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STABILITY OF A KAPCHINSKIJ-VLADIMIRSKIJ BEAM IN A CONTINUOUS-SOLENOID TRANSPORT SYSTEM

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The oscillation modes of a Kapchinskij-Vladimirskij beam[1)] have been analysized by Gluckstern[2)] for the case of a continuous (constant) focussing system. His results permit one to examine the stability of such a beam subject to small (infinitesimal) perturbations of various forms (including perturbations that are not azimuthally symmetrical). We have made preliminary calculations to this end for several cases characterized by different values of the indices m, j, including m > o (Gluckstern's notation[2)], with results presented in attached tabulations. Work sheets and related graphs are also attached.

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[1)] I.M. Kapchinskij and V.V. Vladimirskij, Proc. Internat. Conf. High Energy Accelerators, CERN (Geneva, Switzerland; 1959); p. 274.

[2)] R.L. Gluckstern, Pros. Proton Linear Accelerator Conference, Fermi National Accelerator Laboratory (Batavia, Illinois; 1970), p. 811.

For a perturbation expressed by

$$G(r,\theta) = r^m \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} {}_2F_1\,(-j,\, m+j,\, m+1;\, r^2) \quad \text{[See Eqn.(18), p. 814]},$$

the controlling equation is

$$\delta_{j,o} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\, \frac{\nu^2}{\omega_p{}^2} = -i\lambda \int_o^\infty (\cos s)^m\, {}_2F_1\,(-j, m+j, m+1; \cos^2\theta)\, e^{i\lambda s}\, ds$$

[Eqn. (19), p. 814], where $\nu^2 = \nu_o{}^2 - \omega_p{}^2$, ${}_2F_1$ is the hypergeometric function, and $\delta_{j,o}$ is the Kronecker delta. Evaluation of the integral, for any specified

Footnote 2) continued.

pair of indices, leads to a result of the form

$$\frac{\nu^2}{\omega_p^{\,2}} = \mathrm{Fcn}(\lambda^2).$$

A plot of Fcn(λ^2) vs. λ^2 (for positive values of λ^2) serves to indicate whether a full set of positive roots λ^2 exists for every positive value of $\nu^2/\omega_p^{\,2}$ (unconditional stability) or, otherwise, reveals the lower bound to $\nu^2/\omega_p^{\,2}$ values for which a full set of positive roots λ^2 will occur (condition for stability).

K-V BEAM IN CONTINUOUS SOLENOID

Stability (for *all* Intensities), or Stability Condition

m \ j	0	1	2	3	4
0	—	r^2: STABLE	r^4: $\frac{\nu}{\nu_0} > \frac{1}{\sqrt{17}} \doteq 0.2425356$	r^6: $\frac{\nu}{\nu_0} > 0.3858759$	r^8: $\frac{\nu}{\nu_0} > 0.3985085$
1	$r\cos\theta$: STABLE	$r^3\cos\theta$: STABLE	$r^5\cos\theta$: STABLE	$r^7\cos\theta$: $\frac{\nu}{\nu_0} > 0.2873514$	
2	$r^2\cos 2\theta$: STABLE	$r^4\cos 2\theta$: STABLE	$r^6\cos 2\theta$: $\frac{\nu}{\nu_0} > \frac{1}{\sqrt{33}} \doteq 0.1740777$	$r^8\cos 2\theta$: $\frac{\nu}{\nu_0} > 0.2581845$	
3	$r^3\cos 3\theta$: STABLE	$r^5\cos 3\theta$: STABLE	$r^7\cos 3\theta$: $\frac{\nu}{\nu_0} > 0.2183735$		
4	$r^4\cos 4\theta$: STABLE	$r^6\cos 4\theta$: STABLE	$r^8\cos 4\theta$: $\frac{\nu}{\nu_0} > \sqrt{\frac{5}{261}} \doteq 0.1384091$		
5	$r^5\cos 5\theta$: STABLE	$r^7\cos 5\theta$: STABLE			

m	j	MODES: $r^{m+2j}\cos m\theta$		
0	0	— — —	OMIT	
0	1	r^2	$\nu^2/\omega_p^2 = \frac{2}{\lambda^2-4}$	STABLE
0	2	r^4	$\nu^2/\omega_p^2 = -\frac{1}{\lambda^2-4} + \frac{3}{\lambda^2-16}$	Stable only for $\nu^2/\omega_p^2 > \frac{1}{16}$, or $\nu/\nu_0 > \frac{1}{\sqrt{17}} = 0.2425356$
0	3	r^6	$\nu^2/\omega_p^2 = \frac{1}{4}\left[\frac{1}{\lambda^2-4} - \frac{8}{\lambda^2-16} + \frac{15}{\lambda^2-36}\right]$	Stable only for $\nu^2/\omega_p^2 > 0.17495034$ or $\nu/\nu_0 > 0.3858759$
0	4	r^8	$\nu^2/\omega_p^2 = \frac{1}{16}\left[-\frac{3}{\lambda^2-4} + \frac{10}{\lambda^2-16} - \frac{45}{\lambda^2-36} + \frac{70}{\lambda^2-64}\right]$	Stable only for $\nu^2/\omega_p^2 > 0.188790594$ or $\nu/\nu_0 > 0.3985085$
1	0	$r\cos\theta$	$\nu^2/\omega_p^2 = \frac{1}{\lambda^2-1}$	STABLE
1	1	$r^3\cos\theta$	$\nu^2/\omega_p^2 = \frac{1}{4}\left[-\frac{1}{\lambda^2-1} + \frac{9}{\lambda^2-9}\right]$	STABLE
1	2	$r^5\cos\theta$	$\nu^2/\omega_p^2 = \frac{1}{8}\left[-\frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$	STABLE
1	3	$r^7\cos\theta$	$\nu^2/\omega_p^2 = \frac{1}{64}\left[-\frac{1}{\lambda^2-1} + \frac{9}{\lambda^2-9} - \frac{125}{\lambda^2-25} + \frac{245}{\lambda^2-49}\right]$	Stable only for $\nu^2/\omega_p^2 > 0.0900023693$ or $\nu/\nu_0 > 0.2873514$
2	0	$r^2\cos 2\theta$	$\nu^2/\omega_p^2 = \frac{1}{\lambda^2-4}$	STABLE
2	1	$r^4\cos 2\theta$	$\nu^2/\omega_p^2 = \frac{2}{\lambda^2-16}$	STABLE
2	2	$r^6\cos 2\theta$	$\nu^2/\omega_p^2 = \frac{1}{16}\left[-\frac{5}{\lambda^2-4} - \frac{8}{\lambda^2-16} + \frac{45}{\lambda^2-36}\right]$	Stable only for $\nu^2/\omega_p^2 > \frac{1}{32}$, or $\nu/\nu_0 > \frac{1}{\sqrt{33}} = 0.1740777$
2	3	$r^8\cos 2\theta$	$\nu^2/\omega_p^2 = \frac{1}{8}\left[\frac{1}{\lambda^2-4} - \frac{4}{\lambda^2-16} - \frac{9}{\lambda^2-36} + \frac{28}{\lambda^2-64}\right]$	Stable only for $\nu^2/\omega_p^2 > 0.0714200435$ or $\nu/\nu_0 > 0.2581845$
3	0	$r^3\cos 3\theta$	$\nu^2/\omega_p^2 = \frac{1}{4}\left[\frac{1}{\lambda^2-1} + \frac{3}{\lambda^2-9}\right]$	STABLE
3	1	$r^5\cos 3\theta$	$\nu^2/\omega_p^2 = \frac{1}{16}\left[-\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$	STABLE
3	2	$r^7\cos 3\theta$	$\nu^2/\omega_p^2 = \frac{1}{64}\left[\frac{1}{\lambda^2-1} - \frac{45}{\lambda^2-9} + \frac{25}{\lambda^2-25} + \frac{147}{\lambda^2-49}\right]$	Stable only for $\nu^2/\omega_p^2 > 0.0500748996$ or $\nu/\nu_0 > 0.2183735$
4	0	$r^4\cos 4\theta$	$\nu^2/\omega_p^2 = \frac{1}{2}\left[\frac{1}{\lambda^2-4} + \frac{1}{\lambda^2-16}\right]$	STABLE
4	1	$r^6\cos 4\theta$	$\nu^2/\omega_p^2 = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} + \frac{8}{\lambda^2-16} + \frac{9}{\lambda^2-36}\right]$	STABLE
4	2	$r^8\cos 4\theta$	$\nu^2/\omega_p^2 = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} - \frac{6}{\lambda^2-16} + \frac{9}{\lambda^2-36} + \frac{14}{\lambda^2-64}\right]$	Stable only for $\frac{\nu^2}{\omega_p^2} > \frac{5}{256}$ or $\frac{\nu}{\nu_0} > \sqrt{\frac{5}{261}} = 0.1384091$
5	0	$r^5\cos 5\theta$	$\nu^2/\omega_p^2 = \frac{1}{16}\left[\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{5}{\lambda^2-25}\right]$	STABLE
5	1	$r^7\cos 5\theta$	$\nu^2/\omega_p^2 = \frac{1}{64}\left[-\frac{5}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{75}{\lambda^2-25} + \frac{49}{\lambda^2-49}\right]$	STABLE

Work sheets, where $F(\cos^2 s)$ denotes

$$ {}_2F_1(a, b, c; \cos^2 s) $$

$$ = {}_2F_1(-j,\ m+j,\ m+1;\ \cos^2 s) $$

in terms of the hypergeometric function.

Here, then,

$$ F(z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a\,b}{c}\frac{(a+1)(b+1)}{(c+1)}\frac{z^2}{2!} + \cdots . $$

$$ \cos^2 s = \tfrac{1}{2}\left[1 + \cos 2s\right] $$

$$ \cos^4 s = \tfrac{1}{8}\left[3 + 4\cos 2s + \cos 4s\right] $$

$$ \cos^6 s = \tfrac{1}{32}\left[10 + 15\cos 2s + 6\cos 4s + \cos 6s\right] $$

$$ \cos^8 s = \tfrac{1}{128}\left[35 + 56\cos 2s + 28\cos 4s + 8\cos 6s + \cos 8s\right] $$

$$ \cos^3 s = \tfrac{1}{4}\left[3\cos s + \cos 3s\right] $$

$$ \cos^5 s = \tfrac{1}{16}\left[10\cos s + 5\cos 3s + \cos 5s\right] $$

$$ \cos^7 s = \tfrac{1}{64}\left[35\cos s + 21\cos 3s + 7\cos 5s + \cos 7s\right] $$

We take $\int_0^\infty e^{i\lambda s}\,ds = \frac{i}{\lambda}$ & $\int_0^\infty e^{i\lambda s}\cos ks\,ds = \frac{i\lambda}{\lambda^2 - k^2}$

For the hypergeometric function $F(z) = {}_2F_1(a,b,c;z) = {}_2F_1(-j, m+j, m+1; z)$:

$F(0) = 1$, $\left.\frac{d}{dz}F\right|_{z=0} = \frac{ab}{c} = -\frac{j(m+j)}{m+1}$, and it should satisfy identically the second-order ordinary differential equation

$$ z(1-z)\frac{d^2F(z)}{dz^2} + (m+1)(1-z)\frac{dF(z)}{dz} + j(j+m)F = 0. $$

$m = 0$ $j = 0$ | $a = -j = 0$ $b = m + j = 0$ $c = m + 1 = 1$

$F(z) = 1$

$\cos^m s \cdot F(\cos^2 s) = 1$

OMIT

$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!} \frac{\nu^2}{\omega_p^2} =$ Infinite

$m = 0$ $\quad j = 1$ $\quad a = -j = \underline{-1}$ $\quad b = m+j = \underline{1}$ $\quad c = m+1 = \underline{1}$

$$F(\tfrac{z}{c}) = 1 - z$$

$$\cos^{2m} s \cdot F(\cos^2 s) = 1 - \cos^2 s$$
$$= \tfrac{1}{2} - \tfrac{1}{2}\cos 2s.$$
$$= \tfrac{1}{2}\left[1 - \cos 2s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \tfrac{1}{2}\left[1 - \cos 2s\right] e^{i\lambda s}\, ds$$
$$= -\frac{i\lambda}{2}\left[\frac{i}{\lambda} - \frac{i\lambda}{\lambda^2-4}\right] = \frac{1}{2}\left[1 - \frac{\lambda^2}{\lambda^2-4}\right] = -\frac{2}{\lambda^2-4}$$

$$\frac{\nu^2}{\omega_p^2} = +\frac{2}{\lambda^2-4}$$

ONE POSITIVE ROOT FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE STABLE.

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = -1\,\frac{\nu^2}{\omega_p^2}$$

$m = 0$ $\qquad j = 2$ $\qquad a = -j = \underline{-2}$ $\qquad b = m + j = \underline{2}$ $\qquad c = m + 1 = \underline{1}$

$$F(z) = 1 - 4z + 3z^2$$

$$\cos^m s \cdot F(\cos^2 s) = 1 - 4\cos^2 s + 3\cos^4 s$$

$$= \frac{1}{8} - \frac{1}{2}\cos 2s + \frac{3}{8}\cos 4s$$

$$= \frac{1}{8}\left[1 - 4\cos 2s + 3\cos 4s\right]$$

$$2\frac{\nu^2}{\omega_p^2} = -i\lambda\int_0^\infty \frac{1}{8}\left[1 - 4\cos 2s + 3\cos 4s\right]e^{i\lambda s}\,ds$$

$$= -\frac{i\lambda}{8}\left[\frac{i}{\lambda} - 4\frac{i\lambda}{\lambda^2-4} + 3\frac{i\lambda}{\lambda^2-16}\right] = \frac{1}{8}\left[1 - 4\frac{\lambda^2}{\lambda^2-4} + 3\frac{\lambda^2}{\lambda^2-16}\right] = \frac{1}{8}\left[-\frac{16}{\lambda^2-4} + \frac{48}{\lambda^2-16}\right] = -\frac{2}{\lambda^2-4} + \frac{6}{\lambda^2-16}$$

$$\frac{\nu^2}{\omega_p^2} = -\frac{1}{\lambda^2-4} + \frac{3}{\lambda^2-16}$$

FOR STABILITY, REQUIRE $\frac{\nu^2}{\omega_p^2} > \frac{1}{16}$, or $\nu/\nu_0 > \frac{1}{\sqrt{17}} \cong 0.2425356$

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = +2\frac{\nu^2}{\omega_p^2}$$

$m = 0$ $\qquad j = 3$ $\qquad a = -j = \underline{-3}$ $\qquad b = m + j = \underline{3}$ $\qquad c = m + 1 = \underline{1}$

$$F(z) = 1 - 9z + 18z^2 - 10z^3$$

$$\cos^m s \cdot F(\cos^2 s) = 1 - 9\cos^2 s + 18\cos^4 s - 10\cos^6 s$$

$$= \tfrac{1}{8} - \tfrac{3}{16}\cos 2s + \tfrac{3}{8}\cos 4s - \tfrac{5}{16}\cos 6s$$

$$= \tfrac{1}{16}\left[2 - 3\cos 2s + 6\cos 4s - 5\cos 6s\right]$$

$$-3\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \tfrac{1}{16}\left[2 - 3\cos 2s + 6\cos 4s - 5\cos 6s\right] e^{i\lambda s}\,ds$$

$$= -\frac{i\lambda}{16}\left[2\frac{i}{\lambda} - 3\frac{i\lambda}{\lambda^2-4} + 6\frac{i\lambda}{\lambda^2-16} - 5\frac{i\lambda}{\lambda^2-36}\right] = \frac{1}{16}\left[2 - 3\frac{\lambda^2}{\lambda^2-4} + 6\frac{\lambda^2}{\lambda^2-16} - 5\frac{\lambda^2}{\lambda^2-36}\right] = \frac{1}{16}\left[-\frac{12}{\lambda^2-4} + \frac{96}{\lambda^2-16} - \frac{180}{\lambda^2-36}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{4}\left[\frac{1}{\lambda^2-4} - \frac{8}{\lambda^2-16} + \frac{15}{\lambda^2-36}\right]$$

3 stable roots for positive λ^2 only
for $\nu^2/\omega_p^2 > 0.1749534$, or $\nu/\omega_p > 0.3858759$

$$\delta_{j0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = -3\frac{\nu^2}{\omega_p^2}$$

$m = 0$ $\qquad$ $j = 4$ $\qquad$ $a = -j = \underline{-4}$ $\qquad$ $b = m + j = \underline{4}$ $\qquad$ $c = m + 1 = \underline{1}$

$$F(z) = 1 - 16z + 60z^2 - 80z^3 + 35z^4$$

$$\cos^m s \cdot F(\cos^2 s) = 1 - 16\cos^2 s + 60\cos^4 s - 80\cos^6 s + 35\cos^8 s$$

$$= \frac{9}{128} - \frac{3}{16}\cos 2s + \frac{5}{32}\cos 4s - \frac{5}{16}\cos 6s + \frac{35}{128}\cos 8s$$

$$= \frac{1}{128}\left[9 - 24\cos 2s + 20\cos 4s - 40\cos 6s + 35\cos 8s\right]$$

$$4\frac{\nu^2}{\omega_p^2} = -i\lambda\int_0^\infty \frac{1}{128}\left[9 - 24\cos 2s + 20\cos 4s - 40\cos 6s + 35\cos 8s\right]e^{i\lambda s}\,ds$$

$$= -\frac{i\lambda}{128}\left[9\frac{i}{\lambda} - 24\frac{i\lambda}{\lambda^2-4} + 20\frac{i\lambda}{\lambda^2-16} - 40\frac{i\lambda}{\lambda^2-36} + 35\frac{i\lambda}{\lambda^2-64}\right] = \frac{1}{128}\left[9 - 24\frac{\lambda^2}{\lambda^2-4} + 20\frac{\lambda^2}{\lambda^2-16} - 40\frac{\lambda^2}{\lambda^2-36} + 35\frac{\lambda^2}{\lambda^2-64}\right] = \frac{1}{128}\left[-\frac{96}{\lambda^2-4} + \frac{320}{\lambda^2-16} - \frac{1440}{\lambda^2-36} + \frac{2240}{\lambda^2-64}\right]$$

$$= \frac{1}{4}\left[-\frac{3}{\lambda^2-4} + \frac{10}{\lambda^2-16} - \frac{45}{\lambda^2-36} + \frac{70}{\lambda^2-64}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{16}\left[-\frac{3}{\lambda^2-4} + \frac{10}{\lambda^2-16} - \frac{45}{\lambda^2-36} + \frac{70}{\lambda^2-64}\right]$$

Plot

Four stable roots for positive λ^2 only

for $\nu^2/\omega_p^2 > 0.18879059$, or $\nu/\nu_0 > 0.3985085$

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = \frac{\nu^2}{\omega_p^2}$$

$m = 1$ | $j = 0$ | $a = -j = \underline{0}$ | $b = m + j = \underline{1}$ | $c = m + 1 = \underline{2}$

$F(\zeta) = 1.$

$\cos^m s \cdot F(\cos^2 s) = \cos s$

$$1 + \frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \cos s \; e^{i\lambda s} \, ds$$

$$= -i\lambda \left[\frac{i\lambda}{\lambda^2 - 1} \right] = \frac{\lambda^2}{\lambda^2 - 1} = 1 + \frac{1}{\lambda^2 - 1}$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{\lambda^2 - 1}$$

ONE POSITIVE ROOT FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>.

$$\delta_{j,0} + \frac{(-1)^j \, m! \, j!}{(m+j-1)!} \frac{\nu^2}{\omega_p^2} = 1 + \frac{\nu^2}{\omega_p^2}$$

$m = 1$ $\quad$ $j = 1$ $\quad$ $a = -j = \underline{-1}$ $\quad$ $b = m + j = \underline{2}$ $\quad$ $c = m + 1 = \underline{2}$

$F(z) = 1 - z$

$$\cos^m s \cdot F(\cos^2 s) = \cos s - \cos^3 s = +\frac{1}{4}\cos s - \frac{1}{4}\cos 3s = \frac{1}{4}\left[\cos s - \cos 3s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{4}\left[\cos s - \cos 3s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{4}\left[\frac{i\lambda}{\lambda^2-1} - \frac{i\lambda}{\lambda^2-9}\right] = \frac{1}{4}\left[\frac{\lambda^2}{\lambda^2-1} - \frac{\lambda^2}{\lambda^2-9}\right] = \frac{1}{4}\left[\frac{1}{\lambda^2-1} - \frac{9}{\lambda^2-9}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{4}\left[-\frac{1}{\lambda^2-1} + \frac{9}{\lambda^2-9}\right]$$

TWO POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>.

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = -\frac{\nu^2}{\omega_p^2}.$$

$m = 1$ $\qquad j = 2$ $\qquad a = -j = \underline{-2}$ $\qquad b = m + j = \underline{3}$ $\qquad c = m + 1 = \underline{2}$

$$F(\zeta) = 1 - 3\zeta + 2\zeta^2$$

$$\cos^m x \cdot F(\cos^2 x) = \cos x - 3\cos^3 x + 2\cos^5 x = \frac{1}{8}\left[-\cos 3x + \cos 5x\right]$$

$$\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{8}\left[-\cos 3x + \cos 5x\right] e^{i\lambda x}\,dx$$

$$= -\frac{i\lambda}{8}\left[-\frac{i\lambda}{\lambda^2-9} + \frac{i\lambda}{\lambda^2-25}\right] = \frac{1}{8}\left[-\frac{\lambda^2}{\lambda^2-9} + \frac{\lambda^2}{\lambda^2-25}\right] = \frac{1}{8}\left[-\frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{8}\left[-\frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$$

TWO POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE STABLE.

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = \frac{\nu^2}{\omega_p^2}$$

$m = 1$ $\qquad j = 3$ $\qquad a = -j = \underline{-3}$ $\qquad b = m + j = \underline{4}$ $\qquad c = m + 1 = \underline{2}$

$$F(z) = 1 - 6z + 10z^2 - 5z^3$$

$$\cos^m s \cdot F(\cos^2 s) = \cos s - 6\cos^3 s + 10\cos^5 s - 5\cos^7 s = \frac{1}{64}\left[\cos s - \cos 3s + 5\cos 5s - 5\cos 7s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda\int_0^\infty \frac{1}{64}\left[\cos s - \cos 3s + 5\cos 5s - 5\cos 7s\right]e^{i\lambda s}\,ds$$

$$= -\frac{i\lambda}{64}\left[\frac{i\lambda}{\lambda^2-1} - \frac{i\lambda}{\lambda^2-9} + 5\frac{i\lambda}{\lambda^2-25} - 5\frac{i\lambda}{\lambda^2-49}\right]$$

$$= \frac{1}{64}\left[\frac{\lambda^2}{\lambda^2-1} - \frac{\lambda^2}{\lambda^2-9} + 5\frac{\lambda^2}{\lambda^2-25} - 5\frac{\lambda^2}{\lambda^2-49}\right] = \frac{1}{64}\left[\frac{1}{\lambda^2-1} - \frac{9}{\lambda^2-9} + \frac{125}{\lambda^2-25} - \frac{245}{\lambda^2-49}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{64}\left[-\frac{1}{\lambda^2-1} + \frac{9}{\lambda^2-9} - \frac{125}{\lambda^2-25} + \frac{245}{\lambda^2-49}\right]$$

Four stable roots for positive λ^2 only

For $\nu^2/\omega_p^2 > 0.0900023693$, or $\nu/\nu_0 > 0.2873514$

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = -\frac{\nu^2}{\omega_p^2}$$

$m = 2$ $\quad j = 0$ $\quad a = -j = \underline{\ 0\ }$ $\quad b = m + j = \underline{\ 2\ }$ $\quad c = m + j + 1 = \underline{\ 3\ }$

$F(z) = 1$

$$\cos^m s \cdot F(\cos^2 s) = \cos^2 s = \frac{1}{2}\left[1 + \cos 2s\right]$$

$$1 + 2\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{2}\left[1 + \cos 2s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{2}\left[\frac{i}{\lambda} + \frac{i\lambda}{\lambda^2 - 4}\right] = \frac{1}{2}\left[1 + \frac{\lambda^2}{\lambda^2-4}\right] = \frac{1}{2}\left[2 + \frac{4}{\lambda^2-4}\right] = 1 + \frac{2}{\lambda^2-4}$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{\lambda^2 - 4}$$

ONE POSITIVE ROOT FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>.

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = 1 + 2\frac{\nu^2}{\omega_p^2}$$

$m = 2$ $\quad j = 1$ $\quad a = -j = \underline{-1}$ $\quad b = m + j = \underline{3}$ $\quad c = m + 1 = \underline{3}$

$F(z) = 1 - z$

$$\cos^m s \cdot F(\cos^2 s) = \cos^2 s - \cos^4 s = \tfrac{1}{8}\left[1 - \cos 4s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \tfrac{1}{8}\left[1 - \cos 4s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{8}\left[\frac{i}{\lambda} - \frac{i\lambda}{\lambda^2-16}\right] = \frac{1}{8}\left[1 - \frac{\lambda^2}{\lambda^2-16}\right] = -\frac{2}{\lambda^2-16}$$

$$\frac{\nu^2}{\omega_p^2} = \frac{2}{\lambda^2-16}$$

ONE POSITIVE ROOT FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE STABLE

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = -\frac{\nu^2}{\omega_p^2}$$

$m = 2$ $\quad j = 2$ $\quad a = -j = \underline{-2}$ $\quad b = m + j = \underline{4}$ $\quad c = m + 1 = \underline{3}$

$$F(z) = 1 - \frac{8}{3} z + \frac{5}{3} z^2$$

$$\cos^m s \cdot F(\cos^2 s) = \cos^2 s - \frac{8}{3} \cos^4 s + \frac{5}{3} \cos^6 s = \frac{1}{48} - \frac{5}{96} \cos 2s - \frac{1}{48} \cos 4s + \frac{5}{96} \cos 6s$$

$$= \frac{1}{96} \left[2 - 5 \cos 2s - 2 \cos 4s + 5 \cos 6s \right]$$

$$\frac{2}{3} \frac{v^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{96} \left[2 - 5 \cos 2s - 2 \cos 4s + 5 \cos 6s \right] e^{i\lambda s} \, ds$$

$$= -\frac{i\lambda}{96} \left[\frac{2i}{\lambda} - 5 \frac{i\lambda}{\lambda^2 - 4} - 2 \frac{i\lambda}{\lambda^2 - 16} + 5 \frac{i\lambda}{\lambda^2 - 36} \right] = \frac{1}{96} \left[2 - 5 \frac{\lambda^2}{\lambda^2 - 4} - 2 \frac{\lambda^2}{\lambda^2 - 16} + 5 \frac{\lambda^2}{\lambda^2 - 36} \right]$$

$$= \frac{1}{96} \left[-\frac{20}{\lambda^2 - 4} - \frac{32}{\lambda^2 - 16} + \frac{180}{\lambda^2 - 36} \right] = \frac{1}{24} \left[-\frac{5}{\lambda^2 - 4} - \frac{8}{\lambda^2 - 16} + \frac{45}{\lambda^2 - 36} \right]$$

$$\frac{v^2}{\omega_p^2} = \frac{1}{16} \left[-\frac{5}{\lambda^2 - 4} - \frac{8}{\lambda^2 - 16} + \frac{45}{\lambda^2 - 36} \right]$$

FOR STABILITY REQUIRE $\frac{v^2}{\omega_p^2} > \frac{1}{32}$, or $v/v_0 > \frac{1}{\sqrt{33}} \cong 0.174077$

$$\delta_{j,0} + \frac{(-1)^j m! \, j!}{(m+j-1)!} \frac{v^2}{\omega_p^2} = \frac{2}{3} \frac{v^2}{\omega_p^2}$$

$m = 2$	$j = 3$	$a = -j = \underline{\ -3\ }$ $b = m+j = \underline{\ 5\ }$ $c = m+1 = \underline{\ 3\ }$

$$F(z) = 1 - 5z + \frac{15}{2} z^2 - \frac{7}{2} z^3$$

$$\begin{aligned}
\cos^m s \cdot F(\cos^2 s) &= \cos^2 s - 5\cos^4 s + \tfrac{15}{2}\cos^6 s - \tfrac{7}{2}\cos^8 s \\
&= \frac{3}{256} - \frac{1}{64}\cos 2s + \frac{1}{64}\cos 4s + \frac{1}{64}\cos 6s - \frac{7}{256}\cos 8s \\
&= \frac{1}{256}\left[3 - 4\cos 2s + 4\cos 4s + 4\cos 6s - 7\cos 8s\right]
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}\frac{\nu^2}{\omega_p^2} &= -i\lambda \int_0^\infty \frac{1}{256}\left[3 - 4\cos 2s + 4\cos 4s + 4\cos 6s - 7\cos 8s\right] e^{i\lambda s}\, ds \\
&= -\frac{i\lambda}{256}\left[3\frac{i}{\lambda} - 4\frac{i\lambda}{\lambda^2-4} + 4\frac{i\lambda}{\lambda^2-16} + 4\frac{i\lambda}{\lambda^2-36} - 7\frac{i\lambda}{\lambda^2-64}\right] \\
&= \frac{1}{256}\left[3 - 4\frac{\lambda^2}{\lambda^2-4} + 4\frac{\lambda^2}{\lambda^2-16} + 4\frac{\lambda^2}{\lambda^2-36} - 7\frac{\lambda^2}{\lambda^2-64}\right] \\
&= \frac{1}{256}\left[-\frac{16}{\lambda^2-4} + \frac{64}{\lambda^2-16} + \frac{144}{\lambda^2-36} - \frac{448}{\lambda^2-64}\right] = \frac{1}{16}\left[-\frac{1}{\lambda^2-4} + \frac{4}{\lambda^2-16} + \frac{9}{\lambda^2-36} - \frac{28}{\lambda^2-64}\right]
\end{aligned}$$

$$\nu^2/\omega_p^2 = \frac{1}{8}\left[\frac{1}{\lambda^2-4} - \frac{4}{\lambda^2-16} - \frac{9}{\lambda^2-36} + \frac{28}{\lambda^2-64}\right]$$

4 Stable roots for positive λ^2 only

for $\nu^2/\omega_p^2 > 0.07142 00435$, or $\nu/\nu_0 > 0.258 1845$

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = -\frac{1}{2}\frac{\nu^2}{\omega_p^2}$$

$m = 3$	$j = 0$	$a = -j = \underline{\ 0\ }$ $\quad b = m+j = \underline{\ 3\ }$ $\quad c = m+1 = \underline{\ 4\ }$

$F(z) = 1$

$$\cos^m s \cdot F(\cos^2 s) = \cos^3 s$$
$$= \frac{1}{4}\left[3\cos s + \cos 3s\right]$$

$$1 + 3\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{4}\left[3\cos s + \cos 3s\right] e^{i\lambda s}\, ds$$
$$= -\frac{i\lambda}{4}\left[3\frac{i\lambda}{\lambda^2-1} + \frac{i\lambda}{\lambda^2-9}\right] = \frac{1}{4}\left[3\frac{\lambda^2}{\lambda^2-1} + \frac{\lambda^2}{\lambda^2-9}\right] = 1 + \frac{1}{4}\left[\frac{3}{\lambda^2-1} + \frac{9}{\lambda^2-9}\right]$$
$$= 1 + \frac{3}{4}\left[\frac{1}{\lambda^2-1} + \frac{3}{\lambda^2-9}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{4}\left[\frac{1}{\lambda^2-1} + \frac{3}{\lambda^2-9}\right]$$

TWO POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>.

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = 1 + 3\frac{\nu^2}{\omega_p^2}$$

$m = 3$ $\quad j = 1$ $\quad a = -j = \underline{-1}$ $\quad b = m + j = \underline{4}$ $\quad c = m + 1 = \underline{4}$

$F(z) = 1 - z$

$$\cos^m s \cdot F(\cos^2 s) = \cos^3 s - \cos^5 s$$
$$= \frac{1}{16}\left[2 \cos s - \cos 3s - \cos 5s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{16}\left[2 \cos s - \cos 3s - \cos 5s\right] e^{i\lambda s}\, ds$$
$$= -\frac{i\lambda}{16}\left[2\frac{i\lambda}{\lambda^2-1} - \frac{i\lambda}{\lambda^2-9} - \frac{i\lambda}{\lambda^2-25}\right]$$
$$= \frac{1}{16}\left[2\frac{\lambda^2}{\lambda^2-1} - \frac{\lambda^2}{\lambda^2-9} - \frac{\lambda^2}{\lambda^2-25}\right] = \frac{1}{16}\left[\frac{2}{\lambda^2-1} - \frac{9}{\lambda^2-9} - \frac{25}{\lambda^2-25}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{16}\left[-\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$$ Plot . . . . .

3 POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!} \frac{\nu^2}{\omega_p^2} = -\frac{\nu^2}{\omega_p^2}.$$

$m = 3$ $\qquad j = 2$ $\qquad a = -j = \underline{-2}$ $\qquad b = m + j = \underline{5}$ $\qquad c = m + 1 = \underline{4}$

$$F(z) = 1 - \frac{5}{2} z + \frac{3}{2} z^2$$

$$\cos^m s \cdot F(\cos^2 s) = \cos^3 s - \frac{5}{2}\cos^5 s + \frac{3}{2}\cos^7 s$$

$$= \frac{1}{128}\left[\cos s - 5\cos 3s + \cos 5s + 3\cos 7s\right]$$

$$\frac{1}{2}\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{128}\left[\cos s - 5\cos 3s + \cos 5s + 3\cos 7s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{128}\left[\frac{i\lambda}{\lambda^2-1} - 5\frac{i\lambda}{\lambda^2-9} + \frac{i\lambda}{\lambda^2-25} + 3\frac{i\lambda}{\lambda^2-49}\right] = \frac{1}{128}\left[\frac{\lambda^2}{\lambda^2-1} - 5\frac{\lambda^2}{\lambda^2-9} + \frac{\lambda^2}{\lambda^2-25} + 3\frac{\lambda^2}{\lambda^2-49}\right]$$

$$= \frac{1}{128}\left[\frac{1}{\lambda^2-1} - \frac{45}{\lambda^2-9} + \frac{25}{\lambda^2-25} + \frac{147}{\lambda^2-49}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{64}\left[\frac{1}{\lambda^2-1} - \frac{45}{\lambda^2-9} + \frac{25}{\lambda^2-25} + \frac{147}{\lambda^2-49}\right]$$

4 Stable roots for positive λ^2 only
for $\nu^2/\omega_p^2 > 0.050074899\dot{6}$, or $\nu/\nu_0 > 0.218373\dot{5}$

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = \frac{1}{2}\frac{\nu^2}{\omega_p^2}$$

$m = 4$	$j = 0$	$a = -j = \underline{\ 0\ }$ $\quad b = m+j = \underline{\ 4\ }$ $\quad c = m+1 = \underline{\ 5\ }$

$F(z) = 1$

$$\cos^m s \cdot F(\cos^2 s) = \cos^4 s = \frac{1}{8}\left[3 + 4\cos 2s + \cos 4s\right]$$

$$1 + 4\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{8}\left[3 + 4\cos 2s + \cos 4s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{8}\left[3\frac{i}{\lambda} + 4\frac{i\lambda}{\lambda^2-4} + \frac{i\lambda}{\lambda^2-16}\right] = \frac{1}{8}\left[3 + 4\frac{\lambda^2}{\lambda^2-4} + \frac{\lambda^2}{\lambda^2-16}\right] = 1 + \frac{1}{8}\left[\frac{16}{\lambda^2-4} + \frac{16}{\lambda^2-16}\right]$$

$$= 1 + 2\left[\frac{1}{\lambda^2-4} + \frac{1}{\lambda^2-16}\right].$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{2}\left[\frac{1}{\lambda^2-4} + \frac{1}{\lambda^2-16}\right]$$

TWO POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE STABLE.

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = 1 + 4\frac{\nu^2}{\omega_p^2}$$

$m = 4$ $\quad j = 2$ $\quad a = -j = \underline{-2}$ $\quad b = m + j = \underline{6}$ $\quad c = m + 1 = \underline{5}$

$$F(z) = 1 - \frac{12}{5} z + \frac{7}{5} z^2$$

$$\cos^m s \cdot F(\cos^2 s) = \cos^4 s - \frac{12}{5}\cos^6 s + \frac{7}{5}\cos^8 s = \frac{1}{640}\left[5 - 8\cos 2s - 12\cos 4s + 8\cos 6s + 7\cos 8s\right]$$

$$\frac{2}{5}\frac{v^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{640}\left[5 - 8\cos 2s - 12\cos 4s + 8\cos 6s + 7\cos 8s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{640}\left[5\frac{i}{\lambda} - 8\frac{i\lambda}{\lambda^2-4} - 12\frac{i\lambda}{\lambda^2-16} + 8\frac{i\lambda}{\lambda^2-36} + 7\frac{i\lambda}{\lambda^2-64}\right]$$

$$= \frac{1}{640}\left[5 - 8\frac{\lambda^2}{\lambda^2-4} - 12\frac{\lambda^2}{\lambda^2-16} + 8\frac{\lambda^2}{\lambda^2-36} + 7\frac{\lambda^2}{\lambda^2-64}\right] = \frac{1}{640}\left[-\frac{32}{\lambda^2-4} - \frac{192}{\lambda^2-16} + \frac{288}{\lambda^2-36} + \frac{448}{\lambda^2-64}\right]$$

$$= \frac{1}{20}\left[-\frac{1}{\lambda^2-4} - \frac{6}{\lambda^2-16} + \frac{9}{\lambda^2-36} + \frac{14}{\lambda^2-64}\right]$$

$$\frac{v^2}{\omega_p^2} = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} - \frac{6}{\lambda^2-16} + \frac{9}{\lambda^2-36} + \frac{14}{\lambda^2-64}\right]$$

4 Stable roots for positive λ^2 only -Pot.
for $v^2/\omega_p^2 > \frac{5}{256}$, or $v/v_0 > \sqrt{\frac{5}{261}} \doteq 0.1384091$

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\frac{v^2}{\omega_p^2} = \frac{2}{5}\frac{v^2}{\omega_p^2}$$

$m = 4$ $\qquad j = 1$ $\qquad a = -j = \underline{-1}$ $\qquad b = m+j = \underline{5}$ $\qquad c = m+1 = \underline{5}$

$F(\zeta) = 1 - \zeta$

$$\cos^m s \cdot F(\cos^2 s) = \cos^4 s - \cos^6 s = \frac{1}{32}\left[2 + \cos 2s - 2\cos 4s - \cos 6s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{32}\left[2 + \cos 2s - 2\cos 4s - \cos 6s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{32}\left[2\frac{i}{\lambda} + \frac{i\lambda}{\lambda^2-4} - 2\frac{i\lambda}{\lambda^2-16} - \frac{i\lambda}{\lambda^2-36}\right]$$

$$= \frac{1}{32}\left[2 + \frac{\lambda^2}{\lambda^2-4} - 2\frac{\lambda^2}{\lambda^2-16} - \frac{\lambda^2}{\lambda^2-36}\right] = \frac{1}{32}\left[\frac{4}{\lambda^2-4} - \frac{32}{\lambda^2-16} - \frac{36}{\lambda^2-36}\right]$$

$$= \frac{1}{8}\left[\frac{1}{\lambda^2-4} - \frac{8}{\lambda^2-16} - \frac{9}{\lambda^2-36}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} + \frac{8}{\lambda^2-16} + \frac{9}{\lambda^2-36}\right]$$

THREE POSITIVE ROOTS FOR λ^2

FOR ANY POSITIVE ν^2/ω_p^2. HENCE STABLE

$$\delta_{j,0} + \frac{(-1)^j\, m!\, j!}{(m+j-1)!}\,\frac{\nu^2}{\omega_p^2} = -\frac{\nu^2}{\omega_p^2}$$

$m = 5$ $\quad j = 0$ $\quad a = -j = \underline{0}$ $\quad b = m + j = \underline{5}$ $\quad c = m + 1 = \underline{6}$

$F(z) = 1.$

$$\cos^m s \cdot F(\cos^2 s) = \cos^5 s = \frac{1}{16}\left[10 \cos s + 5 \cos 3s + \cos 5s\right]$$

$$1 + 5\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{16}\left[10 \cos s + 5 \cos 3s + \cos 5s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{16}\left[10\frac{i\lambda}{\lambda^2-1} + 5\frac{i\lambda}{\lambda^2-9} + \frac{i\lambda}{\lambda^2-25}\right] = \frac{1}{16}\left[10\frac{\lambda^2}{\lambda^2-1} + 5\frac{\lambda^2}{\lambda^2-9} + \frac{\lambda^2}{\lambda^2-25}\right] = 1 + \frac{1}{16}\left[\frac{10}{\lambda^2-1} + \frac{45}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$$

$$= 1 + \frac{5}{16}\left[\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{5}{\lambda^2-25}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{16}\left[\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{5}{\lambda^2-25}\right]$$

THREE POSITIVE ROOTS FOR λ^2 ....
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE</u>.

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!}\frac{\nu^2}{\omega_p^2} = 1 + 5\frac{\nu^2}{\omega_p^2}$$

$m = 5$ $\qquad j = 1$ $\qquad a = -j = \underline{-1}$ $\qquad b = m + j = \underline{6}$ $\qquad c = m + 1 = \underline{6}$

$F(z) = 1 - z$

$$\cos^m s \cdot F(\cos^2 s) = \cos^5 s - \cos^7 s = \frac{1}{64}\left[5 \cos s - \cos 3s - 3 \cos 5s - \cos 7s\right]$$

$$-\frac{\nu^2}{\omega_p^2} = -i\lambda \int_0^\infty \frac{1}{64}\left[5 \cos s - \cos 3s - 3 \cos 5s - \cos 7s\right] e^{i\lambda s}\, ds$$

$$= -\frac{i\lambda}{64}\left[5 \frac{i\lambda}{\lambda^2-1} - \frac{i\lambda}{\lambda^2-9} - 3 \frac{i\lambda}{\lambda^2-25} - \frac{i\lambda}{\lambda^2-49}\right] = \frac{1}{64}\left[5\frac{\lambda^2}{\lambda^2-1} - \frac{\lambda^2}{\lambda^2-9} - 3\frac{\lambda^2}{\lambda^2-25} - \frac{\lambda^2}{\lambda^2-49}\right]$$

$$= \frac{1}{64}\left[\frac{5}{\lambda^2-1} - \frac{9}{\lambda^2-9} - \frac{75}{\lambda^2-25} - \frac{49}{\lambda^2-49}\right]$$

$$\frac{\nu^2}{\omega_p^2} = \frac{1}{64}\left[-\frac{5}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{75}{\lambda^2-25} + \frac{49}{\lambda^2-49}\right]$$

4 POSITIVE ROOTS FOR λ^2
FOR ANY POSITIVE ν^2/ω_p^2. HENCE <u>STABLE.</u>

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!} \frac{\nu^2}{\omega^2} = -\frac{\nu^2}{\omega_p^2}$$

$m =$ $j =$ | $a = -j =$ ______ $b = m + j =$ ______ $c = m + 1 =$ ______

$F(z) =$

$\cos^m s \cdot F(\cos^2 s) =$

$$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!} \frac{\nu^2}{\omega_p^2} =$$

$m =$ $j =$ | $a = -j =$ ______ $b = m + j =$ ______ $c = m + 1 =$ ______

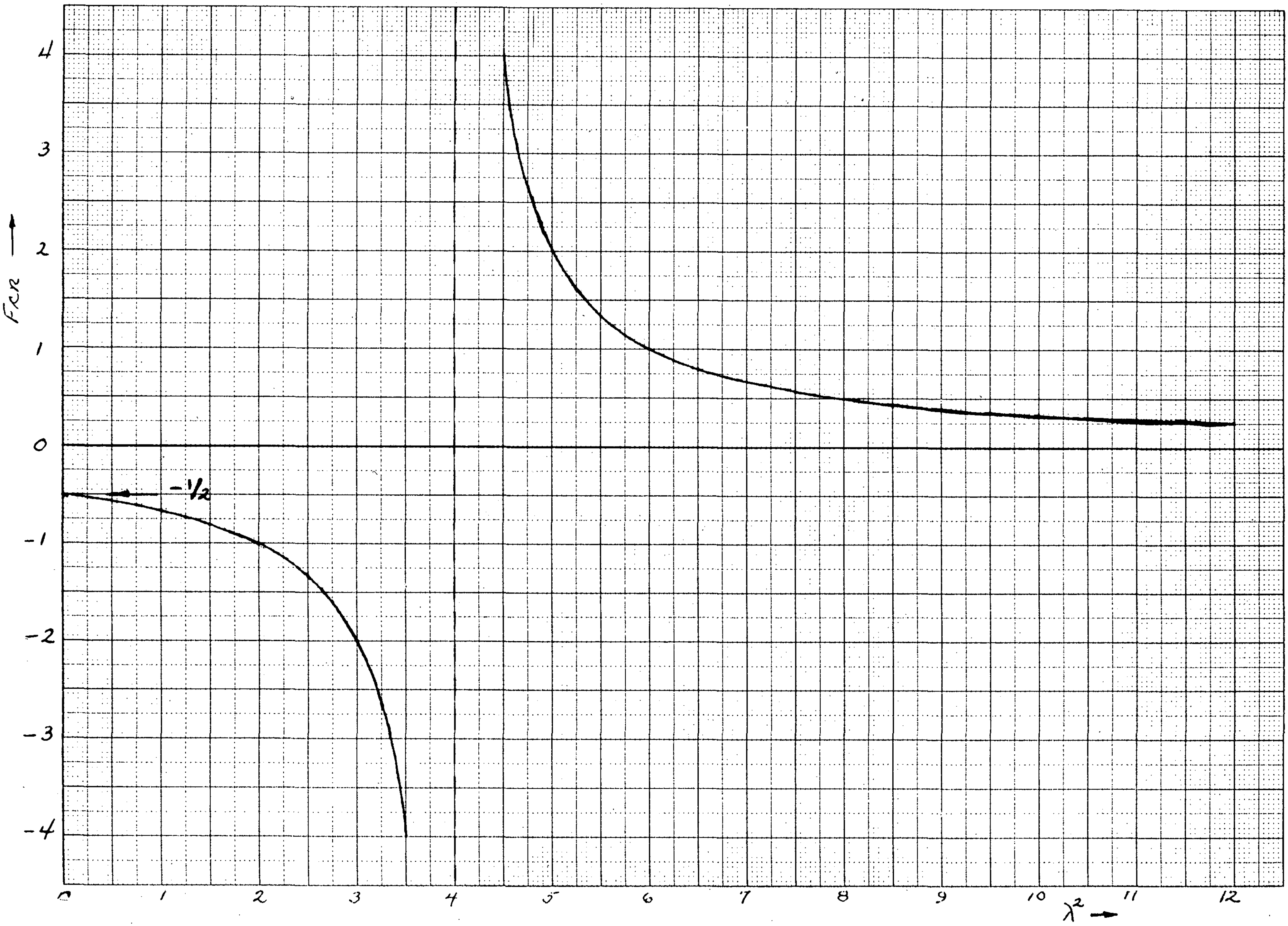
$F(z) =$

$\cos^m s \cdot F(\cos^2 s) =$

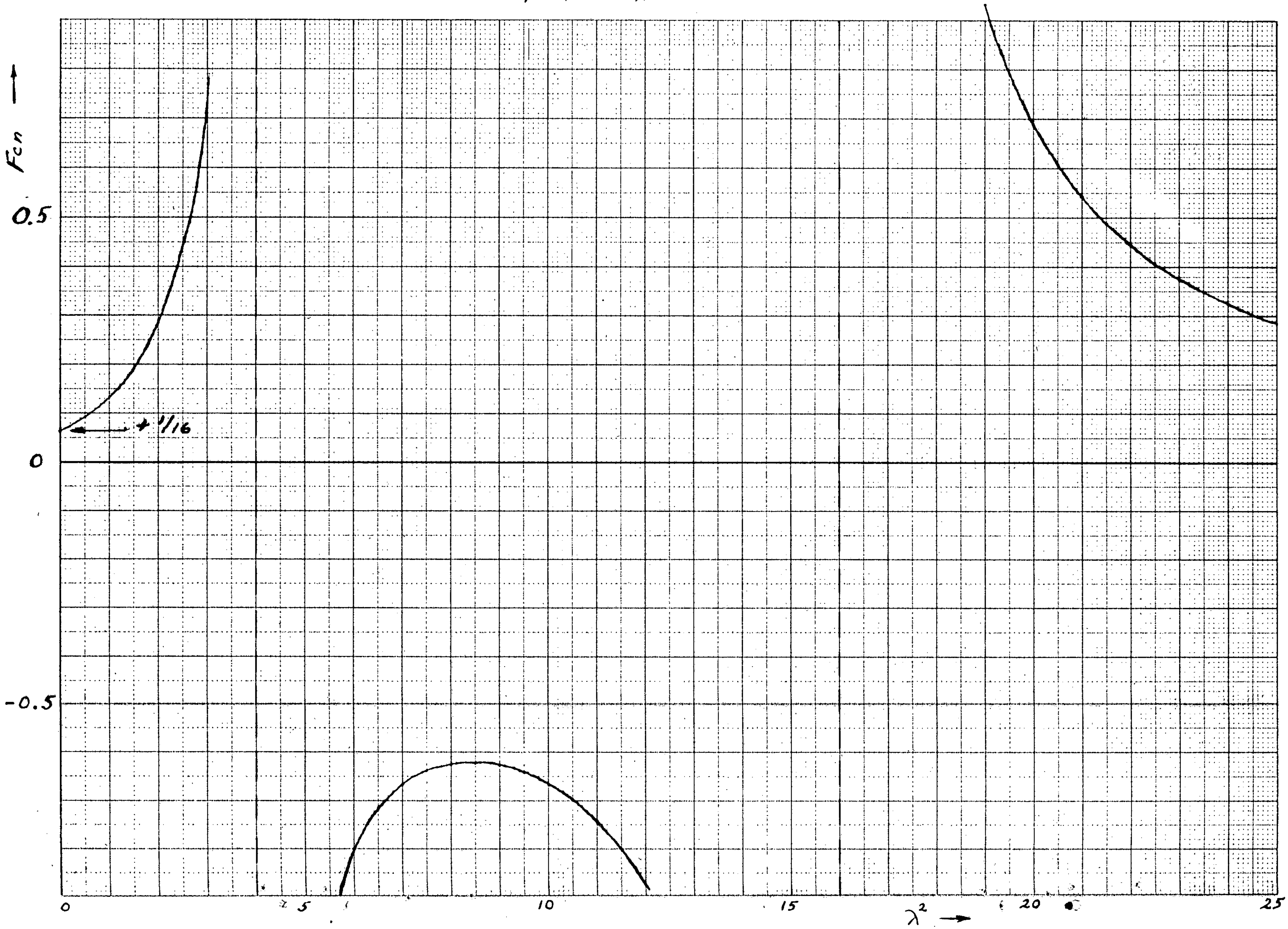
$\delta_{j,0} + \frac{(-1)^j m!\, j!}{(m+j-1)!} \frac{\nu^2}{\omega^2} =$

$m=0, \quad j=1 \qquad F_{cn} = \dfrac{2}{\lambda^2-4}$

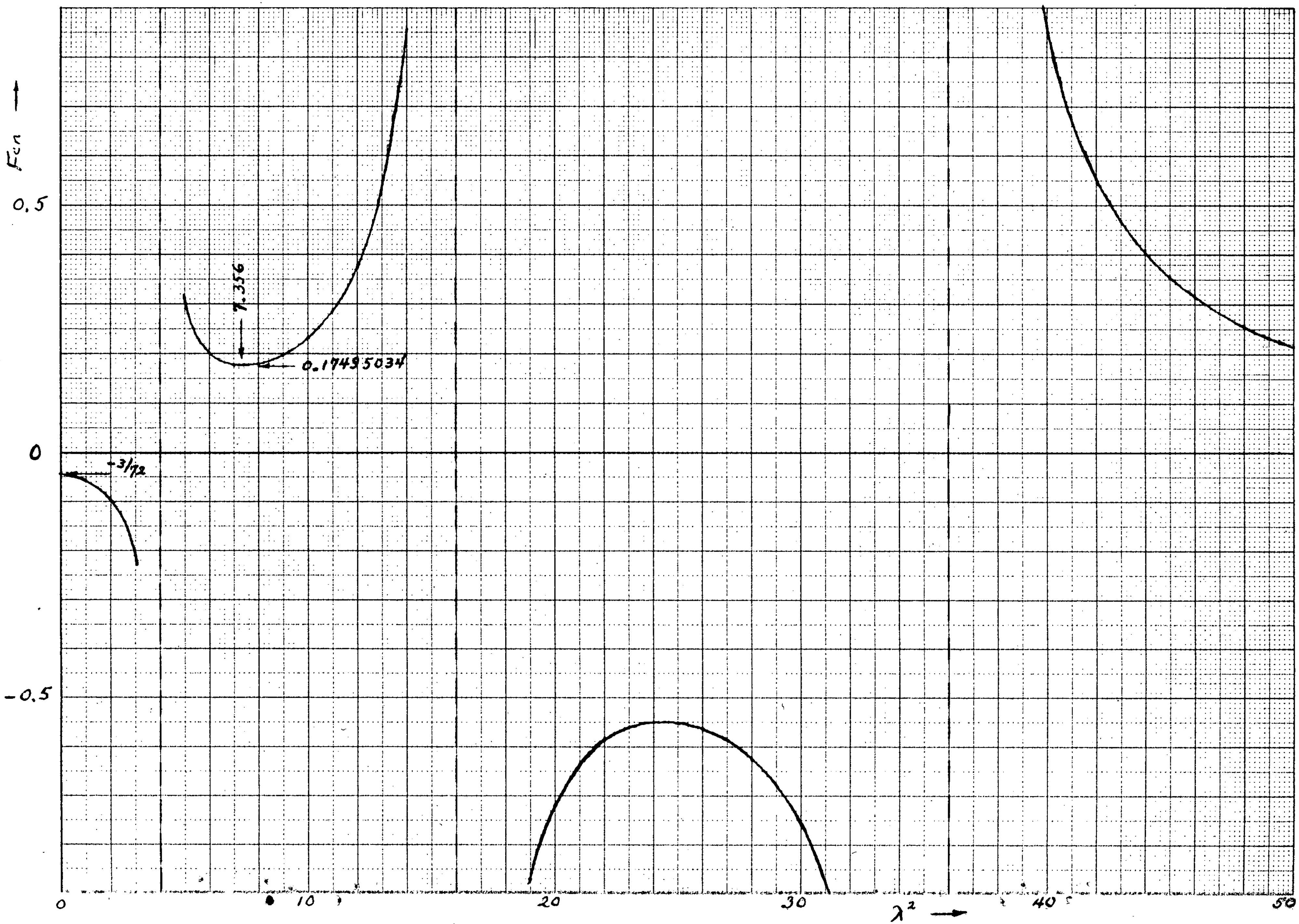
STABLE



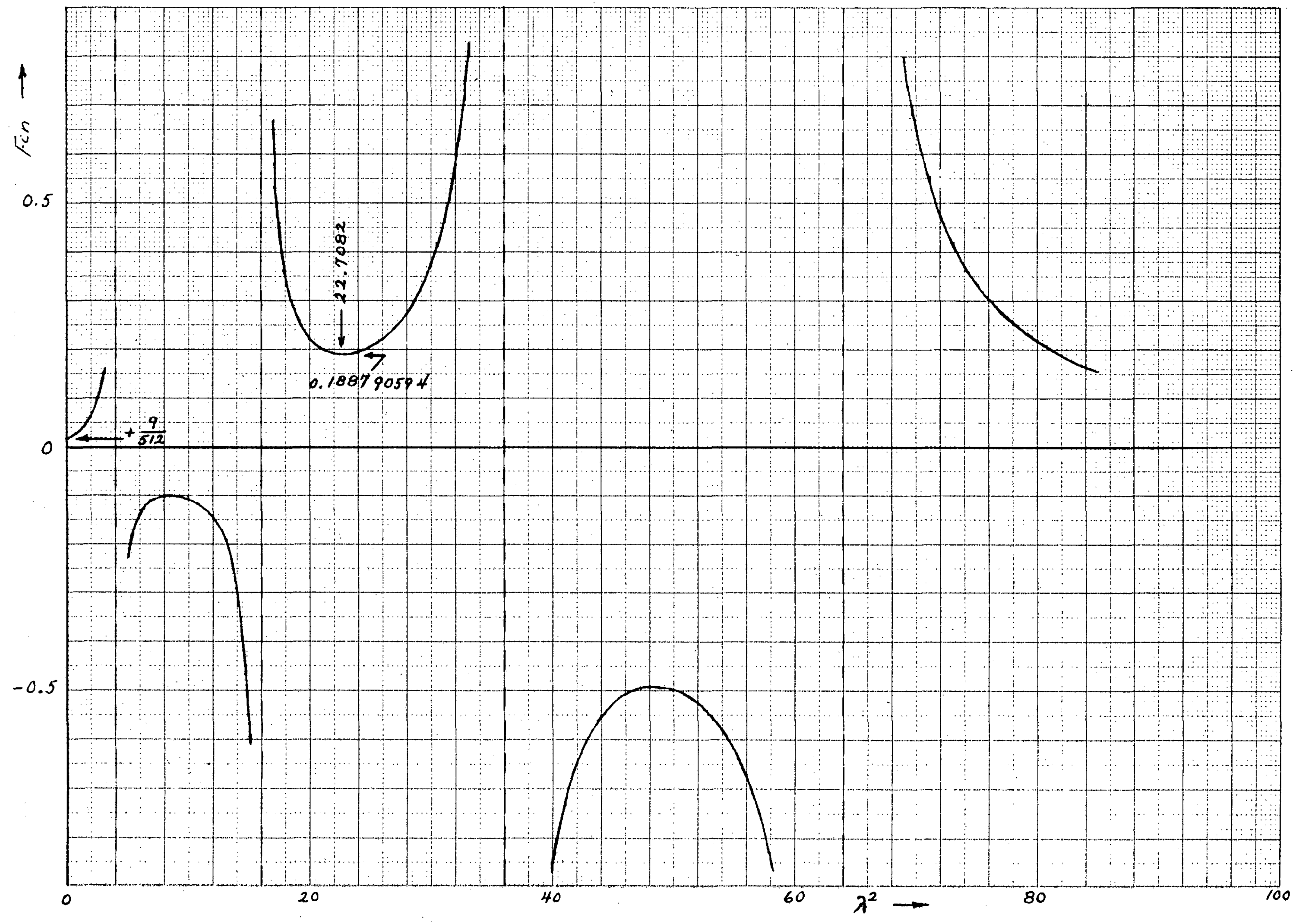
$m = 0 \quad j = 2 \qquad F_{cn} = -\frac{1}{\lambda^2 - 4} + \frac{3}{\lambda^2 - 16}$



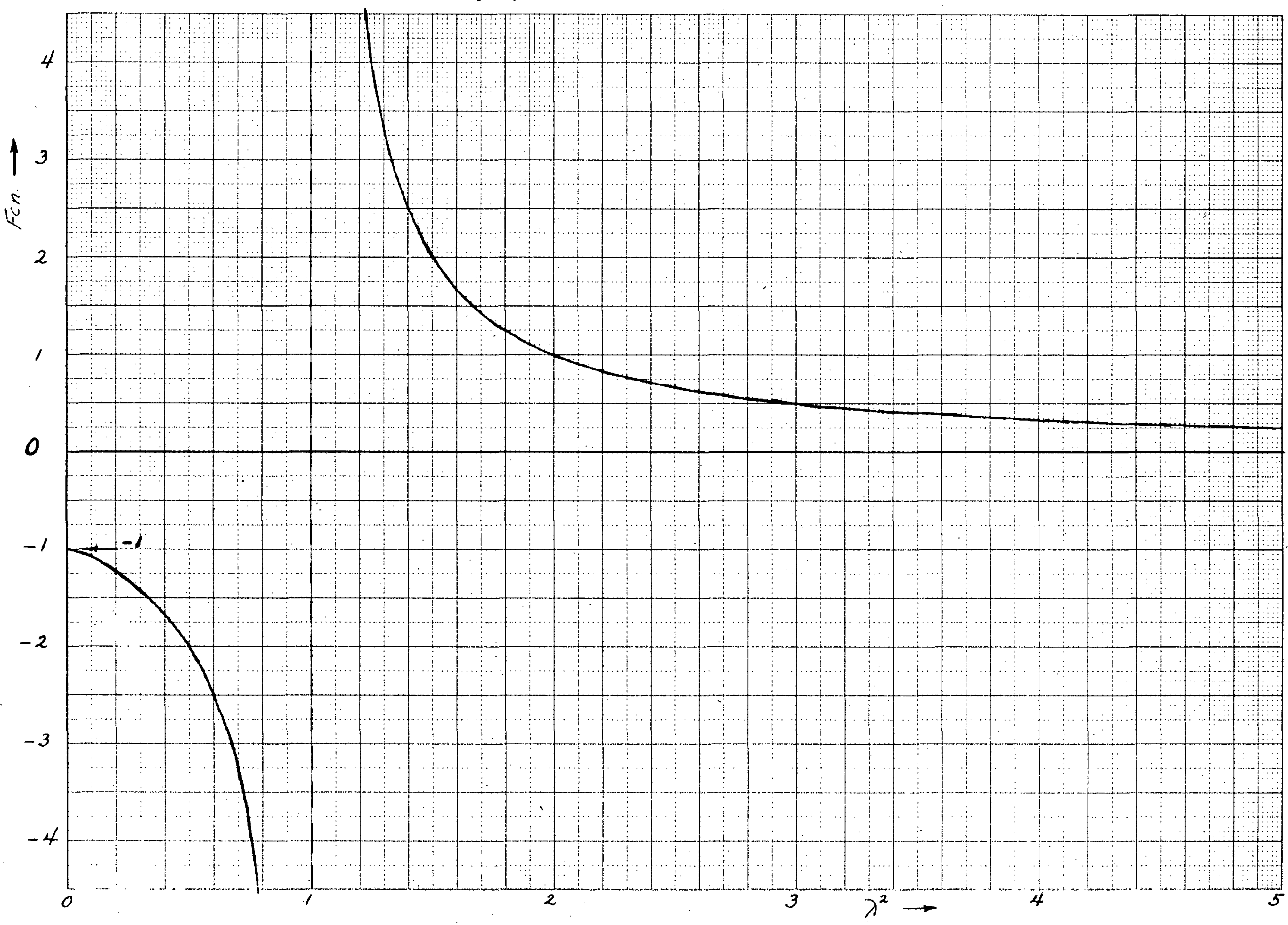
$m = 0 \qquad j = 3 \qquad F_{cn} = \frac{1}{4}\left[\frac{1}{\lambda^2-4} - \frac{8}{\lambda^2-16} + \frac{15}{\lambda^2-36}\right]$



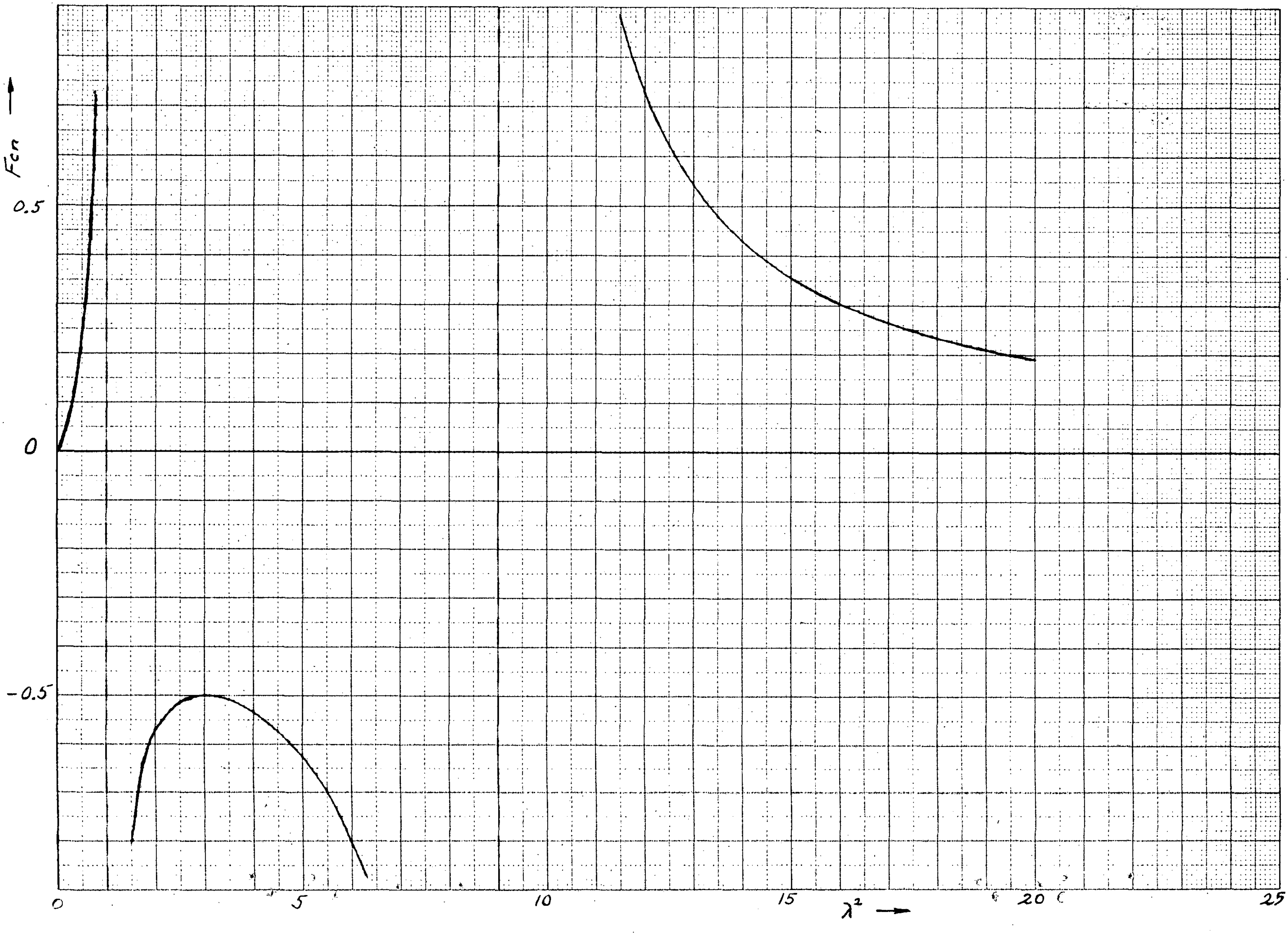
$m = 0$ $j = 4$ $Fcn. = \frac{1}{16}\left[-\frac{3}{\lambda^2-4} + \frac{10}{\lambda^2-16} - \frac{45}{\lambda^2-36} + \frac{70}{\lambda^2-64}\right]$



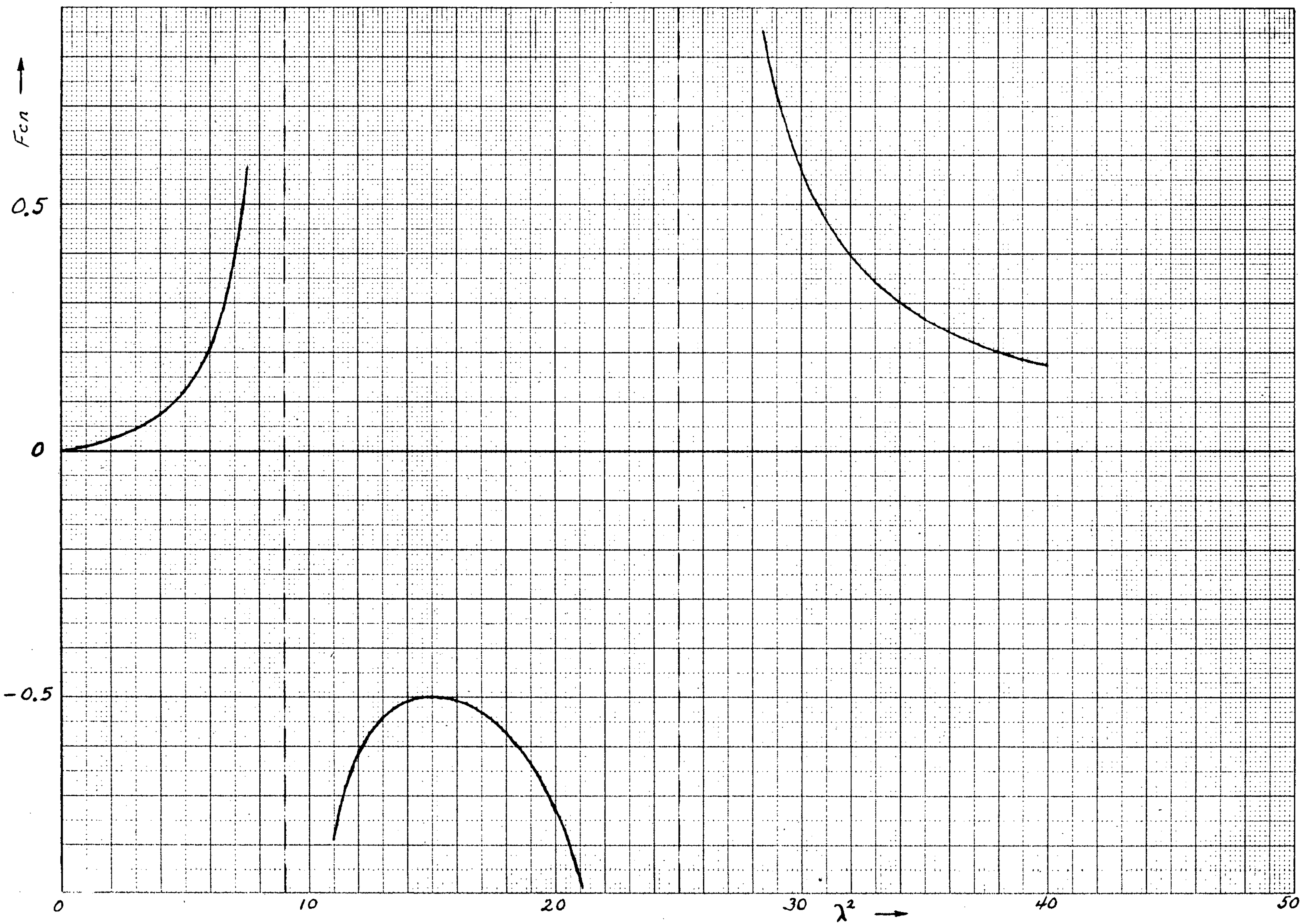
$m = 1 \qquad j = 0 \qquad Fcn = \frac{1}{\lambda^2 - 1}$ STABLE



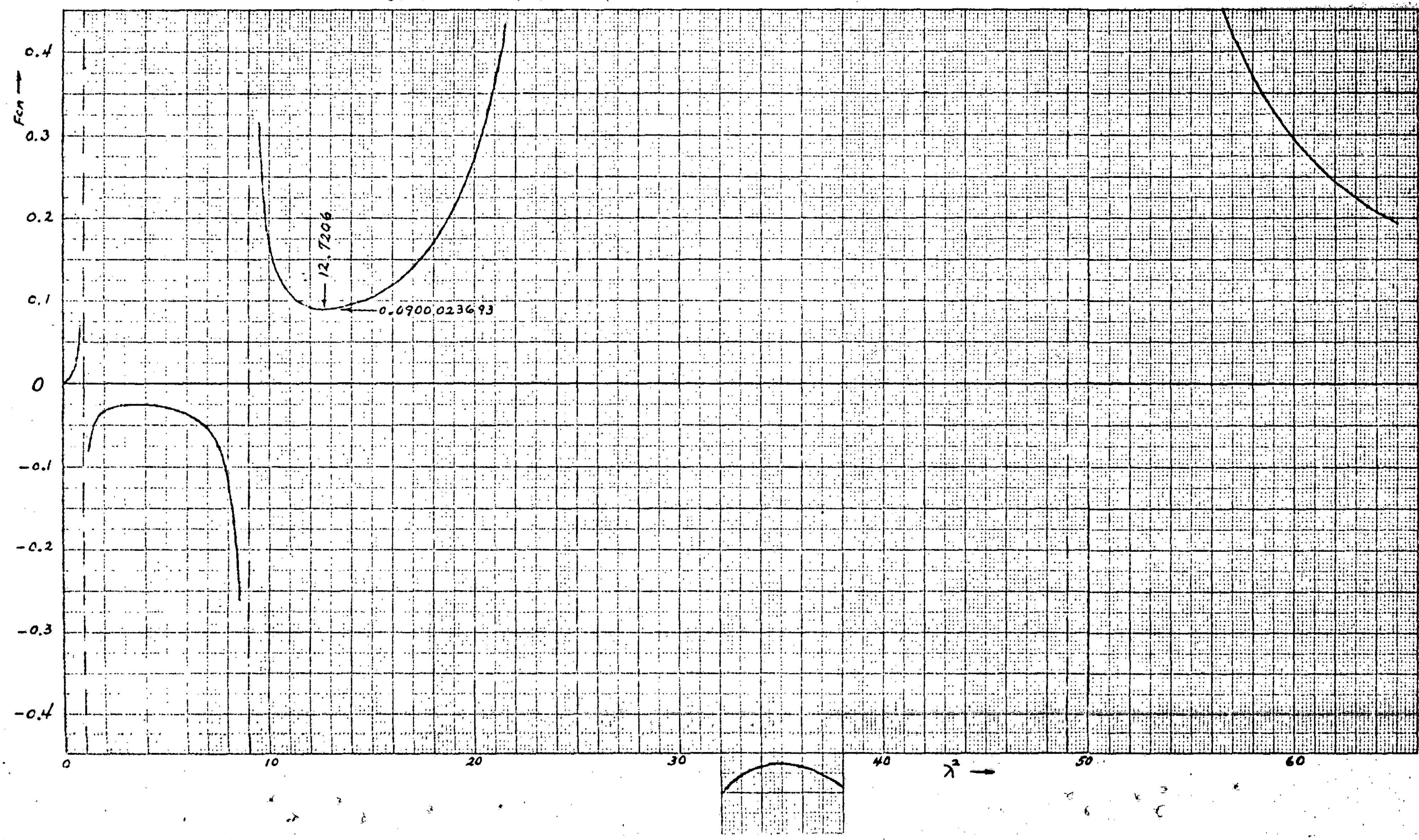
$m = 1 \qquad j = 1 \qquad F_{cn} = \frac{1}{4}\left[-\frac{1}{\lambda^2 - 1} + \frac{9}{\lambda^2 - 9} \right]$ STABLE



$m = 1 \quad j = 2 \qquad F_{cn} = \frac{1}{8}\left[-\frac{9}{\lambda^2 - 9} + \frac{25}{\lambda^2 - 25} \right]$ STABLE

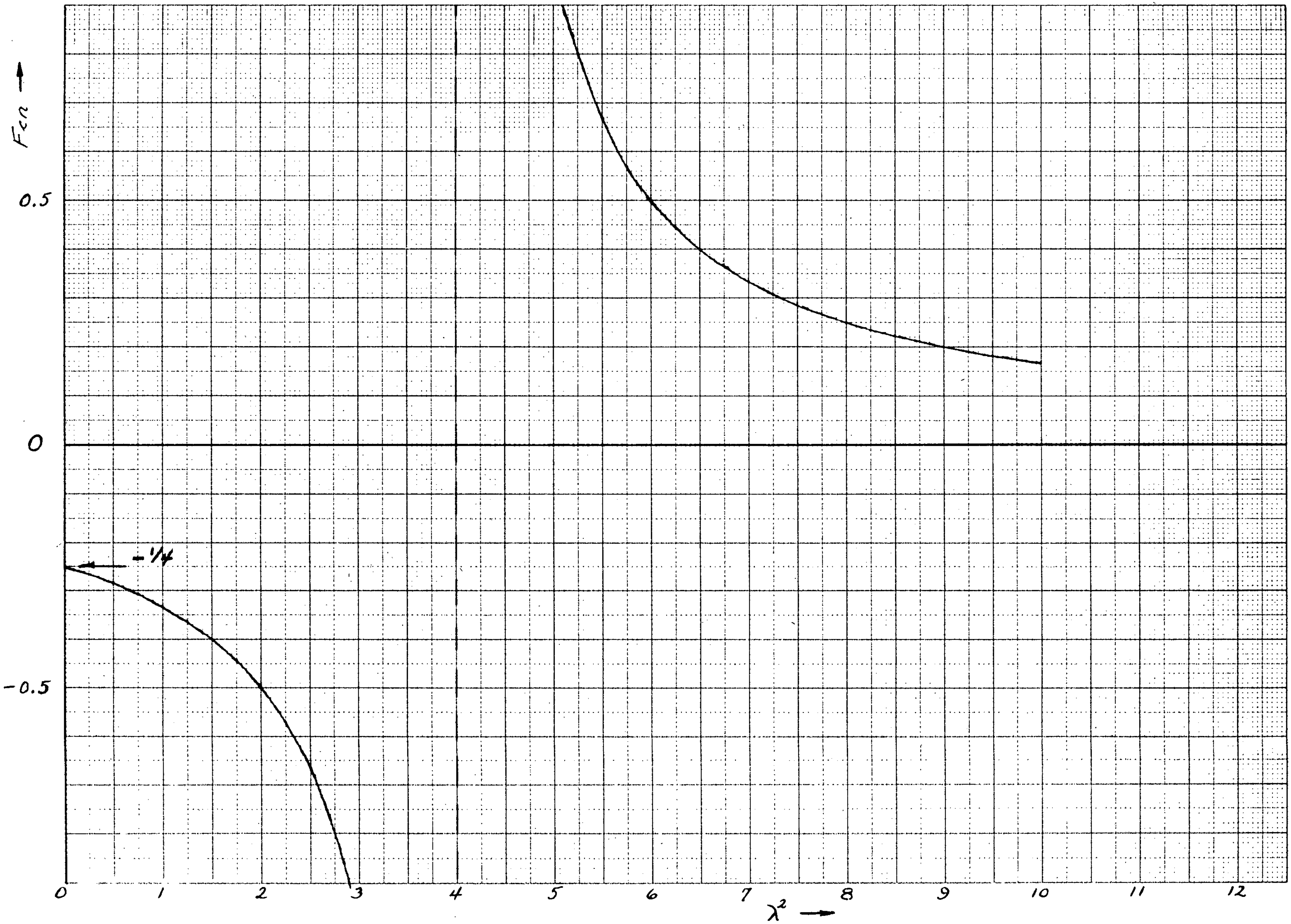


$m = 1 \qquad j = 3 \qquad F_{cn} = \frac{1}{64}\left[-\frac{1}{\lambda^2-1} + \frac{9}{\lambda^2-9} - \frac{125}{\lambda^2-25} + \frac{245}{\lambda^2-49}\right]$

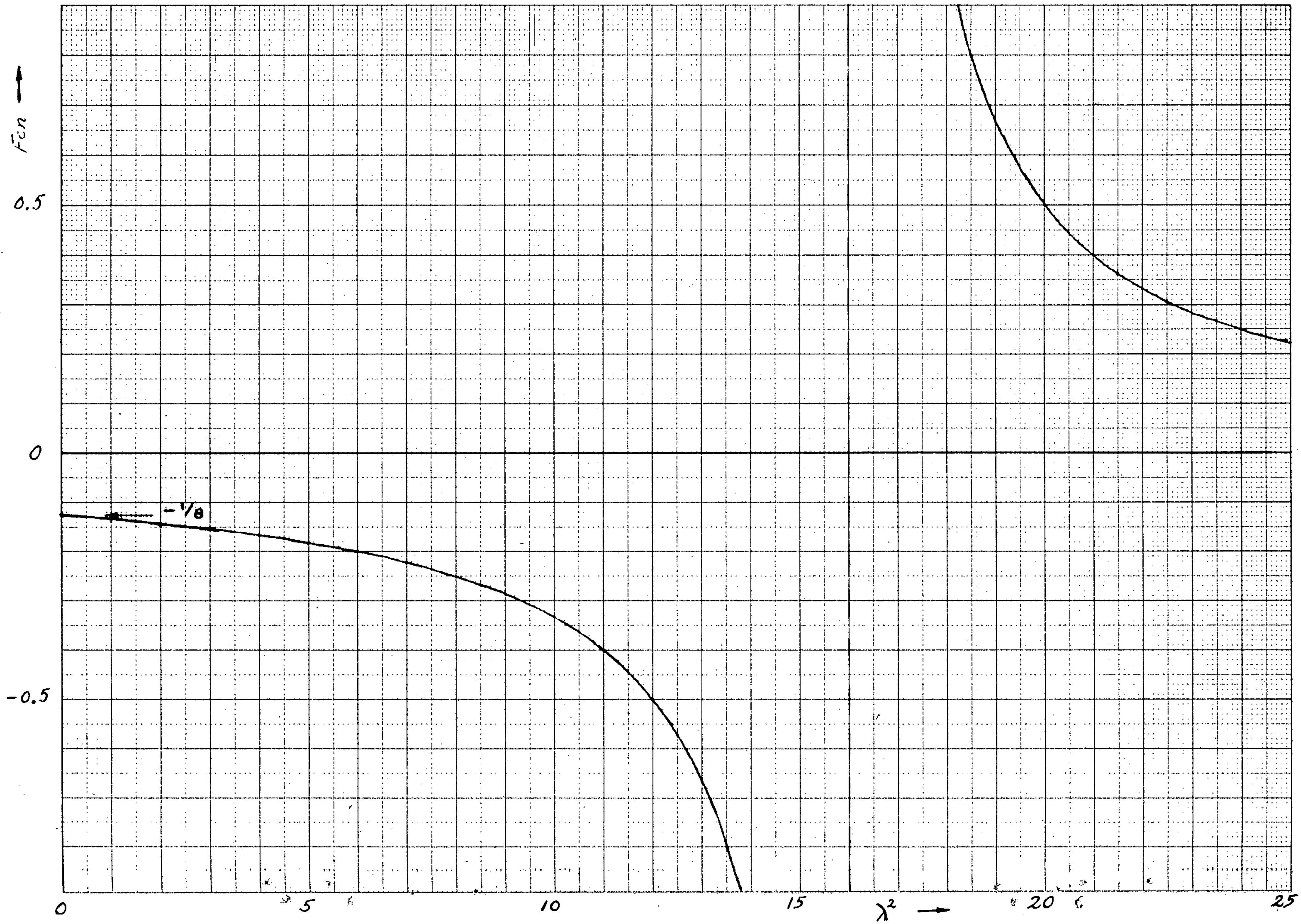


$m = 2 \quad j = 0 \quad F_{cn} = \frac{1}{\lambda^2 - 4}$

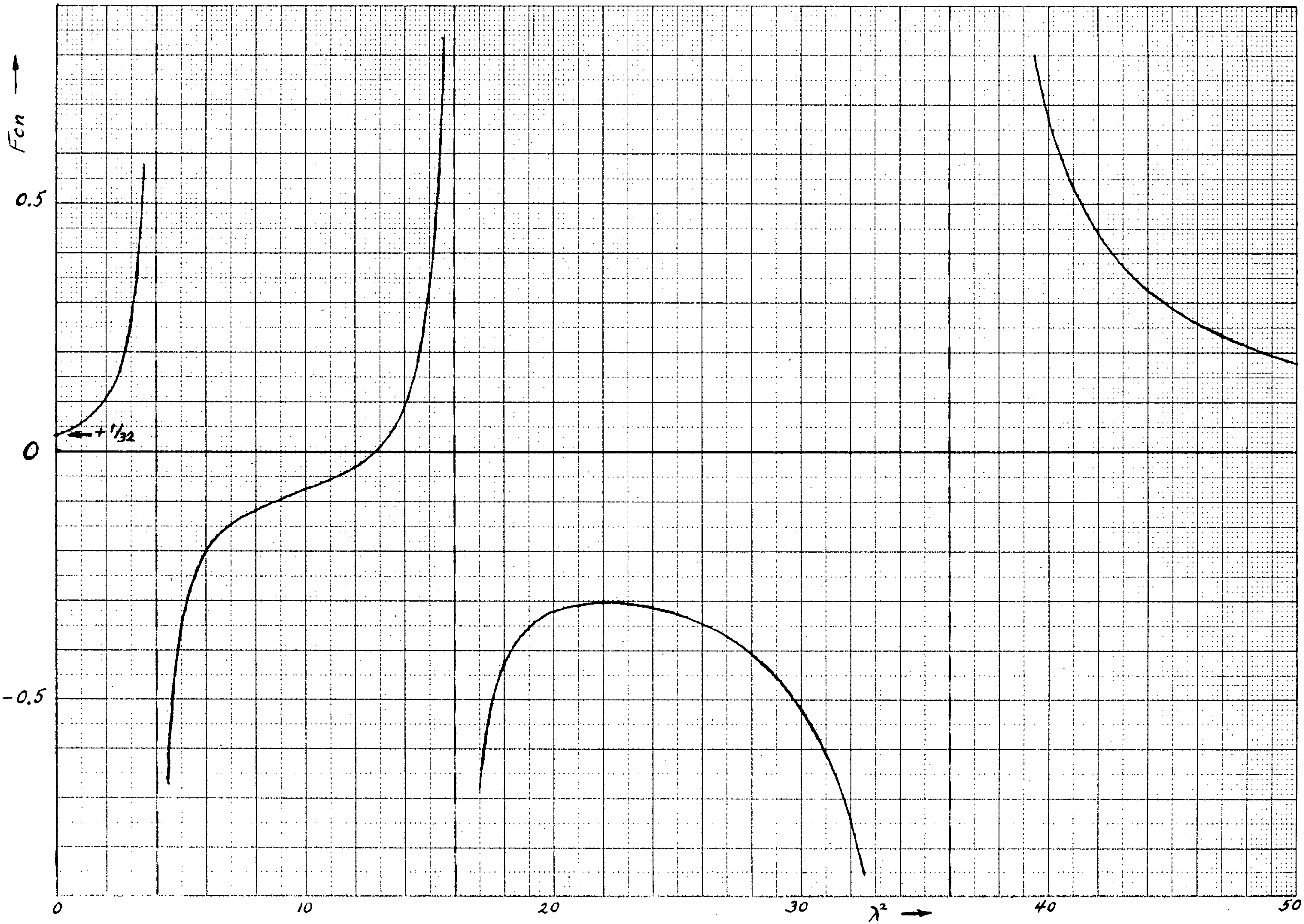
STABLE



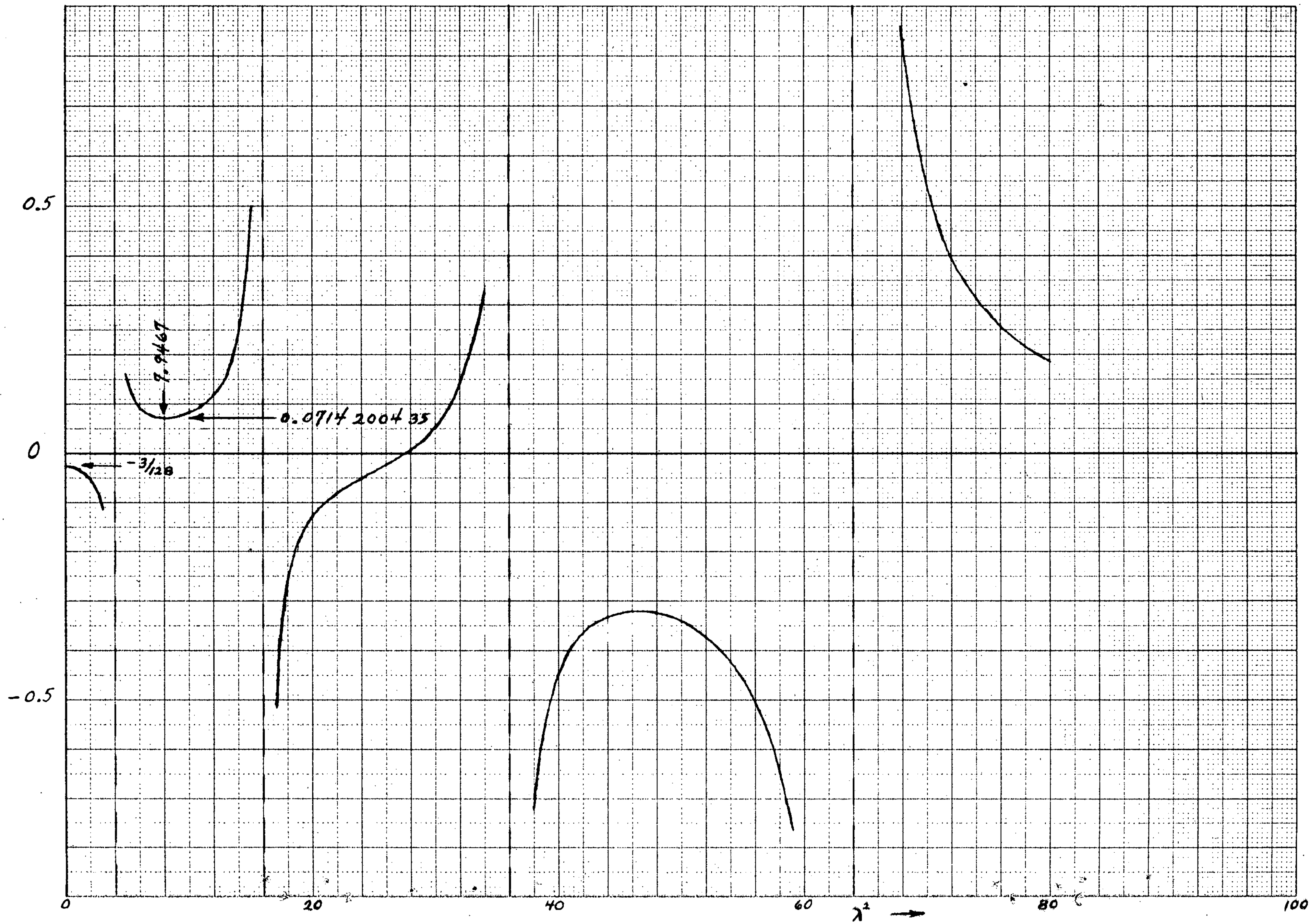
$m = 2 \quad j = 1 \quad F_{cn} = \frac{2}{\lambda^2 - 16}$ STABLE



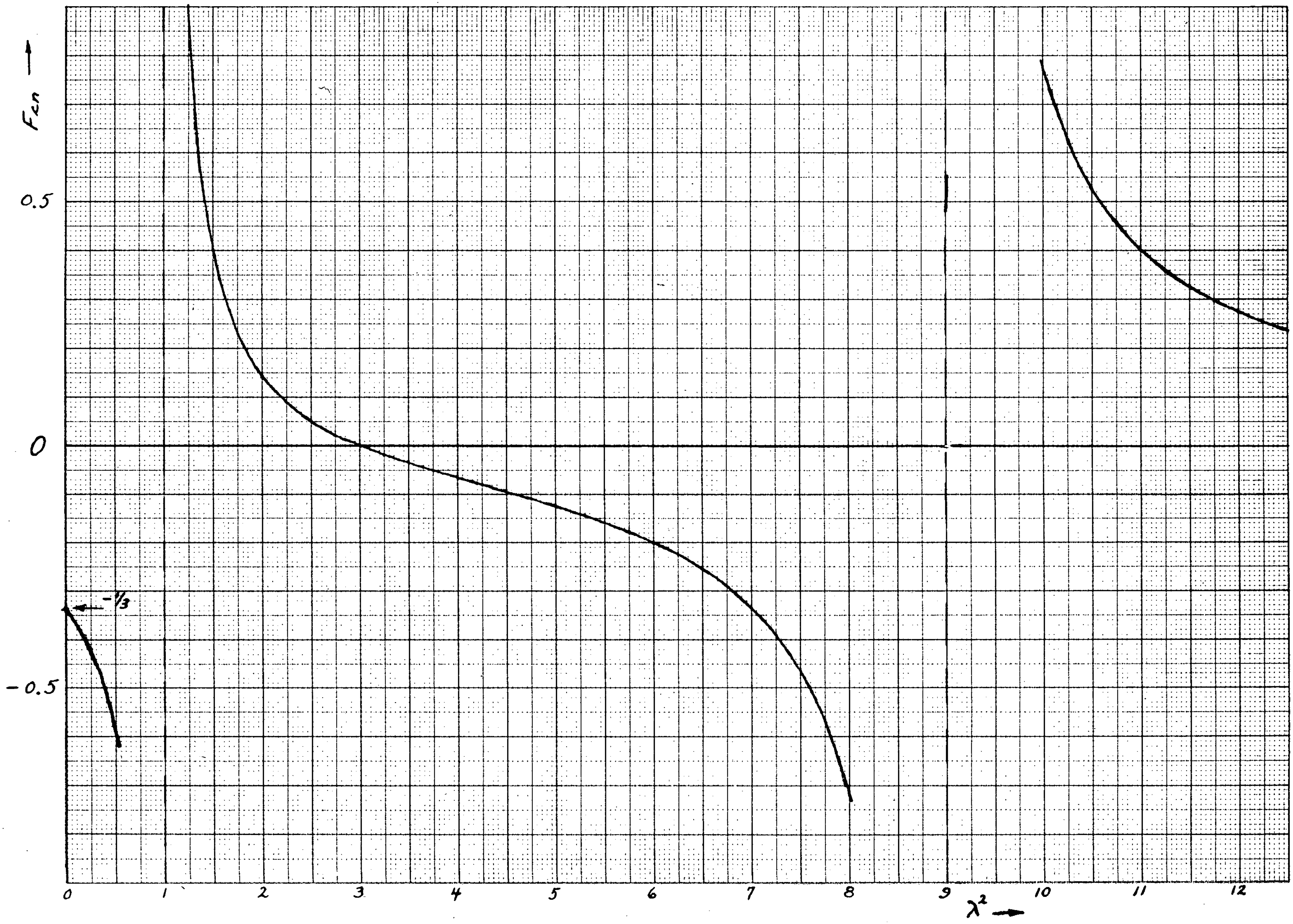
$m = 2 \qquad j = 2 \qquad F_{cn} = \frac{1}{16}\left[-\frac{5}{\lambda^2-4} - \frac{8}{\lambda^2-16} + \frac{45}{\lambda^2-36}\right]$



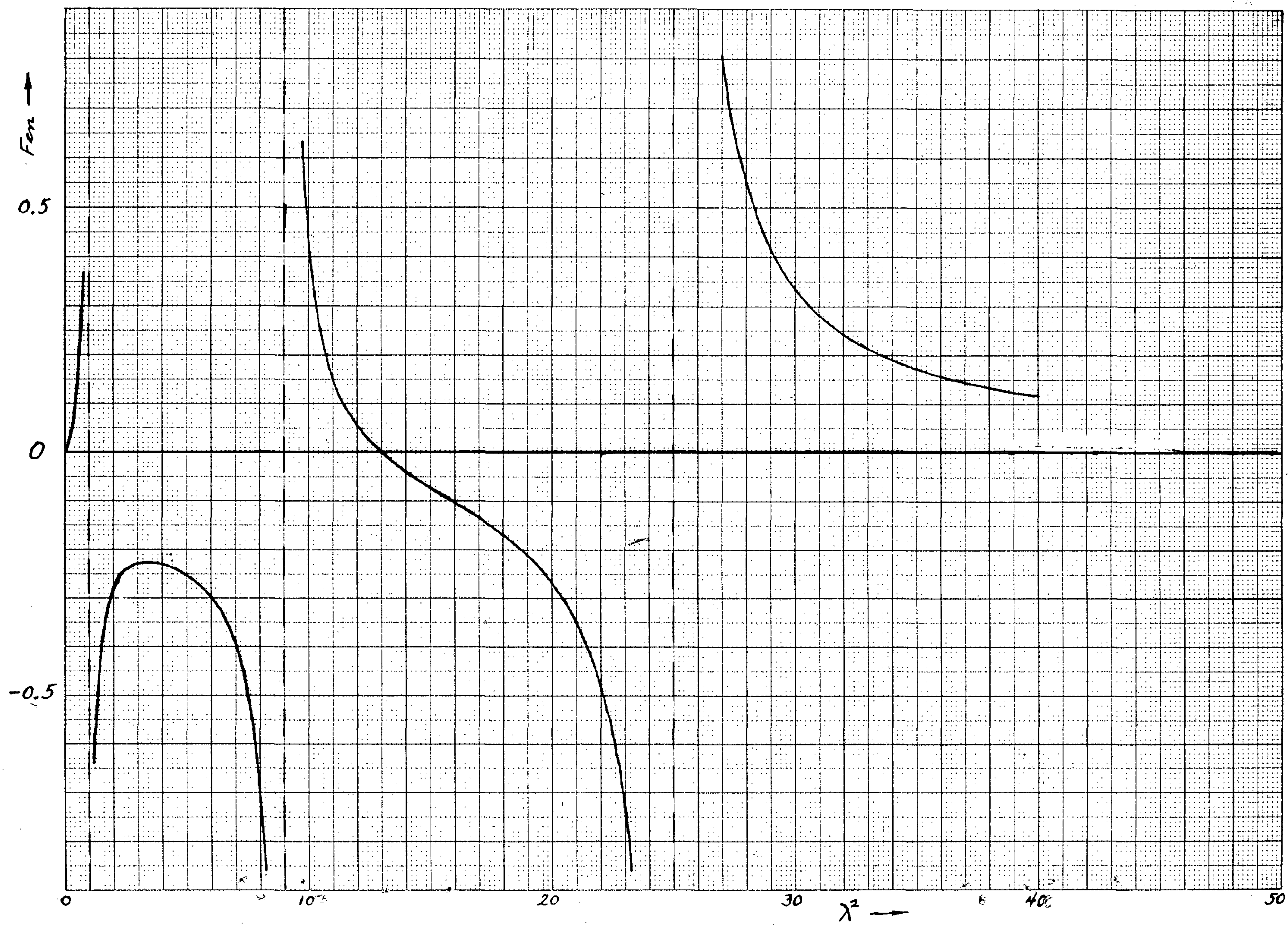
$m = 2 \qquad j = 3 \qquad Fcn = \frac{1}{8}\left[\frac{1}{\lambda^2-4} - \frac{4}{\lambda^2-16} - \frac{9}{1-36} + \frac{28}{\lambda^2-64}\right]$

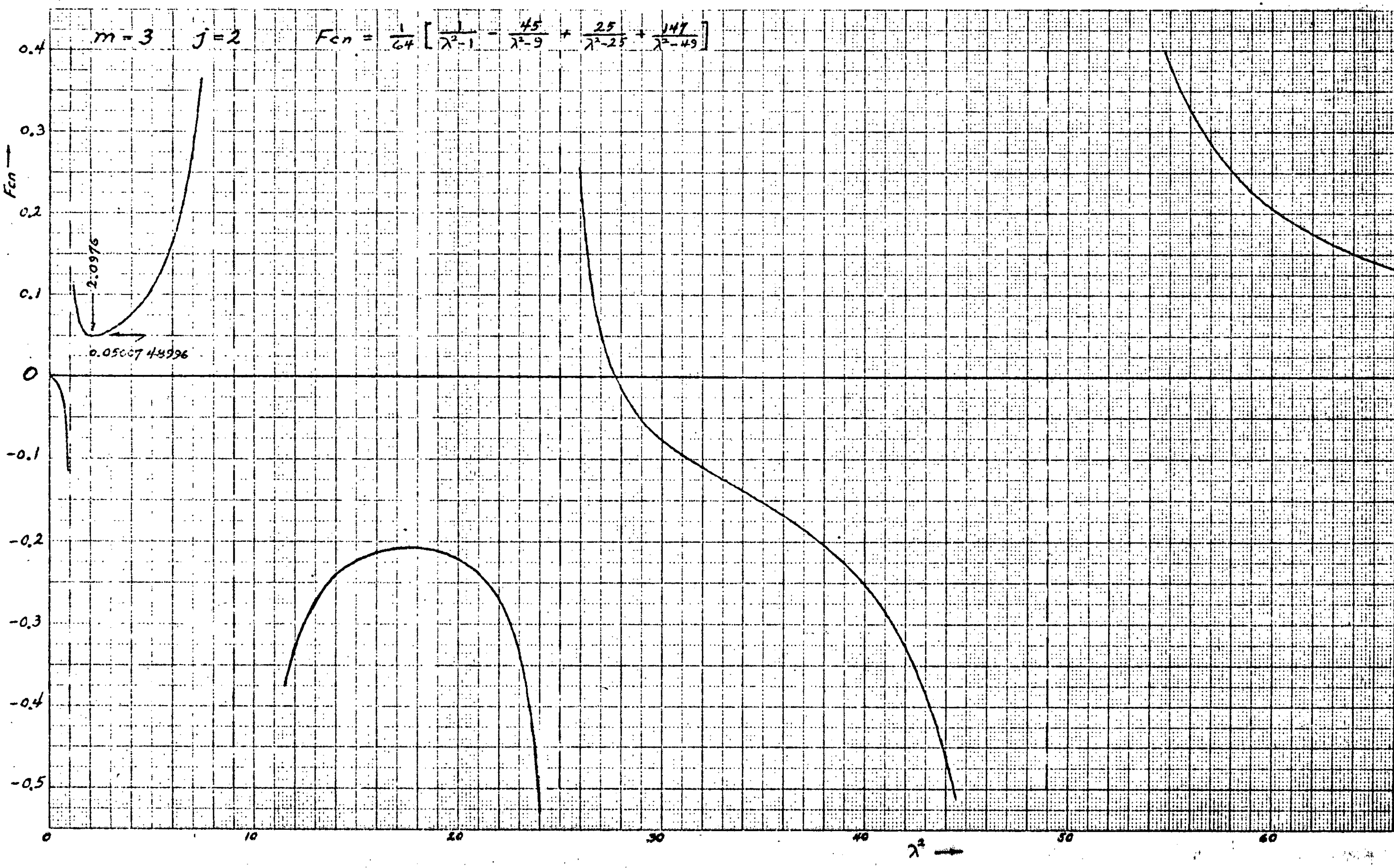


$m = 3 \qquad j = 0 \qquad F_{cn} = \frac{1}{4}\left[\frac{1}{\lambda^2 - 1} + \frac{3}{\lambda^2 - 9}\right]$ STABLE



$m = 3 \qquad j = 1 \qquad F_{cn} = \frac{1}{16}\left[-\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{25}{\lambda^2-25}\right]$ STABLE

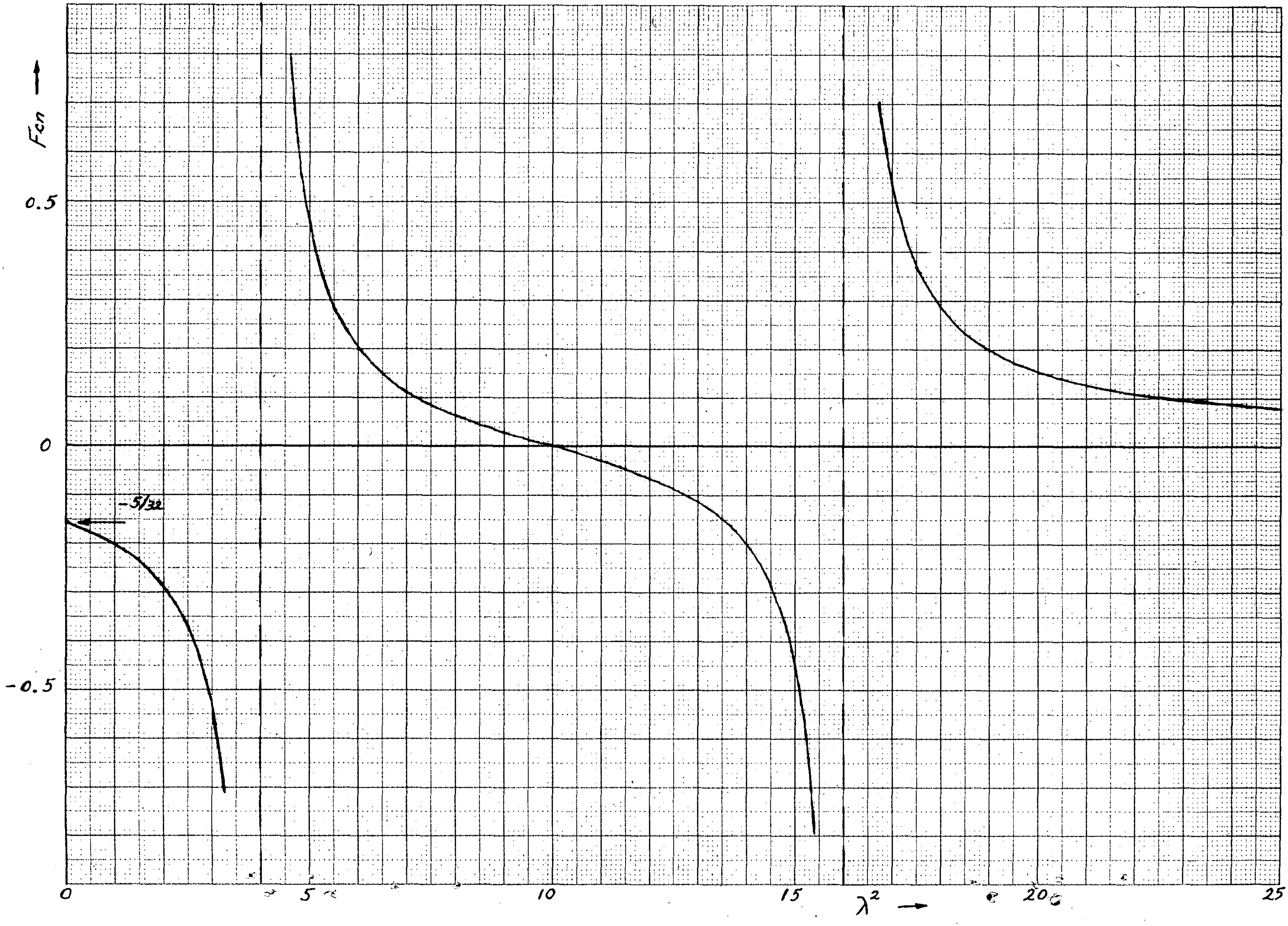




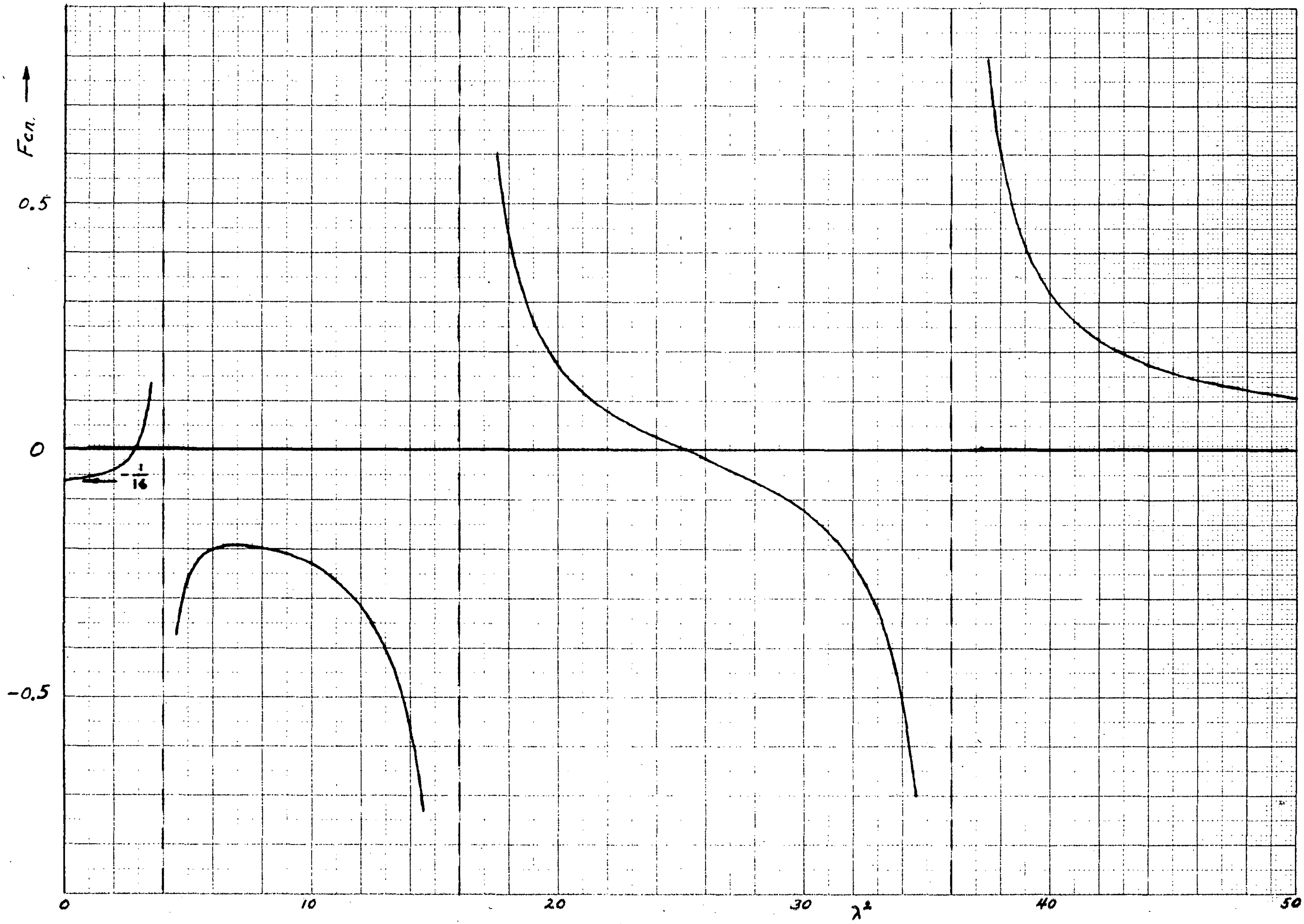
m = 3 j = 2
Fcn = 1/64 [1/(λ²-1) - 45/(λ²-9) + 25/(λ²-25) + 147/(λ²-49)]
2.0976
0.05007 48996
Fcn
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
10
20
30
40
50
60
λ²

$m = 4 \quad j = 0 \qquad F_{cn} = \frac{1}{2}\left[\frac{1}{\lambda^2-4} + \frac{1}{\lambda^2-16}\right]$

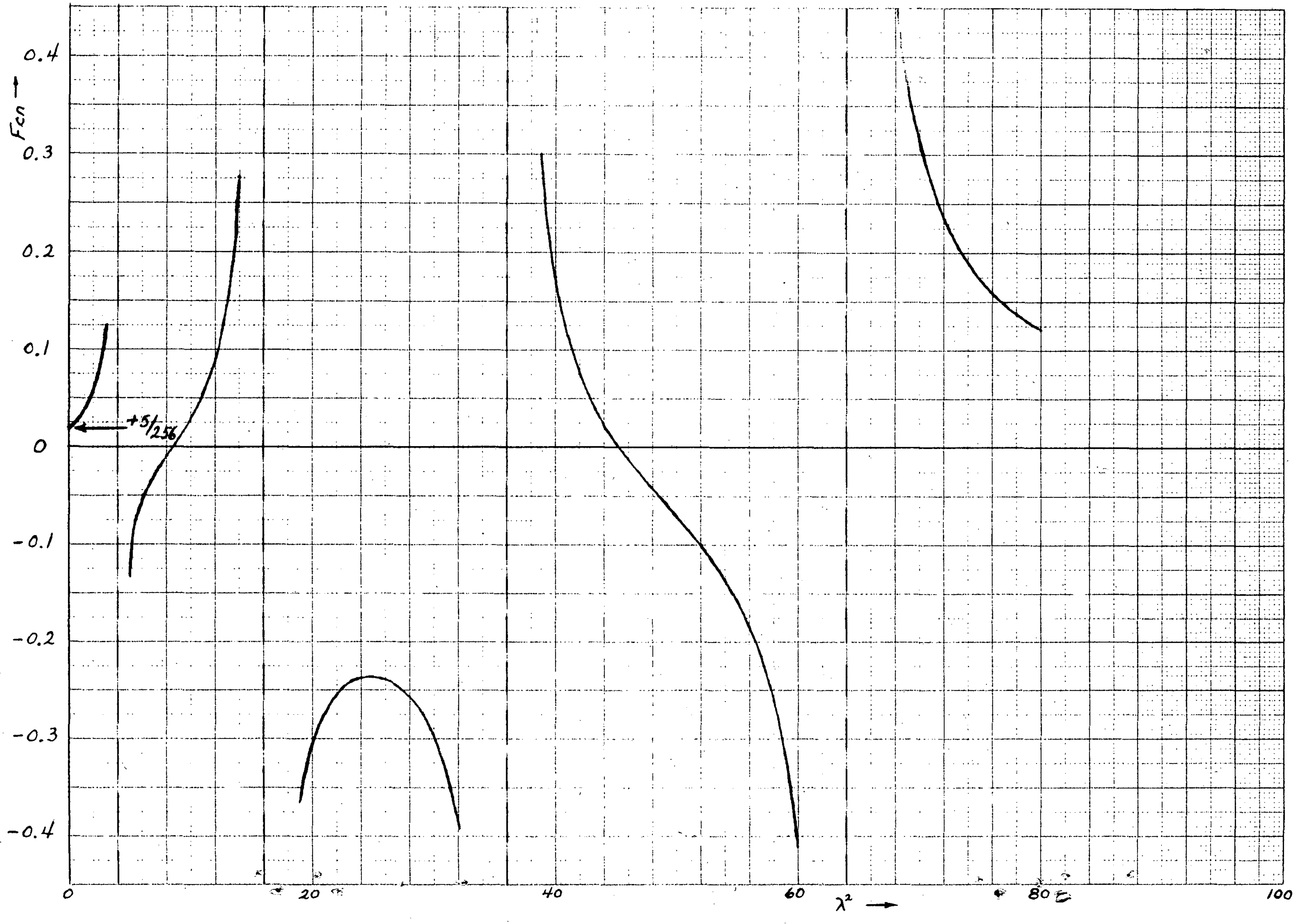
STABLE



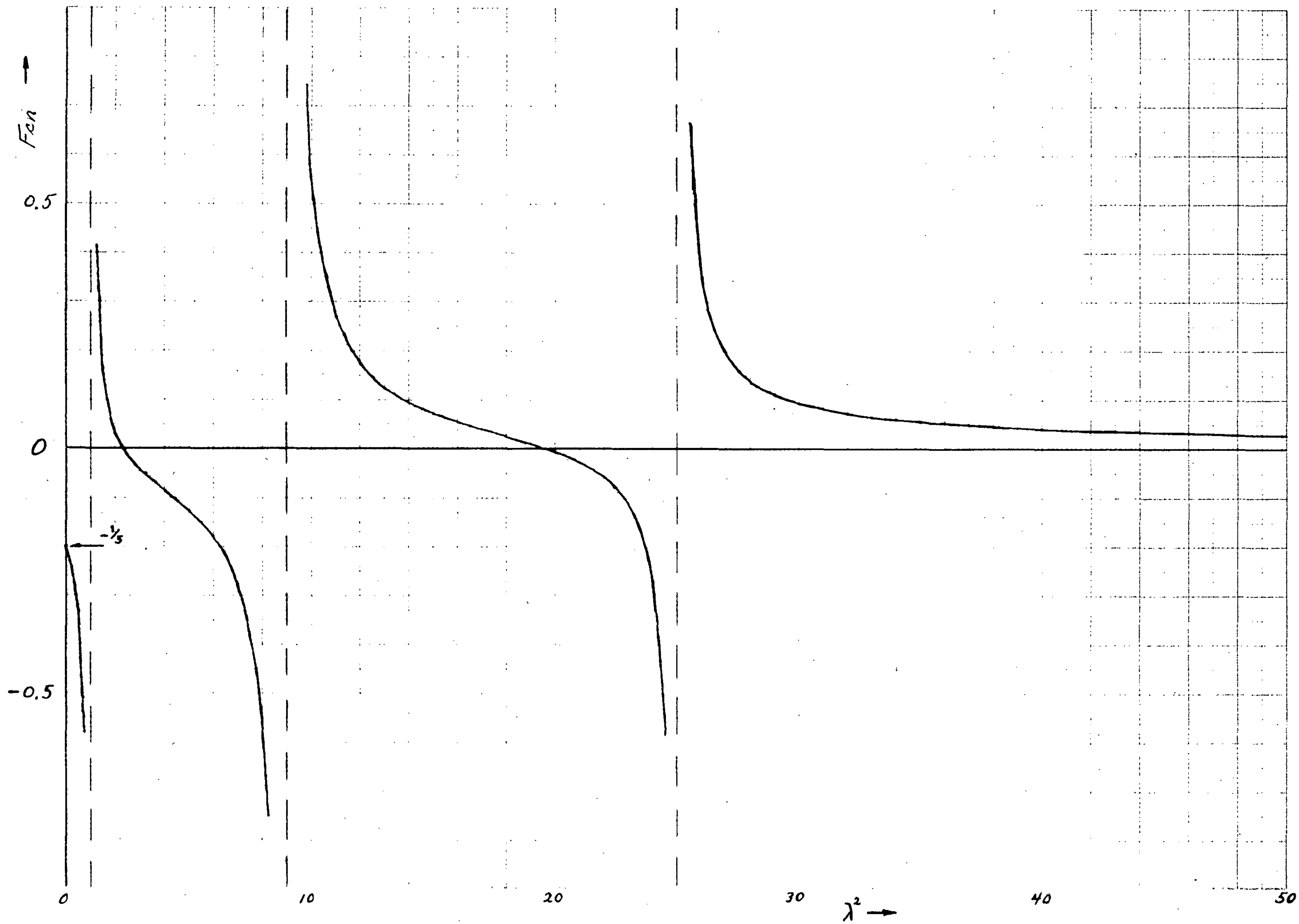
$m = 4 \qquad j = 1 \qquad F_{cn} = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} + \frac{8}{\lambda^2-16} + \frac{9}{\lambda^2-36}\right]$ STABLE

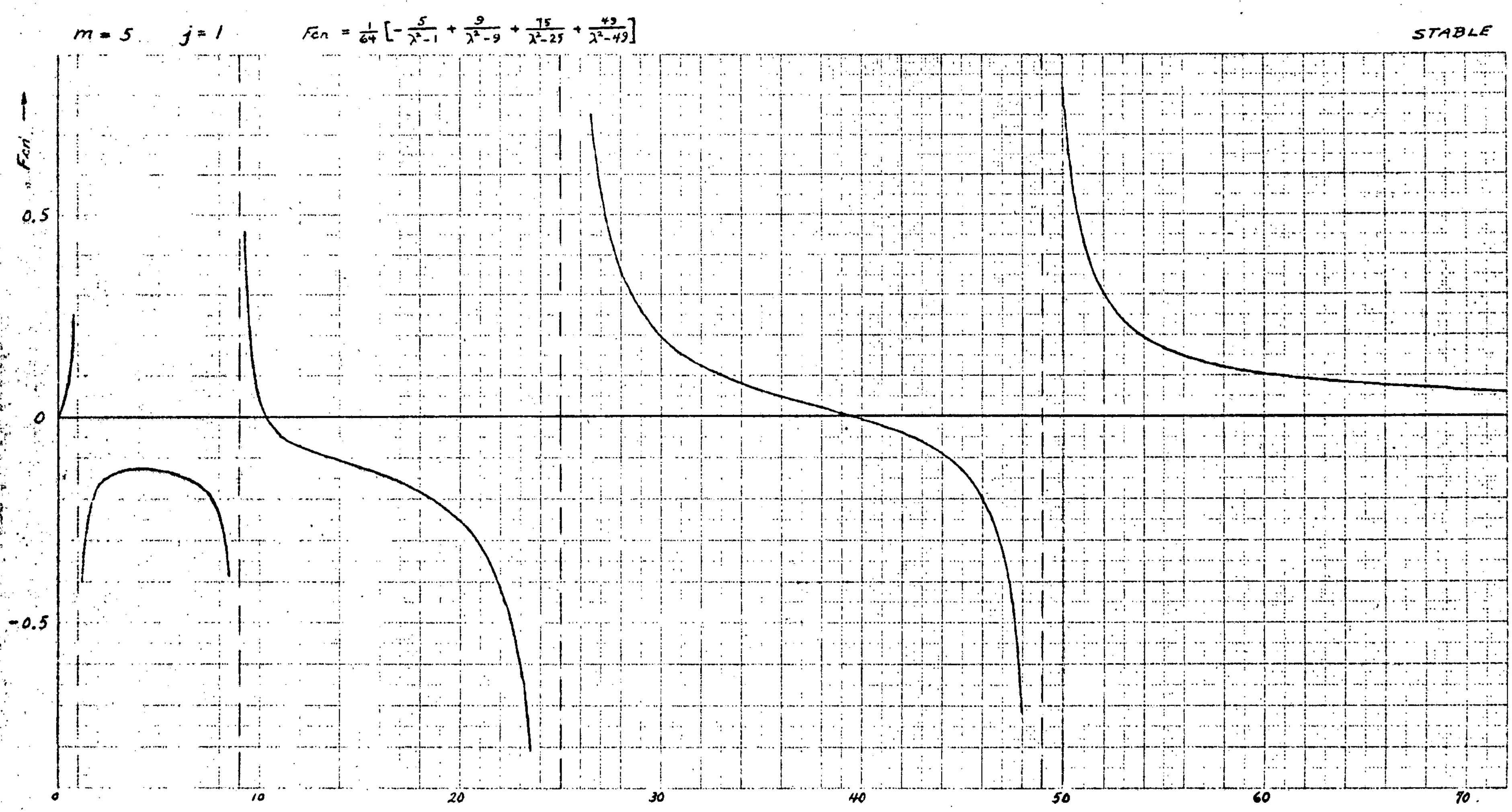


$m = 4 \qquad j = 2 \qquad F_{cn} = \frac{1}{8}\left[-\frac{1}{\lambda^2-4} - \frac{6}{\lambda^2-16} + \frac{9}{\lambda^2-36} + \frac{14}{\lambda^2-64}\right]$



$m = 5 \qquad j = 0 \qquad F_{cn} = \frac{1}{16}\left[\frac{2}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{5}{\lambda^2-25}\right]$ STABLE





$m = 5 \quad j = 1 \qquad Fcn = \frac{1}{64}\left[-\frac{5}{\lambda^2-1} + \frac{9}{\lambda^2-9} + \frac{75}{\lambda^2-25} + \frac{49}{\lambda^2-49}\right]$
STABLE
Fcn
0.5
0
-0.5
0
10
20
30
40
50
60
70