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UNIVERSITY OF CALIFORNIA SAN DIEGO

**Acoustic Nonreciprocity via Boundary-driven Spatiotemporal
Self-modulation Enabled by Nonlinear Mechanical Mode Interaction**

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Jihad Al-Gassimi

Committee in charge:

Professor Nicholas S. Boechler, Chair
Professor Michael J. Frazier
Professor James R. Friend
Professor Michael D. Todd

2024

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The dissertation of Jihad Al-Gassimi is approved,
and it is acceptable in quality and form for publi-
cation on microfilm.

University of California San Diego

2024

DEDICATION

To my beloved parents,...

EPIGRAPH

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

"In the name of God, the Most Gracious, the Most Merciful"

وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ

"And above every person of knowledge, there is one more learned (Allah)"

اللَّهُمَّ أَعِنِّي عَلَى ذِكْرِكَ، وَشُكْرِكَ، وَحُسْنِ عِبَادَتِكَ

"Oh God, Help me to Remember You, and Thank You, and Worship You in the Best of Manners"

اللَّهُمَّ إِنِّي أَسْأَلُكَ عِلْمًا نَافِعًا، وَرِزْقًا طَيِّبًا، وَعَمَلًا مَتَقَبَّلًا

"O God indeed I ask You for beneficial knowledge, and Pure Bestowed, and deeds which are accepted"

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As a side note: Chapter 2, is partially a reprint of a prepared material to be published by Jihad Al-Gassimi, Kai Qian, Nicolas Herard and Nicholas Boechler. The dissertation author was the primary investigator and author of this paper.

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ABSTRACT OF THE DISSERTATION

Acoustic Nonreciprocity via Boundary-driven Spatiotemporal Self-modulation Enabled by Nonlinear Mechanical Mode Interaction

by

Jihad Al-Gassimi

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California San Diego, 2024

Professor Nicholas S. Boechler, Chair

Inspired by examples from electromagnetics, acoustic nonreciprocity refers to the unusual phenomenon where waves (sound, elastic, or mechanical) propagate through a material differently in one direction versus the opposite. Analogously to the electronic diode, acoustic nonreciprocity is of particular interest in the context of enabling acoustic logic and signal processing devices, as well as low-loss (backscattering immune) signal propagation. One approach to disrupt reciprocity is to break time-reversal symmetry via spatiotemporal (ST) manipulation of a material's properties through which a wave is propagating. However, all hitherto experimentally demonstrated examples of acoustic nonreciprocity via ST modulation have achieved said modulation by active, distributed control of the material. Such active, distributed control is problematic for implementations involving high

frequencies, small length scales, or many modulated elements. Here, we present two related variations of an alternative approach, namely self-modulation via nonlinear mechanical mode interaction, induced by driving from the material boundary.

The first system we study consists of a discrete material composed of designer nonlinear elements, wherein a longitudinal mechanical wave excited from the material boundary modulates the shear stiffness of the material such that nonreciprocal shear wave propagation should be achieved. For this system, we formulate the equations of motion for the discrete, nonlinear ST modulated spring-mass system, and calculate the wave dispersion therein approximated to several harmonic orders. Additionally, we outline geometric design constraints of the nonlinear springs and masses comprising the material, that enable control over the relative speed and coupling strength of longitudinal and shear waves, along with the nonreciprocal band structure. To validate our theoretical framework, we present both experimental measurements and numerical simulations.

The second approach is based on the Aubry-André-Harper (AAH) model, originally described in the context of quantum electronics, wherein a lattice is modulated by periodic on-site potentials. In this study, we introduce a mechanical analog of the AAH model. The structure comprises a one-dimensional lattice featuring a zero-energy mode parametrized by an angle. Depending on the value of this angle, the system exhibits finite-frequency edge modes, as a consequence of the non-zero Chern numbers of the bulk bands. The nonlinear interaction between these finite frequency edge modes and the zero mode results in a “self-pumping effect”. This phenomenon is where an excitation from the end with the edge mode activates the zero mode, subsequently transporting (pumping) the excitation from that side to the opposite side of the system via a bulk mode, leading to nonreciprocity. We begin our investigation by deriving the equations of motion for the system to map out the spectrum of the system as a function of the angle of its last rotor. Then we used discrete element model to simulate the time response of the system, later we use fast Fourier transform to analyze the system in the frequency domain.

Chapter 1

Introduction

1.1 List of Important Acronyms

AAH Aubry-André-Harper

BZ Brillouin Zone

DEM Discrete Element Model

FDTD Finite-Difference Time-Domain

FEM Finite Element Model

GSP Gaussian Modulated Sinusoidal Pulse

LDV Laser Doppler Vibrometer

PCs Photonic Crystals

PE and KE Potential and Kinetic Energy

R-T Reflection-Transmission

STM Spatiotemporal Modulation

ST Spatiotemporal

TI Topological Insulators

1.2 Introduction to Acoustic Reciprocity

1.2.1 Definition and Principles of Acoustic Reciprocity

The principle of reciprocity originated in the field of optics through the pioneering research of Stokes [1] and Helmholtz [2] in the year 1859. These pioneers laid forth the fundamental laws that govern the reversible interactions between light waves and materials. The characteristic matrix used in the analysis of interference filters and mirrors demonstrates reciprocity, as it remains unaltered when the source and observation point are interchanged [3]. Later, the concept of reciprocity in acoustics was formulated by the work of Rayleigh in 1873 in his paper "Some general theorems relating to vibrations" [4] and later in "The Theory of Sound" [5]. Rayleigh's reciprocity theorem essentially states that the transmission of sound between two points is equivalent in either direction regardless of swapping the source with the receiver [4, 6]. The proof of acoustic wave reciprocity involves several key assumptions. It assumes lossless, linear, time-invariant systems, meaning the system's properties do not change with time, and the relationship between input and output is linear [7, 8]. Additionally, the system must not have any odd external biases, such as external magnetic fields or directional flows that could introduce nonreciprocal behavior [9]. The medium through which the acoustic waves propagate can be complex, *i.e.*, homogeneous (uniform properties throughout) or isotropic (properties are the same in all directions). The key factor in ensuring reciprocity is the symmetry of the reflection-transmission (R-T) matrix of the assumed systems [9, 10]. For practical purposes, we often assume small perturbations in the medium, meaning the variations in the properties of the medium are small enough to be considered linear perturbations [11]. The proofs typically assume that the domain is bounded and has well-defined boundaries, such as rigid or free boundaries [12]. While time reversal symmetry is often assumed in these proofs, it should be recognized that the introduction of linear damping, while breaking time-reversal symmetry, does not necessarily violate reciprocity. In such scenarios,

the system preserves the fundamental symmetry of the R-T matrix and maintains identical responses for waves propagating in opposite directions, provided the damping is uniform and does not introduce directional bias [13]. Analytically, reciprocal response is defined as

$$G(r_1, r_2) = G(r_2, r_1), \quad (1.1)$$

where $G(r_1, r_2)$ is Green's function, which represents the acoustic response from the initial point r_1 to the final point r_2 [14].

1.2.2 Acoustic symmetry

It is important to distinguish between the two related but distinct concepts in wave physics, asymmetric propagation and nonreciprocal propagation. Asymmetric propagation refers to the phenomenon where waves propagate differently depending on the direction, which can occur due to structural or material asymmetry without necessarily breaking reciprocity [9]. For example, when waves pass through a spatially asymmetric object such as a triangle, the transmission can appear highly asymmetric. However, despite this asymmetry, the transmission remains reciprocal. This can be verified by examining the reflection-transmission scattering matrix [9]. Nonreciprocal propagation, on the other hand, occurs when the system's response is direction-dependent in a way that cannot be reversed by simply inverting the wave direction (rather the transmission coefficients change when switching input and output channels), often requiring nonlinearity and spatial asymmetry or external energy influences such as magnetic fields [15]. This nonreciprocal behavior can be facilitated through several mechanisms, such as magneto-optical effects, where materials introduce a phase shift depending on the wave direction, and time-varying media that break time-translation symmetry [16, 17]. Additionally, nonreciprocal transmission can arise in systems characterized by nonlinear and non-mirror-symmetric properties [18]. Mechanical metamaterials have been used to demonstrate this nonreciprocity via nonlinearity and spatial asymmetry concept [19].

1.2.3 Reciprocity outside acoustics and optics

The principle of reciprocity can extend into other fields as well in addition to acoustics and optics. To name a few, in the field of electrodynamics reciprocity attracted the attention of Lorentz in 1896 and his theory became known as the Lorentz reciprocity theorem [20]. This theorem states that the relationship between an oscillating current and the resulting electric field remains unchanged if the locations where the current is applied and where the field is measured are interchanged [21]. The analogue for this in mechanics is known as the Maxwell-Betti theorem [22], which originated in 1872 and which states that the deflection at one point due to a load at another point is equal to the deflection at the latter point due to a load at the former.

1.2.4 Usefulness of Acoustic Reciprocity

There are numerous applications for reciprocal acoustic systems that take advantage of the interchangeability and consistency of these acoustic signals. By swapping the transmission and reception phases under specific conditions, predictable data can be collected to confirm accuracy within experiments and validate results. For example, reciprocity is often used to ensure precision in electromagnetic systems and electronics, especially when calibrating acoustic measurement devices such as emission sensors or ear probes [23, 24]. This property is very useful for detecting errors in a system [25, 26]. This has been suggested to be useful, *e.g.*, when considering the wave propagation across flawed channels in antennas [27]. Additionally, reciprocity has also been suggested for measuring low-frequency sound sources [28]. Furthermore, reciprocity has been employed in vibrating structures for finding the radiation efficiency, providing improved flexibility and accuracy compared to direct methods, particularly in complex setups [29] by allowing microphones and loudspeakers to be interchanged during transfer function measurements [30]. This principle maintains similar importance in both theoretical and practical applications related to acoustic waves [31], as it streamlines sound wave modeling and analysis in vehicles and engines [32], as well as noise field measurements in submarines [31]. Other example is time reversal focusing, which is a

technique that uses acoustic reciprocity to accurately refocus waves on their original source, even in complex, inhomogeneous media. This is achieved by capturing and re-emitting waves in reverse order, a time-reversal mirror system compensates for distortions, ensuring precise wave convergence on the target [33].

1.3 Breaking Acoustic Reciprocity

1.3.1 Definition Acoustic Nonreciprocity

As described previously, acoustic nonreciprocity is the phenomenon where sound waves exhibit different scattering properties when source and receiver are interchanged, violating the symmetry typically described by the R–T reciprocity theorem [9]. This means that in a nonreciprocal acoustic system, sound can pass more easily in one direction than the other, often due to mechanisms like time-reversal symmetry breaking, external biases, or nonlinear effects within the medium. [34].

1.3.2 Methods to Achieve Acoustic Nonreciprocity

Throughout the years, researchers have identified various techniques to break acoustic reciprocity [35]. These techniques can be categorized into three main groups as discussed next. Before exploring each subcategories, it is important to keep in mind that not every asymmetric propagation leads to nonreciprocity, as explained in Sec. 1.2.3. It is also important to note that particular examples and systems that achieve nonreciprocity may be attributed to different categories in other works, compared to their attribution in this work. Similarly, various systems may fall into multiple categories. In this thesis, we adopt the nonreciprocity framework based on the R-T reciprocity theorem as described by Maznev et al. in Ref. [9]. The three main categories are outlined as follows:

- **Break Time Reversal Symmetry and Spatial Asymmetry:**

Time-reversal symmetry (TRS) is a fundamental concept in physics, where the laws governing a system remain invariant when the direction of time is reversed [36]. Breaking this symmetry can lead to nonreciprocity [37]. This

breaking can occur spontaneously, through phase transitions that result in asymmetry, or be induced by external activity such as magnetic fields or specific interactions in quantum systems [38]. Such broken symmetry can be caused via various techniques, here we list a few:

- Spatially asymmetric gain or loss (activity): Spatially asymmetric gain or loss, or more generally, activity can result in acoustic nonreciprocity. This can be seen as equivalent to the quantum Hall effect [39]. Manifestations of this have been seen using the following mechanisms. Magneto- and electro-mechanical coupling has been shown to alter the mechanical response depending on the direction of wave propagation [40, 41]. Odd elasticity has also been used to show nonreciprocity using this mechanism [42, 43]. Other activity approach involved the use of a fluid-filled acoustic ring cavity coupled with three acoustic waveguides, where rotating the ring cavity resulted in acoustic isolation by splitting the ring’s azimuthal resonant modes [44]. Building on this, a flexible rotating ring was proposed to produce nonreciprocal responses due to Coriolis acceleration and material damping [45]. Additionally, a non-reciprocal acoustic phase shifter was developed, in which forward and backward wave transmissions experienced different phase shifts when the fluid in the waveguide had a constant moving velocity [46].
- Spatiotemporal Modulation (STM): Spatiotemporal or space-time modulation involves periodically varying the properties of a material or system over time and space [35]. When the modulation is caused by external energy input, this category could be grouped with the above category of asymmetry gain. This dynamic variation causes wave properties such as speed and direction to change, thereby breaking the inherent symmetry and enabling unidirectional wave propagation [47]. As example, beams with periodically varying elastic properties, STM breaks mechanical reciprocity by shifting the Brillouin zone limits and forming directional band gaps [47]. Another study used STM in a spring-mass

chain to break parity and time-reversal symmetries, demonstrating the versatility of mechanical modulation in creating direction-dependent acoustic devices [48]. Other realizations include the use of cascaded resonators that are dynamically driven to achieve nonreciprocal sound propagation [44]. The use of STM method will be our primary research focus for achieving nonreciprocal systems, which we will discuss in detail in a later section.

- **System Nonlinearity and Spatial Asymmetry:**

Acoustic nonreciprocity can also be realized in passive systems where material properties remain constant over time. The most common reason for nonreciprocity in such systems is due to the nonlinear effects that can lead to the interaction of waves within materials, especially when combined with geometrical asymmetries or other non-uniform properties [49, 35, 50]. Here we name two of these mechanisms which led to the development of acoustic diodes. The first example utilizes a nonlinear medium that generates multiple harmonics of a wave, resulting in nonreciprocal behavior depending on the nature of the nonlinearity and the frequency of the incident wave, as demonstrated both theoretically and experimentally by combining a nonlinear acoustic medium with a superlattice made from glass and water layers [51, 52]. The second example demonstrated an acoustic rectifier by the use of chaos and bifurcations which is defined as a sudden shift in the system's response due to minor changes in a parameter [53]. This system consists of a one-dimensional granular array of steel particles under static compression, where stiffness is modeled nonlinearly through Hertzian contacts [54]. This system is designed with a light mass defect near the boundary, allowing for tunable operational frequencies through adjustments in static load and the material's geometric properties [53]. These previous approaches encountered limitations due to size-to-frequency constraints, making them unsuitable for audio and low-frequency applications. These limitations were addressed by employing two subwavelength Helmholtz cavities tuned to different frequen-

cies, creating the necessary asymmetry for nonreciprocal behavior, which has been validated experimentally [55]. It is important to note that the earlier use of granular systems has inspired the work presented in this thesis, particularly in the design of a nonlinear spring capable of breaking reciprocity in a more stable system, thereby addressing the inherent instability of granular systems.

- **Topological Insulators:**

Topological insulators (TI) are materials that act as insulators in their interior while supporting conductive states on their surfaces or edges [56]. One intriguing application of topological insulators is in achieving nonreciprocity via robust edge states that are immune to backscattering from impurities and defects, ensuring unidirectional wave transmission [57, 58].

1.3.3 Application of Acoustic nonreciprocity

The nonreciprocal acoustic response has significant implications across a wide range of applications, from communication technologies to critical medical implementations. In this section, we provide an overview of some practical applications of acoustic nonreciprocity in daily life. Additional application can be found in Ref. [59]. Breaking acoustic reciprocity can enhance control over signal transport, reduce interference, protect sources from back-propagation [60], and facilitate energy harvesting [61].

In the context of communications, acoustic diodes can be used to create noise barriers that block sound from entering a specific area while allowing it to exit. This application is particularly useful in urban environments and medical facilities to reduce noise pollution [62, 63]. For instance, if the room one is trying to make quieter has noise sources on the inside (as well as outside), a conventional noise blocking material could make the room act as a resonant cavity. Additionally, in underwater acoustic communication, nonreciprocal devices can establish one-way communication channels that reduce interference and improve the clarity of signals transmitted over long distances [64].

In terms of sensing, acoustic rectifier demonstrates a nonlinear response, where small signals keep the system in a low-amplitude state, while larger signals cause a transition to a high-amplitude state. [65], which can be instrumental in developing advanced sensing technologies. Such sensors may be crucial in preventing catastrophic events, such as through the early detection of earthquakes or tsunamis [66]. Other potential applications include seismic imaging [67] and nondestructive evaluation [68].

Regarding medical applications, acoustic diodes show promise in ultrasound imaging and therapy. Since acoustic diodes can create unidirectional sound barriers [69], they could enhance the precision and effectiveness of ultrasonic procedures, leading to improved imaging quality and more accurate diagnostics. This improvement is achieved by reducing noise and reflections that typically degrade image clarity [70]. Additional medical applications include therapeutic uses, such as focused ultrasound surgery or lithotripsy (a treatment for kidney stones). In these cases, acoustic diodes could direct acoustic waves more precisely to the target area, minimizing damage to surrounding tissues and enhancing the efficiency of energy delivery [71].

1.4 Nonreciprocity via Spatiotemporal Modulation

1.4.1 Development of Spatiotemporal Modulation

In this section we will briefly review the history of STM, which will lead to our current research questions. The initial observation of STM started in the 1950s in the context of parametric amplification of traveling electromagnetic waves, where nonreciprocity was demonstrated using magnet-based isolators [72, 73, 74]. However, the magnet-based isolators approach, although dominant at the time, faced challenges in small-scale applications due to the bulkiness of the equipment and limited frequency range [75]. For instance, early nonreciprocal devices such as ferrite isolators were large and heavy, often requiring significant power to operate

effectively. This made them unsuitable for many emerging technologies, particularly those requiring compact and lightweight components, such as radar systems in aircraft and portable communication devices [76]. As the need for smaller, more efficient systems grew, particularly during the post-war period, there was increasing demand for miniaturization in electronic components that could not be met. The development of transistor technology in the late 1940s and early 1950s, for example, underscored the importance of reducing the size and power requirements of electronic systems [77].

This era also saw the exploration of new modulation techniques that could be implemented on a smaller scale, leading to innovations in both circuit design and materials used in these systems. The limitations of bulky equipment spurred research into alternative solutions, such as the development of time modulation and pulse modulation techniques that were more compatible with smaller, integrated systems [75]. These advancements laid the groundwork for the later development of STM technologies, which offered greater flexibility and compatibility with miniaturized electronic circuits [78].

In the 1960's a new type of electronically tunable acousto-optical filter was proposed. This filter utilized a collinearly propagating acoustic beam such that an incident optical signal of a polarized crystal in anisotropic optical media at a chosen orientation is diffracted into orthogonal polarization collinearly propagating beam [79]. Later in the 1990's, a formulation was presented to demonstrate how the transition between different modes in photonic crystals (PCs) occurs, a process driven by varying dielectric constant in space and time [80]. This formulation established the foundation for developing the use of STM to achieve full isolation in optical systems through the use of refractive index [81]. While previous methods of optical isolation relied on nonlinear optical processes, these were limited to specific power ranges [82]. STM was not constrained by this limitation, and therefore offered a more versatile and scalable approach to isolation, paving the way for broader applications in optical systems [83].

Later in the 2000's Blekhman and Lurie proposed a unique type of smart material which varies in both space and time, which they dubbed dynamic mate-

rials [84]. Researchers began to explore how variations in these dynamic material parameters could influence their dynamic responses, expanding the understanding of material behavior under varying conditions [85, 86]. In 2006, Lurie and Weekes discussed elastic wave propagation in one-dimensional spatiotemporal bi-periodic materials [87], inspiring further studies on spatiotemporal topology optimization by Jensen [88]. Subsequent research employed the finite-difference time-domain (FDTD) method to investigate the effects of time-varying material parameters on acoustic wave propagation [89]. Later, in 2008 Wright and Cobbold formalized the idea of time-varying PCs, combining parametric dynamic modulation with PCs and demonstrating significant changes in acoustic wave transmission due to temporal variations in pipe radius [90]. Later, Li et al. examined unidirectional mode switching and signal separation in dynamic PCs [91].

Applying these ideas to mechanical (elastic) systems, in 2015, Shui et al. proposed a modified multiscale homogenization method to study the dynamic characteristics of spatiotemporal metamaterials, showing unidirectional elastic wave propagation due to time-symmetry breaking [92]. Sorokin and Thomsen explored wave propagation in moving ropes with periodic structures, highlighting limitations of traditional forbidden bands [93]. Trainiti and Ruzzene investigated wave propagation in beams with elastic properties that vary periodically in both space and time [47]. His work to solve discrete systems with spatiotemporally varying stiffness for one degree of freedom [94] is used as the base for our analytical method used in this research with the expansion of multi degree of freedom system and the addition of bending stiffness.

1.4.2 Achieving Spatiotemporal Modulation in Diverse Systems

In this section, we explore how STM can be integrated into various other systems, leading to significant improvements in existing applications and opening avenues for new innovations. As a preliminary overview, we briefly discuss nonreciprocity induced by STM in various domains, including thermal [95], electromagnetic [96, 97], and photonic [98] systems. We note that the examples explored are

non-exhaustive.

Nonreciprocity of thermal waves was achieved by STM of the system's thermal conductivity and volumetric heat capacity [99, 100, 101]. This mechanism introduce a phase difference between the two STM parameters can be tuned to control nonreciprocity in thermal waves, which are inherently transient and governed by diffusion equations was analytically verified and supported by FEM model [99]. Thermal nonreciprocity offers significant potential for wide range applications to name a few, nondestructive thermal detection [102] and imaging [103, 104], as well as thermal cloaking [105, 106, 107], and thermal diodes [108, 109].

In other systems, such as electromagnetic, space-time gradient metasurfaces have been used to modulate surface impedance, resulting in a nonreciprocal electromagnetically induced transparency effect with high efficiency [110, 111]. Other researchers have utilized STM techniques to overcome the drawbacks of magnetically biased ferrites, such as bulkiness, heaviness, and incompatibility with integrated circuits. For example, a nonreciprocal mixer-duplexer-antenna system employing an STM microstrip leaky-wave antenna with an array of varactors to achieve frequency upconversion, downconversion, and beam scanning has been presented [112]. The researchers demonstrated successful nonreciprocal operation, efficient duplexing, and beam scanning, making the system suitable for nonreciprocal metasurfaces [113] and nonreciprocal antennas [114]. Additionally, graphene-based devices have shown that STM of conductivity can create nonreciprocal plasmonic components suitable for terahertz frequencies, enhancing applications in biosensing and interchip communications [115].

In photonics, nonreciprocity was achieved by mimicking the Aharonov-Bohm effect, by inducing a phase modulation in the refractive index of a photonic waveguide, which introduced an effective gauge potential for photons [116, 117]. Other method is STM the permittivity of a medium to break Lorentz reciprocity, resulting in asymmetric band gaps that allow for high levels of isolation [118]. These methods were applied to design a nonmagnetic optical isolator on-chip, achieving significant nonreciprocal light propagation with a high isolation contrast [119, 120].

1.4.3 Experiments in Acoustic Systems

In this section, we present three experimental setups that were developed to verify the effect of STM properties on acoustic or elastic waves. Other experimental setups are also possible and can be found in review articles such as Ref. [35] and Ref. [121].

In an early experimental study by Tessier et al. [122], the researchers designed an experiment to investigate nonreciprocal wave propagation in a one-dimensional piezoelectric phononic crystal. The crystal was composed of 113 piezoelectric rings, each separated by thin brass electrodes. The electrical boundary conditions of these electrodes were STM by using external circuits controlled by a digital signal processor, which enabled them to dynamically switch the electrodes between grounded and floating states. This modulation affected the mechanical waves propagating through the crystal and the electric fields generated within the piezoelectric material. The mechanical waves were generated by applying a voltage difference across selected electrodes, and their propagation was measured using a data acquisition system. Their experimental results showed strong agreement with theoretical predictions and confirmed that by increasing the modulation speed, the system bandgaps progressively narrowed and eventually closed, confirming the impact of the modulation on wave dispersion.

A second experimental work by Mehrvarz et al. [123], involved using an aluminum beam with its stiffness dynamically modulated by a closed-loop feedback control system that continuously adjusts the voltage applied to piezoelectric actuator patches based on the signals received from sensor patches. This modulation is applied in both space and time, allowing for precise control over the stiffness distribution along the beam. The experiment measures the beam's response to wave propagation using a laser vibrometer, with the goal of analyzing how waves behave under different modulation conditions. The results show that the spatiotemporal modulation successfully creates nonreciprocal bandgaps, where wave propagation is allowed in one direction but blocked in the other. This confirms the effectiveness of the setup in achieving tunable, broadband mechanical nonreciprocity demonstrated analytically by Ref. [47].

A third experimental setup by Zhou and Baz [124], involved a 1D acoustic duct segmented into periodic cells, each containing an acoustic cavity with collocated sensor-actuator pairs (microphones and speakers). The feedback gain applied to each actuator is modulated both in time and space through a LabVIEW-controlled system, which introduces phase differences between adjacent cells, inducing STM that leads to nonreciprocal wave propagation. The system’s performance was evaluated by measuring sound pressure along the duct, revealing that the STM effectively created nonreciprocal band gaps. Within these band gaps, sound wave propagation can be controlled and tuned by adjusting the modulation parameters, such as the gain and phase difference.

In summary, the previous examples demonstrate the ability to achieve acoustic or elastic nonreciprocal behavior through spatiotemporal modulation of system properties. This field has significant potential for growth and improvement, given that there are no strict limitations on which properties can be subjected to STM and what mechanism could be used to induce the modulation.

1.5 Motivation

As previously mentioned, the use of granular systems to achieve nonreciprocal behavior was one of the primary inspirations for this thesis. Given that the contact stiffness of steel particles in these systems is modeled nonlinearly through Hertzian contacts, the question arose: Can we design a simpler nonlinear spring stiffness that its shear stiffness dependent upon the longitudinal strain? Could such a spring be implemented to improve mechanical stability compared to granular systems, but produce similar nonreciprocal behavior? The second motivation was related to scalability, both in terms of size and speed. A close examination of previous experimental setups reveals that while STM can be achieved experimentally, it has required active element by element control. This approach may limit the range of frequencies and the overall system size (how many elements controlled simultaneously). This observation led to one of the key questions driving this thesis: Is it possible to achieve acoustic nonreciprocity via STM without the need for

continuous external control?

Both of these questions are explored in detail in this thesis.

1.6 Research Objectives and Thesis Structure

The objective of this research is to explore the feasibility of achieving nonreciprocal wave transport through spatiotemporal modulation and induced geometric nonlinearity in two boundary-driven systems, without relying on any additional external distributed control. Additionally, the simplicity of the design ensures ease of implementation without any complex fabrication processes or specialized materials.

In in Chapter 2, we introduce the spatiotemporal modulated one dimensional spring mass system, the interaction between transverse and axial propagating waves tend to have asymmetric dispersion relation, this is due the shear stiffnesses are nonlinearly dependent on the longitudinal strain. We begin by mathematically deriving the equations of motion for the system using Lagrange's energy approach. Subsequently, utilize the Discrete Element Model (DEM) to simulate the system and develop a Finite Element Model (FEM) to validate the results. Design, manufacture and experimentally test the normal and shear stiffness of the nonlinear spring. Later, design an experimental configuration to validate the findings. Lastly, simplify the system by assuming zero bending stiffness and show the ability for nonreciprocal response both analytically and numerically by DEM simulations.

Afterwards, in Chapter 3, we briefly introduce the main concept of Aubry-André-Harper (AAH) model. Then, we introduce a mechanical analog of AAH model as our second system, where nonreciprocity arises from adiabatic pumping. We mathematically derive the one-dimensional system, illustrating the variation of the zero-energy mode with an angle parameter. Through the utilization of DEM simulations, we demonstrate the interaction that occurs between the edge modes and the zero mode. Via frequency domain analysis of the temporal response, we illustrate the phenomenon known as the "self-pumping effect", where excitation

from one end with the edge mode stimulates the zero mode, subsequently transporting the excitation to the opposite end via the bulk mode, thereby inducing nonreciprocal behavior. Additionally, we estimate the damping characteristics of an actual unit cell through free oscillation tests and incorporate these findings into our simulations of the damped system, providing a foundation for the future development of an experimental setup.

Finally in Chapter 4, we outline several proposed advancements and modifications aimed at enhancing the system's response and accuracy. These improvements are designed to address current limitations and optimize performance. This organization is outlined in Fig. 1.1.

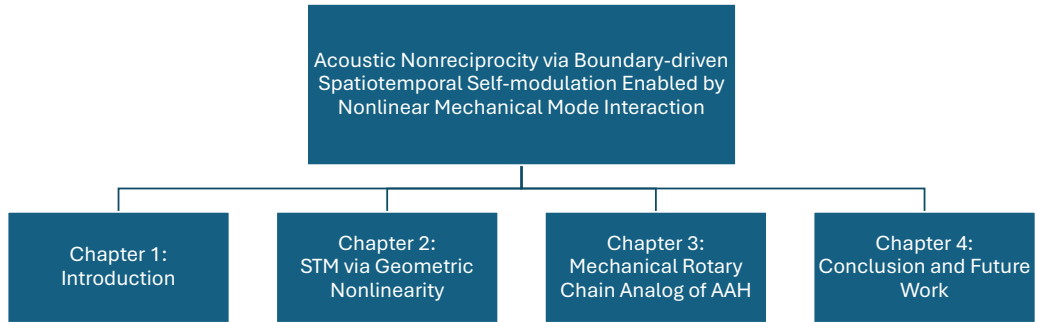


Figure 1.1: Brief outline of the organization of the dissertation

Chapter 2

Exploring the Possibility of Nonreciprocity via Geometric-Nonlinearity-Enabled All-Acoustic Spatiotemporal Modulation

2.1 Introduction

Reciprocity of wave transport is typical in acoustic systems and refers to the characteristic that the transmission of waves between two points remains the same regardless of the source and the receiver being interchanged [121] (or, more restrictively [9], for lossless systems the scattering matrix is symmetric ¹). Wave nonreciprocity, in contrast, has been suggested to be useful for source protection [125], logic and computing [126], and energy harvesting [127]. Common ways that have been shown to break the wave reciprocity include combining asymmetry with nonlinearity [9, 35, 121] or activity [128, 129].

Within the category of nonreciprocity via activity, spatiotemporal modula-

¹which distinguishes asymmetric from nonreciprocal systems

tion, in which one or more constitutive parameters vary with respect to both space and time, has emerged as an approach of interest, including within the domains of electronic circuits, optics, and acoustics [72, 73, 74, 79, 80, 81, 130, 131]. Considering acoustics and mechanics, experimentally demonstrated methods to induce spatiotemporal modulation of a material’s properties have included distributed electronic control by piezoelectric arrays [132, 133, 134, 122], electromagnets [135, 136], or relays [137]. A potential drawback of such an approach is that active element by element control may limit the range of frequencies and system size (both dimension and number of unit cells) to which this approach can be applied, and may be at odds with one of the potential advantages of mechanical computing, namely functionality in extreme, electronically noncompatible environments.

An alternative approach for nonreciprocity through spatiotemporal modulation that overcomes these drawbacks is to leverage the interaction between two waves propagating at different speeds [81, 138]. These two waves do not necessarily need to be of the same “type” (*e.g.*, electromagnetic and acoustic), but there must be a dependence of a physical property affecting one wave on the other wave. Nonreciprocity enabled by spatiotemporal modulation via wave mixing in mechanical systems (*i.e.* “all-acoustic spatiotemporal modulation”) has previously been explored theoretically, for instance using mechanisms such as torsional waves modulating the stiffness affecting longitudinal waves [139], large deformation waves modulating the stiffness affecting small deformation waves [48], and longitudinal waves modulating the shear stiffnesses affecting transverse-rotational waves via Hertzian contacts in granular chains [140].

In this work, we analytically, computationally (through reduced order, discrete element model [DEM] and finite element method [FEM] simulations), and experimentally study a quasiperiodic one dimensional nonlinear system, where the shear stiffness depends on longitudinal strain, with the aim that nonreciprocity of transverse-rotational waves could be enabled by the simultaneous propagation of a longitudinal wave injected from the boundary. Reasonable agreement is found between our analytical predictions, and our DEM and FEM simulations. A key observed difference observed is that leakage from the longitudinal wave is observed

in the FEM, but not DEM simulations, which we attribute to mechanical instability that is not captured by the DEM. While our experiments also show signs of nonreciprocity, we observe that they do so in a qualitatively different manner than in our simulations. In particular, we observe even more leakage between the longitudinal and transverse-rotational modes in experiment, which we suggest may be related to the disorder and asymmetry not present in the idealized models.

2.2 Theoretical System and Equations of Motions

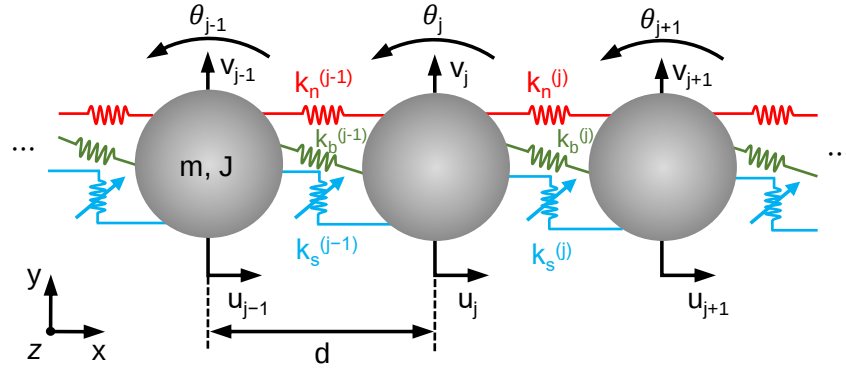


Figure 2.1: Theoretical model of one-dimensional (1D) mass-and-spring system with linear normal, linear bending, and nonlinear shear springs, wherein the shear stiffness depends on the normal strain.

We consider a 1D mass-and-spring model with N unit cells, as shown in Fig. 2.1, where each mass has two translational degrees of freedom u and v (along the x and y directions, respectively) and one rotational degree of freedom θ (about the z direction). The adjacent masses are connected by linear normal springs k_n , linear bending springs k_b , and nonlinear shear springs k_s . Such transverse-rotational models have been extensively studied previously (*e.g.*, [141, 142]). Herein, the nonlinear shear springs have stiffness that depends on the normal strain at the same location. The equations of motion for the j th mass, derived from a Lagrangian, as detailed below To derive the equations of motion for

the j th mass (particle), we can write its Lagrangian in the form of $\mathcal{L} = KE^j - PE^j$, where KE and PE are the particle's kinetic energy and the potential energy of the neighboring springs, respectively, such that

$$KE^j = \frac{m}{2}(\dot{u}_j)^2 + \frac{m}{2}(\dot{v}_j)^2 + \frac{I}{2}(\dot{\theta}_j)^2 \quad (2.1)$$

and

$$PE^j = \sum_{i=j-1,j} \left(\int_0^{u_{i+1}-u_i} (k_n^{(i)} u) du + \int_0^{(v_{i+1}-\frac{d}{2}\sin\theta_{i+1})-(v_i+\frac{d}{2}\sin\theta_i)} (k_s^{(i)} v) dv \right. \\ \left. + \int_0^{\frac{d}{2}(\sin\theta_{i+1}-\sin\theta_i)} (k_b^{(i)} \alpha) d\alpha \right), \quad (2.2)$$

where $k_n^{(j)}$ is the normal stiffness, $k_s^{(j)} = k_0(1 + B_0(u_{j+1} - u_j)/S_0)$ is the shear stiffness, where S_0 is the unstretched length of the springs, and $k_b^{(j)}$ is the bending stiffness. By using the Euler–Lagrange equation

$$(\partial_t \partial_{\dot{q}_j} \mathcal{L}) - (\partial_{q_j} \mathcal{L}) = 0, \quad (2.3)$$

where q is the generalised degree of freedom representing u , v , and θ . As, a result of evaluating Eq. 2.3 we obtain equations of motion for the j th mass

$$m\ddot{u}_j = k_n^{(j-1)}(u_{j-1} - u_j) + k_n^{(j)}(u_{j+1} - u_j) - \\ \frac{B_0 k_0 \left(\frac{d}{2}(\sin(\theta_{j-1}) + \sin(\theta_j)) + v_{j-1} - v_j \right)^2}{2S_0} + \\ \frac{B_0 k_0 \left(\frac{d}{2}(\sin(\theta_j) + \sin(\theta_{j+1})) + v_j - v_{j+1} \right)^2}{2S_0}, \quad (2.4)$$

$$m\ddot{v}_j = k_s^{(j-1)} \left(\frac{d}{2}(\sin(\theta_{j-1}) + \sin(\theta_j)) + v_{j-1} - v_j \right) - \\ k_s^{(j)} \left(\frac{d}{2}(\sin(\theta_j) + \sin(\theta_{j+1})) + v_j - v_{j+1} \right), \quad (2.5)$$

and

$$J\ddot{\theta}_j = (d/2)\cos(\theta_j) \left(k_b^{(j-1)}(d/2)(\sin(\theta_{j-1}) - \sin(\theta_j)) \right. \\ \left. + k_b^{(j)}(d/2)(\sin(\theta_{j+1}) - \sin(\theta_j)) \right. \\ \left. - k_s^{(j-1)}((d/2)(\sin(\theta_{j-1}) + \sin(\theta_j)) + v_{j-1} - v_j) \right. \\ \left. - k_s^{(j)}((d/2)(\sin(\theta_j) + \sin(\theta_{j+1})) + v_j - v_{j+1})) \right). \quad (2.6)$$

By rearranging and assuming small angle approximation ($\sin \theta \approx \theta$ and $\cos \theta \approx 1$) we obtain equations of motions

$$m\ddot{u}_j = -k_n^{(j-1)}(u_j - u_{j-1}) + k_n^{(j)}(u_{j+1} - u_j) + \frac{B_0 k_0 (v_j - v_{j-1} - (d/2)(\theta_j + \theta_{j-1}))^2}{2S_0} - \frac{B_0 k_0 (v_{j+1} - v_j - (d/2)(\theta_{j+1} + \theta_j))^2}{2S_0}, \quad (2.7)$$

$$m\ddot{v}_j = -k_s^{(j-1)} \left(v_j - v_{j-1} - \frac{d}{2}(\theta_j + \theta_{j-1}) \right) + k_s^{(j)} \left(v_{j+1} - v_j - \frac{d}{2}(\theta_{j+1} + \theta_j) \right), \quad (2.8)$$

$$J\ddot{\theta}_j = \frac{d}{2}k_s^{(j-1)} \left(v_j - v_{j-1} - \frac{d}{2}(\theta_j + \theta_{j-1}) \right) + \frac{d}{2}k_s^{(j)} \left(v_{j+1} - v_j - \frac{d}{2}(\theta_{j+1} + \theta_j) \right) + \left(\frac{d}{2} \right)^2 \left(-k_b^{(j-1)}(\theta_j - \theta_{j-1}) + k_b^{(j)}(\theta_{j+1} - \theta_j) \right). \quad (2.9)$$

We further simplify the previous expression by assuming constant normal and bending stiffness such that $k_n^{(j-1)} = k_n^{(j)} = k_n$, $k_b^{(j-1)} = k_b^{(j)} = k_b$ which results in simplified equations of motions

$$m\ddot{u}_j = k_n(u_{j-1} - 2u_j + u_{j+1}) - \frac{S_0}{B_0 k_0} \frac{\Gamma_1^{(j)}}{2} (\Gamma_2^{(j)} + \Gamma_3^{(j)}), \quad (2.10)$$

$$m\ddot{v}_j = \frac{S_0}{B_0} \Gamma_1^{(j)} + \Gamma_2^{(j)}(u_j - u_{j-1}) + \Gamma_3^{(j)}(u_j - u_{j+1}), \quad (2.11)$$

$$J\ddot{\theta}_j = \frac{d}{2}(k_0(v_{j+1} - v_{j-1}) - u_j \Gamma_1^{(j)} + u_{j-1} \Gamma_2^{(j)} - u_{j+1} \Gamma_3^{(j)}) + \left(\frac{d}{2} \right)^2 ((k_b - k_0)(\theta_{j-1} + \theta_{j+1}) - 2(k_b + k_0)\theta_j) \quad (2.12)$$

and

$$\begin{aligned} \Gamma_1^{(j)} &= \frac{B_0 k_0}{S_0} \left(v_{j-1} - 2v_j + v_{j+1} + \frac{d}{2}(\theta_{j-1} - \theta_{j+1}) \right), \\ \Gamma_2^{(j)} &= \frac{B_0 k_0}{S_0} \left(v_{j-1} - v_j + \frac{d}{2}(\theta_{j-1} + \theta_{j+1}) \right), \\ \Gamma_3^{(j)} &= \frac{B_0 k_0}{S_0} \left(v_j - v_{j+1} + \frac{d}{2}(\theta_{j-1} + \theta_{j+1}) \right). \end{aligned} \quad (2.13)$$

where m is the mass, d is the lattice spacing, J is the moment of inertia, $k_s^{(j)}$ is a function of the longitudinal strain defined as $k_0(1 + B_0(u_{j+1} - u_j)/S_0)$, in which k_0 is the constant part of the shear stiffness, B_0 is an experimentally determined nonlinear coefficient, and S_0 is the unstretched spring length, noting that in our physical realization of the system, the portion that we denote as the “spring” comprises a shorter length than that of the unit cell (*i.e.*, it does not include the parts of the unit cell associated with a more rigid “mass”).

2.3 Nonlinear Spring Design

Before analyzing the dynamic response of our system, we design a physical spring, in which we can obtain a shear stiffness dependent on longitudinal strain. We choose a beam-based diamond structure as the nonlinear spring, as shown in Fig. 2.2. The rationale behind choosing this design, is that when the diamond has

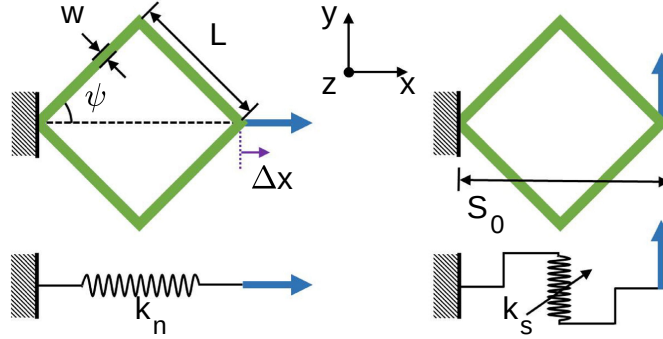


Figure 2.2: Beam-based, diamond-shaped nonlinear spring design (in green) and properties with its effective normal (k_n) and shear (k_s) spring models.

a high aspect ratio (tall in y and narrow in x), we expect high shear stiffness and low normal stiffness (and vice versa for low aspect ratio structures). Geometrically nonlinear, finite deformation applied to the spring will change the aspect ratio of the diamond and thus its stiffness. The normal and shear stiffnesses of the diamond spring are estimated assuming small, elastic deformation of long-thin prismatic

beams around its initial and longitudinally deformed position, respectively, using Castigliano's second theorem [143], as described in the next section.

2.3.1 Effective Normal Stiffness Based on Beam Theory and Energy Methods

Consider the beam model in Fig. 2.3(a) as a quarter of the beam-based diamond shaped spring in Fig. 2.2. The equilibrium equations of static equilibrium can be written as

$$\sum F_x = P - R_{B,x} = 0, \quad (2.14)$$

$$\sum F_y = R_{A,y} = 0, \quad (2.15)$$

and

$$\sum M_A = M_{R,A} + M_{R,B} + R_{B,x}L \sin \psi = 0, \quad (2.16)$$

where the geometric and force and moment variables are defined as in Fig. 2.3. In Eqs. 2.14 to 2.16 there are four unknowns, $R_{B,x}$, $R_{A,y}$, $M_{R,A}$, and $M_{R,B}$, in three equations, which makes this problem statically indeterminate, although we can still obtain

$$R_{A,y} = 0 \quad (2.17)$$

and

$$R_{B,x} = P. \quad (2.18)$$

To solve for the remaining unknowns, we will use the Castigliano's theorem [143]. We first apply a cut as shown in Fig. 2.3(b). Based on Fig. 2.3(b), we write the equations of translational static equilibrium in the x' and y' directions as

$$\sum F_{x'} = P \cos \psi + N(l) = 0 \quad (2.19)$$

and

$$\sum F_{y'} = -P \sin \psi + V(l) = 0, \quad (2.20)$$

where $N(l)$ and $V(l)$ are the normal and shear forces at the cut point, at a distance of l from the point A along the beam. From Eqs. 2.19 and 2.20, we find

$$N(l) = -P \cos \psi, \quad (2.21)$$

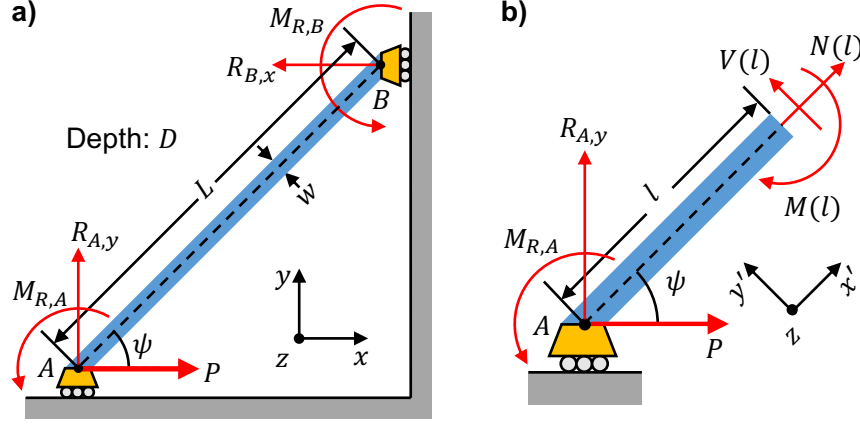


Figure 2.3: (a) Free-body diagram of a long, thin beam model (cross section is a rectangular shape) of length L , width w , depth D (in the z direction), and angle ψ relative to the horizontal line, with roller boundary conditions applied at both ends A and B in x and y directions, respectively. A horizontal load P is applied at the point A . (b) Cut diagram of loaded side from the cut point. Axes x' and y' indicate the axial and shear directions of the beam, and l represents the distance from the point A along x' direction.

and

$$V(l) = P \sin \psi. \quad (2.22)$$

We write the static equation of moment equilibrium at the cutting point as

$$\sum M_{x'=l} = -M(l) + M_{R,A} + Pl \sin \psi = 0, \quad (2.23)$$

leading to

$$M(l) = M_{R,A} + Pl \sin \psi. \quad (2.24)$$

Since the strain energy stored in the beam due to the action of axial force (U_a), bending moment (U_b), direct shear force (U_s), and transverse shear force (U_{ts}) are given by

$$U_a = \int_0^L \frac{N(l)^2}{2AE} dl, \quad (2.25)$$

$$U_b = \int_0^L \frac{M(l)^2}{2EI_A} dl, \quad (2.26)$$

$$U_s = \int_0^L \frac{V(l)^2}{2AG} dl, \quad (2.27)$$

and $U_{ts} = c_{ts}U_s$, where A and I_A are the cross section area and the area moment of inertia of the beam, respectively, E and G are the elastic and shear modulus of the material, respectively, and c_{ts} is the form factor, which is dependent on the shape of the cross section. The total strain energy reads $U = U_a + U_b + U_s + U_{ts}$. According to Castigliano's second theorem, the rotational deflection at point A can be written as

$$\delta\psi_A = \frac{\partial U}{\partial M_{R,A}} = \frac{\partial U_b}{\partial M_{R,A}} = \frac{L^2 P \sin \psi + 2LM_{R,A}}{2EI_A} = 0, \quad (2.28)$$

where $\delta\psi_A$ is set to zero due to the roller boundary. From Eq. 2.28, we obtain

$$M_{R,A} = -\frac{LP \sin \psi}{2}, \quad (2.29)$$

and by substituting Eqs. 2.18 and 2.29 into Eq. 2.16, we find

$$M_{R,B} = -\frac{LP \sin \psi}{2}. \quad (2.30)$$

Since we apply a horizontal load P at the point A, as shown in Fig. 2.3, the effective stiffness in the x direction, $k_{\text{eff},x}$, can be defined as P/δ_x , where δ_x is the total deflection in the x direction, which is the sum of axial (δ_a), bending (δ_b), direct shear (δ_s), and transverse shear deflections (δ_{ts}). To find out each deflection, we use Castigliano's second theorem for each strain energy, leading to

$$\delta_a = \frac{\partial U_a}{\partial P} = \frac{LP \cos^2 \psi}{AE}, \quad (2.31)$$

$$\delta_b = \frac{\partial U_b}{\partial P} = \frac{L^3 P \sin^2 \psi}{12EI_A}, \quad (2.32)$$

$$\delta_s = \frac{\partial U_s}{\partial P} = \frac{LP \sin^2 \psi}{AG}, \quad (2.33)$$

and $\delta_{ts} = c_{ts}\delta_s$. By substituting $A = Dw$, $I_A = Dw^3/12$, and $G = E/(2(1+\nu))$, where ν is the Poisson's ratio, into Eqs. 2.31, 2.32, and 2.33, we obtain

$$\delta_a = \left(\frac{P \cos^2 \psi}{ED} \right) \left(\frac{L}{w} \right), \quad (2.34)$$

$$\delta_b = \left(\frac{P \sin^2 \psi}{ED} \right) \left(\frac{L}{w} \right)^3, \quad (2.35)$$

$$\delta_s = \left(\frac{2P(1+\nu) \sin^2 \psi}{ED} \right) \left(\frac{L}{w} \right). \quad (2.36)$$

As we consider a long, thin beam model ($w/L \ll 1$), we have

$$\frac{\delta_s}{\delta_b} = 2(1+\nu) \left(\frac{w}{L} \right)^2 \rightarrow 0, \quad (2.37)$$

and $\delta_{ts}/\delta_b \rightarrow 0$, leading us to ignore the direct and transverse shear deflections in the total deflection. The ratio between axial and bending deflections reads

$$\frac{\delta_a}{\delta_b} = \left(\frac{w}{L} \right)^2 \cot^2 \psi, \quad (2.38)$$

which depends the choice of ψ besides w/L , especially when ψ is small, leading us to keep both axial and bending deflections in the total deflection

$$\delta_x = \delta_a + \delta_b. \quad (2.39)$$

By using Eqs. 2.34, 2.35, and 2.39, the effective stiffness in the x direction can then be written as

$$k_{\text{eff},x} = \frac{P}{\delta_x} = \frac{EDw^3}{Lw^2 \cos^2 \psi + L^3 \sin^2 \psi} \quad (2.40)$$

Thus, the normal stiffness k_n of the rhombus beam structure can be written as

$$k_n = 2 \left(\frac{1}{2} k_{\text{eff},x} \right) = k_{\text{eff},x} = \frac{EDw^3}{Lw^2 \cos^2 \psi + L^3 \sin^2 \psi}. \quad (2.41)$$

2.3.2 Effective Shear Stiffness Based on Beam Theory and Energy Methods

Consider the beam model in Fig. 2.4(a) as the upper half of the beam-based diamond shaped spring. Using the notation in Fig. 2.4(a), we write the equations of static equilibrium as

$$\sum F'_x = R'_{A,x} - R'_{B,x} = 0, \quad (2.42)$$

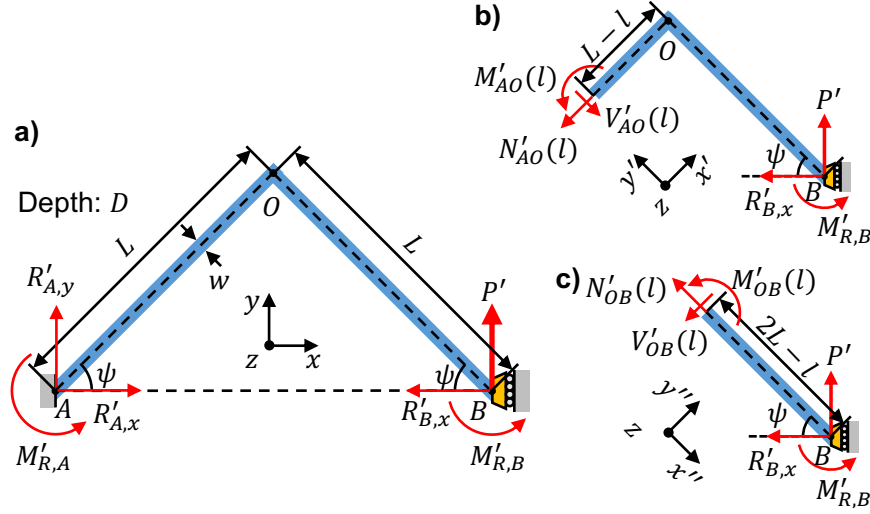


Figure 2.4: (a) Free-body diagram of a symmetric V-shaped, long, thin beam model (cross section is a rectangular shape) of length $2L$, width w , depth D (in the z direction), and angle ψ relative to the horizontal line, with a fixed boundary condition applied at the end A and a roller boundary condition at the end B in y directions. A vertical load P' is applied at the point B . (b-c) Cut diagrams as the cut point in the section of AO and OB , respectively.

$$\sum F'_y = R'_{A,y} + P' = 0, \quad (2.43)$$

$$\sum M'_A = M'_{R,A} + M'_{R,B} + 2P'L \cos \psi = 0. \quad (2.44)$$

Here again, we have five unknowns, $R'_{A,x}$, $R'_{B,x}$, $R'_{A,y}$, $M'_{R,A}$, and $M'_{R,B}$ in three equations, which makes this beam problem statically indeterminate.

We again use Castigliano's theorem to solve for the beam response. We first consider two cuts applied in the sections AO and OB , as shown in Fig. 2.4(b) and (c), respectively. Considering section AO ($0 \leq l \leq L$), we write the equations of static translational equilibrium in the x' and y' directions as

$$\sum F'_{x'} = P' \sin \psi - N'_{AO}(l) = 0, \quad (2.45)$$

$$\sum F'_{y'} = P' \cos \psi - V'_{AO}(l) = 0, \quad (2.46)$$

where $N'_{AO}(l)$ and $V'_{AO}(l)$ are the normal and shear forces at the cut point. From Eqs. 2.45 and 2.46, we find

$$N'_{AO}(l) = P' \sin \psi, \quad (2.47)$$

and

$$V'_{AO}(l) = P' \cos \psi. \quad (2.48)$$

We write the equation for static moment equilibrium at point B as

$$\begin{aligned} \sum M'_B = M'_{AO}(l) + M'_{R,B} + (L \sin 2\psi) N'_{AO}(l) + \\ (L \cos 2\psi + L - l) V'_{AO}(l) = 0. \end{aligned} \quad (2.49)$$

By substituting Eqs. 2.47 and 2.48 into Eq. 2.49, we have

$$M'_{AO}(l) = -M'_{R,B} - P'(2L - l) \cos \psi. \quad (2.50)$$

Considering section OB ($L \leq l \leq 2L$), we write the equations of static translational equilibrium equations in the x' and y' directions as

$$\sum F'_{x''} = -P' \sin \psi - N'_{OB}(l) = 0, \quad (2.51)$$

$$\sum F'_{y''} = P' \cos \psi - V'_{OB}(l) = 0, \quad (2.52)$$

where $N'_{OB}(l)$ and $V'_{OB}(l)$ are the normal and shear forces at the cut point. From Eqs. 2.51 and 2.52, we obtain

$$N'_{OB}(l) = -P' \sin \psi, \quad (2.53)$$

and

$$V'_{OB}(l) = P' \cos \psi. \quad (2.54)$$

We write the equation for static moment equilibrium at point B as

$$\sum M'_B = M'_{OB}(l) + M'_{R,B} + (2L - l) V'_{OB}(l) = 0. \quad (2.55)$$

By using Eq. 2.54 in Eq. 2.55, we obtain

$$M'_{OB}(l) = -M'_{R,B} - P'(2L - l) \cos \psi. \quad (2.56)$$

We then write the strain energy stored in the entire beam due to the action of axial force (U'_a), bending moment (U'_b), direct shear force (U'_s), and transverse shear force (U'_{ts}) as

$$U'_a = \int_0^L \frac{N'_{AO}(l)^2}{2AE} dl + \int_L^{2L} \frac{N'_{OB}(l)^2}{2AE} dl, \quad (2.57)$$

$$U'_b = \int_0^L \frac{M'_{AO}(l)^2}{2EI_A} dl + \int_L^{2L} \frac{M'_{OB}(l)^2}{2EI_A} dl, \quad (2.58)$$

$$U'_s = \int_0^L \frac{V'_{AO}(l)^2}{2AG} dl + \int_L^{2L} \frac{V'_{AO}(l)^2}{2AG} dl, \quad (2.59)$$

and $U'_{ts} = c_{ts}U'_s$. The total strain energy is $U' = U'_a + U'_b + U'_s + U'_{ts}$.

We use Castigliano's second theorem and the roller boundary condition to write the rotational deflection at the point B as

$$\delta\psi'_B = \frac{\partial U'}{\partial M'_{R,B}} = \frac{\partial U'_b}{\partial M'_{R,B}} = \frac{2L^2 P' \cos \psi + 2LM'_{R,B}}{EI_A} = 0. \quad (2.60)$$

From Eq. 2.60, we find

$$M'_{R,B} = -\frac{3LP' \cos \psi}{2}, \quad (2.61)$$

and by using Eq. 2.61 in Eq. 2.44, we find

$$M'_{R,A} = -\frac{LP' \cos \psi}{2}. \quad (2.62)$$

We write the sum of moments at the cut point, for example, in the OB section, as $M'_{OB}(l) + M'_{R,B} + P'(2L - l) \cos \psi - R'_{B,x}(2L - l) \sin \psi = 0$, and by using Eqs. 2.42, 2.56, and 2.61, we obtain $R'_{A,x} = R'_{B,x} = 0$.

Since the vertical load P' is applied at the point B , as shown in Fig. 2.4, the effective stiffness in the y direction, $k_{\text{eff},y}$, can be defined as P'/δ'_y , where δ'_y is the total deflection in the y direction, which is the sum of axial (δ'_a), bending (δ'_b), direct shear (δ'_s), and transverse shear deflections (δ'_{ts}). To find each deflection, we use Castigliano's second theorem for each strain energy, leading to

$$\delta'_a = \frac{\partial U'_a}{\partial P'} = \frac{2LP \sin^2 \psi}{AE}, \quad (2.63)$$

$$\delta'_b = \frac{\partial U'_b}{\partial P'} = \frac{7L^3 P' \cos^2 \psi}{6EI_A}, \quad (2.64)$$

$$\delta'_s = \frac{\partial U'_s}{\partial P'} = \frac{2LP' \cos^2 \psi}{AG}, \quad (2.65)$$

and $\delta'_{ts} = c_{ts}\delta'_s$, which simplify to

$$\delta'_a = \left(\frac{2P' \sin^2 \psi}{ED} \right) \left(\frac{L}{w} \right), \quad (2.66)$$

$$\delta'_b = \left(\frac{14P' \cos^2 \psi}{ED} \right) \left(\frac{L}{w} \right)^3, \quad (2.67)$$

$$\delta'_s = \left(\frac{4P'(1+\nu) \cos^2 \psi}{ED} \right) \left(\frac{L}{w} \right), \quad (2.68)$$

and $\delta'_{ts} = c_{ts}\delta'_s$. As we consider a long, thin beam model ($w/L \ll 1$), we have

$$\frac{\delta'_s}{\delta'_b} = \left(\frac{2(1+\nu)}{7} \right) \left(\frac{w}{L} \right)^2 \rightarrow 0, \quad (2.69)$$

and $\delta'_{ts}/\delta'_b \rightarrow 0$, resulting in our dropping both direct and transverse shear deflections in the total deflection. The ratio between axial and bending deflections thus reads

$$\frac{\delta'_a}{\delta'_b} = \left(\frac{1}{7} \right) \left(\frac{w}{L} \right)^2 (\tan^2 \psi), \quad (2.70)$$

which depends the choice of ψ besides w/L , especially when ψ is large, leading us to retain both axial and bending deflections in the total deflection, such that

$$\delta'_y = \delta'_a + \delta'_b. \quad (2.71)$$

Using Eqs. 2.66, 2.67, and 2.71, the effective stiffness in the y direction can then be written as

$$k_{\text{eff},y} = \frac{P'}{\delta'_y} = \frac{EDw^3}{2Lw^2 \sin^2 \psi + 14L^3 \cos^2 \psi}. \quad (2.72)$$

As such, the shear stiffness k_s of the diamond shaped spring can be written as

$$k_s = 2(k_{\text{eff},y}) = \frac{EDw^3}{Lw^2 \sin^2 \psi + 7L^3 \cos^2 \psi}. \quad (2.73)$$

With Eq. 2.41, we obtain the ratio between k_n and k_s as

$$\frac{k_n}{k_s} = \frac{w^2 \sin^2 \psi + 7L^2 \cos^2 \psi}{w^2 \cos^2 \psi + L^2 \sin^2 \psi}. \quad (2.74)$$

2.3.3 Effective Shear Stiffness as a Function of the Normal Strain

To estimate the dependence of the effective shear stiffness k_s on the normal (horizontal) strain $\Delta x/L$, we consider the scenario shown in Fig. 2.5. By assuming horizontal deformation only cause the angle ψ to change from its initial angle ψ_0 , *e.g.*, as a mechanism, we can write

$$\Delta x = 2L(\cos \psi_0 - \cos \psi), \quad (2.75)$$

which leads to

$$\psi(\Delta x) = \arccos\left(\cos \psi_0 - \frac{\Delta x}{2L}\right). \quad (2.76)$$

By substituting Eq. 2.76 into Eq. 2.73, we have a relation between the effective shear stiffness and the normal strain in the simplified form of

$$k_s(\Delta x) = \frac{EDw^3}{Lw^2 \sin^2 \psi_0 + 7L^3 \cos^2 \psi_0 - \frac{\Delta x^2}{4L^2}}. \quad (2.77)$$

Both Eqs. 2.74 and 2.77 are evaluated and represented in Fig. 2.8 for $E = 2.1$ GPa, $D = 6.35$ mm, $w = 0.45$ mm, $L = 16$ mm and $\psi_0 = 50^\circ$.

2.3.4 Finite Element Method (FEM) Verification of Effective Stiffness

We used FEM simulations via COMSOL Multiphysics to validate the derived Eqs. 2.41, 2.73, 2.74, and their dependence on Δx . A two-dimensional (2D) plane stress model of a linearly elastic material was developed with a geometry and mesh shown in Fig. 2.6. The model was meshed based on a COMSOL physics-controlled mesh type with “extremely fine” mesh size. The material was specified to have a Young’s modulus of 2.1 GPa and a Poisson’s ratio of 0.35, similar to polycarbonate. A fixed boundary condition was specified at the left end. A varied boundary was prescribed on the right depending on the scenario. All remaining boundaries were left free. Geometric nonlinearity was included in all simulations.

To verify the normal stiffness, a prescribed displacement of 1 mm was applied in the x direction, with no motion allowed at the boundary in the y direction.

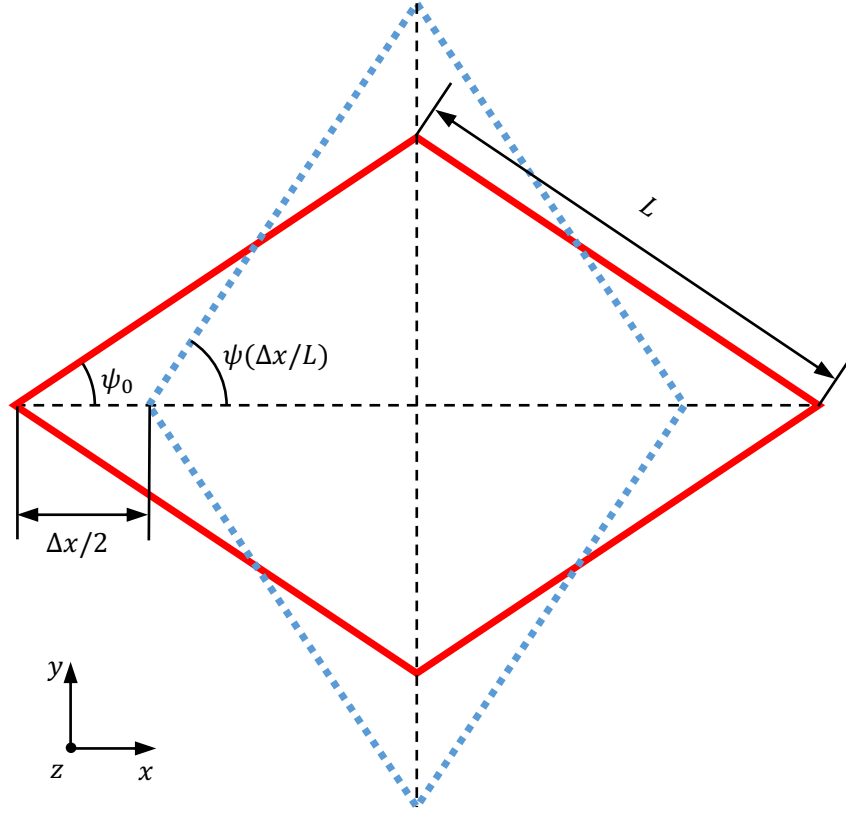


Figure 2.5: Model for deriving effective shear stiffness k_s as a function of axial strain $\Delta x/L$, where ψ_0 and ψ describe the geometry of the undeformed and longitudinally deformed structure, respectively, with the assumption of the structure deforming longitudinally as a mechanism.

The normal stiffness was calculated by using the reaction force in the x direction on the right boundary (extracted via line integration) and dividing by the applied displacement, which was found to be 515.81 N/m (505.27 N/m from beam theory).

To verify the shear stiffness without longitudinal compression (*i.e.*, $\Delta x = 0$), a prescribed displacement of 1mm was applied in the y direction, with no motion allowed at the boundary in the x direction. The shear stiffness was calculated by using the reaction force in the y direction on the right boundary (extracted via line integration) and again dividing by the applied displacement, which was found to be 121.97 N/m (102.56 N/m from beam theory).

To verify the dependence of shear stiffness on the longitudinal strain (as per Fig. 2.8(b)), prescribed displacements of $\Delta x = 0, 2, 4, 6, 8$ mm in the x direction were applied on the right boundary with 1 mm in the y direction, where the geometric nonlinearity was included in the Study Setting of COMSOL. Similarly, for each Δx case, the shear stiffness was calculated by using the reaction force in the y direction on the right boundary (extracted via line integration) and dividing by the applied y displacement.

To obtain the bending stiffness, we used different boundary conditions. The right end was set to be a roller boundary condition in the y direction, with no motion allowed in the x direction. The left end was set to be a roller-pin boundary in the x direction, with no motion allowed in the y direction, and with a prescribed rotation angle around the center of the left end swept over a specific range, achieved by using COMSOL rigid connector. The resulting reaction moments in the z direction was found via line integration of the right end. Bending stiffness was calculated by dividing the reaction moments by the prescribed rotation. By further dividing by the square of moment arm r^2 , we have bending stiffness as 105.24 N/m. However, we choose a different value for our DEM simulation instead of this one, which is explained in later Sec. 2.5.4 Eigenfrequency Analysis.

2.3.5 Experimental Verification of Normal and Nonlinear Shear Stiffness

To experimentally characterize our spring's effective stiffness, we use a MTS Exceed mechanical load frame. We test the same spring geometry as used in the dynamic tests and described in before.

To verify the normal stiffness, we clamped both ends of the spring where the particles would be located, and applied a slow displacement-controlled pull up to approximately 1 mm in the normal direction. The effective normal stiffness was obtained as the slope from linear fitting the measured force versus displacement data.

To verify the dependence of shear stiffness on longitudinal strain, we designed and used the geometry shown in Fig. 2.7. To reduce boundary effects, an

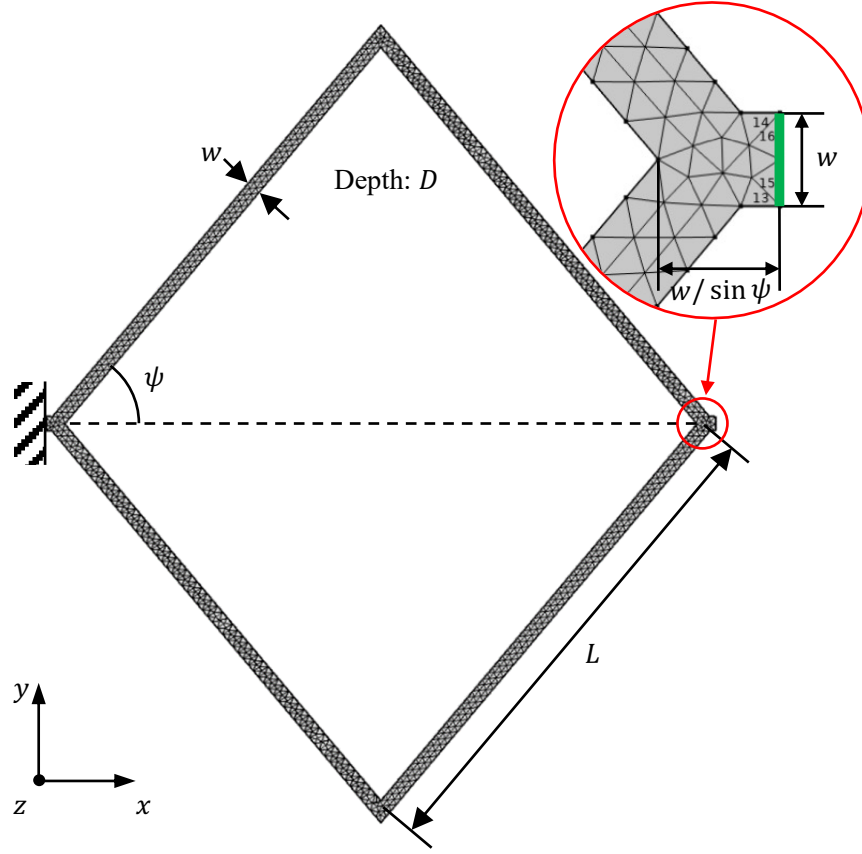


Figure 2.6: Beam-based diamond-shaped structure model (with meshed structures) used in COMSOL simulations for verifying effective normal and shear stiffness, where $L = 16$ mm, $w = 0.45$ mm, $D = 6.35$ mm, and $\psi = 50^\circ$. Fixed boundary condition is illustrated on the left side. Green line in the zoom-in area indicates the right boundary where prescribed displacements are applied. The rest of boundaries are free.

additional frame was added around the spring. The long shadowed beams were added to help accurately set longitudinal compression Δx (indicated by the red hollowed arrows). The desired longitudinal compression was achieved when the shadowed beam region came in contact with the opposite white region. After application of the longitudinal compression, both sides of the system were clamped into the load frame and inspected to ensure that the shadowed beam and white region were still in contact. Whilst keeping the system clamped on both sides, the

shadowed beams were removed to avoid any friction during the shear test. The top of the spring was then vertically displaced at a slow and constant speed (indicated by the blue arrow) up to about 3 mm and the resulting force measured. The effective shear stiffness was obtained as the slope from linear fitting the measured force versus displacement data.

Quantitatively, we experimentally found effective normal and shear stiffness for our spring without longitudinal compression to be 524.23 N/m and 101.7 N/m, respectively.

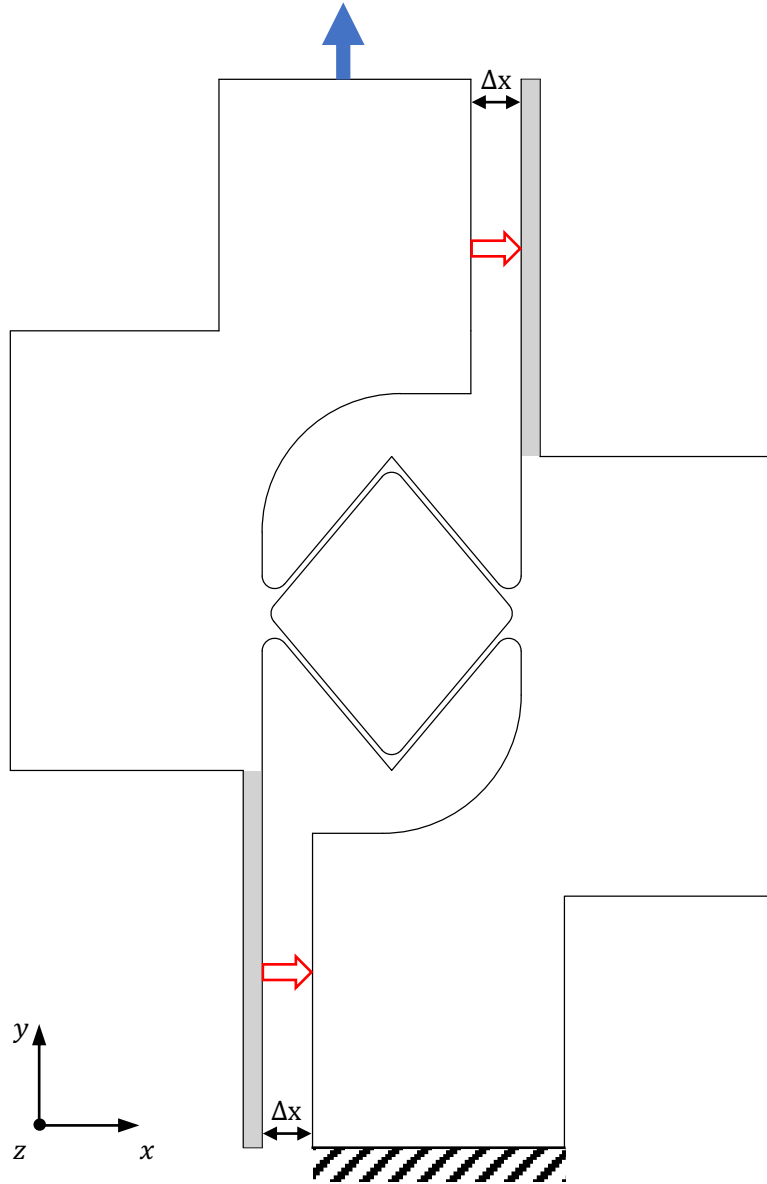


Figure 2.7: Experimental model verifying of effective shear stiffness as a function of the normal strain, where $L = 16$ mm, $w = 0.45$ mm, $D = 6.3$ mm, and $\psi = 50^\circ$. A fixed boundary condition (bottom) and a load (top) are illustrated. Shaded regions are designed for precompression alignment and removed in the tension test.

2.3.6 Summary and Representation of Normal and Shear Stiffness

Recalling the effective normal and shear stiffness derived analytically from beam theory, where the shear stiffness estimate of Eq. 2.79 assumes that under longitudinal deformation of the diamond, the beams forming the diamond retain their length and shape, but the aspect ratio of the diamond changes similar to a hinged mechanism. While the normal stiffness estimate of Eq. 2.78 neglects nonlinear affects that may be incurred during longitudinal deformation.

$$k_n = \frac{EDw^3}{Lw^2 \cos^2 \psi_0 + L^3 \sin^2 \psi_0} \quad (2.78)$$

and

$$k_s = \frac{EDw^3}{Lw^2 \sin^2 \psi + 7L^3 \cos^2 \psi}, \quad (2.79)$$

where ψ_0 is the undeformed spring half angle, ψ is the half angle after longitudinal deformation, E is the elastic modulus of the material, w the beam width, L the diamond's edge length, and D its depth.

Later, to verify the stiffness estimates Eqs. 2.78 and 2.79, we conducted mechanical testing and finite element method (FEM) simulations. The physical springs were machined from polycarbonate. Additional details of the simulation setup and experimental test procedures are described in Sec. 2.3.4 and 2.3.5, respectively. Figure 2.8(a) shows the compared small-deformation stiffness ratio k_n/k_s of the diamond spring as a function of ψ . Figure 2.8(b) shows the variation of shear stiffness k_s for a diamond of $\psi_0 = 50^\circ$ as a function of a large, longitudinal compressive strain $\epsilon_x = \Delta x/S_0$, where the analytical curves use $\psi(\Delta x) = \arccos(\cos \psi_0 - \Delta x/S_0)$ with Eqs. 2.78 and 2.79 as described in Sec. 2.3.3. We also evaluate the slope of a linear line relating $k_s/k_s(\psi = 50^\circ) = B_0 (\Delta x / (2L \cos \psi_0)) + 1$ as shown in Fig. 2.8(b) which was found to be $B_0 = 2.5$. However, the chosen values for our DEM simulation are selected based on our Eigenfrequency analysis, which is explained in later Sec. 2.5.4.

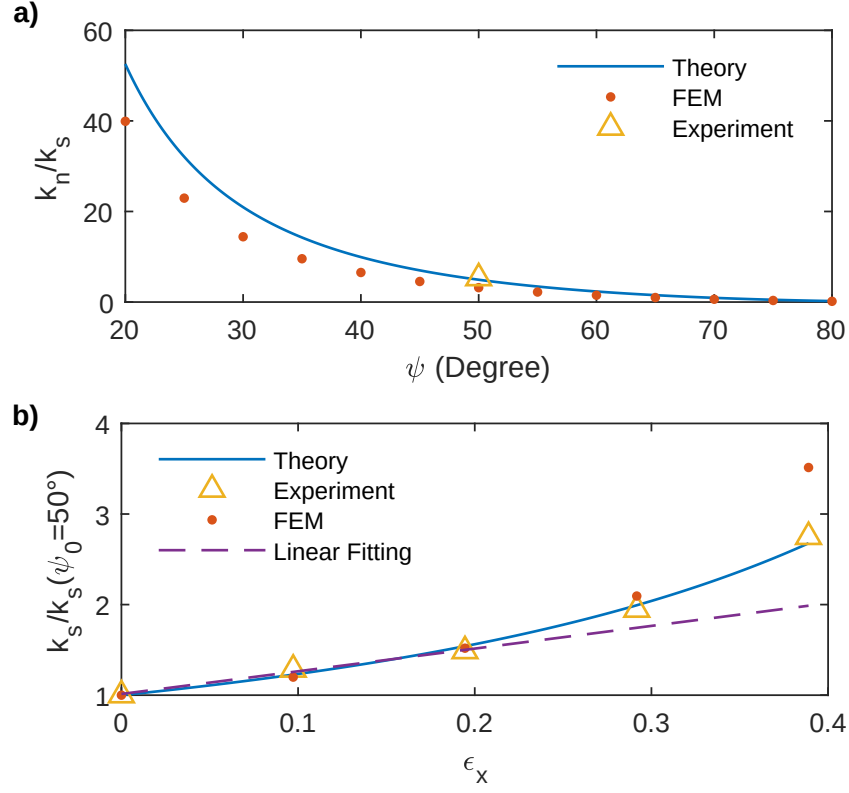


Figure 2.8: Nonlinear spring stiffnesses evaluated via analytics, FEM simulation, and experiment. (a) Small deformation stiffness ratio k_n/k_s for different springs with varied half angle ψ , where $L = 16$ mm, $w = 0.45$ mm, and $D = 6.35$ mm. (b) Shear stiffness (normalized by shear stiffness of the undeformed structure at $\psi_0 = 50^\circ$) versus longitudinal strain $\epsilon_x = \Delta x / (2L \cos \psi_0)$. The dashed purple line is fitted to the first three experimental points and has slope $B_0 = 2.5$.

2.4 Theoretical Dynamical Response

Returning now to the dynamic response of the system of Fig. 2.1, we consider first the u degree of freedom and apply the traveling wave solution $u_j = Ue^{i(\alpha jd - \omega t)}$ in Eq. 2.10, neglecting the coupling terms. The dispersion relation for longitudinal waves is found to be

$$\omega_m(\alpha) = 2 \sqrt{\frac{k_n}{m}} \left| \sin \frac{\alpha d}{2} \right|, \quad (2.80)$$

where $\alpha = 2\pi/\lambda$ is the wavenumber for the u motion, with λ the wavelength. We now define a new “unit cell” for transverse-rotational motion based on the modulation wavelength formed by the longitudinal wave, where here, for simplicity, we assume λ to be an integer multiple of d . The number of masses in the unit cell is $R = \lambda/d$, wherein $r = 1, 2, \dots, R$ is the mass index within the unit cell. The equations of motion of Eqs. 2.11 and 2.12 can then be reformulated to equations of motion for a unit cell, and written as

$$\begin{aligned} m\ddot{v}_r = & - \left(k_s^{(r-1)} + \gamma_1[r] \right) \left((v_r - v_{r-1} + \gamma_2[r, v]) - \frac{d}{2} (\theta_r + \theta_{r-1} - \gamma_2[r, \theta]) \right) + \\ & k_s^{(r)} \left((v_{r+1} - v_r - \gamma_3[r, v]) - \frac{d}{2} (\theta_r + \theta_{r+1} - \gamma_3[r, \theta]) \right) \end{aligned} \quad (2.81)$$

and

$$\begin{aligned} J\ddot{\theta}_r = & \frac{d}{2} \left(k_s^{(r-1)} + \gamma_1[r] \right) \left((v_r - v_{r-1} + \gamma_2[r, v]) - \frac{d}{2} (\theta_r + \theta_{r-1} - \gamma_2[r, \theta]) \right) + \\ & \frac{d}{2} k_s^{(r)} \left((v_{r+1} - v_r - \gamma_3[r, v]) - \frac{d}{2} (\theta_{r+1} + \theta_r - \gamma_3[r, \theta]) \right) + \\ & \left(\frac{d}{2} \right)^2 (-k_b(\theta_r - \theta_{r-1} + \gamma_2[r, \theta]) + k_b(\theta_{r+1} - \theta_r - \gamma_3[r, \theta])), \end{aligned} \quad (2.82)$$

where k_b is constant, $k_s^{(r)}$ (denoted by the (r) superscript) is a function of time, and functions γ_1, γ_2 , and γ_3 are defined as

$$\begin{aligned} \gamma_1[r] &= \delta_{r,1} \left(k_s^{(R)} - k_s^{(r-1)} \right), \\ \gamma_2[r, Q] &= \delta_{r,1} \left(Q_{r-1} - Q_R e^{-i\mu} \right), \\ \gamma_3[r, Q] &= \delta_{r,R} \left(Q_{r+1} - Q_1 e^{i\mu} \right), \end{aligned}$$

and used to enforce periodicity, with Q the degree of freedom (either v or θ), $\delta_{r,n}$ the Kronecker delta (where n is either 1 or R), and $\mu = \kappa\lambda$ the non-dimensional wavenumber (and κ the wavenumber) for v and θ motions.

2.4.1 Modulated Dispersion Relation for Transverse and Rotational Motion

To obtain the dispersion relations of transverse-rotational waves given the longitudinal wave modulation of the shear stiffnesses, we follow Ref. [94]. We first assume our shear stiffness takes the form of a rightward traveling wave such that

$$k_s^{(r)}(t) = k_0 (1 + \beta \cos(\omega_m t - \phi_r)), \quad (2.83)$$

with all variables as defined in previous sections and phase shift of $\phi_r = 2\pi r/R$. We rewrite Eq. 2.83 as a Fourier series in terms of ω_m , such that

$$k_s^{(r)}(t) = \sum_{q=-\infty}^{\infty} K_{q,r} e^{iq\omega_m t}, \quad (2.84)$$

with Fourier coefficients $K_{q,r}$ defined as

$$K_{q,r} = \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} (k_s^{(r)}(t) e^{-iq\omega_m t}) dt. \quad (2.85)$$

We truncate the Eq. 2.84 to

$$k_s^{(r)}(t) = \sum_{q=-P}^P K_{q,r} e^{iq\omega_m t}, \quad (2.86)$$

with $P = 1$, as three coefficients completely describe Eq. 2.83, which give Fourier coefficients $K_{0,r} = k_0$, $K_{-1,r} = k_0\beta e^{i\phi_r}/2$, and $K_{1,r} = k_0\beta e^{-i\phi_r}/2$.

To obtain the dispersion relation for shear waves given the longitudinal wave modulation of the shear stiffness, we assume ansatzes

$$v_r(t) = V_r(t) e^{i(r\mu - \omega t)} \quad (2.87)$$

and

$$\theta_r(t) = \Theta_r(t) e^{i(r\mu - \omega t)}, \quad (2.88)$$

where $V_r(t)$ and $\Theta_r(t)$ are temporally modulated amplitudes and μ defined as non-dimensional wave number. We then expand $V_r(t)$ and $\Theta_r(t)$ as a Fourier series, assuming modulation frequency ω_m corresponding to the chosen λ , defined (as per Eq. 2.7), such that

$$V_r(t) = \sum_{p=-\infty}^{\infty} V_{p,r} e^{ip\omega_m t}, \quad (2.89)$$

$$\Theta_r(t) = \sum_{p=-\infty}^{\infty} \Theta_{p,r} e^{ip\omega_m t}. \quad (2.90)$$

Substituting Eqs. 2.86, 2.87, and 2.88 into Eqs. 2.81 and 2.82, then rewriting the result in matrix form, we obtain

$$\begin{aligned} & m\mathbf{I} \sum_{p=-\infty}^{\infty} (\omega - p\omega_m)^2 (\vec{V}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} \\ & - \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(1)} \sum_{p=-\infty}^{\infty} (\vec{V}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} \\ & - \left(\frac{d}{2}\right) \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(2)} \sum_{p=-\infty}^{\infty} (\vec{\Theta}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} = 0, \\ & J\mathbf{I} \sum_{p=-\infty}^{\infty} (\omega - p\omega_m)^2 (\vec{\Theta}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} \\ & - \left(\frac{d}{2}\right) \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(3)} \sum_{p=-\infty}^{\infty} (\vec{V}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} \\ & - \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(4)} \sum_{p=-\infty}^{\infty} (\vec{\Theta}_p e^{ip\omega_m t}) e^{i(r\mu - \omega t)} = 0, \end{aligned} \quad (2.91)$$

where $\vec{V}_p$ and $\vec{\Theta}_p$ are $V_{p,r}$ and $\Theta_{p,r}$ arranged into column vectors (respectively),

and matrix $\mathbf{K}$ is defined for the case of $R = 3$ as

$$\mathbf{K} = \begin{bmatrix} -(k_s^{(1)} + k_s^{(3)}) & k_s^{(1)} & k_s^{(3)} e^{-i\mu} & -d(k_s^{(1)} - k_s^{(3)})/2 & & \\ k_s^{(1)} & -(k_s^{(1)} + k_s^{(2)}) & k_s^{(2)} & dk_s^{(1)}/2 & & \\ k_s^{(3)} e^{i\mu} & k_s^{(2)} & -(k_s^{(2)} + k_s^{(3)}) & -dk_s^{(3)} e^{i\mu}/2 & & \\ -d(k_s^{(1)} - k_s^{(3)})/2 & dk_s^{(1)}/2 & -dk_s^{(3)} e^{-i\mu}/2 & -d^2(k_s^{(1)} + k_s^{(3)} + 2k_b)/4 & \dots & \\ -dk_s^{(1)}/2 & d(k_s^{(1)} - k_s^{(2)})/2 & dk_s^{(2)}/2 & -d^2(k_s^{(1)} - k_b)/4 & & \\ dk_s^{(3)} e^{i\mu}/2 & -dk_s^{(2)}/2 & d(k_s^{(2)} - k_s^{(3)})/2 & -d^2(k_s^{(3)} - k_b) e^{i\mu}/4 & & \\ & -dk_s^{(1)}/2 & dk_s^{(3)} e^{-i\mu}/2 & & & \\ & d(k_s^{(1)} - k_s^{(2)})/2 & -dk_s^{(2)}/2 & & & \\ & dk_s^{(2)}/2 & d(k_s^{(2)} - k_s^{(3)})/2 & & & \\ \dots & -d^2(k_s^{(1)} - k_b)/4 & -d^2(k_s^{(3)} - k_b) e^{-i\mu}/4 & & & \\ & -d^2(k_s^{(1)} + k_s^{(2)} + 2k_b)/4 & -d^2(k_s^{(2)} - k_b)/4 & & & \\ & -d^2(k_s^{(2)} - k_b)/4 & -d^2(k_s^{(2)} + k_s^{(3)} + 2k_b)/4 & & & \end{bmatrix} \quad (2.92)$$

and $k_s^{(1)}$, $k_s^{(2)}$, and $k_s^{(3)}$ are defined in Eq. 2.86. For simplicity, we can decompose $\mathbf{K}$ into four quadrants, where, by substituting Eq. 2.86, it can be rewritten as $\mathbf{K}_q^{(\eta)}$ (with $\eta = 1, 2, 3, 4$), where each matrix is of size $R \times R$ defined as

$$\mathbf{K}_q^{(\eta)} = k_{s,0} \left(\delta_{q,0} \hat{\mathbf{K}}_0^{(\eta)}(\mu) + \delta_{q,\pm p} \frac{\beta}{2} \hat{\mathbf{K}}_q^{(\eta)}(\mu) \right) \forall p \neq 0, \quad (2.93)$$

where $\delta_{q,p}$ is Kronecker Delta, and for the $R = 3$ unit cell, $\hat{\mathbf{K}}_q^{(\eta)}$ is

$$\hat{\mathbf{K}}_q^{(1)}(\mu) = \begin{bmatrix} e^{iq\phi_3} + e^{iq\phi_1} & -e^{iq\phi_1} & -e^{i(q\phi_3 - \mu)} \\ -e^{iq\phi_1} & e^{iq\phi_1} + e^{iq\phi_2} & -e^{iq\phi_2} \\ -e^{i(q\phi_3 + \mu)} & -e^{iq\phi_2} & e^{iq\phi_2} + e^{iq\phi_3} \end{bmatrix}, \quad (2.94)$$

$$\hat{\mathbf{K}}_q^{(2)}(\mu) = \begin{bmatrix} e^{iq\phi_1} - e^{iq\phi_3} & e^{iq\phi_1} & -e^{i(q\phi_3 - \mu)} \\ -e^{iq\phi_1} & e^{iq\phi_2} - e^{iq\phi_1} & e^{iq\phi_2} \\ e^{i(q\phi_3 + \mu)} & -e^{iq\phi_2} & e^{iq\phi_3} - e^{iq\phi_2} \end{bmatrix}, \quad (2.95)$$

$$\hat{\mathbf{K}}_q^{(3)}(\mu) = \begin{bmatrix} e^{iq\phi_1} - e^{iq\phi_3} & -e^{iq\phi_1} & e^{i(q\phi_3 - \mu)} \\ e^{iq\phi_1} & e^{iq\phi_2} - e^{iq\phi_1} & -e^{iq\phi_2} \\ -e^{i(q\phi_3 + \mu)} & e^{iq\phi_2} & e^{iq\phi_3} - e^{iq\phi_2} \end{bmatrix}, \quad (2.96)$$

and

$$\begin{aligned} \hat{\mathbf{K}}_q^{(4)}(\mu) = & \begin{bmatrix} e^{iq\phi_3} + e^{iq\phi_1} & e^{iq\phi_1} & e^{i(q\phi_3-\mu)} \\ e^{iq\phi_1} & e^{iq\phi_1} + e^{iq\phi_2} & e^{iq\phi_2} \\ e^{i(q\phi_3+\mu)} & e^{iq\phi_2} & e^{iq\phi_2} + e^{iq\phi_3} \end{bmatrix} + \\ & \delta_{q,0} \frac{k_b}{k_{s,0}} \begin{bmatrix} 2 & -1 & -e^{-i\mu} \\ -1 & 2 & -1 \\ -e^{i\mu} & -1 & 2 \end{bmatrix}. \end{aligned} \quad (2.97)$$

To simplify Eqs. 2.91 further, we cancel out $e^{-i\omega t}$ to get

$$\begin{aligned} m\mathbf{I} \sum_{p=-\infty}^{\infty} (\omega - p\omega_m)^2 (\vec{V}_p e^{ip\omega_m t}) e^{ir\mu} - \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(1)} \sum_{p=-\infty}^{\infty} (\vec{V}_p e^{ip\omega_m t}) e^{ir\mu} \\ - \left(\frac{d}{2}\right) \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(2)} \sum_{p=-\infty}^{\infty} (\vec{\Theta}_p e^{ip\omega_m t}) e^{ir\mu} = 0, \\ J\mathbf{I} \sum_{p=-\infty}^{\infty} (\omega - p\omega_m)^2 (\vec{\Theta}_p e^{ip\omega_m t}) e^{ir\mu} - \left(\frac{d}{2}\right) \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(3)} \sum_{p=-\infty}^{\infty} (\vec{V}_p e^{ip\omega_m t}) e^{ir\mu} \\ - \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P e^{iq\omega_m t} \mathbf{K}_q^{(4)} \sum_{p=-\infty}^{\infty} (\vec{\Theta}_p e^{ip\omega_m t}) e^{ir\mu} = 0. \end{aligned} \quad (2.98)$$

We then pre-multiply Eqs. 2.98 by $\omega_m e^{ih\omega_m t}/(2\pi)$, where h is an arbitrary integer, and integrate the result over the modulation period, from $-\pi/\omega_m$ to π/ω_m , to get

$$\begin{aligned}
& m\mathbf{I} \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p))} (e^{i2\pi(h+p)} - 1) (\omega - p\omega_m)^2 \vec{V}_p}{2\pi(h+p)} \\
& - \sum_{q=-P}^P \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p+q))} (e^{i2\pi(h+p+q)} - 1) \mathbf{K}_q^{(1)} \vec{V}_p}{2\pi(h+p+q)} - \\
& \left(\frac{d}{2}\right) \sum_{q=-P}^P \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p+q))} (e^{i2\pi(h+p+q)} - 1) \mathbf{K}_q^{(2)} \vec{\Theta}_p}{2\pi(h+p+q)} = 0, \\
& J\mathbf{I} \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p))} (e^{i2\pi(h+p)} - 1) (\omega - p\omega_m)^2 \vec{\Theta}_p}{2\pi(h+p)} \\
& - \left(\frac{d}{2}\right) \sum_{q=-P}^P \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p+q))} (e^{i2\pi(h+p+q)} - 1) \mathbf{K}_q^{(3)} \vec{V}_p}{2\pi(h+p+q)} - \\
& \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P \sum_{p=-\infty}^{\infty} \frac{ie^{i(r\mu-\pi(h+p+q))} (e^{i2\pi(h+p+q)} - 1) \mathbf{K}_q^{(4)} \vec{\Theta}_p}{2\pi(h+p+q)} = 0.
\end{aligned} \tag{2.99}$$

Replacing the summation variable in Eqs. 2.99 for the first term by $(p \rightarrow h)$, second and third terms by $(p \rightarrow h - q)$ leads to

$$\begin{aligned}
& m\mathbf{I} \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) (\omega - h\omega_m)^2 \vec{V}_h}{4\pi h} - \\
& \sum_{q=-P}^P \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) \mathbf{K}_q^{(1)} \vec{V}_{h-q}}{4\pi h} - \\
& \left(\frac{d}{2}\right) \sum_{q=-P}^P \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) \mathbf{K}_q^{(2)} \vec{\Theta}_{h-q}}{4\pi h} = 0, \\
& J\mathbf{I} \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) (\omega - h\omega_m)^2 \vec{\Theta}_h}{4\pi h} - \\
& \left(\frac{d}{2}\right) \sum_{q=-P}^P \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) \mathbf{K}_q^{(3)} \vec{V}_{h-q}}{4\pi h} - \\
& \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P \sum_{h=-\infty}^{\infty} \frac{ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) \mathbf{K}_q^{(4)} \vec{\Theta}_{h-q}}{4\pi h} = 0.
\end{aligned} \tag{2.100}$$

Canceling the exponential $ie^{i(r\mu-2\pi h)} (e^{i4\pi h} - 1) / (4\pi h)$ in Eqs. 2.100 from all

terms results in

$$\begin{aligned}
& m\mathbf{I} \sum_{h=-\infty}^{\infty} (\omega - h\omega_m)^2 \vec{V}_h - \sum_{h=-\infty}^{\infty} \sum_{q=-P}^P \mathbf{K}_q^{(1)} \vec{V}_{h-q} - \\
& \left(\frac{d}{2}\right) \sum_{h=-\infty}^{\infty} \sum_{q=-P}^P \mathbf{K}_q^{(2)} \vec{\Theta}_{h-q} = 0, \\
& J\mathbf{I} \sum_{h=-\infty}^{\infty} (\omega - h\omega_m)^2 \vec{\Theta}_h - \left(\frac{d}{2}\right) \sum_{h=-\infty}^{\infty} \sum_{q=-P}^P \mathbf{K}_q^{(3)} \vec{V}_{h-q} - \\
& \left(\frac{d}{2}\right)^2 \sum_{h=-\infty}^{\infty} \sum_{q=-P}^P \mathbf{K}_q^{(4)} \vec{\Theta}_{h-q} = 0.
\end{aligned} \tag{2.101}$$

Leveraging the orthogonality of harmonic functions, we drop the summation in h in Eqs. 2.101 and obtain the following equation, valid for all values of h , then interchange the h to p we obtain approximate dispersion relations from Eqs. 2.81 and 2.82 for the case of a $R = 3$ mass unit cell (considered here onward)

$$\begin{aligned}
& m\mathbf{I}(\omega - p\omega_m)^2 \vec{V}_p - \sum_{q=-P}^P \mathbf{K}_q^{(1)} \vec{V}_{p-q} - \left(\frac{d}{2}\right) \sum_{q=-P}^P \mathbf{K}_q^{(2)} \vec{\Theta}_{p-q} = 0, \\
& J\mathbf{I}(\omega - p\omega_m)^2 \vec{\Theta}_p - \left(\frac{d}{2}\right) \sum_{q=-P}^P \mathbf{K}_q^{(3)} \vec{V}_{p-q} - \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P \mathbf{K}_q^{(4)} \vec{\Theta}_{p-q} = 0.
\end{aligned} \tag{2.102}$$

where ω is the frequency of the transverse-rotational modes, p denotes the order of the terms in the truncated Fourier series approximation, $\mathbf{I}$ is the identity matrix, $\vec{V}_p$ and $\vec{\Theta}_p$ are column vectors of length R of amplitudes v_r and θ_r (corresponding to Fourier order p), and $\mathbf{K}_q^{(\eta)}$ (with $\eta = 1, 2, 3, 4$) are $R \times R$ matrices defined as in Eq. 2.93 in terms of stiffness coefficients k_0 , nonlinear coefficient β and non-dimensional wavenumber μ . Under the assumption of the longitudinal wave is excited such that the shear stiffnesses take the form of a rightward traveling wave, as in Eq. 2.83. Organizing Eqs. 2.102 as a quadratic eigenvalue problem in the form

$$(N\omega^2 + Y\omega + H)X = 0, \tag{2.103}$$

we numerically obtain $4R(2P+1)$ eigenfrequencies for each μ , where half are mirrored to negative frequencies around $\omega = 0$. We note the range of solution produced will be for the reduced Brillouin zone (BZ) $\kappa d \in [-\pi/R, \pi/R]$.

2.4.2 Mapping Nonlinear Shear Stiffness Coefficient from Experiment to Theory

To obtain the theoretical nonlinear stiffness coefficient β in Eq. 2.83 we assume the relationship between the normalized shear stiffness and normal strain is

$$\frac{k_s}{k_0} = 1 + B_0 \epsilon_x, \quad (2.104)$$

where k_0 is the shear stiffness when there is no longitudinal strain at $\psi_0 = 50^\circ$, ϵ_x is the longitudinal strain, and B_0 is obtained from experiment. To determine the value of B_0 , we compare the linear relationship of Eq. 2.104 with a fitted line (denoted by the purple dashed line) in the linear region, up to a strain of 0.2 (the first three experimental data points) in Fig. 2.8(b), resulting in $B_0 = 2.5$. As such, in DEM simulations we use a shear stiffness in the form of

$$k_s^{(j)} = k_0 (1 + B_0 (u_{j+1} - u_j) / S_0). \quad (2.105)$$

For the sinusoidal displacement applied to the boundary used in the DEM simulations (amplitude of the applied signal remaining constant for all cases), we obtained a maximum strain of $\epsilon_{x,max} = 0.1$. Therefore, working towards the form of the shear stiffness in our analytical model, we write

$$\frac{k_s}{k_0} = 1 + B_0 \epsilon_{x,max} \cos(\omega_m t). \quad (2.106)$$

Recalling the theoretical form of the shear stiffness given in previous sections,

$$k_s^{(r)}(t) = k_0 (1 + \beta \cos(\omega_m t - \phi_r)), \quad (2.107)$$

we compare Eq. 2.106 and Eq. 2.107, to conclude that

$$\beta = B_0 \epsilon_{x,max}. \quad (2.108)$$

This results in $\beta = 0.25$, which will be used later to generate the dispersion curves.

2.4.3 Modulated Dispersion Relation Examples

In Fig. 2.9, we show an example dispersion relation from Eqs. 2.102 in the limit of zero modulation amplitude ($\beta = 0$) but finite modulation wavelength (blue

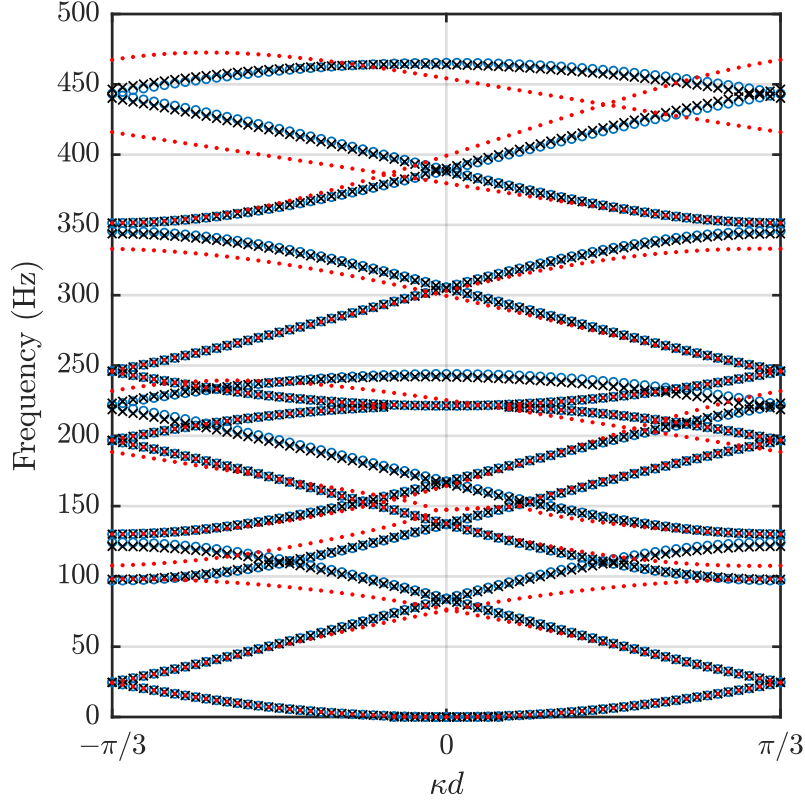


Figure 2.9: Dispersion relations of stiffness modulated chain from Eqs. 2.102 for $P = 1$, $R = 3$, $k_n = 538.3$ N/m, $k_s = 121.965$ N/m, and $k_b = 69.55$ N/m with zero ($\beta = 0$, blue circles), finite ($\beta = 0.25$, black crosses), and higher finite ($\beta = 0.75$, red dots) coupling strength between longitudinal strain and shear stiffness.

circles), from which we observe $2R$ branches (one for each degree of freedom v and θ) with integer shifts of ω_m between the sets. With the nonzero β , besides the shifts, we observe left-right asymmetry of dispersion relations around the $\kappa = 0$ axis and bandgap formation at crossing points between branches. The value of $\beta = 0.25$ is chosen based on the slope B_0 of the fitted, linearized response of the $\psi_0 = 50^\circ$ diamond spring in Fig. 2.8(b), and an assumed longitudinal wave amplitude $U = 1$ mm.

2.5 Dynamical Simulation

2.5.1 FEM Simulations Details

Next, we conduct FEM simulations via COMSOL Multiphysics that account for geometric nonlinearity using masses in the form of a rectangular prism of length and width of 10 mm and thickness of 6.35 mm. The dimensions of the diamond-shaped nonlinear spring are $L = 16$ mm, $w = 0.45$ mm, $D = 6.35$ mm, and $\psi_0 = 50^\circ$, which is the same as shown in Fig. 2.8. A 2D plane stress model of a linearly elastic material was used. The model is meshed based on a COMSOL physics-controlled mesh type with “finer” mesh size. The material was specified to have a Young’s modulus of 2.1 GPa and a Poisson’s ratio of 0.35, similar to polycarbonate. The model has 39 masses with a fixed boundary condition specified at the right end. At the left end, a sinusoidal prescribed displacement is applied in the longitudinal direction for the u motion (starting from $t = 0$ s) in the longitudinal (x) direction with amplitude 1 mm and frequency 226.125 Hz, with no restriction made on v motion. This prohibits rotation of the left boundary, but allows v motion. The longitudinal displacement amplitude is chosen to induce a maximum u strain ϵ_u of 0.1 upto the range of 0.2 to match $\beta = 0.25$ of Fig. 2.9. We then stimulate the 20th mass using a Gaussian-shaped force excitation in the transverse (y) direction, applied as a point load at the bottom of the middle mass centred at 5.37 ms for short chain, with standard deviation of 0.1 ms and amplitude 0.05 N. The applied load was specified as a COMSOL piece-wise function of time (t) with constant extrapolation and no smoothing. The load took the form of a monopolar pulse, which takes the form

$$F e^{-0.5((t-t_0)/\sigma)^2}, \quad (2.109)$$

where F is the excitation amplitude, σ is the standard deviation, and t_0 is the centered time. A time-dependent study was conducted with output times every 20 μ s with total time as 0.1 s. A default time-dependent solver was selected with the nonlinear method modified to “Automatic highly nonlinear (Newton) ” and the termination technique modified to “iterations or tolerance ”. The obtained

velocities were taken from the center point of each mass's lower edge for both the u and v degrees of freedom. Angular velocity, on the other hand, is computed based on the relative v motion of two points separated equally from the center on the bottom edge, such that

$$\dot{\theta}_i = \arctan\left(\frac{\dot{v}_{i,left} - \dot{v}_{i,right}}{r}\right) \quad (2.110)$$

where r is the distance between the points, and $\dot{v}_{i,left}$ and $\dot{v}_{i,right}$ are the v velocity for the left and right points.. Simulation results are compared with DEM simulation and discussed in later section.

2.5.2 DEM Simulations Details

Similarly, we numerically integrate Eqs. 2.10, 2.11 and 2.12 (which we refer to as our DEM simulations), using the same stiffnesses reported in Fig. 2.9. Boundary conditions were set to be fixed for all degrees of freedom for the right mass. For the left mass, the u motion was fully prescribed as a sinusoidal function with an amplitude of 1 mm and a frequency of 226.125 Hz, with no restrictions on the other degrees of freedom. The transverse excitation was applied to the 20th mass with the same forcing profile, amplitude, standard deviation, and center as described in the FEM simulation. In addition to the previous setup described, we note the following parameters. The output time step was defined as 5 ms in MATLAB. The built-in function *ode45* in MATLAB was used as the integrator with relative and absolute error tolerance both set to 10^{-10} . Energy conservation was confirmed, including when all three degrees of freedom were excited simultaneously, by checking that the total energy was constant to within a degree of certainty of $\pm 1.71 \times 10^{-12}$ J. For simplicity, we list all the basic parameters used in the DEM simulation: The mass of the particle was measured to be $m = 0.8$ grams, and the mass moment of inertia was calculated based on mass shape to be a rectangular prism with a length and width of 10 mm and a thickness of 6.35 mm, $J = 13.33 \text{ g}\cdot\text{mm}^2$. Given the spring beam length $L = 16$ mm and an initial angle $\psi_0 = 50^\circ$, the resultant unit cell spacing is $d = 30.6$ mm, and the unscratched spring length is $S_0 = 20.6$ mm. Finally, the linear normal and bending stiffness

were $k_n = 538.3$ N/m and $k_b = 69.55$ N/m, respectively. While, in the nonlinear shear stiffness in the form of Eq. 2.105, has $k_0 = 121.965$ N/m and $B_0 = 2.5$. DEM and FEM simulation are presented and discussed in the next section.

2.5.3 FEM and DEM Results and Dissection

In Fig. 2.10 and Fig. 2.11, we show the system's dynamic response for the described FEM and DEM simulations, respectively, in terms of particle velocity (translational for u and v , and angular for θ). In both, we plot the spatiotemporal response (top rows) and its two-dimensional (2D) Fourier transform (FT, bottom rows) for all degrees of freedom. All spatiotemporal data and Fourier spectra are normalized by their individual maxima. The color bars on panels (b,c) are truncated to improve visibility. The Fourier spectra are overlaid by the analytically predicted dispersion from Eqs. 2.102, wherein the Brillouin zone as shown in Fig. 2.9 is shifted by $\pm 2\pi/R$ to obtain the range $\kappa d \in [-\pi, \pi]$.

Both the FEM and DEM simulations yield similar overall results, with a few distinct characteristics. In all cases, we see good agreement between the analytically predicted band structure and the Fourier spectra. Regarding nonreciprocity, a slight asymmetry is evident in the Fourier spectra corresponding to the θ degree of freedom, as can be seen in panel (f), indicated by the black arrow of Fig. 2.10. Due to the relatively short chain, the transverse excitation occurs in the presence of longitudinal waves being reflected from the end. Despite this, the nonreciprocal response is retained. Additional simulated cases are shown in Appendix 2.7 for longer chain lengths and different shear excitation, including cases that show qualitatively similar, but stronger nonreciprocity. In panel (a), corresponding to the longitudinal motion, a lower frequency transient component can be seen propagating ahead of the main modulation. Concerning the differences between the FEM and DEM simulated results, considering the spectra of the v and θ motion in panels (e) and (f), respectively, we see that the top transverse-rotational branch, corresponding to rotation dominant motion (see next Sec. 2.5.4), is more strongly excited in the FEM simulations. We also see leakage from the longitudinal motion via the energy at the same frequency and wavenumber as the u motion, which only appears visible in the FEM simulations (Fig. 2.10). We hypothesize that the increased presence of energy leakage from the longitudinal motion may stem from the 1D nature of our system and the underlying modulation mechanism requiring large amplitude longitudinal compression. In the FEM model it is possible that

small levels of numerical noise are sufficient to break the axial symmetry of the chain and result in the translation of longitudinal motion into energetically favorable deformation modes such as rotation. This could be considered analogous to localized buckling of the chain. Such a mechanism is not accessible via the DEM simulations.

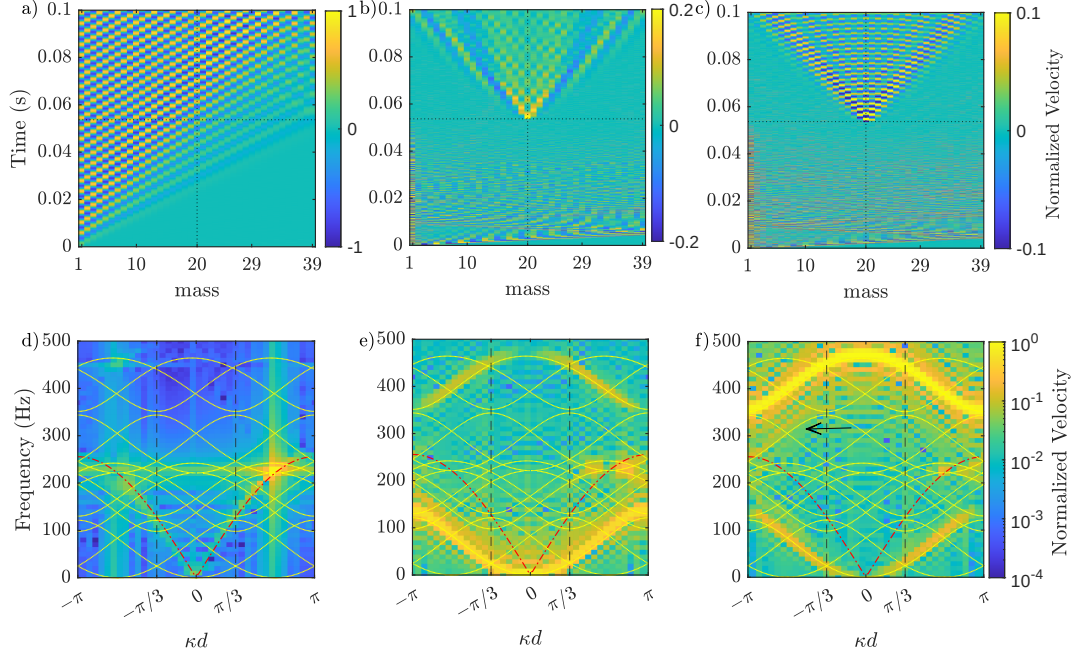


Figure 2.10: COMSOL simulation of nonreciprocal transverse-rotational wave propagation via longitudinal wave stiffness modulation, corresponding to the same parameters as Fig. 2.9. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. The crossed dotted black lines denote the transverse excitation time. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. The black arrow in (f) highlights an asymmetric mode.

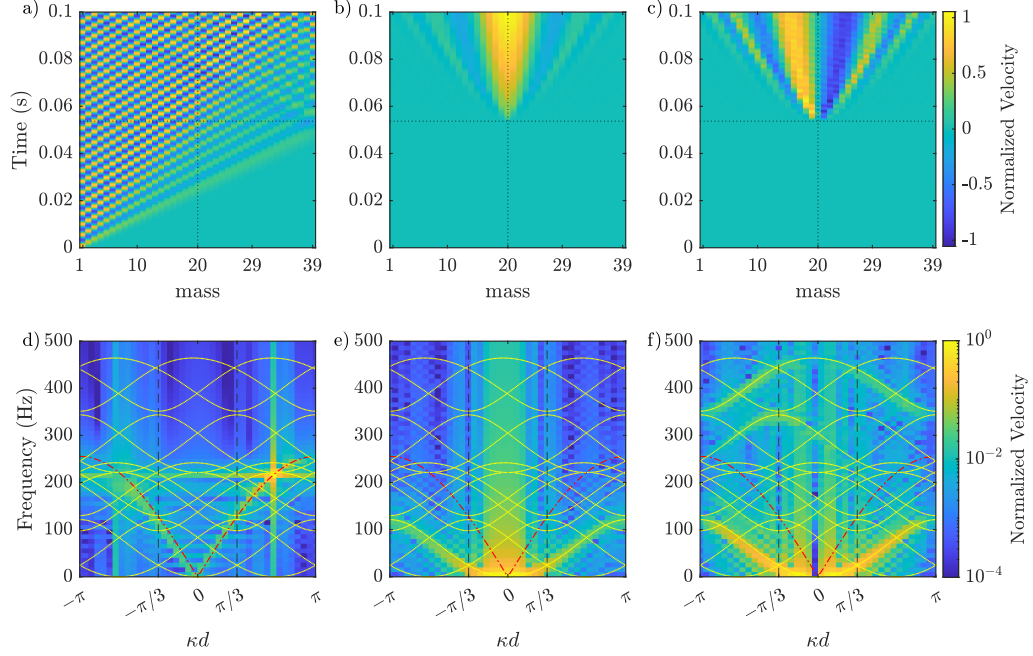


Figure 2.11: Reduced-order, DEM simulation of nonreciprocal transverse-rotational wave propagation via longitudinal wave stiffness modulation, corresponding to the same parameters as Fig. 2.9. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. The crossed dotted black lines denote the transverse excitation time. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell.

2.5.4 Eigenfrequency Analysis

Eigenfrequency analysis was performed on a single unit cell in COMSOL Multiphysics. A 2D plane stress and linearly elastic material was used. The model was meshed based on a COMSOL physics-controlled mesh type with “extremely fine” mesh size. The material was specified to have a Young’s modulus of 2.1 GPa and a Poisson’s ratio of 0.35, similar to polycarbonate. Periodic boundary conditions were applied on the left and right edges with “Floquet periodicity” selected. The x -component of the k-vector for Floquet periodicity was κx with a y -component of 0. A parametric sweep varying κx from 0 to π/a was performed to create a dispersion relation. A default “ARPACK” eigenfrequency solver with geometric nonlinearity was used with a search method for three eigenfrequencies around 1 Hz. The three eigenmodes found are illustrated in Fig. 2.12. The lower mode predominantly exhibits a transverse motion with a little bit of rotation. The second mode corresponds to the longitudinal mode. The third mode shows predominantly rotational motion with a little bit of shear translation. Because of discrepancy between the modes predicted using beam theory analysis, and what was observed in the FEM eigenfrequency analysis, we adjusted all the stiffness values used in the DEM and analytics to align with the eigenfrequency analysis. This adjustment yielded effective stiffness values of $k_n = 538.3$ N/m, $k_b = 69.55$ N/m, and $k_0 = 121.965$ N/m from Eq. 2.105.

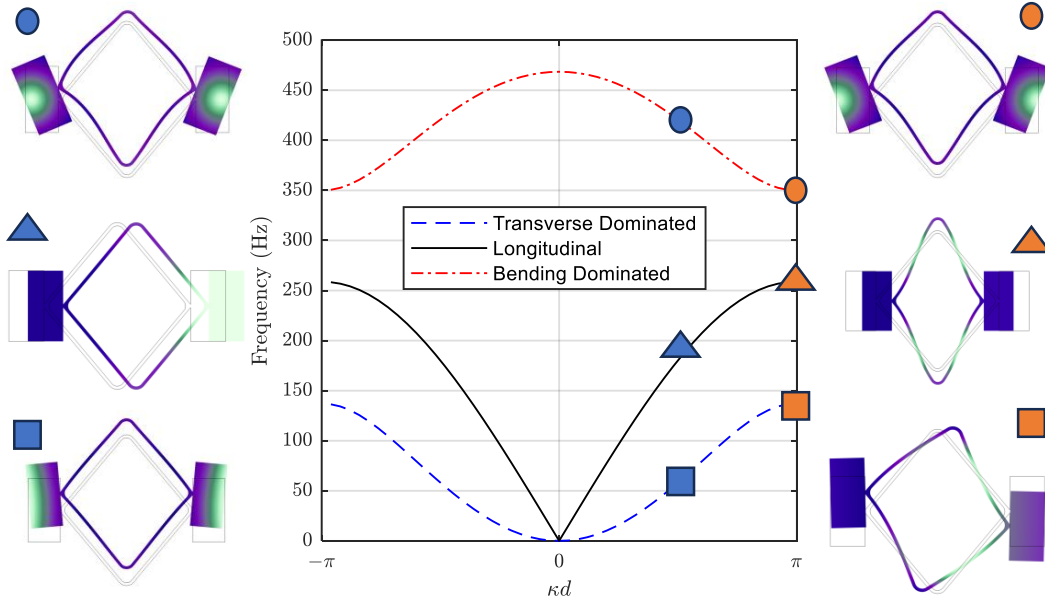


Figure 2.12: Dispersion relation of FEM COMSOL simulation with mode shapes. Blue and orange markers correspond to frequencies of $\pi/2$ and π respectively. Squares represent the transverse dominant modes, triangles represent longitudinal modes, and circles represent the rotation dominant modes.

2.6 Dynamic Testing

2.6.1 Experimental Setup Details

Figure 2.13(a) shows details of the experimental setup. The top end of the lattice was fixed by clamping the 40th particle. We assume that gravitational effects can be neglected for our system and parameters, as displacement of the first mass induced by the shaker is an order of magnitude larger than the the maximum spring stretch (between 1st and 2nd mass) due to gravity. A thumb screw was glued to the bottom end of the lattice and connected to the threaded stinger installed on a vertically placed electrodynamic exciter (S1, Brüel & Kjær Mini-shaker Type 4810), which provided longitudinal excitation with a power amplifier (PA, Brüel & Kjær Type 2718). The top end of the stinger was kept at the same height as the bottom end of the lattice such that additional stretching or compression was minimized. A second electrodynamic exciter (S2, The Modal Shop SmartShaker K2007E01) was placed horizontally at the middle point of the lattice, where a sharp-tip stinger was used with its end kept at about 1 mm away from the middle point of the side surface of particle 20 to provide the transverse pulse excitation. A two-channel function generator (FG1, Tektronix AFG3022C) was set to the single burst mode to provide MATLAB-programmed excitation signals to S1 and S2. A scanning laser Doppler vibrometer (LDV, Polytec PSV-400) was placed at a distance of 1.95 m towards the side of the lattice to measure the transverse velocity at the center point of the side surface of each particle. The sample rate was set to be 12800 Hz and the data was collected via a computer (PC). To minimize the influence of air current during experiments, the other three sides of the lattice were shielded by cardboard. For consistency, a second function generator (FG2, Tektronix AFG1022) was used to send out a pulse signal as a trigger (Trig) to both FG1 and LDV in each measurement. For each particle, the result was averaged in the time domain over 5 measurements. Between each measurement, a 60 s delay was imposed in order to allow the lattice to settle back to its equilibrium state. Velocity measurements were conducted at points a , b , and c , as illustrated in Fig. 2.13(a), for each mass. Points a and b are strategically positioned away from

the corners to ensure measurements of the intended mass during system motion. The experimental results for the transverse degree of freedom are derived from the average of velocity measurements at points a and b . Rotational degrees of freedom are calculated based on Eq. 2.110, with a representing $v_{i,left}$ and b representing $v_{i,right}$. Longitudinal velocity measurements are obtained from the middle sides of each spring at point c , as indicated in Fig 2.13(a). Figure. 2.13(b) show the spring-mass system suspended from a vertically movable stage at its fixed upper end. The lower end is attached to the stinger of Shaker 1, which is also mounted on both vertical and horizontal moving stages. Additionally, the figure shows the second shaker, mounted on a horizontal moving stage positioned behind the system, providing transverse excitation to mass 20. Surrounding the system is cardboard, which serves to minimize interference from air currents..

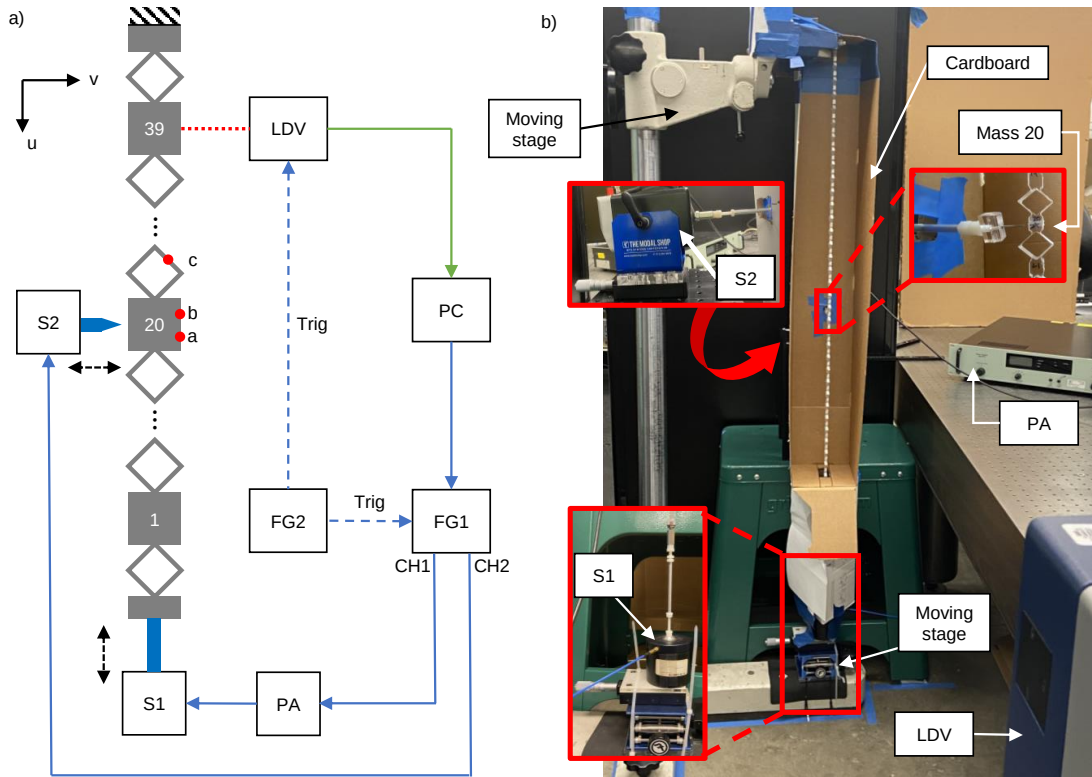


Figure 2.13: Detailed view of the experimental setup (a) Schematic diagram illustrates the connections and layout. (b) Physical system displays key components and the actual assembly of the system as configured for testing.

2.6.2 Experimental Results

To experimentally evaluate the dynamic response of the described system, multiple unit cell segments of springs and masses were machined out of polycarbonate, glued together, and suspended as shown in Fig. 2.14(a), using the same geometry as simulated in the FEM. In our experimental dynamic response, shown in Fig. 2.14, significant qualitative differences are observed from the simulations. Much stronger energy leakage from longitudinal motion can be seen in the transverse-rotational response than even in the FEM simulations. This leakage is evident in both the spatiotemporal response (Fig. 2.14(b,c)) and the associated Fourier spectra (Fig. 2.14(d,e)). Further, this leakage is accompanied with what appears energy at approximately twice the modulation frequency, which may be due to second harmonic generation in the longitudinal motion. The enhancement of such leakage may be due to increased disorder and asymmetry in the physical system compared to the idealized setting of simulation. In the Fourier spectra, we also further see only clear excitation of the lower, transverse-dominant branch, with no discernable excitation of the upper, rotation-dominant branch, in contrast to the simulated response. A weak nonreciprocity in this lower branch can be seen, however, we note that this nonreciprocity is at a different location in the band structure than was observed in simulation. The origin of this difference remains unclear, however we note that differences in the v excitation were observed to excite different nonreciprocal modes in additional simulations (see next section), which may be further exacerbated by the observed leakage and hypothesized role of disorder and asymmetry.

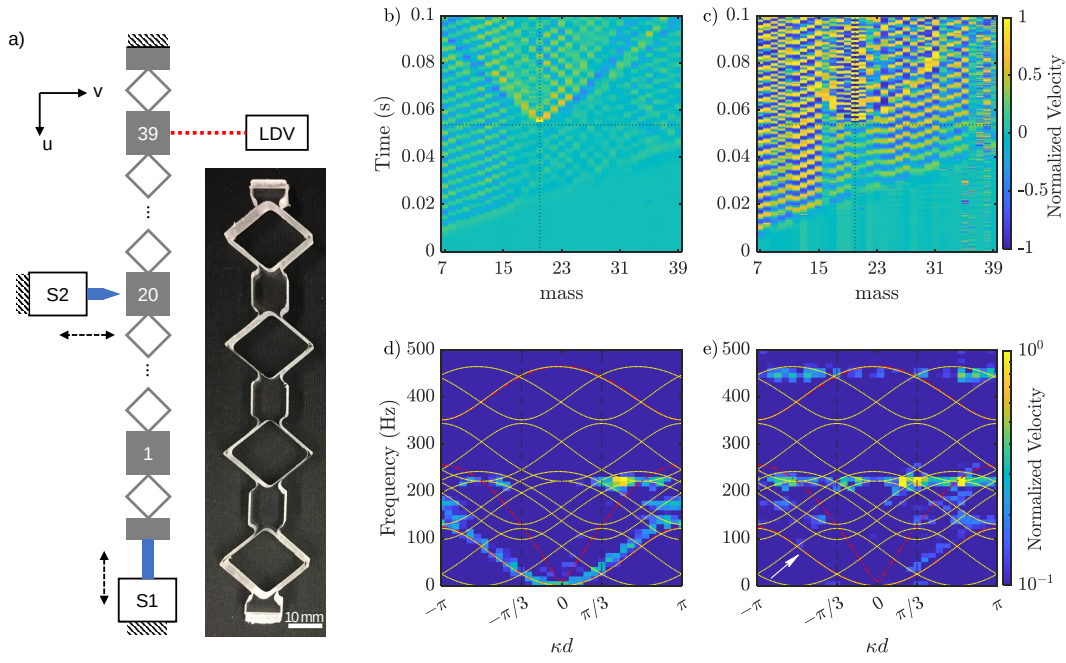


Figure 2.14: Experimental characterization of nonreciprocal transverse-rotational wave propagation via longitudinal wave stiffness modulation (corresponding to the system modeled in Figs. 2.10 and 2.11). (a) Drawing of experimental setup and photograph of a multi-unit-cell segment. The two shakers (S1 and S2, with their direction of excitation) and the LDV are denoted in the drawing. (b,c) Spatiotemporal response of v and θ degrees of freedom, from experiment, respectively. (d,e) Fourier transform of each spatiotemporal field directly above. Overlaid yellow lines are analytically predicted transverse-rotational dispersion relations, while the red lines are longitudinal and transverse-rotational dispersion relations from FEM Eigenfrequency analysis. Vertical dashed black lines in (d,e) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude. The white arrow in (e) highlights the asymmetry observed.

2.6.3 Additional Dynamic Experiments

Additional experiments on a non-modulated lattice were performed using the same setup as shown in Fig. 2.13 (see sec. 2.6.1), except first (longitudinal) shaker (S1) was not activated. This is done to confirm the systems' behavior under the influence of transverse excitation only. The resulting spatiotemporal transverse response and the Fourier transform of that response are shown in Fig. 2.15.

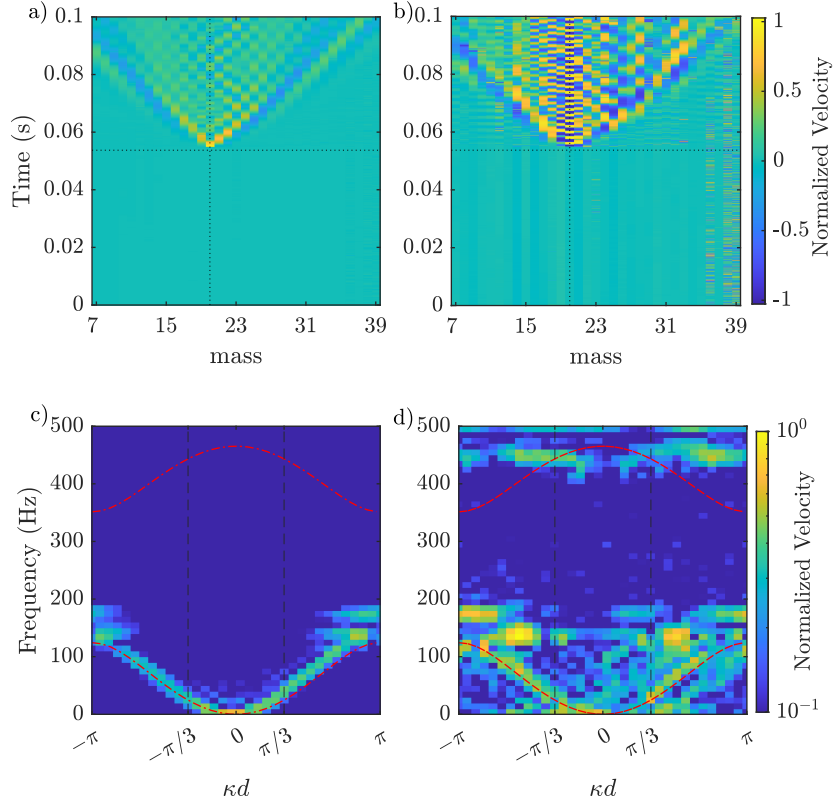


Figure 2.15: Experimental characterization of the dynamic response a 39 particle chain of the same properties as Fig. 2.14, except there is no longitudinal modulation. (a) and (b) Spatiotemporal response of the v and θ motion, respectively. Only the result of particles 7–39 are shown. (c) and (d) are Fourier transform of (a) and (b) over the time range of 0–0.1 s for all 39 particles. Overlaid red dash-dotted lines are the analytically predicted transverse-rotational dispersion relation. All plots are normalized by their respective maximum velocity amplitude.

To clarify the degree of nonrecirpocity induced by the longitudinal modulation in experiment, in Fig. 2.16, we divide the normalized Fourier transform of the experiment without longitudinal modulation (Fig. 2.15(c) and (d)) by the normalized Fourier transform of the experiment with longitudinal modulation (Fig. 2.14(d) and (e)). The color bars on panels are truncated to improve visibility.

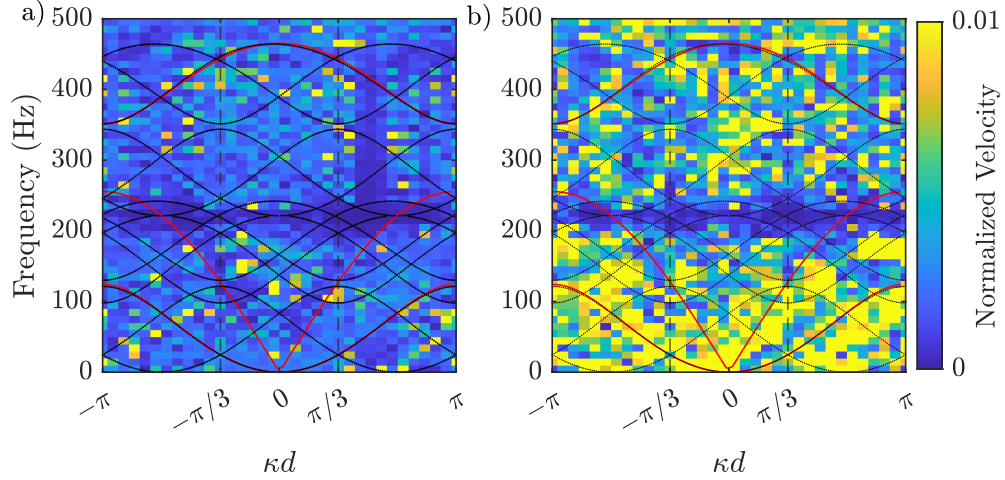


Figure 2.16: Fourier transform of the experimentally measured dynamic response (v and θ motion) for the case without longitudinal modulation (Fig. 2.15(c) and (d)) divided by the normalized Fourier transform of the experiment with longitudinal modulation (Fig. 2.14(d) and (e)). The modulated analytical dispersion relation is overlaid on top of the spectrum as black lines, while the red lines represent the non-modulated dispersion obtained from FEM eigenfrequency analysis.

2.7 Additional FEM and DEM Simulated Scenarios

In this section we simulate several other cases for the one-dimensional (1D) system shown in Fig. 2.1. Both FEM and DEM simulations in this section follow the exact same parameters and details mentioned in Sec. 2.5, unless specified otherwise. The v excitation will take the form of either a monopolar Gaussian pulse as in Eq. 2.109, a bipolar Gaussian pulse with the form

$$F(t_0 - t)e^{-0.5((t-t_0)/\sigma)^2}, \quad (2.111)$$

or a Gaussian modulated sinusoidal pulse (GSP) which takes the form

$$Fe^{-0.5((t-t_0)/\sigma)^2} \sin(2\pi ft), \quad (2.112)$$

where F is the excitation amplitude, f is the excitation frequency, σ is the standard deviation and t_0 is the centered time. The analysed system will be either a 299 masses which we refer to it as long system or a similar length to our experimental system consisting of 39 masses, which we call the short system. Each simulation scenario will have the excitation form and parameters used stated for each case.

2.7.1 Longer Chain Simulation with Additional, Different Excitations

We simulate the response of a system with 299 particles, in which case no longitudinal modulation is applied and a monopolar Gaussian pulse is applied on the center of the chain (mass 150) in the transverse direction in the form of Eq. 2.109, characterized by standard deviation of 0.1 ms, centred at 0.537 s and amplitude of 0.05 N. The spatiotemporal response of the transverse-rotational degrees of freedom, and their respective spectra are shown in Fig. 2.17. The dominant shear mode in the DEM results matches the analytical dispersion derived from numerically solving Eqs. 2.102. We can clearly note the absence of the bending mode, as shown in Fig 2.17(c). In contrast, both the shear and bending modes are clearly visible in the rotational degree of freedom, as illustrated in Fig 2.17(d).

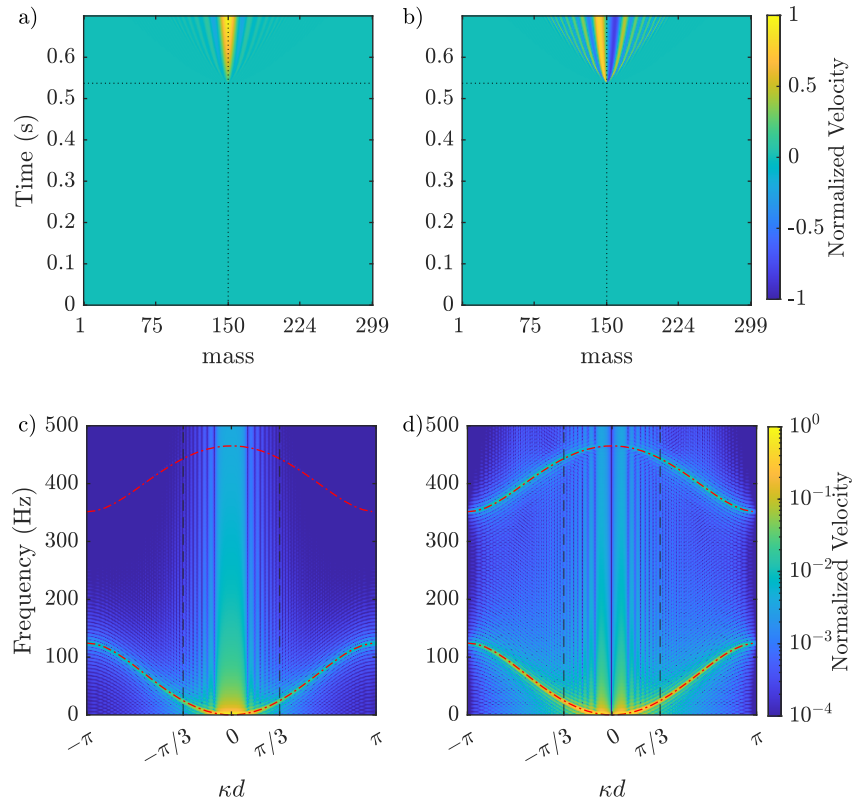


Figure 2.17: DEM numerical simulation with 299 particles and no longitudinal modulation. (a,b) Spatiotemporal velocity responses for v and θ . (c,d) Fourier transform of each corresponding spatiotemporal responses above, with the analytical dispersion relation overlaid on top as a red dash-dotted line.

In Fig. 2.18, we simulate the same system as Fig. 2.17, but add the longitudinal sinusoidal excitation in u direction, with modulation frequency of 226.125 Hz, chosen based on Eq. 2.80 to form a $R = 3$ modulated unit cell. This is the analog to Fig. 2.11, but with a longer chain. Due to the modulation, distinct nonreciprocal branches are observed in the rotational degree of freedom, as seen in Fig.2.18(f). In contrast, there is no significant change in the transverse degree of freedom, as shown in Fig. 2.18(e), when compared to the case without modulation, Fig. 2.17(c).

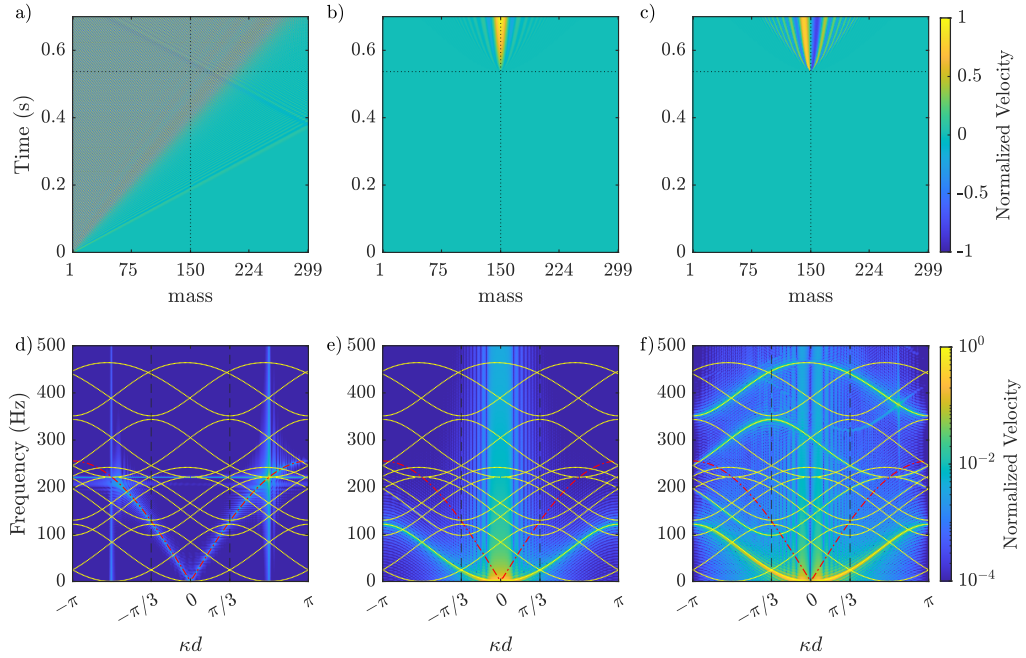


Figure 2.18: DEM numerical simulation results for 299 particles (same properties as Fig. 2.17) with modulation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitudes.

In Fig. 2.19, we change the transverse excitation to a GSP as in Eq. 2.112.

In this case, $f = 410$ Hz, $F = 0.05$ N, $t_0 = 0.537$ s, and $\sigma = 0.01$ s, while maintaining the other parameters from Fig. 2.18. In Fig. 2.19(e,f), the upper branch, representing the rotation dominant mode, is excited alongside the shear-dominant mode and multiple other branches stemming from the modulation, and asymmetric behavior is clearly observed. There is also a notable concentration of energy on the right side compared to the left. Despite, the rotational degree of freedom is excited predominantly at the bending-dominant mode, with the transverse-dominant mode being absent, asymmetric behaviors is observed as will. Despite only altering the transverse excitation force while keeping all other parameters constant, we conclude that the system's nonreciprocity depends on the type of transverse excitation force applied.

We can briefly summarize few key points from the long system analysis. Altering the profile of the transverse excitation results in the appearance of the bending-dominant mode in the transverse degree of freedom, along with presence of the shear-dominant mode, as seen in the DEM simulations panel (e) of Figs. 2.19 in comparison with 2.18. In the case of GSP excitation, most of the energy are concentrated in the bending-dominant modes and at two other branches in DEM as seen in Fig. 2.19(e). These locations are predicted by the analytical dispersion, causing an asymmetric dispersion relation. In the rotational degree of freedom, the system is excited at the bending-dominant mode, with the shear-dominant mode absent as shown in Fig. 2.19(f).

2.7.2 Short Chain Simulation with Different Forcing

We now change to using the same system parameters as in Fig. 2.11, including a chain length of 39 particles. In Fig. 2.20, we excite the transverse direction with a monopolar Gaussian pulse at mass 20 characterized by standard deviation of 0.1 ms, amplitude 0.05 N and centred at 0.0537 s, without any longitudinal modulation. Both DEM and FEM simulations yield similar outcomes, notably the absence of the bending-dominant mode in the transverse degree of freedom, as shown in Fig. 2.20 and Fig. 2.21 panel (c). In contrast, both the shear-dominant and bending-dominant modes are present in the rotational degree of freedom, as

seen in panel (d). One key difference is that the DEM bending mode aligns precisely with the analytical bending mode,

In Fig. 2.21, we show FEM simulations corresponding to the same parameters as Fig. 2.10, but without modulation. Similar to DEM simulation, bending dominant mode only visible in the rotational degree of freedom, with a slight downward shift in FEM, compared to DEM and the analytical predictions. This suggests that the system's shear stiffness is lower in the absence of modulation. We can estimate the shear stiffness by fitting the analytical dispersion curves. As a result, the stiffness is found to be 102.9 N/m, which is close to the calculated shear stiffness using the beam energy method.

In Fig. 2.22 and Fig. 2.23, we excite the same system as Fig. 2.20 and Fig. 2.21 (simulated in DEM and FEM, respectively) in the transverse direction with a bipolar Gaussian pulse in the form of Eq. 2.111, characterized by standard deviation of 0.1 ms, amplitude 0.05 N centred at 0.0537 s, with longitudinal modulation.

In Fig. 2.24 and Fig. 2.25, we simulate the same system as Fig. 2.20 through Fig. 2.21 (simulated in DEM and FEM, respectively) but change the transverse excitation to a GSP for transverse direction with excitation frequency of 410 Hz, amplitude 0.05 N, standard deviation 0.01 s and centred at 0.0537 s. Additionally, in DEM, there are two other excitation locations similar to those observed in the long chain under similar forcing conditions, as presented in Fig. 2.25(f). Meanwhile, FEM shows weak energy at one other branch, as seen in Fig. 2.25(f).

2.8 Strain Amplitude for u Based on DEM and FEM

Here we simulate the strain generated in the u direction (ϵ_x) throughout the chain in response to the longitudinal excitation used in all previous modulated cases by dividing the u response by the unstretched spring length ($S_0 = 2L \cos \psi$). Since, all modulated DEM and FEM simulations are conducted using a prescribed u displacement with amplitude of 1 mm and modulated frequency of 226.125 Hz,

one strain representation plot for each DEM and FEM simulation will be sufficient, as shown in Fig. 2.26. The resulted strain representation looks similar for both simulation method. The importance of this plot is to ensure that our u strain is within the fitted experimental data points range as presented in the main text in Fig. 2.8(b). We also note that the maximum strain amplitude is $\epsilon_{x,max} = 0.1$, where is substitute in Eq. 2.108 with $B_0 = 2.5$ to result in $\beta = 0.25$ as used in Eq. 2.103 to calculate the analytical dispersion relation.

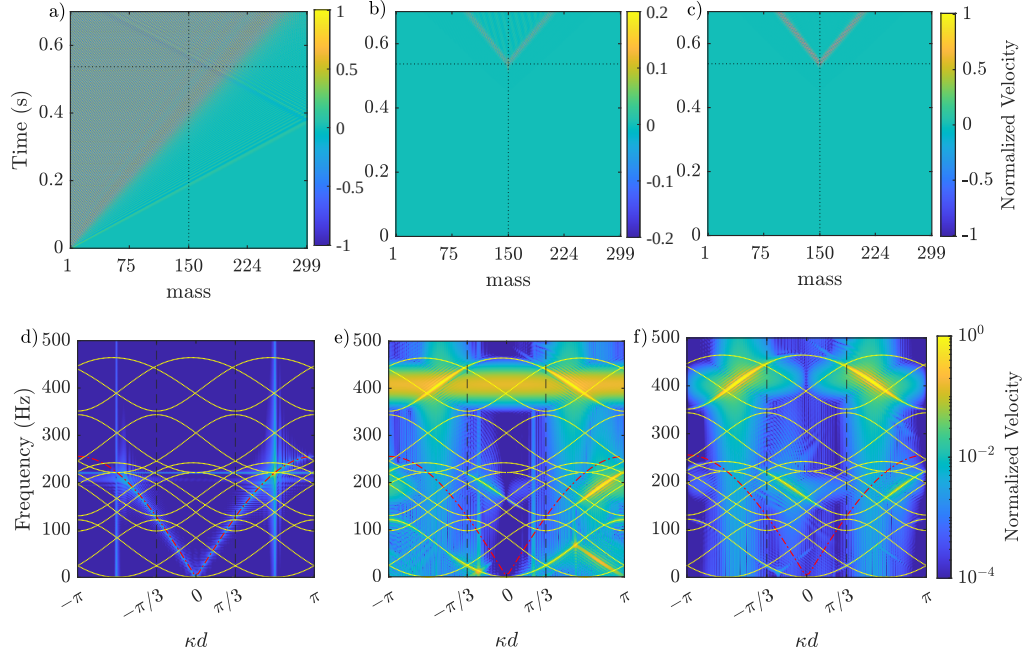


Figure 2.19: DEM numerical simulation results for 299 particles, with the same properties as Fig. 2.18 (including modulation), except the transverse Gaussian excitation has been replaced by a GSP excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

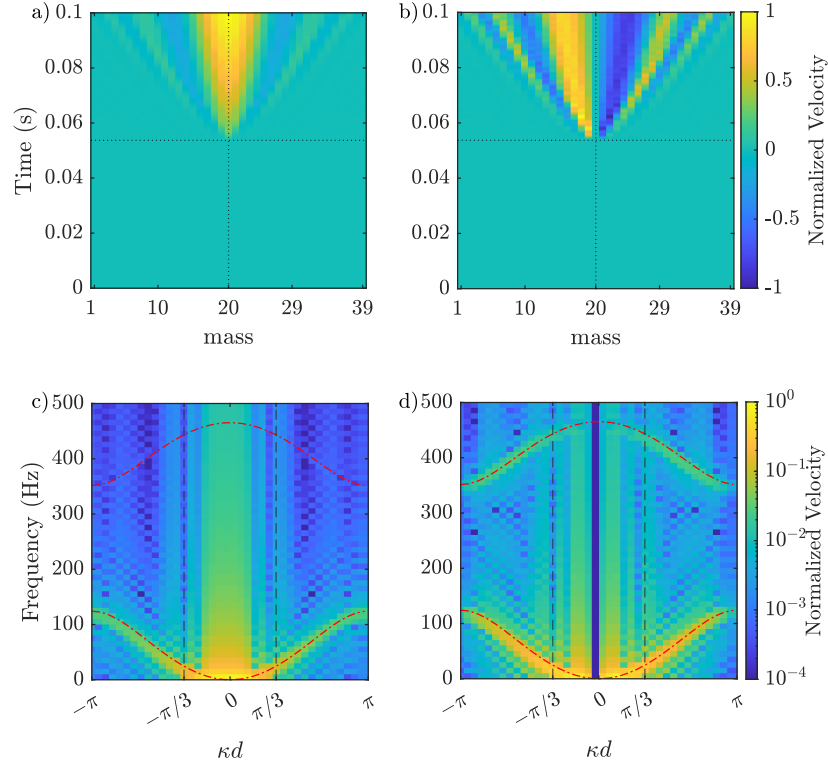


Figure 2.20: DEM numerical simulation results for 39 particles, corresponding to the same system parameters as in Fig. 2.11, but without any longitudinal modulation. (a,b) Spatiotemporal response for v and θ . (c,d) Fourier transform of each corresponding spatiotemporal response above, with the analytical dispersion relation overlaid on top as a red dash-dotted line. All plots are normalized by their respective maximum velocity amplitude.

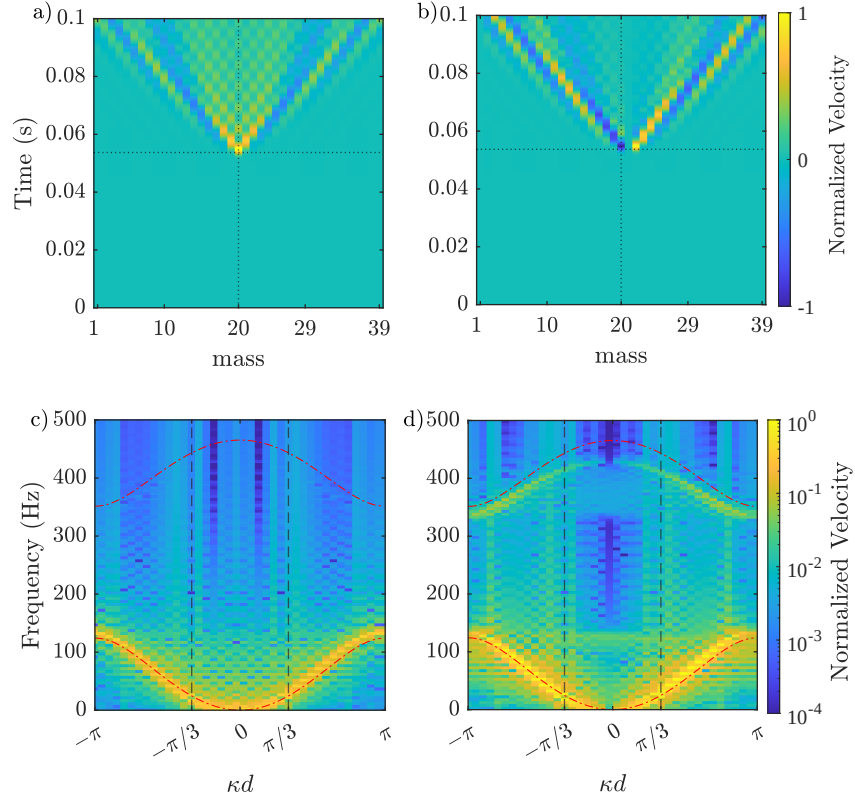


Figure 2.21: FEM COMSOL simulation results for 39 particles, corresponding to the same system parameters as in Fig. 2.10, but without any longitudinal modulation. (a,b) Spatiotemporal response for v and θ . (c,d) Fourier transform of each corresponding spatiotemporal response above, with the analytical dispersion relation overlaid on top as a red dash-dotted line. All plots are normalized by their respective maximum velocity amplitude.

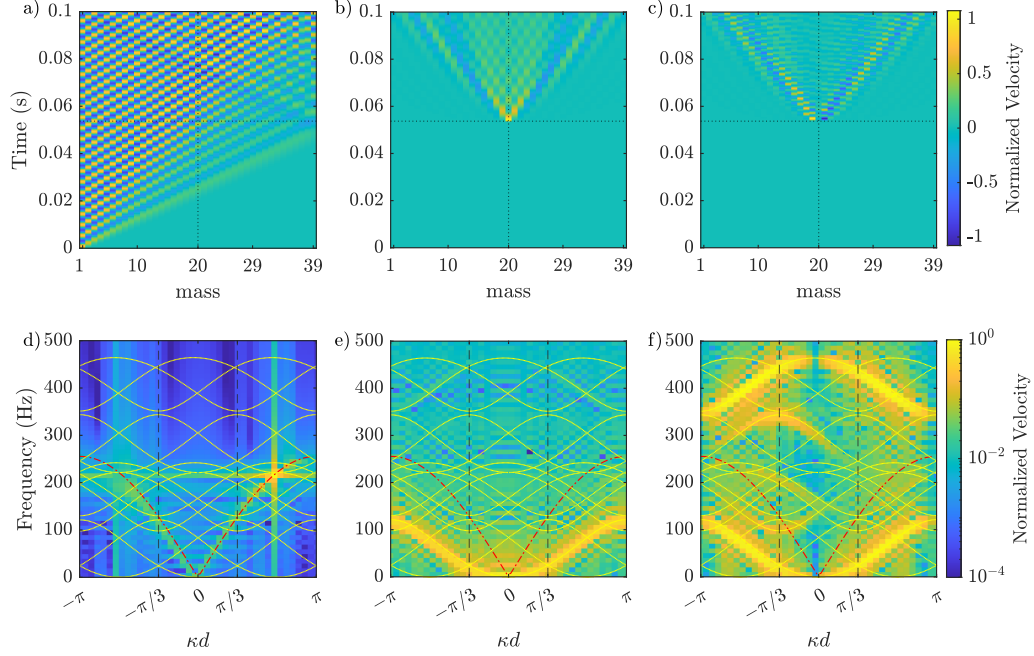


Figure 2.22: DEM numerical simulation results for 39 particles corresponding to the same system as in Fig. 2.20, and the same system and modulation as in Fig. 2.11, but with a bipolar pulse transverse excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

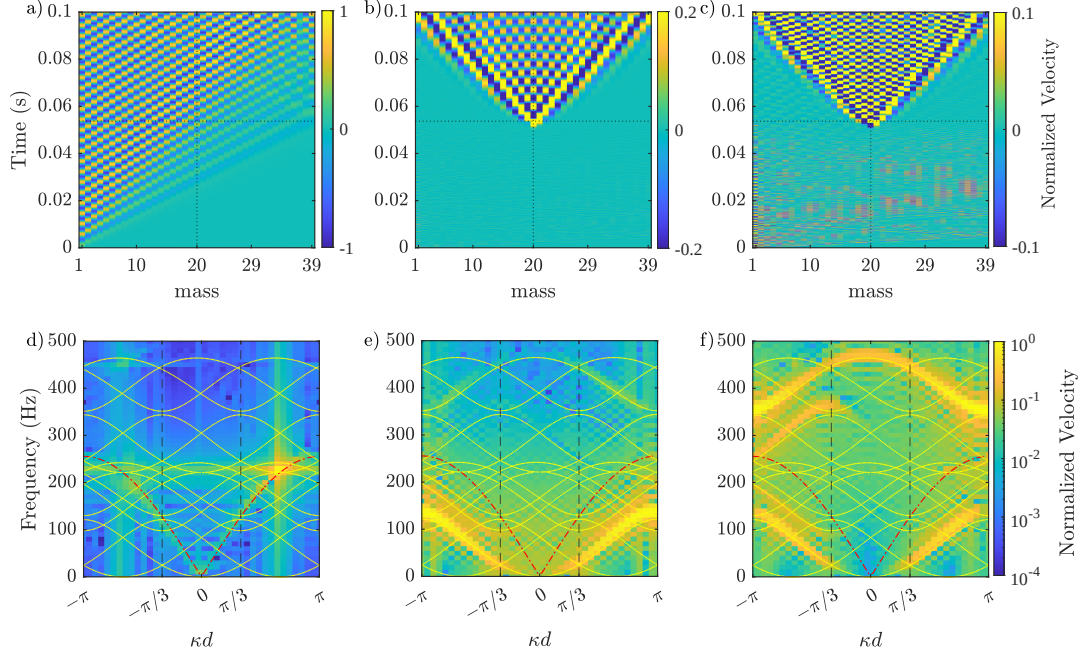


Figure 2.23: FEM COMSOL simulation results for 39 particles corresponding to the same system as in Fig. 2.22, and the same system and modulation as in Fig. 2.10, but with a bipolar pulse transverse excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

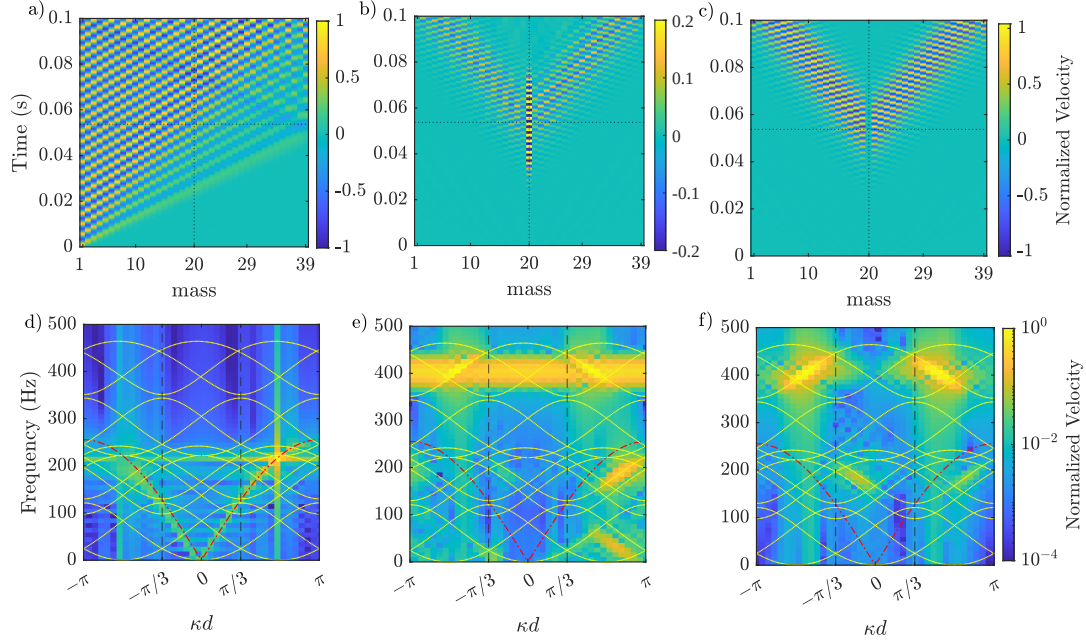


Figure 2.24: DEM numerical simulation results for 39 particles corresponding to the same system as in Fig. 2.20, and the same system and modulation as in Fig. 2.11, but with a GSP transverse excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

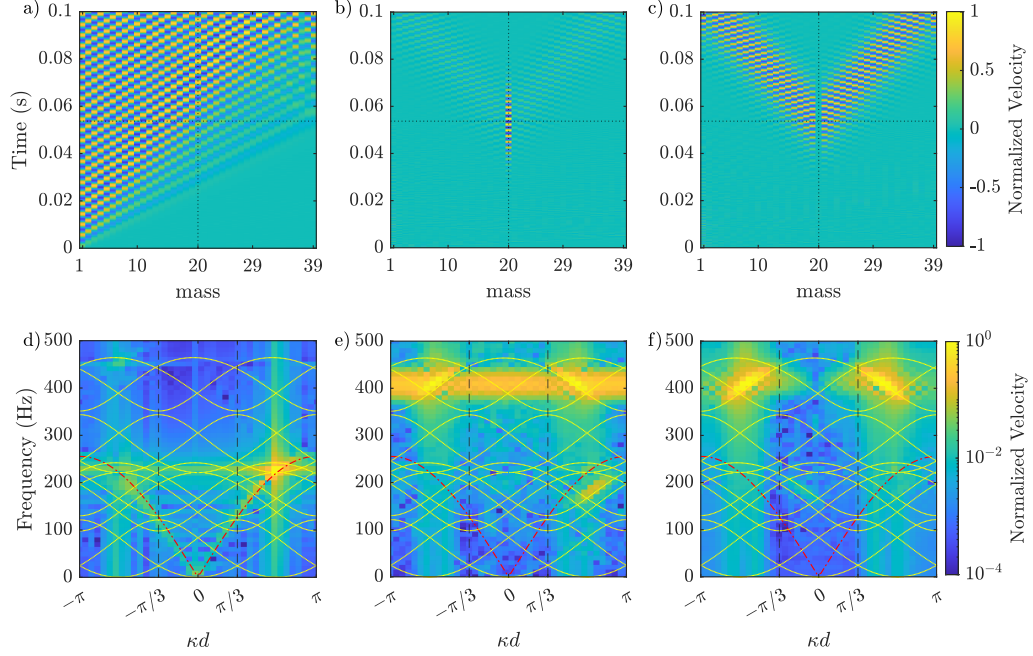


Figure 2.25: FEM COMSOL simulation results for 39 particles corresponding to the same system as in Fig. 2.20, and the same system and modulation as in Fig. 2.10, but with a GSP transverse excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (d,e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

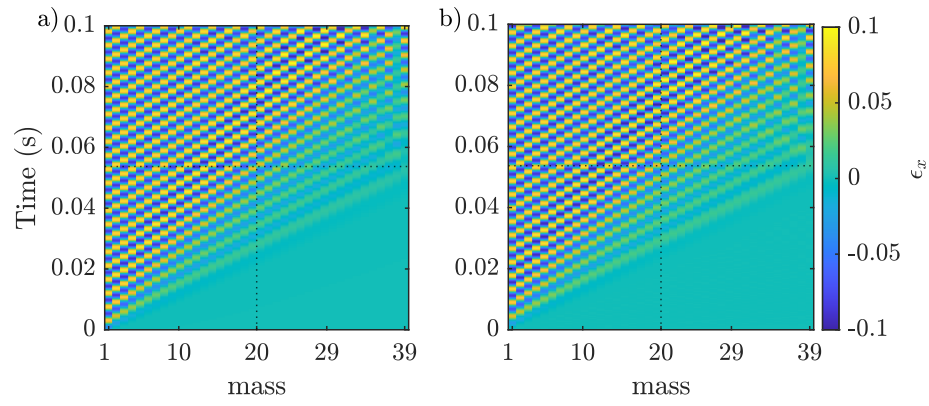


Figure 2.26: Strain in u direction (ϵ_x) calculated for 39 mass, throughout the simulation time of form 0 to 0.1 s (a) Strain for DEM simulation. (b) Strain for FEM simulation.

2.9 System's without Bending Stiffness

In this section, we further simplify the system by eliminating the bending stiffness (i.e., $k_b = 0$) but keeping all other parameters the same. Again we consider a 1D mass-and-spring model with N unit cells, as shown in Fig. 2.27, where each particle has two translational degrees of freedom u and v (along the x and y directions, respectively) and one rotational degree of freedom θ (about the z direction). The adjacent particles are connected by linear normal springs k_n and nonlinear shear springs k_s . The nonlinear shear springs have stiffness that depends on the normal strain at the same location, same as before.

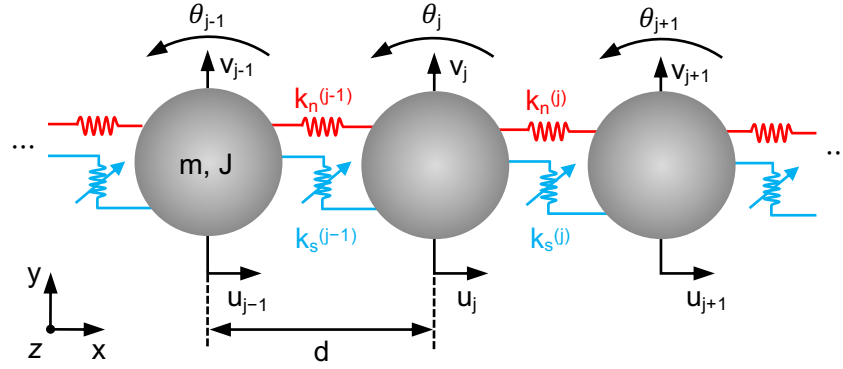


Figure 2.27: Theoretical model of one-dimensional (1D) mass-and-spring system with linear normal and nonlinear shear springs only, wherein the shear stiffness depends on the normal strain.

2.9.1 Equation of Motion and Dispersion Relation

To derive the equations of motion we follow the same steps described in Sec. 2.2, by using Lagrangian method for the j th particle the longitudinal degree of freedom can be written as

$$\begin{aligned}
 m\ddot{u}_j = & -k_n^{(j-1)}(u_j - u_{j-1}) + k_n^{(j)}(u_{j+1} - u_j) + \\
 & \frac{\beta k_0(v_j - v_{j-1} - (d/2)(\theta_j + \theta_{j-1}))^2}{2S_0} - \\
 & \frac{\beta k_0(v_{j+1} - v_j - (d/2)(\theta_{j+1} + \theta_j))^2}{2S_0},
 \end{aligned} \tag{2.113}$$

and the equations for the coupled transverse and rotational degrees of freedom are

$$\begin{aligned}
m\ddot{v}_j &= -k_s^{(j-1)} \left(v_j - v_{j-1} - \frac{d}{2}(\theta_j + \theta_{j-1}) \right) + \\
&\quad k_s^{(j)} \left(v_{j+1} - v_j - \frac{d}{2}(\theta_{j+1} + \theta_j) \right), \\
J\ddot{\theta}_j &= \frac{d}{2}k_s^{(j-1)} \left(v_j - v_{j-1} - \frac{d}{2}(\theta_j + \theta_{j-1}) \right) + \\
&\quad \frac{d}{2}k_s^{(j)} \left(v_{j+1} - v_j - \frac{d}{2}(\theta_{j+1} + \theta_j) \right),
\end{aligned} \tag{2.114}$$

where m is the mass of the particle, d is the lattice spacing, J is the moment of inertia, $k_n^{(j)}$ is the normal stiffness and $k_s^{(j)}$ is shear stiffness as a function of the longitudinal strain as in Eq. 2.105. By following a similar as in Sec. 2.2 we now define a new “unit cell” for transverse-rotational motion based on the modulation wavelength formed by the longitudinal wave, where here, for simplicity, we assume λ to be an integer multiple of d . The number of particles in the unit cell is $R = \lambda/d$, wherein $r = 1, 2, \dots, R$ is the particle index within the unit cell. The equations of motion of Eq. 2.114 can then be reformulated to equations of motion for a unit cell, and written as

$$\begin{aligned}
m\ddot{v}_r &= - \left(k_s^{(r-1)} + \gamma_1[r] \right) \\
&\quad \left((v_r - v_{r-1} + \gamma_2[r, v]) - \frac{d}{2}(\theta_r + \theta_{r-1} - \gamma_2[r, \theta]) \right) + \\
&\quad k_s^{(r)} \left((v_{r+1} - v_r - \gamma_3[r, v]) - \frac{d}{2}(\theta_r + \theta_{r+1} - \gamma_3[r, \theta]) \right),
\end{aligned} \tag{2.115}$$

$$\begin{aligned}
J\ddot{\theta}_r &= \frac{d}{2} \left(k_s^{(r-1)} + \gamma_1[r] \right) \\
&\quad \left((v_r - v_{r-1} + \gamma_2[r, v]) - \frac{d}{2}(\theta_r + \theta_{r-1} - \gamma_2[r, \theta]) \right) + \\
&\quad \frac{d}{2} k_s^{(r)} \left((v_{r+1} - v_r - \gamma_3[r, v]) - \frac{d}{2}(\theta_r + \theta_{r+1} - \gamma_3[r, \theta]) \right),
\end{aligned} \tag{2.116}$$

where $k_s^{(r)}$ (denoted by the (r) superscript) is a function of time, and functions

γ_1, γ_2 , and γ_3 are defined as

$$\begin{aligned}\gamma_1[r] &= \delta_{r-1,0} \left(k_s^{(R)}(t) - k_s^{(r-1)}(t) \right), \\ \gamma_2[r, Q] &= \delta_{r-1,0} \left(Q_{r-1} - Q_R e^{-i\mu} \right), \\ \gamma_3[r, Q] &= \delta_{r,R} \left(Q_{r+1} - Q_1 e^{i\mu} \right),\end{aligned}$$

and used to enforce periodicity, with Q the degree of freedom (either v or θ), $\delta_{r,n}$ the Kronecker delta (where n is either 0 or R), and $\mu = \kappa\lambda$ is the non-dimensional wavenumber (and κ the wavenumber) for v and θ motions.

To obtain the dispersion of transverse-rotational waves given the longitudinal wave modulation of the shear stiffnesses, we follow Ref. [94]. We obtain approximate dispersion relations for the case of a $R = 3$ particle unit cell (considered here onward)

$$\begin{aligned}(\omega - p\omega_m)^2 m \mathbf{I} \vec{V}_p - \sum_{q=-P}^P \mathbf{K}_q^{(1)} \vec{V}_{p-q} - \left(\frac{d}{2}\right) \sum_{q=P}^P \mathbf{K}_q^{(2)} \vec{\Theta}_{p-q} &= 0, \\ (\omega - p\omega_m)^2 J \mathbf{I} \vec{\Theta}_p - \left(\frac{d}{2}\right) \sum_{q=-P}^P \mathbf{K}_q^{(3)} \vec{V}_{p-q} - \left(\frac{d}{2}\right)^2 \sum_{q=-P}^P \mathbf{K}_q^{(4)} \vec{\Theta}_{p-q} &= 0,\end{aligned}\tag{2.117}$$

where ω is the frequency of the transverse-rotational mode, p denotes the order of the terms in the truncated Fourier series approximation, $\mathbf{I}$ is the identity matrix, $\vec{V}_p$ and $\vec{\Theta}_p$ are column vectors of length R of amplitudes v_r and θ_r (corresponding to Fourier order p), and $\mathbf{K}_q^{(\eta)}$ (with $\eta = 1, 2, 3, 4$) are $R \times R$ matrices defined as

$$\mathbf{K}_q^{(\eta)} = k_{s,0} \left(\delta_{q,0} \hat{\mathbf{K}}_0^{(\eta)}(\mu) + \delta_{q,\pm p} \frac{\beta}{2} \hat{\mathbf{K}}_q^{(\eta)}(\mu) \right) \forall p \neq 0, \tag{2.118}$$

where $\delta_{q,p}$ is Kronecker Delta, and for the $R = 3$ unit cell, $\hat{\mathbf{K}}_q^{(\eta)}$ is

$$\hat{\mathbf{K}}_q^{(1)}(\mu) = \begin{bmatrix} e^{iq\phi_3} + e^{iq\phi_1} & -e^{iq\phi_1} & -e^{i(q\phi_3-\mu)} \\ -e^{iq\phi_1} & e^{iq\phi_1} + e^{iq\phi_2} & -e^{iq\phi_2} \\ -e^{i(q\phi_3+\mu)} & -e^{iq\phi_2} & e^{iq\phi_2} + e^{iq\phi_3} \end{bmatrix}, \tag{2.119}$$

$$\hat{\mathbf{K}}_q^{(2)}(\mu) = \begin{bmatrix} e^{iq\phi_1} - e^{iq\phi_3} & e^{iq\phi_1} & -e^{i(q\phi_3-\mu)} \\ -e^{iq\phi_1} & e^{iq\phi_2} - e^{iq\phi_1} & e^{iq\phi_2} \\ e^{i(q\phi_3+\mu)} & -e^{iq\phi_2} & e^{iq\phi_3} - e^{iq\phi_2} \end{bmatrix}, \tag{2.120}$$

$$\hat{\mathbf{K}}_q^{(3)}(\mu) = \begin{bmatrix} e^{iq\phi_1} - e^{iq\phi_3} & -e^{iq\phi_1} & e^{i(q\phi_3 - \mu)} \\ e^{iq\phi_1} & e^{iq\phi_2} - e^{iq\phi_1} & -e^{iq\phi_2} \\ -e^{i(q\phi_3 + \mu)} & e^{iq\phi_2} & e^{iq\phi_3} - e^{iq\phi_2} \end{bmatrix}, \quad (2.121)$$

and

$$\hat{\mathbf{K}}_q^{(4)}(\mu) = \begin{bmatrix} e^{iq\phi_3} + e^{iq\phi_1} & e^{iq\phi_1} & e^{i(q\phi_3 - \mu)} \\ e^{iq\phi_1} & e^{iq\phi_1} + e^{iq\phi_2} & e^{iq\phi_2} \\ e^{i(q\phi_3 + \mu)} & e^{iq\phi_2} & e^{iq\phi_2} + e^{iq\phi_3} \end{bmatrix}. \quad (2.122)$$

Assuming the longitudinal wave is excited such that the shear stiffnesses take the form of a rightward traveling wave, where $k_s^{(r)} = k_0(1 + \beta \cos(\omega_m t - 2\pi r/R))$ with t as time, matrices $\mathbf{K}_q^{(\eta)}$ are defined in terms of stiffness coefficients k_0 and β , and non-dimensional wavenumber μ . Organizing Eqs. 2.117 as a quadratic eigenvalue problem, we numerically obtain $4R(2P+1)$ eigenfrequencies for each μ , where half are mirrored to negative frequencies around $\omega = 0$.

In Fig. 2.28, we show an example dispersion relation for evaluating Eqs. 2.117 with parameters from Sec. 2.5.2 in the limit of zero modulation amplitude but finite modulation wavelength (blue circles). We observe R branches with integer shifts of ω_m between the sets, and clear left-right asymmetry around the $\kappa = 0$ axis. We omit $2R$ flat branches that arise at each of $\pm\omega_m$ and $\omega = 0$, which would be suppressed the presence of a fixed boundary in a finite physical system. Choosing a finite $\beta = 0.25$, we obtain the dispersion shown by the black crosses in Fig. 2.28. Due to the finite modulation strength, we observe hybridization and the formation of gaps at the crossing points between branches. The value of β here was chosen based on the slope B_0 of the fitted, linearized response of the $\psi = 50^\circ$ diamond spring in Fig. 2.8(b) (denoted by the purple dashed line, and fitted based on the first three experimental data points), and an assumed longitudinal wave amplitude U , where $\beta = B_0\epsilon_x$.

2.9.2 Dynamical Simulated Scenarios

Similar to the system with bending stiffness, here we conduct several simulation with long (299 masses) and short (39 masses) chains. We numerically

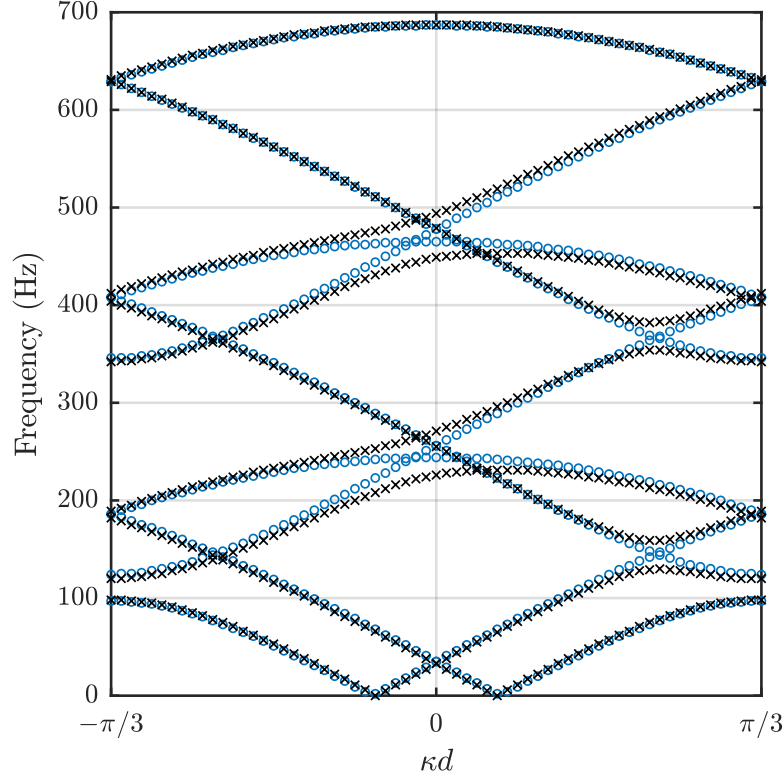


Figure 2.28: Dispersion relations of stiffness modulated chain from Eq. 2.117 for $P = 1$, $R = 3$, $m = 0.8$ grams, $k_n = 538.3$ N/m, $\sqrt{k_n/k_s} = 2.1$ and $\sqrt{md^2/J} = 7.487$ with zero ($\beta = 0$, blue circles) and finite ($\beta = 0.25$, black crosses) coupling strength between longitudinal strain and shear stiffness.

integrate Eqs. 2.113, 2.115 and 2.116 in MATAB using the *ode45* function with relative and absolute error tolerance both set to 10^{-10} with output time steps of 5 ms. Energy conservation was confirmed, by checking the total energy to be constant to within a degree of certainty of $\pm 2.43 \times 10^{-12}$ J. For simplicity, we list all the basic parameters used in the DEM simulation: The mass of the particle was measured to be $m = 0.8$ grams, and the mass moment of inertia was calculated based on mass shape to be a rectangular prism with a length and width of 10 mm and a thickness of 6.35 mm, $J = 13.33$ g $\cdot$ mm². Given the spring beam length $L = 16$ mm and an initial angle $\psi_0 = 50^\circ$, the resultant unit cell spacing is $d = 30.6$ mm, and the unscratched spring length is $S_0 = 20.6$ mm. Finally, the linear nor-

mal and bending stiffness were $k_n = 538.3$ N/m and $k_b = 69.55$ N/m, respectively. While, in the nonlinear shear stiffness in the form of Eq. 2.105, has $k_0 = 121.965$ N/m and $B_0 = 2.5$. DEM and FEM simulation are presented and discussed in the next section. The forcing on the v direction follow the same forms of as in Sec. 2.5

Longer Chain with Additional, Different Forcing Conditions

We simulate the response of a system with 299 masses, $\sqrt{k_n/k_s} = 2.1$ and $\sqrt{md^2/J} = 7.487$. No longitudinal modulation is applied and a monopolar Gaussian pulse is applied at the middle of the system in the transverse direction, characterized by standard deviation of 0.1 ms, centred at 0.537 s and amplitude of 0.05 N. The spatiotemporal behavior of the transverse and rotational degrees of freedom, along with their associated spectra, are depicted in Fig. 2.29.

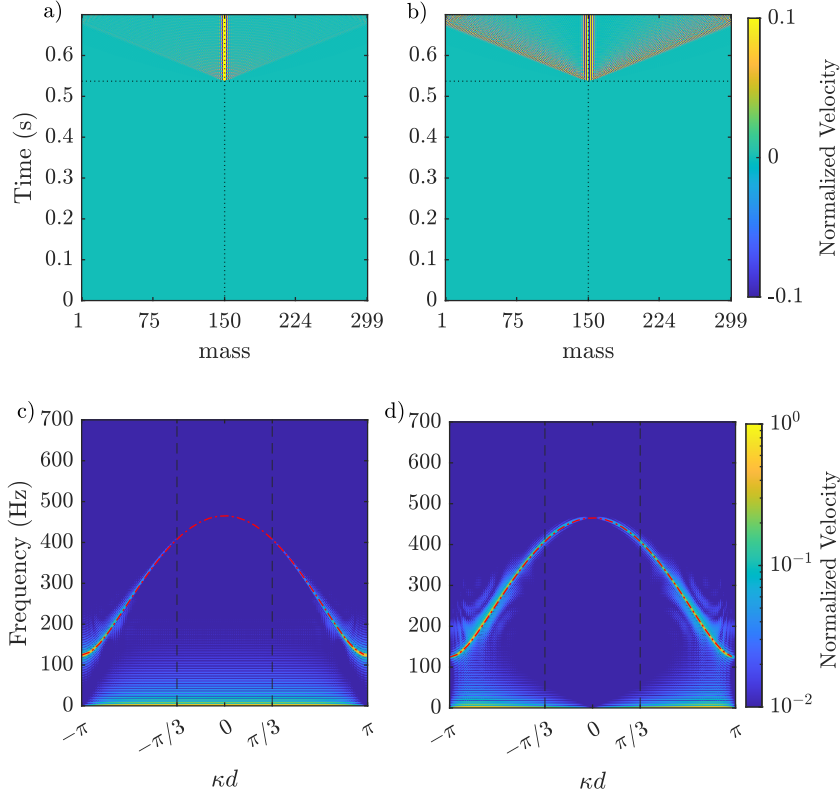


Figure 2.29: DEM numerical simulation for 299 masses, $\sqrt{k_n/k_s} = 2.1$, $\sqrt{md^2/J} = 7.487$, and no longitudinal modulation. (a,b) Spatiotemporal response for v and θ . (c,d) Fourier transform of each corresponding spatiotemporal response above, with the analytical dispersion relation overlaid on top as a red dash-dotted line. All plots are normalized by their respective maximum velocity amplitude.

In Fig. 2.30, we simulate the same system as Fig. 2.29, but add the prescribed displacement longitudinal modulation excitation in u direction in the form of a sine function, with amplitude of 1 mm and modulation frequency of 226.125 Hz, chosen to form a $R = 3$ modulated unit cell.

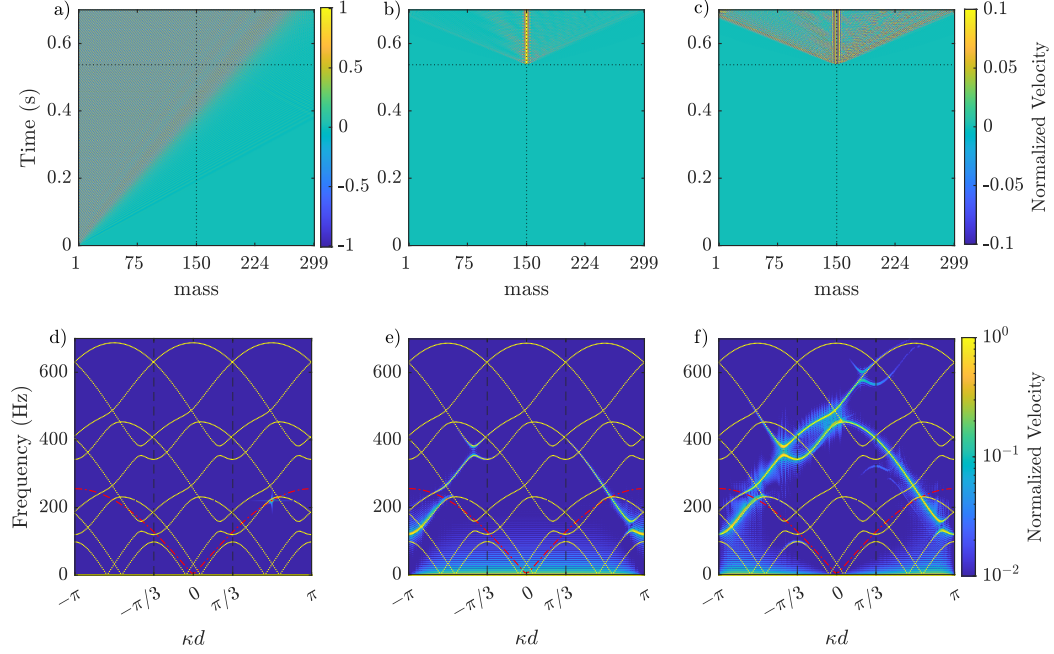


Figure 2.30: DEM numerical simulation results for 299 masses (same properties as Fig. 2.29) with modulation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

In Fig. 2.31, we again change the transverse excitation to a GSP with frequency of 300 Hz, amplitude 0.05 N, and standard deviation 0.01 s, while maintaining the other parameters from Fig. 2.30.

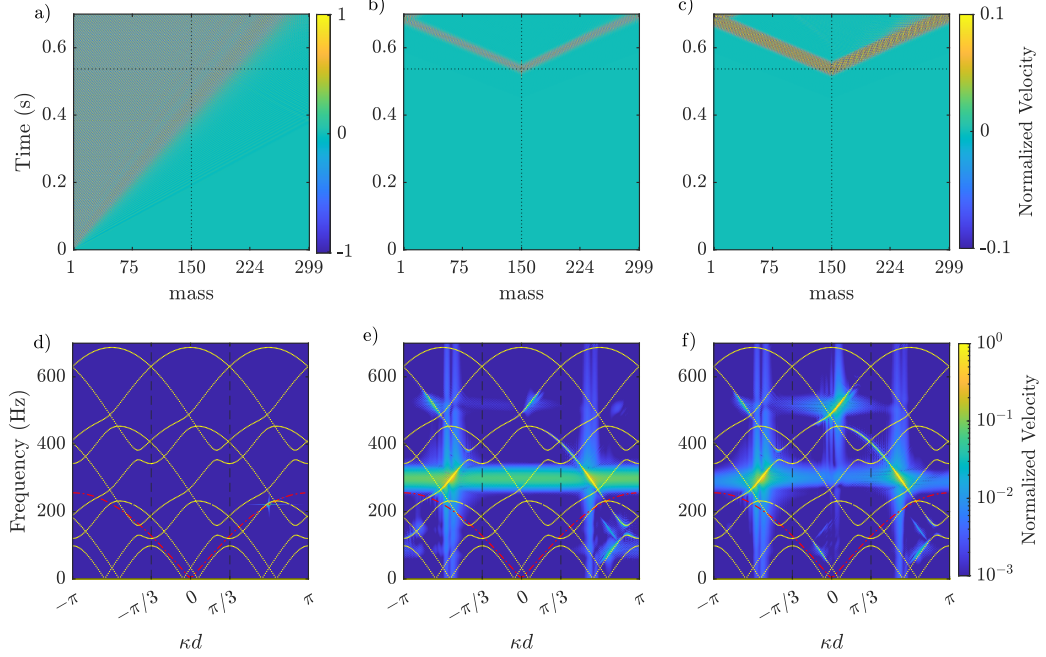


Figure 2.31: DEM numerical simulation results for 299 masses, with the same properties as Fig. 2.30 (including modulation), except the transverse Gaussian excitation has been replaced by a GSP excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

Short Chain with Additional, Different Forcing Conditions

In Fig. 2.32 we use similar system as in Figs. 2.29 ($\sqrt{k_n/k_s} = 2.1$, and $\sqrt{md^2/J} = 7.487$) but for shorter system consist of 39 masses. The transverse excitation is monopolar Gaussian pulse is applied at the middle of the system, characterized by standard deviation of 0.1 ms, centred at 0.0537 s and amplitude of 0.05 N, without any longitudinal modulation. .

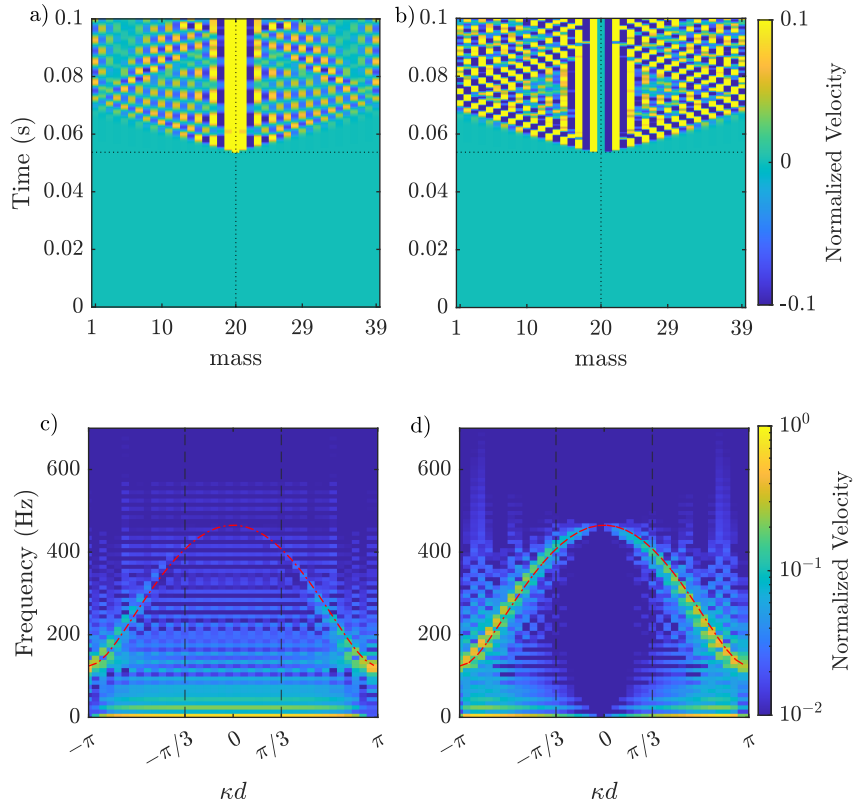


Figure 2.32: DEM numerical simulation results for 39 masses, $\sqrt{k_n/k_s} = 2.1$ and $\sqrt{md^2/J} = 7.487$, without any longitudinal modulation. (a,b) Spatiotemporal response for v and θ . (c,d) Fourier transform of each corresponding spatiotemporal response above, with the analytical dispersion relation overlaid on top as a red dash-dotted line. All plots are normalized by their respective maximum velocity amplitude.

In Fig. 2.33, we simulate the same system as Fig. 2.32, but add the longitudinal sine modulation excitation as prescribed displacement in u direction with amplitude of 1 mm and modulation frequency of 226.125 Hz, chosen to form a $R = 3$ modulated unit cell. This is the analog to Fig. 2.30, but with a shorter chain.

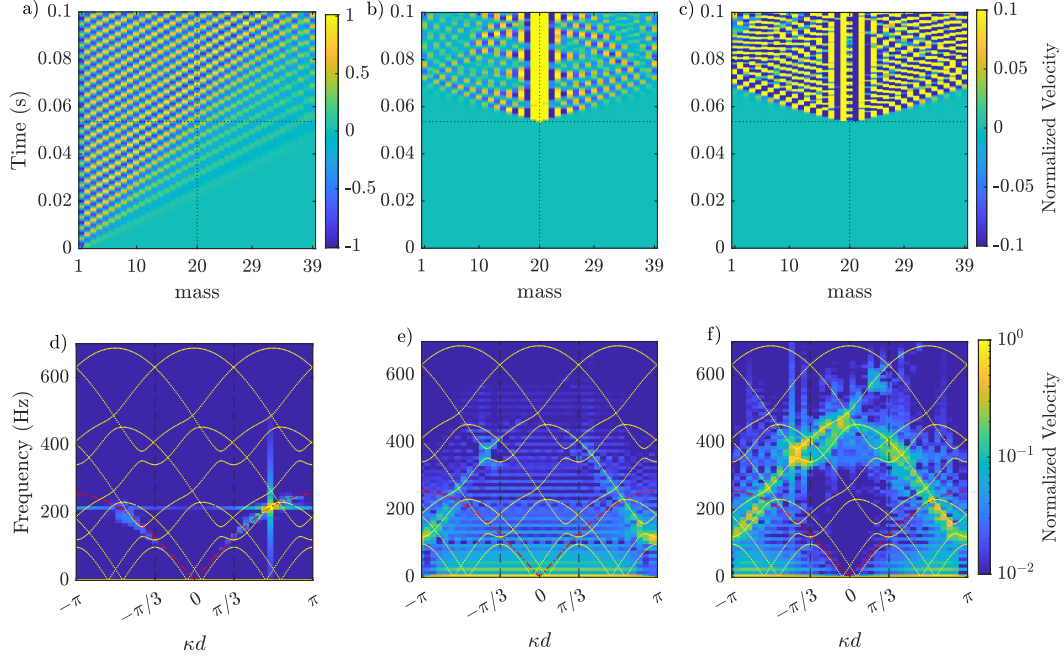


Figure 2.33: DEM numerical simulation results for 39 masses (same properties as Fig. 2.32) with modulation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

Finally, in Fig. 2.34, we simulate the same system as in Fig. 2.33, but change the transverse excitation to a GSP in the v direction with frequency of 300 Hz, amplitude 0.05 N, and standard deviation 0.01 s.

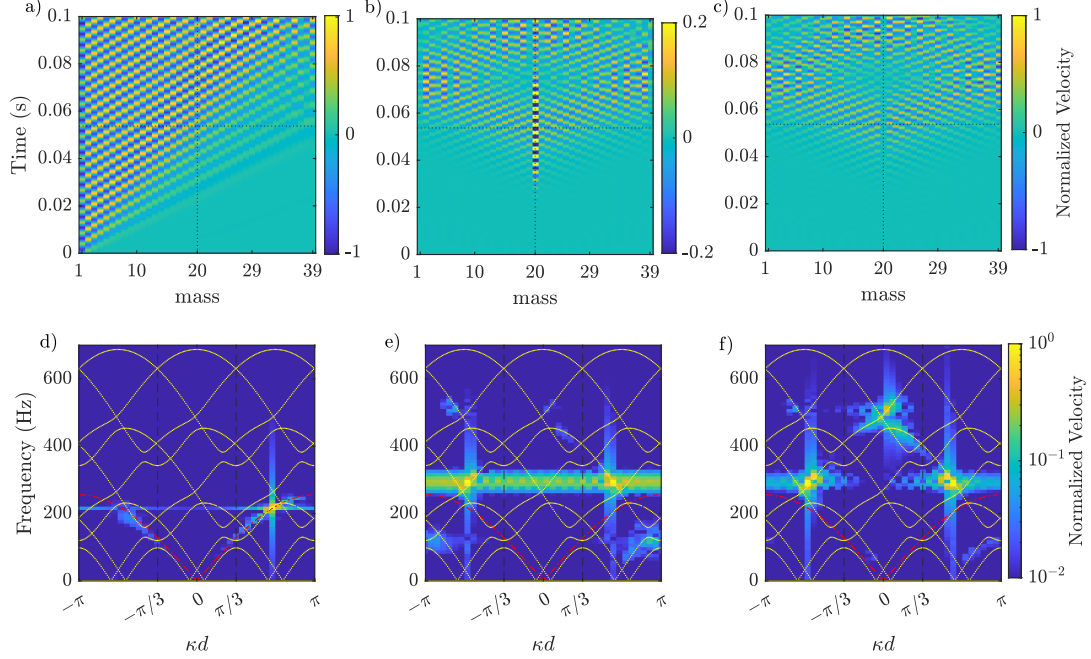


Figure 2.34: DEM numerical simulation results for 39 masses, and the same system and modulation as in Fig. 2.33, but with a GSP transverse excitation. (a,b,c) Spatiotemporal response of u , v , and θ degrees of freedom, respectively. (d,e,f) Fourier transform of each spatiotemporal field directly above. Overlaid red and yellow lines are the analytically predicted longitudinal and transverse-rotational dispersion relations, respectively. Vertical dashed black lines in (e,f) denote the Brillouin zone edge for the modulated $R = 3$ unit cell. All plots are normalized by their respective maximum velocity amplitude.

2.9.3 Folded Branch and Overlapping Conditions

The most pronounced nonreciprocal wave propagation for a given frequency excitation occurs when one side of the dispersion yields a band gap and the other does not. In our modulated system, band gaps emerge as a result of hybridization between crossed bands. Further, our system's dispersion relation can be considered as sets of folded branches shifted in frequency by $p\omega_m$, as is shown in Fig. 2.28 and Fig. 2.35 below. We define quantity $\Delta\omega$ as the difference between the initial and final values of the transverse-rotational branch in the dispersion curve (*e.g.* $\Delta\omega$) (also shown in Fig. 2.35).

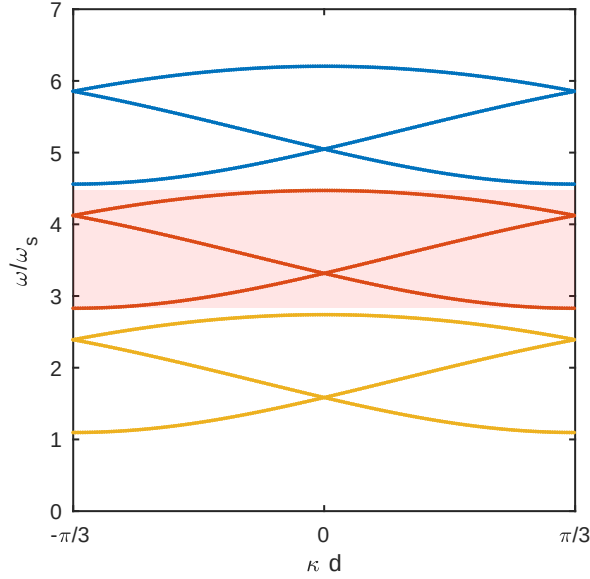


Figure 2.35: Dispersion curve for transverse rotational motion where each folded branch shifted by $p\omega_m$ does not overlap. The shaded area represents a single folded branch, with “unfolded” frequency range of $\Delta\omega$.

To ensure crossing of the shifted dispersion curves we need to ensure that

$$|\Delta\omega| > \omega_m. \quad (2.123)$$

To calculate $\Delta\omega$, we solve for the transverse-rotational dispersion with non-modulated shear stiffness by setting $k_s^{(j)} = k_0$ in Eqs. 2.113 and 2.114. We assume

traveling wave solutions

$$\begin{aligned} v_j &= V e^{i(\kappa j d - \omega t)}, \\ \theta_j &= \Theta e^{i(\kappa j d - \omega t)}. \end{aligned} \quad (2.124)$$

Substituting Eqs. 2.124 into Eqs. 2.114 and converting to matrix form, and solving the resulting eigenvalue problem, we obtain dispersion relation

$$\omega_{v\theta} = \sqrt{\frac{k_0}{m} \frac{(md^2 + 4J) + (md^2 - 4J) \cos d\kappa}{2J}}. \quad (2.125)$$

We then normalize Eq. 2.80 and Eqs. 2.125 by dividing by $\sqrt{k_0/m}$ where k_0 is the undeformed shear stiffness, to obtain

$$\omega_u(\kappa) = 2R_{k_{ns}} \sqrt{\sin^2 \frac{d\kappa}{2}} \quad (2.126)$$

and

$$\omega_{v\theta} = \sqrt{\frac{(R_{mJ}^2 + 4) + (R_{mJ}^2 - 4) \cos d\kappa}{2}}, \quad (2.127)$$

respectively, where $R_{mJ} = \sqrt{md^2/J}$ and $R_{k_{ns}} = \sqrt{k_n/k_0}$.

Using Eq. 2.127 to evaluate $\Delta\omega = \omega_{v\theta}(0) - \omega_{v\theta}(\pi/d)$ in Eq. 2.123, leading to the inequality

$$|R_{mJ} - 2| > c \cdot R_{k_{ns}}, \quad (2.128)$$

where coefficient $c = 2 \sin(\pi/R)$. We can further define the maximum overlapping percentage, as

$$\frac{|\Delta\omega| - \omega_m}{|\Delta\omega|} = \frac{|R_{mJ} - 2| - c \cdot R_{k_{ns}}}{|R_{mJ} - 2|}. \quad (2.129)$$

Figure 2.36 shows several cases for different overlapping percentages without modulation for an artificially defined unit cell of $R = 2$.

In Fig. 2.37, we show cases for an artificially defined unit cell of $R = 3$, again without any modulation. In particular, we note that a change in concavity of the branches can be observed at $R_{mJ} = 2$, as can be seen in Fig. 2.37(b-d). Moreover, in Fig. 2.38 we show a generalized plot to determine the limit of the square root of mass to inertia ratio, by choosing R and d values and substituting in Eq. 2.128.

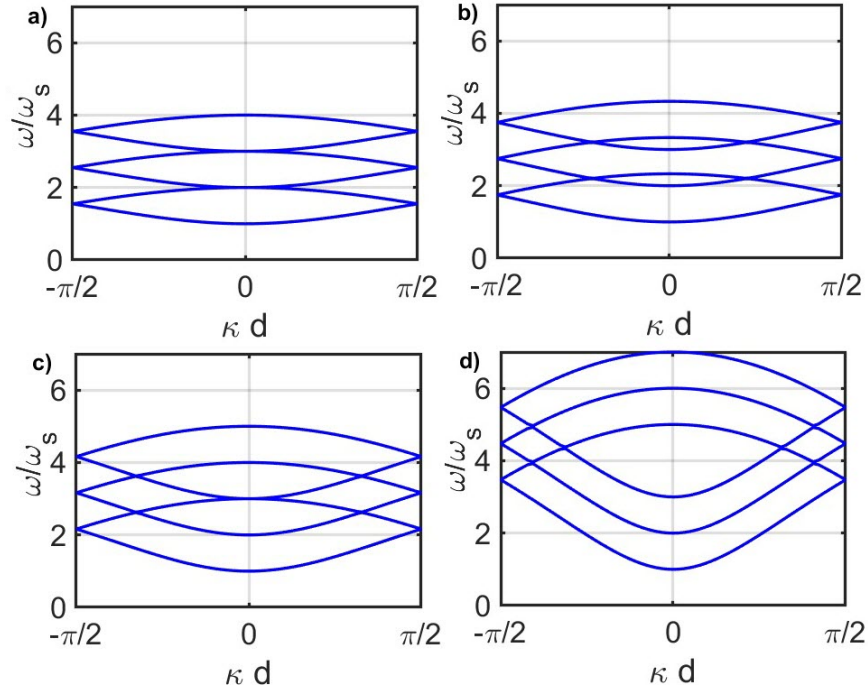


Figure 2.36: Different cases of transverse-rotational dispersion relation branch overlapping for $R = 2$ and $R_{k_{ns}} = 1/2$, where (a) $R_{mJ} = 3$, (b) $R_{mJ} = 3.33$, (c) $R_{mJ} = 4$, and (d) $R_{mJ} = 6$.

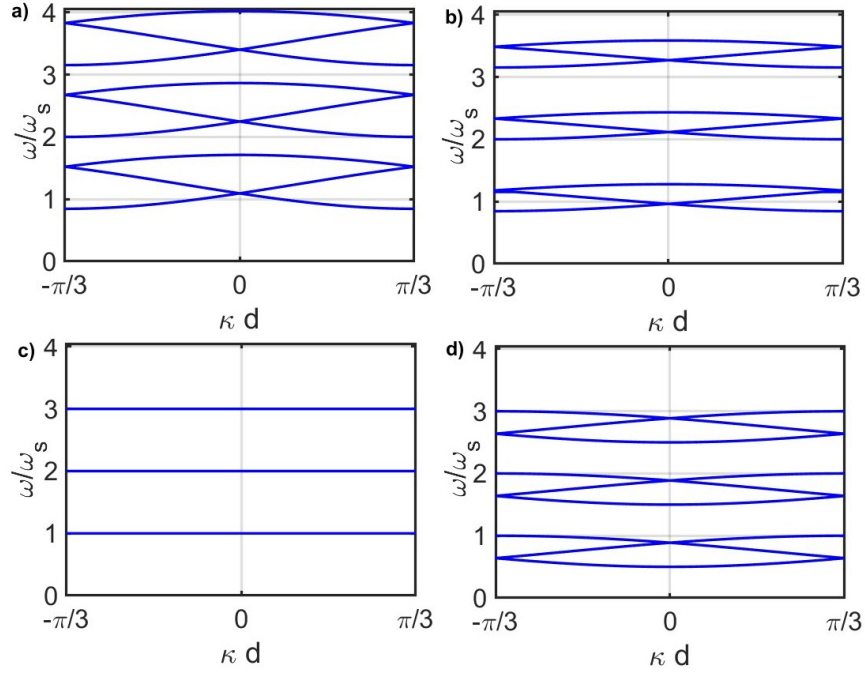


Figure 2.37: Different cases of transverse-rotational dispersion relation branch overlapping for $R = 3$ and $R_{k_{ns}} = 1/2$, where (a) $R_{mJ} = 2.87$, (b) $R_{mJ} = 2.43$, (c) $R_{mJ} = 2$, and (d) $R_{mJ} = 1.5$.

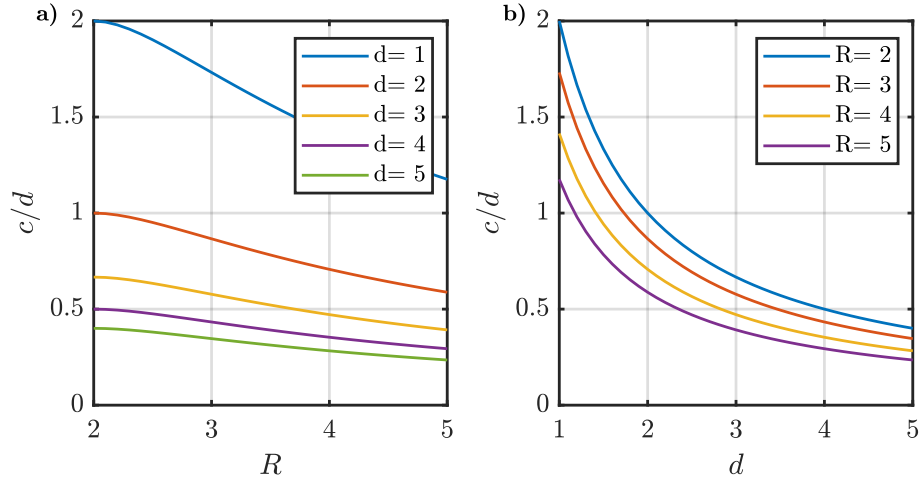


Figure 2.38: generalized plot to determine the limit of overlapping condition for (a) Unitcell spacing d from 1 to 5. (b) Unitcell size R of 2 to 5.

2.10 Conclusion

In this work, we analytically, computationally, and experimentally explored the possibility of achieving nonreciprocal acoustic wave propagation via spatiotemporal modulation by engineering the nonlinear interaction between a signal (transverse-rotational) wave and a modulation (longitudinal) wave. We observed only relatively weak nonreciprocity using this approach, and differences between our FEM and DEM simulations, and between the simulated and experimental realizations. We attributed the leakage between longitudinal and transverse-rotational motion observed in the FEM, but not DEM, simulations to mechanical instability resulting from the large longitudinal compression and 1D geometry of the system. Such leakage was seen to be further amplified in the experimental results, which we suggest may be due to the influence of disorder and asymmetry, coupled with the aforementioned instability. We also observed the signatures of nonreciprocity to occur at different locations in the measured spectra between experiments and simulation, which we speculate may be due to a combination of differing system excitation and the described role of instability and disorder. Via future modification of the design, it is foreseeable that such mechanical-instability-driven mode conversion mechanisms could be suppressed. One proposed modification was to simplify the system by assuming zero bending stiffness. This can be achieved by altering the design of the single unit cell to minimize its effect compared to shear stiffness. Numerical analysis shows that the simplified system behaves similarly to the original system in terms of band gap presence. We were also able to control the band gap opening via the overlapping conditions by either adjusting the mass-to-inertia ratio or by varying the stiffnesses of the normal and shear springs. We further note that neither our analytics or simulations included damping [136]. Regarding the latter, the presence of damping in related spatiotemporally modulated systems has, for instance, been recently predicted to yield universal one-way transport [144]. Potential topics of future study include consideration of these issues. In addition, considering the weak nonreciprocity observed, future studies may be valuable that consider how to maximize the nonreciprocity (and its upper limits using such a technique), as well as how to minimize the modulation

energy cost compared to that of transmitted signal. Such optimization is likely possible through optimization of the systems band structure, nonlinearity [145], and their interplay [146]. Further extensions may include extrapolation to higher dimensions (including more complex modal interactions), the design of higher dimensional analogs of the nonlinear springs introduced herein, and the use of such a modulation strategy in highly nonlinear regimes such as transition wave control [147].

2.11 Acknowledgements

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Chapter 3

Mechanical Rotary Chain Analog of Aubry-André-Harper Systems

3.1 Introduction

3.1.1 Background

The Harper model (due to P.G. Harper (1955) [148]) is a tight-binding model describing an electron in a two dimensional lattice with a magnetic flux, leading to a discrete Schrodinger equation now known as the Harper equation [149]. In the Harper model the eigenstates form Landau levels, which have nontrivial Chern numbers. Consequently, the bulk energy bands of the Harper model are also topological in nature and have nonzero Chern numbers. Due to the topological nature of the bulk bands, this system supports topologically protected edge states [150], which are robust against certain perturbations [151] as explained in the next section. These topologically protected edge states can be used in developing new quantum materials and devices [152] due to their inherent stability [153], which helps reduce decoherence and environmental noise, thus enhancing applications in quantum computation [154] and communication [155]. Additionally, their unique properties improve precision in quantum sensors [156] and increase efficiency in electronic devices by reducing power consumption and heat generation through their conductivity and dissipation characteristics [157].

The Harper model under open boundary condition on one side and periodic on the other, such that it forms a semi infinite model which can be mapped to a problem of electrons on a one dimensional lattice with a commensurate or incommensurate sinusoidal on-site potential (depending on whether or not the crystal lattice can be rationally divided into integer multiples of sublattices [158, 159, 160, 161, 162, 163]), revealing insights into the nature of electronic wave functions in periodic and quasiperiodic structures (see Ch. 5 of [164] for a pedagogical discussion of the model). In 1980, Aubry and André considered this one dimensional model [165] and demonstrated that the model exhibits an Anderson localization transition [166] between extended and localized states depending on the strength of the quasiperiodic potential, which is essential for understanding the behavior of electrons in disordered media [163]. This duality between extended and localized states is a hallmark of this model that is now known as the Aubry-André-Harper (AAH) model [165, 167].

The AAH system has numerous applications across various fields of physics and engineering, showcasing its versatility and significance. In quantum transport [168], the AAH model is used to study nonequilibrium quantum transport in systems connected to external baths, helping researchers understand how environmental noise can either enhance or suppress transport, which is critical for designing efficient quantum devices [169]. The generalized AAH model has been used to study transitions between topologically trivial and nontrivial states [170], providing insights into materials with robust edge states useful in quantum computing [149], aiding in the development of materials with specific electronic properties such as semiconductors [171]. A generalized version of the AAH model with p -wave superconducting pairing has also been used to study the interplay between localization and superconductivity, revealing new phases and transitions significant for understanding high temperature superconductors [162]. The AAH model is also employed to study adiabatic pumping, where particles are transported across a system by slowly varying parameters, important for developing nano scale devices that rely on precise control of particle movement [163, 172]. It is this feature of the AAH model that we are interested in this work as explained in a later section.

3.1.2 Review of topological properties of systems with non-trivial Chern bands

To further understand the AAH system, we begin by introducing the topological properties for nontrivial systems. This foundational knowledge will provide the necessary context for comprehending the subsequent details of the AAH system. Let us assume we have a periodic system (with two degrees of freedom per unit cell for simplicity) in both direction x and y as presented in Fig. 3.1(a). Due to the periodicity, we can use Fourier transformation to analyze the spectrum as a function of wave vector $\mathbf{k} = \{k_x, k_y\}$. At each $\mathbf{k}$ we can evaluate the dynamical matrix $D(\mathbf{k})$ (or more generally the Hamiltonian, for mechanical systems the Hamiltonian is the dynamical matrix) which can be obtained by applying Fourier transform on the full real space dynamical matrix D . Since each unit cell has two degrees of freedoms (by assumption), the matrix $D(\mathbf{k})$ is a 2×2 matrix. At each value of wave number k_x and k_y we can calculate the Eigen spectrum which result in two eigenvalues. Let the eigenvalues and the eigenvectors are $\omega_{\pm}^2(\mathbf{k})$ and $u_{\pm}(\mathbf{k})$ satisfying $D(\mathbf{k})u_{\pm}(\mathbf{k}) = \omega_{\pm}^2(\mathbf{k})u_{\pm}(\mathbf{k})$, where $\omega_{\pm}^2$ are the larger and smaller eigenvalues. Now under variation of $\mathbf{k}$ in both directions, the two eigenvalues trace out two surfaces, we call these two surfaces the bulk frequency bands (bulk because the systems has no boundary). Let us assume that the two bands do not cross each other anywhere, or in other words, they are isolated from each other. For each of these isolated bands, we can evaluate the Chern number c using the following equation [173]

$$c_{\pm} = \frac{1}{2\pi} \int d^2k \left[\partial_{k_x} u_{\pm}(\mathbf{k})^* \partial_{k_y} u_{\pm}(\mathbf{k}) - \partial_{k_y} u_{\pm}(\mathbf{k})^* \partial_{k_x} u_{\pm}(\mathbf{k}) \right]. \quad (3.1)$$

If the Chern numbers of the two bands are $c = 0$, we call the system topologically trivial, as shown in Fig. 3.1(b), otherwise the system is topological with nontrivial Chern number as in Fig. 3.1(c). Note that the total Chern number of all bands in a system has to be zero, which in our case implies that $c_+ = -c_-$ [173]. If we modify the system in Fig. 3.1(a) to be a semi-infinite system by imposing a periodic condition at one side and fixed or free on the other side. By repeating the same steps as before, we calculate the spectrum of the new geometry as a function

of k_y . Now, for the case of trivial system the spectrum will contain bulk modes as shown in Fig. 3.1(e) (with a possibility of topologically trivial edge modes). In the nontrivial case, in addition to the bulk modes, there are topological edge bands in the bulk gap as shown in Fig. 3.1(f) in red and blue colors. Because the two modes are topologically distinct, they cannot be coupled to each other. Now, the way a topological edge band is different than a trivial edge band can be summarized by two main characters [174], firstly, a topological edge band connects the two bulk bands above and below the gap, whereas a trivial edge band does not connect the two bulk bands, secondly, a topological edge mode is unidirectional having either positive or negative group velocity $\partial_{k_y}\omega$, unlike a trivial edge mode is not unidirectional. This unidirectionality of the topological edge modes also means that these propagation is scattering free [175]. The number of topological edge bands on each side of the system is equal to the Chern number, and all of these topological edge bands on one side of the system have the same sign of group velocity, meaning they propagate in the same direction.

3.1.3 Motivation

The AAH model is of particular interest due to its unique ability to facilitate nonreciprocal, one-way transport of a mechanical wave packet through the system from one side to the other. We briefly explain this by writing the governing relation of the AAH model in a 1D nearest neighbor tight-binding form with a sinusoidally varying on-site potential. This is written as

$$E\psi_n = -t\psi_{n+1} - t\psi_{n-1} - E_0 \cos(n\theta + \phi)\psi_n, \quad (3.2)$$

where, E_0 represents the on-site potential, t denotes the hopping amplitude between two sites, ψ_n represents the orbital amplitude at site number n , while θ is the incommensurate angle, and a magnetic field of ϕ . A plot of the energy spectrum E for varying ϕ under open boundary conditions in Fig. 3.2 (data replotted from Ref. [164]) shows the bulk mode as black regions, and two edge modes marked by the blue and red dots. These modes exhibit topological nontriviality with nonzero Chern numbers [164]. The edge states plotted as blue and red dots to correspond

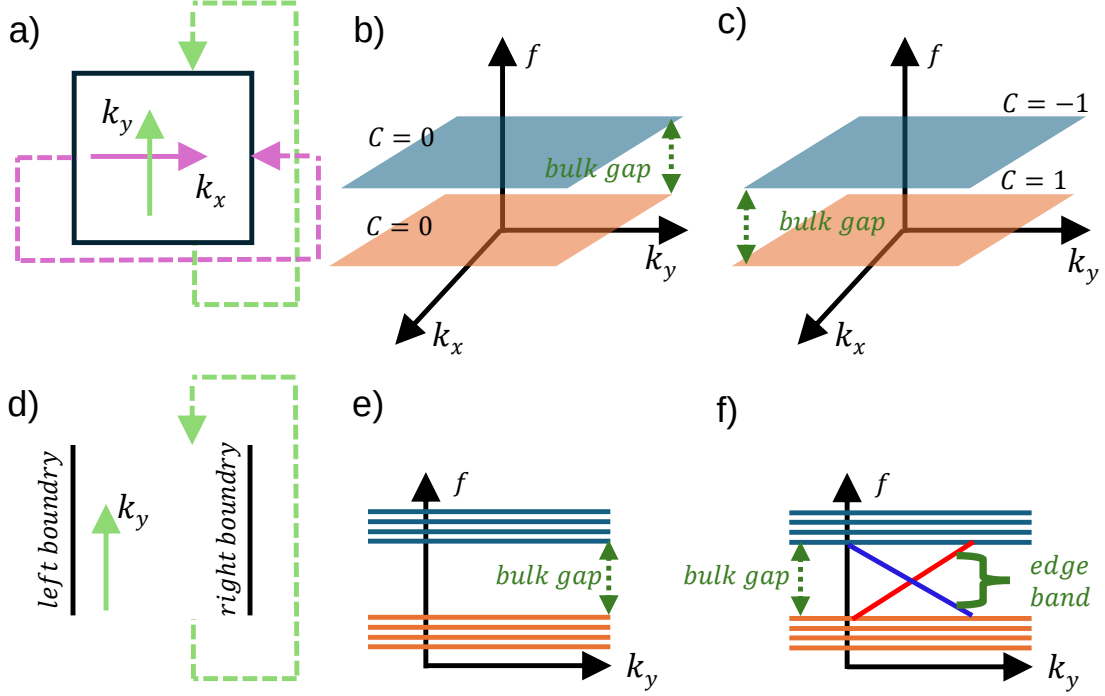


Figure 3.1: (a) Schematic of an infinite periodic system. (b,c) Bulk frequency bands and Chern number for topological trivial and nontrivial infinite system, respectively. (d) Schematic of an semi-infinite periodic system. (e,f) Frequency spectrum for topological trivial and nontrivial semi-infinite system, respectively. Showing the bulk gap in both systems, with the addition of edge bands in the nontrivial topological system.

to the right and left edge modes, respectively [164].

As an example, in a scenario where the system is characterized by a the value of $\phi = 1$, there is only a right edge mode (blue) within the gap (the right vs. left edge mode in the 1D system is the analog to the one way propagating edge mode in the 2D system). If one excites this mode on the right end of the system, while maintaining the value of ϕ as a constant, the response amplitude will be higher in comparison with left end excitation. Now, if we excite the system at the right end, and simultaneously increases the value of ϕ from 1 adiabatically (meaning slowly or quasistatically), then with the increase in the ϕ value, the

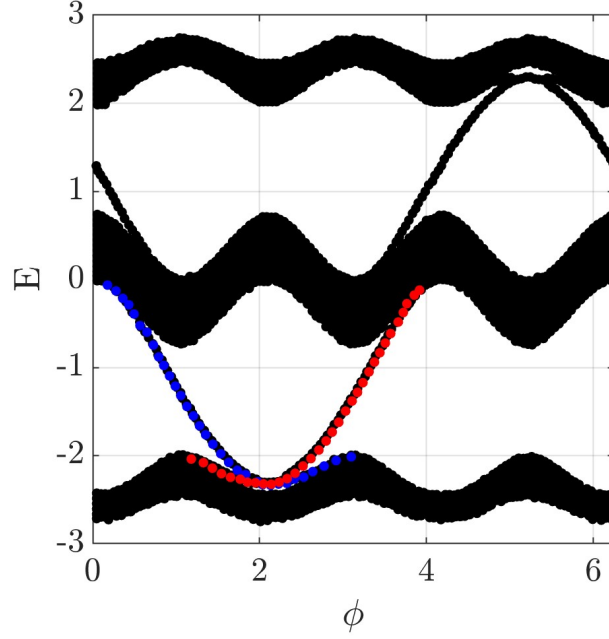


Figure 3.2: Energy spectrum illustrating the variation with respect to ϕ , at values of $t = 1$, $E_0 = 2$, and $\theta = 2\pi/3$. Bulk modes are represented by black regions, and two edge modes (right and left) are marked with blue and red dots, respectively.

excited, localized mode undergoes a transition into the bulk and eventually emerges at the opposite left end, resulting in the existence of only the left edge mode (red) [176]. This unique phenomenon of quasistatically changing the value of ϕ is known as “adiabatic pumping”. The adiabatic pumping phenomena requires an external energy input, which results in external change in the value of ϕ [177]. As the value of ϕ changes, the incommensurate term of Eq. 3.2 also changes with respect to time, thereby breaking time invariance through temporal modulation [178], resulting in nonreciprocal behavior. In the context of my thesis, the introduction of external energy not only alters ϕ but is also considered as an added energy bias to the system. This energy bias manifest as spatiotemporal modulation [179]. The adiabatic pumping phenomena also results in a transfer of energy, as the excited localized mode is moved from the right end to the left end of the system. On the other hand, if the system is initially positioned with $\phi = 1$ and excited on the left

end of the system, then as we gradually increase ϕ to 3, no significant effect will be observed. This is because, during the transition from $\phi = 1$ to 3, the wave packet amplitude decays, due to the absence of a left edge mode, thus the wave packet will not propagate to the other end.

It is important to emphasize that in AAH systems the nonreciprocity can be achieved in one of two ways. The first method is by the external adiabatic pumping, where the nonreciprocal behavior stems from “active energy bias” (or more accurately time invariance breaking) which externally changes ϕ value adiabatically in one direction as seen in the 1D tight-binding model governed by Eq. 3.2. A similar nonreciprocal behavior can be achieved by another method, which is called “self-adiabatic pumping”, due to the system’s nonlinearity. This phenomena does not require an external energy bias other than the signal itself to adiabatically change the value of ϕ ; here, ϕ varies due to a combined effect of edge mode excitation and its coupling to the zero mode via the system’s nonlinearity as we will see later.

Numerous experimental implementations of the AAH model can be found in the literature for various physical domain such as acoustics [180, 181], magneto-mechanical systems [182] and elastic systems [183]. We showcase one experimental example that illustrates the adiabatic pumping mechanism in acoustics [180], comprised of a 3D printed reconfigurable 1D acoustic metamaterial, composed of resonators with adjustable on-site frequencies. This design permits them to experimentally map the topological edge states for sound by simply adjusting piston in the resonators. This capability enables the mimicry of the variation of the pumping mechanism and on-site potential in the Aubry–André–Harper equation in a sinusoidal manner [180]. However, in this and all other experiments, ϕ is either varied spatially by making the system 2D, or temporally via external control.

The question we aim to address is: Can we design a 1D mechanical system capable of nonreciprocal behavior due to adiabatic *self* pumping? To answer this question, we present in the next section a mechanical analogue of the AAH model where adiabatic self pumping can be triggered by the excitation signal or wave packet itself in a nonreciprocal way due to the presence of a zero mode in the

system. The proposed one dimensional lattice structure has one fewer bonds than the number of sites, resulting in the presence of a zero mode which is parameterized by an angle through the system. The zero mode are states that occur at zero energy. Depending on the initial value of this angle along with other constraints, we can calculate the initial configuration of the system, where the angles are spatially changing (spatial modulation) throughout the system. This system supports finite frequency edge modes as a consequence of nonzero Chern numbers of the bulk bands. Interestingly, due to nonlinear interaction between these finite frequency edge modes and the zero mode, we find a “self-pumping effect” meaning that excitation from the end with the edge mode excites the zero mode which in turn transports (pumps) the excitation from that side to the other side of the system via a bulk mode, leading to nonreciprocity via spatiotemporal modulation.

3.2 Chiral chain and Self Pumping Effect

3.2.1 System Configuration

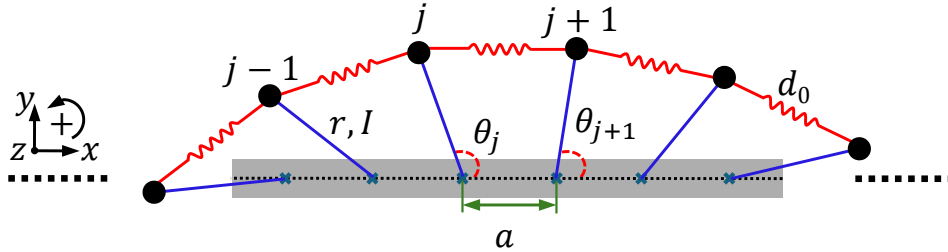


Figure 3.3: Theoretical model of one dimensional (1D) rotary chain system with free-free ends with counter clockwise as positive direction.

Inspired by the work of Kane and Lubensky in the paper titled "Topological boundary modes in isostatic lattices", and the “Kane-Lubensky chain”[184], we consider a 1D rotor chain system with $N_s + 1$ sites (number of rotors) and N_b bonds (number of springs), as shown in Fig. 3.3. The Kane-Lubensky rotor chain is a theoretical model used to explore mechanical topological phases and solitonic

excitations in systems of interconnected rotors [185]. In this model, each rotor can freely rotate and is linked to its neighbors in a way that allows for the study of topological edge states and soliton propagation within mechanical structures. The configuration space of such chains can be designed to control the movement and interaction of solutions, providing a visual and tractable toolkit for mechanism design in mechanical metamaterials and topological mechanics [185]. In our model, rotors are constrained to rotate about fixed pivot points uniformly spaced at intervals of a , depicted as small blue crosses along the central axis, which is illustrated as a black dashed line. Each rotor has a single rotational degree of freedom θ_j about the z direction, with counterclockwise rotation being positive. Each rotor is a uniform rigid structure with a mass moment of inertia denoted as I , radius of r , and connected to adjacent rotor by a red spring with unstretched length of d_0 . It is important to note that since the system lies in the x - y plane, each rotor can be assumed to have two degrees of freedom. This does not alter the fact that the number of sites exceeds the number of bonds by one, as in this case the number of bonds is considered equivalent to the number of springs with the addition of the number of fixed points. From now on and for simplicity, we will assume a single rotational degree of freedom for each rotor, rather than considering both x and y . To calculate d_0 , we start by finding the coordinates of both ends of the spring's in the form of (x, y) results in

$$\begin{aligned} P_j &= (r \cos \theta_j^{(0)}, r \sin \theta_j^{(0)}), \\ P_{j+1} &= (a + r \cos \theta_{j+1}^{(0)}, r \sin \theta_{j+1}^{(0)}), \end{aligned} \tag{3.3}$$

where P represent the end points of the spring at an initial angular position (denoted by the (0) superscript) of rotors j or $j + 1$. By using the distance formula between two point, we can calculate the rest length of the spring d_0 as

$$d_0 = \sqrt{((a + r \cos \theta_{j+1}^{(0)} - r \cos \theta_j^{(0)})^2 + ((r \sin \theta_{j+1}^{(0)}) - r \sin \theta_j^{(0)})^2)}. \tag{3.4}$$

If we substitute a given values of r , a , d_0 , and $\theta_j^{(0)}$ into Eq. 3.4, we can calculate the initial angular position of the adjacent rotor, $\theta_{j+1}^{(0)}$. By repeating this process, we can predict the initial angular configuration of the entire rotary chain. To achieve this, we must ensure a continuous relationship between $\theta_j^{(0)}$ and $\theta_{j+1}^{(0)}$

within the solution range of $[-\pi, \pi]$. Assuming constant values for a , r , d_0 , and $\theta_j^{(0)} = 0$ as an extreme case, we have two possible conditions to consider: either $a > r$ or $a < r$, as illustrated in Fig. 3.4.

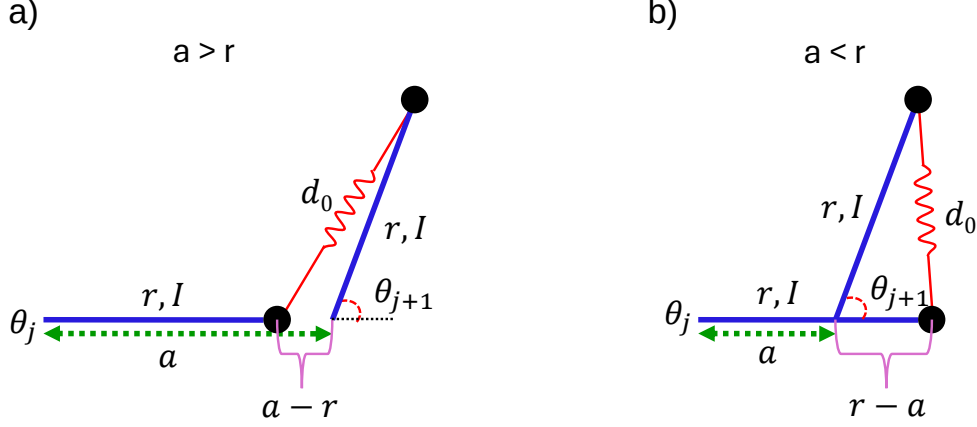


Figure 3.4: Extreme configuration at $\theta_j^{(0)} = 0$ of the 1D theoretical model consist of two rotors for given values of a , r and d_0 where: (a) $a > r$. (b) $a < r$.

These assumptions will result in a triangle in which we can apply the triangle inequality theorem, which states that the sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side. Applying this for $a > r$ and $a < r$ configurations results in the inequalities of Eqs. 3.5 and 3.6, respectively, represented below

$$(a - r) + r > d_0 \implies a > d_0, \quad (3.5a)$$

$$r + d_0 > (a - r) \implies 2r + d_0 > a, \quad (3.5b)$$

$$(a - r) + d_0 > r \implies d_0 > 2r - a, \quad (3.5c)$$

and

$$(r - a) + d_0 > r \implies d_0 > a, \quad (3.6a)$$

$$(r - a) + r > d_0 \implies 2r - a > d_0, \quad (3.6b)$$

$$d_0 + r > (r - a) \implies d_0 > -a. \quad (3.6c)$$

Simplifying the inequalities in Eqs. 3.5 and 3.6 into one inequality each results in opposite relationships such that

$$a > d_0 > 2r - a \quad (3.7)$$

and

$$a < d_0 < 2r - a. \quad (3.8)$$

Assuming values that satisfy Eqs. 3.7 and 3.8 for r , a , d_0 , and $\theta_j^{(0)}$ we can numerically plot in Fig. 3.5 the initial angular position of the adjacent rotor $\theta_{j+1}^{(0)}$ as function of $\theta_j^{(0)}$, respectively. In Fig. 3.5(a), two ranges of discontinuity (highlighted by the yellow regain) are observed in both branches of the solutions. Consequently, the case described by the condition of $a > r$ is not suitable for constructing a continuous system, due to the presence of these discontinuities. Conversely, Fig. 3.5(b) demonstrates that both branches of solutions are continuous over all possible values of $\theta_j^{(0)}$. Therefore, the case of $a < r$ does in fact satisfy a continuous relation, and Eq. 3.8 is considered a constraint for the rotary chain. It is important to note that in a 1D system, rotors can overlap due to Eq. 3.8 spatial constraints. In our experimental setup, we resolve this issue by expanding the system into a new dimension, such as the z -axis.

The solution for $\theta_{j+1}^{(0)}$ from Eq. 3.4 yields two possible continuous solutions, as depicted in Fig. 3.5(b). Since the system's behavior depends on the chain configuration, choosing an initial angular position of the chain to alternate between both branches of the solution will result in a Kane-Lubensky chain, which is analogous to the Su-Schrieffer-Heeger (SSH) system [186]. In our system, the chain configuration follows one branch of the solutions which cause the system chirality and the presence of a zero mode. Depending upon the chosen solutions it will present either a right or left-handed chiral chain for blue and red curves, respectively. When the rotor angles increase from left to right, it is referred to as a right-handed chiral chain. Similarly, if the angles decrease from left to right, it is termed as a left-handed chiral chain.

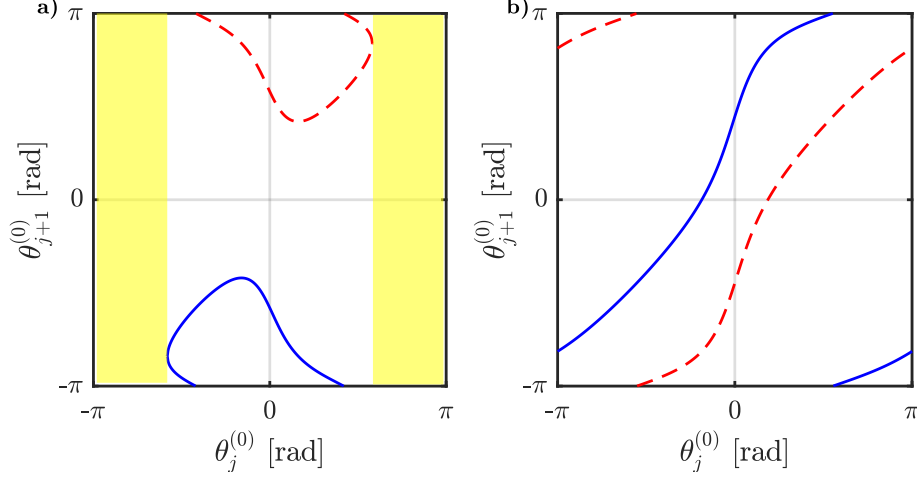


Figure 3.5: Chiral chain $\theta_{j+1}^{(0)}$ prediction for $\theta_j^{(0)} = 0.25$ rad with: (a) $a = 30$ mm and $r = d_0 = 20$ mm. (b) $a = 20$ mm and $r = d_0 = 30$. The blue solid line and red dashed line represent right and left-handed chiral chain, respectively.

The initial angular positions of the entire rotor chain can be accurately calculated by first selecting values that satisfy Eq. 3.8. Subsequently, by assuming a value for $\theta_j^{(0)}$ and utilizing Eq. 3.4, we can determine $\theta_{j+1}^{(0)}$. By repeating this procedure iteratively, the initial configuration of the chain can be established. Intuitively, if we rotate the left most rotor slowly, the other rotors also rotate in such a way that none of the springs is stretched or compressed $\delta d_0 = 0$. Thus, the mechanism of the chiral chain can be parameterized by the angle of the leftmost rotor $\theta_1^{(0)}$, highlighting an important feature of the system. To visualize the chiral chain we select parameters that satisfy the condition of Eq. 3.8 such that $\theta_1^{(0)} = 0.25$ rad, $a = 20$ mm, $r = 30$ mm, and $d_0 = 30$ mm as a result we show the right-handed chiral chain initial configuration in Fig. 3.6. The chain initial positions will be used later once dynamics excitation is added to the system. It is essential to define what is referred to as a “kink” in the chain, which constitutes a specific configuration formed as an interface between topological phases of the system. In our system, we observe three such kinks, these configurations play a pivotal role in the dynamics of the chain, as will be explained later.

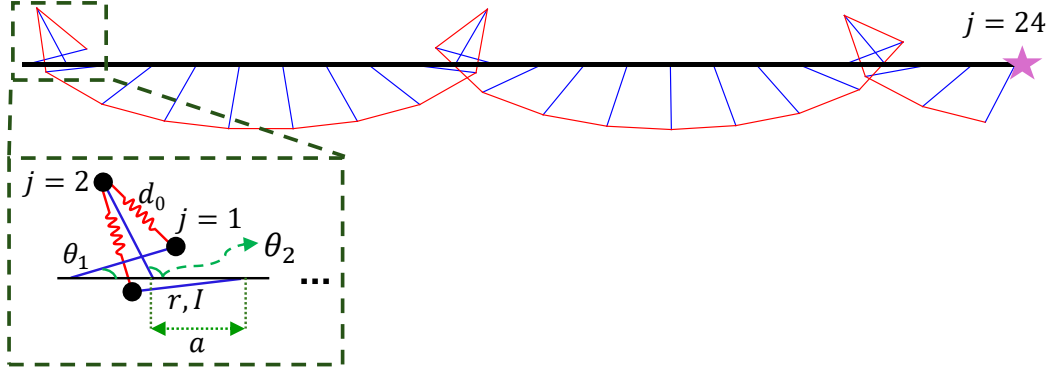


Figure 3.6: Initial angular configuration of a 24 rotary right-handed chiral chain, $\theta_1^{(0)} = 0.25$ rad, $a = 20$ mm, $r = 30$ mm, and $d_0 = 30$ mm. Boundary conditions are free at $j = 1$ and a free-torque controlled motor at $j = 24$ denoted by star for excitation.

3.2.2 Equations of Motion

To derive the equations of motion for the j th rotor within the system we assume 1D chain as presented in Fig. 3.3, and use the Lagrangian method by determining first the kinetic and potential energy of the system as follows

$$KE = \sum_{j=1}^{N_s+1} \frac{I}{2} (\dot{\theta}_j)^2, \quad (3.9)$$

$$PE = \sum_{j=1}^{N_s} \frac{k}{2} (d_j - d_0)^2, \quad (3.10)$$

where k and d_j are spring stiffness and dynamic spring stretch length, respectively. Similar to Eq. 3.4, we calculate the dynamic spring stretch d_j as

$$d_j = \sqrt{(a + r \cos \theta_{j+1} - r \cos \theta_j)^2 + r^2 (\sin \theta_{j+1} - \sin \theta_j)^2}. \quad (3.11)$$

After that, we calculate the Lagrangian as the difference of kinetic and potential energies such that

$$\mathcal{L} = KE - PE. \quad (3.12)$$

At this stage we include an energy dissipation term and external excitation in the Lagrangian equation, then evaluate the following

$$(\partial_t \partial_{\dot{\theta}_j} \mathcal{L}) + (\eta \dot{\theta}_j) - (\partial_{\theta_j} \mathcal{L}) = \tau_j \quad (3.13)$$

where η and τ_j are the system damping coefficient and rotor external torque excitation, respectively. We get the equation of motion for the j th rotors to be

$$I\ddot{\theta}_j + \eta\dot{\theta}_j = rk \left(\frac{(d_j - d_0)(r \sin(\theta_{j-1} - \theta_j) + a \sin \theta_j)}{\sqrt{(a - r \cos \theta_{j-1} + r \cos \theta_j)^2 + r^2(\sin \theta_{j-1} - \sin \theta_j)^2}} - \frac{(d_{j+1} - d_0)(r \sin(\theta_j - \theta_{j+1}) + a \sin \theta_j)}{\sqrt{(a - r \cos \theta_j + r \cos \theta_{j+1})^2 + r^2(\sin \theta_j - \sin \theta_{j+1})^2}} \right) + \tau_j \quad (3.14)$$

Once the equations of motion are formed for the whole chain, we implement them into a discrete element model [DEM] to solve for the angular time response under given excitation. We can further linearize right hand side of the undamped system to find the dynamical matrix by evaluating the second derivative of the system's potential energy (PE) twice with respect to θ_i to determine the diagonal elements of the dynamic matrix. The off-diagonal elements are also obtained by taking the second derivative of PE firstly with respect to θ_i then either θ_{i-1} or θ_{i+1} , which assumes the following form:

$$\begin{aligned} \frac{\partial^2 PE}{\partial \theta_j^2} &= k \left(\frac{d_0 A_j^2}{d_j^3} + \left(1 - \frac{d_0}{d_j}\right) \gamma_1 + \frac{d_0 B_j^2}{d_{j-1}^3} + \left(1 - \frac{d_0}{d_{j-1}}\right) \gamma_3 \right), \\ \frac{\partial^2 PE}{\partial \theta_j \partial \theta_{j+1}} &= k \left(\frac{d_0 A_j B_{j+1}}{d_j^3} + \gamma_2 \left(1 - \frac{d_0}{d_j}\right) \right), \\ \frac{\partial^2 PE}{\partial \theta_j \partial \theta_{j-1}} &= k \left(\frac{d_0 A_{j-1} B_j}{d_{j-1}^3} + \gamma_4 \left(1 - \frac{d_0}{d_{j-1}}\right) \right), \end{aligned} \quad (3.15)$$

where functions A_j , B_j and γ_i (with $i = 1, 2, 3, 4$) are given as

$$\begin{aligned} A_j &= r(r \sin(\theta_j - \theta_{j+1}) + a \sin(\theta_j)), \\ \gamma_1 &= \frac{\partial A_j}{\partial \theta_j}, \quad \gamma_2 = \frac{\partial A_j}{\partial \theta_{j+1}}, \\ B_j &= r(r \sin(\theta_j - \theta_{j-1}) - a \sin(\theta_j)), \\ \gamma_3 &= \frac{\partial B_j}{\partial \theta_j}, \quad \gamma_4 = \frac{\partial B_j}{\partial \theta_{j-1}}. \end{aligned}$$

Now, we expand Eq. 3.15 using Taylor expansion around the static equilibrium of the total angle $\theta_i = \theta_i^{(0)} + \delta\theta_j$, where $\theta_i^{(0)}$ is the equilibrium angle and $\delta\theta_j$ is the change in equilibrium angle, which simplify to

$$\begin{aligned}\left.\frac{\partial^2 PE}{\partial \theta_j^2}\right|_{\delta\theta_i=0} &= k \left(\frac{A_j^2 + B_j^2}{d_0^2} \right), \\ \left.\frac{\partial^2 PE}{\partial \theta_j \partial \theta_{j+1}}\right|_{\delta\theta_i=0} &= k \left(\frac{A_j B_{j+1}}{d_0^2} \right), \\ \left.\frac{\partial^2 PE}{\partial \theta_j \partial \theta_{j-1}}\right|_{\delta\theta_i=0} &= k \left(\frac{A_{j-1} B_j}{d_0^2} \right).\end{aligned}\tag{3.16}$$

Now, we can rewrite Eq. 3.16 in a matrix form, such that

$$\mathbf{I}\ddot{\vec{\Theta}} + \mathbf{K}\vec{\Theta} = \vec{T},\tag{3.17}$$

where $\mathbf{I}$ and $\mathbf{K}$ are inertia and stiffness matrices (or dynamic matrix), respectively, each of size $N_s \times N_s$ and $\vec{\Theta} = \{\delta\theta_i, \delta\theta_{i+1}, \dots, \delta\theta_{N_s}\}$ and $\vec{T}$ are the angular displacement and torque column vectors, respectively each of size $N_s \times 1$. We next numerically evaluate Eqs. 3.17 to get the rotors' frequency spectrum as a function of the first rotor initial angle $\theta_1^{(0)}$ as discussed in the next section.

3.2.3 Spectrum and Topology of Chiral Chain

Assuming no energy dissipation or external excitation in the system, we set an initial value for $\theta_1^{(0)}$ then we determine the chiral chain initial configuration by using Eq. 3.4; next, we solve for the eigenvalues of Eq. 3.17. We repeat these steps for different initial values of $\theta_1^{(0)}$ until the entire range $[0, 2\pi]$ is covered, with each solution of eigenvalue represented as a vertical section in Fig. 3.7 resembling the pattern shown Fig. 3.2. Plotting a vertical line in Fig. 3.7, indicates the initial angular position of the first rotor (e.g. 0.25 rad for blue or 4.4 rad for blue and red dashed lines, respectively). The intersection of the curves with that vertical line represents the expected frequency of the edge or bulk mode at that initial position.

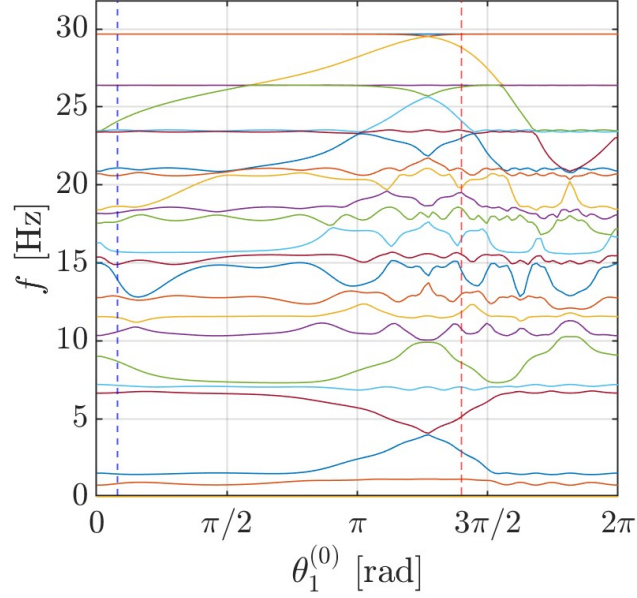


Figure 3.7: Rotary chain spectrum (eigenfrequencies) for 24 rotor, $a = 20$ mm, $r = 30$ mm, $d_0 = 30$ mm, $k = 269$ N/m, and $I = 2.4596 \times 10^{-5}$ kg.m². The vertical dashed lines indicate the initial angular position of the first rotor location at 0.25 rad and 4.4 rad as blue and red, respectively.

3.2.4 Mechanism of Edge-Bulk Mode Transition

In this section, we explain the cycle of edge to bulk mode transition by examining the spectrum of the chiral chain system depicted previously. As a first step, we start by showing in Fig. 3.8(a) a zoomed-in region for the spectrum depicted in Fig. 3.7 for the range of 0 to 7 Hz and marked with two points of interest of $\theta_1^{(0)}$ at 1.4 and 4.4 rad as blue and red marks, respectively. Next, we inspect the respective modeshape for each of the point of interest as shown in Fig. 3.8(b) and (c) to check where the edge is localized or if it is extended. In the case of $\theta_1^{(0)} = 4.4$ rad, it is clear that the mode shape is localized at rotor 24, while at $\theta_1^{(0)} = 1.4$ rad it is distributed along the chain.

Next, in Fig. 3.9(a) we show a zoomed-in region of Fig. 3.7 for the range of 0 to 7 Hz, and consider a chiral chain with initial angular position of 4.4 rad, as

marked by the red dashed line. It is important to note that, although we initially selected $\theta_1^{(0)} = 4.4$ rad to explore the edge to bulk mode transition, this mechanism can also be observed for other values of $\theta_1^{(0)}$, as shown in a later section. Once

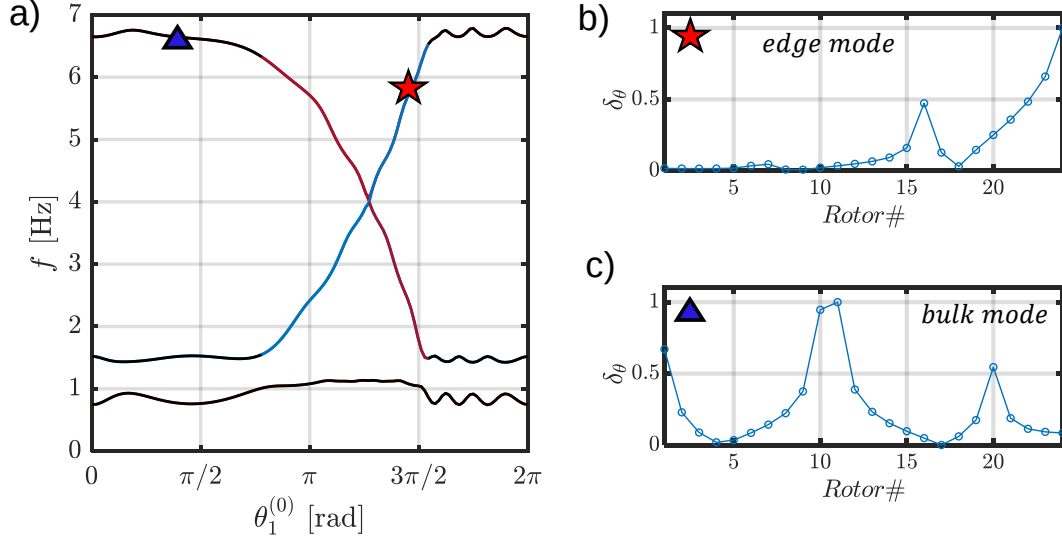


Figure 3.8: (a) Enlarged section of the rotary chain spectrum from Fig. 3.7 for the frequency range of 0 to 7 Hz. Showing blue triangle and red star that corresponds to $\theta_1^{(0)}$ at 1.4 and 4.4 rad, respectively. (b) and (c) Modeshapes of the chiral chain for mode number 22 at the marked points in (a).

the initial location of the first rotor is selected as $\theta_1^{(0)} = 4.4$ rad, we can identify two edge modes located at two frequencies 2.87 Hz and 5.15 Hz. In Fig. 3.9(b), we plot the normalized modeshape associated with each frequency. For $f = 2.87$ Hz, the edge mode is localized at rotor 24, while at $f = 5.15$ Hz, it is localized at rotor 1 as identified by the location of the maximum particle amplitude. Since our excitation is located at rotor 24, we will select the excitation frequency to be $f = 2.87$ Hz, as indicated by y in Fig. 3.9(a), to initiate the edge to bulk mode transition. Once the excitation is applied at the frequency indicated by y , some of the energy input will excite the zero mode due to the system's nonlinearity, leading to a change in the angle θ_1 (as well as the subsequent rotors in the chain). As θ_1 changes in one direction, the frequency of the edge mode also goes to the lower

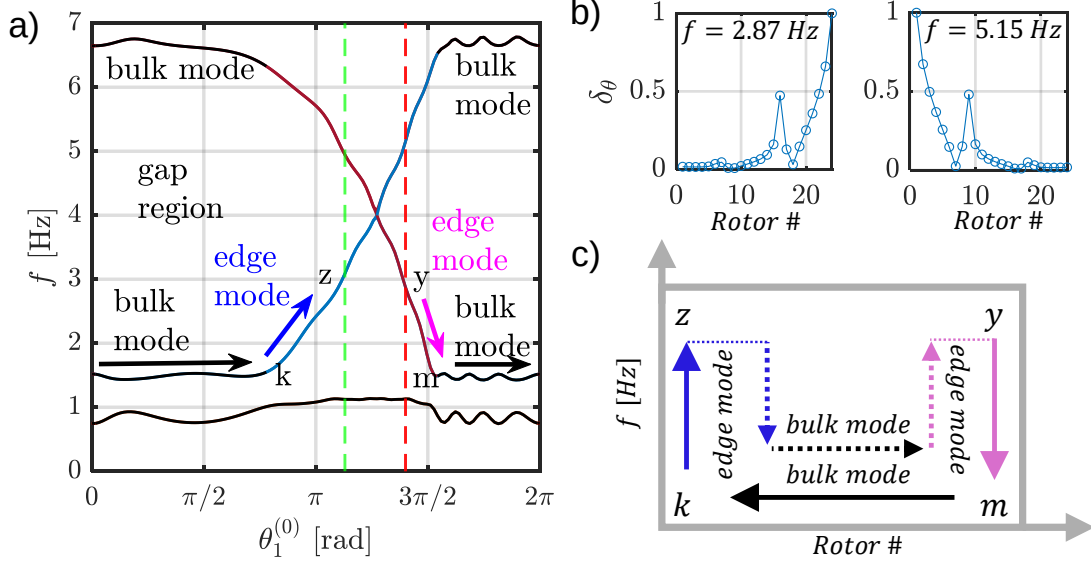


Figure 3.9: (a) Enlarged section of the rotary chain spectrum from Fig. 3.7 for the frequency range of 0 to 7 Hz. Black arrows indicate the bulk modes, while blue and magenta arrows denote edge modes. The green and red vertical dashed lines mark the positions of $\theta_1^{(0)}$ at 3.55 and 4.4, respectively. (b) Modeshapes of the chiral chain at initial position of 4.4 rad for tow frequencies of 2.87 Hz and 5.15 Hz. (c) Schematic of the edge to bulk mode transition in the rotary chain. Letters a to d indicate the locations of the wave packet, with c and d representing the first rotor, and a and b representing the last rotor.

frequency at the new angle, causing the wave packet to decrease its frequency. This process occurs continuously over time as indicated by the magenta arrow until the frequency of the wave packet reaches the bulk mode indicated by point m . As the wave packet reaches the bulk mode, the system further rotates due to the chiral chain mechanism, continuing to change the angle of the first rotor and the state of the rest of the chain. This change causes the wave packet to travel through the chiral chain via the bulk mode (indicated by the black arrow) until first rotor angle θ_1 reaches k . Similarly, the wave packet then starts to increase its frequency until it reaches point z via the edge mode as indicated by the blue arrow.

Since the transition from edge modes to bulk modes is influenced by the

selected excitation frequency and energy input to the system, which is dictated mainly by the amplitude of the excitation [187], it is crucial to select an appropriate amplitude. An appropriate amplitude allows the system to oscillate back and forth due via the zero mode under the assumption of no energy dissipation, causing the wave packet to cycle between point y and point z , as discussed earlier. If the energy input into the system is high, this cycle will be disrupted since the wave packet will follow the edge mode beyond point z in Fig. 3.9(a), resulting in reaching the upper bulk mode. Figure 3.9(c) summarises this cycle schematically.

3.2.5 Dynamical Simulation without Damping

In this section, we model the rotary chain using a DEM, numerically integrating Eqs. 3.14 for long and short systems consisting of 185 and 24 rotors, respectively, with free boundaries on both sides. In both systems, we specify the angular position of the first rotor in order to calculate the rotary chain initial angular position based on Eq. 3.4. A torque excitation was applied at the last rotor free end (*i.e* θ_{185} or θ_{24}), taking the form of a Gaussian modulated sine pulse (GSP) characterized by torque amplitude F , forcing frequency f , a shift of the Gaussian's center of t_0 , and a standard deviation of σ , which takes the form of

$$F \sin(2\pi ft) e^{-0.5((t-t_0)/\sigma)^2}. \quad (3.18)$$

The resulting angular response of the first and last rotors are plotted and compared for several different cases, as seen in the next few sections. As an additional note, the output time step was defined as 10 ms. MATLAB's *ode45* was used as the integrator with relative and absolute error tolerance both set to 10^{-8} . Energy conservation was confirmed, by checking that the total energy to be constant to within a degree of certainty of $\pm 1.57 \times 10^{-10}$ J.

Case 1: Long chain with edge mode excitation at rotor 185 with $\theta_1 = 4.8$ rad

The frequency spectrum (eigenfrequencies) reported in Fig. 3.10, was calculated for a long chain consisting of 185 rotors, $a = 50$ mm, $r = 85$ mm, $d_0 = 53.55$

mm, $k = 269$ N/m, and $I = 2.4596 \times 10^{-5}$ kg.m². Two edge mode were identified as red and blue curves, which correspond to left and right edge modes, respectively. The first rotor was initially positioned at 4.8 rad, and the GSP excitation in the form of Eq. 3.18 was evaluated at $F = 97.176$ μ N.m, $\sigma = 0.7955$ s, $t_0 = 7.11$ s, and forcing frequency of $f = 1.36$ Hz (selected based on the intersection of the vertical red dashed line in Fig. 3.10 with the right edge mode). In Fig. 3.11(a),

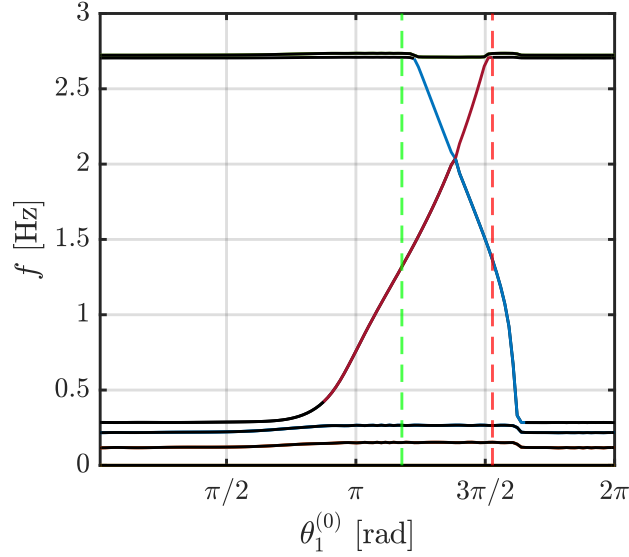


Figure 3.10: Enlarged section of the rotary chain spectrum for the frequency range of 0 to 3 Hz showing the first 5 modes. The green and red vertical dashed lines mark the positions of $\theta_1^{(0)}$ at 3.70 rad and 4.80 rad, respectively.

the initial configuration of the long rotary chain is depicted, corresponding to the parameters presented in Fig. 3.10. We note in Fig. 3.11(a) the existence of loop-like structures in the chain configuration which what we refer to as “loops”. The loop configuration usually characterized by a sequence of four consecutive rotors, comprising three with positive angles and one with a negative angle. In the following, we will show these coincide with important points in the spatiotemporal chain response. For this case study, the geometric parameters have been carefully chosen to increase the number of rotors between any two loops, which allows for a clearer visualization of wave motion prior to any interaction with the loops. Addition-

ally, the figure includes two magnified segments that highlight the initial angular positions of the first few rotors in the chain, as well as the loop configuration.

In Fig. 3.11(b) and (c), we plot the time response of the angular position of the first and last rotor in the chain, respectively. Upon closer examination of Fig. 3.11(c), we can observe the wave packet caused by the applied GSP centered at 7.11 s being injected into the system. As time progress, the wave packet exits the time response of the last rotor, with minimum amplitude at around 170 s. Simultaneously, by looking at the first rotor time response in Fig. 3.11(b), we note that there is no wave packet at 7.11 s, as the wave is located at the last rotor. However, as time advances, the wave packet becomes visible in the response of the first rotor at time between 140 to 175 s.

The edge mode transportation between first and last rotor can be observed more clearly in Fig. 3.12(a). In this panel we plot the spatiotemporal response of the normalised angular velocity with respect to its maximum value. The wave packet travels through the system in the direction indicated by the white arrow (though the spatiotemporal trajectory is not linear). To understand the transition of the wave, in Fig. 3.12(b) we plot a zoomed-in region from 0 to 10 s for all rotors with a truncated color bar set to $\pm 5 \times 10^{-3}$ for enhanced visualization. In panel (b), we can clearly see the wave propagation divided into sections, correlated with the loop positions. It is observed that the wave packet propagates to the next section after a delay when interacting with the loops. Additionally, we zoom further into the time range of 5.5 to 6 s for rotors from 135 to 185 in Fig. 3.12(c). Following the red arrow, the wave packet's propagation allows us to estimate a linear wave speed of 12.33 m/s.

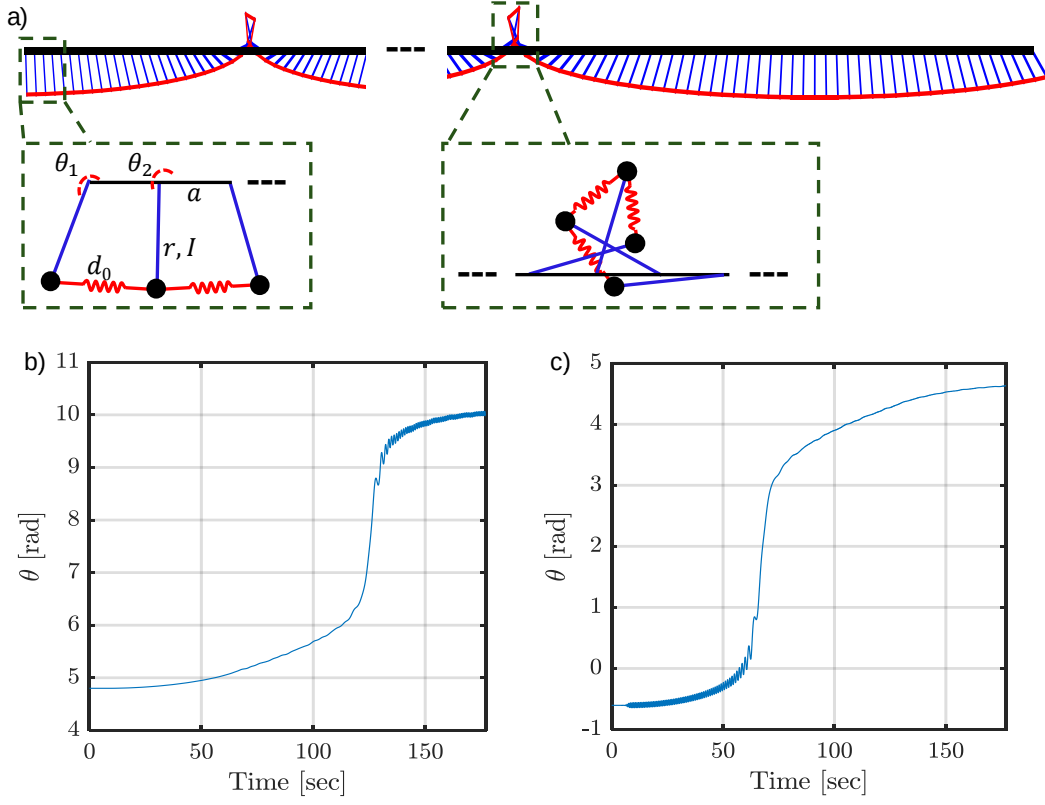


Figure 3.11: (a) Rotary chain initial configuration for 185 rotors, $\theta_1^{(0)} = 4.8$ rad, $a = 50$ mm, $r = 85$ mm, $d_0 = 53.55$ mm, $k = 269$ N/m, and $I = 2.4596 \times 10^{-5}$ kg.m². Two detailed segments depicting the initial section of the chain and the configuration of the loops. Time domain response for torque excitation of $F = 97.176$ μ N.m, $\sigma = 0.7955$ s, $t_0 = 7.11$ s, and forcing frequency of $f = 1.36$ Hz. for: (b) Rotor 1, and (c) Rotor 185.

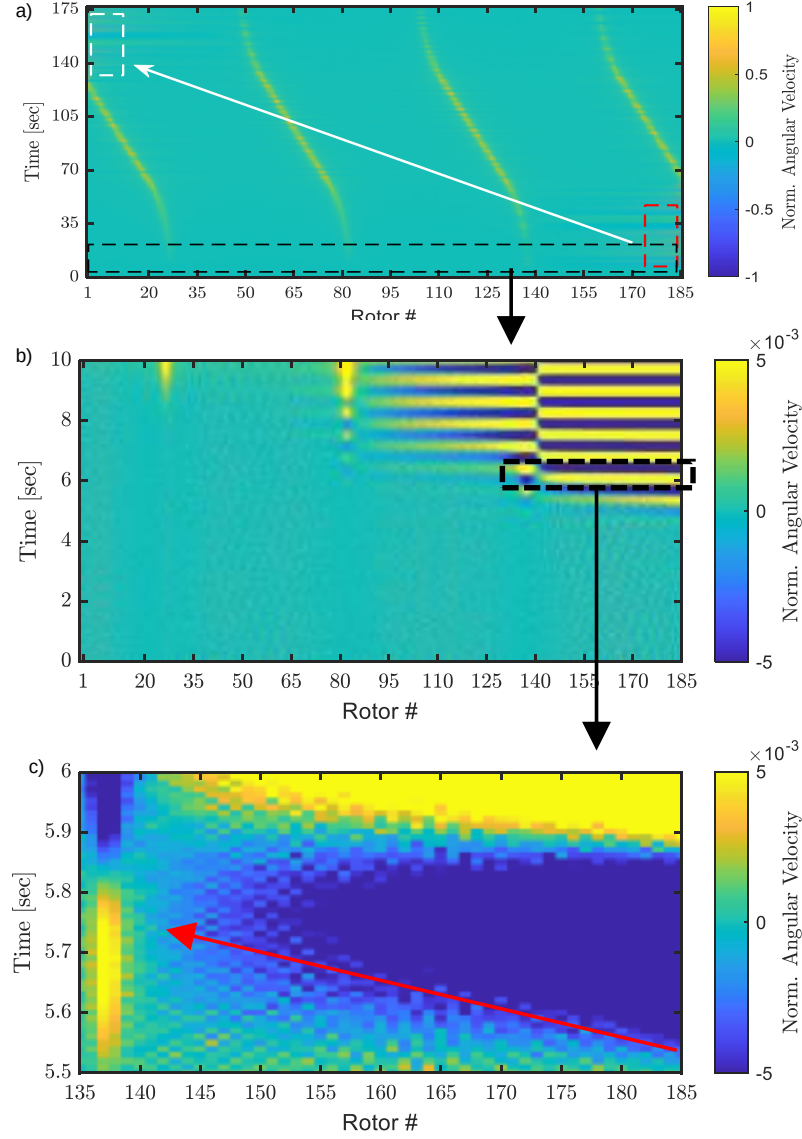


Figure 3.12: (a) Spatiotemporal response of the chiral chain angular velocity normalized by maximum angular velocity amplitude. Red and white rectangles highlight the difference in angular velocity response for rotor 185 and 1, respectively. The white arrow show the direction where the wave packet is traveling through the system. (b) and (c) Show zoom-in truncated color bar for spatiotemporal response.

To further clarify wave packet and loop interaction, we show Fig. 3.13. In

Fig. 3.13(b), the normalized angular velocity at time 20 s, as denoted by the blue horizontal dotted line, is plotted and compared with the linearized chiral chain modeshape at the corresponding angular position, as depicted in Fig. 3.13(a). We note the loop position is correlated with the zero mode amplitude in the modeshapes. We suggest that at these points there is effectively a stiffness defect in the rotary chain due to its configuration. As an observation, these points largely prevent wave crossing, resulting in the waves reflecting back and forth temporarily, as illustrated in Fig. 3.12(c). For completeness, we have also compared the linear modeshape with the normalized angular velocity at time 172 s, indicated by the blue dashed line, which shows a strong correlation between the zero amplitude point in the mode shape and the loop position. This observation suggests that, as time progresses, the system's velocity transitions from the right edge mode to the left edge mode, thereby shifting the location of the zero velocity points and allowing the wave to propagate after a delay. A more in depth investigation of this transition should be conducted in subsequent analyses.

In addition to analyzing the spatiotemporal response, we monitored the motion of the loops over time as indicated by the red dots in Fig. 3.13(b), observing that all loops initially have zero velocity at the start of the simulation. As time progresses, they accelerate until approximately 60 s, in which their motion shifts to constant speed, continuing until the wave packet reaches the opposite end and the loops decelerate once again. For visual illustration, Fig. 3.13(c) shows the full rotary chain configuration at two times, 50 and 120 s, respectively, each indicated by horizontal black dashed and dotted line in the spatiotemporal diagram. We also note that the zero mode does not exhibit the same angular velocity across all rotors. Specifically, the angular velocity of the zero mode in the loop configuration is significantly higher than that of the middle rotors situated between two loops. This discrepancy arises because the loop consists of three or four rotors that must complete a full rotation, whereas the middle rotors only rotate within a smaller range. As the zero mode rotates the chain, the loop configuration begins to shift over time, resulting in a corresponding shift in the region high angular velocity, which can be observed as the yellow pattern in Fig. 3.12(a).

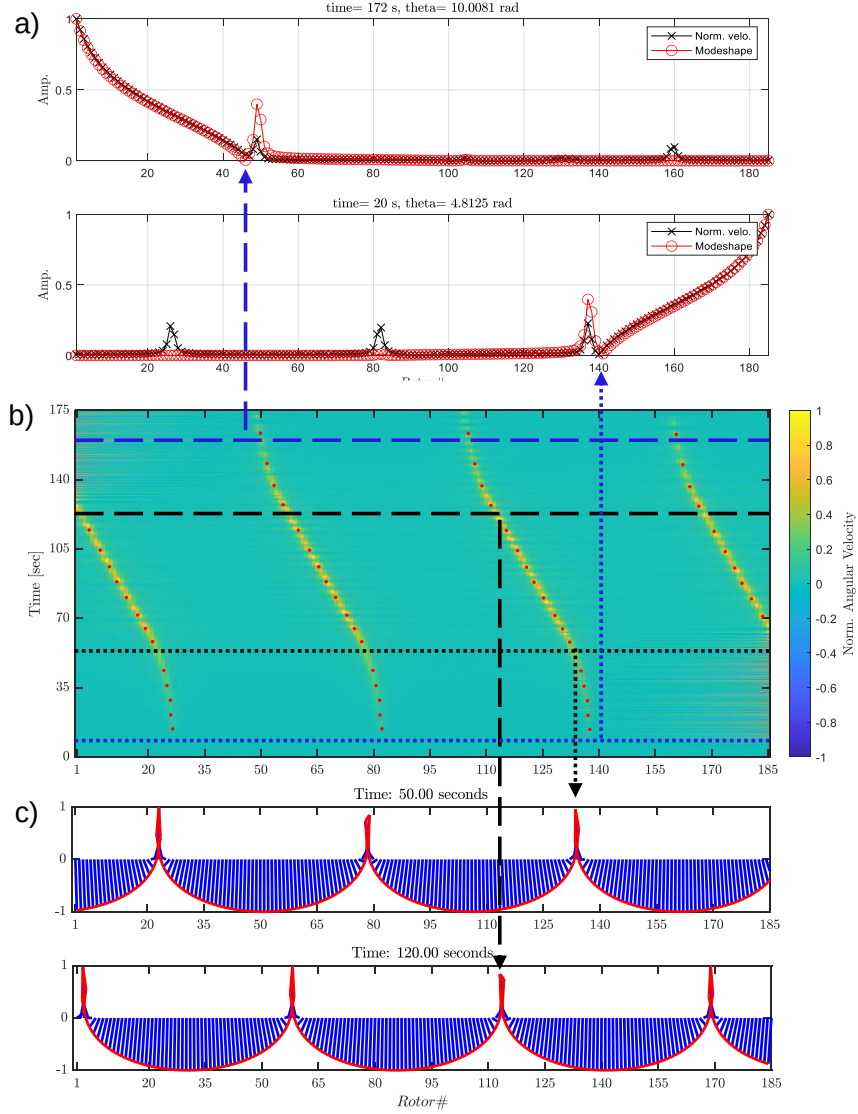


Figure 3.13: (a) Modeshapes comparison with normalized angular velocity at two different time. (b) Spatiotemporal response of the chiral chain angular velocity normalized by maximum angular velocity amplitude. The red dots are tracing the loop motion over time. Horizontal blue dashed and dotted lines are time sections at 172 and 20 s, respectively, while, black dashed and dotted lines are for time at 120 and 50 s, respectively. (c) Show chain configuration highlighting the location of a loop at two time.

To further clarify the system's dynamics, we perform the FFT of the angu-

lar velocity, as represented in Fig. 3.14. This figure shows the FFT at three time ranges for the entire rotary chain. On top of each FFT, we plot a black dashed line that represents the expected frequency of edge and bulk modes, based on the frequency spectrum shown in Fig. 3.10. Upon closer examination of Fig. 3.14(a), it is observed that when the system is initially excited at a forcing frequency of 1.36 Hz within the time range of $1 \sim 60$ s, this excitation manifests as a high color intensity in the frequency domain at the excited rotor. This observation aligns with the expectation, given that the right edge mode, which is localized at rotor 185, is being excited, as indicated by the top eigenmode. As the excitation is applied, part of the energy is transferred due to the system's nonlinearity into the zero mode, as indicated by the white arrow and the red dashed squares. When the energy reaches the zero mode, the chain begins to rotate, causing the wave packet to lower its frequency until most of the energy is accumulated at the zero and bulk modes, as shown in Fig. 3.14(b). In this figure, which displays the FFT for the time range of $60 \sim 120$ s, we observe that the energy is primarily transferred into the zero and bulk modes, highlighted by the red dashed squares, which induces the rotation of the system. As the first rotor reaches a critical angular position where the eigenmode exhibits a left localized edge mode, the energy gradually transfers back to the edge mode, causing the zero and bulk modes to lose energy and decelerating the rotational motion of the chain. This phenomenon is evident from the concentrated color intensity in the FFT shown in Fig. 3.14(c) for the time range of $130 \sim 170$ s. It is notable that although most of the energy returns to the edge mode, a portion of the energy remains in the zero and bulk modes, initiating the cycle in the reverse direction until it reaches the right edge mode. This follows the transition cycle from edge to bulk mode as discussed in Sec. 3.2.4.

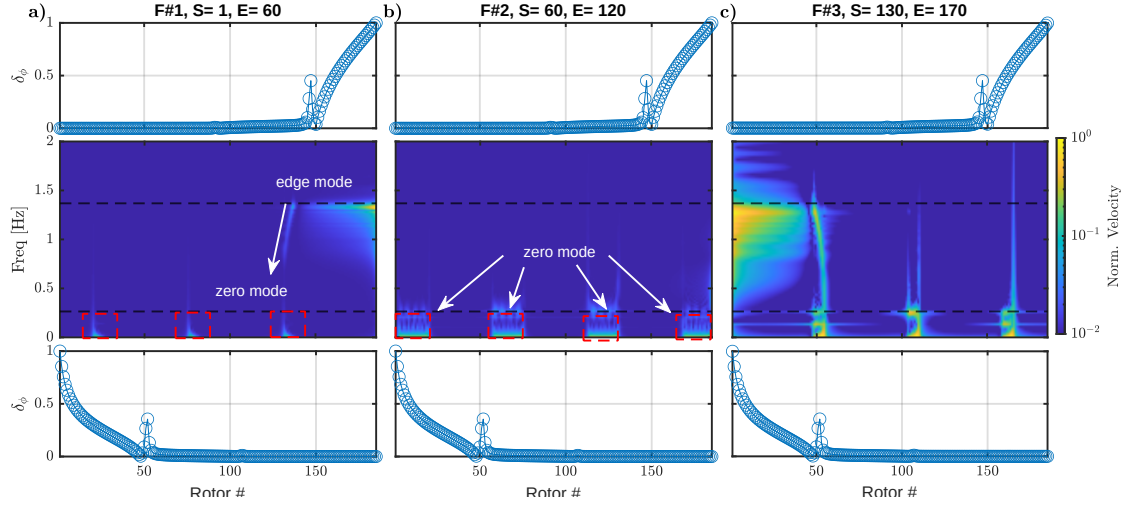


Figure 3.14: Frequency domain response of Fig. 3.12, with chain eigenmode on top and bottom at positions of $\theta_1 = 4.8$ rad and $\theta_1 = 9.98$ rad, respectively for the following time ranges: (a) 1 to 60 s. (b) 60 to 120 s. (c) 130 to 170 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the red and green lines with the edge and bulk modes in Fig. 3.10.

Another interesting way to study the system, is by analyzing the behavior of the springs strain calculated using

$$\epsilon_i = \frac{(d_i - d_0)}{d_0} * 100. \quad (3.19)$$

Figure 3.15(a) displays the spring strain across the entire system and throughout the simulation period, highlighting fluctuations in sections between the last rotor and the nearest loop as the wave packet excites the system. As the system rotates and the edge to bulk mode transition occurs, the wave packet propagates gradually, reaching the first rotor. For enhanced visibility, the color bar is truncated in Fig 3.15(b), which also offers a zoomed-in view from 0 to 10 s, showcasing wave propagation through sections defined by the system's zero point. The nonconstant wave speed within these sections is distinctly visible in the zoomed-in segment of Fig 3.15(b) illustrated by the dotted red curve.

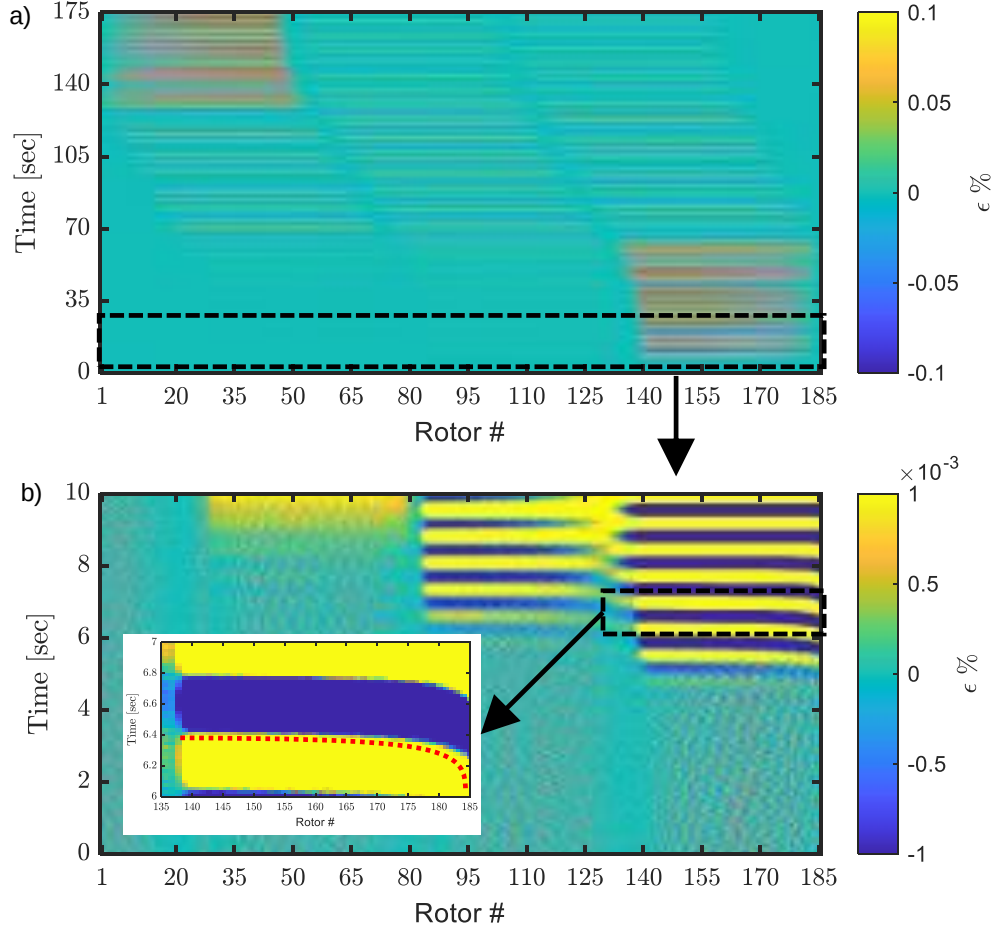


Figure 3.15: (a) Spring strain over the course of the simulation. (b) Color bar truncated time zoom of the region outlined by the dashed black rectangle in (a). Panel (b) also includes a detailed zoom on both time and rotor index with dotted red curve showing the nonlinear wave speed.

Case 2: Long chain with bulk mode excitation at rotor 185 with $\theta_1 = 4.8$ rad

To enhance our understanding of the system, we extend the analysis by investigating the behavior of the same rotary chain as in first case, with excitation frequency to be in bulk mode rather than edge mode. The forcing was applied at rotor 185 with $F = 97.176 \mu\text{N.m}$, $\sigma = 3.081$ s, $t_0 = 7.11$ s, and forcing frequency of $f = 0.265$ Hz. In Fig .3.16(a) and (b) illustrates the time response of the first and

last rotors, respectively. While wave propagation is observed from the last rotor to the first, the range of motion for the first rotor is limited to ± 0.01 rad, whereas in first case, the range was ± 5 rad. We also show in Fig .3.16(c) the linearized modeshapes of the rotary chain at the bulk mode for the initial position of the chain. This mode is clearly extended throughout the rotors, and it will play an important role in the behavior of the system, as we will see later.

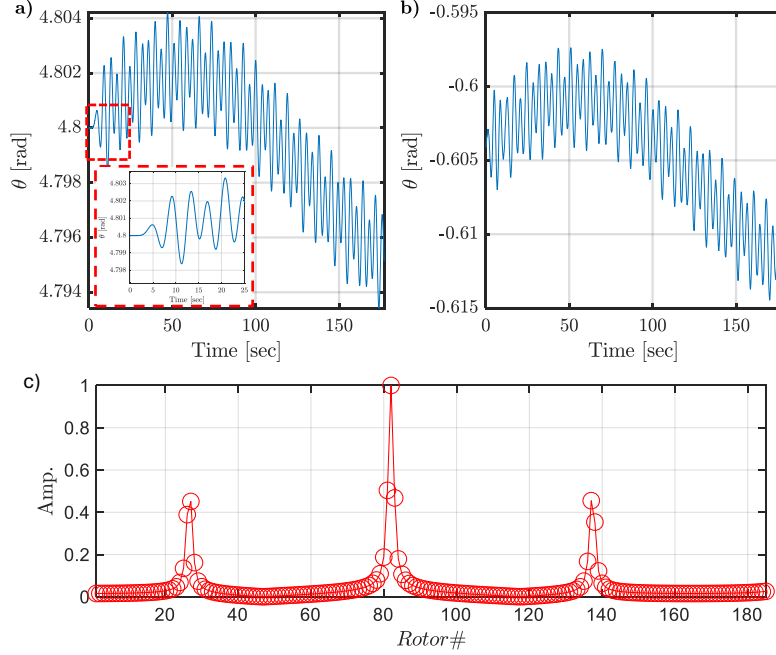


Figure 3.16: Spatiotemporal of the normalized angular velocity for all rotors (a) Full time range 0 to 175 s, and (b) Time range 0 to 10 s. (c) Is the linearized modeshape at the bulk mode for first rotor initial angular position of $\theta_1^{(0)} = 4.8$ rad.

To provide a clearer visualization of the system, Figure 3.17(a) presents the spatiotemporal response of the normalized angular velocity for all rotors. The motion of the loops in the system, indicated by vertical black dots, has also been tracked. It is evident that the system's zero velocity points remain in their initial positions, supporting the expectation that no edge to bulk mode transition occurs. Figure 3.17(b) offers a zoomed-in view of the spatiotemporal response from 0 to 10

s, with a truncated color bar to enhance visual clarity. This zoomed-in view clearly shows wave propagation, and the group velocity is approximately linear, estimated at 2.395 m/s, which suggests that the wave would require 3.8 s to traverse the entire chain while in the first case, the wave took about 80 s to reach the first rotor.

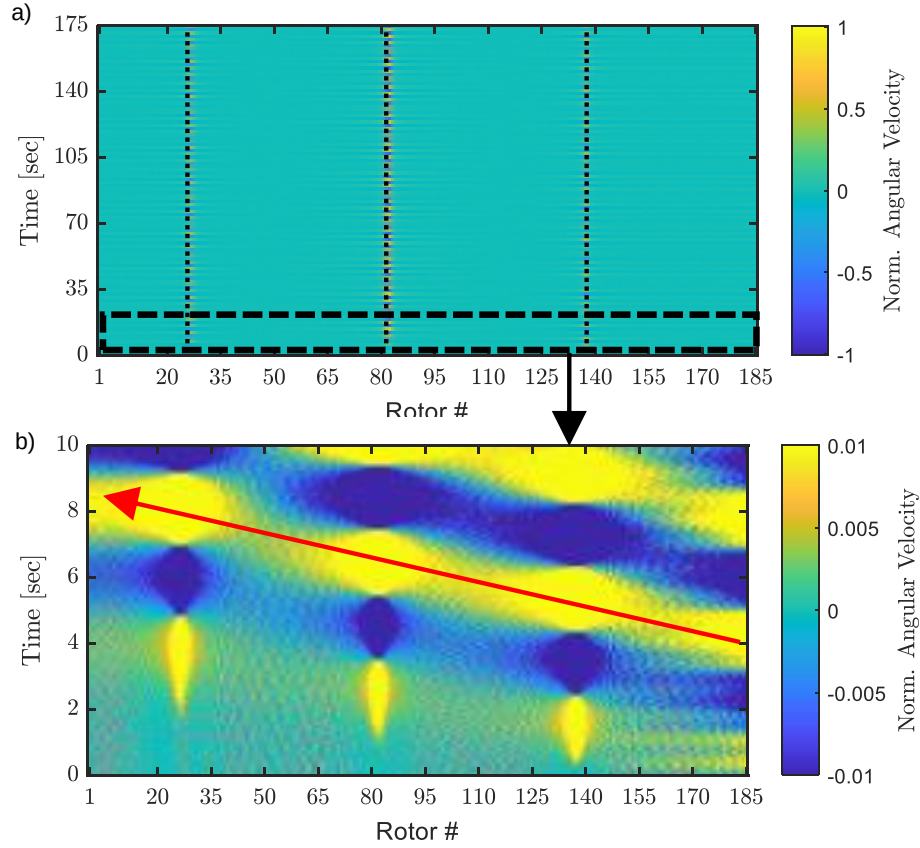


Figure 3.17: Time domain response for torque excitation of $F = 97.176 \mu\text{N.m}$, $\sigma = 3.081 \text{ s}$, $t_0 = 7.11 \text{ s}$, and forcing frequency of $f = 0.265 \text{ Hz}$. for: (a) Rotor 1, and (b) Rotor 185. The vertical black dots are the race of the loops motions over time.

Next, we analyzed the rotors motion in the frequency domain for three time ranges, as presented in Fig. 3.18. When the system is initially excited at a forcing frequency of 0.265 Hz, the energy is concentrated at the three peaks of the extended bulk modeshape, as indicated by the red dashed squares in Fig. 3.18(a). Since the

modeshapes are extended throughout the chain, the system does not require any rotational motion induced by the zero mode to propagate; thus, no bulk mode excitation is observed in the FFT plot, and the energy is fully concentrated in the bulk mode, as shown in Figs. 3.18(b) and (c).

The key distinction between these two cases studies lies in the frequency of the wave excitation injected into the system. In the first case, the wave can only propagate through the chain via activation of the zero mode, which is slower and dependent on the system's nonlinearity to rotate the chain, leading to a longer traversal time from one side to the other. In contrast, when the system is excited at a forcing frequency that aligns with the extended bulk mode, the wave travels rapidly and efficiently across the system in a linear fashion, without involvement of the zero mode. This highlights the crucial role of mode selection, or more specifically, the choice of excitation forcing frequency, in determining both the speed and the nature of wave propagation mechanisms.

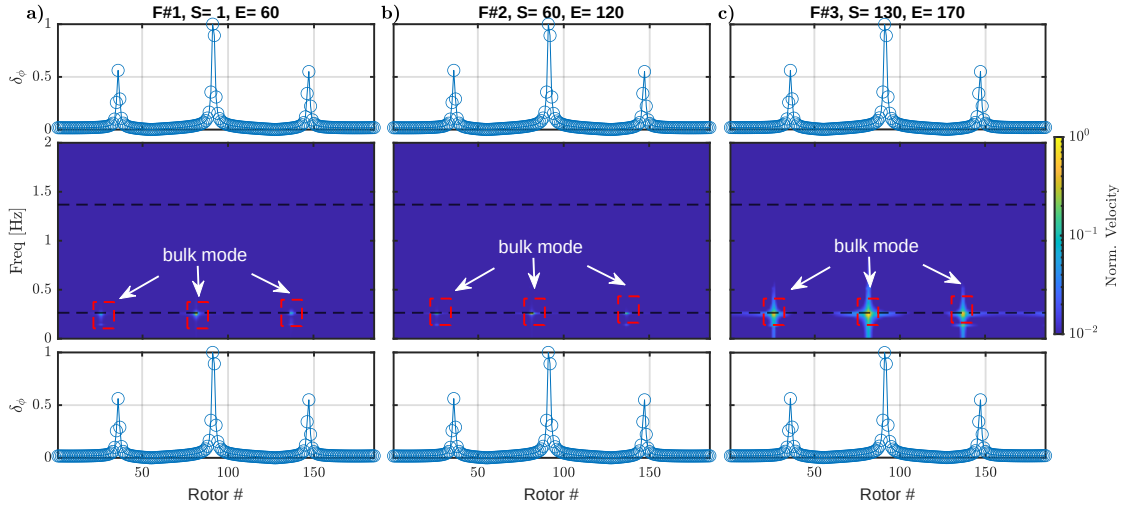


Figure 3.18: Frequency domain response of Fig. 3.17, with chain eigenmode on top and bottom at positions of $\theta_1 = 4.8$ rad and $\theta_1 = 4.81$ rad, respectively for the following time ranges: (a) 1 to 60 s. (b) 60 to 120 s. (c) 130 to 170 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the red and green lines with the edge and bulk modes in Fig. 3.10.

Case 3: Short chain with edge mode excitation at rotor 24 with $\theta_1 = 4.4$ rad

In this and the subsequent case studies, the number of rotors is reduced, and the geometric parameters are intentionally adjusted from those in the first two cases to enhance the feasibility of future experimental implementation. A key distinction in these modifications is the increased loop density, leading to a higher number of loops per unit distance. In Fig. 3.19, we simulate the response of the rotary chain system with 24 rotors and initial position of the first rotor as $\theta_1^{(0)} = 4.4$ rad. The initial configuration of the rotary chiral chain was calculated based on $a = 20$ mm, $r = 30$ mm, and $d_0 = 30$ mm, as seen in Fig. 3.6. The spring stiffness and rotors moment of inertia are $k = 269$ N/m and $I = 2.4596 \times 10^{-5}$ kg.m², respectively. A GSP in the form of Eq. 3.18 was applied on rotor 24 centered at $t_0 = 20$ s with $F = 24.2$ μ N.m, $\sigma = 3.19$ s, and forcing frequency of $f = 2.87$ Hz (selected based on the intersection of the vertical red dashed line in Fig. 3.7 with the edge mode).

In Fig. 3.19(a) through (c), we plot the time response of the angular position of the first, middle and last rotor in the chain, respectively, with a zoomed-in section below each plot for the indicated dashed black rectangle. On closer analysis of Fig. 3.19(c), we can observe the wave packet caused by the applied GSP centered at 20 s. Over time, the wave packet gradually diminishes, reaching minimal amplitude around 60 seconds. Simultaneously, by looking at the first rotor response in Fig. 3.19(a) and (c), we note that there is no wave packet at 20 s, as the wave is located at the last rotor. However, as time advances, the wave packet becomes visible in the response of the middle rotor between approximately 30 and 40 seconds as in Fig. 3.19(b). Later, in Fig. 3.19(a) at time between 45 to 65 seconds, the wave start to appear in the response of the first rotor. The edge mode transportation between first and last rotor can be observed more clearly in Fig. 3.19(d). In this panel we plot the spatiotemporal response of the normalised angular velocity with respect to its maximum value. Again, we see the wave packet at time 20 second located at the last rotor (rotor 24, red dashed box). As the wave packet travels through the system, the direction can be indicated by

the white arrow. Once it reaches the first rotor, we can see the wave in the range bounded by the white rectangle. As explained before, as the zero mode rotates the chain, the loop configuration begins to shift over time, resulting in a corresponding shift in the high angular speed, which can be observed as the yellow pattern in Fig. 3.19(d). When the wave packet reaches the opposite end of the rotary chain, the loop configuration reverses its direction, shifting backward as the edge to bulk mode returns to its original starting point. This reversal is depicted by the blue pattern in Fig. 3.19(d).

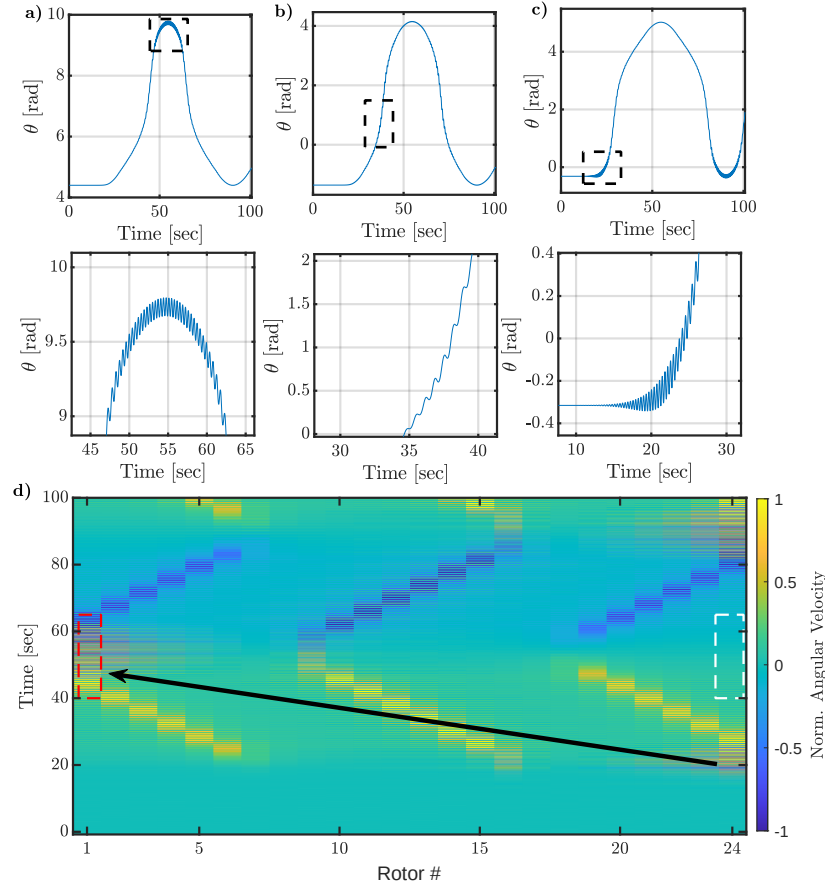


Figure 3.19: Time domain response and zoom-in region for torque excitation of $F = 24.2 \mu\text{N.m}$, $f = 2.87 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$. for: (a) Time response of rotor 1. (b) Time response of rotor 24. (c) Spatiotemporal response of the chiral chain angular velocity normalized by maximum angular velocity amplitude. The black arrow show the direction where the wave packet is traveling through the system.

For a clear visual of the transition between the edge to bulk mode, we plot the two dimensional Fourier transform of the rotary chain normalized velocity to its maximum value as in Fig. 3.20 for three time range intervals. Initially for the time range of $20 \sim 30 \text{ s}$, the edge mode is localized at rotor 24 on the right side at frequency of 2.87 Hz as shown in Fig. 3.20(a). For the second time interval $35 \sim 45$, the wave has transmitted to the bulk mode as in Fig. 3.20(b) with a frequency of 1.5 Hz , next the wave reaches the left side (rotor 1) via edge mode as in Fig. 3.20(c)

at frequency of 2.87 Hz. On top and bottom of the frequency domain plots are the eigenmodes of rotor 1 at initial and maximum response positions (e.g. $\theta_1 = 9.8$ rad), respectively. For simplicity, if we take a closer look at the color intensity of the frequency domain in (a), a clear match is noted with the eigenmodes on top. Similarly, a match is observed between (c) and the eigenmodes on the bottom. Regarding (b), since the wave packet is in the transition phase between the right to left edge modes, the frequency domain exhibits a mixture of the top and bottom eigenmodes.

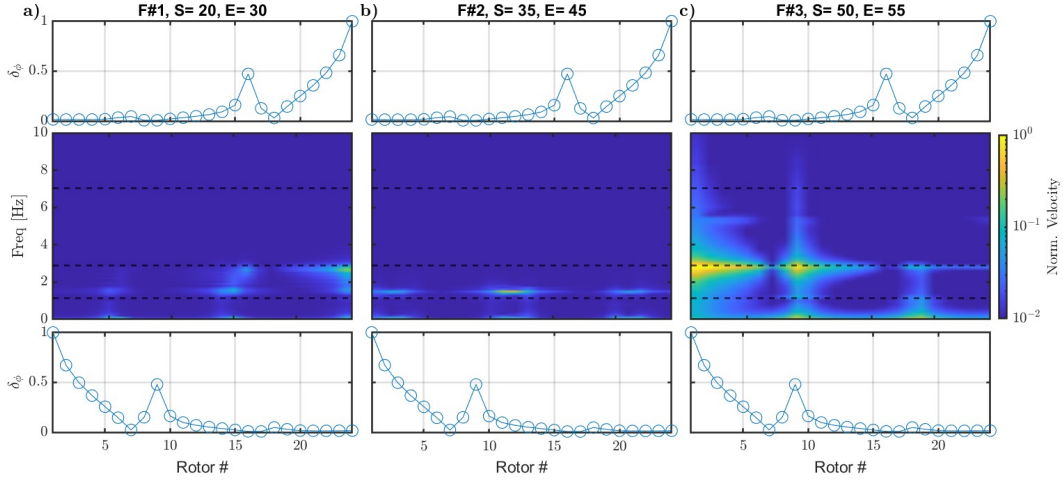


Figure 3.20: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 2.87$ Hz, and $\sigma = 3.19$ s, with eigenmode of first rotor on top and bottom at positions of $\theta_1 = 4.4$ rad and $\theta_1 = 9.8$ rad, respectively, for the following time ranges: (a) 20 to 30 s. (b) 35 to 45 s. (c) 50 to 55 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the red line with the curves in Fig. 3.7.

Case 4: Short chain with edge mode excitation at rotor 1 with $\theta_1 = 4.4$ rad

Here, we investigate the effect of swapping the excitation to the first rotor on the edge mode. For the same rotary chain system as before, we applied the same GSP to rotor 1, with an initial position of $\theta_1 = 4.4$ rad. In Fig. 3.21, we show the

simulated time response of the first and last rotor angler positions. Since the edge mode is localized at rotor 24, exciting the system at the opposite end is expected to prevent the wave packet from propagating between rotors. In both time responses, we can see the rotational amount is minimizes by at least two order of magnitude compared to the previous case. In Fig. 3.22 we plot the frequency response of the rotary chain for three time intervals to show if there is any transportation of the edge mode from the right to left side. Similarly, on top and bottom of the frequency domain plots are the eigenmode of rotor 1 at initial and maximum response positions of 4.425 rad, respectively. In the first time range of $20 \sim 35$ s, we can see the wave packet frequency starts at 2.87 Hz. As time progresses to the second and third time intervals, we observe that the wave packet indeed maintains its frequency and location as expected.

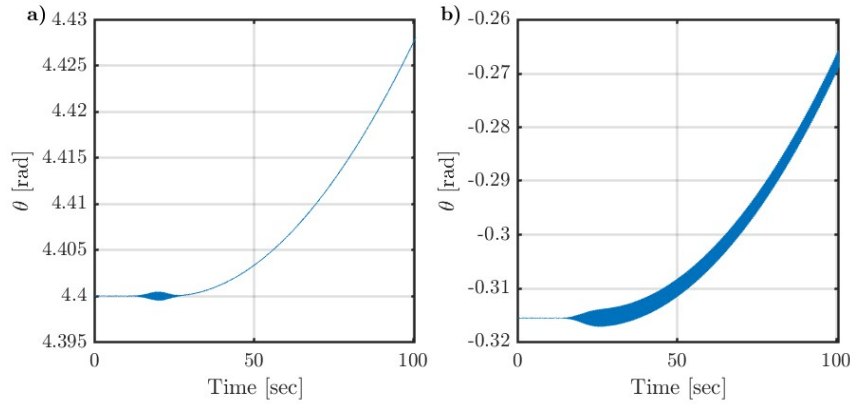


Figure 3.21: Time domain response with torque excitation of $F = 24.2 \mu\text{N.m}$, $f = 2.87$ Hz, and $\sigma = 3.19$ s. for: (a) Time response of rotor 1. (b) Time response of rotor 24.

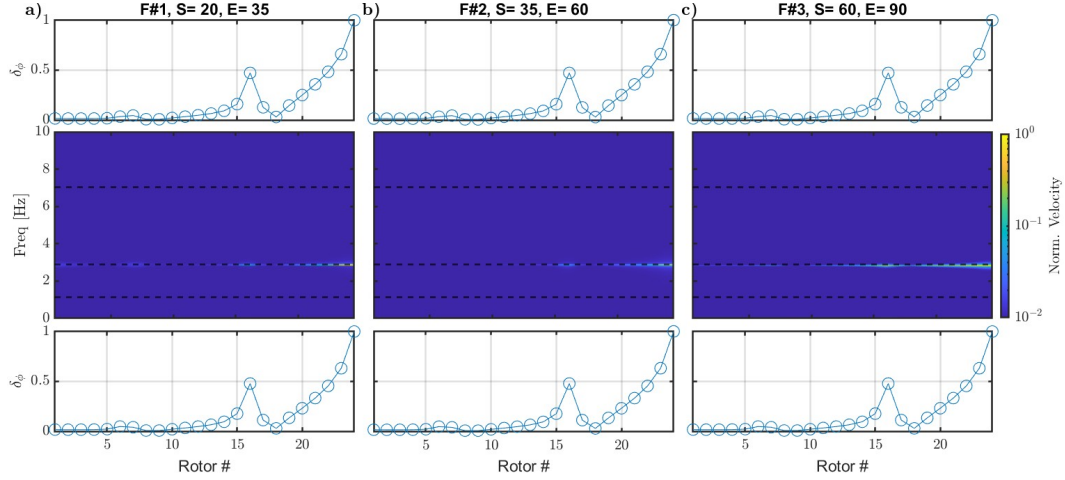


Figure 3.22: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 2.87 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$, with eigenmode of rotor 1 on top and bottom at positions of $\theta_1 = 4.4 \text{ rad}$ and $\theta_1 = 4.425 \text{ rad}$, respectively, for the following time ranges: (a) 20 to 35 s. (b) 35 to 60 s. (c) 60 to 90 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the red line with the curves in Fig. 3.7.

Case 5: Short chain with edge mode excitation at rotor 24 with $\theta_1 = 0.25 \text{ rad}$

This case study replicates the conditions of the third case study, with one key modification in the initial angle. The adjustment in the initial angle is made to avoid the effects of second harmonic generation in the system when conducting future experiments on the physical setup. This change ensures more reliable results by mitigating the influence of higher order harmonics on the system's behavior. In Fig. 3.23, we simulate the response of the rotary chain system with 24 rotors. A GSP is applied at rotor 24 and initial position of the first rotor as $\theta_1 = 0.25 \text{ rad}$. We can observe the wave packet being excited at $t = 20 \text{ s}$ in Fig.3.23(b), with a minimum amplitude in the time range of 35 to 45 s. Conversely, the wave packet can be detected in the same time range for the first rotor response, as shown in Fig.3.23(a). For a clear visualization of the wave packet transition, we plot in Fig.3.23(c) the spatiotemporal velocity response normalised by its maximum value

for the full chain. We can see the applied GSP excitation at rotor 24 when the time is 20 seconds. As time progresses, the wave packet follows the direction indicated by the black arrow to reach the first rotor, as highlighted in the white rectangle. Similarly, in Fig. 3.24 we plot the frequency response of the rotary chain for three time intervals to show the wave transportation via the edge modes from the right side as in Fig. 3.24(a) to the bulk mode as in Fig. 3.24(b) till it reaches the left side via edge mode as in Fig. 3.24(c). Above and below the frequency domain plots are the eigenmodes of rotor 1 at the initial and maximum response positions of 3.57 radians, respectively. By examining the color intensity of the frequency spectrum, a clear match is observed between panel (a) and the top eigenmode, and between panel (c) and the bottom eigenmode, while panel (b) displays a mixture of both eigenmodes.

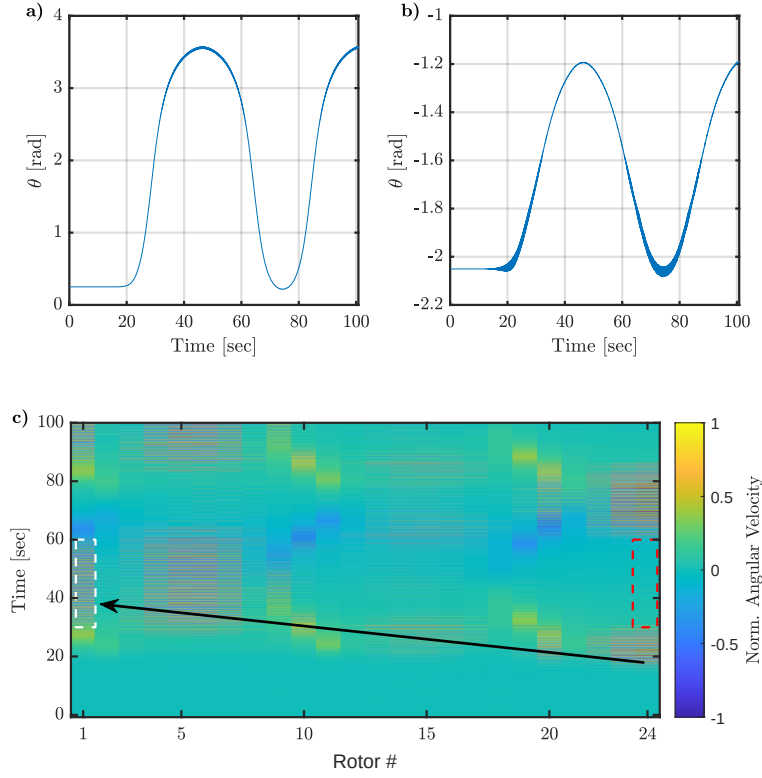


Figure 3.23: Time domain response with torque excitation of $F = 24.2 \mu\text{N.m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$. for: (a) Time response of rotor 1. (b) Time response of rotor 24. (c) Spatiotemporal response of the chiral chain angular velocity normalized by maximum angular velocity amplitude. Red and white rectangles highlight the difference in angular velocity response for rotor 1 and 24, respectively. The black arrow show the direction where the wave packet is traveling thought the system.

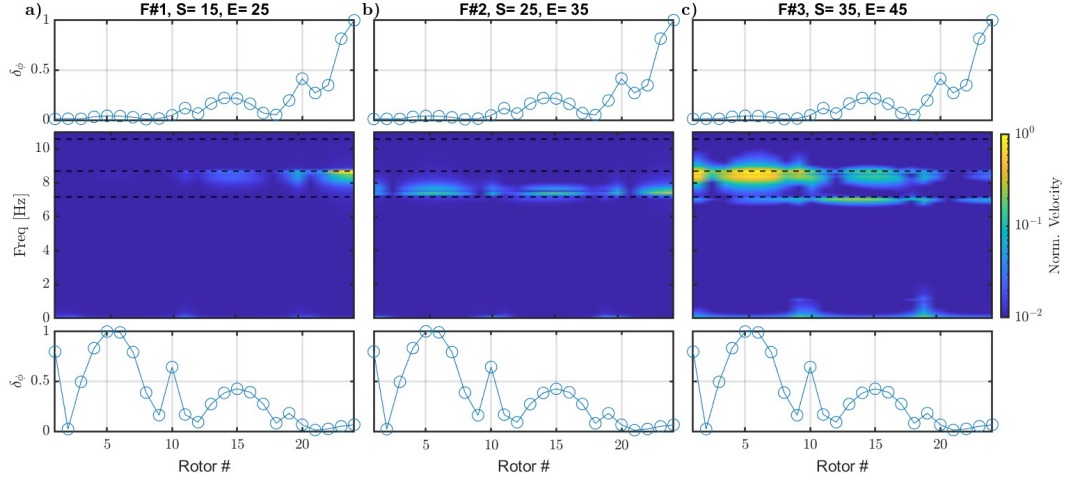


Figure 3.24: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$, with eigenmode of rotor 1 on top and bottom at positions of $\theta_1 = 0.25 \text{ rad}$ and $\theta_1 = 3.57 \text{ rad}$, respectively, for the following time ranges: (a) 15 to 25 s. (b) 25 to 35 s. (c) 35 to 45 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the blue line with the curves in Fig. 3.7.

Case 6: Short chain with edge mode excitation at rotor 1 with $\theta_1 = 0.25 \text{ rad}$

In this case study, we examine the impact of transferring the excitation to the first rotor on the edge mode. Using the same rotary chain system as previously, we applied an identical GSP to rotor 1, with an initial position of $\theta_1 = 0.25 \text{ radians}$. Figure. 3.25 illustrates the simulated time response of the angular positions of the first and last rotors. As expected, the wave packet does not propagate between rotor 1 (Fig.3.25(a)) and rotor 24 (Fig.3.25(b)). For further analysis we plot the frequency response of the rotary chain for three time intervals as in Fig. 3.26. Similarly, on top and bottom of the frequency domain plots are the eigenmode of rotor 1 at initial and maximum response positions at 0.253 rad , respectively. Throughout the three time intervals, the wave packet frequency is consistently maintained at 8.69 Hz , indicating that the wave did not propagate through the system.

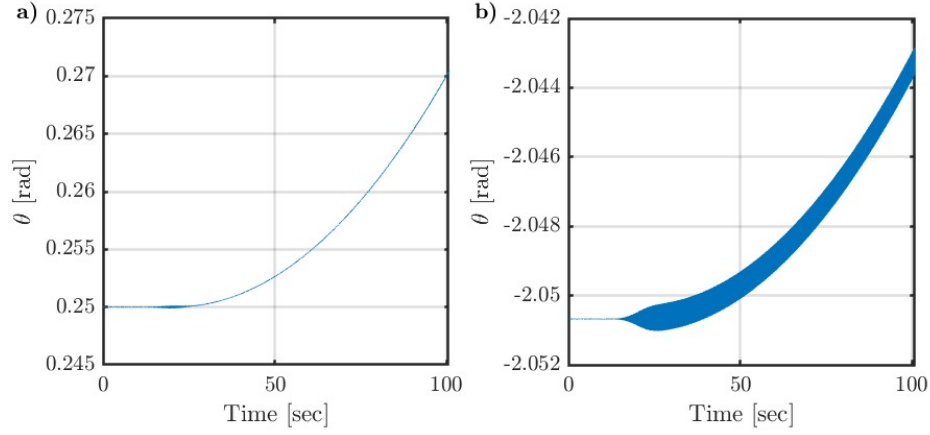


Figure 3.25: Time domain response with torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$. for: (a) Time response of rotor 1. (b) Time response of rotor 24.

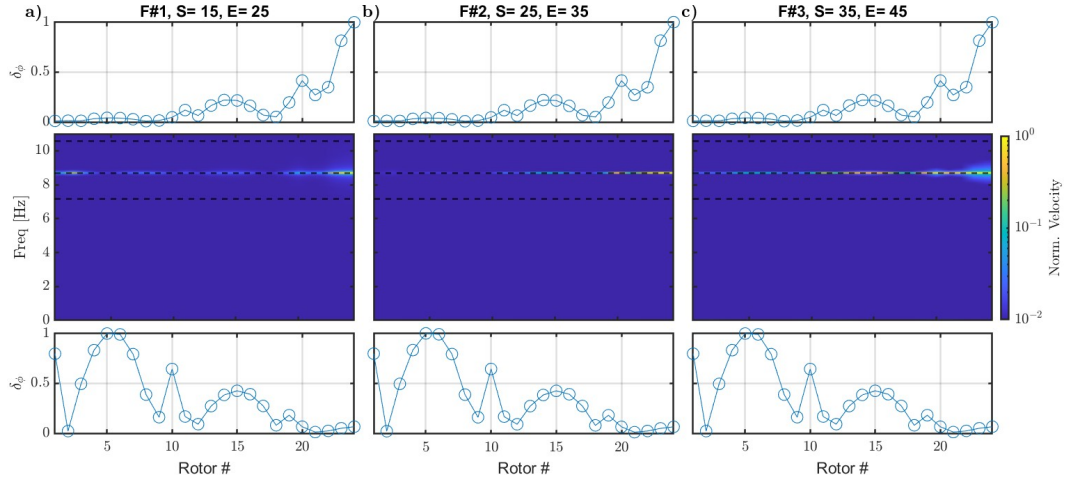


Figure 3.26: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$, with eigenmode of rotor 1 on top and bottom at positions of $\theta_1 = 0.25 \text{ rad}$ and $\theta_1 = 0.253 \text{ rad}$, respectively, for the following time ranges: (a) 15 to 25 s. (b) 25 to 35 s. (c) 35 to 45 s. The black dashed horizontal lines represent the expected frequencies determined by the intersections of the blue line with the curves in Fig. 3.7.

3.2.6 Damping Estimation

To estimate the damping in the system, we examine the time response of two rotors where the bottom rotor is fixed, while the top rotor is free to vibrate, as shown in Fig. 3.27(a). The top rotor is initially rotated to a known angle of $\theta = 0.43$ radians and then released to vibrate freely. Concurrently, a slow motion camera captures the motion of the top rotor's red and green markers, as presented in Fig.3.27(b).

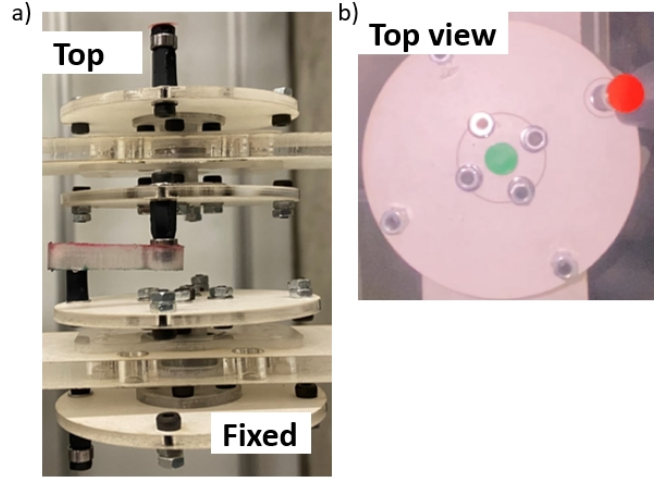


Figure 3.27: Experimental system used to estimate damping ratio consist of a fixed-free rotors connected by a spring. (a) Side view. (b) Top view.

Using digital image processing, we extract the time response of the top rotor, as illustrated in dashed blue in Fig.3.28(a). We then analyze the data in the frequency domain by calculating the FFT. By fitting the resulting spectrum with a Lorentzian function and assessing the quality factor of the fit, we determine the quality factor to be $Q = 20.73$. By using Eqs. 3.20, we estimate the damping ratio to be $\zeta = 0.02412$.

$$Q = \frac{1}{2\zeta} \quad (3.20)$$

To implement the damping effect into Eqs. 3.14, we calculate the damping coefficient based on Eqs. 3.21, which results in $\eta = 0.105107 \text{ g.m}^2/\text{s}$.

$$\eta = 2I\zeta\omega_n \quad (3.21)$$

Considering the experimental damping coefficient value in our numerical model and plotting the time response as shown in Fig. 3.28(a), we can see the agreement between the experimental and numerical time response. The addition of damping to the system induces notable changes in the chiral chain frequency spectrum illustrated in Fig. 3.7.

3.3 The Impact of Damping on System Response

To explore these alterations further, we proceed by numerically constructing the frequency spectrum of the damped system. Our methodology involves initiating rotor 1 at an initial position of $\theta_1^{(0)} = 0.05$ radians. Subsequently, we excite the system with a square pulse and employ Fourier fast transformation to analyze the time response of rotor 1. This process is iteratively conducted, incrementally adjusting the initial position of rotor 1 until it reaches $\theta_1^{(0)} = 2\pi$ radians. All FFTs are plotted using MATLAB's *pcolor()* function, with the frequency spectrum without damping overlaid on top to form the damped frequency spectrum as seen in Fig. 3.29. The comparison generally reveals a close correspondence, slight shifting of certain regions occurs, compared to the undamped system. These shifts are bounded by the following equation

$$\omega_\mu = \omega \left(1 \pm \sqrt{1 + \frac{\eta \sqrt{3}}{\omega}} \right), \quad (3.22)$$

where ω_μ represent damped frequency spectrum, ω denotes the frequency spectrum without damping, and η is the damping coefficient.

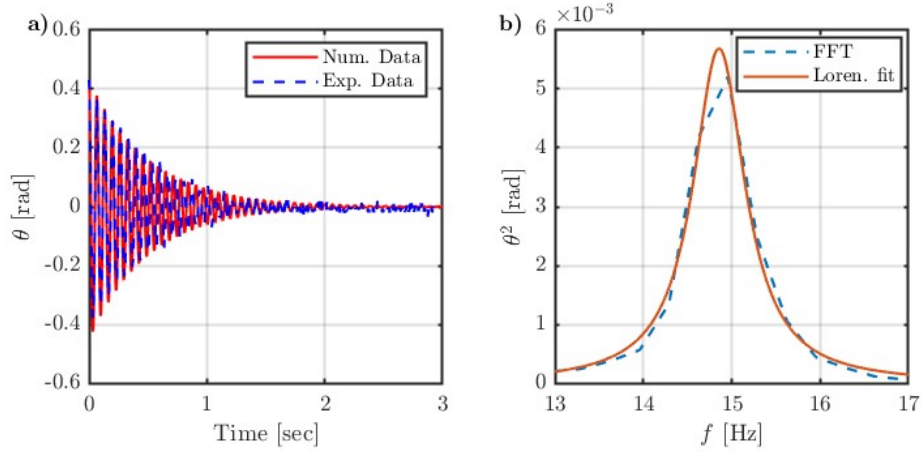


Figure 3.28: Experimental results were obtained for estimating damping ratio for free oscillation start from $\theta = 0.43$ radians, with: (a) The numerical and experimental time responses are depicted in dashed blue and solid red lines, respectively. (b) The experimental FFT is illustrated in dashed blue, accompanied by a Lorentzian fit represented by a solid orange line.

3.3.1 Preliminary Dynamical Simulation with Damping

Case 1: Excitation at Rotor 24 with $\theta_1 = 0.25$ rad

In Fig. 3.30, we simulate the response of the rotary chain system with 24 rotors, with the addition of the experimental damping coefficient found in the previous section, which is $\eta = 0.105107 \text{ g.m}^2/\text{s}$. A GSP is applied at rotor 24 and initial position of rotor 1 as $\theta_1 = 0.25$ rad. Due to the presence of the damping, the edge mode transportation is not clearly observed between rotor 1 and rotor 24, as is shown in Fig. 3.30(a) and (b). In Fig. 3.30(c), we show the xt diagram for the damped response, we note the similar effect of the loops preventing the wave propagation to the other side of the chain. For a clear visualization in Fig. 3.31, we plot the frequency response of the rotary chain for three time intervals to show the transportation of the edge mode from the right side (Fig. 3.31(a)) to the bulk mode (Fig. 3.31(b)), until it reaches the left side (Fig. 3.31(c)). On top of the

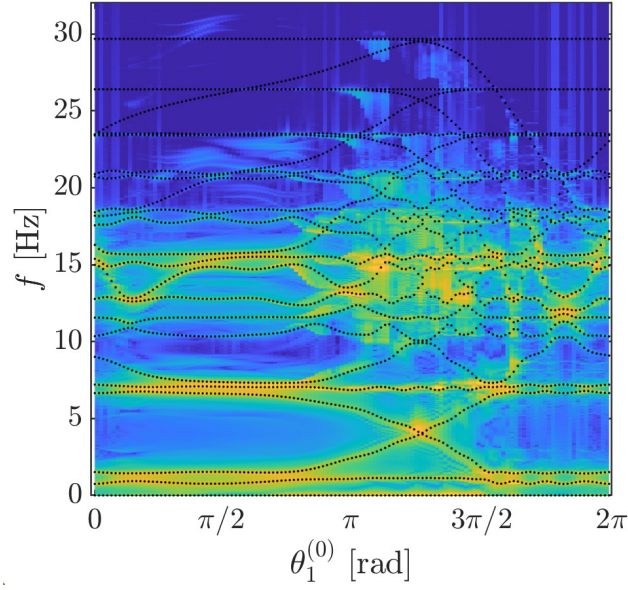


Figure 3.29: Damped rotary chain spectrum for 24 rotor (pcolor), $a = 20$ mm, $r = 30$ mm, $d_0 = 30$ mm, $k = 269$ N/m, $I = 2.4596 * 10^{-5}$ kg.m², and $\eta = 0.105107$ g.m²/s with undamped spectrum on top as black dots.

frequency domain plots are the eigenmode of rotor 1, showing right localized edge mode evaluated at $\theta_1 = 0.25$ rad. On the bottom of each frequency domain plots are the eigenmode for the left localized edge mode evaluated at $\theta_1 = 2.55$ rad which correspond to maximum response positions. In each panel we annotate the similarity of the frequency domain and the eigenmodes, a clear match is noted indicate the edge transition from right to left edge mode as expected.

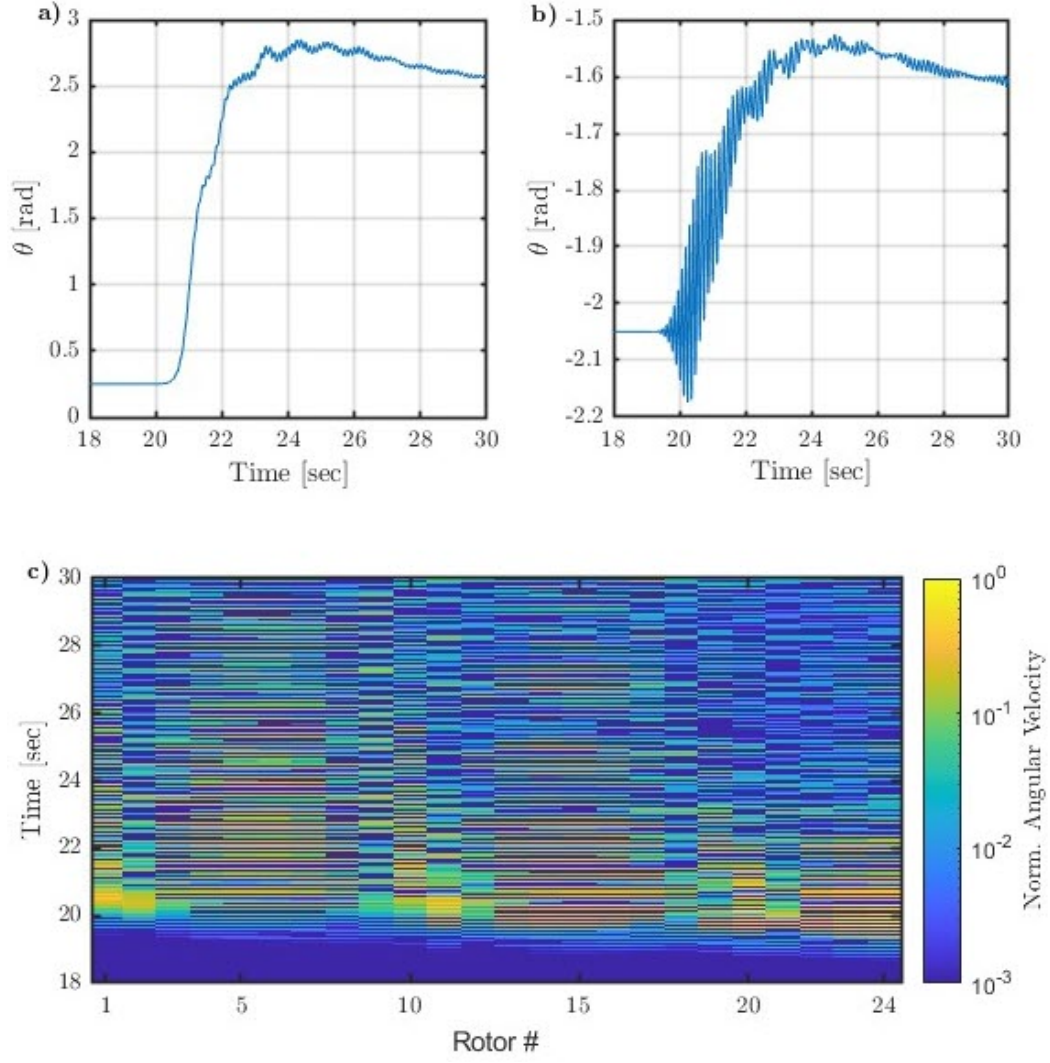


Figure 3.30: Time domain response with torque excitation of $F = 24.2 \mu\text{N.m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$. for: (a) Time response of rotor 1. (b) Time response of rotor 24. (c) Spatiotemporal response of the chiral chain angular velocity normalized by maximum angular velocity amplitude.

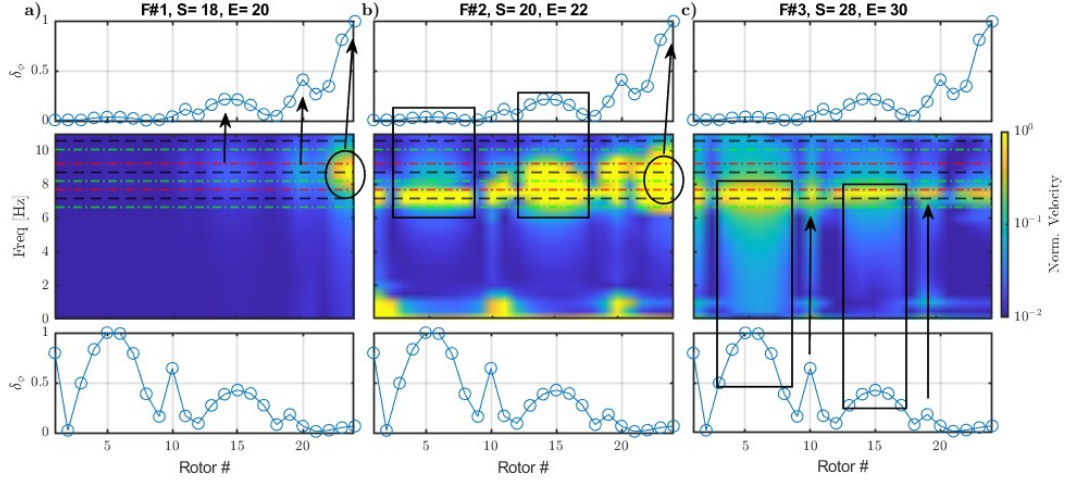


Figure 3.31: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$, with eigenmode of rotor 1 on top and bottom at positions of $\theta_1 = 0.25 \text{ rad}$ and $\theta_1 = 2.55 \text{ rad}$, respectively, for the following time ranges: (a) 18 to 20 s, (b) 20 to 22 s, and (c) 28 to 30 s. The black dashed horizontal lines represent the expected frequencies of 7.14 and 10.58 Hz for bulk modes and 8.69 Hz for right localized edge mode. These values determined by the intersections of the blue line with the curves in Fig. 3.7. Red and green dashed lines indicate the upper and lower limits of the frequencies, respectively, based on Eq. 3.22

Case 2: Excitation at Rotor 1 with $\theta_1 = 0.25 \text{ rad}$

In Fig. 3.32, we simulate the response of the rotary chain system with 24 rotors, with the addition of the experimental damping in same conditions as the previous case. A GSP is applied at rotor 1 and initial position of rotor 1 as $\theta_1 = 0.25 \text{ rad}$. The edge mode does not transport between rotor 1 Fig. 3.32(a) and rotor 24 Fig. 3.32(b). In Fig. 3.33 we plot the frequency response of the rotary chain for three time intervals to show if there is any transportation of the edge mode from the right to left side as in Fig. 3.33(a-c). Similarly, on top of the frequency domain plots are the eigenmode of rotor 1, showing right localized edge mode evaluated at $\theta_1 = 0.25 \text{ rad}$. On the bottom of each frequency domain plots are the eigenmode evaluated at $\theta_1 = 0.27 \text{ rad}$ which correspond to the maximum response positions.

Since no transition is anticipated, both eigenmodes remain quite similar, with the slight positional change being negligible.

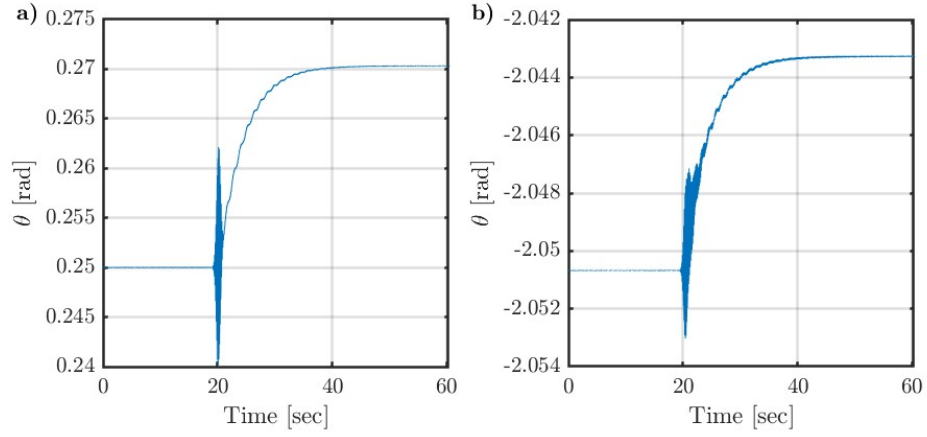


Figure 3.32: Time domain response with torque excitation of $F = 24.2 \mu\text{N.m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$. for: (a) Time response of rotor 1. (b) Time response of rotor 24.

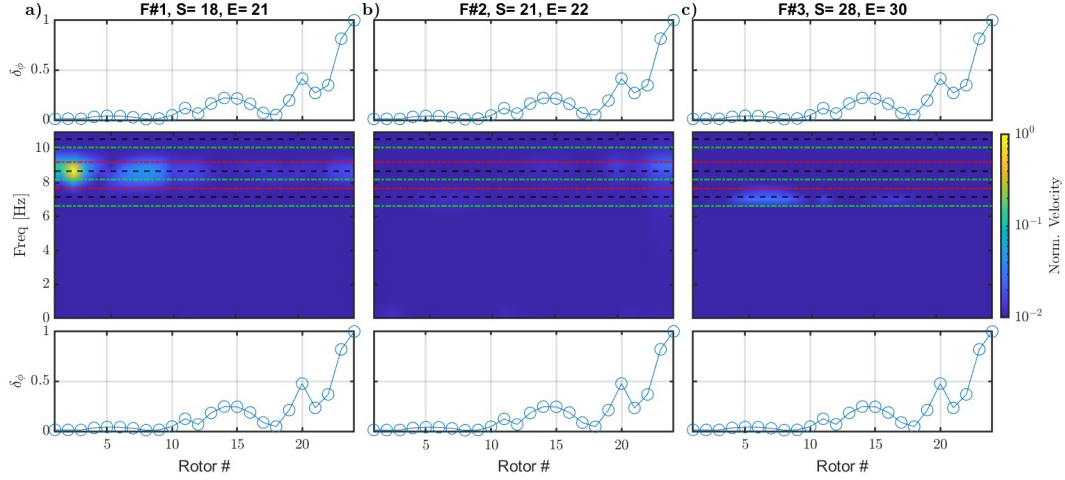


Figure 3.33: Frequency domain response for torque excitation of $F = 24.2 \mu\text{N}\cdot\text{m}$, $f = 8.69 \text{ Hz}$, and $\sigma = 3.19 \text{ s}$, with eigenmode of rotor 1 on top and bottom at positions of $\theta_1 = 0.25 \text{ rad}$ and $\theta_1 = 0.27 \text{ rad}$, respectively, for the following time ranges: (a) 18 to 21 s, (b) 21 to 22 s, and (c) 28 to 30 s. Similar to the previous case, the black dashed horizontal lines at 7.14 and 10.58 Hz represent bulk modes, while 8.69 Hz represent a right localized edge mode. These frequencies are determined by the intersections of the blue line with the curves in Fig. 3.7. Red and green dashed lines indicate the upper and lower limits of the frequencies, respectively, based on Eq. 3.22.

In summary, by analyzing the undamped long rotary chain as presented in case study 1, we demonstrated the transition of a wave packet from the right to the left localized edge mode through an edge-bulk mode transition. This transition was also observed in the undamped shorter chain discussed in case study 3, despite the high loop density in the system. These loops are associated with zero-velocity points; however, as the wave propagates through the bulk mode, the primary effect of these points is to delay the wave's propagation rather than prevent it. Upon analyzing the response in the frequency domain, the transition becomes evident despite the presence of zero-velocity points and the delay.

3.4 Conclusion and Future Work

3.4.1 Conclusion

In conclusion, we have presented a novel mechanical analog of the Aubry-André-Harper (AAH) model, showcasing the ability of adiabatic self pumping to induce nonreciprocal behavior. Our study focuses on a 1D lattice configuration characterized by a zero energy mode, controlled by an angle parameter. By varying this angle, we observed the emergence of finite frequency edge modes attributed to nonzero Chern numbers within the bulk bands. Particularly intriguing is the interaction between these finite frequency edge modes and the zero mode, leading to what we term a "self-pumping effect". This effect elucidates how excitation originating from one end of the system, characterized by the presence of an edge mode, triggers the activation of the zero mode, subsequently facilitating the transport (pumping) of excitation to the opposite end of the system via a bulk mode. This intricate interplay underscores the manifestation of nonreciprocal behavior within the system. Furthermore, we conducted simulations of the system's equations of motion to illustrate the time and frequency response of the chiral chain under various excitation scenarios. These simulations yielded results consistent with theoretical predictions, demonstrating the transition of edge modes from one side of the chain to the other under appropriate excitation conditions. Additionally, the introduction of damping in the simulations aimed to emulate physical world conditions, yet still showcased the transportation of edge modes. Thus, our findings underscore the system's inherent capability to exhibit nonreciprocity without the need for external control.

3.4.2 Future Work

The next critical phase of this project involves experimental validation, a pivotal step that can be divided into two key components.

First, the initial validation process must mirror the methodology and conditions employed to generate Fig.3.29. This entails positioning the top rotor (θ_1) at a specific initial rotational angle while applying square pulse torque to excite the

system at rotor 24 (θ_{24}) on the opposite end of the rotor chain. Subsequently, time response data will be gathered from all rotors, followed by FFT and visualization as a color plot. This results in characterization of the eigenmode spectrum as a function of the first rotor position.

The second component involves validating both case 1 and case 2 of the damped dynamical simulation. This validation process serves to confirm the accuracy and reliability of the simulation under varying conditions and parameters, thus ensuring the robustness of our model in capturing the system's behavior.

Chapter 4

Recommendations and Future Developments

In this section, we outline some proposed advancements and modifications for each system. These suggestions aim to improve the system's response, accuracy, complexity, and applicability to physical systems.

4.1 Nonreciprocity via Geometric-Nonlinearity-Enabled All-Acoustic Spatiotemporal Modulation

- **Development of a Smaller System**

As part of our future work, we plan to develop a smaller system by using a reduced mass size. By this adjustment it is expected, to achieve higher natural frequencies, allowing us to explore a broader range of frequency responses. This is crucial for applications requiring precise frequency control. Additionally, extracting responses from smaller systems can be challenging. To address this, we employ image processing tools, which enhance the accuracy of our measurements. Understanding the behavior of smaller systems also facilitates scalability, aiding in the development of larger, more complex

systems based on these foundational insights.

- **Expansion to Higher Dimensions**

Currently, our system operates as a 1D chain. For future development, we propose expanding this to two dimensional (2D) layers and even three dimensional (3D) structures. This expansion will enable a more comprehensive analysis of system dynamics. Introducing a layer structure allows us to study interactions within a plane, providing insights into applications like surface wave propagation and material interfaces. Extending to three dimensions enables the simulation and analysis of volumetric interactions, applicable to physical systems such as lattice structures in materials science. This multidimensional approach will help us understand complex interactions and behaviors, making our models more realistic and applicable to various engineering and scientific fields.

- **Incorporation of Imperfect Mass Spring Values**

In reality, the spring constants may not be uniform due to manufacturing tolerances and material inconsistencies. As a future enhancement, we will introduce imperfections in the spring values of our system equations of motions. Each spring in the system will have a different stiffness value, reflecting realistic conditions where materials are not perfectly manufactured. This variability allows us to study the effects of these imperfections on system performance, stability, and behavior. By accounting for non-uniformity in material properties, we can develop more accurate models that better predict the behavior of physical systems under various conditions. This enhancement will provide deeper insights into the robustness and reliability of the system in practical applications.

- **Addition of Damper Components**

Our current system lacks damping mechanisms, meaning there is no significant energy dissipation apart from material damping. In future work, we will integrate damper components to study their impact on the system.

Dampers are crucial for controlling and dissipating energy, reducing oscillations, and stabilizing the system. By incorporating dampers, we can derive the equations of motion to include damping effects, leading to a more thorough understanding of the system's physics. Comparative studies of the system's behavior with and without dampers will highlight the importance of energy dissipation in maintaining stability and performance. This addition will make our models more comprehensive and applicable to physical scenarios where damping is a critical factor.

Implementing these proposed suggestions will greatly improve our understanding and capabilities of the experimental system. By creating a smaller system, expanding into higher dimensions, incorporating variable mass spring values, and integrating damper components, we strive to develop a more realistic and comprehensive model. In addition to improving our current system, our recommendations intends to lay the groundwork for more intricate and useful applications.

4.2 Mechanical Rotary Chain Analog of Aubry-André-Harper Systems

- **Experimentally Reproduce Frequency Spectrum**

As an important step to this project is to experimentally reproducing the frequency spectrum, this will validate our theoretical models and provide empirical data to support our hypotheses. This step is crucial for demonstrating the practical applicability and reliability of our theoretical framework.

- **Add Friction into Our Equations of Motion and Inconsistent Damping**

Our current system assumes constant damping throughout, which may not be accurate. Additionally, we are assuming zero friction. As a future enhancement, we will incorporate friction into our equations of motion and consider inconsistent damping. This adjustment will provide a more realistic

model, reflecting the actual conditions more accurately and improving the predictive capability of our system.

- **Elastic or Electronic AAH Chain**

We propose developing an elastic or electronic chain instead of the mechanical chain. This approach will eliminate the use of bearings, thereby reducing the friction in the system and resulting in a clearer response. An elastic or electronic AAH chain can enhance the system's performance and provide a more accurate representation of the intended dynamics.

- **Miniaturization**

Scaling the system down to a smaller size can alter the frequency range and reach higher modes. Miniaturization is an essential step to explore the behavior of the system at different scales and frequencies, providing deeper insights and potentially leading to new applications.

- **Higher Dimension**

Expanding the system to higher dimensions may provide insights into more complex systems or even serve as the basis for new applications depending on the outcomes. By exploring two dimensional or three dimensional configurations, we can better understand the interactions and behaviors in more intricate setups.

These proposed enhancements will substantially advance our knowledge of the rotary AAH chain. By experimentally reproducing the frequency spectrum, introducing friction and variable damping, developing elastic or electronic chains, miniaturizing the system, and exploring higher dimensions, we aim to create a more realistic and comprehensive model. This future work is intended to not only improve our current system but also open avenues for more complex and practical applications. Through these developments, we seek to refine the system's accuracy, robustness, and applicability, thereby offering significant insights into the dynamics of AAH systems.

Appendix A

Important Parameters List for Chapter 2

In this appendix we list the most important parameters for chapter 2, along with there perspective definition.

$a(t)$ and $b(t)$	are periodic amplitude functions in time.
d	is the distance between neighboring masses.
D	is the mass and spring's beam depth.
$\delta_{q,n}$	is the Kronecker Delta function.
E	is the Young's modulus.
F	is the forcing amplitude.
$\gamma_1, \gamma_2, \gamma_3$	are functions to enforce periodic boundary conditions.
$\mathbf{I}$	is the identity matrix $\in \mathbb{R}^{R \times R}$
J	is the mass moment of inertia of the particle.
k_b	is the linear bending stiffness.
k_n	is the linear normal stiffness.
k_s	is the nonlinear shear stiffness.
$\kappa = 2\pi/\lambda_m$	is the wavenumber.

L	is the spring's beam length.
$\lambda_m = R \cdot d$	is the spatial wavelength of the modulation.
$\mathcal{L}$	is the Lagrangian.
K	represent the stiffness matrices.
m	is the mass of the particle.
$\mu = \kappa \lambda_m$	is the non-dimensional wavenumber limited to $\mu \in [-\pi, +\pi]$.
$\Omega = \omega/\omega_0$	is the non-dimensional angular frequency.
$\Omega_m = \omega_m/\omega_0$	is the non-dimensional modulation angular frequency.
P	is the truncation (harmonic) order or the number of branches in DC.
PE^j	is the particle's potential energy.
Q and q	are generalized degrees of freedom.
R	is the number of degrees of freedom (masses) in a unit cell.
$r = 1, \dots, R$	is the number of masses in one unit cell.
S_0	is the unstretched spring length.
t_0	is the centering of the forcing Gaussian pulse.
T_m	is the temporal period of the modulation.
w	is the spring beam width.
ω	is the angular frequency.
$\omega_0^2 = k_0/m$	is the characteristic frequency.
$\omega_m = 2\pi/T_m$	is the frequency associated with the temporal modulation.
$\beta_m = k_m/k_0$	is the non-dimensional modulated stiffness.
ϵ_x	is the longitudinal strain.
$\phi_r = 2\pi r/R$	is the spatial phase shift.
ψ	is the nonlinear diamond spring inner angle.

Appendix B

Supplemental Information for Chapter 3

The initial system configuration parameters, as depicted in Fig. 3.6, are presented here. Subsequently, we outline the electronic circuit and connection setup for the motor, which will be employed in the experimental verification phase.

B.1 Chiral chain initial configuration

Table B.1: Initial angler configuration in radians for a 24 rotors forming a right-handed chiral chain system with $a = 20$ mm, $r = 30$ mm and $d_0 = 30$ mm. Rotor sequence indicated by the subscript j , starting from left to right rotors.

j	$\theta_j^{(0)}$	j	$\theta_j^{(0)}$	j	$\theta_j^{(0)}$	j	$\theta_j^{(0)}$
1	0.25000	7	-1.40644	13	-2.70406	19	-0.39142
2	2.06019	8	-1.05835	14	-2.25908	20	0.36182
3	-3.03421	9	-0.66038	15	-1.89386	21	2.26845
4	-2.48939	10	-0.12013	16	-1.55627	22	-2.96135
5	-2.08996	11	1.04404	17	-1.21754	23	-2.44187
6	-1.74134	12	2.87349	18	-0.84809	24	-2.05068

B.2 Motor connection

This section outlines the connection between the Encoder and other components. Primarily, these connections serve to transmit encoder data to relevant components, other modules, or to power and ground sources. To ensure safety from potential ground loops or other issues, optical isolators are employed for transmission between the net connected to the encoder and Odrive, as well as those of other modules or the microcontroller. For the Odrive and Teensy microcontroller, and their respective nets, the appropriate power supply for the boards should be used (24V for Odrive, 5V for Teensy). The nets associated with each board within a module should be electrically isolated and also isolated from other modules. Regarding signal initialization connections, they primarily facilitate synchronization between modules to commence their tests. These connections pass between modules via optical isolators to ensure safety from potential ground loops or any other arising issues. All previous connections are surmised in Table. B.2. Next, we show connections that are primarily between motor modules to synchronize the start of their tests. They pass between modules via optical isolators, to ensure safety from potential ground loops or any other issues that may arise as demonstrated in Table. B.3.

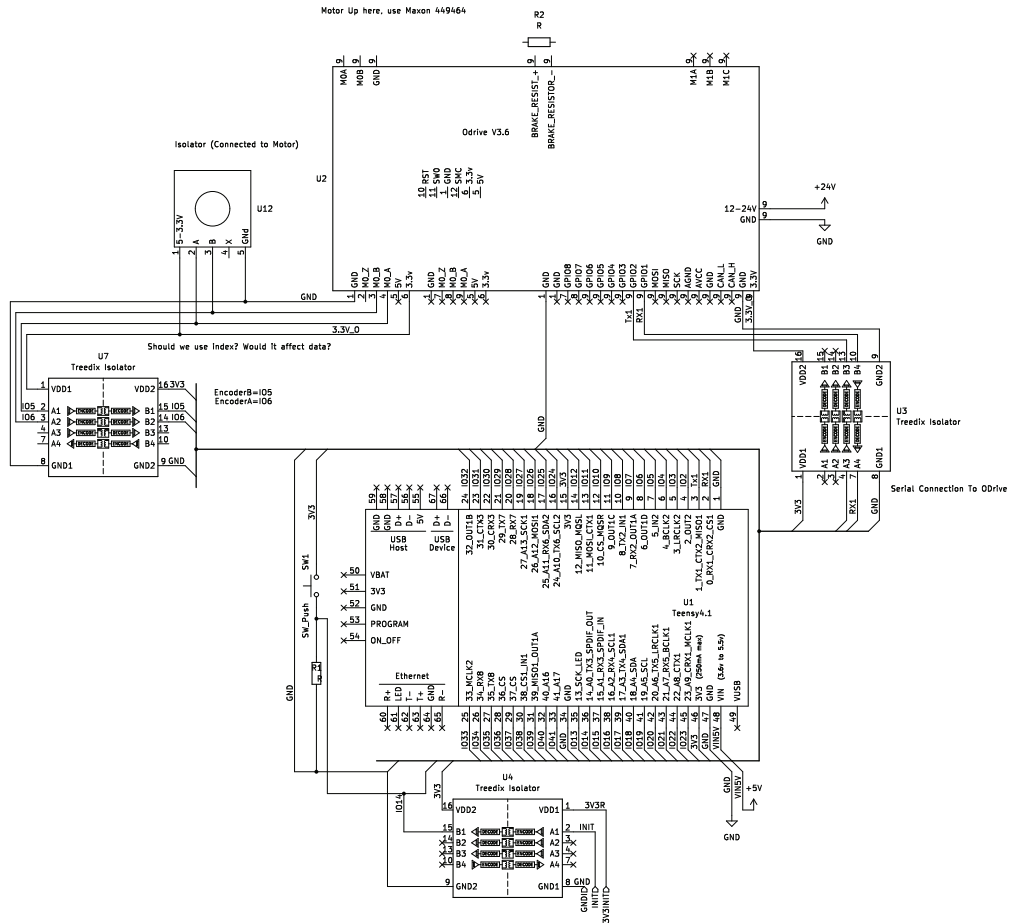
Table B.2: Encoder data and initialization signal connections

Category	From Device	Channel	To Device	Channel
Encoder Data	Encoder (U3)	3.3V	Odrive (U2)	Axis 0 3.3V
			Treedix Isolator(U4)	VDD1
	Encoder (U3)	GND	Odrive (U2)	Axis 0 GND
			Treedix Isolator(U4)	GND1
	Encoder (U3)	A	Odrive (U2)	M0_A
			Treedix Isolator(U4)	A1
	Encoder (U3)	B	Odrive (U2)	M0_B
			Treedix Isolator(U4)	B1
	Treedix Isolator(U4)	VDD2	Teensy (U1)	3v3
	Treedix Isolator(U4)	B1	Teensy (U1)	IO2
	Treedix Isolator(U4)	B2	Teensy (U1)	IO3
	Treedix Isolator(U4)	GND	Teensy (U1)	GND
Initialization Signal	Teensy (U1)	3V3	Push Button (SW1)	Side A
	Push Button SW1	Side B	Resistor (R1)	Side A
			Teensy (U1)	IO 12
	Resistor (R1)	Side B	Teensy (U1)	GND

Table B.3: Serial communications connection

Category	From Device	Channel	To Device	Channel
Serial Communications	Teensy (U1)	3v3	Treedix Isolator (U5)	VDD1
	Teensy (U1)	RX1 (IO0)	Treedix Isolator (U5)	A4
	Teensy (U1)	TX1 (IO1)	Treedix Isolator (U5)	A3
	Teensy (U1)	GND	Treedix Isolator (U5)	GND
	Treedix Isolator (U5)	B4	Odrive (U2)	GPIO 1
	Treedix Isolator (U5)	B3	Odrive (U2)	GPIO 2
	Treedix Isolator (U5)	VDD2	Odrive (U2)	3v3
	Treedix Isolator (U5)	GND	Odrive (U2)	GND

B.3 Motor Circuit Board



Module Core Diagram for Jihad, no photo overlay, 1/29/23

RES Module Core, Boechler Research Group, Gustav Blankenberg
8/7/2023

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