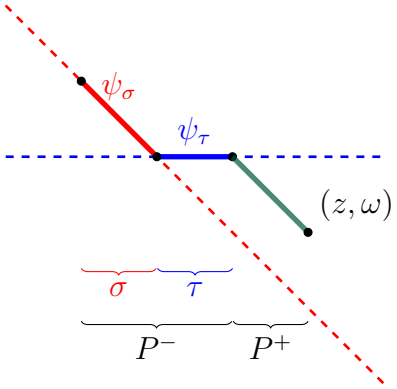
(b) $P^+ = \mathrm{Conv}(z, P^0)$ with the subdivision $\mathcal{S}^+ = \mathrm{Cone}(z, \mathcal{S}^0)$.



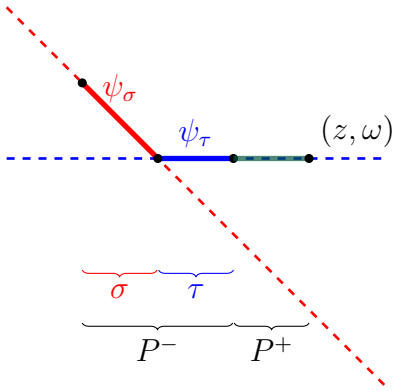
(c) The glued polytope $P = P^- \cup P^+$ with subdivision $\mathcal{S} = \mathcal{S}^- \cup \mathcal{S}^+$.

Figure 2.4: Gluing subdivisions along a common hyperplane.

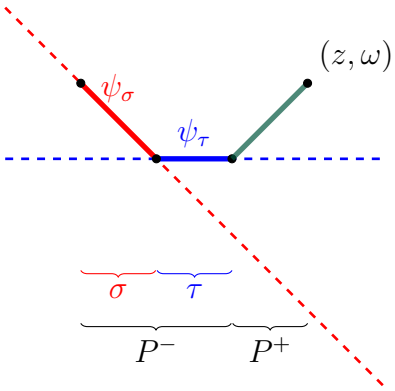
Proof. As in the proof of Lemma 2.4.6, we let $\psi^- : P^- \longrightarrow \mathbb{R}$ witness the regularity of $\mathcal{S}$ and seek to extend ψ^- linearly to some $\psi : P \longrightarrow \mathbb{R}$ which will witness the regularity of



(a) The graph of ψ if we choose ω strictly below $\psi_\sigma(z)$ or $\psi_\tau(z)$. ψ is not convex.



(b) The graph of ψ if we choose ω non-strictly above $\psi_\sigma(z)$ and $\psi_\tau(z)$. ψ is non-strictly convex.



(c) The graph of ψ if we choose ω strictly above $\psi_\sigma(z)$ and $\psi_\tau(z)$. ψ is strictly convex.

Figure 2.5: Various choices of ω in the proof of Lemma 2.4.7 and the resulting convexity of ψ.

Cone$(z, \mathcal{S})$. This time, we cannot choose $\psi(z) = \omega$ so freely, however. For each cell σ of $\mathcal{S}^-$, let ψ_σ be the linear function which agrees with ψ^- on σ. Let ω be any rational number greater than $\psi_\sigma(z)$ for all cells σ of $\mathcal{S}^-$. Define $\psi : P \longrightarrow \mathbb{R}$ to agree with ψ^- on P^-, take the value ω on z, and linearly interpolate in between. See Figure 2.5 for a visual of the way different choices of ω affect the convexity of ψ.

ψ is continuous, and as ψ^- witnessed the regularity of $\mathcal{S}^-$, $\psi^-(\mathbb{Z}^n \cap P^-) \subseteq \mathbb{Q}$. So as $\omega \in \mathbb{Q}$, we have $\psi(\mathbb{Z}^n \cap P) \subseteq \mathbb{Q}$. We are left to show strict convexity with respect to $\mathcal{S}$.

First, we show that the condition $\psi(z) > \psi_\sigma(z)$ for all cells σ of $\mathcal{S}^-$ implies that $\psi(y) \geq \psi_\sigma(y)$ for any $y \in P^+$. Indeed, suppose $\psi(y) < \psi_\sigma(y)$ for some cell $\sigma \in \mathcal{S}^-$ and $y \in P^+$. We may assume $y \neq z$ and $y \notin P^0$. Then the line connecting z and y intersects the hyperplane H at a unique point x. We have the following inequalities

$$(\psi - \psi_\sigma)(x) \geq 0$$
$$(\psi - \psi_\sigma)(y) < 0$$
$$(\psi - \psi_\sigma)(z) > 0$$

where x, y, z are distinct and collinear. Both ψ_σ and ψ are linear on the line segment ℓ from x to y to z. The linear function $(\psi - \psi_\sigma)|_\ell$ cannot simultaneously satisfy all three of these inequalities, so $\psi(y) \geq \psi_\sigma(y)$ for all points $y \in P^+$ and cells $\sigma \in \mathcal{S}^-$.

We now show convexity itself. Let $\psi^+ = \psi|_{P^+}$. ψ^- is convex by hypothesis, and ψ^+ is convex by Lemma 2.4.6. Hence, we need only check convexity for line segments connecting points $x^- \in P^-$ and $x^+ \in P^+$ which are not in P^0. Let $\ell : [-1, 1] \longrightarrow P$ be the line segment so that $\ell(-1) = x^-$ and $\ell(1) = x^+$. We seek to show $\psi \circ \ell$ has increasing slopes. Let $t_0 \in (-1, 1)$ be the unique value where $\ell(t_0) \in P^0$. By convexity of $\psi^\pm$, $\psi \circ \ell$ has increasing slopes on $[-1, t_0]$ and on $[t_0, 1]$. We claim that the slope of $\psi \circ \ell$ at $t_0 - \varepsilon$ is less than the slope of $\psi \circ \ell$ at $t_0 + \varepsilon$ for small ε. For all sufficiently small ε, $\ell(t_0 - \varepsilon)$ is in some fixed cell $\sigma \in \mathcal{S}^-$, so $\psi(\ell(t)) = \psi_\sigma(\ell(t))$ for $t \in [t_0 - \varepsilon, t_0]$. As discussed above, we know that $\psi(y) \geq \psi_\sigma(y)$ for any point $y \in P^+$ and cell $\sigma \in \mathcal{S}^-$. Hence, $\psi(\ell(t)) \geq \psi_\sigma(\ell(t))$ for $t \in [t_0, t_0 + \varepsilon]$. As such, $\psi \circ \ell$ has larger slope at $t_0 + \varepsilon$ than $t_0 - \varepsilon$, proving convexity.

Finally, we show the convexity is strict with respect to $\mathcal{S}$. Certainly, ψ is linear on the cells of $\mathcal{S}$. For cells σ_1, σ_2 in $\mathcal{S}$, let ψ_{σ_i} denote the linear function which agrees with ψ on σ_i. Suppose $\psi_{\sigma_1} = \psi_{\sigma_2}$. If both σ_i are in P^- then by regularity of $\mathcal{S}^-$, $\sigma_1 = \sigma_2$. If σ_1 is in P^- and σ_2 is in P^+, then we have by definition that $\psi_{\sigma_2}(z) = \psi(z) > \psi_{\sigma_1}(z)$, contradicting the hypothesis that $\psi_{\sigma_1} = \psi_{\sigma_2}$ Finally, suppose both σ_i are in P^+. Then $\sigma_i = \mathrm{Conv}(z, \tau_i)$ for

unique cells τ_i of $\mathcal{S}^0$. As $\mathcal{S}^0$ is regular and $\psi_{\sigma_1}|_H = \psi_{\sigma_2}|_H$, it follows that $\tau_1 = \tau_2$. Hence, $\sigma_1 = \sigma_2$ and we conclude strict convexity. □

Now, consider a regular, unimodular, star-shaped triangulation $\mathcal{T}$ of $P_2^{(n)}$. By Lemma 2.4.6, $\text{Cone}\left(e_n^{(n+1)}, \mathcal{T}\right)$ is a regular triangulation of $\text{Conv}\left(e_n^{(n+1)}, P_2^{(n)}\right)$. By Lemma 2.4.7, gluing this to $\text{Conv}\left(w_1^{(n+1)}, P_2^{(n)}\right)$ yields a regular triangulation $\mathcal{T}'$ of $\text{Conv}\left(w_1^{(n+1)}, \text{Conv}\left(e_n^{(n+1)}, P_2^{(n)}\right)\right)$, which is exactly $P_1^{(n+1)}$. By Lemma 2.4.5, $\mathcal{T}'$ is unimodular. Furthermore, star-shapedness is preserved by taking cones. We have shown therefore that to find a regular, unimodular, star-shaped triangulation of $P_1^{(n+1)}$ it suffices to find a regular, unimodular, star-shaped triangulation of $P_2^{(n)}$. That is, Proposition 2.4.1 is reduced to the following.

Proposition 2.4.8. *For all n, there is a regular, unimodular, star-shaped triangulation of $P_2^{(n)}$.*

In the following section, we will construct these triangulations.

2.4.2 Triangulating P_2

We seek a recursive argument to find regular, unimodular, star-shaped triangulations of the family P_2. Observe that the family of polytopes P_2 all sit inside each other as hyperplane sections setting the last coordinate equal to 0. That is,

$$\{x_n = 0\} \cap P_2^{(n+1)} = P_2^{(n)} \times \{0\},$$

which can be argued similarly to Lemma 2.4.2. Unlike the similar relationship between P_1 and P_2 in Lemma 2.4.2, where precisely two new lattice points are added, $P_2^{(n+1)}$ has many more lattice points than $P_2^{(n)}$. As such, a recursive argument using this containment is inconvenient. Instead, we note that P_2 is (remarkably) isomorphic to its polar dual as a lattice polytope, and that the dual polytopes satisfy a much cleaner recurrence.

Lemma 2.4.9. *For all n, the linear map $T : P_2^{(n)} \longrightarrow \breve{P}_2^{(n)}$ given by*

$$e_i \mapsto (-1, \ldots, s_i - 1, \ldots, -1),$$

where $s_i - 1$ is in the ith position (indexed from 0), is an isomorphism of lattice polytopes.

Proof. We first ensure that T actually sends $P_2^{(n)}$ to $\breve{P}_2^{(n)}$. Recall that $P_2^{(n)}$ has vertices $e_0, \ldots, e_{n-1}, w_2$. We claim then that $T(e_0), \ldots, T(e_{n-1}), T(w_2)$ are the vertices of $\breve{P}_2^{(n)}$. By polar duality (Proposition 2.2.4), this means we must show that these vertices define the half-spaces cutting out $P_2^{(n)}$.

We first compute $T(w_2) = (-1, \ldots, -1)$. Indeed, by linearity of T, this means we must show

$$-1 = -\frac{d_2}{s_i}(s_i - 1) + \sum_{j \neq i} \frac{d_2}{s_i}$$

for all $0 \leq i \leq n-1$, where $d_2 = d_2^{(n)} = s_n - 1$. Sylvester's sequence satisfies the following recurrence:

$$\sum_{i=0}^{n-1} \frac{d_2}{s_i} = d_2 - 1.$$

Hence, for all i,

$$\begin{aligned} -\frac{d_2}{s_i}(s_i - 1) + \sum_{j \neq i} \frac{d_2}{s_i} &= (d-1) - \left(\frac{d}{s_i} + \frac{d}{s_i}(s_i - 1) \right) \\ &= -1, \end{aligned}$$

as desired.

We now seek to show that $T(e_0), \ldots, T(e_{n-1}), T(w_2)$ form the half-spaces cutting out $P_2^{(n)}$. Indeed, to show a simplex $\mathrm{Conv}(v_0, \ldots, v_n)$ is cut out by the $n+1$ half-spaces $\{\langle x, u_i \rangle + 1 \geq 0\}$, we must show that each v_i satisfies equality in n of these inequalities and satisfies the strict inequality in the last. In our context, $v_0 = e_0, \ldots, v_{n-1} = e_{n-1}, v_n = w_2$, and $u_i = T(v_i)$.

First, consider v_i for $i < n$. Then $\langle v_i, u_j \rangle$ is the ith component of u_j, where we index components starting from 0. For $i \neq j$, the ith component of u_j is -1, hence we have equality $\langle v_i, u_j \rangle + 1 = 0$. For $i = j$, the ith component of u_i is $s_i - 1$, so $\langle v_i, u_i \rangle + 1 > 0$.

Now, consider v_n. For u_n, we have

$$\begin{aligned} \langle v_n, u_n \rangle &= \sum_{i=0}^{n-1} \frac{d_2}{s_i} \\ &> 0. \end{aligned}$$

For u_j with $j < n$, the above computation that $T(v_n) = (-1, \ldots, -1)$ showed exactly that $\langle v_n, u_j \rangle + 1 = 0$.

We have computed the vertices of $\check{P}_2^{(n)}$ to be precisely the image of the vertices of $P_2^{(n)}$ under T. To conclude that these simplices are isomorphic as lattice polytopes, we must show that T has determinant ± 1. The matrix representation of T is

$$\begin{pmatrix} 1 & -1 & \ldots & -1 \\ -1 & 2 & & -1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \ldots & s_{n-1} - 1 \end{pmatrix}.$$

We compute this determinant by simultaneous row and column operations to clear the first row and column. For instance, doing this once yields:

$$\det(T) = \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & \dots & -2 \\ 0 & -2 & 3 & & -1 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & -2 & -1 & \dots & s_{n-1} - 2 \end{pmatrix}$$
$$= \det \begin{pmatrix} 1 & -2 & \dots & -2 \\ -2 & 3 & & -1 \\ \vdots & & \ddots & \vdots \\ -2 & -1 & \dots & s_{n-1} - 2 \end{pmatrix},$$

where we are positioned again to clear the first row and column. Performing this process k times from T yields

$$\det(T) = \det \begin{pmatrix} 1 & 1 - s_k & \dots & 1 - s_k \\ 1 - s_k & s_{k+1} + (1 - s_k) & & 1 - s_k \\ \vdots & & \ddots & \vdots \\ 1 - s_k & 1 - s_k & \dots & s_{n-1} + (1 - s_k) \end{pmatrix},$$

which can be verified inductively using the following relation of Sylvester's sequence:

$$(1 - s_k) - (1 - s_k)^2 = 1 - s_{k+1}.$$

Taking $k = n - 1$ we therefore determine $\det(T) = 1$, proving that T yields an isomorphism of lattice polytopes $P_2^{(n)} \xrightarrow{\sim} \check{P}_2^{(n)}$. □

We have therefore reduced the proof of Proposition 2.4.8 (and hence of Proposition 2.4.1) to the following.

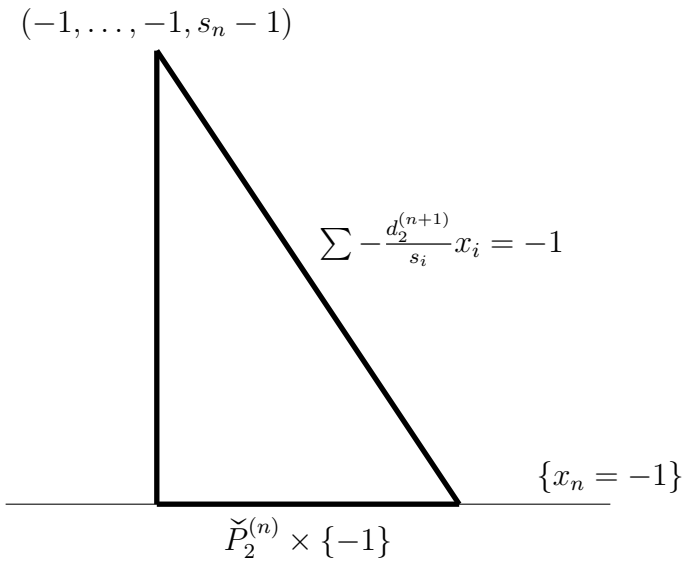


Figure 2.6: A schematic representation of $\breve{P}_2^{(n+1)}$ and its recursive face structure. Literally speaking, the triangle depicted is $\breve{P}_2^{(2)}$.

Proposition 2.4.10. *For all n, there is a regular, unimodular, star-shaped triangulation $\mathcal{T}^{(n)}$ of $\breve{P}_2^{(n)}$.*

We will proceed to construct these triangulations through the remainder of this section. This is notably simpler than working directly with P_2 because the family $\breve{P}_2$ has a much cleaner recursive structure. First, its faces are simple to describe. As $e_0,\ldots,e_n$ are vertices of $P_2^{(n+1)}$, it follows that the coordinate hyperplanes $\{x_i = -1\}$ define all but one of the faces of $\breve{P}_2^{(n+1)}$. The final "slanted" face is defined by $\left\{\sum_{i=0}^{n} -\frac{d_2^{(n+1)}}{s_i}x_i = -1\right\}$. Furthermore, the map $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ taking $x \mapsto (x,-1)$ embeds $\breve{P}_2^{(n)}$ isomorphically as the face $\{x_n = -1\} \cap \breve{P}_2^{(n+1)}$. We refer to this as the "bottom" face of $\breve{P}_2^{(n+1)}$. See Figure 2.6 for a visual of this relationship. This follows directly from the computation in the proof of Lemma 2.4.9 of the vertices of

$\breve{P}_2^{(n+1)}$. Namely, we showed that

$$\breve{P}_2^{(n+1)} = \text{Conv}\left(\begin{pmatrix}-1\\-1\\\vdots\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\\vdots\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\\vdots\\-1\end{pmatrix}, \ldots, \begin{pmatrix}-1\\-1\\\vdots\\s_n-1\end{pmatrix}\right),$$

and setting $x_n = -1$ yields $\breve{P}_2^{(n)}$.

An immediate benefit of the recursive structure of $\breve{P}_2$ is that we can understand the structure of the lattice points within these simplices systematically.

Lemma 2.4.11. *Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be projection onto the first n coordinates. For every $x \in \breve{P}_2^{(n+1)}$, $\pi(x) \in \breve{P}_2^{(n)}$.*

Proof. Let $x \in \breve{P}_2^{(n+1)}$. Then we may write x as a convex combination $x = \sum \lambda_i u_i$, where the u_i are the vertices of $\breve{P}_2^{(n+1)}$ computed above. Take $u_n = (-1, \ldots, -1, s_n - 1)$ and $u_0 = (-1, \ldots, -1)$. Then $\pi(u_n) = \pi(u_0), \ldots, \pi(u_{n-1})$ are the vertices of $\breve{P}_2^{(n)}$. Hence, $\pi(x) = \sum \lambda_i \pi(u_i)$ is a convex combination of vertices of $\breve{P}_2^{(n)}$. So $\pi(x)$ is in $\breve{P}_2^{(n)}$, as desired. □

For $x \in \breve{P}_2^{(n+1)}$, we say x lies "above" $(\pi(x), -1)$. From Lemma 2.4.11, lattice points of $\breve{P}_2^{(n+1)}$ come in columns above the lattice points of $\breve{P}_2^{(n)} \times \{-1\}$. We determine now the highest point on each such column, ordered by x_n-coordinate (recall $\breve{P}_2^{(n+1)}$ is a polytope in $\mathbb{R}^{n+1}$ with coordinates $(x_0, \ldots, x_n)$).

Lemma 2.4.12. *Let $y \in \breve{P}_2^{(n)}$. Let x be the lattice point of $\breve{P}_2^{(n+1)}$ satisfying $\pi(x) = y$ with x_n maximal.*

(a) If $y = (-1, \ldots, -1)$ then $x = (-1, \ldots, -1, s_n - 1)$.

(b) Otherwise, x satisfies

$$x_n = -\sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} x_i.$$

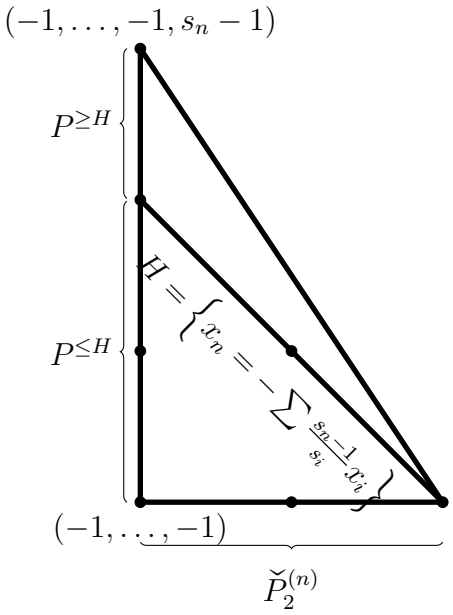


Figure 2.7: A schematic representation of $\check{P}_2^{(n+1)}$ showing its lattice points and how it is cut by the hyperplane H. Literally speaking, the triangle depicted is $\check{P}_2^{(2)}$.

See Figure 2.7.

Proof. (a) Any $z \in \check{P}_2^{(n+1)}$ is of the form $\sum \lambda_i u_i$ where $\lambda_i \geq 0$, $\sum \lambda_i = 1$, and $u_0, \ldots, u_{n+1}$ are the vertices of $\check{P}_2^{(n+1)}$ computed in the course of proving Lemma 2.4.9. Take $u_{n+1} = (-1,\ldots,-1,s_n-1)$ so that the other u_i have last component -1. Hence, the last component of z is $\lambda_n(s_n-1) - \sum_{i=0}^{n-1} \lambda_i$, which is at most $s_n - 1$. So in fact $x = u_{n+1}$ is the highest point on the entirety of $\check{P}_2^{(n+1)}$, and is thus the highest point in its column.

(b) The equation given specifies the last coordinate in terms of the previous ones. Indeed, write

$$x = \left(y_0, \ldots, y_{n-1}, -\sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} y_i \right).$$

We seek to show that x is indeed the highest point above y.

First, we show that $x \in \check{P}_2^{(n+1)}$ by checking that all $x_i \geq -1$ and $-\sum \frac{s_{n+1}-1}{s_i} x_i \geq -1$.

Let's begin with the coordinate inequalities. For $i < n$, we defined $x_i = y_i$, and we

know $y_i \geq -1$ as $y \in \breve{P}_2^{(n)}$. Now, as $y \in \breve{P}_2^{(n)}$ we have

$$-\sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} y_i \geq -1.$$

The left hand side is precisely x_n, so $x_n \geq -1$.

Now, we show the last inequality

$$-\sum_{i=0}^{n} \frac{s_{n+1} - 1}{s_i} x_i \geq -1.$$

Indeed,

$$\begin{aligned} -\sum_{i=0}^{n} \frac{s_{n+1} - 1}{s_i} x_i &= s_n \sum_{i=0}^{n-1} -\frac{s_n - 1}{s_i} x_i - (s_n - 1)x_n \\ &= s_n x_n - (s_n - 1)x_n \\ &= x_n \qquad (*) \\ &\geq -1, \end{aligned}$$

so we have shown $x \in \breve{P}_2^{(n+1)}$.

Finally, we show that nothing above x is in $\breve{P}_2^{(n+1)}$. Indeed, consider $x' = x + e_n$. By assumption, one of the x_i for $i < n$ is strictly greater than -1. Hence,

$$-\sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} x_i < \sum_{i=0}^{n-1} \frac{s_n - 1}{s_i}.$$

The left side of this inequality is x_n. The right side can be computed as

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{s_n - 1}{s_i} &= (s_n - 1)\left(1 - \frac{1}{s_n - 1}\right) \\ &= s_n - 2. \end{aligned}$$

So we conclude that $x_n < s_n - 2$.

Now to show $x' \notin \breve{P}_2^{(n+1)}$ we will show

$$-\sum_{i=0}^{n} \frac{s_{n+1}-1}{s_i} x_i' < -1.$$

Using $(*)$, the left hand side may be computed as

$$\begin{aligned} -\sum_{i=0}^{n} \frac{s_{n+1}-1}{s_i} x_i' &= x_n - \frac{s_{n+1}-1}{s_n} \\ &= x_n - (s_n - 1) \\ &< (s_n - 2) - (s_n - 1) \\ &= -1. \end{aligned}$$

We have shown $x' \notin \breve{P}_2^{(n+1)}$, so x is indeed the highest element in its column.

□

With this precise recursive relationship between the lattice points of $\breve{P}_2^{(n)}$ and $\breve{P}_2^{(n+1)}$, we seek a recursive algorithm to find a regular, unimodular, star-shaped triangulation of $\breve{P}_2^{(n+1)}$ given such a triangulation of $\breve{P}_2^{(n)}$. To do so, we use the notion of the *pulling refinement* due to Haase, Paffenholz, Piechnik, and Santos in [32].

Definition 2.4.13. *Let $\mathcal{S}$ be a subdivision of a polytope $P \subseteq \mathbb{R}^d$ and $m \in P \cap \mathbb{Z}^d$. The* pulling refinement $\text{pull}_m(\mathcal{S})$ *is defined by replacing every $F \in \mathcal{S}$ containing m by $\text{Conv}(m, F')$ for every face $F' \leq F$ which does* not *contain m.*

In [32], the authors prove the following two key properties of the pulling refinement, which we restate here for convenience.

Lemma 2.4.14 (Lemma 2.1 in [32])**.** *If $\mathcal{S}$ is a regular subdivision of a polytope P and $m \in P$ a lattice point, then $\text{pull}_m(\mathcal{S})$ is also a regular subdivision.*

The proof of this proceeds by taking a witness ψ to the regularity of $\mathcal{S}$ and producing a witness ψ' to the regularity of $\text{pull}_m(\mathcal{S})$. ψ' is the piecewise linear function which agrees with ψ on every lattice point of P except m, and has $\psi'(m) = \psi(m) - \varepsilon$ for some sufficiently small $\varepsilon > 0$ (this, incidentally, explains why this is referred to as "pulling" m – we take the graph of ψ and pull $(m, \psi(m))$ down by ε). To ensure $\mathbb{Q}$-linearity, we can simply take ε to be of the form $1/N$ for some sufficiently large N.

Definition 2.4.15. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear and let P be a polytope in $\mathbb{R}^n$. Set $Q = T(P)$. For a polytope $\sigma \subseteq Q$, we define its* pullback *to be*

$$T^*(\sigma) = T^{-1}(\sigma) \cap P,$$

and for a subdivision $\mathcal{S}$ of Q, we define the pullback subdivision *to be*

$$T^*\mathcal{S} = \{T^*(\sigma) \mid \sigma \in \mathcal{S}\}.$$

Proposition 2.4.16 (Theorem 2.8 in [32])**.** *Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be a projection so that we have $\pi(\mathbb{Z}^{n+1}) = \mathbb{Z}^n$. Let P be a lattice polytope in $\mathbb{R}^{n+1}$ and set $Q = \pi(P)$. Suppose $\mathcal{T}$ is a unimodular triangulation of Q such that $\pi^*\mathcal{T}$ is a* lattice *subdivision. Let $\mathcal{T}'$ be a refinement of $\pi^*\mathcal{T}$ arising by pulling $\mathcal{S}$ at all lattice points in P in any order. Then $\mathcal{T}'$ is a unimodular triangulation of P.*

Combining these, we may use pulling refinements to lift regular, unimodular triangulations along projections. (We will see in Lemma 2.4.19 and the discussion afterwards how to preserve the star-shaped condition.) We are tempted then to simply apply this to $P = \check{P}_2^{(n+1)}$ and $Q = \check{P}_2^{(n)}$. However, given a unimodular triangulation $\mathcal{T}$ of $\check{P}_2^{(n)}$ and the map $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ omitting the last coordinate, $\pi^*\mathcal{T}$ is not a *lattice* subdivision. Indeed, the topmost face of $\check{P}_2^{(n+1)}$, defined by $\left\{\sum -\frac{d_2}{s_i} x_i = -1\right\}$, has no lattice points other than its vertices. This can be verified by the formula for the highest lattice points per column given

in Lemma 2.4.12.

Nevertheless, this method does get us quite close to the top. Consider the hyperplane $H = \left\{x_n = -\sum \frac{s_n-1}{s_i} x_i\right\} \subseteq \mathbb{R}^{n+1}$ introduced in Lemma 2.4.12 and let

$$P^{=H} = \left\{ x \in \breve{P}_2^{(n+1)} \,\middle|\, x_n = -\sum_{i=0}^{n} \frac{s_n - 1}{s_i} x_i \right\}$$
$$P^{\leq H} = \left\{ x \in \breve{P}_2^{(n+1)} \,\middle|\, x_n \leq -\sum_{i=0}^{n} \frac{s_n - 1}{s_i} x_i \right\}$$
$$P^{\geq H} = \left\{ x \in \breve{P}_2^{(n+1)} \,\middle|\, x_n \geq -\sum_{i=0}^{n} \frac{s_n - 1}{s_i} x_i \right\}.$$

See Figure 2.7.

π sends $P^{=H}$ isomorphically as a lattice polytope to $\breve{P}_2^{(n)}$ and $\pi(P^{\leq H}) = \breve{P}_2^{(n)}$. Now, we see that a lattice triangulation $\mathcal{T}$ of $\breve{P}_2^{(n)}$ induces a *lattice* subdivision $\pi^*\mathcal{T}$ of $P^{\leq H}$, so Proposition 2.4.16 applies.

The only lattice point in $P^{\geq H}$ which is not on H is $z = (-1, \ldots, -1, s_n - 1)$. And furthermore, $P^{\geq H} = \text{Conv}(z, P^{=H})$. Indeed, P is convex and the condition $x_n \geq -\sum_{i=0}^{n} \frac{s_n-1}{s_i} x_i$ is preserved by convex combinations, so $P^{\geq H}$ is convex and hence contains $\text{Conv}(z, P^{=H})$. For any $y \in P^{\geq H}$ other than z, consider the line connecting z to y. This line will intersect $P^{=H}$ at a unique point x. y is therefore a convex combination of $x \in P^{=H}$ and z. We have thus proven the equality $P^{\geq H} = \text{Conv}(z, P^{=H})$, so we may triangulate $P^{\geq H}$ via $\text{Cone}(z, (\pi^*\mathcal{T})|_H)$.

Lemma 2.4.17. *If $\mathcal{T}$ is unimodular, the subdivision $\text{Cone}(z, (\pi^*\mathcal{T})|_H)$ of $P^{\geq H}$ is unimodular.*

Proof. The linear transformation

$$T(x_0, \ldots, x_n) = \left(x_0, \ldots, x_{n-1}, x_n + \sum_{i=0}^{n} \frac{s_n - 1}{s_i} x_i \right)$$

has determinant 1 and hence preserves unimodularity. For $x \in H$, $T(x) = (\pi(x), 0)$, so T sends $P^{=H}$ isomorphically to $\breve{P}_2^{(n)} \times \{0\}$ and $(\pi^*\mathcal{T})|_H$ isomorphically to $\mathcal{T} \times \{0\}$. Furthermore,

$T(-1,\dots,-1,s_n-1)=(-1,\dots,-1,1)$. We deduce therefore that

$$T\big(P^{\geq H}\big)=\mathrm{Conv}\Big((-1,\dots,-1,1),\check{P}_2^{(n)}\times\{0\}\Big).$$

Hence, T sends $\mathrm{Cone}(z,(\pi^*\mathcal{T})|_H)$ isomorphically to $\mathrm{Cone}((-1,\dots,-1,1),\mathcal{T}\times\{0\})$. This is unimodular by Lemma 2.4.3, as $\mathcal{T}$ is unimodular. □

We have thus far subdivided $P^{\leq H}$ via $\pi^*\mathcal{T}$ and $P^{\geq H}$ via $\mathrm{Conv}(z,(\pi^*\mathcal{T})|_H)$. Gluing these together yields a subdivision $\mathcal{S}$ of $\check{P}_2^{(n+1)}=P^{\leq H}\cup P^{\geq H}$. See Figure 2.8.

Lemma 2.4.18. *If $\mathcal{T}$ is regular then the subdivision $\mathcal{S}$ obtained by gluing $\pi^*\mathcal{T}$ and $\mathrm{Conv}(z,\pi^*\mathcal{T}|_H)$ along H is regular.*

Proof. First, we show that $\pi^*\mathcal{T}$ is regular. Indeed, let $\psi:\check{P}_2^{(n)}\longrightarrow\mathbb{R}$ witness regularity of $\mathcal{T}$. That is, ψ is a continuous, piecewise $\mathbb{Q}$-linear function which is strictly convex with respect to $\mathcal{T}$. Then $\psi\circ\pi:P^{\leq H}\longrightarrow\mathbb{R}$ is continuous, piecewise $\mathbb{Q}$-linear, and strictly convex with respect to $\pi^*\mathcal{T}$, and thus witnesses its regularity.

Now we are in the situation of Lemma 2.4.7, letting $P^-=P^{\leq H}$ subdivided via $\pi^*\mathcal{T}$ and $z=(-1,\dots,-1,s_n-1)$. We therefore conclude that $\mathcal{S}$ is regular. □

So far, we have shown that given a regular, unimodular, star-shaped triangulation $\mathcal{T}^{(n)}$ of $\check{P}_2^{(n)}$ we can form a regular subdivision $\mathcal{S}$ of $\check{P}_2^{(n+1)}$ whose restriction $\mathcal{S}|_{P^{\geq H}}$ is a unimodular triangulation. See Figure 2.8.

To conclude our construction, we can refine $\mathcal{S}$ to a unimodular triangulation $\mathcal{T}^{(n+1)}$ by using the pulling refinement. To ensure that the triangulation is star-shaped, we pull the origin first. See Figure 2.9. More precisely, we can show the following.

Lemma 2.4.19. *Any refinement of the subdivision $\mathrm{pull}_0\big(\pi^*\mathcal{T}^{(n)}\big)$ of $P^{\leq H}$ is star-shaped.*

Proof. The cells of $\pi^*\mathcal{T}^{(n)}$ are pullbacks $\pi^*(\sigma)$ of cells $\sigma\in\mathcal{T}^{(n)}$. Each such σ is a unimodular

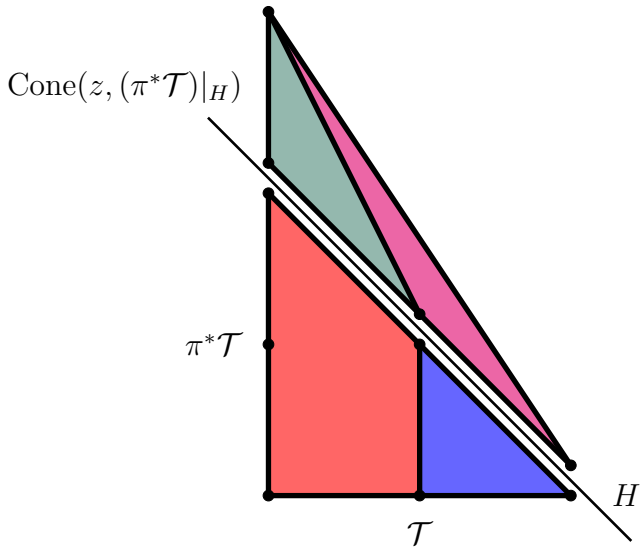
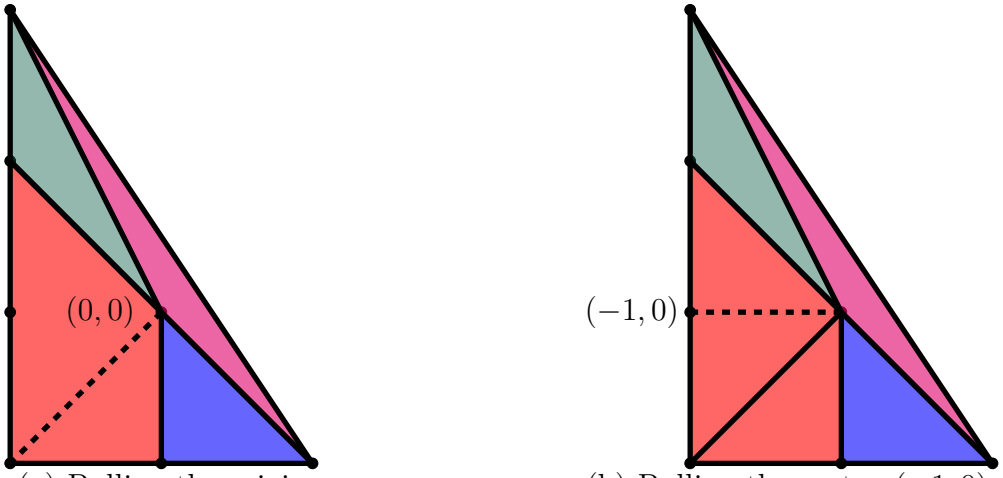


Figure 2.8: $\breve{P}_2^{(n+1)} = P^{\leq H} \cup P^{\geq H}$ with its regular subdivision $\mathcal{S}$ glued from $\pi^*\mathcal{T}$ and Cone$(z, (\pi^*\mathcal{T})|_H)$ along H.

simplex containing the origin (in $\mathbb{R}^n$) as a vertex. It suffices to focus our attention on any such cell $\pi^*(\sigma)$.

The facets of the column $\pi^*(\sigma)$ are the "bottom cap" $\sigma \times \{-1\}$, the "top cap" $\pi^*(\sigma) \cap H$, and the pullbacks $\pi^*(F)$ of facets F of σ. As σ is a simplex, there is precisely one facet $F_0 \leq \sigma$ which does not contain the origin (in $\mathbb{R}^n$). The pullback $\pi^*(F_0)$ and the bottom cap $\sigma \times \{-1\}$ are the only facets of $\pi^*(\sigma)$ not containing the origin (in $\mathbb{R}^{n+1}$). The effect on $\pi^*(\sigma)$ of pulling the origin is thus to replace the facets $\sigma \times \{-1\}$ and $\pi^*(F_0)$ with their cones to the origin Conv$(0, \sigma \times \{-1\})$ and Conv$(0, \pi^*(F_0))$.

Conv$(0, \sigma \times \{-1\})$ is a unimodular simplex, so it cannot be subdivided further, and it already contains the origin. We are thus left to show that for any subdivision of Conv$(0, \pi^*(F_0))$, the origin is a vertex of every cell. Crucially, we have that for a lattice point m of Conv$(0, \pi^*(F_0))$, either $m = 0$ or m lies in the facet $\pi^*(F_0)$. Indeed, as σ is unimodular, its only lattice points are its vertices, which are either the origin itself or are vertices of F_0. Suppose m is not in $\pi^*(F_0)$. Then m must lie above $(0, \ldots, 0, -1)$. Only $(0, \ldots, 0, -1)$ and $(0, \ldots, 0, 0)$ lie above this point due to Lemma 2.4.12, so m must equal one of these two points. A representation of



(a) Pulling the origin. (b) Pulling the vertex $(-1, 0)$.

Figure 2.9: The result of pulling the subdivision $\mathcal{S}$ as in Figure 2.8 at the origin (a) then at the remaining vertices (b). The new edges are depicted with dotted lines.

m as a convex combination of 0 and the vertices of $\pi^*(F_0)$ must contain a positive coefficient for 0, as m is not in $\pi^*(F_0)$. Thus, the last coordinate of m must be strictly greater than -1, so m cannot be $(0, \dots, 0, -1)$. We conclude that $m = 0$.

Now, let Σ be a cell of a subdivision of $\mathrm{Conv}(0, \pi^*(F_0))$. For dimension reasons, Σ cannot be wholly contained in $\pi^*(F_0)$. Hence, it must have some vertex outside of $\pi^*(F_0)$. As discussed above, the only possibility is the origin. We have thus shown that every cell of $\mathrm{pull}_0(\pi^*\mathcal{T}^{(n)})$ contains the origin as a vertex, as desired. □

Now, we pull all the remaining vertices in any order to define our triangulation $\mathcal{T}^{(n+1)}$. By Lemma 2.4.14, this is regular. By Proposition 2.4.16, $\mathcal{T}^{(n+1)}|_{P^{\leq H}}$ is unimodular. And as shown above, $\mathcal{T}^{(n+1)}|_{P^{\geq H}} = \mathcal{S}|_{P^{\geq H}}$ is unimodular. Finally, by Lemma 2.4.19, $P^{\leq H}$ is star-shaped, and $P^{\geq H}$ is star-shaped as it is a cone of $(\pi^*\mathcal{T}^{(n)})|_H$, which is isomorphic to the star-shaped triangulation $\mathcal{T}^{(n)}$. We have therefore found a regular, unimodular, star-shaped triangulation $\mathcal{T}^{(n+1)}$ of $\check{P}_2^{(n+1)}$ that extends $\mathcal{T}^{(n)}$ of $\check{P}_2^{(n)}$. See Figure 2.9. The base case is trivial, as $\check{P}_2^{(1)} = [-1, 1]$.

The above construction proves Proposition 2.4.10, which as discussed proves Proposition 2.4.8. Finally, as shown in Section 2.4.1, this proves Proposition 2.4.1. As discussed in

Section 2.3, we conclude with a family of toric, projective, crepant resolutions of the weighted projective spaces $\mathbb{P}_1^{(n)}$ (as well as a compatible family of toric, projective, crepant resolutions of the $\mathbb{P}_2^{(n)}$).

2.5 Resolving the hypersurface within

Consider the quasi-smooth weighted projective hypersurface

$$X_1^{(n)} = \{x_0^{s_0} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1}x_{n+1} + x_{n+1}^d = 0\} \subseteq \mathbb{P}_1^{(n+1)},$$

where $d = d_1^{(n+1)} = 2s_n - 2$. (Recall that a subvariety of a weighted projective space is called *quasi-smooth* if its affine cone is smooth away from the origin.) For simplicity, we write $X = X_1^{(n)}$ and $\mathbb{P} = \mathbb{P}_1^{(n+1)}$. X is a Calabi–Yau variety of dimension n from which Esser, Totaro, and Wang in [18] define a klt Calabi–Yau pair of dimension n and a terminal Calabi–Yau variety of dimension $n+1$ with conjecturally maximal indices.

We recall these constructions. Let $m = (s_n - 1)(2s_n - 3)$. Then μ_m acts on X via

$$\zeta[x_0, \ldots, x_{n+1}] = \left[\zeta^{d/(2s_0)}x_0 : \cdots : \zeta^{d/(2s_{n-1})}x_{n-1} : x_n : \zeta^{d/2}x_{n+1}\right].$$

The quotient X/μ_m is a klt Calabi–Yau pair of dimension n with standard coefficients and index m by [18, Proposition 3.7]. This is the conjecturally maximal index for such an n-fold.

For the terminal example, consider a μ_m-equivariant terminalization $Z \longrightarrow X$. Then let E be an elliptic curve and $p \in E$ be a point of order m. Let μ_m act on $Z \times E$ via

$$\zeta(x, y) = (\zeta x, y + p).$$

Then $(Z \times E)/\mu_m$ is a terminal Calabi–Yau variety of dimension $n+1$ with index m, which is conjecturally maximal among such varieties.

We seek a smooth, projective Calabi–Yau variety of dimension $n+1$ and index m. To do so, it suffices to find a μ_m-equivariant, crepant, projective resolution $\widetilde{X} \longrightarrow X$ and repeat the above construction with the elliptic curve. Furthermore, this crepant resolution $\widetilde{X}$ will be our smooth, projective Calabi–Yau variety with large sums of Betti numbers.

In Section 2.4, we constructed a toric, projective, crepant resolution $\pi : \widetilde{\mathbb{P}} \longrightarrow \mathbb{P}$. Now, as X is a quasi-smooth hypersurface in $\mathbb{P}$, its only singularities are the singularities of the ambient space, so we are led to believe that by resolving the singularities of the ambient space, we have resolved the singularities of X. More precisely,

Theorem 2.5.1. *Let L be the linear system in $\mathbb{P}$ generated by the monomials*

$$x_0^{s_0}, \ldots, x_{n-1}^{s_{n-1}}, x_n^{d-1}x_{n+1}, x_{n+1}^d.$$

Then for a generic element $X \in L$, $\pi^{-1}X \longrightarrow X$ is a projective, crepant, μ_m-equivariant resolution of singularities.

To work with the canonical divisors of these hypersurfaces, we will need to use the adjunction formula. As $\widetilde{\mathbb{P}}$ is smooth, the adjunction formula is known. However, the adjunction formula is not true for every weighted projective hypersurface. The technical condition that assures this holds is *well-formedness*. We refer to [38] for details.

Definition 2.5.2. *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space and $X \subseteq \mathbb{P}$ a hypersurface.*

(i) $\mathbb{P}$ *is called* well-formed *if for all i we have* $\gcd(a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 1$.

(ii) $X \subseteq \mathbb{P}$ *is called* well-formed *if* $\mathbb{P}(a_0, \ldots, a_n)$ *is well-formed and X contains no codimension 2 singular coordinate stratum of* $\mathbb{P}(a_0, \ldots, a_n)$.

We first note that the weights of a weighted projective space are not uniquely determined, and in fact every weighted projective space is isomorphic to a well-formed one. For

instance, using the Veronese map, we have that $\mathbb{P}(da_0, \ldots, da_n) \cong \mathbb{P}(a_0, \ldots, a_n)$. Hence, we can reduce the weights of a weighted projective space to have no common factors. Furthermore, if $a_0, \ldots, a_n$ have no common factor and $d = \gcd(a_1, \ldots, a_n)$ then $\mathbb{P}(a_0, \ldots, a_n) \cong \mathbb{P}(a_0, a_1/d, \ldots, a_n/d)$. Repeatedly applying this allows the weights of any weighted projective space to be reduced to be well-formed.

Finally, as discussed in [38, 6.14], the adjunction formula holds for well-formed, quasi-smooth weighted projective hypersurfaces. We may now proceed to prove Theorem 2.5.1.

Proof of Theorem 2.5.1. Projectivity of $\pi^{-1}X \longrightarrow X$ follows from projectivity of π and X.

Let $\pi^{-1}L$ be the total transform of the linear system L. Let $\widetilde{p}$ be a base point of $\pi^{-1}L$. Then $p = \pi(\widetilde{p})$ is a base point of L, hence $p = [0 : \cdots : 0 : 1 : 0]$. This is a smooth point of $\mathbb{P}$, corresponding to the cone in $\mathbb{R}^{n+1}$ spanned by the unimodular simplex $\mathrm{Conv}(e_0, \ldots, e_{n-1}, w_1)$. The triangulation of $P_1^{(n+1)}$ defining π leaves the cone $\mathrm{Conv}(e_0, \ldots, e_{n-1}, w_1)$ unchanged, so π is an isomorphism near p. Hence, $\widetilde{p}$ is the only base point of $\pi^{-1}L$. By Bertini's theorem, a generic element of $\pi^{-1}L$ is therefore smooth away from $\widetilde{p}$. Furthermore, p is a smooth point of X, as X is a quasi-smooth hypersurface in $\mathbb{P}$ and $p \in \mathbb{P}$ is a smooth point. As π is an isomorphism near p, it follows that $\widetilde{p}$ is a smooth point of $\pi^{-1}X$. We have shown that every point of $\pi^{-1}X$ is smooth, for generic $X \in L$.

To show crepancy of $\pi^{-1}X \longrightarrow X$, we will need to use the adjunction formula. As discussed above, we will need to show a generic $X \in L$ is well-formed.

First, we show that $\mathbb{P} = \mathbb{P}\left(d_1^{(n+1)}/s_0, \ldots, d_1^{(n+1)}/s_{n-1}, 1, 1\right)$ is well formed. Indeed, there are two weights equal to 1, so if we omit only one weight the remaining $n+1$ terms contain a 1, and as such their greatest common divisor is 1.

We now show that a generic $X \in L$ does not contain any singular codimension 2 stratum of $\mathbb{P}$. Indeed, the base locus of L is, as above, just the point $p = [0 : \cdots : 0 : 1 : 0]$. So if $\dim(\mathbb{P}) \geq 3$, we can ensure that there is no codimension 2 stratum whatsoever in a generic X. When $\dim(\mathbb{P}) = 2$, the base locus $\{p\}$ is codimension 2, so any $X \in L$ contains p. However,

as mentioned above, p is a smooth point of $\mathbb{P}$, so there is no *singular* codimension 2 stratum for X generic.

We therefore apply the adjunction formula and write $K_X = (K_{\mathbb{P}} + X)|_X$. Using this and crepancy of $\pi : \widetilde{\mathbb{P}} \longrightarrow \mathbb{P}$, we will show that $\pi^{-1}X \longrightarrow X$ is crepant. Indeed,

$$\begin{aligned} K_{\pi^{-1}X} &= (K_{\widetilde{\mathbb{P}}} + \pi^{-1}X)|_{\pi^{-1}X} \\ &= \pi^{-1}(K_{\mathbb{P}} + X)|_X \\ &= \pi^{-1}K_X. \end{aligned}$$

Finally, π is equivariant and maps the torus of $\widetilde{\mathbb{P}}$ isomorphically to the torus of $\mathbb{P}$, so the subgroup μ_m of $T_{\mathbb{P}} = T_{\widetilde{\mathbb{P}}}$ fixing X also fixes $\pi^{-1}X$. □

Repeating the above construction then yields our smooth, projective Calabi–Yau variety of large index.

Corollary 2.5.3. *For every n, there is a smooth, projective Calabi–Yau variety of dimension $n+1$ with index $(s_n - 1)(2s_n - 3)$.*

We have now proven our first main theorem Theorem 2.1.1. However our examples are not fully explicit due to the genericity in Theorem 2.5.1. We now show that the generic examples defined there are isomorphic to an explicitly chosen one. Suppose we have an element $X \in L$ given by

$$\{a_0 x_0^{s_0} + \cdots + a_n x_n^{d-1} x_{n+1} + a_{n+1} x_{n+1}^d = 0\}$$

where all a_i are nonzero. Then consider the toric automorphism of $\mathbb{P}$ given by

$$[x_0 : \cdots : x_n] \mapsto \left[a_0^{(1/s_0)} x_0 : \cdots : (a_n a_{n+1}^{-1/d})^{1/(d-1)} x_n : (a_{n+1}^{1/d}) x_{n+1}\right].$$

This restricts to an isomorphism

$$X \xrightarrow{\sim} X_1^{(n)} = \{x_0^{s_0} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1}x_{n+1} + x_{n+1}^d = 0\} \subseteq \mathbb{P}_1^{(n+1)}.$$

Furthermore, as $T_{\widetilde{\mathbb{P}}} = T_{\mathbb{P}}$ this extends to a toric automorphism of $\widetilde{\mathbb{P}}$ which restricts to an isomorphism of their total transforms.

$$\pi^{-1}X \xrightarrow{\sim} \pi^{-1}X_1^{(n)}.$$

As this isomorphism is given via translation by a torus element, it preserves stabilizers and is hence μ_m-equivariant. The above construction then yields *explicit* smooth, projective Calabi–Yau varieties of large index, up to the choice of the order in which we pull vertices to conclude the triangulation discussed at the end of Section 2.4.2.

Furthermore, as mentioned in Section 2.1 the crepant resolutions thus constructed have extreme topological properties. Theorem 5.1 in [18] computes the *orbifold* Hodge numbers of $X_1^{(n)}$. By [67], these are known to agree with the actual Hodge numbers of a crepant resolution of $X_1^{(n)}$, which we have just constructed. Furthermore, en route to crepantly resolving $\mathbb{P}_1$, we crepantly resolved $\mathbb{P}_2$. In [18], the authors define the quasi-smooth weighted projective hypersurfaces

$$X_2^{(n)} := \{x_0^{s_0} + \cdots + x_n^{s_n} + x_{n+1}^{s_{n+1}-1} = 0\} \subseteq \mathbb{P}_2^{(n+1)}$$

which, by an argument as in Theorem 2.5.1, admit projective, crepant resolutions. Let $\widetilde{X}_i^{(n)}$ denote the projective, crepant resolutions of $X_i^{(n)}$ we have thus constructed. We have then an analogous result to Theorem 5.1 in [18].

Theorem 2.5.4. *Let $n \geq 1$.*

(i) $h^{p,q}(\widetilde{X}_i^{(n)}) = 0$ for $i = 1, 2$ unless $p = q$ or $p + q = n$.

(ii) The sum of the Betti numbers of $\widetilde{X}_i^{(n)}$ *is*

$$H = 2(s_0 - 1) \cdots (s_n - 1),$$

which equals the Euler characteristic when n *is even.*

(iii) If n *is odd then*

$$h^n(\widetilde{X}_i^{(n)}) = \begin{cases} (s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 4) & i = 1 \\ (s_0 - 1) \cdots (s_n - 1) & i = 2, \end{cases}$$

and the Euler characteristic is

$$\chi(\widetilde{X}_i^{(n)}) = \begin{cases} -(s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 6) & i = 1 \\ 0 & i = 2. \end{cases}$$

Proof. As $\widetilde{X}_i^{(n)} \longrightarrow X_i^{(n)}$ is a crepant resolution, we have that $h^{p,q}(\widetilde{X}_i^{(n)}) = h^{p,q}(X_i^{(n)})$ by [67]. The result then follows from Theorem 5.1 in [18], where they compute these exact values as the orbifold Hodge/Betti numbers of $X_i^{(n)}$. □

This shows our second main theorem, Theorem 2.1.4. Additionally, see Figure 2.10 for the Hodge diamonds of our 3 and 4 dimensional examples, as computed in [18].

Recall Conjecture 2.1.5, which states that these examples have the largest Betti number sums and for odd n, that the Euler characteristics are as negative as possible. There is a notable omission here. In [18], the authors also constructed a third quasi-smooth weighted projective hypersurface $X_3^{(n)} \subseteq \mathbb{P}_3^{(n+1)}$, which they conjectured to have the largest possible orbifold Euler characteristic among projective varieties with quotient singularities and trivial canonical class in odd dimensions. $X_3^{(n)}$ is mirror to $X_1^{(n)}$ (also, $X_2^{(n)}$ is self-mirror), and its orbifold Euler characteristic is negative to that of $X_1^{(n)}$. We have not constructed a crepant

| | | |

resolution of $X_3^{(n)}$ so we are as of yet unable to extend these computations and conjectures to actual Betti numbers in the smooth case.

CHAPTER 3

The topology of the Hilbert scheme

3.1 Introduction

The *Hilbert scheme* (as developed by Grothendieck in [31]) of a scheme X parametrizes the closed subschemes of X. (Here, all schemes are locally Noetherian.) It appears centrally in the construction of many other moduli spaces, such as the moduli space of curves and, more recently, the moduli space of K-semistable Fano varieties, as in [8, Thm. 1.3].

More formally, we define the *Hilbert functor* of X, denoted $\mathcal{Hilb}(X) : \mathsf{Schemes}^{\mathrm{op}} \longrightarrow \mathsf{Sets}$, via

$$Y \mapsto \{Z \subseteq X \times Y \text{ a closed subscheme} \mid \pi : Z \longrightarrow Y \text{ is flat}\}.$$

That is, the Y-points of this functor are the flat families of closed subschemes $\pi^{-1}(y)$ parametrized by $y \in Y$. In particular, for a field k, $\mathcal{Hilb}(X)(k)$ is the set of closed subschemes of X_k.

When this functor is representable, we call its representing object the *Hilbert scheme* of X, denoted by $\mathrm{Hilb}(X)$. It follows from Yoneda's lemma that there is then a universal family $U = \{(x, [F]) \in X \times \mathrm{Hilb}(X) \mid x \in F\}$ associated to $\mathrm{id} \in \mathrm{Hom}(\mathrm{Hilb}(X), \mathrm{Hilb}(X)) \cong \mathcal{Hilb}(X)(\mathrm{Hilb}(X))$, so that any flat family Z of closed subschemes of X parametrized by Y (i.e. a Y-point of $\mathcal{Hilb}(X)$) arises as the pullback of a unique morphism $Y \longrightarrow \mathrm{Hilb}(X)$.

$$\begin{array}{ccc} Z & \dashrightarrow & U \\ \downarrow & \ulcorner & \downarrow \\ Y & \overset{\exists!}{\dashrightarrow} & \mathrm{Hilb}(X) \end{array}$$

This is very similar to the idea of classifying spaces in topology, such as the Grassmannian $\mathrm{Gr}_d(\mathbb{R}^\infty)$, where there exists a universal family (the tautological rank d vector bundle) so that the set of vector bundles of rank d on a space X corresponds to the homotopy classes of maps $f : X \longrightarrow \mathrm{Gr}_d(\mathbb{R}^\infty)$ by pulling back the universal family along f.

Perhaps the simplest example of this phenomenon in algebraic geometry is the Hilbert

scheme of plane conics, i.e. curves in $\mathbb{P}^2$ cut out by a single quadratic equation

$$\{a_0x_0^2 + a_1x_0x_1 + a_2x_1^2 + a_3x_0x_2 + a_4x_1x_2 + a_5x_2^2 = 0\}.$$

A conic is determined by the coefficients of this equation (at least one of which must be nonzero), up to scaling. The Hilbert scheme of plane conics is thus $\mathbb{P}^5$ with coordinates $[a_0 : \cdots : a_5]$. The universal family is

$$U = \{([x], [a]) \in \mathbb{P}^2 \times \mathbb{P}^5 \mid a_0x_0^2 + a_1x_0x_1 + a_2x_1^2 + a_3x_0x_2 + a_4x_1x_2 + a_5x_2^2 = 0\}.$$

The fiber of $U \longrightarrow \mathbb{P}^5$ over $[a_0 : \cdots : a_5]$ is precisely the conic defined by these coefficients. Furthermore, a morphism $Y \longrightarrow \mathbb{P}^5$ is exactly a collection of algebraic morphisms $[a_0(y) : \cdots : a_5(y)]$ for $y \in Y$, and hence an algebraically varying family of conics

$$\{a_0(y)x_0^2 + a_1(y)x_0x_1 + a_2(y)x_1^2 + a_3(y)x_0x_2 + a_4(y)x_1x_2 + a_5(y)x_2^2 = 0\}.$$

The pullback $Z = U \underset{\mathbb{P}^5}{\times} Y$ is the graph of this family over Y.

Grothendieck proved in [31, Thm 3.1] that the Hilbert functor of $\mathbb{P}^n$ is representable. Moreover, the Hilbert polynomial is locally constant in flat families of closed subschemes of projective space, so the Hilbert scheme decomposes into

$$\mathrm{Hilb}(\mathbb{P}^n) = \coprod_p \mathrm{Hilb}_p(\mathbb{P}^n),$$

where $\mathrm{Hilb}_p(\mathbb{P}^n)$ consists of those closed subschemes with Hilbert polynomial p. For example, a plane conic has Hilbert polynomial $p(m) = 2m - 1$, so the Hilbert scheme of plane conics is denoted $\mathrm{Hilb}_{2m+1}(\mathbb{P}^2)$, which we stated above is isomorphic to $\mathbb{P}^5$. Grothendieck also proved in [31, Thm. 3.2] that each of these pieces $\mathrm{Hilb}_p(\mathbb{P}^n)$ are projective over $\mathrm{Spec}(\mathbb{Z})$, and in [34, Cor. 5.9], Hartshorne proved that $\mathrm{Hilb}_p(\mathbb{P}^n)$ is connected, so that these pieces are precisely

the connected components of the Hilbert scheme of $\mathbb{P}^n$.

Representability of the Hilbert scheme is a crucial first step in the construction of moduli spaces, as it provides an (enormous) parameter space for projective varieties. For instance, the moduli space of smooth, projective curves of genus $g \geq 2$, denoted $\mathcal{M}_g$, can be constructed as follows.

(i) Show that for any such curve C, the complete linear system $|3K_C|$ embeds C into $\mathbb{P}^{5g-6}$, and thus can be viewed as an element of $\mathrm{Hilb}(\mathbb{P}^{5g-6})$.

(ii) Show that the set of smooth, projective curves of genus g embedded tricanonically as above forms a locally closed subset U_g of this Hilbert scheme.

(iii) The ambiguity in picking a tricanonical embedding $C \hookrightarrow \mathbb{P}^{5g-6}$ is in picking a basis for $H^0(C, 3K_C)$, which is $5g-5$ dimensional, so we conclude by forming the quotient U_g/PGL_{5g-5}.

See [33, Ch. 2, §C] for more details.

Many other moduli spaces, such as the moduli space of K-semistable Fano varieties, are constructed similarly as quotients of locally closed subsets of Hilbert schemes, reflecting the central role the Hilbert scheme plays in moduli theory.

Consider the moduli of K-semistable Fano varieties. Step (i) requires boundedness, as shown by Jiang in [12]. Step (ii) requires proving openness of K-semistability, as done by Blum, Liu, and Xu in [8]. In step (iii), the group we quotient by should be reasonable, and Alper, Blum, Halpern-Leistner, and Xu proved reductivity in [1]. Each step is a deep result, so this program is by no means easy to execute, but the existence of the Hilbert scheme remains a fundamental starting point in constructing moduli spaces.

It should come as no surprise, then, that omnipresence begets deep complexity. Vakil proved in [65] that many Hilbert schemes, such as the Hilbert scheme of nonsingular curves

in projective space, satisfy "Murphy's law", i.e. that they contain every possible singularity type which is finite type over $\mathbb{Z}$.

The simplest closed subschemes are the 0-dimensional ones, which have constant Hilbert polynomials. As we are interested in the topology of Hilbert schemes, the simplest example for us will be $\mathrm{Hilb}_d(\mathbb{A}^n)$, which we call the *Hilbert scheme of d points in* $\mathbb{A}^n$. This is the (étale) local model for Hilbert schemes of d points on any smooth variety. It, too, is an extremely pathological object. Jelisiejew proved Murphy's law up to retractions for $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16})$ in [42], which implies for instance that it has entirely non-reduced components. Farkas, Pandharipande, and Sammartano prove in [19] that there are irrational irreducible components of the Hilbert scheme of points in affine space.

For low values of d or n, we can sometimes get a concrete handle on $\mathrm{Hilb}_d(\mathbb{A}^n)$. In every case, there is a morphism, called the *Hilbert–Chow morphism*

$$\mathrm{Hilb}_d(\mathbb{A}^n) \xrightarrow{\mathrm{HC}} \mathrm{Sym}_d(\mathbb{A}^n) = \frac{(\mathbb{A}^n)^d}{S_d},$$

which takes a closed subscheme $Z \subseteq \mathbb{A}^n$ to its support, counted with multiplicity:

$$\mathrm{HC}([Z]) = \sum_{p \in \mathbb{A}^n} \mathrm{length}(\mathcal{O}_{Z,p})p.$$

The open subset of $\mathrm{Hilb}_d(\mathbb{A}^n)$ consisting of d distinct points maps isomorphically to the open subset of $\mathrm{Sym}_d(\mathbb{A}^n)$ defined in the same way. $\mathrm{Hilb}_d(\mathbb{A}^n)$ can have many irreducible components, whereas $\mathrm{Sym}_d(\mathbb{A}^n)$ is clearly irreducible. See [10, 41] for more discussion. Therefore, HC is not birational, unless it happens to be the case that $\mathrm{Hilb}_d(\mathbb{A}^n)$ is irreducible, such as for $d = 2$, due to [20, Thm. 2.4] or $\mathrm{Hilb}_{11}(\mathbb{A}^3)$, due to [15].

For $d = 1$ and any scheme X, $\mathrm{Hilb}_1(X) = \mathrm{Sym}_1(X) = X$, with universal family $\Delta \subseteq X \times X$.

For $d = 2$, there are two possible types of length two subschemes Z of a smooth variety

X over an algebraically closed field. One possibility is that Z consists of two distinct points p, q. The only point in the fiber $\mathrm{HC}^{-1}(p+q)$ is the reduced subscheme $\{p, q\}$. The other possibility is that Z is isomorphic to $k[\varepsilon]/(\varepsilon^2)$ and consists of one point p with a chosen tangent direction in $\mathbb{P}(T_pX)$. The fiber $\mathrm{HC}^{-1}(2p)$ is thus identified with $\mathbb{P}(T_p(X))$. It follows that $HC : \mathrm{Hilb}_2(X) \longrightarrow \mathrm{Sym}_2(X)$ is the blowup along the diagonal.

For X a smooth curve, $\mathrm{HC} : \mathrm{Hilb}_d(X) \longrightarrow \mathrm{Sym}_d(X)$ is an isomorphism. Indeed, the only length d subscheme concentrated at a smooth point of a curve is $\operatorname{Spec} k[t]/(t^d)$.

In striking contrast to the higher dimensional case, the Hilbert scheme of points on a smooth, connected surface is smooth and irreducible, due to Fogarty in [20, Thm. 2.4]. In fact, the Hilbert–Chow morphism is a crepant resolution of singularities. See [53] for more on Hilbert schemes of points on surfaces

In general, the structure of $\mathrm{Hilb}_d(\mathbb{A}^n)$ is very difficult to handle. The structure of its irreducible components, and even its dimension, is unknown in general. See [39, 10, 15, 41].

We generally work over an arbitrary field k in the Zariski topology, but when discussing the homology of the Hilbert scheme we will take $k = \mathbb{C}$ and use the analytic topology. For clarity, we will write $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ in the latter situation.

The first result on the topology of $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ comes from Hartshorne's proof of the connectedness of the Hilbert scheme in [34]. Since then, the range of i, n, d for which $H_i(\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C}); \mathbb{Z})$ is known has grown. See Table 3.1 for a summary of what is known. More details will be provided in Section 3.3.

Authors	Year	i	n	d
Hartshorne, [34]	1966	0	all	all
Ellingsrud, Strømme, [16]	1987	all	2	all
Hoyois, Jelisiejew, Nardin, Totaro, Yakerson, [35]	2024[1]	$i \leq 2n - 2d + 2$	$n \geq d$	all
Khan, [46]	2022	all	all	$d \leq 4$

Table 3.1: Known homology groups for the Hilbert scheme of points in affine space.

Our main theorem in this chapter extends Khan's work and computes $H_i(\mathrm{Hilb}_5(\mathbb{A}^n)(\mathbb{C});\mathbb{Z})$ for all i and n. This will be shown in Section 3.4.

Theorem 3.1.1 (S.). *The integral homology of* $\mathrm{Hilb}_5(\mathbb{A}^n)(\mathbb{C})$ *is finitely generated, torsion free, concentrated in even degrees, and has the following Poincaré polynomial:*

$$z^{2(3n-3)}g_{1,n}(z) + z^{4(n-1)}(1+z^2)g_{2,n}(z) + z^{2(2n-4)}g_{1,3}(z)g_{2,n}(z) + z^{2(n-3)}g_{1,6}(z)g_{3,n}(z) + g_{4,n}(z),$$

where $g_{k,n}$ *is the Poincaré polynomial of* $\mathrm{Gr}_k(\mathbb{C}^n)$.

3.2 Preliminaries

3.2.1 The (punctual) Hilbert scheme of points

As discussed in Section 3.1, our object of study is the Hilbert scheme of d points in $\mathbb{A}^n$.

Definition 3.2.1. *The Hilbert scheme of* d *points in* $\mathbb{A}^n$ *is the representing object to the functor*

$$Y \mapsto \{Z \subseteq Y \times \mathbb{A}^n \mid Z \longrightarrow Y \text{ is finite locally free of degree } d\},$$

denoted by $\mathrm{Hilb}_d(\mathbb{A}^n)$

For more information, such as the representability of this functor, see [61, Tag 0B94, Lemma 44.2.5].

Let k be a field. By the Nullstellensatz, a k-point of $\mathrm{Hilb}_d(\mathbb{A}^n)$ can be represented as a closed subscheme X of length d in $\mathbb{A}^n_k$ or as an ideal $I \subseteq k[x_1,\ldots,x_n]$ so that $k[x_1,\ldots x_n]/I$ is d-dimensional as a k-vector space. We denote the point of the Hilbert scheme associated to a closed subscheme $X \subseteq \mathbb{A}^n_k$ by $[X]$, and the point associated to an ideal $I \subseteq k[x_1,\ldots,x_n]$ by $[I]$.

[1]This article was put on the arXiv in 2020 as arXiv:2002.11439, so we place it in the table chronologically before Khan's work, which refers to this article.

Definition 3.2.2. *The punctual Hilbert scheme of d points in $\mathbb{A}^n$ is the reduced closed subscheme of* $\mathrm{Hilb}_d(\mathbb{A}^n)$*, denoted by* $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$*, consisting of those closed subschemes which are supported at the origin.*

In other words, $[X] \in \mathrm{Hilb}_d(\mathbb{A}^n, 0)$ if and only if $\mathrm{HC}([X]) = d \cdot 0 \in \mathrm{Sym}_d(\mathbb{A}^n)$. The condition that a closed subscheme X of $\mathbb{A}^n$ is supported at the origin is equivalent to its ideal I satisfying $\sqrt{I} = (x_1, \ldots, x_n)$.

Totaro proved the following in [64, Cor. 2.3].

Theorem 3.2.3 (Totaro)**.** *The inclusion* $\mathrm{Hilb}_d(\mathbb{A}^n, 0)(\mathbb{C}) \longrightarrow \mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ *is a homotopy equivalence.*

(Totaro conjectured that they are, in fact, $\mathbb{A}^1$-homotopy equivalent, and proved this up to suspension by $S^{3,1}$ in characteristic 0 in [64, Thm. 8.1]). We therefore reduce to computing the homology of the punctual Hilbert scheme of points. This is notably simpler than the whole Hilbert scheme of points. For instance, we have the following.

Proposition 3.2.4. *The punctual Hilbert scheme* $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ *is a projective variety.*

Proof. Denote $R = k[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$.

Let $[I] \in \mathrm{Hilb}_d(\mathbb{A}^n, 0)$. Then as R/I is d-dimensional over k, it follow that $\mathfrak{m}^d \subseteq I$. We may this view I as an ideal in $R/\mathfrak{m}^d$. I is, in particular, a codimension d k-linear subspace of the finite dimensional vector space $R/\mathfrak{m}^d$. Let $\mathrm{Gr}^d(R/\mathfrak{m}^d)$ be the Grassmannian variety of such subspaces. We have an inclusion $\mathrm{Hilb}_d(\mathbb{A}^n, 0) \longrightarrow \mathrm{Gr}^d(R/\mathfrak{m}^d)$.

As the Grassmannian is projective, via the Plücker embedding, we are left to show that this inclusion is closed. Indeed, for a vector subspace $V \in \mathrm{Gr}^d(R/\mathfrak{m}^d)$ to be an ideal of $k[x_1, \ldots, x_n]/\mathfrak{m}^d$, it suffices to check that $x_i V \subseteq V$ for all i. This imposes a finite set of polynomial equations on the Plücker coordinates of V, so the image is (Zariski) closed. □

This is similar to, but vastly simpler than, the proof Grothendieck gave that $\mathrm{Hilb}_p(\mathbb{P}^n)$ is projective by embedding it into products of Grassmannians. That every ideal in $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$

contains a bounded power of $\mathfrak{m}$ (namely $\mathfrak{m}^d$) was a critical step, and Grothendieck had to prove a much more complicated boundedness result to show projectivity in general.

Projectivity imposes strong topological conditions on a complex variety in its analytic topology. For instance, the homology $H_{2i}(X(\mathbb{C});\mathbb{R})$ is never 0 for $0 \leq i \leq \dim(X)$ when X is projective. In fact, the homology class $[V]$ of any algebraic subvariety $V \subseteq X$ of dimension i is nonzero, as $A^i.[V] > 0$ for any ample divisor A. Subvarieties of all dimensions exist by embedding X into projective space and intersecting with an appropriate dimensional and sufficiently general linear subspace.

Another exceedingly important feature of the punctual Hilbert scheme is the following stratification. This is commonly used in analyzing the punctual Hilbert scheme, such as in [10, 24].

Definition 3.2.5. *Let A be a finite dimensional local k-algebra with maximal ideal $\mathfrak{m}$. Its* Hilbert–Samuel function $h_A : \mathbb{N} \longrightarrow \mathbb{N}$ *sends* $i \mapsto \dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1})$.

Definition 3.2.6. *Fix a function $h : \mathbb{N} \longrightarrow \mathbb{N}$. The* Hilbert–Samuel stratum *associated to h in* $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ *is the reduced subscheme*

$$H_h^n = \{X \in \mathrm{Hilb}_d(\mathbb{A}^n, 0) \mid h_{\mathcal{O}(X)} = h\}.$$

Lemma 3.2.7. *The* Hilbert–Samuel strata H_h^n *are Zariski locally closed subsets of* $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$.

Proof. Let $R = k[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Denote by R_i the degree i homogeneous polynomials in R. Let $[I] \in \mathrm{Hilb}_d(\mathbb{A}^n, 0)$. The maximal ideal $\overline{\mathfrak{m}}$ of R/I is $(\overline{x_1}, \ldots, \overline{x_n})$, where $\overline{x_i}$ denotes the image in R/I. $\overline{\mathfrak{m}}^i$ is the image of the composition $R_i \longrightarrow R \longrightarrow R/I$. To impose $\dim_k(\overline{\mathfrak{m}}^i) \leq j$ is therefore to impose that the rank of this linear map is at most j, which comes down to various determinants vanishing, and is hence a Zariski closed condition. The Hilbert–Samuel stratum H_h^n is a finite intersection of these rank conditions and their open complements, and is hence Zariski locally closed. □

We therefore have a decomposition of $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ into locally closed subsets

$$\mathrm{Hilb}_d(\mathbb{A}^n, 0) = \coprod_{|h|=d} H_h^n,$$

where $|h| = \sum_{i \geq 0} h(i)$. We want to piece together topological information of these strata. To do so, we will use *Borel–Moore homology*, whose definition and first properties we recall.

3.2.2 Borel–Moore homology

Throughout this section, all spaces are homeomorphic to locally-finite, finite-dimensional CW complexes.

The most concrete definition of Borel–Moore homology is in terms of *locally-finite* chains.

Definition 3.2.8. *Let X be homeomorphic to a locally-finite, finite-dimensional CW complex. A* locally-finite *i-chain on X is an infinite formal sum*

$$\sum_{\sigma : \Delta^i \longrightarrow X} a_\sigma \sigma,$$

where Δ^i is the i-simplex and $a_\sigma \in \mathbb{Z}$, so that for each compact subset K, $a_\sigma \neq 0$ for only finitely many σ meeting K.

Denote the group of locally-finite i chains as $C_i^{\mathrm{BM}}(X)$. This forms, in the usual way, a chain complex $C_\bullet^{\mathrm{BM}}(X)$.

Definition 3.2.9. *The* Borel–Moore homology *of X is the homology of the complex of locally-finite chains $C_\bullet^{\mathrm{BM}}(X)$. The rank of $H_i^{\mathrm{BM}}(X; \mathbb{Z})$ is the ith* Borel–Moore Betti number *of X, and is denoted $b_i^{\mathrm{BM}}(X)$. Furthermore, $p_X^{\mathrm{BM}}(z) = \sum b_i^{\mathrm{BM}}(X) z^i$ is the* Borel–Moore Poincaré polynomial *of X.*

This definition is given in [60, §7] and [36, Def. 3.1]. It follows immediately that Borel–Moore homology agrees with the usual singular homology for a compact space. Moreover, it is

proven in [36, Prop. 7.15] that if Y is the one-point compactification of X, then $H_i^{\mathrm{BM}}(X;\mathbb{Z}) = H_i(Y,\{\infty\};\mathbb{Z})$, given some tameness conditions at infinity.

Another equivalent definition, which is useful for formal properties of Borel–Moore homology, uses the dual of the complex computing compactly supported cohomology. Indeed, let $\Gamma_c : \mathsf{Sh}_X \longrightarrow \mathsf{Ab}$ take a sheaf F to its compactly supported sections, i.e. $\Gamma_c = \pi_!$ for $\pi : X \longrightarrow \{*\}$. Let $R\Gamma_c$ be its right derived functor.

Proposition 3.2.10. *The Borel–Moore homology $H_i^{\mathrm{BM}}(X;\mathbb{Z})$ of a space X is realized as*

$$H_i^{\mathrm{BM}}(X;\mathbb{Z}) = H^{-i}(X;\mathrm{Hom}(R\Gamma_c(\mathbb{Z}),\mathbb{Z}))$$

where $\mathbb{Z}$ is viewed as a constant sheaf in degree 0.

This is the definition given by Borel and Moore in [9], where they studied locally compact Hausdorff spaces. This is further explained in [40, Ch. IX]. The equivalence with the definition given above via locally-finite chains is shown in [60, §8], with additional assumptions on X (say, that it is paracompact, Hausdorff, and locally contractible). These hypotheses are satisfied in the situations we are interested in here.

This formulation automatically implies that Borel–Moore homology carries formal properties much like compactly supported cohomology. For one, Borel–Moore homology is covariantly functorial for proper maps and contravariantly functorial for open inclusions $U \hookrightarrow X$.

The crucial property of Borel–Moore homology which makes it useful for analyzing spaces with stratifications is the localization sequence, as with compactly supported cohomology.

Proposition 3.2.11. *Let $Y \subseteq X$ be closed and $U = X \setminus Y$. There is a long exact sequence*

$$\ldots \longrightarrow H_i^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow H_i^{\mathrm{BM}}(X;\mathbb{Z}) \longrightarrow H_i^{\mathrm{BM}}(U;\mathbb{Z}) \longrightarrow H_{i+1}^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow \ldots.$$

By Verdier duality, there is a right adjoint to $R\pi_! = R\Gamma_c : D(X) \longrightarrow D(\{*\}) = D(\mathsf{Ab})$,

denoted by $\pi^! : D(\mathsf{Ab}) \longrightarrow D(X)$, as in [40, Ch VI, §1] (there, it is written as $(-)^!$). Let $\omega_X = \pi^!(\mathbb{Z})$ be the *dualizing complex*.

Below, Hom and $\otimes$ are derived. We compute

$$\begin{aligned} H_i^{\mathrm{BM}}(X;\mathbb{Z}) &= H^{-i}(X;\mathrm{Hom}(R\Gamma_c(\mathbb{Z}),\mathbb{Z})) \\ &= H^{-i}(X;\mathrm{Hom}(\mathbb{Z},\pi^!(\mathbb{Z}))) \\ &= H^{-i}(X;\omega_X). \end{aligned}$$

Moreover, if we replace $\mathbb{Z}$ by a locally constant sheaf of finitely generated abelian groups V, we may define Borel–Moore homology with local coefficients.

Definition 3.2.12. *For a space X and a finitely generated local system V on X, the Borel–Moore homology of X with coefficients in V is defined by*

$$H_i^{\mathrm{BM}}(X;V) = H^{-i}(X;\mathrm{Hom}(R\Gamma_c(V^\vee),\mathbb{Z})),$$

where $V^\vee = \mathrm{Hom}(V,\mathbb{Z})$.

We may compute again

$$\begin{aligned} H_i^{\mathrm{BM}}(X;V) &= H^{-i}(X;\mathrm{Hom}(R\Gamma_c(V^\vee),\mathbb{Z})) \\ &= H^{-i}(X;\mathrm{Hom}(V^\vee,\pi^!\mathbb{Z})) \\ &= H^{-i}(X;(V^\vee)^\vee \otimes \pi^!\mathbb{Z}) \\ &= H^{-i}(X;V\otimes\omega_X). \end{aligned}$$

In the case that X is a smooth, oriented (not necessarily compact!) manifold of dimension n, $\pi^!(\mathbb{Z}) = \mathbb{Z}[n]$, so we have the Poincaré duality isomorphism

$$H^i(X;\mathbb{Z}) \cong H_{n-i}^{\mathrm{BM}}(X;\mathbb{Z}).$$

More generally, for V a finitely generated local system on X, we have

$$H^i(X;V) \cong H^{\mathrm{BM}}_{n-i}(X;V).$$

This yields, for example, the Borel–Moore homology of $\mathbb{R}^n$.

$$H^{\mathrm{BM}}_i(\mathbb{R}^n;\mathbb{Z}) = \begin{cases} 0 & i \neq n \\ \mathbb{Z} & i = n. \end{cases}$$

We remark here that, crucially, Borel–Moore homology is therefore *not* a homotopy invariant (and indeed, nor is compactly supported cohomology). However, *properly homotopic* maps (i.e. the boundaries of a *proper* map $X \times [0,1] \longrightarrow Y$) induce the same map on Borel–Moore homology. Furthermore, we have the isomorphism

$$H^{\mathrm{BM}}_i(X;\mathbb{Z}) \cong H^{\mathrm{BM}}_{i+1}(X \times \mathbb{R};\mathbb{Z}).$$

We will analyze each Hilbert–Samuel stratum via fibrations, so we will also need the following.

Proposition 3.2.13. *Let*

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

be a locally trivial fibration with fiber F, such that B is an oriented, smooth manifold. Then there is a homological spectral sequence

$$E^2_{ij} = H^{\mathrm{BM}}_i(B; H^{\mathrm{BM}}_j(F;\mathbb{Z})) \implies H^{\mathrm{BM}}_{i+j}(X;\mathbb{Z}),$$

where the coefficients $H^{\mathrm{BM}}_j(F;\mathbb{Z})$ are viewed as a local system over B. We refer to this as the Serre spectral sequence for Borel–Moore homology.

Proof. By the Leray spectral sequence, we have

$$E_2^{ij} = H^i(B; R^j\pi_*(\omega_X)) \implies H^{i+j}(X; \omega_X).$$

As discussed above, $H^{i+j}(X;\omega_X) = H^{\mathrm{BM}}_{-i-j}(X;\mathbb{Z})$. Furthermore, $R^j\pi_*(\omega_X)$ is the sheafification of the presheaf

$$U \mapsto H^j(\pi^{-1}(U); \omega_X|_U).$$

We have that $\omega_X|_U = \omega_U$, so this is equal to

$$U \mapsto H^{\mathrm{BM}}_{-j}(\pi^{-1}(U);\mathbb{Z}).$$

As π is locally trivial, this is locally constant. It suffices to compute $H^{\mathrm{BM}}_{-j}(\pi^{-1}(U);\mathbb{Z})$ in terms of the Borel–Moore homology of the fiber F. Indeed, for a sufficiently small open set U, we can insist that $U \cong \mathbb{R}^n$ (where $n = \dim(B)$) and that $\pi^{-1}(U) \cong U \times F$. Then we have

$$\begin{aligned} H^{\mathrm{BM}}_{-j}(\pi^{-1}(U);\mathbb{Z}) &\cong H^{\mathrm{BM}}_{-j}(\mathbb{R}^n \times F;\mathbb{Z}) \\ &\cong H^{\mathrm{BM}}_{-j-n}(F;\mathbb{Z}). \end{aligned}$$

By Poincaré duality (for local systems) as above, we have

$$H^i(B; H^{\mathrm{BM}}_{-j-n}(F;\mathbb{Z})) \cong H^{\mathrm{BM}}_{n-i}(H^{\mathrm{BM}}_{-j-n}(F;\mathbb{Z})).$$

Thus, the Leray spectral sequence becomes

$$E_2^{ij} = H^{\mathrm{BM}}_{n-i}(B; H^{\mathrm{BM}}_{-j-n}(F;\mathbb{Z})) \implies H^{\mathrm{BM}}_{-i-j}(X;\mathbb{Z}).$$

Upon changing variables $i = n - i$ and $j = -j - n$, we conclude with the homological

spectral sequence we desire:

$$E^2_{ij} = H^{\mathrm{BM}}_i(B; H^{\mathrm{BM}}_j(F;\mathbb{Z})) \implies H^{\mathrm{BM}}_{i+j}(X;\mathbb{Z}).$$

□

Finally, we remark that Borel–Moore homology carries many of the same formal properties as Chow groups, and indeed there is a cycle class map

$$CH_i(X) \longrightarrow H^{\mathrm{BM}}_{2i}(X;\mathbb{Z})$$

for a complex variety X, as explained in [21, Ch. 19].

3.3 Known results

We describe now what is known about the homology of the Hilbert scheme of points in affine space. A quick summary was given in Table 3.1. Recall that for results on homology, we work over $k = \mathbb{C}$ and in the analytic topology.

One of the first results on the topology of the Hilbert scheme was by Hartshorne in [34].

Theorem 3.3.1 (Hartshorne)**.** $\mathrm{Hilb}_p(\mathbb{P}^n)$ *is connected for any Hilbert polynomial* p*.*

Said another way, $H_0(\mathrm{Hilb}_p(\mathbb{P}^n)(\mathbb{C});\mathbb{Z}) = \mathbb{Z}$. The same is true of the Hilbert scheme of points in affine space.

The next major development in computing the homology of the Hilbert scheme of points was by Ellingsrud and Strømme in [16], where they fully computed the integral homology (and, in fact, the motive) of the Hilbert scheme of points in the plane using the Białynicki-Birula decomposition. The case of $\mathbb{P}^2$ itself is instructive. A generic $\mathbb{G}_m$-action on $\mathbb{P}^2$ has three fixed points. For a fixed point p, define $U_p = \{q \in \mathbb{P}^2 \mid \lim_{t\to 0} tq = p\}$. This stratification

of $\mathbb{P}^2$ is called its Białynicki-Birula decomposition (with respect to the $\mathbb{G}_m$-action), and will consist of $\mathbb{A}^0$, $\mathbb{A}^1$, and $\mathbb{A}^2$. That is, we recover the CW decomposition of $\mathbb{P}^2$ (much as in Morse theory) and fully read off the integral homology of $\mathbb{P}^2$ by counting the cells of this decomposition in each dimension.

The same principle holds for any smooth, projective variety acted on by $\mathbb{G}_m$ with finitely many fixed points. Ellingsrud and Strømme applied this to $\mathrm{Hilb}_d(\mathbb{P}^2)(\mathbb{C})$ by taking a generic $\mathbb{G}_m$-action on $\mathbb{P}^2$ and extending it to $\mathrm{Hilb}_d(\mathbb{P}^2)(\mathbb{C})$, which is smooth (due to [20, Thm. 2.4]) and projective (due to [31, Thm. 3.2]). The fixed points of this action are monomial ideals, and it remains to compute the dimension of the cells associated to these fixed points and count the number of cells in each dimension. Other Hilbert schemes, such as $\mathrm{Hilb}_d(\mathbb{A}^2)(\mathbb{C})$ and $\mathrm{Hilb}_d(\mathbb{A}^2, 0)(\mathbb{C})$ (and $\mathrm{Hilb}_d(\mathbb{A}^2, \mathbb{A}^1)(\mathbb{C})$) arise as unions of cells of the decomposition of $\mathrm{Hilb}_d(\mathbb{P}^2)(\mathbb{C})$ itself, so their homology can be computed in turn.

Much is known for other smooth, projective surfaces S. In [23], Göttsche used the Weil conjectures to determine a recursive relationship between the Betti numbers of $\mathrm{Hilb}_d(S)(\mathbb{C})$ ranging over all d.

Theorem 3.3.2 (Göttsche)**.** *F S a smooth, projective surface, we have*

$$\sum_{d\geq 0} p_{\mathrm{Hilb}_d(S)(\mathbb{C})}(z)t^d = \prod_{k\geq 1}\prod_{i=0}^{4}(1-(-z)^{2k-2+i}t^k)^{(-1)^{i+1}b_i(S(\mathbb{C}))}.$$

Here, for a space X, $p_X(z) = \sum b_i(X)z^i$ is the *Poincaré polynomial*.

Note that the right hand side depends only on the topology of $S(\mathbb{C})$. For instance, plugging in $z = -1$ yields the equation

$$\sum_{d\geq 0} \chi(\mathrm{Hilb}_d(S)(\mathbb{C}))q^{d-\chi(S)/24} = \eta(\tau)^{-\chi(S)}$$

where $q = e^{2\pi i\tau}$, χ denotes topological Euler characteristic, and η is Dirichlet's eta function $\eta = \Delta^{1/24}$ for Δ the discriminant.

Nakajima [52, Thm. 3.5] and Grojnowski [28, Thm. 7] further explained this relationship by considering the total cohomology

$$H = \bigoplus_{d \geq 0} H^*(\mathrm{Hilb}_d(S)(\mathbb{C}); \mathbb{Q}),$$

and showing that it is an irreducible module over the Heisenberg algebra.

Totaro further studied the topology of the Hilbert scheme of points on a surface, and proved the following result on its *integral* homology in [63, Thm. 3.1], upgrading Göttsche's computation of the Betti numbers.

Theorem 3.3.3 (Totaro)**.** *Let S be a smooth, projective surface whose integral cohomology $H^*(S(\mathbb{C}); \mathbb{Z})$ contains no ℓ-torsion, for a prime ℓ. Then $H^*(\mathrm{Hilb}_d(S)(\mathbb{C}); \mathbb{Z})$ has no ℓ-torsion.*

Returning to affine space, not much was known about the topology of $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ for $n > 2$ since Hartshorne's proof of connectedness, until a striking advance by Hoyois, Jelisiejew, Nardin, Totaro, and Yakerson in [35, Thms. 2.1, 8.2].

Theorem 3.3.4 (HJNTY)**.** *(i) $\mathrm{Hilb}_d(\mathbb{A}^\infty)$ is $\mathbb{A}^1$-homotopy equivalent to $\mathrm{Gr}_{d-1}(\mathbb{A}^\infty)$.*

(ii) The morphism $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C}) \longrightarrow \mathrm{Hilb}_d(\mathbb{A}^\infty)(\mathbb{C})$ is $(2n-2d+2)$-connected in the analytic topology.

(iii) The morphism $\Sigma\,\mathrm{Hilb}_d(\mathbb{A}^n) \longrightarrow \Sigma\,\mathrm{Hilb}_d(\mathbb{A}^\infty)$ is $\mathbb{A}^1$-$(n-d+1)$-connected.

Corollary 3.3.5. *(i) $H^*(\mathrm{Hilb}_d(\mathbb{A}^\infty)(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_{d-1}]$ with $|c_i| = 2i$.*

(ii) The map $\mathbb{Z}[c_1, \ldots, c_{d-1}] \cong H^*(\mathrm{Hilb}_d(\mathbb{A}^\infty)(\mathbb{C}); \mathbb{Z}) \longrightarrow H^*(\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C}); \mathbb{Z})$ is an isomorphism for $i \leq 2n - 2d + 2$.

(iii) The map $H^*(BGL_{d-1}; \mathbb{Z}(i)) \longrightarrow H^*(\mathrm{Hilb}_d(\mathbb{A}^n); \mathbb{Z}(i))$ on motivic cohomology is an isomorphism for $i \leq n - d + 1$.

One also learns that $\mathrm{Hilb}_d(\mathbb{A}^n)$ is $\mathbb{A}^1$-connected for $n \geq d-1 \geq 0$, which is a substantial strengthening of connectedness in the analytic topology. In fact, Totaro proved that this is true for all n, d when working over an infinite field in [64, Thm. 7.1].

Still, this result does not compute the full integral homology of any $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$, which was last done by Ellingsrud and Strømme for $n = 2$ and all d. In his thesis [46], Khan fully computes the integral homology of $\mathrm{Hilb}_d(\mathbb{A}^n, 0)(\mathbb{C})$ for low values of the degree d and for all dimensions n.

Theorem 3.3.6 (Khan)**.** *The integral homology $H_i(\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C}); \mathbb{Z})$ is finitely generated, torsion free, and concentrated in even degrees for all i, n and for $d \leq 4$.*

Let $g_{k,n}(z)$ be the Poincaré polynomial of $\mathrm{Gr}_k(\mathbb{C}^n)$.

d	Poincaré polynomial of $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$
1	1
2	$g_{1,n-1}(z)$
3	$g_{2,n+1(z)}$
4	$z^{2n} g_{2,n}(z) + g_{3,n+2}(z)$

Table 3.2: Poincaré polynomials for $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ with $d \leq 4$

Definition 3.3.7. *A space X has* Tate-type homology *if its homology $H_*^{\mathrm{BM}}(X; \mathbb{Z})$ is finitely generated, torsion free, and concentrated in even degrees.*

For instance, $\mathbb{A}^n$, $\mathbb{P}^n$, and $\mathrm{Gr}_k(\mathbb{A}^n)$ all have Tate-type homology. Ellingsrud and Strømme showed that $\mathrm{Hilb}_d(\mathbb{A}^2)(\mathbb{C})$ has Tate-type homology. By Göttsche and Totaro, if a smooth projective surface $S(\mathbb{C})$ has Tate-type homology, then so does $\mathrm{Hilb}_d(S)(\mathbb{C})$. Applying the universal coefficient theorem to the results of Hoyois, Jelisiejew, Nardin, Totaro, and Yakerson shows that $\mathrm{Hilb}_d(\mathbb{A}^\infty)(\mathbb{C})$ has Tate-type homology, and that $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ has Tate-type homology in the range $i \leq 2n - 2d + 2$. Khan showed that $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$ has Tate-type

homology for $d \leq 4$. Our main result Theorem 3.1.1 shows that $\mathrm{Hilb}_5(\mathbb{A}^n)(\mathbb{C})$ has Tate-type homology, and computes the groups fully.

The follow two lemmas are crucial in determining the homology of a space via stratifications and fibrations.

Lemma 3.3.8. *Let X be a space, $Y \subseteq X$ a closed subspace, and $U = X \setminus Y$. If U and Y have Tate-type homology, then so does X. In fact, $H_i^{\mathrm{BM}}(X;\mathbb{Z}) \cong H_i^{\mathrm{BM}}(U;\mathbb{Z}) \oplus H_i^{\mathrm{BM}}(Y;\mathbb{Z})$.*

Proof. The key tool is the localization sequence from Proposition 3.2.11.

$$\ldots \longrightarrow H_i^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow H_i^{\mathrm{BM}}(X;\mathbb{Z}) \longrightarrow H_i^{\mathrm{BM}}(U;\mathbb{Z}) \longrightarrow H_{i+1}^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow \ldots$$

Let $H_m = H_m^{\mathrm{BM}}(X;\mathbb{Z})$. We first show that these groups vanish for m odd. Indeed, as U and Y have Tate-type homology, $H_m^{\mathrm{BM}}(U;\mathbb{Z}) = H_m^{\mathrm{BM}}(Y;\mathbb{Z}) = 0$. Hence, we have an exact sequence

$$0 \longrightarrow H_m^{\mathrm{BM}}(X;\mathbb{Z}) \longrightarrow 0,$$

so $H_m^{\mathrm{BM}}(X;\mathbb{Z}) = 0$.

Similarly, for $m \gg 0$, $H_m^{\mathrm{BM}}(U;\mathbb{Z}) = H_m^{\mathrm{BM}}(Y;\mathbb{Z}) = 0$, so $H_m^{\mathrm{BM}}(X;\mathbb{Z}) = 0$ for large m.

For m even, we show that $H_m^{\mathrm{BM}}(X;\mathbb{Z})$ is finitely generated and free. Indeed, localization gives us an exact sequence:

$$H_{m-1}^{\mathrm{BM}}(U;\mathbb{Z}) \longrightarrow H_m^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow H_m^{\mathrm{BM}}(X;\mathbb{Z}) \longrightarrow H_m^{\mathrm{BM}}(U;\mathbb{Z}) \longrightarrow H_{m+1}^{\mathrm{BM}}(Y;\mathbb{Z}).$$

As $m \pm 1$ is odd, the outer groups vanish, giving the short exact sequence:

$$0 \longrightarrow H_m^{\mathrm{BM}}(Y;\mathbb{Z}) \longrightarrow H_m^{\mathrm{BM}}(X;\mathbb{Z}) \longrightarrow H_m^{\mathrm{BM}}(U;\mathbb{Z}) \longrightarrow 0.$$

As $H_m^{\mathrm{BM}}(U;\mathbb{Z})$ is free, this sequence splits and we conclude

$$H_m^{\mathrm{BM}}(X;\mathbb{Z}) \cong H_m^{\mathrm{BM}}(U;\mathbb{Z}) \oplus H_m^{\mathrm{BM}}(Y;\mathbb{Z}).$$

□

Lemma 3.3.9. *Let*

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$

be a locally trivial fiber bundle with B an oriented, smooth manifold of dimension n. Suppose that the monodromy action of $\pi_1(B)$ on $H_*^{\mathrm{BM}}(F;\mathbb{Z})$ is trivial. If F and B have Tate-type homology, then so does X. In fact, we then have the Leray–Hirsch isomorphism:

$$H_m^{\mathrm{BM}}(X;\mathbb{Z}) \cong \bigoplus_{i+j=m} H_i^{\mathrm{BM}}(B;\mathbb{Z}) \otimes H_j^{\mathrm{BM}}(F;\mathbb{Z}).$$

Proof. We will use the Serre spectral sequence for Borel–Moore homology constructed in Proposition 3.2.13:

$$E_{ij}^2 = H_i^{\mathrm{BM}}(B; H_j^{\mathrm{BM}}(F;\mathbb{Z})) \implies H_{i+j}^{\mathrm{BM}}(X;\mathbb{Z}).$$

Here, $H_j^{\mathrm{BM}}(F;\mathbb{Z})$ is just an abelian group, as we assumed the monodromy action was trivial. As B and F have Tate-type homology, the odd rows and columns of the E^2 page are all 0. All differentials on pages E^2 and beyond therefore vanish, so $E^2 = E^\infty$. See Figure 3.1.

There is hence a filtration $F_\bullet H_m^{\mathrm{BM}}(X;\mathbb{Z})$ with $F_{-1} = 0$, $F_m = H_m^{\mathrm{BM}}(X;\mathbb{Z})$, and so that the graded pieces satisfy

$$\mathrm{gr}_i H_m^{\mathrm{BM}}(X;\mathbb{Z}) = E_{i,m-i}^2.$$

For simplicity of notation, let $H_m = H_m^{\mathrm{BM}}(X;\mathbb{Z})$. If m is odd, then for all i, one of i or $m-i$ is odd. As the odd rows and columns of E^2 are 0, $\mathrm{gr}_i H_m = E_{i,m-i}^2 = 0$ for all i Hence,

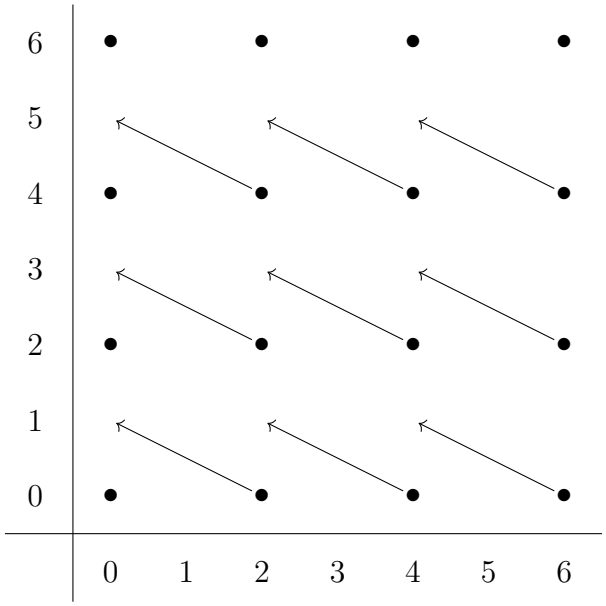


Figure 3.1: The E^2 page of the Serre spectral sequence in Borel–Moore homology for a fibration with Tate-type homology on the base and fiber.

$H_m = 0$ for all odd m.

Furthermore, only finitely many E^2_{ij} are nonzero, so for $m \gg 0$ the same argument implies that $H_m = 0$.

Finally, for m even we wish to show that H_m is finitely generated and free. Each graded piece is finitely generated, so H_m is as well. As for the torsion, recall that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of abelian groups with A and C torsion free, then B is torsion free as well. This allows us to induct on the filtration, as F_{-1} is torsion free and each graded piece is isomorphic to some E^2_{ij}, which are all torsion free by virtue of the homologies of B and F being Tate-type. So X has Tate-type homology.

As for the Leray–Hirsch isomorphism, because all the groups involved are finitely gener-

ated and free, it suffices to show that the ranks are equal. Indeed,

$$\begin{aligned}
\operatorname{rank}\left(H_m^{\mathrm{BM}}(X;\mathbb{Z})\right) &= \sum_i \operatorname{rank}\left(\operatorname{gr}_i H_m^{\mathrm{BM}}(X;\mathbb{Z})\right) \\
&= \sum_{i+j=m} \operatorname{rank}\left(E_{ij}^2\right) \\
&= \sum_{i+j=m} \operatorname{rank}\left(H_i^{\mathrm{BM}}\left(B; H_j^{\mathrm{BM}}\left(F;\mathbb{Z}\right)\right)\right) \\
&= \sum_{i+j=m} \operatorname{rank}\left(H_i^{\mathrm{BM}}(B;\mathbb{Z})^{\oplus \operatorname{rank}\left(H_j^{\mathrm{BM}}(F;\mathbb{Z})\right)}\right) \\
&= \sum_{i+j=m} \operatorname{rank}\left(H_i^{\mathrm{BM}}(B;\mathbb{Z})\right) \operatorname{rank}\left(H_j^{\mathrm{BM}}(F;\mathbb{Z})\right) \\
&= \operatorname{rank}\left(\bigoplus_{i+j=m} H_i^{\mathrm{BM}}(B;\mathbb{Z}) \otimes H_j^{\mathrm{BM}}(F;\mathbb{Z})\right).
\end{aligned}$$

□

Corollary 3.3.10. *Under these hypotheses,* $p_X^{\mathrm{BM}}(z) = p_B^{\mathrm{BM}}(z) p_F^{\mathrm{BM}}(z)$.

The sums and products in the Poincaré polynomials of Theorem 3.3.6 are explained by Lemma 3.3.8 and Lemma 3.3.9 respectively.

3.4 The Hilbert scheme of five points

In his thesis [46], Khan computed the homology of all but one Hilbert–Samuel stratum of $\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$, but his methods were unable to compute the final missing piece.

The Hilbert–Samuel function of a finite local algebra of dimension d must satisfy $h(0) = 1$ and $\sum_i h(i) = d$. As there are 5 partitions of 4, there are at most 5 possible Hilbert–Samuel functions for $d = 5$.

$$(1,1,1,1,1)$$
$$(1,2,1,1)$$

$$(1,2,2)$$

$$(1,3,1)$$

$$(1,4)$$

In fact, all of these are realized by actual local algebras. The isomorphism types of finite local $\mathbb{C}$-algebras of dimension ≤ 6 were fully classified, along with their Hilbert–Samuel functions, by Poonen in [55]. Of these, Khan computed the topology in every case but $h = (1,2,1,1)$ via iterated fibrations.

Proposition 3.4.1 (Khan)**.** *$H_h^n(\mathbb{C})$ has Tate-type homology for $h = (1,1,1,1,1)$, $(1,2,2)$, $(1,3,1)$, and $(1,4)$, with the following Borel–Moore Poincaré polynomials.*

h	Fibration structure	$p^{\mathrm{BM}}_{H_h^n(\mathbb{C})}(z)$
(1, 1, 1, 1, 1)	$\mathbb{A}^{3n-3} \longrightarrow H_h^n \downarrow \mathbb{P}^{n-1}$	$z^{2(3n-3)} g_{1,n}(z)$
(1, 2, 2)	$\mathbb{A}^{2n-4} \longrightarrow H_h^n \downarrow$ $\mathbb{P}^2 \longrightarrow \mathbb{P}(\mathcal{E}) \downarrow \mathrm{Gr}_2(\mathbb{A}^n)$	$z^{2(2n-4)} g_{1,3}(z) g_{2,n}(z)$
(1, 3, 1)	$\mathbb{A}^{n-2} \longrightarrow H_h^n \downarrow$ $\mathbb{P}^5 \longrightarrow \mathbb{P}(\mathcal{F}) \downarrow \mathrm{Gr}_3(\mathbb{A}^n)$	$z^{2(n-3)} g_{1,6}(z) g_{3,n}(z)$
(1, 4)	$\mathbb{A}^0 \longrightarrow H_h^n \downarrow \sim \mathrm{Gr}_4(\mathbb{A}^n)$	$g_{4,n}(z)$

Table 3.3: Fibration structure and Borel–Moore Poincaré polynomials of most Hilbert–Samuel strata of $\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$ by Khan

Here, $\mathcal{E}$ and $\mathcal{F}$ are vector bundles over the respective Grassmannians, and any diagram of the shape

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \\ & & B \end{array}$$

is a Zariski locally trivial fibration $X \longrightarrow B$ with fiber F.

To investigate the missing stratum $H^n_{(1,2,1,1)}$, we need a new idea beyond Khan's techniques. We first note that Graffeo, Monavari, Moschetti, and Ricolfi record the following fibration structure on Hilbert–Samuel strata in [24, Thm. 3.4].

Lemma 3.4.2. *Fix some $n \geq 1$ and some finitely supported Hilbert–Samuel function h. Let $\tau : H^n_h \longrightarrow \mathrm{Gr}_{n-h(1)}(R_1)$ via $[I] \mapsto \mathrm{in}(I)_1$, where R_1 is the set of linear forms in $R = k[x_1, \ldots, x_n]$. Then τ is a Zariski locally trivial fibration.*

Here, $\mathrm{in}(I)$ denotes the *initial ideal* of I, which is the (homogeneous) ideal generated by the initial forms of every element of I. The *initial form* of a polynomial is the sum of its lowest degree terms. For instance, $\mathrm{in}(x + y + x^2) = x + y$. (Note that this definition is distinct from the identically named initial ideal used in the theory of Gröbner bases.) Letting $X = V(I)$, a basis for $\mathrm{in}(I)_1$ consists of $n - h(1) = n - \dim T_0(X)$ linear equations cutting out $T_0(X) \subseteq T_0(\mathbb{A}^n)$.

The authors also prove that for a fiber F of this morphism, $[F] = [\mathbb{A}^{(n-h(1))(d-h(1)-1)}][H^{h(1)}_h]$ in the Grothendieck ring of varieties. This is, however, insufficient to use to compute homology. The following is therefore the key technical lemma we prove, where we upgrade the "dimension-reduction" technique used to compute the class of F in the Grothendieck ring to furthermore determine its topology.

Lemma 3.4.3. *Let F be a fiber of the morphism τ in Lemma 3.4.2. There is a morphism $\pi : F \longrightarrow H^{h(1)}_h$ which is a vector bundle of rank $(n - h(1))(d - h(1) - 1)$.*

We therefore have the following diagram of Zariski locally trivial fibrations.

$$\begin{array}{ccccc} \mathbb{A}^{(n-h(1))(d-h(1)-1)} & \longrightarrow & F & \longrightarrow & H_h^n \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ & & H_h^{h(1)} & & \mathrm{Gr}_{n-h(1)}(R_1) \end{array}$$

The upshot is that to determine the homology of $H_h^n(\mathbb{C})$, we are reduced to the homology of an affine space, a Grassmannian, and $H_h^{h(1)}(\mathbb{C})$ (and, potentially, much homological algebra). $H_h^{h(1)}$ has a lower dimensional ambient space, hence our reference to this as a "dimension-reduction" technique. Before moving onto the proof, let us remark that in the case of $H^n_{(1,2,1,1)}$, we reduce to $H^2_{(1,2,1,1)}$ via this lemma, which places us in an extremely well-positioned situation due to the great understanding we have of Hilbert schemes of points in the plane.

The idea of the proof is well illustrated in the case of $h = (1, 1, \ldots, 1)$. In this case, $H^n_{(1,1,\ldots,1)}$ consists of the "curvilinear" subschemes. That is, the finite subschemes $X \subseteq \mathbb{A}^n$ which are supported on a smooth curve. (For instance, (x^2, xy, y^2) is not supported on any smooth curve, but $(y - x^2, x^3)$ is.)

For any such X of length d, $\mathcal{O}(X) \cong k[t]/(t^{d-1})$. Let us further specialize, for the sake of illustration, to the case $n = 2$. Consider $\tau : H^2_{(1,1,\ldots,1)} \longrightarrow \mathbb{P}^1$ via $[X] \mapsto T_0(X)$. (Technically, the τ of Lemma 3.4.2 sends $[X]$ to the equations defining $T_0(X)$, not $T_0(X)$ itself.) The fiber F over the x-axis $V = \{y = 0\}$ consists of all curvilinear X with fixed tangent space V. View $H^1_{(1,1,\ldots,1)}$ as containing subschemes supported in $V = \mathbb{A}^1_x$. We form $\pi : F \longrightarrow H^1_{(1,1,\ldots,1)}$ taking $[X]$ to its projection onto the x-axis V. On the level of ideals, this is $[I] \mapsto [I \cap k[x]]$ – the elimination ideal.

There is only only element of $H^1_{(1,1,\ldots,1)}$, which is the closed subscheme $\operatorname{Spec} k[x]/(x^d) \subseteq \mathbb{A}^1_x$. To pick an element $I \in F$ lying over this point, we must define a germ of a smooth curve with tangent space $\{y = 0\}$. Such a curve is formally defined by $\{y = f(x)\}$ for some

power series $f(x)$ with no constant or linear term. As $k[x,y]/I \cong k[t]/(t^d)$ as local algebras, we know that $x^d \in I$. Hence, we can truncate f to be of degree less than d and write $I = \left(y - \sum_{i=2}^{d-1} a_i x^i, x^d\right)$. The coefficients a_i may be chosen freely, which yields a fiber $\mathbb{A}^{d-2}$. The diagram following Lemma 3.4.3 in this case is the following.

$$\begin{array}{ccccc} \mathbb{A}^{d-2} & \longrightarrow & F & \longrightarrow & H^2_{(1,1,\ldots,1)} \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ & & \{*\} & & \mathbb{P}^1 \end{array}$$

We follow this basic outline below, closely using the technique of [24, Thm. 3.4] in computing the class of F in the Grothendieck ring.

Proof of Lemma 3.4.3. Let $R = k[x_1, \ldots, x_n]$. For any homogeneous ideal $I \subseteq R$ (including R itself), let I_a denote the ath graded piece.

We seek to analyze the fibers of τ. As the Grassmannian is connected, it suffices to analyze a single fiber. We pick coordinates $x_1, \ldots, x_{h(1)}, y_1, \ldots, y_{n-h(1)}$ on $\mathbb{A}^n$ and consider the fiber over $\langle y_1, \ldots, y_{n-h(1)} \rangle$, denoted $F \subseteq H^n_h$. For notational convenience, we denote $\vec{x}$ as the tuple $(x_1, \ldots, x_{h(1)})$ and $\vec{y}$ as $(y_1, \ldots, y_{n-h(1)})$, and use these whenever we would write $x_1, \ldots, x_{h(1)}$ or $y_1, \ldots, y_{n-h(1)}$.

Let $V = \{\vec{y} = 0\} = \mathbb{A}^{h(1)}_{\vec{x}}$ and let

$$H_h(V) = \{[X] \in \mathrm{Hilb}_d(V, 0) \mid h_{\mathcal{O}(X)} = h\}.$$

Here, the elements of $\mathrm{Hilb}(V)$ are closed subschemes of V, which is an affine space of dimension $h(1)$, so $H_h(V) = H^{h(1)}_h$.

For $[X]$ to be in F means that $\mathrm{in}(I(X))_1 = \langle \vec{y} \rangle$, so that the tangent space of X at the origin is exactly equal to V. For $[X]$ to be in $H_h(V)$ means X is wholly contained in $T_0(X) = V$. We therefore define a morphism $\pi : F \longrightarrow H_h(V)$ which takes $[X]$ to the projection $\pi([X])$

onto its tangent space V. On the level of ideals, this is defined as $[I] \mapsto [I \cap k[\vec{x}]]$ – the elimination ideal.

We seek to prove that π is a vector bundle of rank $\mathbb{A}^{(n-h(1))(d-h(1)-1)}$. Note that π has a natural section $H_h(V) \longrightarrow F$, where we view a closed subscheme X of V as a closed subscheme of $\mathbb{A}^n$ via the inclusion $V \subseteq \mathbb{A}^n$. On ideals, this is defined by $[J] \mapsto [(\vec{y}) + R \cdot J]$. This will be the 0-section in the coming discussion. It extends to an inclusion $\mathrm{Hilb}(V) \subseteq \mathrm{Hilb}(\mathbb{A}^n)$.

First, we must show that $I \cap k[\vec{x}]$ is actually in $H_h(V)$ when I is in F. That is, we must show that its Hilbert–Samuel function is h. In fact, we will show the isomorphism

$$\frac{k[\vec{x}]}{I \cap k[\vec{x}]} \cong \frac{k[\vec{x}, \vec{y}]}{I}.$$

Indeed, the inclusion $k[\vec{x}] \subseteq k[\vec{x}, \vec{y}]$ induces an injection

$$\frac{k[\vec{x}]}{I \cap k[\vec{x}]} \hookrightarrow \frac{k[\vec{x}, \vec{y}]}{I}.$$

We must show this is onto. Indeed, let $g(\vec{x}, \vec{y}) \in k[\vec{x}, \vec{y}]$. It suffices to show that g is equivalent modulo I to a polynomial only in terms of $\vec{x}$. By hypothesis, $\mathrm{in}(I)_1 = \langle \vec{y} \rangle$. I therefore contains elements $y_i - f_i(\vec{x}, \vec{y})$ where f_i only has terms of degree at least 2. Write $\vec{f} = (f_1, \ldots, f_{n-h(1)})$. In $k[\vec{x}, \vec{y}]/I$, we thus have that $\vec{y} = \vec{f}(\vec{x}, \vec{y})$. We may therefore compute

$$\begin{aligned} g(\vec{x}, \vec{y}) &= g(\vec{x}, \vec{f}(\vec{x}, \vec{y})) \\ &= g(\vec{x}, \vec{f}(\vec{x}, \vec{f}(\vec{x}, \vec{y}))) \\ &= \ldots \end{aligned}$$

in $k[\vec{x}, \vec{y}]/I$. As the terms of $\vec{f}$ are all of degree at least 2, each iteration of this process will strictly increase the degree of any term of g containing some y_i. As $k[\vec{x}, \vec{y}]/I$ is finite dimensional, every term with some y_i will thus become 0 somewhere in this process. Hence,

$g(\vec{x}, \vec{y})$ can be reduced modulo I to land in $k[\vec{x}]$, and we conclude our desired isomorphism.

Now, fix some $[J] \in H_h(V)$. We want to determine what precise freedom we have in building an element of $\pi^{-1}([J])$ from J. The easiest choice is via the section we described above: take $I_0 = (\vec{y}) + R \cdot J$. More generally, we can replace each y_i here by $y_i - f_i(\vec{x}, \vec{y})$. If we take $f_i(\vec{x}, \vec{y}) \in (\vec{x}, \vec{y})^2$, $\mathrm{in}(I)_1$ and the projection $I \cap k[\vec{x}]$ are unchanged. That is, let $\vec{f} = (f_1, \ldots, f_{n-h(1)})$ and form $I = (\vec{y} - \vec{f}(\vec{x}, \vec{y})) + R \cdot J$.

If we choose $f_i \in I_0$, then $I = I_0$. We therefore want to perturb the y_i by elements of R/I_0 to get new ideals. More formally, we can choose monomials $m_1, \ldots, m_{d-h(1)-1}$ of degree at least 2 so that $B = \{1, x_1, \ldots, x_{h(1)}, m_1, \ldots, m_{d-h(1)-1}\}$ is a basis of R/I_0. Let $\vec{m} = (m_1, \ldots, m_{d-h(1)-1})$. As we range over the matrices $U \in \mathbb{A}^{(n-h(1))\times(d-h(1)-1)}$, $U\vec{m}$ ranges over the $\vec{f} = (f_1, \ldots, f_{n-h(1)})$ that we may use to perturb $\vec{y}$. That is, we have a morphism

$$\mathbb{A}^{(n-h(1))\times(d-h(1)-1)} \xrightarrow{\phi_J} \pi^{-1}(J)$$

$$U \longmapsto [(\vec{y} - U\vec{m}) + R \cdot J].$$

As in [24, Thm. 3.4], ϕ_J is a bijection.

Furthermore, we let

$$U_B = \{[J] \in \mathrm{Hilb}_d(\mathbb{A}^n, 0) \mid B \text{ projects onto a basis of } R/J\},$$

and observe that the definition of ϕ_J is uniform on $H_h(V) \cap U_B$, where we are viewing $H_h(V)$ as being in $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ via the inclusion $V \subseteq \mathbb{A}^n$. Letting B vary, the U_B form a Zariski open cover of the $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$, so $F_B = F \cap U_B$ and $H_h(V)_B = H_h(V) \cap U_B$ form open covers of the respective closed subsets.

We claim indeed that $\pi : F \longrightarrow H_h(V)$ is trivialized over this open cover. For simplicity

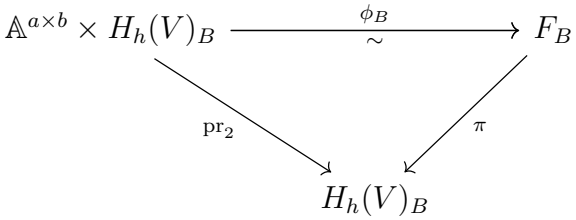
of notation, let $a = n - h(1)$ and $b = d - h(1) - 1$. We define the morphism

$$\mathbb{A}^{a\times b} \times H_h(V)_B \xrightarrow{\quad\phi_B\quad} F_B$$

$$(U, J) \longmapsto (\vec{y} - U\vec{m}) + R \cdot J.$$

(Notice that the section $J \mapsto (\vec{y}) + R \cdot J$ is defined by $U = 0$ in these coordinates.)

We claim ϕ_B is an isomorphism making the following diagram commute.



Note that in [24, Theorem 3.4], the authors only show that ϕ_B is a bijection, which is sufficient to prove $[F] = \mathbb{L}^{ab}[H_h(V)]$ in the Grothendieck ring of varieties, but does not suffice for our topological aims.

Let us now construct an inverse $F_B \longrightarrow \mathbb{A}^{a\times b} \times H_h(V)_B$. The component $F_B \longrightarrow H_h(V)_B$ must be π for the diagram to commute, so we need only construct the component $F_B \longrightarrow \mathbb{A}^{a\times b}$. Indeed, let $[I] \in F_B$. By definition of F, we have $\mathrm{in}(I)_1 = \langle \vec{y} \rangle$. In particular, $\mathrm{in}(I)$ contains each y_i. By definition, there is some $f_i(\vec{x}, \vec{y})$ with no linear or constant terms so that $y_i - f_i(x, y) \in I$. Working modulo I, we have $y_i \equiv f_i(\vec{x}, \vec{y}) \pmod{I}$. By definition of U_B, B projects to a basis of R/I. Hence, we may write

$$f_i(\vec{x}, \vec{y}) \equiv \sum_{j=1}^{b} U_{ij} m_j \pmod{I}$$

for unique $(U_{ij})_j \in \mathbb{A}^b$. The coefficients U_{ij} of this expansion form the ith row of the matrix $U \in \mathbb{A}^{a\times b}$ we map $[I]$ to, so that f_i is the ith entry of $U\vec{m}$. It suffices to show that we can extract the coefficients U_{ij} as algebraic functions in $[I]$.

Let $\overline{R} = R/\mathfrak{m}^d$, where $\mathfrak{m} = (\vec{x}, \vec{y})$. We recall that as R/I is d-dimensional over k, $\mathfrak{m}^d \subseteq I$, and we therefore view I as an ideal in $\overline{R}$. Let $W_0 = \mathrm{span}(B) \subseteq \overline{R}$, a k-linear subspace of dimension d. We define

$$\mathrm{Gr}_B^d\big(\overline{R}\big) = \big\{W \in \mathrm{Gr}^d\big(\overline{R}\big) \mid W \cap W_0 = 0\big\},$$

where $\mathrm{Gr}^d\big(\overline{R}\big)$ is the Grassmannian of codimension d subspaces of $\overline{R}$. The condition $W \cap W_0 = 0$ is Zariski open in W, and is equivalent to B being a basis of $\overline{R}/W$. This is the same condition defining the open cover U_B of $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ described above, so that $\mathrm{Hilb}_d(\mathbb{A}^n, 0) \cap \mathrm{Gr}_B^d\big(\overline{R}\big) = U_B$ via the inclusion used to prove Proposition 3.2.4. We consider the morphism

$$\mathrm{Gr}_B^d\big(\overline{R}\big) \longrightarrow \mathrm{Hom}_k(\overline{R}, W_0)$$

where a subspace W is sent to the unique k-linear projection $\overline{R} \longrightarrow W_0$ with kernel W. Write $B = \{v_1, \ldots, v_d\}$ for ease of notation, and pick some basis $w_{d+1}, \ldots, w_N$ of W, where $N = \dim_k \overline{R}$. The matrix

$$\begin{pmatrix} | & & | & | & & | \\ v_1 & \ldots & v_d & 0 & \ldots & 0 \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} | & & | & | & & | \\ v_1 & \ldots & v_d & w_{d+1} & \ldots & w_N \\ | & & | & | & & | \end{pmatrix}^{-1}$$

defines this projection. The resulting matrix is, by uniqueness, independent of the choice of basis of W, and varies algebraically in the Plücker coordinate $w_{d+1} \wedge \cdots \wedge w_N$ of W.

The composition

$$\mathrm{Gr}_B^d\big(\overline{R}\big) \longrightarrow \mathrm{Hom}_k\left(\overline{R}, W_0\right) \xrightarrow{\mathrm{ev}_{y_i}} W_0 \xrightarrow{\sim} \mathbb{A}^d$$

is thus algebraic, where ev_{y_i} is evaluation at the element $y_i \in R$ and $W_0 \xrightarrow{\sim} \mathbb{A}^d$ is the isomorphism taking an element of $W_0 = \mathrm{span}(B)$ to its column vector representation in the

basis B. This composition takes a subspace $W \in \mathrm{Gr}_B^d(\overline{R})$ to the coordinates of y_i in R/W in the B-basis. Restricting to F_B, and putting these together for all i, this is precisely the morphism $c : F_B \longrightarrow \mathbb{A}^{a\times b}$ we desired.

We now show that the morphisms we have described between $\mathbb{A}^{a\times b} \times H_h(V)_B$ and F_B are inverse isomorphisms. First, consider the composition

$$\begin{array}{ccccc} \mathbb{A}^{a\times b} \times H_h(V)_B & \longrightarrow & F_B & \longrightarrow & \mathbb{A}^{a\times b} \times H_h(V)_B \\ (U, [J]) & \longmapsto & [(\vec{y} - U\vec{m}) + R \cdot J] & & \\ & & [I] & \longmapsto & (c([I]), [I \cap k[\vec{x}]]). \end{array}$$

By definition, $c([I])$ is the matrix (c_{ij}) so that $y_i \equiv \sum_j c_{ij} m_j \pmod{I}$. As $\vec{y} - U\vec{m} \in I$, we have that $\vec{y} \equiv U\vec{m} \pmod{I}$, so $c([I]) = U$. We are left therefore to prove that $I \cap k[\vec{x}] = J$. It's immediate that $J \subseteq I \cap k[\vec{x}]$. We proved above that $k[\vec{x}]/J \cong k[\vec{x}, \vec{y}]/I \cong k[\vec{x}]/I \cap k[\vec{x}]$. Tracing these maps, the composition $k[\vec{x}]/J \longrightarrow k[\vec{x}]/I \cap k[\vec{x}]$ sends $\vec{x} \mapsto \vec{x}$, so $J = I \cap k[\vec{x}]$. Then $\mathbb{A}^{a\times b} \times H_h(V)_B \longrightarrow F_B \longrightarrow \mathbb{A}^{a\times b} \times H_h(V)_B$ is indeed the identity.

Now, we consider

$$F_B \longrightarrow \mathbb{A}^{a\times b} \times H_h(V)_B \longrightarrow F_B,$$

which is the identity by a similar argument.

That

$$\begin{array}{ccc} \mathbb{A}^{a\times b} \times H_h(V)_B & \xleftarrow{\phi_B^{-1}} & F_B \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & H_h(V)_B & \end{array}$$

commutes is trivial, so we have shown that the morphism $\pi : F \longrightarrow H_h(V) = H_h^{h(1)}$ is trivialized over the open cover $H_h(V)_B$ with fiber $\mathbb{A}^{a\times b}$.

We now show that $\pi : F \longrightarrow H_h(V)$ is, in fact, a vector bundle. Given two such bases $B = \{1, \vec{x}, \vec{m}\}$, $B' = \{1, \vec{x}, \vec{m}'\}$, we denote $H_h(V)_{BB'} = H_h(V)_B \cap H_h(V)_{B'}$, $F_{BB'} = F_B \cap F_{B'}$. Also let c' be the component

$$F_{B'} \xrightarrow{\phi_{B'}} \mathbb{A}^{a\times b} \times H_h(V)_{B'} \xrightarrow{\mathrm{pr}_1} \mathbb{A}^{a\times b}.$$

Then we have the transition function

$$\begin{aligned}
\mathbb{A}^{a\times b} \times H_h(V)_{BB'} &\xrightarrow{\phi_B^{-1}} F_{BB'} \xrightarrow{\phi_{B'}} \mathbb{A}^{a\times b} \times H_h(V)_{BB'} \\
(U, [J]) &\longmapsto [(y - U\vec{m}) + R \cdot J] \\
[I] &\longmapsto (c'([I]), [I \cap k[\vec{x}]]) \\
&= (c'([I]), [J]).
\end{aligned}$$

We seek to show linearity. Fix some $J \in H_h(V)_{BB'}$ and consider the fiber

$$\begin{aligned}
\mathbb{A}^{a\times b} &\longrightarrow \mathbb{A}^{a\times b} \\
U &\longmapsto c'([(y - U\vec{m}) + R \cdot J]).
\end{aligned}$$

The entries of $U = c(I)$ are the coefficients of $\vec{y}$ in R/I written in the B basis, whereas the entries of $c'(I)$ are the coefficients of $\vec{y}$ in R/I written in the B' basis. This is manifestly a linear change in coordinates. Also, the section $H_h(V) \longrightarrow F$ sending $J \mapsto (\vec{y}) + R \cdot J$ is, in any of these coordinate neighborhoods $\mathbb{A}^{a\times b} \times H_h(V)_B$, defined by $U = 0$. This is therefore the 0-section of the vector bundle $\pi : F \longrightarrow H_h(V)$. □

With this result in hand, we have the following diagram of Zariski locally trivial fibrations

for the missing Hilbert–Samuel stratum $H^n_{(1,2,1,1)}$.

$$\begin{array}{ccccc} \mathbb{A}^{(n-2)\cdot 2} & \longrightarrow & F & \longrightarrow & H^n_{(1,2,1,1)} \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ & & H^2_{(1,2,1,1)} & & \mathrm{Gr}_{n-2}(\mathbb{A}^n) \end{array}$$

We therefore seek to compute the homology of $H^2_{(1,2,1,1)}(\mathbb{C})$.

Definition 3.4.4. *For $h : \mathbb{N} \longrightarrow \mathbb{N}$ with finite support, we let*

$$\mathcal{H}^n_h = \{I \in H^n_h \mid I \text{ is homogeneous}\}$$

with the reduced subscheme structure.

There is a morphism $\mathrm{in} : H^n_h \longrightarrow \mathcal{H}^n_h$ taking $[I] \mapsto [\mathrm{in}(I)]$. In [24, Lemma 2.7], the authors state the following structure of this morphism.

Lemma 3.4.5. *Fix a Hilbert–Samuel function h and let t be the largest integer so that $h(t) \neq 0$. Let h_R be the Hilbert–Samuel function of $R = k[x_1, \ldots, x_n]$, so that $h_R(i)$ is the number of monic monomials of degree i in n variables.*

(i) If $h(i) = h_R(i)$ for $0 \leq i \leq t-3$ then the fibers of $\mathrm{in} : H^{h(1)}_h \longrightarrow \mathcal{H}^{h(1)}_h$ are affine spaces.

(ii) If, in addition, $h(t-2) = h_R(t-2)$, then $\mathrm{in} : H^{h(1)}_h \longrightarrow \mathcal{H}^{h(1)}_h$ is a vector bundle of rank $h(t)(h_R(t-1) - h(t-1))$.

The authors of [24] cite [15, Lemmas 3.2, 3.3] for the argument. [15, Lemma 3.2], which is part (ii) of this lemma, in turn, cites [10, Prop. 4.3] for the argument. However, [10, Prop. 4.3] only proves this for $t = 3$, and does not show that the morphism is a vector bundle or Zariski locally trivial, as they only needed its fibers to be affine spaces of the same dimension for their analysis of irreducible components. For completeness, we will provide a proof for part (ii) of this lemma.

Proof of Lemma 3.4.5 (ii). We follow the method given in [10, Prop. 4.3]. Let $[I] \in H_h^{h(1)}$. As $\mathrm{in}(I)$ is homogeneous, $h(i) = \dim(R/\mathrm{in}(I))_i = \dim R_i - \dim \mathrm{in}(I)_i$. So $h(i) = h_R(i)$ precisely when $\mathrm{in}(I)_i = 0$. Our assumption on h thus asserts that $\mathrm{in}(I)_i = 0$ for $i \leq t-2$. Furthermore, as $h(i) = 0$ for all $i \geq t+1$, it follows that $\mathfrak{m}^{t+1} \subseteq I$. We determine that $\mathrm{in}(I) = \mathrm{in}(I)_{t-1} \oplus \mathrm{in}(I)_t \oplus \mathfrak{m}^{t+1}$.

Fix now some $[J] \in \mathcal{H}_h^{h(1)}$ and consider how to build the ideals $[I] \in H_h^{h(1)}$ with $\mathrm{in}(I) = J$. Similarly to the proof of Lemma 3.4.3, we seek to perturb elements of J by higher order terms in such a way that the initial ideal is unchanged. As $\mathfrak{m}^{t+1} \subseteq I$ for all such I, adding terms of degree $\geq t+1$ makes no change. Furthermore, as the lowest degree appearing in J is $t-1$, no element of I can have any term of degree less than $t-1$. So we cannot add any terms of degree $< t-1$. The freedom in determining an I so that $\mathrm{in}(I) = J$ is thus in adding higher degree terms to generators of J of degree $t-1$ or t. Adding a higher order term to an element of J_t would mean adding a term of degree $\geq t+1$, which leaves the ideal untouched, as discussed.

The freedom which remains is adding terms of degree t to elements of J_{t-1}. However, if we had elements $f \in J_{t-1}$ and $g \in J_t$, then replacing f by $f+g$ leaves the ideal unchanged. To define elements of $\mathrm{in}^{-1}([J])$, we are left to add terms of degree t to elements of J_{t-1}.

More formally, let $f_1, \ldots, f_a$ be a basis for J_t and $g_1, \ldots, g_b$ be elements of R_t descending to a basis for R_t/J_t, where $a = h_R(t-1) - h(t-1)$ and $b = h(t)$. Write $\vec{f} = (f_1, \ldots, f_a)$ and $\vec{g} = (g_1, \ldots, g_b)$. We define a morphism

$$\mathbb{A}^{a \times b} \longrightarrow \mathrm{in}^{-1}([J])$$

$$U \longmapsto \left[(\vec{f} - U\vec{g}) + J_t + \mathfrak{m}^{t+1} \right].$$

We note the similarity of this map to the one given in the proof of Lemma 3.4.3. Following along those lines, the set of $J \in \mathcal{H}_h^{h(1)}$ for which $g_1, \ldots, g_b$ descends to a basis for R_t/J_t is

Zariski open, and as $B = \{g_1, \ldots, g_b\}$ varies, we form an open cover $\{\mathcal{U}_B\}_B$ of $\mathcal{H}_h^{h(1)}$. The morphism $\mathbb{A}^{a\times b} \longrightarrow \mathrm{in}^{-1}([J])$ is defined uniformly for all J in the same $\mathcal{U}_B$, and we may argue very similarly to Lemma 3.4.3 that this proves $\mathrm{in} : H_h^{h(1)} \longrightarrow \mathcal{H}_h^{h(1)}$ is trivialized over $\mathcal{U}_B$, with fiber $\mathbb{A}^{a\times b}$. Furthermore, as in Lemma 3.4.3 we have a clear zero section by the inclusion $\mathcal{H}_h^{h(1)} \subseteq H_h^{h(1)}$, which corresponds in each coordinate neighborhood to $U = 0$. The transition functions are clearly linear, and we conclude that $\mathrm{in} : H_h^{h(1)} \longrightarrow \mathcal{H}_h^{h(1)}$ is a vector bundle of rank $ab = (h_R(t-1) - h(t-1))h(t)$. □

In the remaining case of $H^2_{(1,2,1,1)}$, we have $t = 3$ and the second condition applies, so that $\mathrm{in} : H^2_{(1,2,1,1)} \longrightarrow \mathcal{H}^2_{(1,2,1,1)}$ is a vector bundle of rank 2. We are now left to analyze $\mathcal{H}^2_{(1,2,1,1)}$.

Lemma 3.4.6. *Let k be a field of characteristic not equal to 2. Then $\mathcal{H}^2_{(1,2,1,1)} \cong \mathbb{P}^1$.*

Proof. An application of [10, Prop. 3.4] shows that $\mathcal{H}^2_{(1,2,1,1)}$ is an irreducible curve via a surjection $\phi : \mathbb{P}^1 \longrightarrow \mathcal{H}^2_{(1,2,1,1)}$. This is defined as follows. Let $R^\vee = k[\partial_x, \partial_y]$ act on $R = k[x, y]$ via partial differentiation. We view $\mathbb{P}^1$ as $\mathbb{P}(R_1^\vee)$. For a nonzero linear form $\ell \in R_1^\vee$, let $\phi([\ell]) = (f \in R_2 \mid \ell^2(f) = 0) + (x, y)^4$. (The hypothesis that $\mathrm{char}(k) \neq 2$ is used here to ensure that the pairing $R_2 \otimes R_2^\vee \longrightarrow k$ is nondegenerate.)

As [10, Prop. 3.4] only states that ϕ is surjective, it does not preclude the possibility that $\mathcal{H}^2_{(1,2,1,1)}$ could be, say, a nodal cubic. However, both $\mathbb{P}^1 = \mathbb{P}(R_1^\vee)$ and $\mathcal{H}^2_{(1,2,1,1)}$ carry actions of PGL_2, acting on $\partial x, \partial y$ or x, y. These actions can be chosen compatibly so that ϕ is equivariant. As it is nonconstant, this forces it to be an isomorphism. (Here, we are using that $\mathcal{H}_h^n$ is given the reduced scheme structure.) □

Proposition 3.4.7. *$H^n_{(1,2,1,1)}(\mathbb{C})$ has Tate-type homology with Borel–More Poincaré polynomial*

$$z^{4(n-1)}(1 + z^2)g_{2,n}(z).$$

Proof. We have the shown following diagram of Zariski locally trivial fibrations.

$$\begin{array}{ccccc}
\mathbb{A}^{(n-2)\cdot 2} & \longrightarrow & F & \longrightarrow & H^n_{(1,2,1,1)} \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\
\mathbb{A}^2 & \longrightarrow & H^2_{(1,2,1,1)} & & \mathrm{Gr}_{n-2}(\mathbb{A}^n) \\
 & & \downarrow{\scriptstyle \mathrm{in}} & & \\
 & & \mathbb{P}^1 & &
\end{array}$$

Note also that $\mathbb{P}^1(\mathbb{C})$ and $\mathrm{Gr}_{n-2}(\mathbb{C}^n)$ are simply connected. For instance, use their stratifications into affine spaces to form CW decompositions with only even dimensional cells. Furthermore, as in : $H^2_{(1,2,1,1)}(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ is a vector bundle, it follows that $H^2_{(1,2,1,1)}(\mathbb{C})$ is homotopy equivalent to $\mathbb{P}^1(\mathbb{C})$ and is therefore also simply connected. Hence, the base of every fibration in this diagram is simply connected, so we may use Lemma 3.3.9 three times to show that the homology of $H^n_{(1,2,1,1)}(\mathbb{C})$ is Tate-type and compute its Borel–Moore Poincaré polynomial as follows:

$$\begin{aligned}
p^{\mathrm{BM}}_{H^2_{(1,2,1,1)}(\mathbb{C})}(z) &= z^4 g_{1,2}(z) \\
&= z^4(1+z^2) \\
p^{\mathrm{BM}}_{H^n_{(1,2,1,1)}(\mathbb{C})}(z) &= z^{4(n-2)} p^{\mathrm{BM}}_{H^2_{(1,2,1,1)}(\mathbb{C})}(z) g_{n-2,n}(z) \\
&= z^{4(n-1)}(1+z^2) g_{n-2,n}(z) \\
&= z^{4(n-1)}(1+z^2) g_{2,n}(z).
\end{aligned}$$

□

We have therefore completed the computation of the homology of all Hilbert–Samuel strata of $\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$.

h	Fibration structure	$p^{\mathrm{BM}}_{H^n_h(\mathbb{C})}(z)$
(1, 1, 1, 1, 1)	$\begin{array}{ccc} \mathbb{A}^{3n-3} & \longrightarrow & H^n_h \\ & & \downarrow \\ & & \mathbb{P}^{n-1} \end{array}$	$z^{2(3n-3)} g_{1,n}(z)$
(1, 2, 1, 1)	$\begin{array}{ccccc} \mathbb{A}^{(n-2)\cdot 2} & \longrightarrow & F & \longrightarrow & H^n_h \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ \mathbb{A}^2 & \longrightarrow & H^2_h & & \mathrm{Gr}_{n-2}(\mathbb{A}^n) \\ & & \downarrow{\scriptstyle \mathrm{in}} & & \\ & & \mathbb{P}^1 & & \end{array}$	$z^{4(n-1)}(1+z^2) g_{2,n}(z)$
(1, 2, 2)	$\begin{array}{ccc} \mathbb{A}^{2n-4} & \longrightarrow & H^n_h \\ & & \downarrow \\ \mathbb{P}^2 & \longrightarrow & \mathbb{P}(\mathcal{E}) \\ & & \downarrow \\ & & \mathrm{Gr}_2(\mathbb{A}^n) \end{array}$	$z^{2(2n-4)} g_{1,3}(z) g_{2,n}(z)$
(1, 3, 1)	$\begin{array}{ccc} \mathbb{A}^{n-2} & \longrightarrow & H^n_h \\ & & \downarrow \\ \mathbb{P}^5 & \longrightarrow & \mathbb{P}(\mathcal{F}) \\ & & \downarrow \\ & & \mathrm{Gr}_3(\mathbb{A}^n) \end{array}$	$z^{2(n-3)} g_{1,6}(z) g_{3,n}(z)$
(1, 4)	$\begin{array}{ccc} \mathbb{A}^0 & \longrightarrow & H^n_h \\ & & \downarrow{\scriptstyle \sim} \\ & & \mathrm{Gr}_4(\mathbb{A}^n) \end{array}$	$g_{4,n}(z)$

Table 3.4: Fibration structure and Borel–Moore Poincaré polynomials of all Hilbert–Samuel strata in $\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$

.

Finally, we can prove our main theorem and compute the homology of $\mathrm{Hilb}_5(\mathbb{A}^n)(\mathbb{C})$.

Proof of Theorem 3.1.1. By Proposition 3.4.1 and Proposition 3.4.7, every Hilbert–Samuel stratum of $\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$ has Tate-type homology. By repeated applications of Lemma 3.3.8,

$\mathrm{Hilb}_5(\mathbb{A}^n, 0)(\mathbb{C})$ has Tate-type homology, and its Borel–Moore Poincaré polynomial is the sum of the Borel–Moore Poincaré polynomials of the Hilbert–Samuel strata in Table 3.4. □

d	Poincaré polynomial of $\mathrm{Hilb}_d(\mathbb{A}^n)(\mathbb{C})$
1	1
2	$g_{1,n-1}(z)$
3	$g_{2,n+1}(z)$
4	$z^{2n}g_{2,n}(z) + g_{3,n+2}(z)$
5	$z^{2(3n-3)}g_{1,n}(z) + z^{4(n-1)}(1+z^2)g_{2,n}(z) + z^{2(2n-4)}g_{1,3}(z)g_{2,n}(z) + z^{2(n-3)}g_{1,6}(z)g_{3,n}(z) + g_{4,n}(z)$

Table 3.5: Poincaré polynomials for $\mathrm{Hilb}_d(\mathbb{A}^n)$ with $d \leq 5$

Taking $n = 2$, we determine that

$$p_{\mathrm{Hilb}_5(\mathbb{A}^2,0)(\mathbb{C})}(z) = z^8 + 2z^6 + 2z^4 + z^2 + 1.$$

Ellingsrud and Strømme computed in [16, Thm. 1.1] that

$$b_{2k}(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) = P(k, 5-k),$$

where $P(m, n)$ is the number of sequences of natural numbers $(b_0, \ldots, b_m)$ so that $n \geq b_0 \geq \cdots \geq b_m = 0$ and $\sum b_i = m$. We may compute

$$\begin{aligned}
b_0(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(0, 5) = 1 \\
b_2(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(1, 4) = 1 \\
b_4(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(2, 3) = 2 \\
b_6(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(3, 2) = 2 \\
b_8(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(4, 1) = 1 \\
b_{\geq 10}(\mathrm{Hilb}_5(\mathbb{A}^2, 0)(\mathbb{C})) &= P(\geq 5, \leq 0) = 0,
\end{aligned}$$

which agrees with the Poincaré polynomial we computed.

Furthermore, Graffeo, Monavari, Moschetti, and Ricolfi computed the classes of $\mathrm{Hilb}_d(\mathbb{A}^n, 0)$ in the Grothendieck ring of varieties over $\mathbb{C}$ purely in terms of the classes $[\mathrm{Hilb}_d(\mathbb{A}^i, 0)] \in K_0(\mathsf{Var}_{\mathbb{C}})$ for $i < d$. Indeed, [24, Cor. 5.4] states

$$[\mathrm{Hilb}_d(\mathbb{A}^n, 0)] = \sum_{i=1}^{d-1} (-1)^{d+i+1} \left[\mathrm{Hilb}_d(\mathbb{A}^i, 0)\right] \left[\mathrm{Gr}_{n-d}(\mathbb{A}^{n-i-1})\right] [\mathrm{Gr}_i(\mathbb{A}^n)] \mathbb{L}^{\binom{d-i}{2}}.$$

The class $[\mathrm{Gr}_k(\mathbb{A}^n)]$ is a polynomial in $\mathbb{L}$ given by replacing z^2 with $\mathbb{L}$ in $g_{k,n}(z)$, as the Grassmannian admits a stratification into affine spaces.

This arises via the main theorem [24, Thm. A], which computes the generating series

$$Z_d(t) = \sum_{n \geq 0} \left[\mathrm{Hilb}_d(\mathbb{A}^{n+1}, 0)\right] t^n$$

as

$$Z_d(t) = \zeta_{\mathbb{P}^{d-1}}(t) P_d(t),$$

where

$$\zeta_{\mathbb{P}^{d-1}}(t) = \prod_{i=1}^{d-1} \frac{1}{1 - \mathbb{L}^i t}$$

is the motivic zeta function of $\mathbb{P}^{d-1}$ and

$$P_d(t) = \sum_{i=0}^{d-2} a_i^{(d)} t^i$$

with coefficients

$$a_i^{(d)} = \sum_{j=0}^{i} (-1)^j \left[\mathrm{Hilb}_d(\mathbb{A}^{i-j+1}, 0)\right] \left[\mathrm{Gr}_j(\mathbb{A}^d)\right] \mathbb{L}^{\binom{j}{2}}.$$

They furthermore explicitly compute $P_d(t)$ for $d \leq 8$ in [24, Appendix A]. We have verified that upon substituting $z^2 = \mathbb{L}$, the polynomials $p_{\mathrm{Hilb}_5(\mathbb{A}^n, 0)}(z)$ and $[\mathrm{Hilb}_5(\mathbb{A}^n, 0)]$ agree for $n \leq 300$ using SageMath [58].

CHAPTER 4

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