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More saturated ideals

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MORE SATURATED IDEALS

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§1. In this paper we prove three theorems relating the consistency of huge cardinals with the consistency of saturated ideals on regular cardinals and with the consistency of model theoretic transfer properties.

We prove:

THEOREM. Con(ZFC+"there is a 2-huge cardinal") implies Con(ZFC+"for all $m, n \in \omega$ with $m > n$, $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$").

THEOREM. Con(ZFC+"there is a huge cardinal") implies Con(ZFC+"for all $n \in \omega$, there is a normal, $\aleph_n$-complete, $\aleph_{n+1}$-saturated ideal on $\aleph_n$"+"there is a normal, $\aleph_{\omega+1}$-complete, $\aleph_{\omega+2}$-saturated ideal on $\aleph_{\omega+1}$").

The theorem above contains all the new ideas necessary to prove the following theorem:

THEOREM. Con(ZFC+"there is a huge cardinal") implies Con(ZFC+"every regular cardinal κ carries a κ^+-saturated ideal").

We now make some definitions: Let $\mathcal{L}$ be a countable language with a unary predicate U. An $\mathcal{L}$-structure $\mathfrak{A}$ is said to have **type** (κ, λ) if and only if $|\mathfrak{A}| = \kappa$ and $U^{\mathfrak{A}} = \lambda$. If $\kappa \geq \kappa'$, $\lambda \geq \lambda'$ we say that $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ if and only if every structure $\mathfrak{A}$ of type (κ, λ) has an elementary substructure $\mathcal{B} \prec \mathfrak{A}$ of type (κ', λ').

An ideal $\mathcal{I} \subseteq \mathcal{P}$ is set to be **α-complete** if and only if whenever $\{X_\gamma : \gamma < \beta\} \subseteq \mathcal{I}$ and $\beta < \alpha$, $\bigcup_{\gamma<\beta} X_\gamma \in \mathcal{I}$. A set $A \subseteq \wp(\kappa)$ is said to be **positive** if $A \notin \mathcal{I}$. The ideal $\mathcal{I}$ is **normal** if and only if for every positive set $A \subseteq \kappa$ and every regressive function f defined on A there is a $\beta \in \kappa$ such that $\{\alpha : f(\alpha) = \beta\}$ is positive. The ideal $\mathcal{I}$ is said to be **λ-saturated** if and only if it is normal and $\wp(\kappa)/\mathcal{I}$ has the λ-chain condition. (We shall never consider "non-normal" saturated ideals.) There is an extensive literature on saturated ideals. (*Cf.* [KM78, Kun78].)

Let $j : V \to M$ be an elementary embedding from V into a transitive class M. Let κ_0 be the critical point of j (*i.e.*, the first ordinal moved by j), and let $\kappa_{i+1} = j(\kappa_i)$. We shall call j an n-huge embedding and κ_0 an n-huge

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cardinal if and only if M is closed under κ_n-sequences. (This means that if $\langle x_a : \alpha < \kappa_n\rangle \subseteq M$ then $\langle x_a : a < \kappa_n\rangle \in M$.) An almost n-huge embedding is an embedding j as above such that M is closed under $< \kappa_n$-sequences, *i.e.*, if $\beta < \kappa_n$ and $\{x_\alpha : \alpha < \beta\} \subseteq M$ then $\{x_\alpha : \alpha < \beta\} \in M$. We shall use the notation "crit(j)" for the critical point of j.

If $\kappa_0 \leq \lambda \leq \kappa_1$ and j is at least an almost huge embedding, then j induces a supercompact measure on $\wp_\kappa(\lambda)$. In particular, j induces a normal measure on κ.

PROPOSITION 1.1. Let j be an n-huge embedding. Let U be the measure on κ induced by j. Then there is a set A of measure one for U such that for all α and $\beta \in A$, there is an almost n-huge embedding $j_{\alpha,\beta}$ such that the critical point of $j_{\alpha,\beta}$ is α and $j_{\alpha,\beta}(\alpha) = \beta$.

PROOF. This is a routine reflection argument. $\dashv$ (Proposition 1.1)

We now precisely state the theorems we shall prove:

THEOREM 1.2. Con(ZFC+there is a sequence of huge embeddings $\langle j_n : n \in \omega\rangle$ with j_n (critical point of j_n) =critical point of j_{n+1}) $\Rightarrow$ Con(ZFC+for all $m, n \in \omega$, $m > n$ implies $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$).

THEOREM 1.3. Con(ZFC+there is a sequence of almost huge embeddings $\langle j_n : n \in \omega\rangle$ with j_n (critical point of j_n) = critical point of j_{n+1}) $\Rightarrow$ Con(ZFC+for all $n \in \omega$ there is a normal, $\aleph_n$-complete, $\aleph_{n+1}$-saturated ideal on $\aleph_n$).[1]

THEOREM 1.4. Con(ZFC+there is a huge cardinal) $\Rightarrow$ Con(ZFC + $\aleph_{\omega+1}$ carries a normal $\aleph_{\omega+2}$-saturated ideal and for all $n \in \omega$, $\aleph_n$ carries an $\aleph_{n+1}$-saturated ideal).

We shall assume that the reader is familiar with iterated forcing. (*Cf.* [Bau83] for an excellent exposition.) All of our partial orderings $\mathbb{P}$ will have a unique greatest element, $1_{\mathbb{P}}$. Our notion of "support" will be the standard one and if p is a condition in an iteration we shall write "supp p" for its support. For the inverse limit of a system $\{\mathbb{P}_i : i \in I\}$ we shall write $\varprojlim\{\mathbb{P}_i : i \in I\}$. We shall also use the notion of support for products of partial orderings. If $\{\mathbb{Q}_i : i \in I\}$ is a collection of partial orderings, we let $\prod\langle \mathbb{Q}_i : i \in I\rangle = \{f : f$ is a function with domain I and for all $i \in I$ $f(i) \in \mathbb{Q}_i\}$. The product $\prod\langle \mathbb{Q}_i : i \in I\rangle$ is ordered coordinatewise. If $p \in \prod\langle \mathbb{Q}_i : i \in I\rangle$, then supp $p = \{i : p(i) \neq 1_{\mathbb{Q}_i}\}$. If $K \subseteq \wp(I)$ is an ideal then $\prod_{\operatorname{supp} p \in K}\langle \mathbb{Q}_i : i \in I\rangle = \{f \in \prod\langle \mathbb{Q}_i : i \in I\rangle : \operatorname{supp} f \in K\}$.

If $\langle \mathbb{Q}_i : i \in \omega\rangle$ is a sequence of terms such that $\mathbb{Q}_{i+1} \in V^{\mathbb{Q}_0 * \mathbb{Q}_1 * \cdots * \mathbb{Q}_i}$, we write $\mathop{*}\limits_{i=0}^{n} \mathbb{Q}_i$ for the finite iteration $\mathbb{Q}_0 * \mathbb{Q}_1 * \cdots * \mathbb{Q}_n$. If S is a partial ordering

[1] We note that the model used to prove Theorem 1.2 also satisfies the conclusion of Theorem 1.3.

with a uniform definition, we shall use $S^{\mathbb{P}}$ to denote the partial ordering S defined in $V^{\mathbb{P}}$. To simplify notation, we shall write $\mathbb{P} * \bar{S}$ to mean $\mathbb{P} * S^{\mathbb{P}}$.

If $\mathbb{P}$ is a partial ordering we shall use $\mathcal{B}(\mathbb{P})$ to denote the canonical complete boolean algebra obtained from $\mathbb{P}$. If $\varphi(\dot{t}_1, \ldots, \dot{t}_n)$ is a formula in the forcing language of $\mathbb{P}$, we shall use the notation $p \wedge q$, and $p \vee q$ to be the meet and join of p and q in $\mathcal{B}(\mathbb{P})$. Similarly $\neg p$ will denote the complement of $p \in \mathcal{B}(\mathbb{P})$. If $b \in \mathcal{B}(\mathbb{P})$, we shall say that p "decides" b (in symbols $p \parallel b$) if and only if $p \leq b$ or $p \leq \neg b$. We shall write $p \Vdash b$ if $p \leq b$.

We define $C(\kappa, \gamma) = \langle \{p \ : \ p : \kappa \to \gamma, |p| < \kappa\}, \supseteq \rangle$. $C(\kappa, \gamma)$ is the partial ordering appropriate for making γ have cardinality κ. We shall call this the Levy collapse. Similarly, we shall define the Silver collapse $S(\kappa, \lambda)$ by $p \in S(\kappa, \lambda)$ if and only if

(a) $p : \lambda \times \kappa \to \lambda$
(b) $|p| \leq \kappa$
(c) there is a $\xi < \kappa$, $\operatorname{dom} p \subseteq \lambda \times \xi$
(d) for all $\alpha < \kappa, \gamma < \lambda$, $p(\gamma, \alpha) < \gamma$

$S(\kappa, \lambda)$ is ordered by reverse inclusion: Standard arguments show that for inaccessible λ, $S(\kappa, \lambda)$ makes λ into κ^+.

If $\gamma' < \gamma$ and $\kappa < \gamma'$, $C(\kappa, \gamma')$ is a subset of $C(\kappa, \gamma)$. If $p \in C(\kappa, \gamma)$, we define $p \cap C(\kappa, \gamma')$ to be q, where $\operatorname{dom} q = \{\alpha < \kappa \ : \ p(\alpha) < \gamma'\}$ and for each $\alpha \in \operatorname{dom} q$, $q(\alpha) = p(\alpha)$. It is easy to see that $p \cap C(\kappa, \gamma') \in C(\kappa, \gamma')$. If $p \in C(\kappa, \gamma)$, we define $\sup p$ to be $\sup[\{p(\alpha) + 1 \ : \ \alpha \in \operatorname{dom} p\}]$.

We shall write $\vec{\alpha}$ for a finite sequence of ordinals. If $\alpha_0, \alpha_1, \ldots, \alpha_n$ are mentioned in connection with $\vec{\alpha}$ we shall assume that $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$.

If κ and λ are cardinals, with $\kappa < \lambda$ and $x, y \in \wp_\kappa(\lambda) = \{z \subseteq \lambda \ : \ |z| < \kappa\}$ then we write $x \prec y$ if and only if $x \subseteq y$ and the order type of x is less the the order type of $y \cap \kappa$. If $x \in \wp_\kappa(\lambda)$, let $\operatorname{crit}(x) =$ order type of $x \cap \kappa$.

If $\mathcal{B}$ and $\mathcal{C}$ are complete Boolean algebras and $\pi : \mathcal{B} \to \mathcal{C}$ is an order preserving function, π is called a **projection map** if whenever $G \subseteq \mathcal{B}$ is generic, $\pi''G \subseteq \mathcal{C}$ is generic.

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orderings. If $i : \mathbb{P} \to \mathbb{Q}$ is one to one, order and incompatibility preserving function with the property that whenever $A \subseteq \mathbb{P}$ is a maximal antichain, $i''A \subseteq \mathbb{Q}$ is a maximal antichain, then we say that i is a **neat embedding** from $\mathbb{P}$ into $\mathbb{Q}$, and write $i : \mathbb{P} \hookrightarrow \mathbb{Q}$. Standard theory says that if $i : \mathbb{Q} \hookrightarrow \mathbb{P}$ is a neat embedding then there is a projection map $\pi : \mathcal{B}(\mathbb{Q}) \to \mathcal{B}(\mathbb{P})$ such that for all $p \in \mathbb{P}$, $\pi(i(p)) = p$.

If $e : \mathbb{P} \hookrightarrow \mathbb{Q}$ and $S \in V^{\mathbb{Q}}$ is a partial ordering we define $\mathbb{P} *_e S$ to be the iteration with amalgam $e''\mathbb{Q}$: conditions consist of pairs (p, τ) where $p \in \mathbb{P}$ and τ is a $\mathbb{Q}$-term for an element of S. Let $(p, \tau) \leq (p', \tau')$ if and only if $p \leq_{\mathbb{P}} p'$ and $\pi(p) \Vdash \tau \leq \tau'$. (*Cf.* [For82] for more on this definition.)

We shall say that a partial ordering $\mathbb{P}$ has the **κ-chain condition** (is κ-c.c.) if and only if there is no antichain in $\mathbb{P}$ of size κ. The partial ordering $\mathbb{P}$

is κ-closed if and only whenever $\langle p_\alpha : \alpha < \beta\rangle$ is a decreasing sequence of conditions with $\beta < \kappa$, there is a $q \in \mathbb{P}$ such that for all $\alpha < \beta, q < p_\alpha$.

If $\mathbb{P}$ is a partial ordering and $p \in \mathbb{P}$, then $\mathbb{P}/p$ is defined to be $\{q : q \leq p\}$.

§**2.** In this section we prove Theorems 1.2 & 1.3. We begin by mentioning theorems of Magidor and Kunen that we shall use extensively in this paper.

Theorem (Kunen; [Kun78]). Let $j : V \to M$ be a huge embedding. Suppose $\mathbb{P} * \mathbb{Q}$ is a forcing notion such that

(a) $\mathbb{P}$ is κ_0-c.c.,
(b) $\mathbb{Q}$ is κ_0-closed in $V^{\mathbb{P}}$,
(c) $|\mathbb{P}| = \kappa_0, |\mathbb{P} * \mathbb{Q}| = \kappa_1$,
(d) there is a projection map $\pi : \mathcal{B}(j(\mathbb{P}) * j(\mathbb{Q})) \to \mathcal{B}(\mathbb{P} * \mathbb{Q})$ and a $q \in j(\mathbb{P}) * j(\mathbb{Q})$ such that for all $r \in j(\mathbb{P}) * j(\mathbb{Q}), r \leq q$ implies $j(\pi(r)) \geq r$

then in $V^{\mathbb{P}*\mathbb{Q}}$:

(i) κ_0 carries a normal, κ_0-complete, $j(\kappa_0)$-saturated ideal, and
(ii) $(j(\kappa_0), \kappa_0) \twoheadrightarrow (\kappa_0, < \kappa_0)$.

Magidor proved that to get a normal κ_0-complete, $j(\kappa_0)$-saturated ideal on κ_0, it is enough to have j be almost huge and replace (d) by:

(d′) There is a neat embedding $i : \mathcal{B}(\mathbb{P}' * \mathbb{Q}) \to \mathcal{B}(j(\mathbb{P}'))$.

We shall need the following preservation lemmas.

Lemma 2.1. Let λ be a regular cardinal. Suppose $\mathbb{P}$ is λ-c.c. and $\mathbb{Q}$ is λ-closed. (We do not rule out $\mathbb{P} = \varnothing$.)

(a) If $\kappa < \lambda$ and κ carries a normal λ-saturated ideal in $V^{\mathbb{P}}$, then κ carries a normal λ-saturated ideal in $V^{\mathbb{P}\times\mathbb{Q}}$.
(b) If $\lambda' < \lambda$, $\gamma \leq \kappa \leq \lambda$ and $(\lambda, \lambda') \twoheadrightarrow (\kappa, \gamma)$ in V, then $(\lambda, \lambda') \twoheadrightarrow (\kappa, \gamma)$ in $V^{\mathbb{Q}}$.

Proof. Since $\mathbb{P}$ is λ-c.c., $\mathbb{Q}$ is (λ, ∞)-distributive in $V^{\mathbb{P}}$. Thus, if $\mathcal{I}$ is a normal λ-saturated ideal on κ in $V^{\mathbb{P}}$, $\mathcal{I}$ remains a normal ideal in $(V^{\mathbb{P}})^{\mathbb{Q}}$. Suppose $\mathcal{I}$ is no longer λ-saturated in $V^{\mathbb{P}\times\mathbb{Q}}$. Let $\langle \sigma_\alpha : \alpha < \lambda\rangle$ be terms in $V^{\mathbb{P}\times\mathbb{Q}}$ for an antichain. We shall build a descending sequence $\langle q_\alpha : a < \lambda\rangle \subseteq \mathbb{Q}$ in V with the property that there is a term $\tau_\alpha \in V^{\mathbb{P}}$ such that $\|q_\alpha \Vdash_{\mathbb{Q}} \sigma_\alpha = (\tau_\alpha)^{V^{\mathbb{P}}}\|_{\mathbb{P}} = 1$. We do this by induction on α. Let $q_{-1} = 1_{\mathbb{Q}}$.

Assume we have defined $\{q_\beta : \beta < \alpha\}$. Let $q_\alpha^{-1} \leq \{q_\beta : \beta < \alpha\}$. There is such a q_α^{-1} because $\alpha < \lambda$ and $\mathbb{Q}$ is λ-closed. We shall simultaneously build $\{q_\alpha^\gamma : \gamma < \gamma_\alpha\} \subseteq \mathbb{Q}$ and a maximal antichain $\{p_\gamma : \gamma < \gamma_\alpha\} \subseteq \mathbb{P}$. These will have the property that for each γ there is a term $\tau_\alpha^\gamma \in V^{\mathbb{P}}$ such that $(p_\gamma, q_\alpha) \Vdash_{\mathbb{P}\times\mathbb{Q}} \sigma_a = (\tau_\alpha^\gamma)^{V^{\mathbb{P}}}$. Assume that we have built $\langle q_\alpha^\gamma : \gamma < \xi\rangle$ and $\langle p_\gamma : \gamma < \xi\rangle$ with the above property. Note that $\xi < \lambda$ since $\mathbb{P}$ has the λ-chain condition. If $\langle p_\gamma : \gamma < \xi\rangle$ is not a maximal antichain, pick p_ξ, q_ξ^α such that

$q^\alpha_\xi \leq \langle q^\alpha_\gamma : \gamma < \xi\rangle$ and for some $\tau^\xi_\alpha \in V^{\mathbb{P}}$, $(p_\xi, q^\xi_\alpha) \Vdash_{\mathbb{P}\times\mathbb{Q}} (\tau^\xi_\alpha)^{V^{\mathbb{P}}} = \sigma_\alpha$ and p_ξ is incompatible with each p_γ, $\gamma < \xi$.

Since $\mathbb{P}$ has the λ-c.c., for some $\xi < \lambda$, $\{p_\gamma : \gamma < \xi\}$ is a maximal antichain. Let $q_\gamma \leq \{q^\gamma_\alpha : \gamma < \xi\}$ and $\tau_\alpha \in V^{\mathbb{P}}$ be such that $p_\gamma \Vdash \tau^\gamma_\alpha = \tau_\alpha$. Then $\|q_\alpha \Vdash_{\mathbb{Q}} (\tau^{V^{\mathbb{P}}}_\alpha) = \sigma_\alpha\|_{\mathbb{P}} = 1$.

In $V^{\mathbb{P}}$, $\langle q_\alpha : \alpha < \gamma\rangle$ identifies a sequence $\langle \tau_\alpha : \alpha < \lambda\rangle \subseteq (\wp(\kappa))^{V^{\mathbb{P}}}$. If $\alpha < \beta < \lambda$, then $q_\beta \Vdash_{\mathbb{Q}} \tau_\alpha \cap \tau_\beta \in \mathcal{I}$ and $\tau_\alpha, \tau_\beta \notin \mathcal{I}$. But "$\tau_\alpha \cap \tau_\beta \in \mathcal{I}$" and "$\tau_\alpha \notin \mathcal{I}$" is absolute between $V^{\mathbb{P}}$ and $V^{\mathbb{P}\times\mathbb{Q}}$. Hence in $V^{\mathbb{P}}$, $\langle \tau_\alpha : \alpha < \lambda\rangle$ is an antichain of size λ in $\wp(\kappa)/\mathcal{I}$, a contradiction.

(b) Let $\mathfrak{A} = \langle \lambda\,;\, \lambda', f_i\rangle_{i\in\omega}$ be a fully skolemized structure of type (λ, λ') in $V^{\mathbb{Q}}$. Using the λ-closure of $\mathbb{Q}$, in V we can find a sequence $\langle p_\alpha : \alpha < \lambda\rangle \subseteq \mathbb{Q}$ and a structure $\mathfrak{A}' = \langle \lambda\,;\, \lambda', f'_i\rangle_{i\in\omega}$ such that f'_i are defined on all of λ and for each $\vec{\beta} \in \lambda^{<\omega}$ and each i there is an $\alpha < \lambda$ $p_\alpha \Vdash f_i(\vec{\beta}) = f'_i(\vec{\beta})$. Let $\mathcal{B} \prec \mathfrak{A}'$ be of type (κ, γ). Since $\kappa < \lambda$ and λ is regular there is an α such that for all $\beta \in \mathcal{B}$, $p_\alpha \Vdash f_i(\vec{\beta}) = f'_i(\vec{\beta})$. But then $p_\alpha \Vdash \mathcal{B}$ is closed under all of the f_i's. Hence $p_\alpha \Vdash \mathcal{B} \prec \mathfrak{A}$. ⊣ (Lemma 2.1)

A partial ordering $\mathbb{P}$ will be called **κ-centered** if and only if there is a collection $\{C_\alpha : \alpha < \kappa\}$ of disjoint subsets of $\mathbb{P}$ such that
(a) $\mathbb{P} = \bigcup_{\alpha<\kappa} C_\alpha$.
(b) If $p_1, \dots, p_n \in C_\alpha$ then $\bigwedge_{i=1}^n p_i \neq 0$.
In particular, if $|\mathbb{P}| \leq \kappa$, $\mathbb{P}$ is κ-centered.

Lemma 2.2. Let $\mathcal{I}$ be a κ^{++}-saturated ideal on κ^+. Suppose $\mathbb{P}$ is a κ-centered partial ordering. Then $\mathcal{I}$ generates a normal κ^+-complete, κ^{++}-saturated ideal in $V^{\mathbb{P}.}$ (Laver proved stronger results unknown to the author at the time this work was done. *Cf.* [Lav82A] and [Lav82B].)[2]

Proof. Fix a centering $\langle C_\gamma : \gamma < \kappa\rangle$ of $\mathbb{P}$. Then $\mathbb{P}$ is manifestly κ^+-c.c. Hence $\mathcal{I}$ remains κ^+-complete and normal. We need to see that $\mathcal{I}$ is κ^{++}-saturated. Let $\langle \dot{X}_\alpha : \alpha < \kappa^{++}\rangle$ be a term of an antichain of size κ^{++}. Let $\dot{Y}_{\alpha,\beta} = \dot{X}_\alpha \cap \dot{X}_\beta$. Then $\dot{Y}_{\alpha,\beta} \in V^{\mathbb{P}}$ and $\|\dot{Y}_{\alpha,\beta} \in \bar{\mathcal{I}}\| = 1$. (Here we are using $\bar{\mathcal{I}}$ to stand for the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$.) Hence for each pair α, β

$$\|\exists Y \in V(Y \in \mathcal{I} \text{ and } Y_{\alpha,\beta} \subseteq Y)\| = 1.$$

Since $\mathbb{P}$ is κ^+-c.c. and $\mathcal{I}$ is κ^+-complete, there is a sequence in V, $\langle Y_{\alpha,\beta} : \alpha < \beta < \kappa^{++}\rangle \subseteq \mathcal{I}$ and $\|X_\alpha \cap X_\beta \subseteq Y_{\alpha,\beta}\| = 1$. For each X_α and each $\gamma < \kappa$, let $X_{\alpha,\gamma} = \{\xi :$ there is a $p \in C_\gamma, p \Vdash \xi \in X_\alpha\}$. Since $\|X_\alpha \subseteq \bigcup_{\gamma<\kappa} X_{\alpha,\gamma}\| = 1$ and $\mathcal{I}$ is κ^+-complete, there must be some γ_α with $X_{\alpha,\gamma_\alpha} \notin \mathcal{I}$. By the pigeonhole principle, there is a set $S \subseteq \kappa^{++}$, $S \in V$ with $|S| = \kappa^{++}$ such that for all $\alpha, \beta \in S$, $\gamma_\alpha = \gamma_\beta$. Fix such a set S and such a $\gamma = \gamma_\alpha$. We claim that if $\alpha, \beta \in S$, then $X_{\alpha,\gamma} \cap X_{\beta,\gamma} \subseteq Y_{\alpha,\beta}$.

[2] *Note* (2018): This remark is in the original paper.

Let $\xi \in X_{\alpha,\gamma} \cap X_{\alpha,\beta}$. Since C_γ is a collection of compatible elements we can find a $q \in C_\gamma$, $q \Vdash \xi \in X_{\alpha,\gamma} \cap X_{\beta,\gamma}$. But then $q \Vdash \xi \in \check{Y}_{\alpha,\beta}$. Hence $\xi \in Y_{\alpha,\beta}$. We now derive our contradiction: $\langle X_{\alpha,\gamma} : \alpha \in S\rangle$ is a set of positive elements of $\wp(\kappa^+)$ with respect to $\mathcal{I}$ and if $\alpha, \beta \in S$, $\alpha \neq \beta$, $X_{\alpha,\gamma} \cap X_{\beta,\gamma} \subseteq Y_{\alpha,\beta} \in \mathcal{I}$. Hence $\langle X_{\alpha,\gamma} : \alpha \in S\rangle$ are an antichain of size κ^{++} in the ground model, a contradiction. $\dashv$ (Lemma 2.2)

The following lemma is standard (*Cf.* [KM78]).

Lemma 2.3. Suppose $\lambda' < \kappa' \leq \lambda < \kappa$ are regular cardinals.

(a) If $\mathbb{P}$ has the λ'-c.c. and $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ then in $V^{\mathbb{P}}$, $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$.
(b) If $\mathbb{P}$ has the λ-c.c. and κ carries a κ^+-saturated ideal $\mathcal{I}$, then in $V^{\mathbb{P}}$, $\mathcal{I}$ is a κ^+-saturated ideal.

Let $\langle j_n : n \in \omega\rangle$ be a sequence of huge embeddings. Let $\kappa_n = \mathrm{crit}(j_n)$ and suppose $j_n(\kappa_n) = \kappa_{n+1}$. Suppose $\langle \mathbb{R}_n : n \in \omega\rangle$ is a sequence of terms for partial orderings with $\mathbb{R}_{n+1} \in V^{\overset{n}{\underset{i=0}{*}}\mathbb{R}_i}$ such that

(a) $\overset{n}{\underset{i=0}{*}}\mathbb{R}_i$ has the κ_n-chain condition and in $V^{\overset{n}{\underset{i=0}{*}}\mathbb{R}_i}$, $\mathbb{R}_{n+1}$ is κ_n-closed
(b) In $V^{\overset{n}{\underset{i=0}{*}}\mathbb{R}_i}$, for all $i \leq n$ $\kappa_i = \aleph_{i+2}$
(c) In $V^{\overset{n}{\underset{i=0}{*}}\mathbb{R}_i}$, $\mathbb{R}_{n+1}$ has cardinality κ_{n+1}.
(d) For each n, there is a projection map $\pi : j_n(\overset{n}{\underset{i=0}{*}}\mathbb{R}_i) \to \overset{n}{\underset{i=0}{*}}\mathbb{R}_i * \mathbb{R}_{n+1}$ and a condition $q \in j_n(\overset{n}{\underset{i=0}{*}}\mathbb{R}_i) * j_n(\mathbb{R}_{n+1})$ such that for all $r \leq q$, $r \in j_n(\overset{n}{\underset{i=0}{*}}\mathbb{R}_i) * j_n(\mathbb{R}_{n+1})$, $j_n(\pi(r)) \geq r$. (Thus, if $\mathbb{P}' = \overset{n}{\underset{i=0}{*}}\mathbb{R}_i$ and $\mathbb{Q} = \mathbb{R}_n$, then $\mathbb{P}'$ and $\mathbb{Q}$ satisfy the hypothesis (d) in Kunen's theorem.)

Let $\mathbb{P} = \varprojlim \langle \overset{n}{\underset{i=0}{*}}\mathbb{R}_i : n \in \omega\rangle$. By Kunen's Theorem, in $V^{\overset{n}{\underset{i=0}{*}}\mathbb{R}_i}$ there is κ_{n+1}-saturated ideal on κ_n and $(\kappa_{n+1}, \kappa_n) \twoheadrightarrow (\kappa_n, \kappa_{n-1})$. (Let $\kappa_{-1} = \aleph_1$.) But $\mathbb{P}/\overset{n}{\underset{i=0}{*}}\mathbb{R}_i$ is κ_{n+1}-closed. Hence by Lemma 2.1 in $V^{\mathbb{P}}$, for all $m, n \in \omega$, $m > n > 0$, $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ and $\aleph_m$ carries a normal $\aleph_m$-complete $\aleph_{m+1}$-saturated ideal.

By Lemmas 2.2 & 2.3, in $V^{\mathbb{P} * C(\aleph_0, \aleph_1)}$ for all $m, n \in \omega$, $m > n$, $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ and $\aleph_m$ carries an $\aleph_m$-complete $\aleph_{m+1}$-saturated ideal.

We shall now concentrate our attention on building the $\mathbb{R}$'s. The construction given here is considerably simpler than our original one at the expense of introducing somewhat more technicality.[3] The next few definitions and lemmas explore these technicalities.

[3] *Editors' note*: This wording is from the original published manuscript.

Lemma 2.4. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orderings. Suppose $\pi\colon \mathbb{P} \to \mathbb{Q}$ has the properties:

(1) $p_1 \leq p_2$ implies $\pi(p_1) \leq \pi(p_2)$, and
(2) for all $p \in \mathbb{P}$ there is a $q \leq \pi(p)$ such that for all $q' \leq q$ there is a $p' \leq p$ such that $\pi(p') \leq q'$.

$$
\begin{array}{ccc}
\mathbb{P} & & \mathbb{Q} \\
\hline
 & & \\
p & \xrightarrow{\pi} & \pi(p) \\
 & & \rotatebox[origin=c]{90}{$\leq$} \\
 & & q \\
\rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} \\
 & & q' \\
 & & \rotatebox[origin=c]{90}{$\leq$} \\
p' & \xrightarrow{\pi} & \pi(p')
\end{array}
$$

Then π is a projection map.

Proof. This is standard. ⊣ (Lemma 2.4)

The following was discovered independently by Avraham:

Definition 2.5 (Laver). Let $\mathbb{P}$ be a partial ordering and $\mathbb{Q} \in V^{\mathbb{P}}$ be a term for partial ordering. We define the **termspace** partial ordering $\mathbb{Q}^*$ to be the following partial order:

$$\operatorname{dom}\mathbb{Q}^* = \{\tau : \tau \in V^{\mathbb{P}}, \|\tau \in \mathbb{Q}\| = 1 \text{ and for all } \tau' \in V^{\mathbb{P}}, \\ \text{if } \operatorname{rank}(\tau') < \operatorname{rank}(\tau) \text{ then } \|\tau' = \tau\|_{\mathbb{P}} \neq 1\}.$$

For $\tau, \tau' \in \operatorname{dom}\mathbb{Q}^*$,

$$\tau \leq_{\mathbb{Q}^*} \tau' \quad \text{if and only if} \quad \|\tau \leq_{\mathbb{Q}} \tau'\|_{\mathbb{P}} = 1.$$

(Technically we want to take the universe of $\mathbb{Q}^*$ to be equivalence classes of terms modulo the relation $\tau_1 \sim \tau_2$ if and only if $\|\tau_1 = \tau_2\|_{\mathbb{P}} = 1$. In practice we ignore this distinction.)

The main use of the termspace partial ordering is summed up by:

Lemma 2.6.

(a) Let $\mathbb{Q} \in V^{\mathbb{P}}$ be a term for a partial ordering. There is a projection map $\pi\colon \mathbb{P} \times \mathbb{Q}^* \to \mathbb{P} * \mathbb{Q}$
(b) Let $\langle \mathbb{Q}_i : i \in \omega \rangle \subseteq V^{\mathbb{P}}$ be terms for partial orderings. Then there is a projection map

$$\pi\colon \mathbb{P} \times \prod_{i\in\omega} (\mathbb{Q}_i)^* \to \mathbb{P} * \Big(\prod_{i\in\omega} \mathbb{Q}_i\Big)^{V^{\mathbb{P}}}$$

(c) If $\langle \mathbb{Q}_i : i \in \omega \rangle$, $\langle \mathbb{P}_i : i \in \omega \rangle$ are two sequences of partial orderings and for each $i \in \omega$ $\varphi_i : \mathbb{Q}_i \to \mathbb{P}_i$ is a neat embedding, then there is a $\varphi : \prod_{i \in \omega} \mathbb{Q}_i \to \prod_{i \in \omega} \mathbb{P}_i$ extending each φ_i.

Proof. (a) Define π as follows: If $(p, \tau) \in \mathbb{P} \times \mathbb{Q}^*$, let $\pi(p, \tau) = (p, \tau) \in \mathbb{P} * \mathbb{Q}$. We want to apply Lemma 2.4. It is enough to see that if $(p, \tau) \in \mathbb{P} \times \mathbb{Q}^*$ and $(q, \sigma) \in \mathbb{P} * \mathbb{Q}$ with $(q, \sigma) \leq_{\mathbb{P}*\mathbb{Q}} (p, \tau)$ there is a $\sigma' \in \mathbb{Q}^*$ such that $(q, \sigma') \leq_{\mathbb{P} \times \mathbb{Q}^*} (p, \tau)$ and $(q, \sigma') \leq_{\mathbb{P}*\mathbb{Q}} (q, \sigma)$. Let σ' be a term of minimal rank such that $\|\sigma' = \sigma\|_{\mathbb{P}} \geq q$ and $\|\sigma' = \tau\|_{\mathbb{P}} \geq \neg q$. Then $q \Vdash \sigma' \leq \sigma$, hence $(q, \sigma') \leq_{\mathbb{P}*\mathbb{Q}} (q, \sigma)$. Also $\neg q \leq \|\sigma' = \tau\|$, hence $q \vee (\neg q) \leq \|\sigma' \leq \tau\|$ so $\|\sigma' \leq \tau\| = 1$ and $(q, \sigma') \leq_{\mathbb{P} \times \mathbb{Q}^*} (p, \tau)$ as desired.

The proof of (b) is similar and (c) is standard. $\dashv$ (Lemma 2.6)

If $\mathbb{Q}$ is in $V^{\mathbb{P}}$ we shall use the notation $A(\mathbb{P} ; \mathbb{Q})$ for the termspace partial ordering.

We now make a definition we shall use to get an easy hold on the chain condition of the termspace partial ordering.

Definition 2.7.

(a) Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orderings. $\mathbb{Q}$ is said to be **representable** in $\mathbb{P}$ if and only if there is an incompatibility preserving map $i : \mathbb{Q} \to \mathbb{P}$. (In other words: if q and q' are incompatible in $\mathbb{Q}$, $i(q)$ and $i(q')$ are incompatible in $\mathbb{P}$.)

(b) Let $\mathbb{P}$ and S be partial orderings. Define $\mathcal{F}(\mathbb{P}, S)$ as the partial ordering with domain

$$\{f : f : \mathbb{P} \to S \text{ and f is order preserving}\}$$

The ordering of $\mathcal{F}(\mathbb{P}, S)$ is $f \leq g$ if and only if for all $p \in \mathbb{P}$ $f(p) \leq g(p)$.

If $\mathbb{Q}$ is representable in $\mathbb{P}$ then the chain condition of $\mathbb{P}$ gives an upper bound on the chain condition for $\mathbb{Q}$.

Example 2.8. (a) Let $\mathbb{P}$ and S be partial orderings and $\prod_{p \in \mathbb{P}} S$ be the product of $|\mathbb{P}|$ copies of S, one for each element of $\mathbb{P}$. Then $\mathcal{F}(\mathbb{P}, S)$ is representable in $\prod_{p \in \mathbb{P}} S$, namely map $f \in \mathcal{F}(\mathbb{P}, S)$ to $g \in \prod_{p \in \mathbb{P}} S$ where g has the value $f(p)$ on the pth copy of S (so g is pointwise equal to f).

(b) Let $\mathbb{P}$ be a κ-c.c. partial ordering. Let $S^{\mathbb{P}}(\kappa, \lambda)$ be the Silver collapse of λ to κ^+ defined in $V^{\mathbb{P}}$. Let $S(\kappa, \lambda)$ be the Silver collapse of λ to κ^+ defined in V. Then $(S^{\mathbb{P}}(\kappa, \lambda))^*$ is representable in $\mathcal{F}(\mathbb{P} ; S(\kappa, \lambda))$: define a map $i : (S^{\mathbb{P}}(\kappa, \lambda))^* \to \mathcal{F}(\mathbb{P} ; S(\kappa, \lambda))$ by $i(\tau) = f_\tau$ where

$$f_\tau(p) = \{(\alpha, \beta, \gamma) : p \Vdash \tau(\alpha, \beta) = \gamma.\}$$

Using the κ-c.c. of $\mathbb{P}$ we see that $|f_\tau(p) \leq \kappa|$. Further, $p \Vdash f_\tau(p) \in S^{\mathbb{P}}(\kappa, \lambda)$, so there is a $\xi < \kappa$ such that dom $f_\tau(p) \subseteq \lambda \times \xi$. Hence $f_\tau(p) \in S(\kappa, \lambda)$. If $p \leq_{\mathbb{P}} p'$ and $p' \Vdash \tau(\alpha, \beta) = \gamma$ then $p \Vdash \tau(\alpha, \beta) = \gamma$. Thus f_τ is order preserving and $f_\tau \in \mathcal{F}(\mathbb{P}, S)$.

We must see that i preserves incompatibility. Let $\tau, \tau' \in (S^{\mathbb{P}}(\kappa, \lambda))^*$, τ and τ' incompatible. Then there is a $p \in \mathbb{P}$,

$$p \Vdash_{\mathbb{P}} \tau \text{ and } \tau' \text{ are incompatible in } S^{\mathbb{P}}(\kappa, \lambda).$$

Pick $p' \leq p, \gamma \neq \gamma', p' \Vdash \tau(\alpha, \beta) = \gamma$ and $p' \Vdash \tau'(\alpha, \beta) = \gamma'$. Then $f_\tau(p')$ is incompatible with $f_{\tau'}(p')$, and hence f_τ is incompatible with $f_{\tau'}$.

(In fact if $|\mathbb{P}| = \kappa$ it is possible to see that $(S^{\mathbb{P}}(\kappa, \lambda))^* \cong \{f : f : \mathbb{P} \to S(\kappa, \lambda),$ f is order preserving and there is a $\xi < \kappa$ $\bigcup_{p \in \mathbb{P}} \operatorname{dom} f(p) \subseteq \lambda \times \xi\}$. We leave this to the reader.)

If $\mathbb{Q} \in V^{\mathbb{P}}$ and $\|\mathbb{Q}$ is λ-closed$\|_{\mathbb{P}} = 1$ and $\langle \tau_\alpha : \alpha < \beta \rangle$ $(\beta < \lambda)$ is a descending sequence in $\mathbb{Q}^*$ then $\|$there is a $\tau \in \mathbb{Q},\ \tau \leq \tau_\alpha$ for all $\alpha\|_{\mathbb{P}} = 1$. Let τ be a term for an element of $\mathbb{Q}$ such that for each α $\|\tau \leq \tau_\alpha\|_{\mathbb{P}} = 1$. Then $\tau \leq_{\mathbb{Q}^*} \inf_{\alpha<\beta} \tau_\alpha$. Hence $\mathbb{Q}^*$ is λ-closed.

Lemma 2.9. Let $\mathbb{R} \in V^{\mathbb{P}*\mathbb{Q}}$ be a term for a partial ordering. Then $A(\mathbb{P} ; A(\mathbb{Q} ; \mathbb{R})) \cong A(\mathbb{P} * \mathbb{Q} ; \mathbb{R})$.

Proof. If $\tau \in A(\mathbb{P} ; A(\mathbb{Q} ; \mathbb{R})))$ let $\pi(\tau) \in A(\mathbb{P} * \mathbb{Q} ; \mathbb{R})$ be a term τ^* such that $\|\tau^* = ((\tau)^{V^{\mathbb{P}}})^{V^{\mathbb{Q}}}\|_{\mathbb{P}*\mathbb{Q}} = 1$. We verify that π is an isomorphism. π is clearly order preserving. Suppose $\tau, \sigma \in A(\mathbb{P} ; S(\mathbb{Q} ; \mathbb{R})),\ \|\tau = \sigma\|_{\mathbb{P}} \neq 1$. Let $p \in \mathbb{P}$, $p \Vdash$"$\tau \neq \sigma$ in $A(\mathbb{Q} ; \mathbb{R})$". Pick $q \in V^{\mathbb{P}}$, $\|q \in \mathbb{Q}\| \geq p$, with $p \Vdash$"$q \Vdash (((\tau)^{V^{\mathbb{P}}})^{V^{\mathbb{Q}}}) \neq ((\sigma)^{V^{\mathbb{P}}})^{V^{\mathbb{Q}}}$." Then $(p, q) \in \mathbb{P}*\mathbb{Q}$ and $(p, q) \Vdash \tau^* \neq \sigma^*$. Let $\tau^* \in V^{\mathbb{P}*\mathbb{Q}}$, $\|\tau^* \in \mathbb{R}\|_{\mathbb{P}*\mathbb{Q}} = 1$. In $V^{\mathbb{P}}$, there is a term τ for an element of $V^{\mathbb{P}*\mathbb{Q}}$ such that $\big\|\|\tau = \tau^*\|_{\mathbb{Q}} = 1\big\|_{\mathbb{P}} = 1$. Then $\pi(\tau) = \tau^*$. Thus π is one-to-one, onto and order preserving. $\dashv$ (Lemma 2.9)

The following lemma is standard:

Lemma 2.10. Let λ be a measurable cardinal. Let $\alpha < \lambda$ and $\langle \mathbb{P}_\gamma : \gamma < \lambda \rangle$ be a sequence of λ-c.c. partial orderings. Then $\prod_{\alpha\text{-supports}} \{\mathbb{P}_\gamma : \gamma < \lambda\}$ has the λ-c.c. (The term "α-supports" refers to the ideal of subsets of λ with cardinality $\leq \alpha$.)

We shall be interested in constructing partial orderings with nice embedding properties. We make the following definition, which is similar to one that will appear in [FL88].

Our partial orderings will be in the form $\mathbb{R}(\kappa, \lambda)$ for all regular $\kappa > \omega$ and all measurable $\lambda > \kappa$. The partial orderings $\mathbb{R}(\kappa, \lambda)$ are given by a uniform definition in the next few paragraphs.

If λ is greater than the first measurable above κ, the partial ordering $\mathbb{R}(\kappa, \lambda)$ is the product $\prod_{n\in\omega} S^n(\kappa, \lambda)$, where the partial orderings $S^n(\kappa, \lambda)$ are defined by induction on n.

Assume that we have defined $\mathbb{R}(\beta, \alpha)$ for all regular $\beta, \omega < \beta < \alpha$ and all inaccessible $\alpha < \lambda$. We now define $\mathbb{R}(\beta, \lambda)$.

Case 1. λ is the first measurable above β. Let $\mathbb{R}(\beta, \lambda) = S(\beta, \lambda) =$ the Silver collapse of λ to β^+.

Case 2. Otherwise.

By induction on n, we define $S^n(\alpha, \lambda)$ for all regular $\alpha, \beta \leq \alpha < \lambda$. We shall have a uniform definition of $S^n(\alpha, \lambda)$ no matter which model we define it in, hence we define it only in the ground model. Then $(S^n(\alpha, \lambda))^{V^{\mathbb{R}(\beta,\alpha)}}$ is the partial ordering constructed in $V^{\mathbb{R}(\beta,\alpha)}$ using this definition.

If $n = 0$:

$$\text{Let } S^0(\alpha, \lambda) = S(\alpha, \lambda) = \text{ the Silver collapse of } \lambda \text{ to } \alpha^+.$$

Assume that we have defined $S^n(\alpha, \lambda)$ for all regular α.

For $n + 1$:

Let $S^{n+1}(\alpha, \lambda) =$

$$\prod_{\alpha\text{-supports}} \{A(\mathbb{R}(\alpha, \gamma)\ ;\ S^n(\gamma, \lambda))\ :\ \alpha < \gamma < \lambda \text{ and } \gamma \text{ is measurable}\}.$$

LEMMA 2.11. For all regular $\beta > \omega$ and all measurable λ, $\mathbb{R}(\beta, \lambda)$ is λ-c.c. and has cardinality λ.[4]

PROOF. By Lemma 2.10, it is enough to see that each $S^n(\beta, \lambda)$ has the λ-c.c. We show by induction on λ and n, that for all measurable $\alpha_1, \dots, \alpha_m$ $A(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^n(\alpha_m, \lambda))$ has the cardinality λ and the λ-c.c.

Assume that this is true for all $\lambda' < \lambda$. We first consider the case $n = 0$. By Example 2.8 we know that $A(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^0(\alpha_m, \lambda))$ is representable in $\mathcal{F}(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^0(\alpha_m, \lambda))$. This in turn is representable in the product of α_m copies of $S^0(\alpha_m, \lambda)$. By Lemma 2.10, this has the λ-c.c.

Assume that $A(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^n(\alpha_m, \lambda))$ is λ-c.c. for all $\alpha_1, \dots, \alpha_m$ measurable. We want for each $\alpha_1, \dots, \alpha_m$ measurable, $A(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^{n+1}(\alpha_m, \lambda))$ is λ-c.c.

Fix a sequence $\alpha_1, \dots, \alpha_m$. In $V^{\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i,\alpha_{i+1})}$, $S^{n+1}(\alpha_m, \lambda) =$

$$\prod_{\alpha_m\text{-supports}} \{A(\mathbb{R}(\alpha_m, \beta)\ ;\ S^n(\beta, \lambda))\ :\ \alpha_m < \beta < \lambda \text{ and } \beta \text{ is measurable}\}.$$

Since $\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})$ has the α_m-c.c., if $\tau \in A(\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i, \alpha_{i+1})\ ;\ S^{n+1}(\alpha_m, \lambda))$ there is a set $D \subseteq \lambda$ lying in V such that $|D| = \alpha_m$ and $\|\operatorname{supp}\tau \subseteq D\|_{\mathop{*}_{i=1}^{m-1}\mathbb{R}(\alpha_i,\alpha_{i+1})} = 1$. For each $\beta \in D$, we get a term τ_β such that $\|\tau(\beta) =$

[4] In fact we could have replaced the requirement that λ be measurable by asking that λ be Mahlo throughout, but we leave this to the reader.

$\tau_\beta \|_{\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})} = 1$ and $\tau_\beta \in A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})\ ;\ A(\mathbb{R}(\alpha_m,\beta)\ ;\ S^n(\beta,\lambda)))$. It is easy to verify that the map $\tau \mapsto \langle \tau_\beta\ :\ \beta \in D\rangle$ is an isomorphism between $A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})\ ;\ S^{n+1}(\alpha_m,\lambda))$ and $\prod_{\alpha_m\text{-supports}}\{A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})\ ;\ A(\mathbb{R}(\alpha_m,\beta)\ ;\ S^n(\beta,\lambda)))\ \alpha_m < \beta < \lambda$ and β is measurable$\}$. By Lemma 2.9,

$$A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})\ ;\ A(\mathbb{R}(\alpha_m,\beta)\ ;\ S^n(\beta,\lambda))) \cong A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1}) * \mathbb{R}(\alpha_m,\beta)\ ;\ S^n(\beta,\lambda)).$$

By the induction hypothesis for n, $A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1}) * \mathbb{R}(\alpha_m,\beta)\ ;\ S^n(\beta,\lambda))$ has the λ-c.c. and thus by 2.10 $A(\overset{m-1}{\underset{i=1}{*}}\mathbb{R}(\alpha_i,\alpha_{i+1})\ ;\ S^{n+1}(\alpha_m,\lambda))$ has λ-c.c. $\dashv$ (Lemma 2.11)

We now establish the properties we need to satisfy the conditions of Kunen's Theorem:

Lemma 2.12. For each $\kappa > \omega$, κ regular and each λ measurable:

(a) $\prod_{\substack{n\in\omega\\ n\geq 1}} S^n(\kappa,\lambda)$ is κ^+-closed.

(b) $\mathbb{R}(\kappa,\lambda) = \prod_{n\in\omega} S^n(\kappa,\lambda)$ is κ-closed.

(c) If α is measurable and $\mathrm{id}\colon \mathbb{R}(\kappa,\alpha) \hookrightarrow \mathbb{R}(\kappa,\lambda)$ then there is a map φ extending id such that
$$\varphi\colon \mathbb{R}(\kappa,\alpha) \times \prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda)) \hookrightarrow \mathbb{R}(\kappa,\lambda).$$

(d) If α is measurable and $\mathrm{id}\colon \mathbb{R}(\kappa,\alpha) \hookrightarrow \mathbb{R}(\kappa,\lambda)$ then there is a map ψ extending id,
$$\psi\colon \mathcal{B}(\mathbb{R}(\kappa,\alpha) * \mathbb{R}(\alpha,\lambda)) \hookrightarrow \mathcal{B}(\mathbb{R}(\kappa,\lambda)).$$

Proof. (a) We show by induction on $n > 0$ that for all regular α, $S^n(\alpha,\lambda)$ is α^+-closed.

$n = 1$: $S^1(\alpha,\lambda) = \prod_{\alpha\text{-supports}}\{A(\mathbb{R}(\alpha,\beta)\ ;\ S^0(\beta,\lambda))\ \alpha < \beta < \lambda$ and β is measurable$\}$

By our remarks preceding Lemma 2.9, each $A(\mathbb{R}(\alpha,\beta)\ ;\ S^0(\beta,\lambda))$ is β-closed and hence α^+-closed. Thus $\prod_{\alpha\text{-supports}}\{A(\mathbb{R}(\alpha,\beta)\ ;\ S^0(\beta,\lambda))\ \alpha < \beta < \lambda$ and β is measurable$\}$ is α^+-closed.

Assume that for all regular β, $S^n(\beta,\lambda)$ is β^+-closed and $n \geq 1$. Then, $A(\mathbb{R}(\alpha,\beta)\ ;\ S^n(\beta,\lambda))$ is β^+-closed and hence α^+-closed. Again this implies $\prod_{\alpha\text{-supports}}\{A(\mathbb{R}(\alpha,\beta)\ ;\ S^0(\beta,\lambda))\ \alpha < \beta < \lambda$ and β is measurable$\}$ is α^+-closed.

(b) $\mathbb{R}(\kappa,\lambda) = S^0(\kappa,\lambda) \times \prod_{n\geq 1} S^n(\kappa,\lambda)$ where $S^0(\kappa,\lambda)$ is the Silver collapse of λ to κ^+. The Silver collapse is κ-closed and by (a), $\prod_{n\geq 1} S^n(\kappa,\lambda)$ is κ^+-closed. Hence the product is κ-closed.

(c) Suppose $\mathrm{id}\colon \mathbb{R}(\kappa,\alpha) \hookrightarrow \mathbb{R}(\kappa,\lambda)$. By the definition of $S^{n+1}(\kappa,\lambda)$, if α is measurable then $A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ is a factor of $S^{n+1}(\kappa,\lambda)$. Let φ_n be the map from $A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ to $S^{n+1}(\kappa,\lambda)$ that sends $p \in A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ to the element $q \in S^{n+1}(\kappa,\lambda)$ that is 1 on each factor of $S^{n+1}(\kappa,\lambda)$ except $A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ where it is equal to p.[5] The product, $\prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ can be embedded in $\prod_{n\in\omega} S^n(\kappa,\lambda)$ by the map

$$\langle p_n\ :\ n\in\omega\rangle \xmapsto{\varphi_\omega} \langle q_n\ :\ n\in\omega\rangle$$

where $q_0 = 1$, $q_{n+1} = \varphi_n(p_n)$.

By Lemma 2.6(c), φ_ω is a neat embedding from $\prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ into $\prod_{n\in\omega} S^n(\kappa,\lambda)$.

Each

$$\begin{aligned} S^{n+1}(\kappa,\lambda) = &\prod_{\kappa\text{-supports}} \{A(\mathbb{R}(\kappa,\beta)\ ;\ S^n(\beta,\lambda))\ :\ \kappa<\beta<\alpha \text{ and } \beta \text{ is measurable.}\} \\ &\times \prod_{\kappa\text{-supports}} \{A(\mathbb{R}(\kappa,\beta)\ ;\ S^n(\beta,\lambda))\ :\ \alpha\le\beta<\lambda \text{ and } \beta \text{ is measurable }\}. \end{aligned}$$

Let

$$S_0^{n+1}(\kappa,\lambda) = \prod_{\kappa\text{-supports}} \{A(\mathbb{R}(\kappa,\beta)\ ;\ S^n(\beta,\lambda)) : \kappa<\beta<\alpha \text{ and } \beta \text{ is measurable.}\},$$

$$S_1^{n+1}(\kappa,\lambda) = \prod_{\kappa\text{-supports}} \{A(\mathbb{R}(\kappa,\beta)\ ;\ S^n(\beta,\lambda))\ :\ \alpha\le\beta<\lambda \text{ and } \beta \text{ is measurable}\}.$$

By rearranging our product we get

$$\mathbb{R}(\kappa,\lambda) = S^0(\kappa,\lambda) \times \prod_{\substack{n\in\omega\\ n\ge 1}} S_0^n(\kappa,\lambda) \times \prod_{\substack{n\in\omega\\ n\ge 1}} S_1^n(\kappa,\lambda).$$

For $n \ge 1$, $\mathrm{id}''S^n(\kappa,\alpha) \subseteq S_0^n(\kappa,\lambda)$. Hence, if $\mathrm{id}\colon \mathbb{R}(\kappa,\alpha) \hookrightarrow \mathbb{R}(\kappa,\lambda)$ then

$$\mathrm{id}\colon \mathbb{R}(\kappa,\alpha) \hookrightarrow S^0(\kappa,\lambda) \times \prod_{1\le n\in\omega} S_0^n(\kappa,\lambda)$$

By the definition of φ_n, $\varphi_n'' A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda)) \subseteq S_1^n(\kappa,\lambda)$. So,

$$\varphi_\omega'' \prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda)) \subseteq \prod_{n\in\omega} S_1^n(\kappa,\lambda).$$

Thus we have a map $\varphi = \mathrm{id}\times\varphi_\omega$

$$\begin{aligned} \varphi\colon \mathbb{R}(\kappa,\alpha) \times \Big[\prod_{n\in\omega} A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))\Big] &\hookrightarrow \\ &\Big(S^0(\kappa,\lambda) \times \prod_{1\le n\in\omega} S_0^n(\kappa,\lambda)) \times \Big[\prod_{1\le n\in\omega} S_1^n(\kappa,\lambda)\Big]\Big). \end{aligned}$$

[5]In essence φ_n is the identity map of $A(\mathbb{R}(\kappa,\alpha)\ ;\ S^n(\alpha,\lambda))$ to its factor in $S^{n+1}(\kappa,\lambda)$.

Hence

$$\varphi\colon \mathbb{R}(\kappa,\alpha)\times\Big(\prod_{n\in\omega}A(\mathbb{R}(\kappa,\alpha)\,;\,S^n(\alpha,\lambda))\Big)\hookrightarrow\mathbb{R}(\kappa,\lambda).$$

(d) By Lemma 2.6(b), we have a neat embedding

$$\psi_1\colon \mathcal{B}(\mathbb{R}(\kappa,\alpha)*\mathbb{R}(\alpha,\lambda))\hookrightarrow\mathcal{B}\Big(\mathbb{R}(\kappa,\alpha)\times\prod_{n\in\omega}A(\mathbb{R}(\kappa,\alpha)\,;\,S^n(\alpha,\lambda))\Big)$$

By (c), we have

$$\psi_2\colon \mathcal{B}\Big(\mathbb{R}(\kappa,\alpha)\times\prod_{n\in\omega}A(\mathbb{R}(\kappa,\alpha)\,;\,S^n(\alpha,\lambda))\Big)\hookrightarrow\mathcal{B}(\mathbb{R}(\kappa,\lambda))$$

Composing we get:

$$\psi=\psi_1\circ\psi_2\colon \mathcal{B}(\mathbb{R}(\kappa,\alpha)*\mathbb{R}(\alpha,\lambda))\hookrightarrow\mathcal{B}(\mathbb{R}(\kappa,\lambda)).$$

This completes the definition and verification of the basic properties of the $\mathbb{R}$'s.

⊣ (Lemma 2.12)

Returning to the proofs of Theorems 1.2 & 1.3:
Let

$$\mathbb{P}=\varinjlim\,\langle\mathbb{R}(\aleph_1,\kappa_0)*\overline{\mathbb{R}(\kappa_0,\kappa_1)}*\overline{\mathbb{R}(\kappa_1,\kappa_2)}*\cdots*\overline{\mathbb{R}(\kappa_{n-1},\kappa_n)}\;:\;n\in\omega\rangle.$$

To see in $V^{\mathbb{P}}$:

(a) for all $m>1$, $\aleph_m$ carries an $\aleph_{m+1}$-saturated ideal
(b) for all $m>n\geq 1$, $(\aleph_{m+1},\aleph_m)\twoheadrightarrow(\aleph_{n+1},\aleph_n)$

It is enough to see that $\mathbb{P}'=\mathbb{R}(\aleph_1,\kappa_0)*\cdots*\bar{\mathbb{R}}(\kappa_{n-1},\kappa_n)$ and $\mathbb{Q}=\mathbb{R}^{\mathbb{P}'}(\kappa_n,\kappa_{n+1})$ together with the embedding j_n satisfy the conditions for Kunen's Theorem. Conditions (a), (b) and (c) follow directly from Lemma 2.11 and the remarks before Lemma 2.9.

Since $\mathbb{P}'$ is κ_n-c.c., $j_n\colon\mathbb{P}'\to j(\mathbb{P}')$ is a neat embedding. Since $\mathbb{P}'\subseteq V_{\kappa_n}$, $j_n\restriction\mathbb{P}'=\mathrm{id}$ and hence $\mathrm{id}\colon\mathbb{P}'\hookrightarrow j(\mathbb{P}')$. Unraveling this we get

$$\begin{aligned}\mathrm{id}\colon \mathbb{R}(\aleph_1,\kappa_0)*\bar{\mathbb{R}}(\kappa_0,\kappa_1)*\cdots*\bar{\mathbb{R}}(\kappa_{n-1},\kappa_n)&\\ \hookrightarrow\mathbb{R}(\aleph_1,\kappa_0)*\bar{\mathbb{R}}(\kappa_0,\kappa_1)*\cdots*\bar{\mathbb{R}}(\kappa_{n-1},\kappa_{n+1}).&\end{aligned}$$

By Lemma 2.12 there is a generic object $G*H\subseteq\mathbb{P}'*\mathbb{Q}$ in $V^{j_n(\mathbb{P}')}$. In $V^{j(\mathbb{P}')}$, $H\cong H_0\times H_{>0}$, where $H_0\subseteq S^0(\kappa_n,\kappa_{n+1})$ and $H_{>0}\subseteq\prod_{k>0}S^k(\kappa_n,\kappa_{n+1})$. Still in $V^{j(\mathbb{P}')}$: $j''(H)=j''(H_0)\times j''(H_{>0})\subseteq S^0(\kappa_{n+1},\kappa_{n+2})\times\prod_{k>0}S^k(\kappa_{n+1},\kappa_{n+2})$ and $j''H$ has cardinality κ_{n+1}. By Lemma 2.12, $\prod_{k>0}S^k(\kappa_{n+1},\kappa_{n+2})$ is κ_{n+1}^+-closed and hence there is a $q_{>0}\in\prod_{k>0}S^k(\kappa_{n+1},\kappa_{n+2})$ such that for all $p\in j''H_{>0}$, $q_{>0}\leq p$. Let $q_0=\bigcup_{\tau\in H_0}(j(\tau))^{V^{j(\mathbb{P}')}}$. We show that $q_0\in S^0(\kappa_{n+1},\kappa_{n+2})$. Clearly q_0 is a function satisfying from $\kappa_{n+2}\times\kappa_{n+1}$ all but the cardinality conditions for being an element of $S^0(\kappa_{n+1},\kappa_{n+2})$. To see that it satisfies these, we note that for each $\tau \in (S^0(\kappa_n, \kappa_{n+1}))^{V^{\mathbb{P}'}}$ there is a $\xi < \kappa_n$ such that $|\mathrm{dom}\,\tau \subseteq \kappa_{n+1} \times \xi|_{\mathbb{P}'} = 1$. Hence, $\mathrm{dom}\, q_0 \subseteq \kappa_{n+2} \times \kappa_n$. Further,

$$|\mathrm{dom}\, q_0| = \Big|\bigcup_{\tau \in H_0} \mathrm{dom}\, j(\tau)^{V^{j(\mathbb{P}')}}\Big| = \kappa_{n+1} \times \kappa_{n+1} = \kappa_{n+1}.$$

Hence, $q_0 \in (S^0(\kappa_{n+1}, \kappa_{n+2}))^{V^{j(\mathbb{P}')}}$. Let

$$q = (q_0, q_{>0}) \in S^0(\kappa_{n+1}, \kappa_{n+2}) \times \prod_{k>0} S^k(\kappa_{n+1}, \kappa_{n+2}).$$

It is easy to check that in $V^{j(\mathbb{P})}$, if $p \in H$, $j(p) \geq q$. From this we conclude that $(1, q) \in j(\mathbb{P}') * \mathbb{R}(\kappa_{n+1}, \kappa_{n+2})$ functions as a master condition and satisfies (d) of Kunen's Theorem.

Forcing with $\mathbb{P} \times C(\aleph_0, \aleph_1)$ we get a model as in Theorem 1.2. This completes the proof of Theorem 1.2. For Theorem 1.3 we remark that if our original sequence $\langle j_n : n \in \omega\rangle$ was a sequence of almost huge embeddings then the $\mathbb{R}'s$ satisfy the Magidor conditions and our proof would go through to get the saturated ideals.(Although apparently not the strong transfer principles.)

We end this section with some problems. First, not much is known about transfer principles that go "infinite distances". A problem along this line might be:

Open Problem 2.13. It is consistent to have $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$?

Magidor has remarked that if the GCH holds then $(\aleph_{\omega+1}, \aleph_\omega) \not\twoheadrightarrow (\aleph_{m+1}, \aleph_m)$ for $m > 0$. Along these lines, the author has shown:

Con(ZFC+"there is a 2-huge cardinal") implies

Con(ZFC+"there is a supercompact cardinal κ with $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_1, \aleph_0)$").

This might be relevant to this problem.

In this paper we have "gap one" transfer principles. In [For82] it was shown that Con(ZFC+"there is a 2-huge cardinal") implies Con(ZFC+$(\aleph_3, \aleph_1) \twoheadrightarrow (\aleph_2, \aleph_0)$).

Open Problem 2.14. Is it consistent to have for all $m > n$ $(\aleph_{m+2}, \aleph_m) \twoheadrightarrow (\aleph_{n+2}, \aleph_n)$?

A general solution to this problem would probably solve many problems in this area.

Another problem is:

Open Problem 2.15. Does "for all $m > n$ $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$" imply "$\aleph_\omega$ is Jonsson"?

Finally, since the method of proof for Theorems 1.2 & 1.3 was the same it would be interesting to establish a direct relationship between saturated ideals and transfer properties. Shelah has shown that "there is an $\aleph_2$-complete ideal $\mathcal{I}$ on $\aleph_2$ such that $\wp(\aleph_2)/\mathcal{I}$ has a dense ω-closed subset" implies $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$. The proof however is tied very closely to the cardinals $\aleph_2, \aleph_1$ and $\aleph_0$. What is desired is some more general statement for cardinals bigger than $\aleph_2$. For the sake of concreteness we mention a possibility.

Open Problem 2.16. Does "$\aleph_2$ carries an $\aleph_3$-saturated ideal and $\aleph_3$ carries precipitous ideal" imply $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$?

§**3.** We now turn our attention to proving Theorem 1.4. For this section a "saturated ideal" on κ will mean a normal, κ-complete, κ^+-saturated ideal on κ.

Proof of Theorem 1.4. Let κ be a huge cardinal. Then there is a set $A \subseteq \kappa$ with measure one with respect to the measure on κ induced by the huge embedding, such that for all $\alpha, \beta \in A$, $\beta > \alpha$ implies: There is an almost huge embedding $j_{\alpha,\beta}$ with $\text{crit}(j_{\alpha,\beta}) = \alpha$ and $j_{\alpha,\beta} = \beta$. Fix such a set A. If $\alpha \in A$, let α' be the successor of $\alpha \in A$.

We need to redefine our $\mathbb{R}(\kappa, \lambda)$'s. Assume by induction that we have defined $\mathbb{R}(\kappa, \lambda')$ for all regular $\kappa > \omega$ and all measurable $\lambda' < \lambda$, and that $\mathbb{R}(\kappa, \lambda')$ is closed, has cardinality λ' and is λ'-c.c.

If $X \subseteq \lambda$ is a set for measurable cardinals we define the **standard iteration** I_X as follows:

(a) I_X is an iteration of length $\leq \lambda$ with Easton supports. We shall define the notion of an ordinal in X being mentioned in $(I_X)_\alpha$ by induction on α. To start with, no ordinals are mentioned in $(I_X)_0$.

(b) At stage $\alpha + 1$: If there are fewer than five elements of X not mentioned in $(I_X)_\alpha$, then $I_X = (I_X)_\alpha$.

Otherwise, let $\gamma_0 < \gamma_1 < \cdots < \gamma_4$ be the first five elements of X not mentioned in $(I_X)_\alpha$. If $\alpha \geq \gamma_0$ let,

$$(I_X)_{\alpha+1} = (I_X)_\alpha * [\bar{\mathbb{R}}(\gamma_0, \gamma_1) * \bar{\mathbb{R}}(\gamma_1, \gamma_2) * \bar{\mathbb{R}}(\gamma_2, \gamma_3) * \bar{\mathbb{R}}(\gamma_3, \gamma_4)].$$

The set of elements of X mentioned in $(I_X)_{\alpha+1}$ is defined to be elements of X mentioned in $(I_X)_\alpha$ together with $\gamma_0, \dots, \gamma_4$.

If $\alpha < \gamma_0$ then $(I_X)_{\alpha+1} = (I_X)_\alpha * 1$, and the elements of X mentioned in $(I_X)_{\alpha+1}$ are the same as the elements of X mentioned in $(I_X)_\alpha$.

Informally, this iteration is the result of iterating, five at a time, the elements of X strung together by the $\mathbb{R}$'s.

Proposition 3.1. For every X, I_X is λ-c.c. and $I_X/(I_X)_\alpha$ is γ_0-closed in $V^{(I_X)_\alpha}$, where γ_0 is the least ordinal mentioned in $(I_X)_{\alpha+1}$ not mentioned in $(I_X)_\alpha$.

Proof. This uses standard facts about iterations with Easton support. $\dashv$ (Proposition 3.1)

We can now define our new version of $\mathbb{R}(\kappa,\lambda)$ for all regular $\kappa > \omega$. Again $\mathbb{R}(\kappa,\lambda) = \prod_{n\in\omega} S^n(\kappa,\lambda)$ where $S^n(\alpha,\lambda)$ are defined by induction on n, simultaneously for all regular α, $\omega < \alpha < \lambda$.

Let $S^0(\alpha,\lambda) =$ the Silver collapse of λ to α^+

$$S^{n+1}(\alpha,\lambda) = \prod_{\alpha\text{-supports}} \{A(\mathbb{R}(\alpha,\beta); C(\beta,\gamma) \times [I_X * \bar{S}^n(\gamma,\lambda)]) : \alpha < \beta < \gamma < \lambda, \beta, \gamma \text{ are measurable and } X \subseteq \gamma\backslash\beta \text{ is a set of measurable cardinals.}\}$$

Let $\mathbb{R}(\alpha,\lambda) = \prod_{n\in\omega} S^n(\alpha,\lambda)$.

Using the techniques of §2 it is now easy to verify that if $\omega < \alpha < \gamma < \lambda$, α is regular, β, γ and λ are measurable and $X \subseteq \gamma\backslash\beta$ is a set of measurable cardinals then there is a map ψ

$$\psi\colon \prod_{n\in\omega} A(\mathbb{R}(\alpha,\beta) \,;\, C(\beta,\gamma) \times [I_X * \bar{S}^n(\gamma,\lambda)]) \hookrightarrow \mathbb{R}(\alpha,\lambda).$$

From this (as before) we conclude that if $\mathrm{id}\colon \mathbb{R}(\alpha,\beta) \hookrightarrow \mathbb{R}(\alpha,\lambda)$ then there is a map φ

$$\varphi\colon \mathbb{R}(\alpha,\beta) \times \prod_{n\in\omega} A(\mathbb{R}(\alpha,\beta) \,;\, C(\beta,\gamma) \times [I_X * \bar{S}^n(\gamma,\lambda)]) \hookrightarrow \mathbb{R}(\alpha,\lambda).$$

Again this implies that: if $\mathrm{id}\colon \mathbb{R}(\alpha,\beta) \hookrightarrow \mathbb{R}(\alpha,\lambda)$ then there is a map φ

$$\varphi\colon \mathcal{B}(\mathbb{R}(\alpha,\beta) * [\bar{C}(\beta,\gamma) \times (I_X * \bar{\mathbb{R}}(\gamma,\lambda))]) \hookrightarrow \mathcal{B}(\mathbb{R}(\kappa,\lambda))$$

LEMMA 3.2. If $\kappa > \omega$ is regular and $\lambda > \kappa$ is measurable

(a) $\mathbb{R}(\kappa,\lambda)$ has the λ-c.c.
(b) $\mathbb{R}(\kappa,\lambda)$ is κ-closed.
(c) In $V^{\mathbb{R}(\kappa,\lambda)}$, $\lambda = \kappa^+$.
(d) $|\mathbb{R}(\kappa,\lambda)| = \lambda$.
(e) $\prod_{\substack{n\in\omega\\ n\geq 1}} S^n(\kappa,\lambda)$ is κ^+-closed.
(f) $\prod_{n\in\omega} S^n(\kappa,\lambda)$ is κ-closed.

PROOF. Essentially the same as in Section 2. ⊣ (Lemma 3.2)

To build our model of Theorem 1.3 we start by first doing the following preparatory forcing. (Recall that κ is our huge cardinal, j our huge embedding, and the definition of A.)

Let

$$\mathbb{P}_{\leq\kappa} = I_A * \bar{\mathbb{R}}(\kappa,\kappa') * \bar{\mathbb{R}}(\kappa',\kappa'') * \bar{\mathbb{R}}(\kappa'',\kappa''') * \bar{\mathbb{R}}(\kappa''',\kappa^{\mathrm{iv}}),$$

where I_A is the standard iteration of A and $\kappa', \kappa'', \kappa''', \kappa^{\mathrm{iv}}$ are the first four elements of $j(A)$ above κ.

PROPOSITION 3.3. In $\mathbb{P}_{\leq\kappa}$, we have that

(a) $\kappa' = \kappa^+, \kappa'' = \kappa^{++}, \kappa''' = \kappa^{+3}, \kappa^{\text{iv}} = \kappa^{+4}$,
(b) κ remains κ^{+5} supercompact by a measure μ_5 such that there is a set of μ_5-measure-one worth of $x \in \mathbb{P}_\kappa(\kappa^{+5})$ such that if $\alpha = \operatorname{crit}(x)$, then $\alpha^+, \alpha^{+2}, \alpha^{+3}$ carry saturated ideals, and
(c) $2^\kappa = \kappa^+$.

Proof. (a) I_A is κ-c.c. by earlier comments. In V^{I_A}, $\mathbb{R}(\kappa, \kappa')$ makes κ' into κ^+ and is κ-closed. In the $V^{I_A * \bar{\mathbb{R}}(\kappa,\kappa')}$, $\bar{\mathbb{R}}(\kappa', \kappa'') * \bar{\mathbb{R}}(\kappa'', \kappa''') * \bar{\mathbb{R}}(\kappa''', \kappa^{\text{iv}})$ is κ'-closed. Hence, in $V^{\mathbb{P}_{\leq\kappa}}$, $\kappa' = \kappa^+$. In $V^{I_A * \mathbb{R}(\kappa,\kappa')}$, $\mathbb{R}(\kappa', \kappa'')$ makes κ'' into $(\kappa')^+$. Since $\mathbb{R}(\kappa'', \kappa''') * \bar{\mathbb{R}}(\kappa''', \kappa^{\text{iv}})$ is $< \kappa''$-closed in $V^{I_A * \mathbb{R}(\kappa,\kappa') * \mathbb{R}(\kappa',\kappa'')}$, $\kappa'' = (\kappa')^+$ in $V^{\mathbb{P}_{\leq\kappa}}$. Hence $\kappa'' = \kappa^{+2}$ in $V^{\mathbb{P}_{\leq\kappa}}$. Similarly we see that in $V^{\mathbb{P}_{\leq\kappa}}, \kappa''' = \kappa^{+3}$ and $\kappa^{\text{iv}} = \kappa^{+4}$.

(b) This is a standard argument using the fact that $\mathbb{P}_{\leq\kappa}$ is initial segment of $j(\mathbb{P}_{\leq\kappa})$ and $j(\mathbb{P}_{\leq\kappa})/\mathbb{P}_{\leq\kappa}$ is $(\beth_3(\kappa^{+5}))^{V^{\mathbb{P}_{\leq\kappa}}}$-closed in $V^{\mathbb{P}_{\leq\kappa}}$. (*Cf.* [Bau83]). In fact we can require that the measure μ_5 induces on κ extends the measure j induces on κ. Hence, $\{\alpha$: at some stage γ in the iteration I_A, α is the least ordinal not mentioned in $(I_A)_\gamma\}$ has measure one with respect to the measure μ_5 induces on κ.

At some stage γ, if α is the least ordinal not mentioned in $(I_A)_\gamma$ and $\gamma \geq \alpha$,

$$I_A = (I_A)_\gamma * \bar{\mathbb{R}}(\alpha, \alpha') * \bar{\mathbb{R}}(\alpha', \alpha'') * \bar{\mathbb{R}}(\alpha'', \alpha''') * \bar{\mathbb{R}}(\alpha''', \alpha^{\text{iv}}) * (I_A/(I_A)_{\gamma+1}).$$

By Lemma 2.1, it is enough to see that in $V^{(I_A)_{\gamma+1}}$, $\alpha^+, \alpha^{++}, \alpha^{+3}$ carry saturated ideals.

Since, $\alpha', \alpha'' \in A$, there is an almost huge embedding in V, $j_{\alpha,\alpha'}$ such that $j_{\alpha'}(\alpha') = \alpha''$ and $\operatorname{crit}(j_\alpha) = \alpha'$. By Magidor's modification of the Kunen theorem (with $\mathbb{P}' = (I_A)_\alpha * \mathbb{R}(\alpha, \alpha')$ and $\varphi = \bar{\mathbb{R}}(\alpha', \alpha'')$ and $j_{\alpha',\alpha''}$ the almost huge embedding). There is a saturated ideal on α' in $V^{(I_A)\alpha * \bar{\mathbb{R}}(\alpha,\alpha') * \bar{\mathbb{R}}(\alpha',\alpha'')}$. Since $(I_A)_{\alpha+1}/((I)_\alpha * \mathbb{R}(\alpha, \alpha') * \mathbb{R}(\alpha', \alpha''))$ is α''-closed we can apply Lemma 2.1 to get that there is a saturated ideal on α^+ in $V^{(I_A)_{\alpha+1}}$. Entirely similar arguments work for α^{+2} and α^{+3}.

(c) is standard. ⊣ (Proposition 3.3)

In our final model, κ will become $\aleph_\omega$ using a technique of Magidor [Mag78]. Let $V' = V^{\mathbb{P}_{\leq\kappa}}$.

Lemma 3.4. Let $\gamma < \beta$ be inaccessible elements of A in V'. In $(V')^{C(\gamma^{+4},\beta)}$, γ^{+4} carries a $(\gamma^{+4})^+$-saturated ideal.

Proof. In $(V')^{C(\gamma^{+4},\beta)}$, $(\gamma^{+4})^+ = (\beta^+)^{V'}$. Let α_1 and α_2 be the stages where γ and β are the least elements of A not mentioned in $(I_A)_{\alpha_1}$ and $(I_A)_{\alpha_2}$ respectively.

By Proposition 3.1, $\mathbb{P}_{\leq\kappa}/(\mathbb{P}_{\alpha_2} * \bar{\mathbb{R}}(\beta, \beta'))$ is β'-closed. Further $C(\gamma^{+4}, \beta)^{V'} = C(\gamma^{+4}, \beta)^{V^{\mathbb{P}_{\alpha_1+1}}}$ and has cardinality β. Thus we are in position to apply

Lemma 2.1 in $V^{\mathbb{P}_{\alpha_2}*\mathbb{R}(\beta,\beta')}$. Note $(V')^{C(\gamma^{+4},\beta)}$ is an extension of $V^{\mathbb{P}_{\alpha_2}*\mathbb{P}(\beta,\beta')}$ of the form

$$(\beta'\text{-c.c.}) \times (\beta'\text{-closed})$$
$$(\textit{i.e.}, C(\gamma^{+4},\beta) * \mathbb{P}_{\leq\kappa}/(\mathbb{P}_{\alpha_2} * \mathbb{P}(\beta,\beta')))$$

Thus by Lemma 2.1, it is enough to see that γ^{+4} carries a β'-saturated ideal in $V^{\mathbb{P}_{\alpha_2}*\bar{\mathbb{R}}(\beta,\beta')*\bar{C}(\gamma^{+4},\beta)}$. Let $\mathbb{Q}' = \mathbb{P}_{\alpha_1} * \bar{\mathbb{R}}(\gamma,\gamma') * \bar{\mathbb{R}}(\gamma',\gamma'') * \bar{\mathbb{R}}(\gamma'',\gamma''')$. Then $\mathbb{P}_{\alpha_1+1} = \mathbb{Q}' * \mathbb{R}(\gamma''',\gamma^{\mathrm{iv}})$. Let $j_{\gamma^{\mathrm{iv}},\beta'}$ be the almost huge embedding in V with $\mathrm{crit}(j_{\gamma^{\mathrm{iv}},\beta'}) = \gamma^{\mathrm{iv}}$ and $j_{\gamma^{\mathrm{iv}},\beta'}(\gamma^{\mathrm{iv}}) = \beta'$. Then

$$j_{\gamma^{\mathrm{iv}},\beta'}(\mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}})) = \mathbb{Q}' * \mathbb{R}(\gamma''',\beta').$$

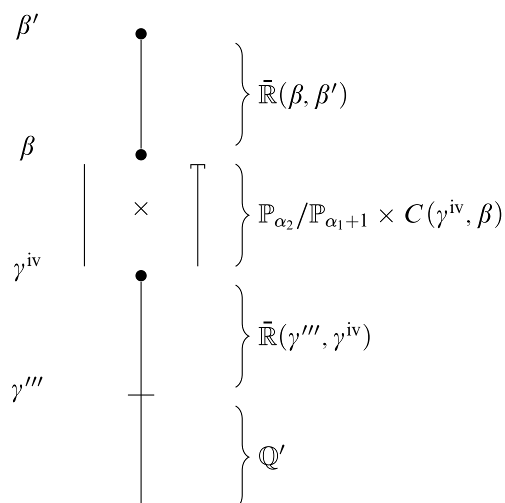
Since $\mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}}) \subseteq V_{\gamma^{\mathrm{iv}}}$ and $\mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}})$ is γ^{iv}-c.c., $j \restriction \mathbb{R}(\gamma''',\gamma^{\mathrm{iv}}) = \mathrm{id}$ and hence

$$\mathrm{id}\colon \mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}}) \hookrightarrow \mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\beta').$$

Thus by our construction of $\bar{\mathbb{R}}(\gamma''',\beta')$, there is an embedding

$$i\colon \mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}}) * [\bar{C}(\gamma^{\mathrm{iv}},\beta) * \{(\mathbb{P}_{\alpha_2}/\mathbb{P}_{\alpha_1+1}) * \bar{\mathbb{R}}(\beta,\beta')\}] \hookrightarrow \mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\beta').$$

Since $\bar{C}(\gamma^{\mathrm{iv}},\beta) \times \{(\mathbb{P}_{\alpha_2}/\mathbb{P}_{\alpha_1+1}) * \bar{\mathbb{R}}(\beta,\beta')\}$ is γ^{iv}-closed in $V^{\mathbb{Q}'*\mathbb{R}(\gamma''',\gamma^{\mathrm{iv}})}$, Magidor's remarks about Kunen's theorem (in Section 2) apply with $\mathbb{P}' = \mathbb{Q}' * \bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}})$ and $\mathbb{Q} = \bar{C}(\gamma^{\mathrm{iv}},\beta) \times \{(\mathbb{P}_{\alpha_2}/\mathbb{P}_{\alpha_1+1}) * \bar{\mathbb{R}}(\beta,\beta')$ and we conclude that in the model $V^{\mathbb{Q}'*\bar{\mathbb{R}}(\gamma''',\gamma^{\mathrm{iv}})*[\bar{C}(\gamma^{\mathrm{iv}},\beta)\times\{(\mathbb{P}_{\alpha_2}/\mathbb{P}_{\alpha_1+1})*\mathbb{R}(\beta,\beta')\}]}$, γ^{iv} carries a $(\gamma^{\mathrm{iv}})^+$-saturated ideal. Pictorially:



$\dashv$ (Lemma 3.4)

We are now ready to define our forcing conditions for the model in Theorem 1.4.

This forcing is a direct variant of the Magidor forcing [Mag78]. However, in our forcing we collapse the points on the Prikry sequence. This necessitates a slightly different argument which we show below.

For the moment, we shall regard V' as our ground model. Note that in V', κ^+ carries a saturated ideal. We shall define two forcings, $\mathbb{P}$ and $\mathbb{P}^\pi$ with a projection map

$$\pi\colon \mathbb{P} \to \mathbb{P}^\pi.$$

Our final model will be $(V')^{\mathbb{P}^\pi}$. We shall use $\mathbb{P}$ to show that $\mathbb{P}^\pi$ has nice Prikry-type properties.

Let μ_3 be the κ^{+3}-supercompact measure on $\wp_\kappa(\kappa^{+3})$ in V' that is induced by μ_5. For $x \in \wp_\kappa(\kappa^{+3})$, let $x \cap \kappa = \kappa_x$. Let

$$N = \{x \;:\; x \in \wp_\kappa(\kappa^{+3}) \text{ and } x \cap \kappa \text{ is an inaccessible cardinal, } x \cap \kappa \in A\}.$$

Standard arguments show that N has μ_3-measure one and that μ_3 is closed under the following kind of diagonal intersection:

Suppose $\rho\colon \wp_\kappa(\kappa^{+3}) \times V_\kappa \to \wp(\wp_\kappa(\kappa^{+3}))$ is such that for all (x, y), $\rho(x, y)$ has the μ_3-measure one. Define $\Delta\rho = \{z \;:\;$ for all $x \prec z$ and for all y $\in V_{\mathrm{crit}(z)}, z \in \rho(x, y)\}$. Then $\Delta\rho$ has μ_3-measure one.

A condition $p \in \mathbb{P}$ will be of the form $p = (x_0, p_0, x_1, p_1, \dots, x_n, p_n, f, B)$ where

(a) $x_i \in N$, $x_i \prec x_{i+1}$,
(b) $p_i \in C(\kappa^{\mathrm{iv}}_{x_i}, \kappa_{x_{i+1}})$,
(c) $B \subseteq \wp_\kappa(\kappa^{+3}) \cap N$ and B has μ_3-measure one, and
(d) $f : B \to V_\kappa$ and for all $x \in B$, $f(x) \in C(\kappa^{\mathrm{iv}}_x, \kappa)$.

If $q = (x_0, q_0, x_1, q_1, \dots, x_n, q_n, y_1, r_1, y_2, r_2, \dots, y_m, r_m, g, C)$ then $q \leq_{\mathbb{P}} p$ if and only if

(a) $y_i \in B$ for all $i \leq m$,
(b) $q_i \leq p_i$ for all $i \leq n$,
(c) $r_i \leq f(y_i)$ for all $i \leq m$,
(d) $C \subseteq B$, and
(e) for all $x \in C$, $g(x) \leq f(x)$.

$\mathbb{P}$ has the Prikry property in the following sense:

Lemma 3.5. Let $b \in \mathcal{B}(\mathbb{P})$.

(a) There is a condition (g, D) that decides b (by which we mean that either $(g, D) \leq b$ or (g, D) is incompatible with b. In symbols: $(g, D) \parallel b$).
(b) If $p = (x_0, p_0, \dots, x_n, p_n, g, D) \in \mathbb{P}$ there is a condition $q \leq p$ such that $q = (x_0, p_0, \dots, x_n, p_n, g', D') \parallel b$ and the length of q is equal to the length of p.

A proof of this appears in Magidor [Mag78]; also cf. [FW91]. The fact that the conditions have a slightly different form does not change the proof.

If $G \subseteq \mathbb{P}$ is generic and $p = (x_0, p_0, \ldots, x_n, p_n, g, D) \in G$, then G_p is defined to be $\{(x_0, q_0, \ldots, q_{n-1}, x_n, q_n) : \text{there is a } q \in G, \mathrm{lh}(q) = \mathrm{lh}(p), q = (x_0, q_0, \ldots, q_{n-1}, x_n, q_n, g', D')\}$. If $\lambda < \kappa$ then $G \restriction \lambda$ is defined to be $\{(x_0, q_0, \ldots, x_n, q_n) : \text{there is a } q \in G, q = (x_0, q_0, \ldots, x_n, q_n, g, D), \text{ and } \sup q_n < \lambda\}$.

Using Lemma 3.5 we can easily establish the following corollary:

COROLLARY 3.6.

(a) Suppose $p \in \mathbb{P}$ and $p \Vdash \dot{\tau} \subseteq \alpha$ and $\alpha < \kappa_x$ for some $x \in \wp_\kappa(\kappa^{+3})$ occurring in p. Then $p \Vdash \dot{\tau} \in V[G_p]$.

(b) Let $p = (x_0, p_0, \ldots, x_n, p_n, g, D)$ and $H \subseteq C(\omega, \kappa_{x_0})$ be generic. If $\alpha < \kappa_{x_n}$, $\tau \in V'[H]^{\mathbb{P}}$ and $p \Vdash_{\mathbb{P}} \dot{\tau} \subseteq \alpha$, then $p \Vdash \dot{\tau} \in V'[\dot{G}_p \times H]$. ($\dot{G}_p$ is a canonical term for the generic object $G \subseteq \mathbb{P}$ restricted to p.)

We now make the following claim.

CLAIM 3.7. If $G \times H \subseteq \mathbb{P} \times C(\aleph_0, \kappa_{x_0})$ is a generic over V', then $V'[G \times H] \models$ for all $n \in \omega$, $\aleph_n$ carries an $\aleph_n$ carries $\aleph_{n+1}$-saturated ideal.

PROOF. Let $\langle \kappa_i : i \in \omega \rangle$ be the sequence of κ_{x_i} such that x_i occurs in some condition in G. By Magidor's arguments, κ becomes $\aleph_\omega$. By the corollary, it is enough to see that for each i,

$V'[G \restriction \kappa_i \times H] \Vdash$ for all $n \in \omega$ ($\aleph_n < \kappa_i$ implies $\aleph_n$ carries a saturated ideal).

Our sequence of $\aleph_n$'s in $V[G \times H]$ is

$$\begin{aligned} \aleph_0 &= \omega \\ \aleph_1 &= \kappa_0' \\ \aleph_2 &= \kappa_0'' \\ \aleph_3 &= \kappa_0''' \\ \aleph_4 &= \kappa_0^{\mathrm{iv}} \\ \aleph_5 &= \kappa_1' \\ &\vdots \\ \aleph_{4n+\kappa} &= \kappa_n^{+k} \quad 1 \le \kappa \le 4 \end{aligned}$$

If $p = (x_1, p_1, \ldots, x_{n-1}, p_{n-1}, g, D)$, then in V', $G \restriction \kappa_n \times H$ is generic over

$$[C(\kappa_1^{\mathrm{iv}}, \kappa_2) \times \cdots \times C(\kappa_{n-1}^{\mathrm{iv}}, \kappa_n)] \times C(\omega, \kappa_0).$$

Hence, if $\aleph_m = \kappa_i^k$, $1 \le k \le 3$, we claim $\aleph_m$ carries a saturated ideal in $V'[G \restriction \kappa_n \times H]$ since it does in V' and our forcing is of the form

$$(\text{cardinality} < \aleph_m) \times (\aleph_{m+1}\text{-closed}).$$

Any forcing of cardinality$< \aleph_m$ preserves a saturated ideal on $\aleph_m$ by Lemma 2.2. Thus by Lemma 2.1 we can conclude that $\aleph_m$ has a saturated ideal in $V'[G \restriction \kappa_n \times$

$H]$. If $\aleph_m = \kappa_i^{\text{iv}}$ for some $i < n$, by Lemma 3.4, $\aleph_m$ carries a saturated ideal in $(V')^{C(\kappa_i^{\text{iv}},\kappa_{i+1})}$. To pick up $V'[G{\restriction}\kappa_n \times H]$ from $(V')^{C(\kappa_i^{\text{iv}},\kappa_{i+1})}$ we need forcing of the form

$$(\text{cardinality} < \aleph_m) \times ((\aleph^{\text{iv}}_{\kappa_{i+1}}, \infty)\text{-distributive})$$

(namely:

$$[C(\omega,\kappa_0) \times C(\kappa_0^{\text{iv}},\kappa_1) \times \cdots \times C(\kappa_{i-1}^{\text{iv}},\kappa_i)] \times \\ [C(\kappa_{i+1}^{\text{iv}},\kappa_{i+2}) \times \cdots \times C(\kappa_{n-1}^{\text{iv}},\kappa_n)]).$$

Thus by Lemma 2.2, $\aleph_m$ carries a saturated ideal in $V'[G \restriction \kappa_n \times H]$. This establishes the claim. ⊣ (Claim 3.7)

Unfortunately, in $V'[G \times H]$, $(\kappa^+)^{V'}$ has cardinality κ, since $\bigcup\{x_i$: x_i occurs in some $p \in G\} = (\kappa^{+3})^{V'}$. We now define a κ^+-c.c. partial ordering $\mathbb{P}^\pi$, a condition $p_0 \in \mathbb{P}$ and a projection map

$$\pi\colon \mathbb{P}/p_0 \to \mathbb{P}^\pi.$$

If $G \times H \subseteq \mathbb{P} \times (\omega, \kappa_{x_0})$ then $V'[G \times H]$ and $V'[\pi''G \times H]$ will have the same bounded subsets of κ.

Let U be the measure on κ induced by μ_3. Let $\mathbb{Q}$ be the following partial ordering: $p \in \mathbb{Q}$ if and only if $p\colon X \to V_\kappa$ for a set $X \subseteq \kappa$, $X \in U$ and if $\alpha \in X$, $p(\alpha) \in \mathcal{B}(C(\alpha^{+4},\kappa))$.

If p and q are elements of $\mathbb{Q}$, $p \leq_{\mathbb{Q}} q$ if and only if $\{\alpha : p(\alpha) \leq q(\alpha)\} \in U$.

If $B \subseteq \wp_\kappa(\kappa^{+3})$ is a set of measure one (for μ_3) and $g\colon B \to V_\kappa$ is a function such that for all $x \in \mathcal{B}$, $g(x) \in C(\kappa_x^{+4},\kappa)$, we can associate an element $\bar{g}_B$ of $\mathbb{Q}$ with domain $\pi(B) = \langle \alpha : \exists x \in B, \kappa_x = \alpha \text{ and } \alpha \in A\rangle$ by

$$\bar{g}_B(\alpha) = \bigvee_{\substack{\kappa_x=\alpha\\ x\in B}} g(x)$$

Thus if $q \in C(\alpha^{+4},\kappa)$ for some $\alpha \in \pi(B)$ and $q \wedge \bar{g}_B(\alpha) \neq 0$, we can find an $x \in B$, $\kappa_x = \alpha$ and $q \wedge g(x) \neq 0$.

A function g as above naturally gives rise to a filter $\mathcal{F}_g$, namely

$$\mathcal{F}_g = \{p \in \mathbb{Q} : \text{ there is a set } B \in \mu_3, g \text{ is defined on } B \text{ and } p \geq \bar{g}_B.\}$$

If $g\colon B \to R_\kappa, h\colon C \to R_\kappa$ are functions as above with $\mu_3(B) = \mu_3(C) = 1$ and $g \leq h$ almost everywhere with respect to μ_3, then $\mathcal{F}_g \supseteq \mathcal{F}_h$. (If $g \leq h$ almost everywhere we shall say $g \leq_{\mu_3} h$.)

Claim 3.8. If $\langle g_\alpha : \alpha < \beta\rangle \subseteq \{h :$ there is a $B \subseteq N$, $\mu_3(B) = 1$, $h\colon B \to R_\kappa$, and for all $x \in B$, $h(x) \in C(\kappa_x^{+4},\kappa)\}$ is a $\leq_{\mu_3}$-descending sequence of length $\leq \kappa^{+3}$ then there is a g_β and a set B_β, $g_\beta\colon B_\beta \to \mathbb{R}_\kappa$ and for all $x \in B_\beta$, $g(x) \in C(\kappa_x^{+4},\kappa)$ and for all $\alpha < \beta$, $g_\beta \leq_{\mu_3} g_\alpha$.

PROOF. We view each g_α as $[g_\alpha] \in V'^{\wp_\kappa(\kappa^{+3})}/\mu_3 = M$. Then $g_\alpha \in C(\kappa^{+4}, j(\kappa))^M$. If $\alpha < \alpha' < \beta$ then $[g_\alpha] \geq [g_{\alpha'}]$. $C(\kappa^{+4}, j(\kappa))$ is κ^{+4}-closed in M, hence, using κ^{+3}-supercompactness, there is a $q \in C(\kappa^{+4}, j(\kappa))^M$. such that $q \leq [g_\alpha]$ for all $\alpha < \beta$. Let $g_\beta \colon \wp_\kappa(\kappa^{+3}) \to R_\kappa$ be such that $[g_\beta] = q$. Let $B_\beta = \{x : g(x) \in C(\kappa_x^{+4}, \kappa)\}$. Then it is easy to verify that q_β and B_β satisfy the conclusions of the claim. ⊣ (Claim 3.8)

We now build a sequence $\langle g_\alpha : \alpha < \kappa^{++}\rangle$ such that if $g \leq_{\mu_3} g_\alpha$ for all α and $h \leq_{\mu_3} g$, then $\mathcal{F}_g = \mathcal{F}_h$.

To do this: At successor stages $\alpha+1$ pick a $g_{\alpha+1} \leq_{\mu_3} g_\alpha$ such that $\mathcal{F}_{g_{\alpha+1}} \supsetneq \mathcal{F}_{g_\alpha}$ if possible. At limit stages β, pick a $g_\beta \leq_{\mu_3} g_\alpha$ for all $\alpha < \beta$. Let $g \leq_{\mu_3} g_\alpha$ for all $\alpha < \kappa^{++}$. Since $2^\kappa = \kappa^+$ in V', we cannot have a strictly increasing chain of filters of order type κ^{++}. Hence for a tail of κ^{++}, $\mathcal{F}_{g_\alpha}$ is constant. From this we conclude that if $h \leq g$, $\mathcal{F}_g = \mathcal{F}_h$. (In fact it is possible to show that $\mathcal{F}_g$ is an ultrafilter.) Fix such a g.

A condition in the projected forcing $\mathbb{P}^\pi$ will be of the form

$$p = (\alpha_0, p_0, \alpha_1, p_1, \ldots, \alpha_n, p_n, b, B)$$

where

(a) $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \in A$,
(b) $p_i \in C(\alpha_i^{+4}, \alpha_{i+1})$,
(c) $B \subseteq \kappa$, $B \in U$, and
(d) $b \in \mathcal{F}_g$ and b is defined on B.

We let

$$(\alpha_0, p_0, \alpha_1, p_1, \ldots, \alpha_n, p_n, b, B) \geq \\ (\alpha_0, p_0', \alpha_1, p_1', \ldots, \alpha_n, p_n', \beta_1, q_1, \ldots, \beta_m, q_m, c, C)$$

if and only if

(a) $p_i' \leq p_i$ for all $i \leq n$,
(b) for all $j \leq m$, $\beta_j \in B$,
(c) for all j, $q_j \leq b(\beta_j)$,
(d) $C \subseteq B$, and
(e) $c \leq b$ in $\mathbb{Q}$.

If $D \in \mu_3$, let $\pi(D) = \{\alpha : \text{there is an } x \in D, \kappa_x = \alpha\}$. Then $\pi(D) \in U$. Let $p_0 = (g, \operatorname{dom} g)$, then $p_0 \in \mathbb{P}$. If $p \in \mathbb{P}$, $p \leq p_0$, $p = (x_0, p_0, x_1, p_1, \ldots, x_n, p_n, h, B)$, let $\pi(p) = (\kappa_{x_0}, p_0, \kappa_{x_1}, p_1, \ldots, \kappa_{x_n}, p_n, \bar{h}_B, \pi(B))$. Then π is clearly order preserving. By Lemma 2.4, to see that π is a projection map we must show that for all $p \in \mathbb{P}/p_0$ there is a $q \in \mathbb{P}^\pi$, $q \leq \pi(p)$ such that for all $q' \leq q$ there is a $p' \leq p$ with $\pi(p') \leq q'$. (It is here that this argument differs slightly from earlier arguments of this type, hence we include it.)

Let $p \in \mathbb{P}/p_0$, $p = (x_0, p_0, \ldots, x_n, p_n, h, B)$, and $\pi(p) = (\kappa_{x_0}, p_0, \ldots, \kappa_{x_n}, p_n, \bar{h}, \pi(B))$. We want to define $q \leq \pi(p)$. We assume that for all $y \in B$,

$x_n \prec y$. We shall build a descending sequence $\langle A_i : i \in \omega \rangle \subseteq \mu_3$ with the property that for all $x \in A_{i+1}$ and for all $\alpha < \kappa_x$, if for some p there is $y \in A_i$, $\alpha = \kappa_y$,

$$p \in C(\alpha^{+4}, \kappa_x)$$
$$p \wedge \bar{h}_{A_i}(\alpha) \neq 0 \quad (\text{in } \mathcal{B}(C(\alpha^{+4}, \kappa)))$$

then there is a $y \in A_i$, $\alpha = \kappa_y$ such that

(a) $p \wedge h(y) \neq 0$,
(b) $y \prec x$, and
(c) $h(y) \in C(\alpha^{+4}, \kappa_x)$.

(It is reasonable to hope that $h(y) \in Col(\alpha^{+4}, \kappa_x)$ because $C(\alpha^{+4}, \kappa_x) \subseteq C(\alpha^{+4}, \kappa)$.)

Let $A_0 = B \cap \{x : \kappa_x > \sup p_n \text{ and } x_n \prec x\}$. Assume that A_i has been defined. For each $\alpha \in \pi(A_i)$ and each regular $\beta > \alpha$, pick a set $Y_\beta \subseteq A_i$ of cardinality β such that

$$\{h(y) \cap C(\alpha^{+4}, \beta) : y \in A_i\} = \{h(y) \cap C(\alpha^{+4}, \beta) : y \in Y_\beta\}.$$

Let $X_\beta = \{x \in A_i : \text{for all } y \in Y_\beta, y \prec x \text{ and } \kappa_x > \sup h(y)\}$. Let $Z_\alpha = \underset{\beta<\kappa}{\triangle} X_\beta = \{x : \text{for all } \beta < \kappa_x,\ x \in X_\beta\}$. Then Z_α has μ_3-measure one and if $x \in Z_\alpha$, $p \in C(\alpha^{+4}, \kappa_x)$ and p is compatible with the $\bigvee\{h(y) : y \in A_i \text{ and } \kappa_y = \alpha\}$, there is a $y \in A_i$, $\kappa_y = \alpha$ and p is compatible with $h(y)$. But $h(y) \cap (\alpha^{+4}, \kappa_x)$ is bounded in κ_x hence there is a $\beta < \kappa_x$, $\beta > \sup p$, $h(y) \cap (\alpha^{+4}, \kappa) = h(y) \cap (\alpha^{+4}, \beta)$. Pick $y' \in Y_\beta$ such that

$$h(y) \cap (\alpha^{+4}, \beta) = h(y') \cap (\alpha^{+4}, \beta).$$

Then $y' \prec x$, $\sup h(y') < x$ and $h(y')$ is compatible with p.

Let $A_{i+1} = \underset{\alpha<\kappa}{\triangle} Z_\alpha = \{x : \text{for all } \alpha < \kappa_x,\ x \in Z_\alpha\}$. Then A_{i+1} has the desired properties (a)–(c).

Let $q = (\kappa_{x_0}, p_0, \kappa_{x_1}, p_1, \ldots, \kappa_{x_n}, p_n, \bar{h}_{\bigcap A_i}, \pi(\bigcap A_i))$. Let $q' \leq q$, and suppose that $q' = (\kappa_{x_0}, q_0, \ldots, \kappa_{x_n}, q_n, \alpha_1, q_{n+1}, \ldots, \alpha_m, q_{n+m}, c, C)$. We must define $p' \in \mathbb{P}$ with $\pi(p') \leq q'$. Pick $y_m \in A_m$, $h(y_m) \wedge q_{n+m} \neq 0$ and $\kappa_{y_m} = \alpha_m$. Pick $p'_{n+m} \leq h(y_m) \wedge q_{n+m}$. Since $y_m \in A_m$ there is a $y_{m-1} \in A_{m-1}$, $\kappa_{y_{m-1}} = \alpha_{m-1}$, $\sup h(y_{m-1}) < \kappa_{y_m}$ and $h(y_{m-1}) \wedge q_{n+m-1} \neq 0$. Pick such a y_{m-1} and $p_{n+m-1} \leq q_{n+m-1} \wedge h(y_{m-1})$. Continue in this way for $0 \leq k \leq m$ to pick $y_{m-(k+1)} \in A_{m-(k+1)}$ with $y_{m-(k+1)} \prec y_{m-k}$, $\kappa_{y_{m-(k+1)}} = \alpha_{m-(k+1)}$, $\sup h(y_{m-(k+1)}) < \kappa_{y_{m-k}}$ and $q_{n+m-(k-1)}$ is compatible with $h(y_{m-(k+1)})$. Pick $p'_{m-(k+1)} \leq q_{n+m-(k+1)} \wedge h(y_{m-(k+1)})$. Let $p' = (x_1, q_1, x_2, q_2, \ldots, x_n, q_n, y_1, p'_{n+1}, y_2, p'_{n+2}, \ldots, y_m, p'_{n+m}, g, D)$ where $g \leq h$, $D \subseteq \cap A_i$ and $\bar{g}_D \leq c$ and $\pi(D) \subseteq C$.

It is clear that $p' \leq p$ and $\pi(p') \leq q'$. Thus $\pi\colon \mathbb{P}/p_0 \to \mathbb{P}^\pi$ is a projection map.

Let $G \times H \subseteq \mathbb{P} \times C(\aleph_0, \kappa_0)$ be V'-generic. Let G^π be $\pi''G$. Then $G^\pi \times H$ is V'-generic for $\mathbb{P}^\pi \times C(\aleph_0, \kappa_0)$. If $\tau \subseteq \alpha < \kappa$, $\tau \in V'[G \times H]$ then $\tau \in V'[G_p \times H]$ for some $p \in G$ and from this we conclude $\tau \in V'[G^\pi \times H]$. Hence $V'[G^\pi \times H] \models$"for all $n \in \omega$, $\aleph_n$ carries a saturated ideal".

To see that $\aleph_{\omega+1}$ carries a saturated ideal it suffices to show that $\mathbb{P}^\pi \times C(\aleph_0, \kappa_0)$ is κ-centered.

To see this we define an equivalence relation on $\mathbb{P}^\pi \times C(\aleph_0, \kappa_0)$ with κ equivalence classes and show that if $p_1, \ldots, p_n$ lie in the same equivalence class then $\bigwedge_{i=1}^n p_i \neq 0$.

If we are given $p = ((\alpha_1, p_1, \ldots, \alpha_n, p_n, c, C), p') \in \mathbb{P}^\pi \times C(\aleph_0, \kappa_0)$ and $q = ((\beta_1, q_1, \ldots, \beta_n, q_n, b, B), q') \in \mathbb{P}^\pi \times C(\aleph_0, \kappa_0)$ set $p \sim q$ if and only if

(a) $n = m$ and for all $k \leq n$ $\alpha_k = \beta_k$, $p_k = q_k$, and
(b) $p' = q'$.

There are only $|\kappa^\omega = \kappa|$ equivalence classes. If we have equivalent elements $p_1, \ldots, p_n$ then to find a $p \leq \bigwedge_{i=1}^n p_i$ we merely take the intersection of the boolean values in q (this is non-zero since they all lie in the filter $\mathcal{F}_g$) and the intersection of the sets of measure one. This establishes Theorem 1.4.
⊣ (Theorem 1.4)

With a little more work, starting from a 2-huge cardinal we can also get $(\aleph_{n+1}, \aleph_n) \twoheadrightarrow (\aleph_{m+1}, \aleph_m)$ for all $m > n$ to hold in the model satisfying Theorem 1.4.

To prove Con(ZFC+"there is a huge cardinal") implies Con(ZFC+"all regular cardinals κ carry a saturated ideal") we do the same preparatory forcing $\mathbb{P}_{\leq\kappa}$. We then do the modified Radin forcing that appears in [FW91].

The auxiliary function again takes its values in the appropriate collapse algebra. There are no essential new ideas needed to do this that do not appear in this paper or in [FW91].

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