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The Norm of the L^p-Fourier Transform III Compact Extensions

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A theorem of Hausdorff Young type is proved for integral operators in the setting of gage spaces. This theorem is used to show that the norm of the L^p-Fourier transform on unimodular groups is stable under compact extension.

1. INTRODUCTION

The problem under consideration in this series has only recently been solved in the case of any locally compact Abelian group G by Beckner [2]. In the setting of Abelian groups, the problem is to compute the smallest constant $A_p = A_p(G)$ for which

$$\| \mathscr{F} f \|_{p'} \leqslant A_p \| f \|_p , \qquad f \in L^p(G). \tag{1.1}$$

Here $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\mathscr{F} f$ is the Fourier transform of f if f is integrable. The fact that A_p exists and is not greater than one is known as the Hausdorff Young theorem.

Beckner's solution consisted in computing the smallest constant A_p in (1.1) in the case of the real line $G = \mathbb{R}$. This completed (by different methods) previous work of Babenko [1] who showed that $A_p(\mathbb{R}) = [p^{1/p}/(p')^{1/p'}]^{1/2}$ if p belongs to the infinite sequence 4/3, 6/5, 8/7,.... Babenko's method required that p' be an even integer. Beckner was able to obtain this formula for all $p \in (1, 2)$. From this followed easily the computation for $\mathbb{R}^n$ and any Abelian group by the well-known structure theorem and [10: Section 43).

The Hausdorff Young theorem was extended to non-Abelian groups by Kunze [13]. Namely, for locally compact unimodular groups, an analog of (1.1) with $A_p = 1$ exists in which the function $\mathscr{F} f$ is replaced by an operator L_f and the norm $\| L_f \|_{p'}$ is defined with respect to a gage space (in the sense of I. E. Segal) canonically constructed from the group.

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For compact groups (Abelian or not) it is easy to see by taking $f = 1$ that $A_p = 1$ is the best constant in (1.1). For noncompact, non-Abelian unimodular groups, the computation of the best constant in the aforementioned analog of (1.1) was not considered prior to [19] which forms part I of this series. In that paper I showed that results of Hewitt, Ross, and Hirschman on maximal functions [10: Section 43] extended verbatim to unimodular groups. I also considered direct and semidirect products and computed the best constant for the class of central topological groups and for the Euclidean groups, i.e., the groups of rigid motions of Euclidean space. In part II [20], I studied this problem for the class of connected, simply connected, real nilpotent Lie groups. Estimates were obtained for most of the known examples of such groups. Some but not all of the estimates in these two papers, and another [21] made use of the author's Hausdorff Young theorem for integral operators. This theorem has been extended to the case of operator valued kernels [8], and it is this extension which is the starting point for the present paper, the contents of which will now be described.

In Section 2 I discuss the analog of the main result of [8] in which the gage space $(\mathscr{H}, \mathscr{B}(\mathscr{H}), tr)$ is replaced by more general ones. As it turns out, the result of [8] and its proof extend word for word to the more general setting. (See Prop. 2.1.) However, for the applications considered in the present paper, the hypothesis of this extended theorem is not satisfied. Therefore, it is necessary to abandon interpolation theory and to adopt a pedestrian approach which uses the result of [8] together with direct integral decompositions of gage spaces. The price paid for this is that the resulting inequality (Theorem 1) is proved for only certain values of the indices, i.e., $p = 4/3, 6/5, 8/7,...$.

Section 3 contains the result (Theorem 2) that the norm of the L^p-Fourier transform is stable under compact extension (for p' an even integer). Precisely, if N is unimodular and of type I and if G is a separable compact extension of N, then the smallest constant in the Hausdorff Young theorem for G is dominated by the one for N. In Section 4 I discuss the implications of Theorem 2 for general linear groups and for Moore groups.

Sections 3 and 4 depend heavily on [12]. Accordingly, the blanket assumption is made that all groups considered in this paper are separable (= second countable). Also, Δ_G denotes the modular function of G, $\hat{G}$ denotes the unitary dual of G, μ_G denotes the Plancherel measure on $\hat{G}$ if G is of type I and unimodular, and $\mathscr{K}(G)$ denotes the collection of continuous complex valued functions on G with compact support.

If $\mathscr{H}$ is a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ denotes the Banach space of bounded linear operators on $\mathscr{H}$ with the usual operator norm. In Section 2 the concepts of gage space and of direct integral decomposition are used. References for these are [16, 22] and [5: chapter II], respectively. The Lebesgue spaces associated with a gage space Γ will be denoted by $L^p(\Gamma)$, $1 \leqslant p \leqslant \infty$. If Γ is a (discrete) gage space of the form $(\mathscr{H}, \mathscr{B}(\mathscr{H}), tr)$, the Lebesgue spaces will be denoted by $C_p(\mathscr{H})$, $1 \leqslant p < \infty$.

2. Integral Operators and Gage Spaces

With X a measure space and $\mathscr{H}$ a complex Hilbert space, let $\mathscr{K}$ denote the Hilbert space $L^2(X; \mathscr{H})$. With k a $\mathscr{B}(\mathscr{H})$-valued measurable function on $X \times X$, let T_k denote the integral operator with kernel k (if it exists):

$$T_k f(x) = \int_X k(x, y) f(y)\, dy, \qquad f \in \mathscr{K}, \text{ a.e. } x \in X. \tag{2.1}$$

If the quantities $M_1 = \operatorname{ess.sup}_y \int_X \| k(x, y)\|_{\mathscr{B}(\mathscr{H})}\, dx$ and $M_2 = \operatorname{ess.sup}_x \int_X \times \| k(x, y)\|_{\mathscr{B}(\mathscr{H})}\, dy$ are both finite, then it can easily be shown, using Schwarz's inequality (several times), that $T_k \in \mathscr{B}(\mathscr{K})$ and that

$$\| T_k \|_{\mathscr{B}(\mathscr{K})} \leqslant (M_1 M_2)^{1/2}. \tag{2.2}$$

Given X and $\mathscr{H}$, consider semifinite von Neumann algebras $\mathscr{M}$ on $\mathscr{H}$ and $\mathfrak{A}$ on $\mathscr{K}$, with faithful normal semifinite traces τ and σ, respectively. Then $\Gamma = (\mathscr{H}, \mathscr{M}, \tau)$ and $\Lambda = (\mathscr{K}, \mathfrak{A}, \sigma)$ are regular gage spaces in the sense of Segal [22, 16]. They are related only by the fact that $\mathscr{K} = L^2(X; \mathscr{H})$.

For $p, q, r \in [1, \infty]$, denote the norm in the Banach space $L^{p,q}(X \times X)(L^r(\Gamma))$, for convenience, by $\| \cdot \|_{r,p,q}$. Recall that if $\mathscr{X}$ is a Banach lattice of real-valued measurable functions on a measure space, and B is a Banach space, then $\mathscr{X}(B)$ is the Banach space of B-valued measurable functions f such that $\| f(\cdot)\|_B \in \mathscr{X}$, with the norm $\| f \|_{\mathscr{X}(B)} = \| \| f(\cdot)\|_B \|_{\mathscr{X}}$ (cf. [4]). Also $L^{p,q}(X \times X)$ denotes a mixed norm space in the sense of [3]. Therefore, for example, if p and q are finite, $\| k \|_{r,p,q} = [\int_X \{\int_X \| k(x, y)\|^p_{L^r(\Gamma)}\, dx\}^{q/p}\, dy]^{1/q}$. Now if k has its values in $\mathscr{M}$ $(= L^\infty(\Gamma))$ and T_k belongs to $\mathfrak{A}$ $(= L^\infty(\Lambda))$ i.e., for each x and y $k(x, y)$ is a bounded measurable operator with respect to Γ and T_k is a bounded measurable operator with respect to Λ, (2.2) may be restated as

$$\| T_k \|_{L^\infty(\Lambda)} \leqslant (\| k \|_{\infty,1,\infty} \| k^* \|_{\infty,1,\infty})^{1/2} \tag{2.3}$$

where $k^*(x, y) = k(y, x)^*$. A possible Hausdorff Young inequality for integral operators in this setting would be the assertion for $1 \leqslant p \leqslant 2$ and $1/p + 1/p' = 1$ that

$$\| T_k \|_{L^{p'}(\Lambda)} \leqslant (\| k \|_{p',p,p'} \| k^* \|_{p',p,p'})^{1/2}. \tag{2.4}$$

The meaning of (2.4), since k is only $L^{p'}(\Gamma)$ valued and therefore the $k(x, y)$ are not necessarily everywhere defined, is that the map $k \to T_k$ defined by (2.1) if the integral in (2.1) makes sense and converges, extends to a mapping of the set of k for which the right side of (2.4) is finite and that $T_k \in L^{p'}(\Lambda)$ and (2.4) holds for such k. In particular T_k must be a (not necessarily bounded) measurable operator with respect to Λ.

The standard procedure for proving an inequality such as (2.4) is to establish its validity at the endpoints $p = 1$ and $p = 2$ and then to use interpolation methods to obtain it for $1 < p < 2$. This has been done for the special case in which $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\mathfrak{A} = \mathcal{B}(\mathcal{K})$ and τ and σ are the ordinary traces [8]. The particular case of this in which $\mathcal{H}$ is one dimensional was treated in [19] and [21] and was instrumental for determining the best constant in the Hausdorff Young theorem for some classes of unimodular groups. As remarked above and shown below this procedure cannot be used in this paper.

The reason for considering, in the present paper, gage spaces more general than $(\mathcal{H}, \mathcal{B}(\mathcal{H}), tr)$ is that the gage space that arises in the study of the Hausdorff Young theorem on unimodular groups is never of this form, and not always decomposable as a direct integral of gage spaces of this form.

The discussion which follows will set the stage for Theorem 1 and Section 3. We are given two gage spaces $\Gamma = (\mathcal{H}, \mathcal{M}, \tau)$ and $\Lambda = (\mathcal{K}, \mathfrak{A}, \sigma)$ which are related only by the fact that $\mathcal{K} = L^2(X; \mathcal{H})$ for some measure space X. For each $p \in (1, 2)$ let $\mathcal{X}_p$ denote the Banach lattice of measurable functions $(L^{p,p'}(X \times X))^{1/2}(L^{p,p'}(X \times X)^*)^{1/2}$ considered in [8 Section 3]. Recall that if $\mathcal{Y}$ is a Banach lattice of measurable functions on $X \times X$ then $\mathcal{Y}^* = \{k: k^* \in \mathcal{Y}\}$ with norm $\| k \|_{\mathcal{Y}^*} = \| k^* \|_{\mathcal{Y}}$. Recall also that if $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are two Banach lattices of measurable functions on some measure space, then $\mathcal{Y}_0^{1-t}\mathcal{Y}_1{}^t$, for $0 < t < 1$, denotes the Banach lattice consisting of all measurable functions f satisfying an inequality of the form $|f| \leqslant \lambda g_0^{1-t} g_1{}^t$ for some positive number λ and non-negative elements $g_i \in \mathcal{Y}_i$ with $\| g_i \|_{\mathcal{Y}_i = 1, i=0,1}$. The norm of f in $\mathcal{Y}_0^{1-t}\mathcal{Y}_1{}^t$ is the infinimum of all λ in the inequality. (See [4: 13.5] or [8: Section 2]). In the following proposition, $\mathcal{E}_p$ denotes $\mathcal{X}_p(L^{p'}(\Gamma))$ for $1 \leqslant p \leqslant 2$.

PROPOSITION 2.1. 1. $\| k \|_{\mathcal{E}_p} \leqslant (\| k \|_{p',p,p'} \| k^* \|_{p',p,p'})^{1/2}$.

2. *$\mathcal{E}_1$, $\mathcal{E}_2$ form an interpolation pair in the sense of* [4] *and the intermediate spaces satisfy* $[\mathcal{E}_1, \mathcal{E}_2]_s = \mathcal{E}_p$ *if* $s = 2/p'$, $1 < p < 2$.

3. *If* $k \in \mathcal{E}_1$, *then* T_k, *defined by* (2.1) *belongs to* $\mathcal{B}(\mathcal{K})$ *and* $\| T_k \|_{\mathcal{B}(\mathcal{K})} \leqslant \| k \|_{\mathcal{E}_1}$. *(Note that* T_k *is not necessarily measurable with respect to* Λ *and that in view of* (1) *this is an improvement over* (2.2), *cf.* (2.3).)

4. *Suppose that* T_k *is a measurable operator for every* $k \in \mathcal{E}_1$ *and suppose also that the map* $k \to T_k$ *defined initially by* (2.1) *extended to a linear map of norm* $\leqslant 1$ *of* $\mathcal{E}_2$ *into* $L^2(\Lambda)$. *Then* $k \to T_k$ *would extend to a bounded linear map of norm* $\leqslant 1$ *of* $\mathcal{E}_p$ *into* $L^{p'}(\Lambda)$ *for every* $p \in (1, 2)$.

Proof. (1) follows from the definition of the norm in the space $\mathcal{E}_p$.[1] (2) and (3)

[1] The very same argument proves the following: let $\mathcal{Y}_0$, $\mathcal{Y}_1$ be Banach lattices on some measure space and let B be any Banach space. For $0 < t < 1$ let $\mathcal{Y}_t = \mathcal{Y}_0^{1-t}\mathcal{Y}_1{}^t$. Then $\| k \|_{\mathcal{Y}_t(B)} \leqslant \| k \|_{\mathcal{Y}_0(B)}^{1-t} \| k \|_{\mathcal{Y}_1(B)}^t$. In this proposition $\mathcal{Y}_0 = L^{p,p'}(X \times X)$, $\mathcal{Y}_1 = \mathcal{Y}_0^*$, $B = L^{p'}(\Gamma)$, $t = \frac{1}{2}$.

are proved in [8: Section 3] for a special case but the same proof works here. (4) follows from (2) and (3) and the general theory of interpolation [4] since $[L^{\infty}(\Lambda), L^2(\Lambda)]_s = L^{p'}(\Lambda)$ for $s = 2/p'$. Here the assumption that T_k be a measurable operator is crucial since $\mathscr{B}(\mathscr{H})$, $L^2(\Lambda)$ do not in general form an interpolation pair so that $[\mathscr{B}(\mathscr{H}), L^2(\Lambda)]_s$ is meaningless.

To place Theorem 1 (to follow) in a proper perspective it is appropriate to consider an important example which is central to this paper and which will be discussed further in Section 3. Let G be a locally compact separable unimodular group and suppose N is a closed normal unimodular subgroup of G such that $G/N = K$ is compact. Let

$$\Gamma = \Gamma_N = (L^2(N), \mathscr{L}(N), m_N) \qquad \text{and} \qquad \Lambda = \Gamma_G = (L^2(G), \mathscr{L}(G), m_G)$$

be the canonical gage spaces of N and G respectively [13, 23] (see Section 3). If $\varphi \in \mathscr{K}(G)$ then the operator L_φ of convolution on the left by φ on $L^2(G)$ is bounded and measurable with respect to Γ_G and can be shown to be unitarily equivalent to an integral operator T_{k_φ} on $L^2(K; L^2(N))$ whose kernel k_φ is $\mathscr{L}(N)$ valued. Therefore $\| T_{k_\varphi} \|_{L^\infty(\Gamma_G)} \leqslant \| k_\varphi \|_{\mathscr{E}_1}$. For $i = 1, 2$ let $\mathscr{D}_i$ be the closure of $\{k_\varphi ; \varphi \in \mathscr{K}(G)\}$ in $\mathscr{E}_i$. By the Plancherel theorem for G, $\| T_{k_\varphi} \|_{L^2(\Gamma_G)} = \| k_\varphi \|_{\mathscr{E}_2}$, but it is not true in general that $\mathscr{D}_i = \mathscr{E}_i$.[2] In any case $\mathscr{D}_1$ is a closed subspace of $\mathscr{E}_1$ and according to the interpolation theory discussed above the best that can be said is that the map $k_\varphi \to T_{k_\varphi}$ will carry $[\mathscr{D}_1, \mathscr{D}_2]_s$ into $L^{p'}(\Gamma_G)$ $(s = 2/p')$. Now generally $[\mathscr{D}_1, \mathscr{D}_2]_s$ is a linear subspace of $[\mathscr{E}_1, \mathscr{E}_2]_s = \mathscr{E}_p$ but does not necessarily have the same norm. Therefore it cannot be asserted that $\| T_{k_\varphi} \|_{L^{p'}(\Gamma_G)} \leqslant \| k_\varphi \|_{\mathscr{E}_p}$ which is the inequality desired. Furthermore even though the map $k_\varphi \to T_{k_\varphi}$ extends to a map $k \to T_k$ of $\mathscr{E}_1$ into $\mathscr{B}(L^2(G))$, as pointed out above there is no guarantee that T_k will be measurable with respect to Γ_G for every $k \in \mathscr{E}_1$. (In this paragraph and the next we have identified T_{k_φ} with L_φ.)

To resolve this dilemma it is only necessary to make the (reasonable) assumption that N be a group of type I. For then, since N is assumed to be unimodular and separable we can quote [6: 18.8.1] to obtain direct integral decompositions

$$L^2(N) = \int_{\hat{N}}^{\oplus} (\mathscr{H}_\lambda \otimes \bar{\mathscr{H}}_\lambda)\, d\lambda \tag{2.5}$$

$$\mathscr{L}(N) = \int_{\hat{N}}^{\oplus} (\mathscr{B}(\mathscr{H}_\lambda) \otimes \mathbb{C})\, d\lambda \tag{2.6}$$

$$m_N = \int_{\hat{N}}^{\oplus} \tau_\lambda\, d\lambda, \qquad \text{where} \quad \tau_\lambda(T \otimes I) = \operatorname{tr}(T). \tag{2.7}$$

[2] If $G = \mathbb{R}$, $N = \mathbb{Z}$, then $\mathscr{D}_1 = \mathscr{E}_1$ implies that an arbitrary integral operator is normal.

Then for $\varphi \in \mathcal{K}(G)$, and $k, h \in K$, as noted above $k_\varphi(k, h) \in \mathcal{L}(N)$ so by (2.6)

$$k_\varphi(k, h) = \int_{\hat{N}}^{\oplus} (k_{\varphi,\lambda}(k, h) \otimes I)\, d\lambda \qquad \text{with} \quad k_{\varphi,\lambda}(k, h) \in \mathcal{B}(\mathcal{H}_\lambda). \tag{2.8}$$

Since $L^2(K)$ is a separable Hilbert space,

$$\begin{aligned}\mathcal{K} &= L^2(K; L^2(N)) = L^2(K) \otimes L^2(N) = L^2(K) \otimes \int_{\hat{N}}^{\oplus} (\mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda)\, d\lambda \\ &= \int_{\hat{N}}^{\oplus} L^2(K) \otimes (\mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda))\, d\lambda \\ &= \int_{\hat{N}}^{\oplus} \mathcal{K}_\lambda\, d\lambda \qquad \text{where} \quad \mathcal{K}_\lambda = L^2(K; \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda).\end{aligned} \tag{2.9}$$

Using (2.8) and (2.9) and [5: chapter II] we shall obtain

$$T_{k_\varphi} = \int_{\hat{N}}^{\oplus} (T_{k_{\varphi,\lambda}} \otimes I)\, d\lambda. \tag{2.10}$$

Then using the Plancherel theorem for G, the compactness of K and the well-known fact

$$\| T_{k_{\varphi,\lambda}} \|^2_{C_2(\mathcal{K}_\lambda)} = \iint \| k_{\varphi,\lambda}(x, y) \|^2_{C_2(\mathcal{H}_\lambda)}\, dx\, dy \text{ (for any } \lambda) \tag{2.11}$$

we shall obtain

$$\| T_{k_\varphi} \|^2_{L^2(\Gamma_G)} = \int_{\hat{N}} \| T_{k_{\varphi,\lambda}} \|^2_{C_2(\mathcal{K}_\lambda)}\, d\lambda. \tag{2.12}$$

This example will be continued in Section 3. Thus far, this example justifies the hypotheses in the following Lemma and Theorem.

LEMMA 2.2. *Let X be a measure space with $L^2(X)$ separable and let $\Gamma = (\mathcal{H}, \mathcal{M}, \tau)$ and $\Lambda = (\mathcal{K}, \mathfrak{A}, \sigma)$ be gage spaces such that $\mathcal{K} = L^2(X; \mathcal{H})$. Suppose that $\Gamma = \int_\Omega^\oplus \Gamma_\lambda\, d_\lambda$ is a direct integral of gage spaces $\Gamma_\lambda = (\mathcal{H}_\lambda, \mathcal{M}_\lambda, \tau_\lambda)$ where $\mathcal{M}_\lambda$ is a factor of type I for a.e. $\lambda \in \Omega$. Suppose that $k \to T_k$ extends to an isometry of $\mathcal{E}_2$ onto $L^2(\Lambda)$. For each $k \in \mathcal{E}_1$, let k_λ be defined by $k(x, y) = \int_\Omega^\oplus k_\lambda(x, y)\, d\lambda$ with $k_\lambda(x, y) \in \mathcal{M}_\lambda$. Then $T_k = \int_\Omega^\oplus T_{k_\lambda}\, d\lambda$, for $k \in \mathcal{E}_1$ and*

$$\| T_k \|^{2n}_{L^{2n}(\Lambda)} = \int_\Omega \| T_{k_\lambda} \|^{2n}_{C_{2n}(\mathcal{K}_\lambda)}\, d\lambda,$$

$$\mathcal{K}_\lambda = L^2(X; \mathcal{H}_\lambda) \qquad \text{for} \quad k \in \mathcal{E}_1 \cap \mathcal{E}_2 \text{ and } n = 1, 2, 3, \ldots. \tag{2.13}$$

Proof. Since $L^2(X)$ is separable [5: Prop. 11, p. 152] tells us that $\mathscr{K} = L^2(X; \mathscr{H}) \simeq L^2(X) \otimes \mathscr{H} = L^2(X) \otimes (\int_\Omega^\oplus \mathscr{H}_\lambda \, d\lambda) \simeq \int_\Omega^\oplus (L^2(X) \otimes \mathscr{H}_\lambda) \, d\lambda \simeq \int_\Omega^\oplus L^2(X; \mathscr{H}_\lambda) \, d\lambda$ via the map defined densely by

$$\sum_{i=1}^n f_i \otimes \alpha_i \rightarrow \int_\Omega^\oplus \left(\sum_{i=1}^n f_i \otimes \alpha_i(\lambda) \right) d\lambda$$

for $f_1, ..., f_n \in L^2(X)$ and $\alpha_1, ..., \alpha_n \in \mathscr{H}$ with $\alpha_j = \int_\Omega^\oplus \alpha_j(\lambda) \, d\lambda$, $1 \leqslant j \leqslant n$. It follows that

$$\mathscr{K} = \int_\Omega^\oplus \mathscr{K}_\lambda \, d\lambda, \; \mathscr{K}_\lambda = L^2(X; \mathscr{H}_\lambda) \text{ and for } F \in \mathscr{K}$$

$$F = \int_\Omega^\oplus F_\lambda \, d\lambda \qquad \text{where} \quad F_\lambda \in \mathscr{K}_\lambda \text{ is defined by}$$

$$F(x) = \int_\Omega^\oplus F_\lambda(x) \, d\lambda \qquad \text{for} \quad x \in X.$$

Now let $(\alpha_i)_{i=1}^\infty \subset \mathscr{H}$ be a fundamental sequence of measurable vector fields [5: Def. 1, p. 141] for $\mathscr{H} = \int_\Omega^\oplus \mathscr{H}_\lambda \, d\lambda$. Letting $(f_1, f_2, ...)$ be an orthonormal base for $L^2(X)$ we know [5: Proof of Prop. 10, p. 152] that the sequence $\{f_i \otimes \alpha_j\}_{i,j=1}^\infty \subset \mathscr{K}$ defined by $f_i \otimes \alpha_j = \int_\Omega^\oplus f_i \otimes \alpha_j(\lambda) \, d\lambda$ is a fundamental sequence of measurable vector fields for $\mathscr{K} = \int_\Omega^\oplus \mathscr{K}_\lambda \, d\lambda$. Let $k \in \mathscr{E}_1$; to prove that $T_k = \int_\Omega^\oplus T_{k_\lambda} \, d\lambda$ it must be shown that $\lambda \rightarrow T_{k_\lambda}$ is a measurable field of operators and that $T_k F = \int^\oplus T_{k_\lambda} F_\lambda \, d\lambda$ for $F = \int^\oplus F_\lambda \, d\lambda \in \mathscr{K}$ [5: Def. 2, p. 160]. Now by [5: Prop. 1, p. 157] $\lambda \rightarrow T_{k_\lambda}$ is a measurable field if and only if $\lambda \rightarrow (T_{k_\lambda}[f_i \otimes \alpha_j](\lambda), [f_k \otimes \alpha_l](\lambda))_{\mathscr{K}_\lambda}$ is a measurable function of λ for all i, j, k, l. This function equals $\iint f_i(y) f_k(x)(k_\lambda(x, y)\alpha_j(\lambda), \alpha_l(\lambda))_{\mathscr{H}_\lambda} \, dx \, dy$ which is measurable as a function of λ since $\lambda \rightarrow k_\lambda(x, y)$ is a measurable field of operators and $(\alpha_i)_{i=1}^\infty$ is a fundamental sequence of measurable vector fields ([5: Prop. 1, p. 157] again). Now if $F = \int^\oplus F_\lambda \, d\lambda \in \mathscr{K}$, then $(T_k F)(x) = \int_X k(x, y) F(y) \, dy = \int_X \int_\Omega^\oplus k_\lambda(x, y) F_\lambda(y) \, d\lambda \, dy = \int_\Omega^\oplus \int_X k_\lambda(x, y) F_\lambda(y) \, dy \, d\lambda = \int_\Omega^\oplus (T_{k_\lambda} F_\lambda)(x) \, d\lambda$ so that $T_k F = \int_\Omega^\oplus T_{k_\lambda} F_\lambda \, d\lambda$ as required.

Now suppose $k \in \mathscr{E}_1 \cap \mathscr{E}_2$. Then $\| T_k \|_{L^2(\Lambda)}^2 = \| k \|_{\mathscr{E}_2}^2 = \| k \|_{2,2,2}^2 = \int_X \int_X \| k(x, y) \|_{L^2(\Gamma)}^2 \, dx \, dy = \int_X \int_X \int_\Omega \| k_\lambda(x, y) \|_{C_2(\mathscr{H}_\lambda)}^2 \, d\lambda \, dx \, dy = \int_\Omega (\int_X \int_X \| k_\lambda(x, y) \|_{C_2(\mathscr{H}_\lambda)}^2 \, dx \, dy) \, d\lambda = \int_\Omega \| T_{k_\lambda} \|_{C_2(\mathscr{K}_\lambda)}^2 \, d\lambda$. This completes the proof for $n = 1$.

With our $k \in \mathscr{E}_1 \cap \mathscr{E}_2$ let h be defined by $T_h = (T_k)^* T_k$. Since $T_k \in L^\infty(\Lambda) \cap L^2(\Lambda)$ we have $T_h \in L^\infty(\Lambda) \cap L^2(\Lambda)$ and thus $h \in \mathscr{E}_2$ and $\| T_h \|_{L^2(\Lambda)} = \| h \|_{\mathscr{E}_2}$. Now it is well known and easy to verify that

$$h(x, y) = \int_X k(z, x)^* \, k(z, y) \, dz. \tag{2.14}$$

But k has its values in $\mathscr{M}$ and $\mathscr{M}$ is weakly closed so that $h(x, y) \in \mathscr{M}$ and therefore $h(x, y) = \int_\Omega^\oplus h_\lambda(x, y)\, d\lambda$, with $h_\lambda(x, y) \in \mathscr{M}_\lambda$. Although it is not known if $h \in \mathscr{E}_1$, we can still repeat the argument in the first part of the proof to conclude that $T_h = \int_\Omega^\oplus T_{h_\lambda}\, d\lambda$. Moreover using Fubini's theorem again with (2.14) tells us that $T_{h_\lambda} = (T_{k_\lambda})^* K_{k_\lambda}$. Therefore, using the argument for $n = 1$ with h we get

$$\begin{aligned}\| T_k \|_{L^4(\Lambda)}^4 &= \| T_h \|_{L^2(\Lambda)}^2 = \| h \|_{\mathscr{E}_2}^2 = \| h \|_{2,2,2}^2 = \int_X \int_X \| h(x, y)\|_{L^2(\Gamma)}^2\, dx\, dy \\ &= \int_X \int_X \int_\Omega \| h_\lambda(x, y)\|_{C_2(\mathscr{H}_\lambda)}^2\, d\lambda\, dx\, dy \\ &= \int_\Omega \left(\int_X \int_X \| h_\lambda(x, y)\|_{C_2(\mathscr{H}_\lambda)}^2\, dx\, dy \right) d\lambda \\ &= \int_\Omega \| T_{h_\lambda} \|_{C_2(\mathscr{H}_\lambda)}^2\, d\lambda = \int_\Omega \| T_{k_\lambda} \|_{C_4(\mathscr{H}_\lambda)}^4\, d\lambda \text{ as required.}\end{aligned}$$

The proof for $n = 3, 4,\ldots$ is similar.

THEOREM 1. *Let X be a measure space with $L^2(X)$ separable and let $\Gamma = (\mathscr{H}, \mathscr{M}, \tau)$ and $\Lambda = (\mathscr{K}, \mathfrak{A}, \sigma)$ be gage spaces such that $\mathscr{K} = L^2(X; \mathscr{H})$. Suppose that the gage space Γ is decomposable into a direct integral of gage spaces $\Gamma_\lambda = (\mathscr{H}_\lambda, \mathscr{M}_\lambda, \tau_\lambda)(\lambda \in \Omega)$, where $\mathscr{M}_\lambda$ is a factor of type I for a.e. λ. Suppose that $k \to T_k$ extends to an isometry of $\mathscr{E}_2$ onto $L^2(\Lambda)$. Then for any $k \in \mathscr{E}_1 \cap \mathscr{E}_2$ and $p \in \{4/3, 6/5, 8/7,\ldots\}$ we have*

$$\| T_k \|_{L^{p'}(\Lambda)} \leqslant (\| k \|_{p',p,p'} \| k^* \|_{p',p,p'})^{1/2}.$$

Proof. For our $k \in \mathscr{E}_1 \cap \mathscr{E}_2$ let k_λ be defined as in Lemma 2.2 by $k(x, y) = \int_\Omega^\oplus k_\lambda(x, y)\, d\lambda$ with $k_\lambda(x, y) \in \mathscr{M}_\lambda$. If $p \in \{4/3, 6/5, 8/7,\ldots\}$ then $p' \in \{4, 6, 8,\ldots\}$ so by Lemma 2.2

$$\| T_k \|_{L^{p'}(\Lambda)}^{p'} = \int_\Omega \| T_{k_\lambda} \|_{C_{p'}(\mathscr{H}_\lambda)}^{p'}\, d\lambda.$$

But by [8: Cor. 2] for all $p \in (1, 2)$ and all λ,

$$\| T_{k_\lambda} \|_{C_{p'}(\mathscr{H}_\lambda)} \leqslant (\| k_\lambda \|_{p',p,p'} \| k_\lambda^* \|_{p',p,p'})^{1/2}.$$

Therefore for $p \in \{4/3, 6/5, 8/7,\ldots\}$

$$\begin{aligned}&\| T_k \|_{L^{p'}(\Lambda)}^{p'} \\ &= \int_\Omega \| T_{k_\lambda} \|_{C_{p'}(\mathscr{H}_\lambda)}^{p'}\, d\lambda \leqslant \int_\Omega (\| k_\lambda \|_{p',p,p'} \| k_\lambda^* \|_{p',p,p'})^{p'/2}\, d\lambda \\ &\leqslant \left(\int_\Omega \| k_\lambda \|_{p',p,p'}^{p'}\, d\lambda \right)^{1/2} \left(\int_\Omega \| k_\lambda^* \|_{p',p,p'}^{p'}\, d\lambda \right)^{1/2}.\end{aligned}$$

Now

$$\int_\Omega \| k_\lambda \|_{p',p,p'}^{p'} \, d\lambda$$

$$= \int_\Omega \int_X \left(\int_X \| k_\lambda(x, y) \|_{C_{p'}(\mathscr{H}_\lambda)}^{p} \, dx \right)^{p'/p} dy \, d\lambda$$

$$\leqslant \int_X \left(\int_X \left(\int_\Omega \| k_\lambda(x, y) \|_{C_{p'}(\mathscr{H}_\lambda)}^{p'} \, d\lambda \right)^{p/p'} dx \right)^{p'/p} dy$$

$$= \int \left(\int \| k(x, y) \|_{L^{p'}(\Gamma)}^{p} \, dx \right)^{p'/p} dy = \| \| k(\cdot, \cdot) \|_{L^{p'}(\Gamma)} \|_{p,p'}^{p'} = \| k \|_{p',p,p'}^{p'}$$

and similarly

$$\int_\Omega \| k_\lambda^* \|_{p',p,p'}^{p'} \, d\lambda \leqslant \| k^* \|_{p',p,p'}^{p'}$$

so that

$$\| T_k \|_{L^{p'}(\Lambda)}^{p'} \leqslant (\| k \|_{p',p,p'}^{p'} \| k^* \|_{p',p,p'}^{p'})^{1/2} \text{ as required.}$$

Remark 2.2. The point of Theorem 1 is that it cannot be assumed that $T_k \in L^\infty(\Lambda)$ for *all* $k \in \mathscr{E}_1$ since this is not satisfied in our applications. By not making this assumption the conclusion is weakened from $\| T_k \|_{L^{p'}(\Lambda)} \leqslant \| k \|_{\mathscr{I}_p}$ (see Prop. 1.1-4) in two ways. First, the right side of this inequality is replaced by $(\| k \|_{p',p,p'} \| k^* \|_{p',p,p'})^{1/2}$ which is larger than $\| k \|_{\mathscr{I}_p}$, and this inequality is proved only for $k \in \mathscr{E}_1 \cap \mathscr{E}_2$. This does not affect any of the applications which follow in this paper in any way. Second, the resulting inequality is proved for the sequence $p = 4/3, 6/5, 8/7, \ldots$ instead of for all $p \in (1, 2)$. It remains a challenging problem in interpolation theory to prove the inequalities obtained in this paper for all $p \in (1, 2)$.

3. Compact Extensions

In this section it will be shown that the L^p-Fourier transform $(1 < p < 2)$ is stable under compact extension (if p' is an even integer).

Let G be a locally compact separable (= second countable) group with closed subgroup N. (It is not assumed yet that N is normal or that G or N is unimodular.) Let dg and dn denote right Haar measures on G and N respectively. Let $K = G/N$ be the homogeneous space of right cosets and give K a quasi-invariant measure dk.

If γ is a continuous unitary representation of N on a Hilbert space $\mathscr{H}_\gamma$, let $\pi = \operatorname{ind}_N{}^G \gamma$ be the representation of G induced by γ. By use of a Borel cross section $s: K \to G$, π can be shown to be unitarily equivalent to a representation $\tilde{\pi}$ which acts on the Hilbert space $L^2(K; \mathscr{H}_\gamma)$ (see [12: Section 3]). Moreover, if $\varphi \in \mathscr{K}(G)$, then $\tilde{\pi}(\varphi) = \int_G \varphi(x)\, \tilde{\pi}(x)\, dx$ is an integral operator T_{k_φ} with kernel k_φ given by

$$k_\varphi(k, h) = \Delta_G(s(k)^{-1})\, q(s(k))^{-1/2} \int_N \varphi(s(k)^{-1}\, ns(h))\, \gamma(n)\, q(ns(h))^{-1/2}\, dn \tag{3.1}$$

for $k, h \in K$ and

$$(T_{k_\varphi}F)(k) = \int_K k_\varphi(k, h)\, F(h)\, dh, F \in L^2(K; \mathscr{H}_\gamma), \qquad k \in K. \tag{3.2}$$

Here q is the continuous function on G to $(0, \infty)$ which arises in the definition of the quasi-invariant measure dk. For notation's sake let

$$f_{\varphi,k,h}(n) = \varphi(s(k)^{-1}\, ns(h))\, q(ns(h))^{-1/2} \tag{3.3}$$

so that

$$k_\varphi(k, h) = \Delta_G(s(k)^{-1})\, q(s(k))^{-1/2}\, \gamma(f_{\varphi,k,h}). \tag{3.4}$$

Note that if γ is the right regular representation of N, then π is the right regular representation of G and thus $\pi(\varphi)$, for $\varphi \in \mathscr{K}(G)$ is unitarily equivalent to an integral operator on $L^2(K; L^2(N)) \simeq L^2(G)$.

If H is any locally compact unimodular group (not necessarily separable) let $\mathscr{F}_p(H)$, for $1 < p < 2$ denote the L^p-Fourier transform on H, i.e., $\mathscr{F}_p(H): L_p(H) \to L^{p'}(\Gamma_H)$ is the map $f \to R_f =$ convolution by f on the right in $L^2(H)$. Here Γ_H is the dual gage space of H and $1/p + 1/p' = 1$. The Hausdorff Young theorem for H is the assertion $\| R_f \|_{L^{p'}(\Gamma_H)} \leqslant \| f \|_{L^p(H)}$, i.e., $\| \mathscr{F}_p(H)\| \leqslant 1$. Note that the "right" gage space is being used here instead of the "left" one described in [13] and in Section 2. This is done to conform with our notation for induced representations and is valid since H is unimodular. Recall that $R_f = \pi(f)$ where π is the right regular representation of H.

THEOREM 2. *Let G be a locally compact separable unimodular group and let N be a closed normal subgroup of G which is unimodular and of type I. Suppose that the quotient group G/N is compact. Then for $1 < p < 2$ and p' an even integer,*

$$\| \mathscr{F}_p(G)\| \leqslant \| \mathscr{F}_p(N)\|. \tag{3.5}$$

Proof. Let $\Gamma_N = (L^2(N), \mathscr{R}(N), m_N)$ and $\Gamma_G = (L^2(G), \mathscr{R}(G), m_G)$ be the dual gage spaces of N and G, respectively, and let $K = G/N$ be equipped with normalized Haar measure. Let $\varphi \in \mathscr{K}(G)$. Then as noted above, the operator R_φ is unitarily equivalent to an integral operator T_{k_φ} acing on $L^2(K; L^2(N))$ with kernel $k_\varphi : K \times K \to \mathscr{R}(N)$ given by $k_\varphi(k, h) = \gamma(f_{\varphi,k,h})$ for $k, h \in K$, where γ is the right regular representation of N, $f_{\varphi,k,h}(n) = \varphi(s(k)^{-1}\, ns(h))$

for $n \in N$ and $s: K \to G$ is a Borel cross section. By the Plancherel formula for G, for all $\varphi \in L^2(G)$,

$$\| R_\varphi \|_{L^2(\Gamma_G)} = \| \varphi \|_{L^2(G)} . \tag{3.6}$$

On the other hand, for $\varphi \in \mathscr{K}(G)$,

$$\begin{aligned} \| k_\varphi \|_{2,2,2}^2 &= \int_K \int_K \| k_\varphi(k, h) \|_{L^2(\Gamma_N)}^2 \, dk \, dh \\ &= \int_K \int_K \| \gamma(f_{\varphi,k,h}) \|_{L^2(\Gamma_N)}^2 \, dk \, dh \\ &= \int_K \int_K \| f_{\varphi,k,h} \|_{L^2(N)}^2 \, dk \, dh \quad \text{(Plancherel formula for } N) \\ &= \int_K \int_K \int_N | \varphi(s(k)^{-1} n s(h)) |^2 \, dn \, dk \, dh \\ &= \int_K \int_G | \varphi(s(k)^{-1} g) |^2 \, dg \, dk = \int_K \| \varphi \|_{L^2(G)}^2 \, dk = \| \varphi \|_{L^2(G)}^2 . \end{aligned}$$

This and (3.6) show that

$$\| R_\varphi \|_{L^2(\Gamma_G)} = \| k_\varphi \|_{2,2,2} \qquad \text{for} \quad \varphi \in L^2(G). \tag{3.7}$$

Now it is well known (see [13]) that R_φ is measurable so that

$$\| R_\varphi \|_{L^\infty(\Gamma_G)} \leqslant \| k_\varphi \|_{\mathscr{E}_1} \qquad \text{for} \quad \varphi \in \mathscr{K}(G). \tag{3.8}$$

Therefore by (3.7), (3.8) [6: 18.8.1] and Theorem 1 we have

$$\| R_\varphi \|_{L^{p'}(\Gamma_G)} \leqslant (\| k_\varphi \|_{p',p,p'} \, \| k_\varphi^* \|_{p',p,p'})^{1/2} \tag{3.9}$$

for $1 < p < 2$, p' an even integer, and $\varphi \in \mathscr{K}(G)$. Now

$$\| k_\varphi \|_{p',p,p'} = \left(\int_K \left(\int_K \| k_\varphi(k, h) \|_{L^{p'}(\Gamma_N)}^p \, dk \right)^{p'/p} dh \right)^{1/p'}$$

for all p, $1 < p < 2$, and

$$\| k_\varphi(k, h) \|_{L^{p'}(\Gamma_N)} = \| \gamma(f_{\varphi,k,h}) \|_{L^{p'}(\Gamma_N)} \leqslant \| \mathscr{F}_p(N) \| \, \| f_{\varphi,k,h} \|_{L^p(N)}$$

for all p, $1 < p < 2$. Therefore for $1 < p < 2$ and p' even,

$$\begin{aligned} \| k_\varphi \|_{p',p,p'} &\leqslant \| \mathscr{F}_p(N) \| \left(\int_K \left(\int_K \| \varphi(s(k)^{-1} \cdot s(h)) \|_{L^p(N)}^p \, dk \right)^{p'/p} dh \right)^{1/p'} \\ &= \| \mathscr{F}_p(N) \| \, \| \varphi \|_{L^p(G)} \text{, and similarly} \end{aligned}$$

$$\| k_\varphi^* \|_{p',p,p'} \leqslant \| \mathscr{F}_p(N) \| \, \| \varphi \|_{L^p(G)} .$$

Using these last two inequalities in (3.9) yields for $1 < p < 2$ and p' even,

$$\| R_\varphi \|_{L^{p'}(\Gamma_G)} \leqslant \| \mathscr{F}_p(N) \| \, \| \varphi \|_{L^p(G)} \text{ for } \varphi \in \mathscr{K}(G).$$

This proves (3.5).

4. Examples

In this section Theorem 2 is used to obtain estimates for the norm of the L^p-Fourier transform on general linear groups and on Moore groups.

Lemma 4.1. *Let G be a locally compact group which is unimodular and of type I and let K be a compact normal subgroup of G. Then for $1 < p < 2$,*

$$\| \mathscr{F}_p(G/K) \| \leqslant \| \mathscr{F}_p(G) \|. \tag{4.1}$$

Proof. Let $j: G \to G/K$ be the canonical homomorphism with adjoint $\hat{j}: (G/K)\hat{} \to \hat{G}$ given by $\hat{j}(\rho)(x) = \rho(j(x))$, for $x \in G$ and $\rho \in (G/K)\hat{}$. Let $H = G/K$ and let $\hat{G}_K = \hat{j}(H) \subset \hat{G}$. By [14: Lemma 5.2] $\hat{G}_K$ is an open closed subset of $\hat{G}$ and $\mu_G \mid \hat{G}_K = \mu_H$. For $\varphi \in \mathscr{K}(H)$ and $\pi \in \hat{H}$, let $\tilde{\pi} = \pi \circ j$ and $\tilde{\varphi} = \varphi \circ j$ so that $\tilde{\pi} \in \hat{G}$ and $\tilde{\varphi} \in L^1(G)$ (by [10: (28.54)(v)]). By [10: (28.54)(v)] again, $\pi(\varphi) = \int_H \varphi(h)\, \pi(h)\, dh = \int_{G/K} \varphi(xK)\, \pi(xK)\, d(xK) = \int_G \tilde{\varphi}(x)\, \tilde{\pi}(x)\, dx = \tilde{\pi}(\tilde{\varphi})$. Thus

$$\begin{aligned} \| R_\varphi \|_{p'}^{p'} &= \int_{\hat{H}} \| \pi(\varphi) \|_{p'}^{p'} \, d\mu_H(\pi) = \int_{\hat{H}} \| \tilde{\pi}(\tilde{\varphi}) \|_{p'}^{p'} \, d\mu_H(\pi) \\ &\leqslant \int_{\hat{G}} \| \rho(\tilde{\varphi}) \|_{p'}^{p'} \, d\mu_G(\rho) = \| R_{\tilde{\varphi}} \|_{p'}^{p'}. \end{aligned} \tag{4.2}$$

On the other hand,

$$\| \tilde{\varphi} \|_p^p = \int_G | \tilde{\varphi}(x) |^p \, dx = \int_{G/K} | \varphi(xK) |^p \, d(xK) = \| \varphi \|_p^p \tag{4.3}$$

(again using [10: (28.54)(v))]). Using (4.2) and (4.3) we have

$$\| R_\varphi \|_{p'} \leqslant \| R_{\tilde{\varphi}} \|_{p'} \leqslant \| \mathscr{F}_p(G) \| \, \| \tilde{\varphi} \|_p = \| \mathscr{F}_p(G) \| \, \| \varphi \|_p$$

and (4.1) follows.

Example (cf. [12: p. 473]). Let F be a locally compact, nondiscrete field with $\operatorname{char}(F) = 0$. Set $G_n = \{ g \in GL(n, F) : \det g \in F^{*n} \}$. Then (as pointed out in [12]) G_n is a closed normal subgroup of finite index in $GL(n, F)$. Therefore by Theorem 2 for p' even,

$$\| \mathscr{F}_p(GL(n, F))\| \leqslant \| \mathscr{F}_p(G_n)\|. \tag{4.4}$$

But (as also pointed out in [12]) the map $((g_{ij}), a) \to (ag_{ij})$ is a continuous open homomorphism of $SL(n, F) \times F^*$ onto G_n with finite kernel K. Thus by Lemma 4.1 for all $p \in (1, 2)$

$$\| \mathscr{F}_p(G_n)\| \leqslant \| \mathscr{F}_p(SL(n, F) \times F^*)\| \tag{4.5}$$

Combining (4.4), (4.5) and [19: Theorem 2] we obtain for p' an even integer that

$$\| \mathscr{F}_p(GL(n, F)\| \leqslant \| \mathscr{F}_p(SL(n, F))\| \, \| \mathscr{F}_p(F^*)\| \tag{4.6}$$

Remark 4.2. Since F^* is Abelian $\| \mathscr{F}_p(F^*)\|$ is known for all $p \in (1, 2)$. It is reasonable to expect that for a semisimple Lie group G with Iwasawa decomposition $G = KAN$ that $\| \mathscr{F}_p(G)\| \leqslant \| \mathscr{F}_p(AN)\|$ where $\| \mathscr{F}_p(AN)\|$ is defined as in [21: Section 4] but so far this has not been proved. However it is an unpublished result of the author and Klein [11] that $\| \mathscr{F}_p(AN)\| \leqslant \| \mathscr{F}_p(\mathbb{R})\|^2$ if p' is an even integer and AN is the "$ax + b$" group. This can be compared with the estimate $\| \mathscr{F}_p(AN)\| \leqslant \| \mathscr{F}_p(\mathbb{R})\|$, valid for all $p \in (1, 2)$ [21: Prop. 19].

A locally compact group G is called a Moore group if each of its continuous irreducible unitary representations is finite dimensional. Moore groups have been characterized in terms of semidirect products [18] and in terms of projective limits [15]. Recall that $G = \text{proj.}\lim(G_\alpha)$ means there is a family (H_α) of normal subgroups of G directed by inclusion such that $G_\alpha = G/H_\alpha$ and $\bigcap_\alpha H_\alpha = 1$, and such that a cofinal set of the H_α are compact.

LEMMA 4.3. *Let* $G = \text{proj.}\lim(G_\alpha)$ *and suppose* G *has no compact open subgroup. Then each member of some cofinal subset of* (G_α) *has no compact open subgroup.*

Proof. If this was false there would be an index β such that for all $\alpha \geqslant \beta$, $G_\alpha = G/H_\alpha$, H_α is a normal subgroup of G, and C_α is an open subgroup of G containing H_α such that C_α/H_α is compact. By the definition of projective limit there is an index $\gamma \geqslant \beta$ such that H_γ is compact. The compactness of H_γ and of C_γ/H_γ imply the compactness of C_γ ([10: (5.25)]), contrary to our assumption.

LEMMA 4.4. *Let* $G = \text{proj.}\lim(G_\alpha)$ *and suppose* G *is unimodular and of type I. Let* $p \in (1, 2)$, $\varphi \in \mathscr{K}(G)$ *and* $\epsilon > 0$. *Then there is an index* β *such that for all* $\alpha \geqslant \beta$

$$\| R_\varphi \|_{p'} \leqslant \| \mathscr{F}_p(G_\alpha)\| \, \| \varphi \|_p + \epsilon.$$

Proof. Since G is given as a projective limit we can use [14: Theorem 5.4] without the assumption that G be almost connected. With the notation of the preceding Lemmas and of [14: Theorem 5.4] we have $\hat{G} = \bigcup_\alpha \hat{G}_\alpha$, $G_\alpha = G/K_\alpha$ so that

$$\int_{\hat{G}-\hat{G}_\alpha} \| \pi(\varphi)\|_{p'}^{p'} \, d\mu_G(\pi) < \epsilon, \qquad \text{for} \quad \alpha \geqslant \beta \tag{4.7}$$

and therefore

$$\| R_\varphi \|_{p'}^{p'} < \epsilon + \int_{\hat{G}_\alpha} \| \pi(\varphi)\|_{p'}^{p'} \, d\mu_G(\pi), \qquad \text{for} \quad \alpha \geqslant \beta. \tag{4.8}$$

Now for any index α, K_α carries its normalized Haar measure and the Haar measure on G_α is chosen so that the integration formula

$$\int_G F(x)\, dx = \int_{G_\alpha} \int_{K_\alpha} F(xk)\, dk\, d\bar{x} \text{ holds.}$$

If $\pi \in \hat{G}$ satisfies $\pi \mid K_\alpha = 1$ let $\tilde{\pi} \in \hat{G}_\alpha$ be defined by $\tilde{\pi}(\bar{x}) = \pi(x)$ if $\bar{x} = xK_\alpha \in G_\alpha$. Also let $\tilde{\varphi}(\bar{x}) = \int_{K_\alpha} \varphi(xk)\, dk$ for $\bar{x} = xK_\alpha \in G_\alpha$. Then

$$\begin{aligned} \pi(\varphi) &= \int_G \varphi(x)\, \pi(x)\, dx = \int_{G_\alpha} \int_{K_\alpha} \varphi(xk)\, \pi(xk)\, dk\, d\bar{x} \\ &= \int_{G_\alpha} \tilde{\varphi}(\bar{x})\, \tilde{\pi}(\bar{x})\, d\bar{x} = \tilde{\pi}(\tilde{\varphi}). \end{aligned} \tag{4.9}$$

Now

$$\begin{aligned} \| \tilde{\varphi} \|_p^p &= \int_{G_\alpha} | \tilde{\varphi}(\bar{x})|^p \, d\bar{x} = \int_{G_\alpha} \left| \int_{K_\alpha} \varphi(xk)\, dk \right|^p d\bar{x} \\ &\leqslant \int_{G_\alpha} \left(\int_{K_\alpha} | \varphi(xk)|\, dk \right)^p d\bar{x} \leqslant \int_{G_\alpha} \int_{K_\alpha} | \varphi(xk)|^p \, dk\, d\bar{x} \\ &= \int_G | \varphi(x)|^p \, dx = \| \varphi \|_p^p . \end{aligned} \tag{4.10}$$

Using (4.9) and (4.10) in (4.8) yields (for $\alpha \geqslant \beta$)

$$\begin{aligned} \| R_\varphi \|_{p'}^{p'} &< \epsilon + \int_{\hat{G}_\alpha} \| \tilde{\pi}(\tilde{\varphi})\|_{p'}^{p'} \, d\mu_{G_\alpha}(\tilde{\pi}) \\ &= \epsilon + \| R_{\tilde{\varphi}} \|_{p'}^{p'} \leqslant \epsilon + \| \mathscr{F}_p(G_\alpha)\|^{p'} \, \| \tilde{\varphi} \|_p^{p'} \\ &\leqslant \epsilon + \| \mathscr{F}_p(G_\alpha)\|^{p'} \, \| \varphi \|_{p'}^{p'} \end{aligned}$$

i.e.,

$$\| R_\varphi \|_{p'} \leqslant (\epsilon + \| \mathscr{F}_p(G_\alpha)\|^{p'} \| \varphi \|_p^{p'})^{1/p'}.$$

The lemma follows.

THEOREM 3. *Let G be a Moore group and let* $p \in (1, 2)$.

(a) *If G has a compact open subgroup, then* $\| \mathscr{F}_p(G)\| = 1$.

(b) *If G is separable and has no compact open subgroups then for* p' *even,* $\| \mathscr{F}_p(G)\| \leqslant \| \mathscr{F}_p(\mathbb{R})\|$.

Proof. (a) is valid for any unimodular group [19: Theorem 1]. To prove (b) consider first the case that G is a Lie group. Then by [15: Theorem 2] G contains an open subgroup A of finite index in G which is a central topological group. Setting $H = \bigcap_{x \in G} xAx^{-1}$, then H is an open subgroup of A hence central [9: Theorem 2.1] and H is normal and of finite index in G. By Theorem 2, $\| \mathscr{F}_p(G)\| \leqslant \| \mathscr{F}_p(H)\|$ for p' even. Since G has no compact open subgroups, neither does H and therefore by [19: Corollary to Theorem 2] $\| \mathscr{F}_p(H)\| = \| \mathscr{F}_p(\mathbb{R})\|^n$ for some positive integer n (and all $p \in (1, 2)$). In the general case, by [15: Theorem 3] $G = \text{proj.} \lim(G_\alpha)$ where G_α is a Moore group and a Lie group. By Lemma 4.3 each member of some cofinal subset of (G_α) has no compact open subgroup so by the first part of the proof, $\| \mathscr{F}_p(G_\alpha)\| \leqslant \| \mathscr{F}_p(\mathbb{R})\|^{n_\alpha}$ for a cofinal set of α, where n_α is a positive integer. By Lemma 4.4, given $\epsilon > 0$ and $\varphi \in \mathscr{K}(G)$ there is an index β such that

$$\| R_\varphi \|_{p'} \leqslant \| \mathscr{F}_p(G_\alpha)\| \| \varphi \|_p + \epsilon \qquad \text{for all } \alpha \geqslant \beta.$$

Thus

$$\| R_\varphi \|_{p'} \leqslant \| \mathscr{F}_p(\mathbb{R})\| \| \varphi \|_p + \epsilon \quad \text{for every } \epsilon > 0$$

and

$$\| R_\varphi \|_{p'} \leqslant \| \mathscr{F}_p(\mathbb{R})\| \| \varphi \|_p \qquad \text{for every } \varphi \in \mathscr{K}(G).$$

Remark 4.5. In [18: Theorem 2] every Moore group is shown to satisfy the following structure theorem: $G = \mathbb{R}^n \times B$, semidirect product with $\mathbb{R}^n$ normal and B a Moore group with compact component of the identity containing a normal subgroup H of finite index such that $\mathbb{R}^n \times H$ is a direct product.

It follows that $\mathbb{R}^n \times H$ is a normal subgroup of finite index in G. Thus if G is separable, Theorem 2 and [19: Theorem 2] imply

$$\| \mathscr{F}_p(G)\| \leqslant \| \mathscr{F}_p(\mathbb{R}^n \times H)\| = \|\mathscr{F}_p(\mathbb{R})\|^n \| \mathscr{F}_p(H)\|$$

(for p' even). This gives another proof of Theorem 3 if $n > 0$.

Remark 4.6. The following result, which is similar in spirit to the results of this section has been obtained recently by Fournier [7]. Let $p \in (1, 2)$; then there is a constant $b_p \in (0, 1)$ such that for any locally compact unimodular group G which does not have a compact open subgroup one has $\| \mathscr{F}_p(G) \| \leqslant b_p$. This result, though universal, is quite crude. For example, if $p = 4/3$ then $.999999 < b_p < 1$.

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