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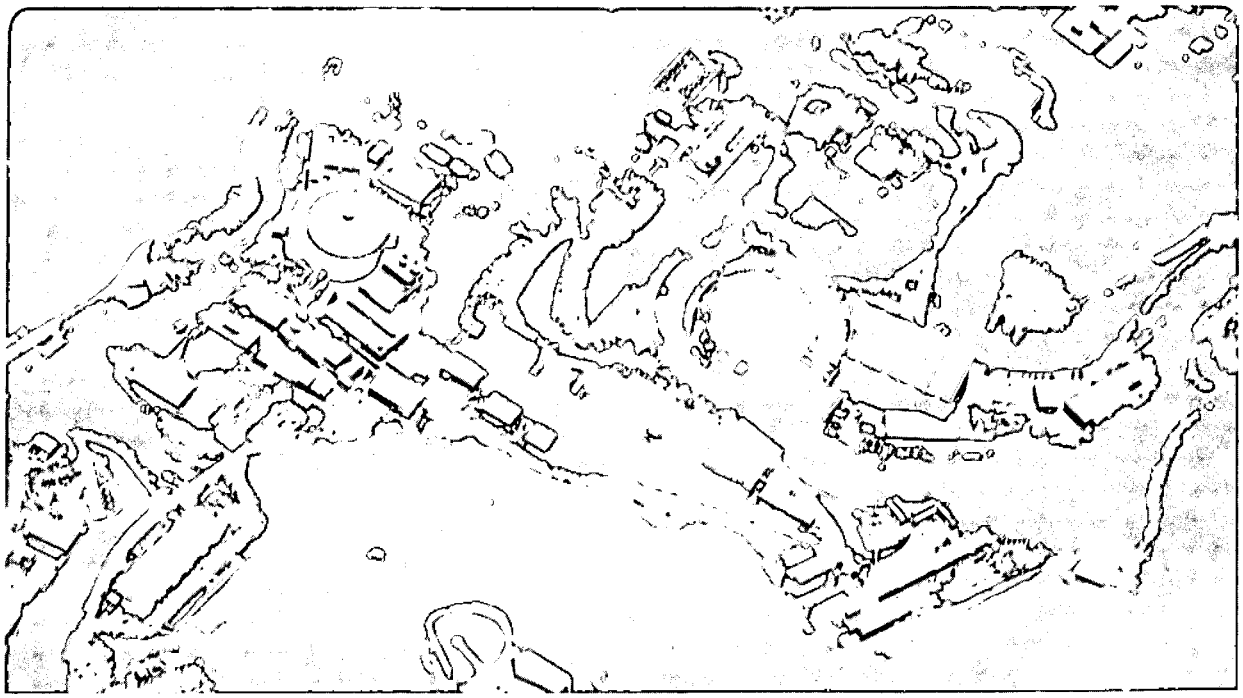

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CONSTRUCTION OF SIMPLE MULTISCALE BASES FOR FAST MATRIX OPERATIONS*

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Bradley K. Alpert

Abstract

Wavelet-like bases are presented with the property that a variety of integral operators are represented in these bases as sparse matrices. The bases differ from standard constructions of wavelets, in that they possess no elemental basis function, yet they retain the multiresolution structure of wavelets. Integral operators arising in problems of potential theory, which possess kernels that are smooth except at diagonal singularities, have sparse matrix representations in these bases. For these matrices, common matrix operations, including application of a matrix or its inverse to a vector, matrix-matrix multiplication, and matrix factorization, are fast.

Key Words. wavelets, integral equations, sparse matrices, fast algorithms

AMS(MOS) subject classifications. 42C15, 45L10, 65R10, 65R20

Several common matrix operations often dominate the computer time in scientific computations in which they occur. These operations include the application of a matrix to a vector (matrix-vector multiplication), the application of the inverse of a matrix to a vector, and the addition or multiplication of two matrices. For arbitrary dense $n \times n$-matrices, these operations each require computer time of order $O(n^2)$ to order $O(n^3)$, and typical scientific computations require many repetitions of these operations. Sparse matrix techniques have been developed to reduce these costs in cases where the matrices involved are sparse.

In recent years a number of numerical algorithms have been developed ([7], [10], [11], [14]) in which operators of the type arising in potential theory are represented with "sparse" constructions. These algorithms effectively achieve application of a dense $n \times n$-matrix to an arbitrary vector in order $O(n)$ operations. Yet more recently, algorithms have been constructed ([3], [6], [8]) which explicitly transform dense matrices representing integral operators to sparse matrices in order $O(n)$ or $O(n \log n)$ operations. The application of these sparse matrices to vectors is similarly fast.

Perhaps of even greater interest is the fact that the inverses of these sparse matrices, when they exist, are also sparse and can be obtained in order $O(n)$ or $O(n \log^2 n)$ operations. As a result, the application of the inverse of a dense matrix to a vector can be made fast, and a variety of integral equations can be solved rapidly.

The transformation of dense matrices to sparse ones is accomplished by a coordinate transformation, in which the revised coordinates, or basis, is a basis of wavelets. For these applications the essential properties of the basis is that it consists of functions that (generally) are

1. Non-zero on finite intervals of various lengths, and

2. Orthogonal to low-order polynomials.

These two properties combine so that "smooth" operators are transformed to sparse matrices. The defining property of wavelet bases, that a basis should consist of a single basic shape which is identical on all scales, is not essential for these applications. In this chapter we outline the construction of bases which retain the multiscale structure of wavelets but in which the requirement of a single basic shape is discarded. The additional flexibility thereby obtained enables construction of simple bases for $\mathcal{L}^2[0,1]$, as well as bases for the finite-dimensional space of functions defined on a set of points $\{x_1, \ldots, x_n\} \subset [0,1]$. In addition, we construct bases (in one dimension) for which the transformed matrices can be factored into sparse lower and upper-triangular matrices. The latter construction leads to the rapid solution of certain first-kind integral equations not treated by other algorithms.

In §1 we present a sketch of the construction of these classes of bases (details may be found in the references). In §2 we state the fundamental analytical properties of the bases which lead to the sparse representation of integral operators. In §3 we present several numerical examples, and in §4 we give a few concluding remarks.

1 Construction of Simple Multiscale Bases

We begin this section with the construction in §1.1 of a class of bases for $\mathcal{L}^2[0,1]$. The class is indexed by k, a positive integer, which denotes the number of vanishing moments of the basis functions: we say a basis $\{b_1, b_2, b_3, \ldots\}$ from this class is of *order* k if

$$\int_0^1 b_i(x)\, x^j\, dx = 0, \qquad j = 0, \ldots, k-1,$$

for each b_i with $i > k$. We will see that in addition to several vanishing moments, most basis functions b_i are non-zero only on small subintervals of $[0,1]$. In §1.2 we show how a slightly different point of view leads to a class of bases for functions defined on a discrete set of points $\{x_1, \ldots, x_n\}$. Finally, in §1.3 we show how the constructions of bases defined in §1.1 and §1.2 can be revised to yield bases supporting sparse factorizations.

1.1 Bases for $\mathcal{L}^2[0,1]$

We employ the multi-resolution analysis framework developed by Mallat [12] and Meyer [13], and discussed in detail by Daubechies [9]. For $m = 0, 1, 2, \ldots$ and $i = 0, 1, \ldots, 2^m - 1$ we define a half-open interval $I_{m,i} \subset [0,1)$ by the formula

$$I_{m,i} = \left[2^{-m} i, 2^{-m}(i+1)\right). \tag{1}$$

For a fixed m, the dyadic intervals $I_{m,i}$ are disjoint and their union is $[0,1)$; also $I_{m,i} = I_{m+1,2i} \cup I_{m+1,2i+1}$. Now we suppose that k is a positive integer and for $m = 0, 1, 2, \ldots$ and $i = 0, 1, \ldots, 2^m - 1$ we define a space $S^k_{m,i}$ of piecewise polynomial functions,

$$S^k_{m,i} = \{f : \begin{array}{l} f : \mathcal{R} \to \mathcal{R}, \text{ the restriction of } f \text{ to the interval } I_{m,i} \text{ is a} \\ \text{polynomial of degree less than } k, \text{ and } f \text{ vanishes elsewhere}\} \end{array} \tag{2}$$

and we further define the space S^k_m by the formula

$$S^k_m = S^k_{m,0} \oplus S^k_{m,1} \oplus \cdots \oplus S^k_{m,2^m-1}.$$

It is apparent that for each m and i the space $S^k_{m,i}$ has dimension k, the space S^k_m has dimension $2^m k$, and

$$S^k_{m,i} \subset S^k_{m+1,2i} \oplus S^k_{m+1,2i+1};$$

thus

$$S^k_0 \subset S^k_1 \subset \cdots \subset S^k_m \subset \cdots.$$

For $m = 0, 1, 2, \ldots$ and $i = 0, 1, \ldots, 2^m - 1$, we define the k-dimensional space $R^k_{m,i}$ to be the orthogonal complement of $S^k_{m,i}$ in $S^k_{m+1,2i} \oplus S^k_{m+1,2i+1}$,

$$S^k_{m,i} \oplus R^k_{m,i} = S^k_{m+1,2i} \oplus S^k_{m+1,2i+1}, \qquad R^k_{m,i} \perp S^k_{m,i},$$

and we further define the space R^k_m by the formula

$$R^k_m = R^k_{m,0} \oplus R^k_{m,1} \oplus \cdots \oplus R^k_{m,2^m-1}.$$

Now we have $S_m^k \oplus R_m^k = S_{m+1}^k$, so we inductively obtain the decomposition

$$S_m^k = S_0^k \oplus R_0^k \oplus R_1^k \oplus \cdots \oplus R_{m-1}^k. \tag{3}$$

Suppose that functions $h_1, \ldots, h_k : \mathcal{R} \to \mathcal{R}$ form an orthogonal basis for R_0^k. Since R_0^k is orthogonal to S_0^k, the first k moments of $h_1, \ldots, h_k$ vanish,

$$\int_0^1 h_i(x)\, x^j\, dx = 0, \qquad j = 0, 1, \ldots, k-1.$$

The space $R_{m,i}^k$ has an orthogonal basis consisting of the k functions $h_1(2^m x - i)$, $\ldots, h_k(2^m x - i)$, which are non-zero only on the interval $I_{m,i}$, and themselves each have k vanishing moments. Introducing the notation $h_{m,i}^j$ for $j = 1, \ldots, k$, $m = 0, 1, 2, \ldots$, and $i = 0, 1, \ldots, 2^m - 1$, by the formula

$$h_{m,i}^j(x) = h_j(2^m x - i), \qquad x \in \mathcal{R},$$

we obtain from decomposition (3) the formula

$$\begin{aligned} S_m^k = S_0^k \oplus \text{linear span } \{h_{m,i}^j : \ & j = 1, \ldots, k; \\ & m = 0, 1, 2, \ldots;\ i = 0, 1, \ldots, 2^m - 1\}. \end{aligned} \tag{4}$$

An explicit construction of $h_1, \ldots, h_k$ is given in [3].

We define the space S^k to be the union of the S_m^k, given by the formula

$$S^k = \bigcup_{m=0}^{\infty} S_m^k, \tag{5}$$

and observe that $\overline{S^k} = \mathcal{L}^2[0,1]$. In particular, S^1 contains the Haar basis for $\mathcal{L}^2[0,1]$, which consists of functions piecewise constant on each of the intervals $I_{m,i}$. Here the closure $\overline{S^k}$ is defined with respect to the $\mathcal{L}^2$-norm. We let $\{u_1, \ldots, u_k\}$ denote any orthogonal basis for S_0^k; in view of (4) and (5), the orthogonal system

$$\begin{aligned} B_k = \quad & \{u_j : \ j = 1, \ldots, k\} \\ & \cup \{h_{m,i}^j : \ j = 1, \ldots, k;\ m = 0, 1, 2, \ldots;\ i = 0, \ldots, 2^m - 1\} \end{aligned}$$

spans $\mathcal{L}^2[0,1]$; we refer to B_k as the *multi-wavelet basis of order k* for $\mathcal{L}^2[0,1]$.

In [3] it is shown that B_k may be readily generalized to bases for $\mathcal{L}^2(\mathcal{R})$, $\mathcal{L}^2(\mathcal{R}^d)$, and $\mathcal{L}^2[0,1]^d$.

1.2 Bases for Discretely Defined Functions

The bases B_k described in §1.1 can be revised somewhat to yield bases for the n-dimensional space of functions defined on a set of points $X = \{x_1, \ldots, x_n\} \subset [0,1]$, where $x_1 < \cdots < x_n$. In the following development, for simplicity we assume that $n = 2^l k$, where l and k are positive integers. Analogous to the intervals $I_{m,i}$ defined in (1), for $m = 0, 1, \ldots, l$ and $i = 0, 1, \ldots, 2^m - 1$ we define the sets $X_{m,i}$ by the formula

$$X_{m,i} = \left\{ x_{n2^{-m}i+1},\ x_{n2^{-m}i+2}, \ldots,\ x_{n2^{-m}(i+1)} \right\}.$$

We again assume that k is a positive integer and for $m = 0, 1, \ldots, l$ and $i = 0, 1, \ldots, 2^m - 1$ we define the k-dimensional space $U^k_{m,i}$ by the formula

$$U^k_{m,i} = \{ f : \ f : X \to \mathcal{R}, \text{ the restriction of } f \text{ to the set } X_{m,i} \text{ is a polynomial of degree less than } k, \text{ and } f \text{ vanishes elsewhere} \}$$

and we further define the $2^m k$-dimensional space U^k_m by the formula

$$U^k_m = U^k_{m,0} \oplus U^k_{m,1} \oplus \cdots \oplus U^k_{m,2^m-1}.$$

Clearly, as for $S^k_{m,i}$, we have the inclusions

$$U^k_{m,i} \subset U^k_{m+1,2i} \oplus U^k_{m+1,2i+1}$$

and

$$U^k_0 \subset U^k_1 \subset \cdots \subset U^k_l.$$

For $m = 0, \ldots, l-1$ and $i = 0, 1, \ldots, 2^m - 1$, we define the k-dimensional space $T^k_{m,i}$ (analogous to $R^k_{m,i}$) to be the orthogonal complement of $U^k_{m,i}$ in $U^k_{m+1,2i} \oplus U^k_{m+1,2i+1}$,

$$U^k_{m,i} \oplus T^k_{m,i} = U^k_{m+1,2i} \oplus U^k_{m+1,2i+1}, \qquad T^k_{m,i} \perp U^k_{m,i},$$

and we further define the space T^k_m by the formula

$$T^k_m = T^k_{m,0} \oplus T^k_{m,1} \oplus \cdots \oplus T^k_{m,2^m-1}.$$

Now we have $U^k_m \oplus T^k_m = U^k_{m+1}$, so we inductively obtain the decomposition

$$U^k_l = U^k_0 \oplus T^k_0 \oplus T^k_1 \oplus \cdots \oplus T^k_{l-1}. \tag{6}$$

This decomposition parallels (3). In both decompositions the constituent subspaces, which reflect various scales, are naturally spanned by "locally" supported basis elements, each orthogonal to low-order polynomials. Although the spaces R^k_m are spanned by the translates and dilates of only k basis functions $h_1, \ldots, h_k$,

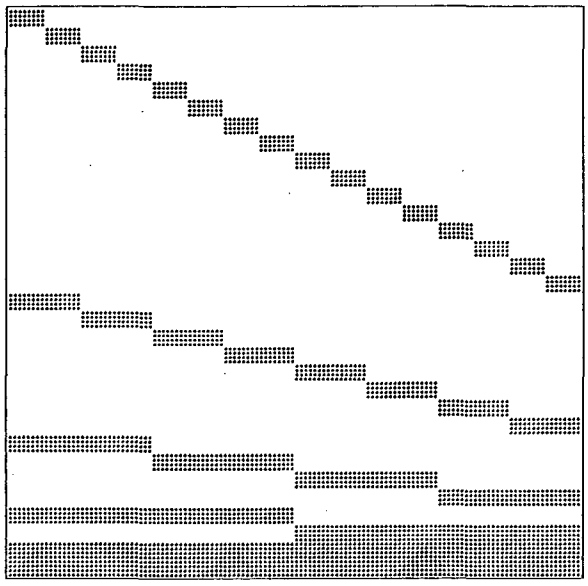

Figure 1: *The matrix represents a basis C_k for the space of functions defined on a set of points $\{x_1, \ldots, x_n\}$, where $n = 128$ and $k = 4$. Each row denotes one basis vector, with the dots depicting non-zero elements. All but the final k vectors have k vanishing moments.*

the spaces T_m^k do not have the same scale-invariance. Nevertheless, a basis for the space U_l^k, which reflects the hierarchical decomposition (6), can be constructed by constructing bases for each of the k-dimensional spaces $T_{m,i}^k$ and combining them with a basis for U_0^k. We refer to such a basis as C_k; the construction is illustrated in Fig. 1.

It is noteworthy that the construction of C_k requires only n/k orthogonalizations of $2k \times 2k$-matrices and is therefore an order $O(n)$ procedure (see [6] for this construction).

1.3 Bases Supporting LU Factorization

In §2 we will see that certain integral operators, whose kernels are smooth (and non-oscillatory) except at diagonal singularities, are represented as sparse matrices in the coordinates B_k or C_k. Also, the inverses of these matrices, when they exist, are sparse. It would be convenient if in addition these matrices could be factored into sparse lower and upper-triangular matrices.

Integral operators expanded in bases B_k and C_k do not directly admit sparse LU-factorizations. Bases with this property do exist, however, and we outline their construction next. We construct bases (analogous to C_k) for the space U_l^k of functions defined on $\{x_1, \ldots, x_n\}$; similar bases exist for the spaces S_m^k.

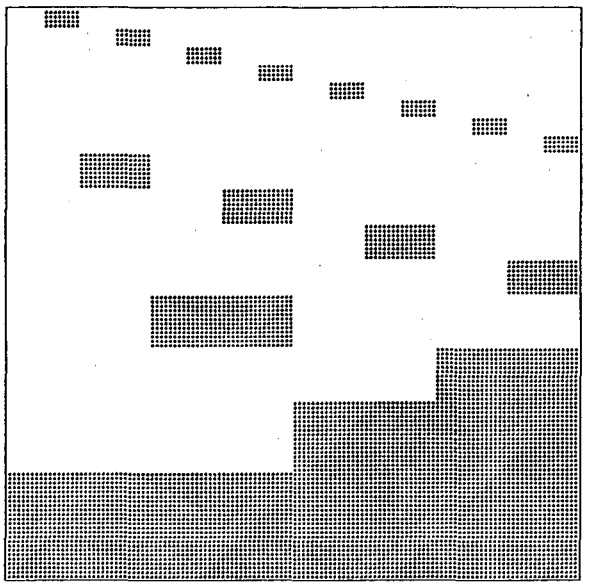

Figure 2: *The matrix represents a basis D_k, for $n = 128$ and $k = 4$ (compare Fig. 1). The limited interaction between basis functions on one scale and between scales leads to sparse LU factorizations of integral operators represented in these bases.*

The idea of the construction is to create subspaces of U_l^k of various scales, analogous to the T_m^k, but whose basis vectors are supported on *separated* sets of points in X; the construction decomposes the spaces T_m^k. For the following definitions, we assume that the points $x_1, \ldots, x_n$ are equispaced. For $m = 1, \ldots, l-1$ and $i = 1, 3, 5, \ldots, 2^m - 1$, we define the space $\tilde{T}_{m,i}^k$ by the formula

$$\tilde{T}_{m,i}^k = T_{m,i}^k \oplus T_{m+1,2i}^k \oplus T_{m+2,4i}^k \oplus \cdots \oplus T_{l-1,2^{l-1-m}i}^k,$$

we define $\tilde{T}_m^k$ by the formula

$$\tilde{T}_m^k = \tilde{T}_{m,1}^k \oplus \tilde{T}_{m,3}^k \oplus \tilde{T}_{m,5}^k \oplus \cdots \oplus \tilde{T}_{m,2^m-1}^k,$$

and we define $\tilde{T}_0^k$ by the formula

$$\tilde{T}_0^k = T_{0,0}^k \oplus T_{1,0}^k \oplus \cdots \oplus T_{l-1,0}^k.$$

We now have the analogue of (6), namely

$$U_l^k = U_0^k \oplus \tilde{T}_0^k \oplus \cdots \oplus \tilde{T}_{l-1}^k, \tag{7}$$

which expresses our intended decomposition. As is the case for (6), which gives rise to the basis C_k, a basis D_k of the space U_l^k that reflects the structure in (7)

can be constructed by constructing bases for the $\tilde{T}^k_{m,i}$. Such a basis is illustrated in Fig. 2. Its basic properties will be discussed in §2. At this point we mention, however, that another such basis can be obtained by a straighforward reordering of the elements of C_k.

2 Analytical Properties of the Bases

We now describe how the wavelet-like bases B_k, C_k, and D_k whose construction is outlined in §1 produce sparse matrix representations for many integral operators. The main idea is the following: a function which is analytic except at a finite set of singularities can be well approximated on intervals separated from the singularities by low-order polynomials. An interval separated from the singularities is a line segment in the complex plane (generally lying on the real axis) whose length is less than its distance to the nearest singularity. We say that a function is well approximated by low-order polynomials on such intervals if the relative error of Chebyshev interpolation on each interval decays exponentially and uniformly in the order of the interpolations. The singularities are assumed to be poles or branch points. An illustration of this notion is given in Fig. 3.

The ability to locally approximate by low-order polynomials any function with a finite number of singularities leads to the efficiency of function representation in the bases B_k, C_k, and D_k. For such a function f, if the order k is chosen based on the required precision, a basis element of B_k whose interval of support is separated from the singularities of f (and is orthogonal to polynomials of degree less than

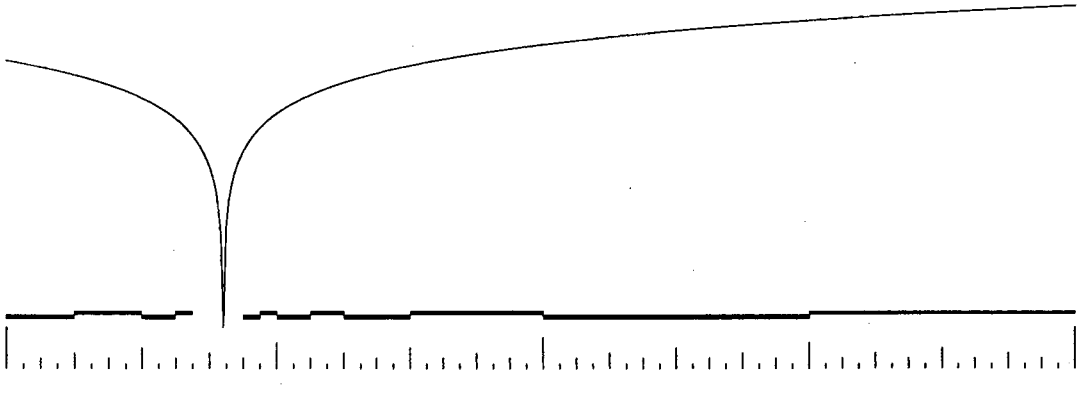

Figure 3: *The function* $f(x) = \ln|x - .2|$ *is graphed on the interval* $[0, 1]$*, which is divided into dyadic subintervals. On each subinterval separated from the singularity (indicated by solid line segments), the function can be represented by a polynomial of order 7 to six-digit accuracy.*

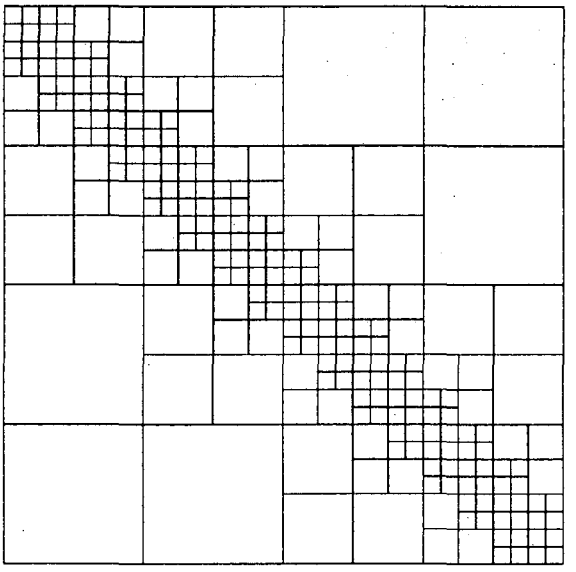

Figure 4: *The matrix represents a discretized integral operator with a kernel that is singular along the diagonal. The matrix is divided into submatrices of rank k (to high precision) and transformed to a sparse matrix with $O(n)$ elements. Here $n/k = 32$.*

k) receives negligible projection from f. Consider now an integral operator $\mathcal{K}$,

$$(\mathcal{K}g)(x) = \int_0^1 K(x,t)\, g(t)\, dx,$$

whose kernel K is an analytic function of x and t except at $x = t$, where it is singular. (Here the function $K(x,t)$, for fixed x or fixed t, is the analogue of the function f.) We can approximate the integral with the trapezoidal rule, which results in a system of linear equations. For a positive integer n and equispaced $x_1, \ldots, x_n$ on $[0,1]$, we define an $n \times n$ matrix $L = L(n)$ with elements L_{ij} given by the formula

$$L_{ij} = \begin{cases} K(x_i, x_j)/n & i \neq j \\ 0 & i = j. \end{cases}$$

The sequence of matrices $\{L(n)\}_{n \in \mathcal{Z}^+}$ converges to $\mathcal{K}$. Although the rate of convergence is low, a change to endpoint-corrected quadratures [4] can be used to greatly accelerate the convergence. Each matrix L can be subdivided into squares separated from the diagonal, as shown in Fig. 4, in which the elements can be approximated by low-order polynomials. As a result, transformation of L to the basis C_k yields a sparse matrix. In particular, given a precision ϵ, the order k can be chosen so that, up to ϵ, the number of non-negligible elements in the matrices $\{L(n)\}$ grows linearly in n.

Next consider two integral operators represented as sparse matrices in the basis C_k, possessing kernels K_1 and K_2 which are analytic except at diagonal singularities. The kernel K_3 of their product, given by the formula

$$K_3(x,t) = \int_0^1 K_2(x,y)\, K_1(y,t)\, dy,$$

is itself analytic except for $x = t$, and therefore has a sparse representation in the basis C_k. The product matrix can be obtained for a cost of order $O(n)$ by simply multiplying the matrices of the two operators! This fact makes the Schulz method, a Newton-iteration-like scheme for inverting matrices, useful in practice [6].

All invertible integral operators of the above type have asymptotically sparse inverses (the number of non-negligible elements of the $n \times n$-matrix representing the inverse operator is proportional to n). For certain operators, however, the number of non-negligible elements in the inverse matrix for moderate values of n may be nearly n^2. This lack of sparseness occurs when inverting operators from first-kind integral equations. One solution to this problem would be to obtain an operator's decomposition into a product of lower and upper-triangular matrices, if these matrices are sparse. Operators represented in the bases D_k have such sparse factorizations, due to two properties:

1. Two basis vectors b_1 and b_2 of the space $\tilde{T}_m^k$ which are non-zero on different sets of points have negligible interaction $b_1\, L\, {b_2}^T$. This follows from the fact that $L\, {b_2}^T$ is negligible outside the $3 \cdot 2^{l-m}k$ points centered on the $2^{l-m}k$ points where b_2 is non-zero.

2. The interaction of basis vectors of different subspaces has "controlled growth" as one moves up the hierarchy. This property is somewhat complicated to state precisely [2], but is illustrated in Fig. 5.

The number of non-negligible elements in $n \times n$-matrices representing integral operators in the basis D_k is potentially of order $O(n \log^2 n)$, but in numerical experiments we have observed slower growth.

Analytical properties of the wavelet-like bases are stated and proved in greater detail in [5].

3 Numerical Examples

In this section we give several numerical examples using the bases defined in §1. We consider a class of integral equations with logarithmic kernel,

$$f(x) - p(x) \int_0^1 \log|x-t|\, f(t)\, dt = g_m(x), \qquad x \in [0,1], \tag{8}$$

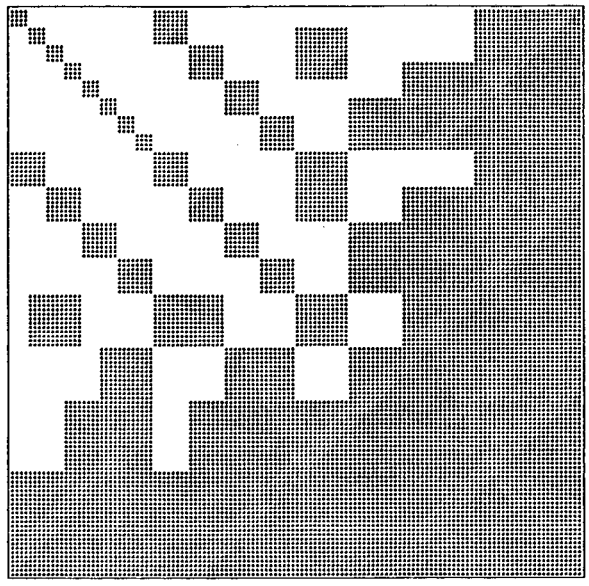

Figure 5: *The potential non-negligible matrix elements are shown for a matrix representing an integral operator in the basis D_k. Note the matrix can be factored into lower and upper-triangular matrices without fill-in (additional elements) provided that pivoting is done only within each block.*

where the right hand side g_m is chosen so that the solution f is given by the formula $f(x) = \sin(mx)$. The integration can be performed explicitly, yielding the formula

$$\begin{aligned}\int_0^1 \log|x-t|\,\sin(mt)\,dt &= \log(x) - \cos(m)\log(1-x) \\ &\quad - \cos(mx)[\mathrm{Ci}(mx) - \mathrm{Ci}(m(1-x))] \\ &\quad - \sin(mx)[\mathrm{Si}(mx) + \mathrm{Si}(m(1-x))],\end{aligned}$$

where Ci and Si are the cosine integral and sine integral (see, *e.g.*, [1], p. 231). Equation (8) clearly requires quadratures with increasing resolution as m increases; for our examples we let $n = m$, which corresponds to 2π points per oscillation of the right hand side g_m. We use quadratures derived from the trapezoidal rule with endpoint corrections [4], [15]; these quadrature rules converge rapidly.

We consider cases with three different values of the coefficient $p(x)$, namely $p(x) = 1$, $p(x) = 100$, and $p(x) = 1 + \frac{1}{2}\sin(100x)$. The first two cases can be solved using the bases of this chapter, while the third case requires a minor elaboration. For the case $p(x) = 1$, both the operator and its inverse are sparse in the bases C_k and D_k, so either the inverse or the LU factorization can be obtained cheaply. The second case, in which $p(x) = 100$, is poorly-conditioned

Table 1: *The integral equation* $f(x) - p(x)\int_0^1 \log|x-t|\, f(t)\, dt = g_m(x)$, *for which an explicit solution is known, is solved for various coefficients* $p(x)$ *and various values of* m *(see text). Precision* $\epsilon = 10^{-3}$ *and order* $k = 4$ *were used. The "bandwidths"* N_1, N_2, *and* N_3 *denote the average number of elements/row in the sparse matrices representing the operator, its inverse, and its LU factorization, while* t_1, t_2, *and* t_3 *denote the time in seconds required to compute these matrices. The condition number of the operator matrix is denoted by* κ, *and* e_2 *and* e_3 *denote the relative* $\mathcal{L}^2$ *errors of the solution obtained by inversion and LU factorization.*

	Operator			Inverse			LU Factorization		
n, m	N_1	t_1	κ	N_2	t_2	e_2	N_3	t_3	e_3
$p(x) = 1$									
64	27.7	3	5	31.3	30	.235E−3	27.7	0	.237E−3
128	31.0	7	4	34.2	80	.169E−3	31.1	1	.171E−3
256	30.6	16	4	33.6	173	.161E−3	30.6	3	.161E−3
512	27.5	33	3	30.2	306	.130E−3	27.5	7	.130E−3
1024	21.7	64	3	24.4	380	.597E−3	21.7	13	.164E−3
2048	15.5	115	3	18.1	487	.479E−3	15.5	15	.117E−3
4096	9.7	199	3	10.6	463	.415E−3	9.7	13	.107E−3
8192	6.0	357	3	7.3	549	.354E−3	6.0	17	.104E−3
$p(x) = 100$									
64	29.1	3	411	63.3	180	.442E−2	29.6	0	.802E−2
128	33.3	8	393	113.0	812	.374E−2	34.3	1	.445E−2
256	34.6	17	386	169.2	3133	.468E−2	35.9	3	.521E−2
512	32.2	38	350	202.1	8991	.337E−2	33.7	6	.352E−2
1024	28.6	78					29.9	13	.433E−2
2048	22.9	159					23.6	28	.599E−2
4096	15.9	316					16.3	47	.516E−2
8192	10.7	628					10.8	32	.382E−2
$p(x) = 1 + \frac{1}{2}\sin(100x)$									
64	36.2	3	4	41.3	52	.228E−2	36.3	0	.228E−2
128	40.8	8	4	47.0	150	.209E−3	40.8	1	.210E−3
256	40.5	18	4	47.3	343	.177E−3	40.5	4	.177E−3
512	34.7	36	4	40.9	574	.125E−3	34.7	9	.125E−3
1024	26.6	69	3	32.5	840	.134E−3	26.7	15	.134E−3
2048	18.7	124	3	22.5	858	.597E−3	18.7	22	.117E−3
4096	12.2	221	3	14.2	910	.529E−3	12.2	26	.100E−3
8192	7.2	394	3	8.4	893	.461E−3	7.2	21	.913E−4

and the inverse is not as sparse as the operator. For this case, it is dramatically cheaper to obtain the LU factorization.

The third case, with an oscillatory coefficient p, is solved by the construction [6] of bases in which the basis elements are orthogonal to the functions $p(x)^{1/2}x^j$, for $j = 0, \ldots, k-1$, rather than simply the moments x^j. This construction leads to the sparse matrix representation of an integral operator with a highly oscillatory coefficient. As we will see, the inverse is also sparse.

Table 1 presents the numerical results for various coefficients $p(x)$ and various right hand sides $g_m(x)$. For each problem instance, the dense matrix representing the integral operator was transformed to wavelet coordinates C_k reordered for sparse LU factorization. The smallest matrix elements were discarded according to an element threshold determined by the chosen precision ϵ, leaving a sparse matrix representing the integral operator. Second, the inverse matrix, which is also sparse, was computed by the Schulz method. As an alternative method, the LU factorization is obtained by direct Gaussian elimination. Third, the solution is obtained and the error is measured, for each of the two methods: application of the inverse to the right hand side and forward and back-substitution using the LU factorization. The table presents the "bandwidths," computation times, and errors associated with the methods, based on our FORTRAN implementation run on a Sun Sparcstation 1+.

We make several observations:

1. The "bandwidths" of the operator, its inverse, and its LU factorization *decrease* with increasing matrix size. In other words, in the range of matrix sizes tabulated, the number of matrix elements grows *sublinearly* in the matrix dimension n.

2. The operator matrix in wavelet coordinates is computed in time that grows nearly linearly in n. The inverse matrix and LU factorization are each computed in time which grows sublinearly in n, due to the decreasing "bandwidths" as n increases.

3. The solution accuracies are generally within the specified precision. Exceptions are attributable to quadrature errors (for $n = 64$) and to poor conditioning of the underlying problem (for $p(x) = 100$). When confronted with large condition numbers, one must obtain the integral operator to higher accuracy than the accuracy required in the solution (as is well known).

4. The times to compute the LU factorizations are much less than those for the inverses, even when the sparsities are comparable. In the case of the poorly-conditioned problem, the inverse matrices were not very sparse (large sizes

exhausted available memory and were not computed), yet LU factorizations remained sparse and inexpensive to compute.

5. The ability to solve problems requiring an 8192-point discretization in a few minutes on a Sparcstation using these methods should be compared to an estimated 78 days for Gauss-Jordan inversion and 26 days for LU decomposition of an 8192×8192-matrix. One dense matrix-vector multiplication of that size requires roughly 19 minutes.

We summarize these observations by remarking that the LU factorization of an integral operator represented in the wavelet coordinates described above is a highly effective method for the numerical solution of integral equations.

4 Summary

In this chapter we have described a class of bases in which integral operators are represented as sparse matrices. More generally, a dense matrix with elements that are a smooth function of their indices is transformed to a sparse matrix in these bases (to high precision). The sparseness results from the fact that the typical basis element has "local" support and is orthogonal to low-order polynomials. A locally smooth function is therefore concisely represented in these bases.

These sparse matrices can be manipulated rapidly. Matrix operations, including application of a matrix to a vector, application of its inverse to a vector, and matrix-matrix multiplication, scale roughly linearly with the dimension of the matrices. We have presented examples demonstrating the effectiveness of these methods for the solution of a variety of integral equations. We anticipate that these bases will be applied successfully to other matrix computations, including some arising from elliptic and hyperbolic partial differential equations.

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