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A NOTE ON CONFORMAL RICCI FLOW

PENG LU, JIE QING†, AND YU ZHENG

Abstract. In this note we study conformal Ricci flow introduced by Arthur Fischer in [Fi]. We use DeTurck's trick to rewrite conformal Ricci flow as a strong parabolic-elliptic partial differential equations. Then we prove short time existences for conformal Ricci flow on compact manifolds as well as on asymptotically flat manifolds. We show that Yamabe constant is monotonically increasing along conformal Ricci flow on compact manifolds. We also show that conformal Ricci flow is the gradient flow for the ADM mass on asymptotically flat manifolds.

1. Introduction

Suppose that M^m is a smooth m-dimensional manifold and that g_0 is a Riemmanian metric on M^m with a constant scalar curvature s_0. The conformal Ricci flow on M^m is defined as follows:

$$\begin{cases} \dfrac{\partial g}{\partial t} + 2\left(Ric - \dfrac{s_0}{m} g\right) = -2pg \text{ in } \mathrm{M}^m \times (0,T) \\ s[g(t)] = s_0 \text{ in } \mathrm{M}^m \times [0,T) \end{cases} \tag{1.1}$$

for a family of metrics $g(t)$ with initial condition $g(0) = g_0$ and a family of functions $p = p(t)$ on $\mathrm{M}^m \times [0,T)$, where $s[g(t)]$ is the scalar curvature of the evolving metric $g(t)$. Conformal Ricci flow (1.1) was introduced by Arthur Fischer in [Fi] as a modified Ricci flow that preserves the constant scalar curvature of the evolving metrics. Because the role of conformal geometry plays in maintaining scalar curvature constant such a modified Ricci flow was named as conformal Ricci flow in [Fi]. It was shown in [Fi] that on compact manifolds conformal Ricci flow is equivalent to

$$\begin{cases} \dfrac{\partial g}{\partial t} + 2\left(Ric - \dfrac{s_0}{m} g\right) = -2pg \text{ in } \mathrm{M}^m \times (0,T) \\ (m-1)\Delta p + s_0 p = -|\mathrm{Ric} - \dfrac{s_0}{m} g|^2 \text{ in } \mathrm{M}^m \times [0,T) \end{cases} \tag{1.2}$$

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with the initial condition $g(0) = g_0$. Based on the fact that conformal Ricci flow (1.2) is of parabolic-elliptic nature analogous to Navier-Stokes equations, the function p is named as conformal pressure function in [Fi]. Using the theory of dynamical systems on infinite dimensional manifolds it was proved in [Fi] that conformal Ricci flow exists at least for short time on compact manifolds of negative Yamabe type. It was also observed in [Fi] that Yamabe constant monotonically increases along conformal Ricci flow on compact manifolds of negative Yamabe type. Therefore it is hopeful that conformal Ricci flow does a good job in constructing Einstein metrics, considering the behavior of Hilbert-Einstein action on the space of Riemannian metrics.

In this note we adopt DeTurck's trick [DT2][DT3] to eliminate the degeneracy of (1.2) from the symmetry of diffeomorphisms and consider the following what we call DeTurck conformal Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} g + 2\left(\mathrm{Ric} - \frac{s_0}{m} g\right) = -2pg + \mathcal{L}_W g \\ (m-1)\Delta p + s_0 p = -|\mathrm{Ric} - \frac{s_0}{m} g|^2 \end{cases} \tag{1.3}$$

for an appropriately chosen vector field W (cf. (3.5)) with an initial metric $g(0) = g_0$ of constant scalar curvature s_0. (1.3) is a strong parabolic-elliptic partial differential equations. We use the contractive mapping theorem to prove the isomorphism property for linearized DeTurck conformal Ricci flow and we use an implicit function theorem to prove short time existence for DeTurck conformal Ricci flow. Therefore we obtain short time existence for confomral Ricci flow based on the discussion in Section 3.1. For parabolic Hölder spaces and the theory of linear and nonlinear parabolic equations we take references mostly from [Lu].

Theorem 1.1. *Let $(\mathrm{M}^m,\ g_0)$ be a compact Riemannian manifold of constant scalar curvature s_0 with no boundary. And suppose that the elliptic operator $(m-1)\Delta[g_0]+s_0$ is invertible. Then there exists a small positive number T such that conformal Ricci flow $g(t)$ with the initial metric g_0 exists for $t \in [0, T]$.*

This extends the existence result in [Fi] to include compact manifolds of positive Yamabe type. We also extend the monotonicity of Yamabe constant in [Fi] as follows:

Theorem 1.2. *Let (M^m, g_0) be a compact Riemannian manifold with no boundary and let $g(t)$, $t \in [0, T)$ be the solution of conformal Ricci flow with $g(0) = g_0$. Suppose that g_0 is the only Yamabe metric in the conformal class $[g_0]$ with $s[g_0] = s_0$ and that $(m-1)\Delta[g_0]+s_0$ is invertible. Then there is $T_0 \in (0, T]$ such that each metric $g(t)$, $t \in [0, T_0)$, is a Yamabe metric and the Yamabe constant $Y[g(t)]$ increases strictly for $t \in [0, T_0)$ unless g_0 is an Einstein metric.*

This theorem indicates that conformal Ricci flow is somehow better family of constant scalar curvature metrics than those obtained in [Ko].

On asymptotically flat manifolds we use weighted Hölder spaces defined in [LP] and define weighted parabolic Hölder spaces based on the similar ones in [Lu] [OW].

Theorem 1.3. *Let* $(\mathrm{M}^m,\ g_0)$ *be scalar flat and asymptotically flat manifold with* $g_0 - g_e \in C^{4,\alpha}_{-\tau}$ *and* $\tau \in (0, m-2)$*, where* g_e *is the standard Euclidean metric. Then there exists a small positive number* T *such that the conformal Ricci flow* $g(t)$ *from the initial metric* g_0 *exists for* $t \in [0,T]$ *and* $g(t) - g_e \in C^{1,2+\alpha}_{-\tau}([0,T]\times \mathrm{M})$.

It is easily seen that (1.1) and (1.2) are equivalent on asymptotically flat manifolds because of the uniqueness of bounded solutions to linear parabolic equations on asymptotically flat manifolds. The scalar flat assumption in Theorem 1.3 is less stringent than it looks. Thanks to [SY, Lemma 3.3, Corollary 3.1] we know that one can always conformally deform an asymptotically flat metric with nonnegative scalar curvature into a scalar flat asymptotically flat metric.

Conformal Ricci flow is the gradient flow for the ADM mass on asymptotically flat manifolds in the following sense.

Theorem 1.4. *Let* $g(t)$ *be the conformal Ricci flow obtained in Theorem 1.3 for* $\tau \in (\frac{m-2}{2}, m-2)$*. Then*

$$\frac{d}{dt}m(g(t)) = -2\int_M |Ric[g(t)]|^2 dvol[g(t)].$$

In particular, the ADM mass $m(g(t))$ *is strictly decreasing under conformal Ricci flow except that* g_0 *is the Euclidean metric.*

As a quick application of Theorem 1.4 one can easily show the rigidity part of the celebrated positive mass theorem of Schoen and Yau [SY]. The monotonicity of the ADM mass along conformal Ricci flow is sharply in contrast to the invariance of the ADM mass [DM] [OW] along Ricci flow on asymptotically flat manifolds.

The organization of the note is as follows: in Section 2 we introduce conformal Ricci flow and establish the monotonicity of Yamabe constant on compact manifolds. In Section 3 we prove short existences for conformal Ricci flow both on compact manifolds and on asymptotically flat manifolds. In Section 4 we recall the definition of the ADM mass and show that conformal Ricci flow on asymptotically flat manifolds is the gradient flow for the ADM mass.

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2. Conformal Ricci Flow

In this section we first introduce conformal Ricci flow and calculate evolution equations for curvatures along conformal Ricci flow. We then discuss the monotonicity of Yamabe quotients and Yamabe constants along conformal Ricci flow.

2.1. **Conformal Ricci Flow.**

Suppose that M^m is a smooth m-dimensional manifold and that g_0 is a Riemmanian metric on M^m with constant scalar curvature s_0. In [Fi], the conformal Ricci flow on M^m is defined by (1.1) for a family of metrics $g(t)$ with initial condition $g(0) = g_0$ and a family of functions $p = p(t)$ on $\mathrm{M}^m \times [0, T)$.

As calculated in [Fi], the normalization condition $s[g(t)] = s_0$ may be replaced by an elliptic equation and rewrite (1.1) as (1.2). The equivalence between (1.1) and (1.2) was proved in [Fi] in cases when M^m is a compact manifold without boundary (cf. [Fi, Proposition 3.2, Proposition 3.4]). Based on the evolution equation for scalar curvature, it is easily seen that (1.1) always implies (1.2) and (1.2) implies (1.1) when the solution to the linear heat equations is unique, which is true in both cases we consider in this note: compact and asymptotically flat.

One important issue for geometric PDE is the scaling property. It is easily seen that

$$g_\lambda(\cdot, t)) = \lambda^{-2} g(\cdot, \lambda^2 t)) \quad \text{and} \quad p_\lambda(\cdot, t) = \lambda^2 p(\cdot, \lambda^2 t) \tag{2.1}$$

solve conformal Ricci flow (1.2) if (g, p) solve conformal Ricci flow (1.2).

2.2. **Curvature Evolution Equations under conformal Ricci flow.**

To understand conformal Ricci flow one often needs to calculate how curvatures behave along conformal Ricci flow. The calculations are straightforward. Suppose that we consider a general geometric flow

$$\frac{\partial}{\partial t} g = -2T. \tag{2.2}$$

Then we recall the evolution equations for curvarures (cf. [CLN] [Be], for example)

$$\frac{\partial}{\partial t}s = 2\Delta\Theta - 2\nabla^i\nabla^j T_{ij} + 2R^{ij}T_{ij}$$
$$\frac{\partial}{\partial t}R_{ij} = \Delta T_{ij} - \nabla_i\nabla^k T_{kj} - \nabla_j\nabla^k T_{ki} + \nabla_i\nabla_j\Theta + 2R_{ikjl}T^{kl} - R_{ik}T^k{}_j - R_{jk}T^k{}_i$$
$$\frac{\partial}{\partial t}R_{ikjl} = \nabla_i\nabla_j T_{kl} - \nabla_i\nabla_l T_{kj} - \nabla_k\nabla_j T_{il} + \nabla_k\nabla_l T_{ij} - R_{ikjm}T^m{}_l - R_{ikml}T^m{}_j$$

where $\Theta := g^{ij}T_{ij}$. Particularly for conformal Ricci flow, where

$$T = \mathrm{Ric} - \frac{s_0}{m}g + pg \text{ and } \Theta = s - s_0 + mp,$$

we may calculate the evolution equations for curvatures under conformal Ricci flow as follows:

$$\frac{\partial}{\partial t}s = \Delta s + 2\frac{s_0}{m}(s - s_0) + 2p(s - s_0) + 2(m-1)\Delta p + 2s_0 p + 2|\mathrm{Ric} - \frac{s_0}{m}g|^2$$
$$\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} + 2R_{ikjl}R^{kl} - 2R_{ik}R^k{}_j + (m-2)\nabla_i\nabla_j p + \Delta p g_{ij}$$
$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta\mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm} + 2\frac{s_0}{m}\mathrm{Rm} - 2p\mathrm{Rm} + T(\nabla^2 p)$$

where operations $*$ stands for contractions of tensors and

$$T(\nabla^2 p)_{ikjl} = g_{kl}\nabla_i\nabla_j p - g_{kj}\nabla_i\nabla_l p - g_{il}\nabla_k\nabla_j p + g_{ij}\nabla_k\nabla_l p.$$

2.3. **Yamabe Constants under Conformal Ricci Flow.** On compact manifolds with no boundary, along conformal Ricci flow, we may calculate

$$\Theta = mp$$

$$\frac{\partial}{\partial t}d\mathrm{vol}[g(t)] = -mpd\mathrm{vol}[g(t)]$$

$$\frac{d}{dt}\mathrm{vol}(\mathrm{M}) = -m\int_{\mathrm{M}} pd\mathrm{vol}[g] = \frac{m}{s_0}\int_{\mathrm{M}} |\mathrm{Ric} - \frac{s_0}{m}g|^2 d\mathrm{vol}[g] \tag{2.3}$$

Given a compact Riemannian manifold (M^m, h) with no boundary, the Yamabe quotient is defined as:

$$Q[h] := \frac{\int_{\mathrm{M}} s[h]d\mathrm{vol}[h]}{\mathrm{vol}[h](\mathrm{M})^{\frac{m-2}{m}}}.$$

and the Yamabe constant is defined as:

$$Y[h] = \inf_{h\in[h]} Q[h].$$

A Riemannian metric h is said to be a Yamabe metric if and only if

$$Q[h] = Y[h].$$

Thus from (2.3) we have

Proposition 2.1. *Suppose that $g(t)$, $t \in [0, T)$, is a solution to conformal Ricci flow (1.2) on a compact manifold with no boundary with $s[g_0] \neq 0$. Then $Q[g(t)]$ increases strictly unless g_0 is an Einstein metric.*

Next we consider the change of Yamabe constants along conformal Ricci flow. As observed in [WZ] [ChL] [And] [Ko] Yamabe constant could behave rather irregular in general among the metrics of positive Yamabe type.

Theorem 2.2. *Let $(\mathrm{M}^m,\ g_0)$ be a compact Riemannian manifold with no boundary and let $g(t)$, $t \in [0, T)$ be the solution of conformal Ricci flow with $g(0) = g_0$. Suppose that g_0 is the only Yamabe metric in the conformal class $[g_0]$ with $s[g_0] = s_0$ and that $(m-1)\Delta[g_0] + s_0$ is invertible. Then there is $T_0 \in (0, T]$ such that each metric $g(t)$, $t \in [0, T_0)$, is a Yamabe metric and the Yamabe constant $Y[g(t)]$ increases strictly for $t \in [0, T_0)$ unless g_0 is an Einstein metric.*

Proof. We use the proof by contradiction. Assume otherwise there is a sequence $t_i \to 0^+$ such that $g(t_i)$ are not Yamabe metrics. Let $\tilde{g}_i$ be a Yamabe metric in the conformal class $[g(t_i)]$. Then by the compactness of Yamabe metrics $\tilde{g}_i$ converges to a Yamabe metric $g_\infty \in [g_0]$ (taking subsequence if necessary). By the assumption that g_0 is the only Yamabe metric in $[g_0]$ we have $g_\infty = g_0$. That is to say that $g(t_i)$ and $\tilde{g}(t_i)$ both converge to g_0, which is a contradiction since $(m-1)\Delta[g_0] + s_0$ is assumed to be invertible. The similar arguments have been used in [WZ] [ChL] [And] [Ko]. □

3. Short Time Existences of conformal Ricci flow

In this section we prove the short time existence of conformal Ricci flow. The first step is to combine the two equations in conformal Ricci flow into one evolution eqution with one non-local term. Hence (1.2) turns into

$$\frac{\partial}{\partial t} g + 2(\mathrm{Ric} - \frac{s_0}{m} g) = -2\mathcal{P}(g) g \text{ on M}, \tag{3.1}$$

where

$$\mathcal{P}(g) = ((m-1)\Delta + s_0)^{-1} |\mathrm{Ric} - \frac{s_0}{m} g|^2,$$

provided that $(m-1)\Delta[g(t)] + s_0$ is invertible for all $t \in [0, T]$. The strategy to prove the short time existence for conformal Ricci flows is similar to the one used in [DT2] [DT3] to prove the short time existence for Ricci flow. We will first prove the short time existence for DeTurck conformal Ricci flow written in one equation:

$$\frac{\partial}{\partial t}g + 2(\mathrm{Ric} - \frac{s_0}{m}g) = -2\mathcal{P}(g)g + \mathcal{L}_W g \text{ on M.} \tag{3.2}$$

To prove the short time existence for (3.2) we calculate the linearization of DeTurck conformal Ricci flow and apply an implicit function theorem.

3.1. **DeTurck's Trick.** The conformal Ricci flow as a system of differential equations is of parabolic-elliptic nature similar to Navier-Stokes equations. The significant difference between conformal Ricci flow and Navier-Stokes equations is that conformal Ricci flow is a geometric flow. Hence we need to find ways to eliminate the degeneracy of conformal Ricci flow arising from the symmetry of diffeomorphisms.

In this subsection we will follow the idea from the improved version [DT3] of the approach to solve the short time existence for the Ricci flow in [DT2] to rid off the degeneracy of diffeomorphisms for conformal Ricci flow. To introduce DeTurck's trick we first recall the following operator G. Let g be a Riemannian metric on M^m. The operator G on symmetric 2-tensor B is defined as:

$$G(B) := B - \frac{1}{2}\mathrm{Tr}_g(B)g. \tag{3.3}$$

Also recall that the divergence operator δ

$$(\delta B)_i := \nabla^j B_{ij} : \Gamma(S^2(\mathrm{M}^m)) \to \Gamma(T^*\mathrm{M}^m)$$

and its adjoint operator δ^*

$$(\delta^*\omega)_{ij} := -\frac{1}{2}(\omega_{i,j} + \omega_{j,i}) : \Gamma(T^*\mathrm{M}^m) \to \Gamma(S^2(\mathrm{M}^m)).$$

Note that, if X is the dual vector field of ω, then $\delta^*\omega = -\mathcal{L}_X g$ where $\mathcal{L}_X$ denotes the Lie derivative in X direction.

According to DeTurck's improved version [DT3] of his approach to the short time existence of Ricci flow [DT2] we consider the following gauge-fixed conformal Ricci flow on M^m

$$\begin{cases} \frac{\partial}{\partial t}g + 2\left(\mathrm{Ric} - \frac{s_0}{m}g\right) = -2pg + 2(\delta^*(\tilde{g}^{-1}\delta G(\tilde{g}))) \\ (m-1)\Delta p + s_0 p = -|\mathrm{Ric} - \frac{s_0}{m}g|^2 \end{cases} \tag{3.4}$$

for a family of metrics $g(t)$ with $g(0) = g_0$ and a family of functions $p(t)$ on $\mathrm{M}^m \times [0, T)$, where $\tilde{g}$ is any fixed metric on M^m.

Suppose that $g(t)$, $t \in [0, T)$, solves (3.4). Then we consider the time dependent vector field W

$$W^k := g^{ij}\left(\Gamma^k_{ij}[g] - \Gamma^k_{ij}[\tilde{g}]\right). \tag{3.5}$$

It turns out that (cf. [Ha2, §6] [Shi])

$$2(\delta^*(\tilde{g}^{-1}\delta G(\tilde{g}))) = \mathcal{L}_W g.$$

Hence we can rewrite (3.4) as follow:

$$\begin{cases} \frac{\partial}{\partial t} g + 2\left(\mathrm{Ric} - \frac{s_0}{m} g\right) = -2pg + \mathcal{L}_W g \\ (m-1)\Delta p + s_0 p = -|\mathrm{Ric} - \frac{s_0}{m} g|^2. \end{cases} \tag{3.6}$$

Conformal Ricci flow (1.2) and DeTurck conformal Ricci flow (3.6) are related to each other by coordinate changes in the following sense. Suppose that $\hat{g}(t)$ solves DeTurck conformal Ricci flow equations (3.6) and W is given as in (3.5). We consider the one-parameter family of diffeomofphisms φ_t generated by W on M^m as:

$$\frac{\partial}{\partial t}\varphi_t(x) = -W(\varphi_t(x), t), \qquad \varphi_0(x) = x \tag{3.7}$$

for some time period $[0, T)$.

Lemma 3.1. *Let $(\hat{g}(t), \hat{p}(t))$, $t \in [0, T)$, be a solution to DeTurck conformal Ricci flow* (3.6) *on manifold M^m with the initial metric g_0. Assume that the solution $\varphi_t(x)$ to* (3.7) *exists for $t \in [0, T)$. Let*

$$g(t) := \varphi_t^* \hat{g}(t) \text{ and } p(t) := \hat{p}(\varphi_t(x), t).$$

Then $(g(t), p(t))$, $t \in [0, T)$, is a solution to conformal Ricci flow (1.2) *on manifold M^m with $g(0) = g_0$.*

Proof. We simply compute by using (3.6) (cf. [CLN, §2.6], for example)

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= \varphi_t^* \left(\frac{\partial}{\partial t}\hat{g}(t)\right) + \frac{\partial}{\partial s}\Big|_{s=0} \left(\varphi_{t+s}^* \hat{g}(t)\right) \\ &= -2\varphi_t^* \left(\mathrm{Ric}[\hat{g}] - \frac{s_0}{m}\hat{g} + \hat{p}\hat{g}\right) + \varphi_t^* \left(\mathcal{L}_W \hat{g}\right) - \mathcal{L}_{\left(\varphi_t^{-1}\right)_* W}(\varphi_t^* \hat{g}) \\ &= -2\left(\mathrm{Ric}[g] - \frac{s_0}{m} g + pg\right). \end{aligned}$$

The second equation for p in (1.2) is readily seen to be true since the scalar curvature under both flows is kept constant as s_0. ∎

This lemma is particularly important to us because it enables us to prove the short time existence of conformal Ricci flow by proving the short time existence of DeTurck conformal Ricci flow. The later will be proven to be a system of parabolic-elliptic equations (cf. Lemma 3.3).

On the other hand, suppose that $(g(t), p(t))$, $t \in [0, T)$, solves conformal Ricci flow (1.2) on M^m with initial metric g_0 and $\tilde{g}$ is any fixed metric on M^m. We then consider the harmonic map flow

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),\tilde{g}}\varphi_t, \qquad \varphi_0 = \mathrm{Id} \tag{3.8}$$

for $\varphi_t : \mathrm{M}^m \to \mathrm{M}^m$, where the nonlinear Laplacian in local coordinates is given as

$$(\Delta_{g_1,g_2} f)^\gamma = \Delta[g_1] f^\gamma + \Gamma^\gamma_{\alpha\beta}[g_2] \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} g_1^{ij}.$$

The following lemma is useful to derive the uniqueness of conformal Ricci flow via the uniqueness of DeTurck conformal Ricci flow. It is therefore readily seen that the uniqueness of conformal Ricci flow with a given initial metric holds at least on compact manifolds with no boundary.

Lemma 3.2. *Let $(g(t), p(t))$, $t \in [0, T)$, be a solution to conformal Ricci flow (1.2) on manifold M^m with initial metric g_0. Assume the solution $\varphi_t : M \to M$ to the harmonic map flow* (3.8) *exists for $t \in [0, T)$. Let*

$$\hat{g}(t) := (\varphi_t^{-1})^* g(t) \ \text{and}\ \hat{p}(x, t) := p(\varphi_t^{-1}(x), t).$$

Then $(\hat{g}(t), \hat{p}(t)), t \in [0, T)$, is a solution to DeTurck conformal Ricci flow (3.6) *on manifold M^m with initial metric g_0.*

Proof. This follows from a calculation similar to the one in the proof of Lemma 3.1 after identifying the vector field $W = \Delta_{g(t),\tilde{g}}\varphi_t$(cf. [CLN, p.117], for example). ☐

3.2. **The linearization of DeTurck conformal Ricci flow.** In this subsection we compute the linearization of DeTurck confomral Ricci flow (3.2). To do so we set

$$g_\lambda(t) = g(t) + \lambda h(t)$$

for a family of symmetric 2-tensors $h(t)$ and for $\lambda \in (-\epsilon, \epsilon)$. We rewrite DeTurck conformal Ricci flow

$$\mathcal{M}(g(t)) = \frac{\partial}{\partial t} g + 2\left(\mathrm{Ric} - \frac{s_0}{m} g\right) + 2\mathcal{P}(g) g - \mathcal{L}_W g := \frac{\partial}{\partial t} g - \mathcal{F}(g(t)) = 0 \tag{3.9}$$

and calculate

$$\frac{d}{d\lambda}|_{\lambda=0}\mathcal{M}(g_\lambda).$$

To compute the linearization of $\mathcal{P}$ we first calculate

$$\frac{d}{d\lambda}|_{\lambda=0}\Delta[g_\lambda]\mathcal{P}(g_\lambda) = -h_{ij}\nabla^i\nabla^j\mathcal{P} - \frac{1}{2}(2\nabla^i h_{ij} - \nabla_j h^i{}_i)\nabla^j\mathcal{P} + \Delta\mathcal{P}'$$

(cf, [CLN, (S.5) on p.547], for example), where $\mathcal{P}' = \frac{d}{d\lambda}|_{\lambda=o}\mathcal{P}(g_\lambda)$. Next we may calculate

$$\frac{d}{d\lambda}|_{\lambda=0}|\mathrm{Ric}[g_\lambda] - \frac{s_0}{m}g_\lambda|^2$$

using the linearization of Ricci curvature (cf. [Be] [CLN])

$$\begin{aligned}\frac{d}{d\lambda}|_{\lambda=0}2R_{ij}[g_\lambda] = &-\Delta h_{ij} - 2R_{ikjl}h_{kl} + R_{ik}h^k{}_j + R_{jk}h^k{}_i \\ &- \nabla_i\nabla_j h^k{}_k + \nabla_i\nabla^k h_{kj} + \nabla_j\nabla^k h_{ki}\end{aligned} \tag{3.10}$$

(see (2.31) in [CLN], for example). In summary we have

$$(m-1)\Delta\mathcal{P}' + s_0\mathcal{P}' - P_1^{ijkl}\nabla_i\nabla_j h_{kl} - P_2^{ijk}\nabla_i h_{jk} + P_3^{ij}h_{ij} = 0,$$

that is

$$\mathcal{P}' = ((m-1)\Delta + s_0)^{-1}(P_1 * \nabla^2 h + P_2 * \nabla h + P_3 * h), \tag{3.11}$$

where P_1, P_2, P_3 are tensors that depend on curvature of $g(t)$ and up to second order spatial derivatives of p.

In the calculation of the linearization of $\mathcal{M}$ the crucial step is to calculate

$$\frac{d}{d\lambda}|_{\lambda=0}\mathcal{L}_{W_\lambda}g_\lambda = -\frac{d}{d\lambda}|_{\lambda=0}\delta^*[g_\lambda]\omega_\lambda,$$

where $(\omega_\lambda)_i := (g_\lambda)_{ik}W^k_\lambda$ and $W_\lambda = -\tilde{g}^{-1}\delta[g_\lambda]G[g_\lambda](\tilde{g})$. In fact the key point of the DeTurck's trick is to collect the part of second order covariant derivatives of h in the above and realize that it cancels the second line in (3.10). To see that we first collect terms involving the first order covariant derivatives of h in the fiollowing

$$\frac{d}{d\lambda}|_{\lambda=0}(\omega_\lambda)_i = \nabla^k h_{ki} - \frac{1}{2}\nabla_i h^k{}_k + \text{ other terms } \cdots .$$

Then we collect the second order covariant derivatives of h in the following

$$\frac{d}{d\lambda}|_{\lambda=0}((\delta_{g_\lambda})^*\omega_\lambda)_{ij} = -\nabla_i\nabla^k h_{ki} - \nabla_j\nabla^k h_{kj} + \nabla_i\nabla_j h^k{}_k + \text{other terms.} \tag{3.12}$$

Therefore

$$\frac{d}{d\lambda}|_{\lambda=0}\mathcal{M}(g_\lambda) = \frac{\partial}{\partial t}h - \Delta h + 2\mathcal{P}'g + M_1^{ijk}\nabla_i h_{jk} + M_2^{ij}h_{ij}, \tag{3.13}$$

where M_1 depends only $g(t)$ and $\mathcal{P}(g)$ and M_2 depends on the curvature of $g(t)$ and $\mathcal{P}(g)$. To summarize we have

Lemma 3.3. *Suppose that $g(t)$, $t \in [0, T]$, is a family of metrics such that the elliptic operator $(m-1)\Delta[g(t)] + s_0$ is invertible for all $t \in [0, T]$. Then the* *linearization of DeTurck conformal Ricci flow equations* (3.2) *at the metrics* $g(t)$ *in the directions of the symmetric 2-tensors* $h(t)$ *is given as*

$$D\mathcal{M}(g)(h) = \frac{\partial}{\partial t}h - \Delta h + 2\mathcal{P}' g + M_1 * \nabla h + M_2 * h \tag{3.14}$$

where

$$\mathcal{P}' = ((m-1)\Delta + s_0)^{-1}(P_1 * \nabla^2 h + P_2 * \nabla h + P_3 * h),$$

P_1, P_2, P_3 *are tensors depending on curvature of* $g(t)$ *and up to the second order derivatives in spatial variables of* $\mathcal{P}(g)$, *and* M_1, M_2 *are tensors depending on curvature of* $g(t)$ *and function* $\mathcal{P}(g)$.

3.3. **Short Time Existence on Closed Manifolds.** Let us first solve conformal Ricci flow on a compact manifold M^m with no boundary. There are many books that are good for references in linear and nonlinar systems of parabolic equations. We will mostly use the book [Lu, §5.1], in particular Theorem 5.1.21 in [Lu] for existence and standard estimates. We adopt definitions of parabolic Hölder spaces from [Lu, p. 175-177]. We use the same notations for parabolic Hölder spaces for functions and tensor fields when there is no confusion. To define the norms for tensor fields we may use the initial metric and local coordinate charts.

3.3.1. *Preliminaries.* Since we deal with systems of parabolic-elliptic equations we need to consider elliptic estimates with time parameter. There is an advantage to use only the super norm in time variable, as indicated by the following lemma.

Lemma 3.4. *Let* $g(t)$, $t \in [0, T]$, *be a family of smooth Riemannian metric on compact manifold* M^m *with no boundary. Suppose operator* $(m-1)\Delta[g(t)] + s_0$ *is invertible for* $t \in [0, T]$. *Then the equation*

$$(m-1)\Delta[g(t)]p(t) + s_0 p(t) = \gamma \tag{3.15}$$

has a unique solution $p \in C^{0,2+\alpha}$ *for each* $\gamma \in C^{0,\alpha}$. *Moreover* p *satisfies the estimate*

$$\|p\|_{C^{0,2+\alpha}} \leq C\|\gamma\|_{C^{0,\alpha}} \tag{3.16}$$

for some constant C *independent of* γ.

Proof. In the light of standard Schauder estimates for elliptic PDE we only need to varify that $p(t)$ is continuous in time variable, which is a consequence of classical Bernstein estimates. □

The following interpolatory inclusion will be useful in the proof of the short-time existences (cf. [Lu, Lemma 5.1.1]).

Lemma 3.5. *There is a constant C independent of T such that for any $t_1, t_2 \in [0,T]$ we have*

$$\|h(t_1,\cdot) - h(t_2,\cdot)\|_{C^{k-2,\alpha}} \leq C \cdot |t_1 - t_2| \cdot \|h\|_{C^{1,k+\alpha}}.$$

for all $h \in C^{1,k+\alpha}([0,T]\times \mathrm{M})$.

3.3.2. *On Linearized DeTurck conformal Ricci Flow.* We first solve the linarized DeTurck conformal Ricci flow

$$\begin{cases} D\mathcal{M}(g)(h) = \dfrac{\partial}{\partial t}h - \Delta h + 2\mathcal{P}'g + M_1 * \nabla h + M_2 * h = \gamma \\ \qquad h(0,\cdot) = 0 \end{cases} \tag{3.17}$$

for appropriately given metrics $g(t)$ for each $\gamma \in C^{0,\alpha}$. Namely,

Proposition 3.6. *Suppose that $g(t)$, $t \in [0,T]$, is a family of metrics such that the elliptic operator $(m-1)\Delta[g(t)] + s_0$ is invertible for all $t \in [0,T]$. Then, for $\gamma \in C^{0,\alpha}$, the initial value problem for* (3.17) *has a unique solution $h \in C^{1,2+\alpha}$. Moreover*

$$\|h\|_{C^{1,2+\alpha}([0,T]\times\mathrm{M})} \leq C\|\gamma\|_{C^{0,\alpha}([0,T]\times\mathrm{M})}. \tag{3.18}$$

Proof. To use contractive mapping type argument we consider the Banach space

$$E_1([0,T^*]) = \{h \in C^{0,2+\alpha} : h(0,\cdot) = 0\}.$$

Given a $\tilde{h} \in E_1([0,T^*])$, based on Theorem 5.1.21 in [Lu], we first solve an usual system of linear parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}h - \Delta h + M_1 * \nabla h + M_2 * h = \tilde{\gamma} \\ h(0,\cdot) = 0, \end{cases} \tag{3.19}$$

where $\tilde{\gamma} = \gamma - 2\mathcal{P}'(\tilde{h})g \in C^{0,\alpha}$ and $\mathcal{P}'(\tilde{h})$ is defined by (3.11). We remark here that it takes some work to extend Theorem 5.1.21 in [Lu] to be applicable to our context, but there is no significant issues in doing so. Hence we may define a map

$$\Psi : E_1([0,T^*]) \to E_1([0,T^*])$$

by $\Psi(\tilde{h}) = h$. Note that, if set

$$v = \Psi(\tilde{h}_1) - \Psi(\tilde{h}_2),$$

then v satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}v - \Delta v + M_1 * \nabla v + M_2 * v = 2(\mathcal{P}'(\tilde{h}_2) - \mathcal{P}'(\tilde{h}_1))g \\ v(0,\cdot) = 0, \end{cases}$$

Due to the fact that

$$\|(\mathcal{P}'(\tilde{h}_2) - \mathcal{P}'(\tilde{h}_1))g\|_{C^{0,2+\alpha}} \leq C\|\tilde{h}_1 - \tilde{h}_2\|_{C^{0,2+\alpha}}$$

from (3.11) and Lemma 3.4, we obtain again from the estimates based on Theorem 5.1.21 in [Lu] that

$$\|v\|_{C^{1,4+\alpha}} \leq C\|\tilde{h}_1 - \tilde{h}_2\|_{C^{0,2+\alpha}}.$$

Hence, in the light of Lemma 3.5, we have

$$\|v(t_1) - v(t_2)\|_{C^{2,\alpha}} \leq C \cdot |t_1 - t_2| \cdot \|\tilde{h}_1 - \tilde{h}_2\|_{C^{0,2+\alpha}}.$$

In particular

$$\|\Psi(\tilde{h}_1) - \Psi(\tilde{h}_2)\|_{C^{0,2+\alpha}} \leq CT^*\|\tilde{h}_1 - \tilde{h}_2\|_{C^{0,2+\alpha}}.$$

To apply contractive mapping theorem we observe that

$$\|\Psi(\tilde{h})\|_{C^{0,2+\alpha}} \leq \|\Psi(0)\|_{C^{0,2+\alpha}} + CT^*\|\tilde{h}\|_{C^{0,2+\alpha}},$$

where

$$\|\Psi(0)\|_{C^{1,2+\alpha}} \leq C_0\|\gamma\|_{C^{0,\alpha}},$$

for some constant C_0, from the estimates based on Theorem 5.1.21 in [Lu]. Thus

$$\Psi : B_R = \{h \in E_1([0, T^*]) : \|h\|_{C^{0,2+\alpha}} \leq R\} \to B_R$$

for $R = 2C_0\|\gamma\|_{C^{0,\alpha}}$ is a contractive mapping when T^* is appropriately small. Then, by the uniqueness of the solution to linear parabolic equations (3.17), one may extend the solution to $[0, T]$ for (3.17) by steps in time of length T^*. The estimate (3.18) then follows from the estimates based on Theorem 5.1.21 in [Lu]. □

To summarize we have established that

$$D\mathcal{M}(g) : C^{1,2+\alpha}([0, T] \times \mathrm{M}) \bigcap \{h(0, \cdot) = 0\} \to C^{0,\alpha}([0, T] \times \mathrm{M})$$

is an isomorphism, provided that $g(t)$ satisfies the assumptions in Proposition 3.6.

3.3.3. *Implicit Function Theorem Argument.* Next we solve DeTurck conformal Ricci flow and then conformal Ricci flow. Our approach is to use an implicit function theorem. Let us start with the following simple implicit function theorem.

Lemma 3.7. *Suppose X and Y are Banach spaces and*

$$\mathcal{H} : X \to Y$$

is a C^1 map. Suppose that there is a point $x_0 \in X$ such that there are positive numbers δ and C, and

$$\|(D\mathcal{H}(x))^{-1}\| \leq C \quad \forall \quad x \in B_\delta(x_0)$$

and

$$\|D\mathcal{H}(x_1) - D\mathcal{H}(x_2)\| \le \frac{1}{2C}, \quad \forall \quad x_1, x_2 \in B_\delta(x_0).$$

Then, if

$$\|\mathcal{H}(x_0)\| \le \frac{\delta}{2C},$$

then there is $x \in B_\delta(x_0)$ *such that*

$$\mathcal{H}(x) = 0.$$

To apply the above implicit function theorem to the map for solving DeTurck conformal Ricci flow

$$\mathcal{M} : C^{1,2+\alpha}([0,T] \times \mathrm{M}) \bigcap \{g(0) = g_0\} \to C^{0,\alpha}([0,T] \times \mathrm{M})$$

we need to show that $\mathcal{M}$ is continuously differentiable. In fact we have

Lemma 3.8. *Suppose that* M^m *is a compact manifold without boundary. Suppose that* $g(t) \in C^{1,2+\alpha}([0,T]\times\mathrm{M})$ *such that the elliptic operator* $(m-1)\Delta[g(t)] + s_0$ *is invertible for all* $t \in [0,T]$*. Then there is* $\delta_0 > 0$ *such that*

$$\|D\mathcal{M}(g_1) - D\mathcal{M}(g_2)\|_{L(C^{1,2+\alpha}, C^{0,\alpha})} \le C\|g_1 - g_2\|_{C^{1,2+\alpha}}$$

for $\|g_i - g\|_{C^{1,2+\alpha}} \le \delta_0$ *in* $C^{1,2+\alpha}([0,T] \times \mathrm{M})$ *and* $i = 1, 2$.

Proof. We calculate, for any $h \in C^{1,2+\alpha} \bigcap \{h(0,\cdot) = 0\}$,

$$\begin{aligned}(D\mathcal{M}(g_1) - D\mathcal{M}(g_2))h = (\Delta[g_2] - \Delta[g_1])h + 2\mathcal{P}'[g_1](g_1 - g_2)\\ +2(\mathcal{P}'[g_1] - \mathcal{P}'[g_2])g_2 + M_1[g_1] * (\nabla[g_1]h - \nabla[g_2]h)\\ +(M_1[g_1] - M_1[g_2]) * \nabla[g_2]h + (M_2[g_1] - M_2[g_2]) * h.\end{aligned}$$

It is easily seen that

$$\|\Delta[g_1]h - \Delta[g_2]h\|_{C^{0,\alpha}} \le C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}}$$

$$\begin{aligned}\|M_1[g_1] * (\nabla[g_1]h - \nabla[g_2]h) + (M_1[g_1] - M_1[g_2]) * \nabla[g_2]h\|_{C^{0,\alpha}}\\ \le C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}}\end{aligned}$$

and

$$\|(M_2[g_1] - M_2[g_2])h\|_{C^{0,\alpha}} \le C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}}.$$

It is also easy to see that

$$\|\mathcal{P}'[g_1](g_1 - g_2)\|_{C^{0,\alpha}} \le C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}}.$$

under the assumption that $\|g_i - g\|_{C^{1,2+\alpha}} \leq \delta_0$ for $i = 1, 2$ from the definition of $\mathcal{P}'$ in (3.11). For the last remaining term we write

$$\begin{aligned}((m-1)\Delta + s_0)(\mathcal{P}'[g_2] - \mathcal{P}'[g_1]) &= (m-1)(\Delta[g_1] - \Delta[g_2])\mathcal{P}'[g_1] \\ &+(P_1[g_2] - P_1[g_1]) * \nabla^2[g_2]h + P_1[g_1] * (\nabla^2[g_2] - \nabla^2[g_1])h \\ &+(P_2[g_2] - P_2[g_1]) * \nabla[g_2]h + P_2[g_1] * (\nabla[g_2] - \nabla[g_1])h \\ &+(P_3[g_2] - P_3[g_1]) * h\end{aligned}$$

and apply Lemma 3.4. Therefore

$$\|\mathcal{P}'[g_1] - \mathcal{P}'[g_2]\|_{C^{0,\alpha}} \leq C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}},$$

which implies

$$\|(\mathcal{P}'[g_1] - \mathcal{P}'[g_2])g\|_{C^{0,\alpha}} \leq C\|g_1 - g_2\|_{C^{1,2+\alpha}}\|h\|_{C^{1,2+\alpha}}.$$

Thus the proof is complete. □

Next, to apply Lemma 3.7, we consider the initial approximate solution as follows:

$$\bar{g}(t) = g_0 + t\mathcal{F}(g_0) \tag{3.20}$$

where $\mathcal{F}$ is introduced in (3.9). We then calculate

$$\begin{aligned}\mathcal{M}(\bar{g}) &= -\mathcal{F}(g_0 + t\mathcal{F}(g_0)) + \mathcal{F}(g_0) \\ &= -t\int_0^1 D\mathcal{F}(g_0 + \theta t\mathcal{F}(g_0))d\theta \cdot \mathcal{F}(g_0).\end{aligned} \tag{3.21}$$

Now we are ready to state and prove the short time existence theorem for conformal Ricci flows.

Theorem 3.9. *Let* M^m *be a compact manifold with no boundary. Suppose that* $g_0 \in C^{4,\alpha}$ *is a Riemannian metric on* M *such that the scalar curvature* $s[g_0] = s_0$ *is a constant and that the elliptic operator* $(m-1)\Delta[g_0] + s_0$ *is invertible. Then there exists a small positive number* T *such that the conformal Ricci flow* $g(t)$ *exists in* $C^{1,2+\alpha}$ *from the initial metric* g_0 *for* $t \in [0, T]$.

Proof. First we notice that Proposition 3.6 holds for the family of metrics $\bar{g}(t) = g_0 + t\mathcal{F}(g_0)$ in $C^{1,2+\alpha}$ for some appropriately small T such that the elliptic operator $(m-1)\Delta[\bar{g}] + s_0$ is invertible for all $t \in [0, T]$. Therefore there is a constant C and a small number δ_0 such that

$$\|(D\mathcal{M}(g))^{-1}\| \leq C$$

and

$$\|D\mathcal{M}(g_1) - D\mathcal{M}(g_2)\| \leq \frac{1}{2C}$$

for all $g, g_1, g_2, \in B(\delta_0)$, where $B(\delta_0) = \{g \in C^{1,2+\alpha} : \|g - \bar{g}\|_{C^{1,2+\alpha}} \leq \delta_0\}$, according to Lemma 3.8. Next, choose even smaller T if necessary, we observe from (3.21) that

$$\|\mathcal{M}(\bar{g})\|_{C^{0,\alpha}} \leq \frac{\delta_0}{2C}.$$

Hence Lemma 3.7 implies that DeTurck conformal Ricci flow $\hat{g}(t)$ exists in $C^{1,2+\alpha}$ with the initial metric g_0. Therefore, applying Lemma 3.1, we obtain the short time existence for conformal Ricci flow from the initial metric g_0, since (3.7) is always solvable at least for short time. □

3.4. **Short Time Existence on Asymptotically Flat Manifolds.** In this subsection we establish the short time existence of conformal Ricci flow on asymptotically flat manifolds. The idea of the proof is the same as the proof in last subsection. We remark here that the short time existence of Ricci flow on asymptotically flat manifolds has been established independently in [DM] and [OW]. The approach in [DM] is to use the short time existence result in [Shi] and the maximum principle to show that Ricci flow in fact remains to be asymptotically flat when starting from asymptotically flat metric; while the approach in [OW] is to establish short time existence of Ricci flow based on weighted function spaces. Our approach is similar to the one in [OW] since no short time existence of conformal Ricci flow on non-compact manifolds is available and no maximum principle for conformal Ricci flow is available.

3.4.1. *Analysis on Asymptotically Flat Manifolds.* We first briefly introduce asymptotically flat manifolds according to [LP] and then construct appropriate parabolic Hölder spaces on asymptotically flat manifolds. The following is from [LP, Definition 6.3].

Definition 3.10. *A Riemannian manifold* M^m *with* C^2*-metric* g *is called asymptotically flat of order* $\tau > 0$ *if there exists a decomposition* $\mathrm{M} = \mathrm{M}_0 \cup \mathrm{M}_\infty$ *(with* M_0 *compact) and a diffeomorphism* $\Psi : \mathrm{M}_\infty \to \mathbb{R}^n \setminus B_{\mathrm{R}}(\vec{0})$*, for some* $\mathrm{R} > 0$*, satisfying:*

$$g(z) = g_e(z) + O(\rho^{-\tau}), \quad \partial_k g(z) = O(\rho^{-\tau-1}), \quad \partial_k \partial_l g(z) = O(\rho^{-\tau-2}),$$

where g_e *is the standard Euclidean metric and* $\rho = \rho(z) = |z| \to \infty$ *in the coordinates* $z = (z^1, \cdots, z^m)$ *induced on* M_∞ *by the diffeomorphism* Ψ*.*

We adopt the definition of weighted Hölder spaces $C^{k,\alpha}_\beta$ from [LP, p. 75]. Again we will use the same notations for weighted Hölder spaces for functions and tensor fields if there is no confusion. We use local coordinate charts and a given metric whenever it is necessary for the definition of Hölder spaces for tensor fields on asymptotically flat manifolds.

Fix a number $T > 0$, analogous to [Lu, p. 175-177], we define parabolic weighted Hölder spaces

$$C_\beta^{0,k+\alpha} := \{h \in C([0,T] \times \mathrm{M}) : h(t) \in C_\beta^{k,\alpha} \text{ and } \max_{t\in[0,T]} \|h(t)\|_{C_\beta^{k,\alpha}} < \infty\}$$

with the norm

$$\|h\|_{C_\beta^{0,k+\alpha}} := \max_{t\in[0,T]} \|h(t)\|_{C_\beta^{k,\alpha}}.$$

Similarly we define

$$C_\beta^{1,k+\alpha} := \{h \in C_\beta^{0,k+\alpha} \text{ and } \partial_t h \in C_{\beta-2}^{0,k-2+\alpha}\}$$

with the norm

$$\|h\|_{C_\beta^{1,k+\alpha}} := \max_{t\in[0,T]} \|h(t)\|_{C_\beta^{k,\alpha}} + \max_{t\in[0,T]} \|\partial_t h(t)\|_{C_{\beta-2}^{k-2,\alpha}}.$$

We now recall the elliptic theory for weighted Hölder spaces, for example, from [LP, Theorem 9.2] in our context.

Lemma 3.11. *Let $(\mathrm{M}^m,\ g(t))$, for $t \in [0,T]$, be a family of asymptotically flat manifolds with $g(t) - g_e \in C_{-\tau}^{0,2+\alpha}$ for $\tau > 0$. Then for $\beta \in (2-m, 0)$*

$$\Delta[g(t)] : C_\beta^{0,2+\alpha} \to C_{\beta-2}^{0,\alpha}$$

is an isomorphism, that is, there is C such that

$$\|u\|_{C_\beta^{0,2+\alpha}} \leq C\|\Delta[g(t)]u\|_{C_{\beta-2}^{0,\alpha}}.$$

Analogous to Lemma 3.5 we have a simple interpolatory inclusion.

Lemma 3.12. *There is a constant C independent of T such that for any $t_1, t_2 \in [0,T]$ we have*

$$\|h(\cdot,t_1) - h(\cdot,t_2)\|_{C_{\beta-2}^{k-2,\alpha}} \leq C \cdot |t_1 - t_2| \cdot \|h\|_{C_\beta^{1,k+\alpha}}.$$

for all $h \in C_\beta^{1,k+\alpha}([0,T] \times \mathrm{M})$.

3.4.2. *Short Time Existence on Asymptotically Flat Manifolds.* In this subsection we assume that the initial metric g_0 on M^m is asymptotically flat and scalar flat. Thanks to [SY, Lemma 3.3, Corollary 3.1] we know that one can always conformally deform an asymptotically flat metric with nonnegative scalar curvature into a scalar flat asymptotically flat metric. We will use the same strategy as in Section 3.3 to prove the short time existence of conformal Ricci flow on asymptotically flat manifolds.

First with changes of notations we are able to prove the isomorphism analogous to Proposition 3.6. An extension of [Lu, Theorem 5.1.21] to the weigted parabolic Hölder spaces on asymptotically flat manifolds may be proven by the standard argument through interior estimates and scaling invariance of the interior estimates (cf. [OW] [Ba] [LP]). The key is to realize that one may move in and out the weight for local estimates.

Proposition 3.13. *Suppose that $g(t)$, $t \in [0, T_0]$, is a family of asymptotically flat metrics with $g(t) - g_e \in C^{0,2+\alpha}_{-\tau}$ with $\tau \in (0, m-2)$. Then there is a $T_* \in (0, T_0]$ such that, for any $T \leq T_*$ and $\gamma \in C^{0,\alpha}_{-\tau-2}$, the initial value problem for* (3.17) *has a unique solution $h \in C^{1,2+\alpha}_{-\tau}$. Moreover*

$$\|h\|_{C^{1,2+\alpha}_{-\tau}([0,T]\times \mathrm{M})} \leq C\|\gamma\|_{C^{0,\alpha}_{-\tau-2}([0,T]\times \mathrm{M})}.$$

This is to say that, for $\tau \in (0, m-2)$,

$$D\mathcal{M}(g) : C^{1,2+\alpha}_{-\tau}([0,T]\times \mathrm{M}) \bigcap \{h(0,\cdot) = 0\} \to C^{0,\alpha}_{-\tau-2}([0,T]\times \mathrm{M})$$

is an isomorphism, provided that $g(t)$ and T satisfiy the assumptions in the above Proposition 3.13. The restriction on the order τ of weight is solely used in solving elliptic equations on weighted spaces in Lemma 3.11.

To obtain a short time existence of DeTurck conformal Ricci flow we again apply the implicit function theorem (Lemma 3.7) to the map

$$\mathcal{M} : \{g(t) : g(t) - g_e \in C^{1,2+\alpha}_{-\tau}([0,T]\times \mathrm{M}) \text{ and } g(0) = g_0\} \to C^{0,\alpha}_{-\tau-2}([0,T]\times \mathrm{M})$$

for any $\tau \in (0, m-2)$ and T given from Proposition 3.13. Finally we arrive at the short time existence of conformal Ricci flow.

Theorem 3.14. *Let $(\mathrm{M}^m,\ g_0)$ be scalar flat and asymptotically flat with $g_0 - g_e \in C^{4,\alpha}_{-\tau}$ and $\tau \in (0, m-2)$. Then there exists a small positive number T such that the conformal Ricci flow $g(t)$ from the initial metric g_0 exists for $t \in [0,T]$ and $g(t) - g_e \in C^{1,2+\alpha}_{-\tau}([0,T]\times \mathrm{M})$.*

Proof. As in Section 3.3.3 we first verify

$$\|D\mathcal{M}(g_1) - D\mathcal{M}(g_2)\|_{L(C^{1,2+\alpha}_{-\tau}, C^{0,\alpha}_{-\tau-2})} \leq C\|g_1 - g_2\|_{C^{1,2+\alpha}_{-\tau}}.$$

The proof goes like the one for Lemma 3.8 with only changes of notations. We then construct

$$\bar{g}(t) = g_0 + t\mathcal{F}(g_0) \in g_e + C^{1,2+\alpha}_{-\tau}$$

as in Section 3.3.3, for $g_0 - g_e \in C^{4,\alpha}_{-\tau}$. Another issue one needs to take care of is solving (3.7) to construct conformal Ricci flow from DeTurck conformal Ricci flow. But, since $W \in C^{0,1+\alpha}_{-\tau-1}$, it is easy to solve (3.7) on the whole manifold M for some short time. The rest of the proof goes like the one in Section 3.3.3 for Theorem 3.9 with little changes except notations. Notice that the equivalence between (1.1) and (1.2) holds because of the uniqueness of the bounded solution to linear parabolic equations on asymptotically flat manifolds. □

4. ADM Mass under Conformal Ricci Flow

Asymptotically flat manifolds are used in general relativity to describe isolated gravitational systems. The fundamental geometric invariant of a asymptotically flat manifold is called the mass of the gravitational system. The so-called ADM mass of an asymptotically flat manifold was first defined in [ADM] in early 1960s.

In general relativity the world is modeled by a 4-dimensional spacetime X^4 with a Lorentzian metric g. The physical law that describes the gravity induced by matters in the spacetime is the famous Einstein equations

$$Ric[g] - \frac{1}{2}s[g]g = T,$$

where T is the energy-momentum-stress tensor that is supposed to reflect the nature and state of the matters in the spacetime. A time slice of a spacetime that represents an isolated gravitational system is an asymptotically flat 3-manifold M^3.

One of the most important solution of Einstein equations is the Schwarzchild spacetime, which represents the gravitational system of a static point particle of mass m and whose a time slice is an asymptotically flat metric

$$g_{Sch} = g_e + \frac{m}{\rho}g_e + O(\rho^{-2})$$

on the punctured $\mathbb{R}^3$. The crucial test to validate the notion of mass in relativity is that its prediction reduces to those of Newtonian gravity under the circumstances when Newtonian theory is known to be valid - specifically, when gravity is weak, motions are much slower than the speed of the light, and material stresses are much smaller than the mass-energy density (cf. [Wald, 4.4]).

We now follow [LP, Definition 8.2] to introduce ADM mass for asymptotically flat manifolds.

Definition 4.1. *Given an asymptotically flat Riemannian manifold* (M^m, g) *with asymptotic coordinates* z*, we define the ADM mass by (if the limit exists)*

$$m(g) = \lim_{R\to\infty} \omega_{m-1}^{-1} \int_{\mathbb{S}_R} (\partial_i g_{ij} - \partial_j g_{ii}) n^j d\sigma$$

where ω_{m-1} *is the volume of the unit sphere* $\mathbb{S}^{m-1}$*,* $\vec{n} = (n^1, \cdots, n^m)$ *is the outward unit normal vector of the sphere* $\mathbb{S}_R = \{z \in \mathbb{R}^m, |z| = R\}$ *and* $d\sigma$ *is the area element of* $\mathbb{S}_R$*.*

Recall from [LP] that

$$\mathcal{M}_\tau := \{g = g_e + h : h \in C^{1,\alpha}_{-\tau} \text{ and } \partial_j\partial_i h_{ij} - \partial_j\partial_j h_{ii} \in L^1(\mathrm{M}, d\mathrm{vol}[g_e])\}.$$

After Definition 4.1 one wonders if the ADM mass is indeed a geometric invariant for the asymptotically flat metric. It was confirmed as follows:

Lemma 4.2. *(*[ADM][Ba]*) Suppose that g is an asymptotically flat metric in $\mathcal{M}_\tau$ on M^m for $\tau > \frac{m-2}{2}$. Then the ADM mass $m(g)$ is indeed independent of the choice of asymptotic coordinates at infinity.*

Another important fact about the ADM mass is observed in [LP, (8.11)] and supported by [LP, Lemma 9.4].

Lemma 4.3. *(*[LP]*) Let $g(t)$ be a smooth family of asymptotically flat metrics in $\mathcal{M}_\tau$ on M^m for $\tau > \frac{m-2}{2}$. Then the mass $m(g(t))$ is differentiable and*

$$\frac{d}{dt}\left(-\int_{\mathrm{M}} s[g(t)]dvol[g(t)] + \omega_{m-1}m(g(t))\right) = \int_M G[g(t)]\cdot\phi(t)dvol[g(t)]$$

where $G[g(t)] = Ric[g(t)] - \frac{1}{2}s[g(t)]g(t)$ is the Einstein tensor and $\phi(t) = \partial_t g(t)$.

Consequently we have, from Theorem 3.14 and Lemma 4.3,

Theorem 4.4. *Let g_0 be a scalar flat and asymptotically flat metric on M^m such that $g_0 - g_e \in C^{4,\alpha}_{-\tau}$ for $\tau \in (\frac{m-2}{2}, m-2)$. Then the conformal Ricci flow $g(t)$ starting with $g(0) = g_0$ exists for some short time and*

$$g(t) \in \mathcal{M}_\tau \quad \text{and} \quad g(t) - g_e \in C^{1,2+\alpha}_{-\tau}.$$

Moreover

$$\frac{d}{dt}m(g(t)) = -2\int_M |Ric[g(t)]|^2 dvol[g(t)].$$

In particular, the ADM mass is strictly decreasing under conformal Ricci flow except that g_0 is the Euclidean metrics.

Proof. To verify that conformal Ricci flow $g(t) \in \mathcal{M}_\tau$ we only need to verify that

$$\partial_j\partial_i g_{ij}(t) - \partial_j\partial_j g_{ii}(t) \in L^1(\mathrm{M}, d\mathrm{vol}[g_e]).$$

Recall [LP, (9.2)]

$$s = \partial_j\partial_i g_{ij} - \partial_j\partial_j g_{ii} + O(\rho^{-2\tau-2})$$

which implies that

$$\partial_j\partial_i g_{ij} - \partial_j\partial_j g_{ii} = O(\rho^{-2\tau-2}) \in L^1(\mathrm{M}, d\mathrm{vol}[g_e]).$$

for $\tau \in (\frac{m-2}{2}, m-2)$. It is easily seen that the ADM mass is strictly decreasing except that g_0 is Ricci flat. Then, using [BKN, Theorem 1.5] and [LP, Proposition 10.2], one concludes that g_0 is the standard Euclidean metric. Therefore the proof is complete. □

A quick application of the above Theorem 4.4 is a simple and direct proof of the rigidity part of Schoen and Yau positive mass theorem. Namely,

Corollary 4.5. *([SY]) Suppose that* $(\mathrm{M}^m,\ g)$ *is asymptotically flat manifold with nonnegative scalar curvature and that* $g - g_e \in C^{4,\alpha}_{-\tau}$ *for* $\tau > \frac{m-2}{2}$. *Then, if the ADM mass* $m(g) = 0$, *then* $(\mathrm{M}^m,\ g)$ *is isometric to the standard Euclidean space* $\mathbb{R}^m$.

Proof. First we know g has to be scalar flat. Otherwise one can conformally deform the metric to scalar flat and decrease the ADM mass to negative, which is impossible due to the first part of the positive mass theorem of Schoen and Yau. Next we invoke Theorem 4.4 and come to the same contradiction if g is not flat. □

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