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On real Johnson-Wilson theories

Author
Averett, Maia Christine

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On real Johnson-Wilson theories

A dissertation submitted in partial satisfaction of the
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by

Maia Christine Averett

Committee in charge:

Professor Nitu Kitchloo, Chair
Professor Justin Roberts, Co-Chair
Professor Russell Impagliazzo
Professor James P. Lin
Professor Doug Smith

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The dissertation of Maia Christine Averett is approved, and it is acceptable in quality and form for publication on microfilm:

Co-Chair

Chair

University of California, San Diego

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DEDICATION

To my parents.

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VITA

2003	B. S. in Mathematics, University of California, Santa Barbara
2003-2008	Graduate Teaching Assistant, University of California, San Diego
2004	M. S. in Mathematics, University of California, San Diego
2007-2008	President's Dissertation Year Fellow, University of California, San Diego
2008	Ph. D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

On real Johnson-Wilson theories

by

Maia Christine Averett
Doctor of Philosophy in Mathematics

University of California San Diego, 2008

Professor Nitu Kitchloo, Chair
Professor Justin Roberts, Co-Chair

The central object of study in this thesis is a family of generalized cohomology theories $ER(n)$, known as real Johnson-Wilson theories. These theories arise as the homotopy fixed points of the classical Johnson-Wilson theories $E(n)$ under the $\mathbb{Z}/2$-action of complex conjugation. The classical Johnson-Wilson theories $E(n)$ are closely related to another family E_n of cohomology theories, the so-called Lubin-Tate or Morava E-theories. A purely obstruction-theoretic argument given by Hopkins and Miller [Rez98] shows that the E_n admit an action of the Morava stabilizer group of automorphisms of the height n Honda formal group law. We relate the real Johnson-Wilson theories $ER(n)$ to homotopy fixed points of the Morava E-theories E_n under an action by a certain subgroup of the Morava stabilizer group. In doing so, we obtain a calculation of the coefficients of the homotopy fixed points of E_n for this subgroup and as a corollary we see that after completion the $ER(n)$ are commutative S-algebras (i.e. E_∞-ring theories). We work entirely at the prime 2.

Chapter 1

Introduction

1.1 Background

Cohomology theories derived from complex cobordism are central objects of study in modern homotopy theory. Due to its geometric nature as the Thom spectrum associated to the unitary group, the spectrum MU representing complex cobordism admits a natural involution by complex conjugation. This gives rise to a theory of $\mathbb{Z}/2$-equivariant or *Real* cobordism.

In 1966, Atiyah [Ati66] noticed that a complex vector bundle over a $\mathbb{Z}/2$-space might admit a complex antilinear $\mathbb{Z}/2$-action compatible with the action on the base. He called such a bundle a Real vector bundle and introduced Real K-theory $KR(X)$ as the Grothendieck group of such bundles over a $\mathbb{Z}/2$-space X. Motivated by this, in 1967 Landweber [Lan67] defined cobordism of Real manifolds and later [Lan68], using the involution of complex conjugation on the Thom spaces $MU(n)$ and equivariant suspension, he defined an associated spectrum $\mathbb{MR}$ indexed on a complete $\mathbb{Z}/2$-universe.[1] Later, Araki [AM78], [Ara79b], [Ara79a], observed that one could carry out many of the classical constructions of complex cobordism for this new Real cobordism. He defined the notion of a Real-oriented spectrum, Real formal group laws, and used Quillen's idempotent to construct a Real analogue $\mathbb{BPR}$ of the Brown-Peterson spectrum BP. Much more recently, in work con-

[1] Note that the equivariant homotopy ring $\mathbb{MR}_\star$ does not calculate the Real cobordism groups of compact Real manifolds, unlike the complex case (see [Hu99]).

nected to constructing a Real version of the Adams-Novikov spectral sequence, Hu and Kriz [HK01] constructed Real versions of Johnson-Wilson theory $E(n)$ and Morava K-theory $K(n)$. Of course, these constructions presuppose a fixed prime p. Throughout this thesis, we work exclusively at $p = 2$.

Given a Real spectrum $\mathbb{E}$, there is a naïve $\mathbb{Z}/2$-spectrum associated to the trivial representation. In this way, we obtain naïve $\mathbb{Z}/2$-spectra associated to Johnson-Wilson theories. Since forgetting the $\mathbb{Z}/2$-action returns the Johnson-Wilson spectra $E(n)$, we write $E(n)$ for the naïve $\mathbb{Z}/2$-spectra as well. Taking homotopy fixed points yields spectra

$$ER(n) := E(n)^{h\mathbb{Z}/2}$$

that go by the name of 'real Johnson-Wilson' theories.[2] The central object of study in this thesis is this family $ER(n)$ of real Johnson-Wilson theories. These were first considered by Kitchloo and Wilson [KW06], who made computations with $ER(n)$ feasible by constructing a fibration

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x(n)} ER(n) \to E(n). \tag{1.1.1}$$

Here $x(n)$ is a distinguished 2-torsion element in $ER(n)_*$ and $\lambda(n)$ is the integer $2^{2n+1} - 2^{n+2} + 1$. The Johnson-Wilson theories are higher K-theories in the sense that $E(1)$ is complex K-theory localized at 2. The $ER(n)$ are similarly related to real K-theory KO and the fibration above is a generalization of the classical fibration

$$\Sigma KO_{(2)} \xrightarrow{\eta} KO_{(2)} \to KU_{(2)}.$$

The fibration (1.1.1) gives rise to an exact couple and a Bockstein spectral sequence measuring $x(n)$-torsion. Kitchloo and Wilson [KW06] have used this spectral sequence to compute $ER(2)$-cohomology of real projective spaces and to prove new bounds for nonimmersions of real projective spaces in Euclidean space. We will use this spectral sequence to calculate the coefficients of $ER(n)$ and related theories.

[2]The terminology is meant to mimic that of Atiyah's: the analogous construction applied to Atiyah's Real K-theory KR returns real K-theory KO. The real Johnson-Wilson theories $ER(n)$ are honest spectra corresponding to Hu and Kriz's Real Johnson-Wilson theories $\mathbb{ER}(n)$. We indicate this by writing 'real' rather than 'Real.'

In this thesis, we will relate the real Johnson-Wilson spectra $ER(n)$ to a much-studied but difficult to access family of spectra known as Morava E-theories. These are a family of theories E_n with coefficients

$$(E_n)_* = W(\mathbb{F}_{2^n})[[u_1, \ldots, u_{n-1}]][u^{\pm}]$$

where $|u_i| = 0, |u| = 2$, and $W(\mathbb{F}_{2^n})$ denotes the Witt vectors of $\mathbb{F}_{2^n}$. They can be constructed using Landweber's exact functor theorem for BP. Let S_n denote the Morava stabilizer group of automorphisms of the height n Honda formal group law Γ_n over $\mathbb{F}_{2^n}$ and write $G(n) = \mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \ltimes S_n$ (see [Rez98]). The Lubin-Tate theory of lifts gives an action of S_n on $(E_n)_*$ and $\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ acts on $(E_n)_*$ by its action on the Witt vectors, so we have an action of $G(n)$ on $(E_n)_*$ (see [HM],[Rez98]). In the early 1990's, Hopkins and Miller used obstruction theory to rigidify this action to an action on the level of spectra. They showed that the space of A_∞-self-maps of E_n is homotopy discrete with components equal to the set $G(n)$. They thus realized the action of $G(n)$ on $(E_n)_*$ by an action of a homotopy discrete monoid on E_n in the category of A_∞-ring spectra (see [Rez98]). Later, Goerss and Hopkins [GH04] improved this by replacing A_∞ by E_∞. These theorems therefore give rise to a great supply of interesting E_∞-ring spectra obtained by taking homotopy fixed points of E_n with respect to various subgroups of $G(n)$.

For example, we might consider the action of $\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \ltimes \mathbb{F}_{2^n}^\times$ and the resulting homotopy fixed points

$$E_n(\mathrm{Gal}) := E_n^{h\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)\ltimes \mathbb{F}_{2^n}^\times}.$$

The action of the Galois group $\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ on the Witt vectors $W(\mathbb{F}_{2^n})$ fixes precisely $\mathbb{F}_2$, so the fixed point set of this action is $W(\mathbb{F}_2) = \widehat{\mathbb{Z}}_2$, the 2-adic integers. Calculating the homotopy fixed points spectral sequence for the action of $\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ gives

$$(E_n^{h\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)})_* = \widehat{\mathbb{Z}}_2[[u_1, \ldots, u_{n-1}]][u^{\pm}].$$

Since $\mathbb{F}_{2^n}^\times$ has odd order, the homotopy fixed point spectral sequence for the action of it on $E_n^{h\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)}$ collapses and gives

$$E_n(\mathrm{Gal})_* = \widehat{\mathbb{Z}}_2[[v_1, \ldots, v_{n-1}]][v_n^{\pm}]$$

where $|v_i| = 2(2^i - 1)$. Recall that

$$E(n)_* = \mathbb{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$$

so

$$E_n(\mathrm{Gal})_* = \widehat{\mathbb{Z}}_2[[v_1, \ldots, v_{n-1}]][v_n^{\pm}] = (E(n)_*)^{\wedge}_{I_n}$$

where I_n is the maximal ideal $(v_0, \ldots, v_{n-1}) \subseteq E(n)_*$ and the notation denotes I_n-adic completion. The Morava stabilizer group S_n again acts on $E_n(\mathrm{Gal})$ and there is a subgroup G_2 of order 2 in S_n generated by the formal inverse. The purpose of this thesis is to relate $ER(n)$ to the homotopy fixed points $E_n(\mathrm{Gal})^{hG_2}$.

Now, we may also consider $K(n)$-localization of $E(n)$

$$\widehat{E(n)} := L_{K(n)}E(n)$$

which also has

$$\widehat{E(n)}_* = (E(n)_*)^{\wedge}_{I_n}.$$

The spectrum $\widehat{E(n)}$ inherits a $\mathbb{Z}/2$-action from the action of complex conjugation on $E(n)$, and so the homotopy fixed points of $\widehat{E(n)}$ with respect to this action may be considered to be a completion of $ER(n)$. We will prove that

$$\widehat{E(n)}^{h\mathbb{Z}/2} \simeq E_n(\mathrm{Gal})^{hG_2}$$

and that the canonical map

$$ER(n) \to E_n(\mathrm{Gal})^{hG_2}$$

induces an algebraic completion on the level of coefficients. This is the content of Theorem 6.2.4. Along the way, we obtain a calculation of $E_n(\mathrm{Gal})^{hG_2}$. Computations of this nature are in general extremely difficult and, unfortunately, ours arises through a very special circumstance that does not generalize to other subgroups of S_n or $G(n)$.

Beginning in the next section, we use the modern terminology of Elmendorf, Kriz, Mandell, and May's S-algebras (see [EKMM97] or Chapter 2 for a very brief summary of the essential definitions and properties). An S-algebra is essentially an A_∞-ring spectrum and a commutative S-algebra is essentially an E_∞-ring spectrum.

1.2 Methods

In order to relate the action of complex conjugation on $\widehat{E(n)}$ to the action of the subgroup generated by the formal inverse on $E_n(\mathrm{Gal})$, we would like to have a map

$$\widehat{E(n)} \to E_n(\mathrm{Gal})$$

that is equivariant with respect to these actions. If this map were also an equivalence, it would induce an equivalence on homotopy fixed points. Since $\widehat{E(n)}$ and E_n are equivalent as S-algebras ([Laz03]) and the equivalence is *homotopy* equivariant, it is natural to attempt to rigidify this homotopy equivariant equivalence to one that is honestly equivariant. A standard trick for this purpose is the following.

Trick 1.2.1. Suppose A and B are spaces or spectra or S-algebras with G-actions and suppose that the space of maps $F(A, B)$ is homotopy discrete. Let $\varphi \in F(A, B)$ be equivariant up to homotopy. Let $F(A, B)_\varphi$ denote the component of φ in $F(A, B)$. Then $A \wedge F(A, B)$ and A are equivalent, $F(A, B)_\varphi$ inherits a G-action via conjugation, and the evaluation map

$$A \wedge F(A, B)_\varphi \to B$$

is honestly equivariant.

In order for this trick to work, one needs the space of maps between the two objects to be homotopy discrete, or at least for the relevant component to be contractible. In general, this is a quite stringent restriction, but if the objects are endowed with extra structure, the space of maps preserving this structure can sometimes be shown to be homotopy discrete. For example, A and B might be S-algebras or commutative S-algebras and we might ask if the space of S-algebra maps from A to B is homotopy discrete. As noted in the Introduction, Hopkins and Miller showed that the space of S-algebra self-maps of E_n is homotopy discrete and its components are the extended Morava stabilizer group $G(n)$.[3] This was later extended by Hopkins and Goerss [GH04] to the space of commutative S-algebra

[3] See Rezk's notes [Rez98] for an account of their work.

maps. More recently, Lazarev [Laz03] proved generalized Hopkins-Miller theorems for a class of theories called strongly $K(n)$-complete. Namely, he showed that the space of S-algebra maps from $\widehat{E(n)}$ to a strongly $K(n)$-complete theory is homotopy discrete with components equal to the set of multiplicative cohomology operations.

We might hope to use this trick for $A = \widehat{E(n)}$ and $B = E_n(\mathrm{Gal})$, together with the results of Hopkins, Miller, and Goerss, to rigidify our homotopy equivariant map. Here we run into a problem: it is not known if the $\mathbb{Z}/2$-action on $\widehat{E(n)}$ arising from complex conjugation on MU is an action by S-algebra maps. That is, the conjugation action on the space of maps from $\widehat{E(n)}$ to $E_n(\mathrm{Gal})$ does not necessarily descend to an action on the subspace of S-algebra maps from $\widehat{E(n)}$ to E_n.

To rectify this problem, we retreat to a situation where we know that the action is via S-algebra maps. Because the commutative S-algebra MU representing complex cobordism is constructed geometrically, the $\mathbb{Z}/2$-action of complex conjugation is via S-algebra maps. Moreover, the natural map

$$MU \to v_n^{-1}MU \to E(n) \to \widehat{E(n)} \to E_n(\mathrm{Gal})$$

is homotopy equivariant, so we might try to rigidify this map. The space of S-algebra maps from MU to $E_n(\mathrm{Gal})$ is not known to be homotopy discrete, but in light of results of Lazarev [Laz03], one might guess that the space of S-algebra maps from $v_n^{-1}\widehat{MU} = L_{K(n)}MU$ is. The standard way of computing the homotopy of such a space of maps is to identify E_2-term of the Bousfield-Kan spectral sequence with Hochschild cohomology and then show that the spectral sequence collapses. This argument falls apart for the space

$$F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$$

because the Hochschild cohomology is not concentrated in a single vertical line. However, we are able to get around this obstruction by means of another trick. We construct an S-algebra T, a wedge of smashes of spheres, with the property that

$$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, \widehat{E(n)}) \simeq F_{S-\mathrm{alg}}(\widehat{E(n)}, \widehat{E(n)}). \qquad (1.2.1)$$

We give $\widehat{E(n)}$ the trivial T-algebra structure via a map $T \to S$. The right hand side of (1.2.1) is homotopy discrete by Hopkins-Miller theory. To prove the equivalence of these mapping spaces, we calculate the Bousfield-Kan spectral sequence for

$$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, \widehat{E(n)})$$

and compare it to the Bousfield-Kan spectral sequence for

$$F_{T-\mathrm{alg}}(\widehat{E(n)} \wedge_S T, \widehat{E(n)}).$$

We must also construct an S-algebra action on $v_n^{-1}\widehat{MU}$ that extends the action of complex conjugation on MU. We use Bousfield localization and the complex conjugation action σ on MU to construct a map

$$\widehat{\sigma} \in F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, v_n^{-1}\widehat{MU})$$

with $\widehat{\sigma}^2 = \mathrm{id}_{v_n^{-1}\widehat{MU}}$. With this result in hand, we are finally able to rigidify our homotopy equivariant map.

Write $G_1 \subseteq F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, v_n^{-1}\widehat{MU})$ for the $\mathbb{Z}/2$ generated by $\widehat{\sigma}$ and $G_2 \subseteq F_{S-\mathrm{alg}}(E_n(\mathrm{Gal}), E_n(\mathrm{Gal}))$ for the homotopy discrete monoid corresponding to the subgroup of S_n generated by the formal inverse, so that G_1 acts on $v_n^{-1}\widehat{MU}$ and G_2 acts on $E_n(\mathrm{Gal})$. Define $\widetilde{\mathbb{Z}/2}$ by the following pullback diagram

$$\begin{array}{ccc}
\widetilde{\mathbb{Z}/2} & \xrightarrow{\tilde{\Delta}} & G_1 \times G_2 \\
\downarrow & & \downarrow{\scriptstyle \pi_0} \\
\mathbb{Z}/2 & \xrightarrow{\Delta} & \mathbb{Z}/2 \times \mathbb{Z}/2
\end{array}$$

where Δ is the diagonal map. We give $E_n(\mathrm{Gal})$ the trivial T-algebra structure, so a T-algebra map

$$E_n(\mathrm{Gal}) \to E_n(\mathrm{Gal})$$

is just an S-algebra map. Then we obtain an action of $\widetilde{\mathbb{Z}/2}$ on

$$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$$

by conjugation and hence a $\widetilde{\mathbb{Z}/2}$ action on

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$$

diagonally. Using Trick 1.2.1, we obtain an equivariant map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal}).$$

where ν is the map $v_n^{-1}\widehat{MU} \to E_n(\mathrm{Gal})$ that is the quotient map on coefficients and the subscript denotes the component of ν.

Next, we show that after taking homotopy fixed points, this map factors through $ER(n)$, resulting in a map

$$ER(n) \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}.$$

We prove that this map is a completion in the sense that it induces an algebraic completion on the level of homotopy. In fact, we see that $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ is equivalent to the homotopy fixed points of $\widehat{E(n)}$ under the action of complex conjugation. This, together with a calculation of the homotopy ring of $ER(n)$, allows us to compute the coefficients of $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$. Because $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ is a commutative S-algebra by Hopkins-Goerss theory, we obtain the result that $ER(n)$ becomes a commutative S-algebra after completion.

1.3 Outline

Now we proceed to outline the basic structure of this thesis. We begin in Chapter 2 by reviewing terminology and constructions that we will need. These include the basics of Elmendorf, Kriz, Mandell, and May's categories of S- and R-algebras, various types of completion, and a quick review of Real spectra. We also define the spectra $ER(n)$ and related spectra that will be considered in this thesis.

Next, in Chapter 3, we use calculations of Hu and Kriz [HK01] and Kitchloo and Wilson [KW06] to calculate the homotopy groups of various spectra, including Kitchloo and Wilson's real Johnson-Wilson theory $ER(n)$ and the homotopy fixed points of a completion of Johnson-Wilson theory.

In Chapter 4, we use Bousfield localization to construct an S-algebra $v_n^{-1}\widehat{MU}$ and an S-algebra map

$$\widehat{\sigma} : v_n^{-1}\widehat{MU} \to v_n^{-1}\widehat{MU}$$

with $\widehat{\sigma}^2 = \mathrm{id}_{v_n^{-1}\widehat{MU}}$. This map $\widehat{\sigma}$ is induced by the action

$$\sigma : MU \to MU$$

of complex conjugation.

In Chapter 5, we construct the S-algebra T and prove that the space of T-algebra maps from $v_n^{-1}\widehat{MU}$ to $\widehat{E(n)}$ is homotopy discrete.

Finally, in Chapter 6, we tie all the results together to compare the actions. In this chapter, we construct the map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal})$$

and prove that it factors through $ER(n)$ after taking homotopy fixed points. This gives rise to the main theorems and corollaries, which are also drawn here.

Chapter 2

Terminology and constructions

In this chapter, we describe the categories in which we work, define the main objects of study, and give basic constructions that we will use throughout this thesis. We begin with a very brief description of the "brave new" structured point-set categories that we work with in place of the category of spectra. The reference for all of this material is the text [EKMM97] of Elmendorf, Kriz, Mandell, and May. They construct a topological closed model category $\mathcal{M}_S$ with a unital symmetric monoidal smash product $\wedge_S$ such that the derived category $\mathcal{D}_S$ of $\mathcal{M}_S$ is equivalent to the classical derived category of spectra. We only provide the basic terminology here and we refer the reader to [EKMM97] for details and proofs. Next, we recall a few useful constructions. We briefly visit Bousfield localization and describe its relationship to completion. We give examples that define and describe $\widehat{E(n)}$ and $v_n^{-1}\widehat{MU}$. After that, we recall the theory of Real spectra as studied by Hu and Kriz [HK01]. We describe the construction of the real Johnson-Wilson theories and related theories that we will study in this thesis.

Terminology. We are strict about our terminology throughout this thesis. We use the terms "R-module" and "R-algebra" to refer to the highly structured notions of [EKMM97] discussed below. The terms "spectrum" or "R-module spectrum" or "R-ring spectrum" refer to the classical homotopical notions.

2.1 $\mathbb{L}$-spectra

All of the highly structured categories of spectra that we work with are built on top of the category of $\mathbb{L}$-spectra. Let $\mathcal{S}U$ denote the category of spectra indexed on some fixed universe U. Let $\mathcal{I}$ denote the category whose objects are universes and whose morphisms are linear isometries. We write $\mathcal{I}(U, U')$ for morphisms in this category. Give each universe a topology as the union of its finite dimensional subspaces and give the set $\mathcal{I}(U, U')$ the function space topology. Then $\mathcal{I}(U, U')$ is contractible.

Let U^j be the direct sum of j copies of U and write $\mathcal{L}(j)$ for the space $\mathcal{I}(U^j, U)$. The space $\mathcal{L}(0)$ is a point and $\mathcal{L}(1)$ contains the identity map on U. The symmetric group Σ_j acts on the left on U^j by permutations and induces a free right action of Σ_j on $\mathcal{L}(j)$. Define maps

$$\mathcal{L}(k) \times \mathcal{L}(j_1) \times \cdots \times \mathcal{L}(j_k) \to \mathcal{L}(j_1 + \cdots + j_k)$$

by $(g, f_1, \ldots, f_k) \mapsto g \circ (f_1 \oplus \cdots \oplus f_k)$. The spaces $\mathcal{L}(j)$ together with these maps form what is called the *linear isometries operad*, which we denote by $\mathcal{L}$. Let $\mathbb{L}$ denote the monad in $\mathcal{S}$ given by $\mathbb{L}E = \mathcal{L}(1) \ltimes E$.[1]

Definition 2.1.1 ([EKMM97], I.4.2)**.** An $\mathbb{L}$-*spectrum* is an algebra over the monad $\mathbb{L}$. Denote the category of $\mathbb{L}$-spectra by $\mathcal{S}[\mathbb{L}]$.

This means that an $\mathbb{L}$-spectrum is a spectrum E together with an action $\mathbb{L}E \to E$ so that the following diagrams commute

$$\begin{array}{ccc} \mathbb{L}\mathbb{L}E & \longrightarrow & \mathbb{L}E \\ \downarrow & & \downarrow \\ \mathbb{L}E & \longrightarrow & E \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E & \longrightarrow & \mathbb{L}E \\ & \searrow & \downarrow \\ & & E \end{array}$$

A morphism $E \to F$ of $\mathbb{L}$-spectra is a map of spectra commuting with the action of $\mathbb{L}$ on E and F. The category of $\mathbb{L}$-spectra has a naturally associative and commutative internal smash product $E \wedge_{\mathcal{L}} F$ defined as the coequalizer

$$(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E \wedge F) \rightrightarrows \mathcal{L}(2) \ltimes (E \wedge F) \to E \wedge_{\mathcal{L}} F$$

[1]The symbol $\ltimes$ denotes the twisted half smash product of a space with a spectrum. See Appendix A of [EKMM97].

Unfortunately, this smash product is only unital on the homotopy category. However, there is a unital symmetric monoidal category whose derived category is equivalent to that of classical spectra. The category is that of so-called S-modules, which we will define in the next section. First, we note that for any $\mathbb{L}$-spectrum E there is a natural map $\lambda : S \wedge_{\mathcal{L}} E \to E$ which is always a weak equivalence of $\mathbb{L}$-spectra. See Section I.8 of [EKMM97] for its construction.

2.2 S-modules

Let S denote the sphere spectrum indexed on a fixed universe U.

Definition 2.2.1. An S-*module* is an $\mathbb{L}$-spectrum E such that $\lambda : S \wedge_{\mathcal{L}} E \to E$ is an isomorphism. Write $\mathcal{M}_S$ for the full subcategory of $\mathbb{L}$-spectra whose objects are S-modules. Define $E \wedge_S F := E \wedge_{\mathcal{L}} F$ and $F_S(E, F) := S \wedge_{\mathcal{L}} F_{\mathcal{L}}(E, F)$.

Elmendorf, Kriz, Mandell, and May justify the name S-modules with the following commutative diagrams

$$\begin{tikzcd} S \wedge_S S \wedge_S E \arrow[r, "\lambda\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\lambda"'] & S \wedge_S E \arrow[d, "\lambda"] \\ S \wedge_S E \arrow[r, "\lambda"] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, "\lambda^{-1}"] \arrow[dr, "\mathrm{id}"'] & S \wedge_S E \arrow[d, "\lambda"] \\ & E \end{tikzcd}$$

The following theorem summarizes the basic properties of $\mathcal{M}_S$.

Theorem 2.2.2 ([EKMM97], II.1.3, II.1.4, II.1.6)**.**

1. *The functor $S \wedge_{\mathcal{L}} - : \mathcal{S}[\mathbb{L}] \to \mathcal{M}_S$ is left adjoint to the functor*

$$F_{\mathcal{L}}(S, -) : \mathcal{M}_S \to \mathcal{S}[\mathbb{L}]$$

   and right adjoint to the inclusion $\mathcal{M}_S \to \mathcal{S}[\mathbb{L}]$.

2. *The category of S-modules is complete and cocomplete, with colimits created in $\mathcal{S}[\mathbb{L}]$ and limits created by applying $S \wedge_{\mathcal{L}} -$ to limits created in $\mathcal{S}[\mathbb{L}]$.*

3. *The category $\mathcal{M}_S$ is symmetric monoidal under $\wedge_S$ and*

$$\mathcal{M}_S(E \wedge_S F, P) \cong \mathcal{M}_S(E, F_S(F, P))$$

for S-modules E, F, and P.

2.3 S-algebras and their modules

Definition 2.3.1. An *S-algebra* is a monoid in $\mathcal{M}_S$ and a *commutative S-algebra* is a commutative monoid in $\mathcal{M}_S$ and we denote the category of S-algebras by $\mathcal{A}_S$. If R is an S-algebra or commutative S-algebra, then a *left (resp. right) R-module* is a left (resp. right) R-object in $\mathcal{M}_S$. Write $\mathcal{M}_R$ for the category of R-modules.

Remark 2.3.2. These notions replace the older notions of A_∞ and E_∞ ring spectra. In fact, an S algebra is just an A_∞ ring spectra which is also an S-module. Similarly in the commutative case. Of course, S is a commutative S-algebra.

Theorem 2.3.3 ([EKMM97], III.1)**.** *Let R be an S-algebra.*

1. *The functors $R \wedge_S -$ and $F_S(R, -)$ from $\mathcal{M}_S$ to $\mathcal{M}_R$ are the free and cofree functors from S-modules to R-modules. They are left and right adjoint to the forgetful functor.*

2. *The category of R-modules is complete and cocomplete. Limits and colimits are created in $\mathcal{M}_S$.*

Definition 2.3.4. The free S-module generated by a spectrum X is

$$\mathbb{F}_S X := S \wedge_{\mathcal{L}} \mathbb{L}X$$

and the free R-module generated by a spectrum X is $\mathbb{F}_R X := R \wedge_S \mathbb{F}_S X$. The m-sphere S-module is the free S-module generated by the m-sphere spectrum, i.e.

$$S_S^m := S \wedge_{\mathcal{L}} \mathbb{L}S^m$$

The m-sphere R-module is $S_R^m := R \wedge_S S_S^m$.

Warning 2.3.5. The term "free" is somewhat inaccurate, since $\mathbb{F}$ is not left adjoint to the forgetful functor. It is only left adjoint to a functor that is naturally equivalent to the forgetful functor. In detail, the functor $\mathbb{F} : \mathcal{S} \to \mathcal{M}_R$ is left ajdoint to the functor that sends an R-module E to the spectrum $F_{\mathcal{L}}(S, E)$ and there is a natural map of R-modules $\mathbb{F}E \to E$ whose adjoint $E \to F_{\mathcal{L}}(S, E)$ is a weak equivalence of spectra. This is enough to ensure that

$$\pi_m(E) := h\mathcal{S}(S^n, E) \cong h\mathcal{M}_S(S^m_S, E) \cong h\mathcal{M}_R(S^m_R, E).$$

Note that for X a wedge of sphere spectra, $\pi_*(\mathbb{F}_R X)$ is the free $\pi_*(R)$-module with a generator of degree m for each m-sphere.

2.4 Model structures

All of the categories described above admit topological closed model structures. In all cases, the weak equivalences are created in the category of spectra in the sense that a map is a weak equivalence if it is a weak equivalence of the underlying spectra.

Theorem 2.4.1 ([EKMM97], VII.4.6)**.** *The category $\mathcal{M}_S$ is a topological model category with weak equivalences created in $\mathcal{S}$. Its q-fibrations are Serre fibrations of S-modules, i.e. maps $f : M \to N$ such that*

$$F(\mathrm{id}, f) : F_{\mathcal{L}}(S, M) \to F_{\mathcal{L}}(S, N)$$

is a Serre fibration of spectra.

We write $\mathcal{D}_S$ and $\mathcal{D}_R$ for the derived categories of S- and R-modules obtained from $\mathcal{M}_S$ and $\mathcal{M}_R$ by inverting the weak equivalences. Ring objects in $\mathcal{D}_S$ are ring spectra and ring objects in $\mathcal{D}_R$ are R-ring spectra.

2.5 Various constructions

In this section, we collect the basic definitions of some commonly occurring constructions.

Homotopy fixed points

Given a space X with a G-action, the fixed point set X^G can be identified with the set of equivariant maps from a point into X, i.e.

$$X^G = F^G(\mathrm{pt}, X).$$

This construction, however, is badly behaved for use in homotopy theory: it need not preserve homotopy equivalences. To rectify this, we replace pt by a free contractible G-space EG and define

$$X^{hG} := F^G(EG, X),$$

the *homotopy fixed points* of X under G. This construction enjoys the following very useful property: given a G-equivariant map $X \to Y$ which is a homotopy equivalence, the induced map

$$X^{hG} \to Y^{hG}$$

is an equivalence. This construction generalizes to spectra, S-modules, etc. in the obvious way.

Inverting a homotopy element

Let R be a commutative S-algebra and let M be an R-module. An element a of the homotopy ring $\pi_*(R)$ gives rise to a self-map of R via the composition

$$R = S \wedge_S R \xrightarrow{a \wedge \mathrm{id}} R \wedge_S R \xrightarrow{\mu} R$$

where μ is multiplication of the S-algebra R. Let

$$R[1/a] := \mathrm{colim}(R \xrightarrow{a} R \xrightarrow{a} R \xrightarrow{a} \cdots).$$

Then $R[1/a]$ is an S-algebra with $(R[1/a])_* = R_*[a^{-1}]$. Let

$$M[1/a] := M \wedge_R R[1/a]$$

so $(M[1/a])_* = M_*[a^{-1}]$. We often write $a^{-1}M$ for $M[1/a]$.

Quotients of R-modules

For a commutative S-algebra R, an R-module M, and an element $a \in \pi_*(R)$, define R/a by the cofiber sequence

$$R \xrightarrow{a} R \to R/a$$

in $\mathcal{D}_R$. Let that $M/a := M \wedge_R R/a$. For a sequence $\mathbf{a} = (a_1, a_2, \ldots)$ of elements $\pi_*(R)$, let

$$M/\mathbf{a} := M \wedge_R \left(\bigwedge_i R/a_i \right)$$

where the product is taken over R. If $\mathbf{a}$ is regular on $\pi_*(M)$, we call $M/\mathbf{a}$ a *regular quotient* of M. If $\pi_i(R) = 0$ for i odd and $\mathbf{a}$ and $\mathbf{a}'$ are regular sequences generating the same ideal I in $\pi_*(R)$, then the regular quotients $M/\mathbf{a}$ and $M/\mathbf{a}'$ are equivalent R-modules (see [EKMM97], Cor. V.2.10). This justifies the notation M/I for $M/\mathbf{a}$. See [Str99], [Wüt05], and [BL01] for many details and properties of such quotients.

2.5.1 Forms of completion

We will be working with various forms of completion of spectra at ideals in their homotopy rings. In this section, we recall a few constructions and outline their respective properties and relationships. First, we recall completions in algebra. Then we recall various ways of mimicking these constructions in homotopy theory. The material in this section is standard.

Algebraic completions

In algebra and algebraic geometry, it is common to study the completion of a ring R at an ideal I. The completion of R at I is the inverse limit

$$R^\wedge_I := \varprojlim_s R/I^s$$

over the inverse system

$$\cdots \supseteq R/I^s \supseteq \cdots \supseteq R/I^2 \supseteq R/I.$$

Similarly, for an R-module M, one can define the completion of M at I as the inverse limit

$$M_I^\wedge := \varprojlim_s M/I^s M.$$

See Chapter 10 of [AM69] or Chapter 7 of [Eis95] for thorough treatments of algebraic completion.

Examples 2.5.2.

1. If $R = \mathbb{Z}$ and $I = (p)$, then the completion is the ring of p-adic integers, $\widehat{\mathbb{Z}}_p$.
2. If $R = S[x_1, \ldots, x_n]$ is a polynomial ring and $I = (x_1, \ldots, x_n)$, then the completion is the power series ring $R_I^\wedge = S[[x_1, \ldots, x_n]]$.

Homotopy I-completion

Let R be a commutative S-algebra and let M be an R-module. Let $a \in \pi_*(R)$. As in [GM95], define the *Koszul spectrum* $K_R(a)$ by the fiber sequence

$$K_R(a) \to R \to R[1/a].$$

Since $R[1/a]$ is an R-module and the map $R \to R[1/a]$ is an R-module map, $K_R(a)$ is an R-module. For a finite sequence of elements $a_1, \ldots, a_k \in \pi_*(R)$, define

$$K_R(a_1, \ldots, a_k) := K_R(a_1) \wedge_R \cdots \wedge_R K_R(a_k)$$

Up to equivalence, $K_R(a_1, \ldots, a_k)$ only depends on the radical of the ideal I generated by $a_1, \ldots, a_k$, so we write $K_R(I)$ for $K_R(a_1, \ldots, a_k)$. For an R-module M, we define the completion

$$M_I^\wedge := F_R(K_R(I), M) \tag{2.5.1}$$

where the right side is R-module maps from $K_R(I)$ to M. Notice that $M_I^\wedge$ is again an R-module and comes equipped with an R-module map $M \to M_I^\wedge$.

Bousfield Localization

We recall the basic definitions necessary for Bousfield localization. We use the modern formulations for S- and R-algebras, as in [EKMM97], though these are just reformulations of the original ideas of Bousfield [Bou79]. Let R be a commutative S-algebra and fix an S-module E. An R-module A is *E-acyclic* if $E \wedge_R A \simeq \mathrm{pt}$ and a map is an *E-equivalence* if its cofiber is E-acyclic. An R-module Y is *E-local* if $F_R(A, Y) \simeq \mathrm{pt}$ for all E-acyclics A. A map $X \to L_E^R X$ is a *Bousfield localization of X with respect to E* if it is an E-equivalence and $L_E^R X$ is E-local. The superscript R may be omitted when clear from the context. Of course, these definitions make sense when $R = S$.

By Theorem VIII.2.1 of [EKMM97], Bousfield localization preserves S- and R-algebra structures. This will play an important role in Chapter 4 and the relevant theorem is cited there as Theorem 4.1.1. The following lemma relates Bousfield localization over different S-algebras.

Lemma 2.5.3 ([May96], XXIII.6.5; [BJ02], Lemma 4.3)**.** *Let R be a commutative S-algebra and let A be an R-algebra. Let E be an R-module and let M be an A-module. The $(A \wedge_R E)$-localization map*

$$M \to L^A_{A \wedge_R E} M$$

in the category of A-modules is a E-localization in the category of R-modules; hence there is a weak equivalence

$$L^A_{A \wedge_R E} M \simeq L^R_E M$$

of R-modules.

Properties

The following theorem relates homotopy I-completion and Bousfield localization.

Theorem 2.5.4 (Greenlees-May, [GM95] Thm 4.2)**.** *Let R be a S-algebra and let M be an R-module. Let I be a finitely generated ideal of $\pi_*(R)$. The map*

$$M \to M_I^\wedge$$

is Bousfield localization in the category of R-modules with respect to the R-module $K(I)$.

Example 2.5.5 ($\widehat{E(n)}$)**.** Consider the $K(n)$-localization $\widehat{E(n)} := L^S_{K(n)}E(n)$ of Johnson-Wilson theory $E(n)$, as in the Introduction. There are several equivalent ways to describe $\widehat{E(n)}$ in terms of localization or completion. Let $I_n = (v_0, \ldots, v_{n-1}) \subseteq E(n)_*$ be the standard ideal. In [BW89], $\widehat{E(n)}$ is constructed as a spectrum representing an algebraic completion of Johnson-Wilson theory. It is defined on finite spectra by

$$\widehat{E(n)}^*(X) = \varprojlim_s (E(n)^*/I_n^s) \otimes_{E(n)^*} E(n)^*(X)$$

and since $\widehat{E(n)}^* = (E(n)^*)^\wedge_{I_n}$ is linearly compact with respect to the I_n-adic topology, this determines the spectrum $\widehat{E(n)}$. Later, Baker [Bak91] constructed $\widehat{E(n)}$ as the I_n-adic completion of $E(n)$ on the level of spectra. He constructed a tower

$$\cdots \to E(n)/I_n^s \to \cdots \to E(n)/I_n^2 \to E(n)/I_n \tag{2.5.2}$$

of $E(n)$-module spectra whose homotopy inverse limit gives the $\widehat{E(n)}$ of [BW89]. Here $E(n)/I_n^s$ is the quotient $E(n)$-module spectrum obtained by killing all monomials $v_0^{i_0}v_1^{i_1}\cdots v_{n-1}^{i_{n-1}}$ with $\sum_{j=0}^{n-1} i_j = s$. The canonical map

$$E(n) \to \operatorname{holim}_s E(n)/I_n^s$$

is Bousfield localization with respect to $K(n)$ by Cor. 6.13 of [Wüt05], so the homotopy limit of this tower is equivalent to $L^S_{K(n)}E(n) = \widehat{E(n)}$ as we considered it in the introduction. Now, by Lemma 2.5.3, there is a weak equivalence

$$\widehat{E(n)} = L^S_{K(n)}E(n) \simeq L^{MU}_{MU\wedge_S K(n)}E(n)$$

of S-modules. In Prop. 4.2 of [BJ02], Baker and Jeanneret show that the natural map $E(n) \to \operatorname{holim}_s E(n)/I_n^s$ is a morphism of MU-algebras. In proving their proposition, they show that $MU \wedge_S K(n)$ is Bousfield equivalent to $v_n^{-1}MU/I_n$, so there is a weak equivalence

$$\widehat{E(n)} \simeq L^{MU}_{v_n^{-1}MU/I_n}E(n) \tag{2.5.3}$$

of S-modules. Baker and Jeanneret go on to explain that Theorem 6.4 of [BL01] shows that (2.5.3) is in fact an equivalence of MU-modules (the tower (2.5.2) can be constructed as one of MU-modules and its limit is shown to be $L^{MU}_{v_n^{-1}MU/I_n}E(n)$).

Each of these various descriptions of $\widehat{E(n)}$ has a slightly different set of obvious properties, so it comes in handy to have them all at our disposal. Since

$$\operatorname{holim}_s E(n)/I_n^s \simeq L^S_{K(n)}E(n) \simeq L^{MU}_{v_n^{-1}MU/I_n}E(n)$$

are (weakly) equivalent as MU-modules, we feel justified in using the notation of $\widehat{E(n)}$ for any of them.

Example 2.5.6 ($v_n^{-1}\widehat{MU}$)**.** Consider MU as a commutative MU-algebra. Since Bousfield localization preserves MU-algebra structures, we obtain a commutative MU-algebra

$$v_n^{-1}\widehat{MU} := L^{MU}_{v_n^{-1}MU/I_n}MU$$

as the Bousfield localization with respect to $v_n^{-1}MU/I_n$ in the category of MU-modules. By the discussion in the previous example, we have weak equivalences of S-modules

$$v_n^{-1}\widehat{MU} = L^{MU}_{v_n^{-1}MU/I_n}MU \simeq L^{MU}_{MU\wedge_S K(n)}MU \simeq L^S_{K(n)}MU$$

As explained in [GM95], the homotopy ring of $v_n^{-1}\widehat{MU}$ is

$$\pi_*(v_n^{-1}\widehat{MU}) = \pi_*(MU)[v_n^{-1}]^{\wedge}_{I_n}.$$

(See also Section 1.1 of [HS99].) Note that applying Bousfield localization with respect to $MU \wedge_S K(n)$ to the MU-algebra map $MU \to E(n)$ gives a MU-algebra map $v_n^{-1}\widehat{MU} \to \widehat{E(n)}$.

2.6 Real cobordism

In this section, we briefly review the definition of a Real spectrum and give examples of Real spectra relevant to this thesis.

A *Real spectrum* $\mathbb{E}$ is a $\mathbb{Z}/2$-equivariant spectrum indexed on the real representation ring $RO(\mathbb{Z}/2)$, i.e. it is a collection of $\mathbb{Z}/2$-spaces $\mathbb{E}_V$ for each $V \in RO(\mathbb{Z}/2)$ such that for every $W \in RO(\mathbb{Z}/2)$ we have compatible maps $\Sigma^W \mathbb{E}(V) \to \mathbb{E}(V \oplus W)$, whose adjoints are homeomorphisms $\mathbb{E}(V) \xrightarrow{\cong} \Omega^W \mathbb{E}(V \oplus W)$. Here Σ^W means smashing with S^W, the one-point compactification of W, and Ω^W means maps out of S^W. Homotopy classes of maps from a Real (i.e. $\mathbb{Z}/2$-) space into a Real spectrum defines a $RO(\mathbb{Z}/2)$-graded cohomology theory.

Since $RO(\mathbb{Z}/2) = \mathbb{Z} \oplus \alpha\mathbb{Z}$, a Real spectrum is secretly a bigraded spectrum $\mathbb{E}_{a,b}$ such that each $\mathbb{E}_{a,b}$ has a $\mathbb{Z}/2$-action and there are maps

$$\Omega \mathbb{E}_{a,b} \to \mathbb{E}_{a-1,b}$$

$$\Omega^\alpha \mathbb{E}_{a,b} \to \mathbb{E}_{a,b-1}$$

which are compatible with the $\mathbb{Z}/2$ actions and whose adjoints are homeomorphisms

$$\mathbb{E}_{a,b} \to \Sigma \mathbb{E}_{a-1,b}$$

$$\mathbb{E}_{a,b} \to \Sigma^\alpha \mathbb{E}_{a,b-1}$$

Note also that for a fixed value of b, the collection of spaces and maps $\{\mathbb{E}_{a,b}\}_{a \in \mathbb{Z}}$ forms a naïve $\mathbb{Z}/2$-spectrum. For $b = 0$, this is the underlying naïve $\mathbb{Z}/2$-spectrum associated to the trivial representation that we referred to in the introduction; it is often denoted $\mathbb{E}_{\{e\}}$.

Remark 2.6.1. We will use blackboard bold to indicate a Real spectrum $\mathbb{E}$. We will use the subscript $\star$ to denote the bigraded coefficients, so that $\mathbb{E}_\star(X) = \mathbb{E}_{*,*}(X)$ for a Real space X.

Example 2.6.2 (Real cobordism)**.** Let $MU(k)$ be the Thom space of the universal k-plane bundle over $BU(k)$. There is an action of $\mathbb{Z}/2$ on $MU(k)$ given by complex conjugation, constructed as follows. Recall that $BU(k)$ is the Grassmannian of k-planes in $\mathbb{C}^\infty$ and define a map

$$BU(k) \to BU(k)$$

by sending $V \mapsto \bar{V}$, where $\bar{V} = \{w \mid \bar{w} \in V\}$. This map is covered by a map on the total space level $EU(k) \to EU(k)$ by $v \mapsto \bar{v}$ on each fiber, so we obtain a map on $MU(k)$. Since conjugating twice gives back the original element, the map $c : MU(k) \to MU(k)$ squares to the identity and hence defines an action of $\mathbb{Z}/2$ on $MU(k)$. In order to define a Real spectrum $\mathbb{MR}$ associated to this action on the prespectrum $\{MU(k)\}_{k\geq 0}$, we need structure maps in the 'trivial' and the 'α' directions. Consider the diagram

$$\begin{tikzcd} (\mathbb{R} \oplus \alpha\mathbb{R}) \oplus EU(k) \arrow[r] \arrow[d] & EU(k+1) \arrow[d] \\ BU(k) \arrow[r, "f"] & BU(k+1) \end{tikzcd}$$

where we identify $\mathbb{C}$ with $\mathbb{R} \oplus \alpha\mathbb{R}$, and f is the classifying map for the bundle on the left. Taking Thom spaces gives a map

$$S^{1+\alpha} \wedge MU(k) = \Sigma^{1+\alpha} MU(k) \to MU(k+1)$$

which is compatible with the $\mathbb{Z}/2$-action by construction. We can then define the zeroth space of the Real spectrum $\mathbb{MR}$ by

$$\mathbb{MR}_{0,0} := \operatorname{colim}_k \Omega^{k(1+\alpha)} MU(k)$$

and the rest of the spaces by

$$\mathbb{MR}_{a,b} := \Omega^{-a-b\alpha} \operatorname{colim}_k \Omega^{k(1+\alpha)} MU(k)$$

This definition makes sense because eventually k will be larger than both a and b. Note that any naïve $\mathbb{Z}/2$-spectrum can be made into a Real spectrum in this way.

Example 2.6.3 (The Real Brown-Peterson spectrum)**.** In [Ara79b], Araki used a Quillen idempotent $\mathbb{MR} \to \mathbb{MR}$ to construct a Real version $\mathbb{BPR}$ of the Brown-Peterson spectrum BP. His construction is analogous to the construction of BP from MU. As explained in [HK01], this gives a splitting

$$\mathbb{MR} = \bigvee_{m_i} \Sigma^{m_i(1+\alpha)} \mathbb{BPR}$$

of Real $\mathbb{MR}$-module spectra, where the m_i range over the dimensions of additive free generators of a free symmetric algebra on generators of dimension $k \neq 2^i - 1$. These m_i are the same as those that give the classical splitting

$$MU = \bigvee_{m_i} \Sigma^{2m_i} BP$$

of MU-module spectra.

Example 2.6.4 (Spectra derived from Real cobordism)**.** Recall that $\pi_*(MU) = MU_* = \mathbb{Z}[x_1, x_2, \ldots]$, where $|x_k| = 2k$, and that many interesting spectra may be constructed by killing a regular sequence of elements MU_* and (or) localizing at elements of MU_*. If E is such a spectrum, we may use the methods of Section 3 of [HK01] to construct a Real spectrum (in fact an $\mathbb{MR}$-module) $\mathbb{ER}$ with the property that the spectrum $\{\mathbb{ER}_{a,0}\}_{a\in\mathbb{Z}}$ is E. In [HK01], Prop. 2.7, it is shown that the natural map $\mathbb{MR}_\star \to MU_*$ splits by a map of rings $MU_* \to \mathbb{MR}_\star$ that sends the generator x_k of degree $2k$ to an element of bidegree (k, k).[2] Thus we may construct $\mathbb{ER}$ by killing off and localizing the lifts of the elements of MU_* used to define E. In this way, we obtain Real versions $\mathbb{KR}(n), \mathbb{ER}(n)$, etc., of the classical spectra $K(n), E(n)$, etc. This method also gives an alternate construction of $\mathbb{BPR}$.

In the classical case, the regularity of the sequence of elements in MU_* used to construct a derived spectrum of MU allows one to easily calculate the coefficients of such a spectrum. In the Real world, the lifts of these elements do not necessarily form a regular sequence in $\mathbb{MR}_\star$, so the calculation of the coefficients $\mathbb{BPR}_\star, \mathbb{KR}(n)_\star$, and $\mathbb{ER}(n)_\star$ is not immediate. However, Real spectra often satisfy a completion theorem that facilitates the computation of their coefficients. The Real spectra $\mathbb{MR}, \mathbb{BPR}, \mathbb{KR}(n)$, and $\mathbb{ER}(n)$ all satisfy completion theorems [HK01]. This fact is used in [Hu01] and [HK01] to compute $\mathbb{BPR}_\star$.

2.7 The main players

[2]In fact, this is a ring isomorphism from MU_* onto the diagonal subring of $\mathbb{MR}_\star$, i.e. the subring of all elements in bidigrees (k, k).

We are now ready to define the objects of central study in this thesis. For a Real spectrum $\mathbb{E}$, we noted that for a fixed b, the collection of spaces and corresponding maps $\{\mathbb{E}_{a,b}\}_{a\in\mathbb{Z}}$ forms a naïve $\mathbb{Z}/2$-spectrum, that is, an ordinary spectrum in which each of the spaces is equipped with a $\mathbb{Z}/2$-action and the structure maps are equivariant. Fixing $b = 0$ and applying this to the Real versions of derived spectra of MU described in Example 2.6.4 gives naïve $\mathbb{Z}/2$-spectra $MU, BP, E(n)$, etc. which are the classical $MU, BP, E(n)$ if we forget the $\mathbb{Z}/2$-action. We can now take homotopy fixed points to obtain spectra $MU^{h\mathbb{Z}/2}, BP^{h\mathbb{Z}/2}, E(n)^{h\mathbb{Z}/2}$, etc.

The central object of study in this thesis is the spectrum

$$ER(n) := E(n)^{h\mathbb{Z}/2}.$$

Kitchloo and Wilson ([KW06], [KW]) refer to this as real Johnson-Wilson theory. Note that this is not a Real spectrum in the sense of the previous section; it is only real in the sense that it is derived from a Real spectrum by taking homotopy fixed points of an underlying naïve $\mathbb{Z}/2$-spectrum, just as real K-theory is derived from Atiyah's Real K-theory. We shall rely on our notation of bold for Real spectra to make this distinction clear.

Chapter 3

Calculating coefficients

In this chapter, we use published calculations of the coefficients of various Real spectra to compute the coefficients of various fixed points of underlying naïve $\mathbb{Z}/2$-spectra.

3.1 $\mathbb{BPR}_\star$ and $\mathbb{ER}(n)_\star$

As described in Section 2.6, there is a Real spectrum $\mathbb{MR}$ associated with complex cobordism and Real spectra $\mathbb{BPR}$ and $\mathbb{ER}(n)$ associated with Brown-Peterson theory and Johnson-Wilson theory. In various papers, Hu and Kriz have calculated the coefficients of $\mathbb{BPR}$ using the Borel spectral sequence. In [KW06], this calculation is modified to give the coefficients of $\mathbb{ER}(n)$. We recall these here.

Theorem 3.1.1 (Hu-Kriz,[HK01])**.**

$$\mathbb{BPR}_\star = \mathbb{Z}_{(2)}[v_k(l), a \mid k \geq 0, l \in \mathbb{Z}]/I$$

where I is the ideal generated by the relations

$$v_0(0) = 2, \quad a^{2^{k+1}-1} v_k(l) = 0, \quad \text{and for } k \leq m, \quad v_m(j) v_k(2^{m-k} l) = v_m(j+l) v_k(0)$$

The bidegrees of the elements are

$$|a| = -\alpha, \quad |v_k(l)| = (2^k - 1)(1 + \alpha) + l 2^{k+1}(\alpha - 1)$$

Here we use the notation $(k, l) = k + l\alpha$.

Remark 3.1.2. Note that BP_* sits inside $\mathbb{BPR}_\star$ as the $v_k(0)$.

Theorem 3.1.3 (Kitchloo-Wilson,[KW06])**.**

$$\mathbb{ER}(n)_\star = \mathbb{Z}_{(2)}[v_k(l), a, v_n^{\pm}, \sigma^{\pm 2^{n+1}} \mid 0 \leq k < n, l \in \mathbb{Z}]/I$$

where $v_k(l) = v_k \sigma^{l2^{k+1}}$ *and* I *is the ideal generated by the relations*

$$v_{0,0} = 2, \quad a^{2^{k+1}-1} v_k(l) = 0, \quad \textit{and for } k \leq m, \quad v_m(j) v_k(2^{m-k} l) = v_m(j+l) v_k(0).$$

The bidegrees of the elements are the same as above, with $|\sigma| = \alpha - 1$ *and* $|v_n| = (2^n - 1)(1 + \alpha)$.

3.2 $ER(n)_*$

Before we calculate the homotopy ring of $ER(n)$, we recall a few key elements of the homotopy rings of related spectra. The diagonal elements

$$v_k(0) \in \mathbb{ER}(n)_{2^k-1, 2^k-1}$$

are invariant under the $\mathbb{Z}/2$-action of complex conjugation and hence give rise to elements in $\mathbb{ER}(n)^{h\mathbb{Z}/2}_{2^k-1,2^k-1}$, which we also call $v_k(0)$. Multiplying each $v_k(0)$ by $\sigma^{-2^k+1} \in \mathbb{ER}(n)_{1-2^k, 2^k-1}$ yields the element

$$v_k = v_k(0) \sigma^{-2^k+1} \in \mathbb{ER}(n)_{2(2^k-1), 0} = E(n)_{2(2^k-1)}.$$

Now, for $v_k \in E(n)_{2(2^k-1)}$, put

$$\hat{v}_k := v_k v_n^{-(2^k-1)(2^n-1)}$$

for $0 \leq k < n$. To facilitate the calculations to come, we reindex the homotopy of $E(n)$ using the $\hat{v}_k$ for $k < n$, rather than v_k. Of course, these generators are just as good as the original generators, since v_n is a unit. From now on, we put

$$E(n)_* = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm 1}].$$

Let $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. In [KW06], Kitchloo and Wilson identify an invertible element

$$y(n) := v_n^{2^n-1}\sigma^{-(2^{n-1}-1)2^{n+1}} \in \mathbb{ER}(n)^{h\mathbb{Z}/2}_{\lambda(n),-1}$$

which then give rise to elements

$$\hat{v}_k(0) := v_k(0) \cdot y(n)^{1-2^k} \in \mathbb{ER}(n)^{h\mathbb{Z}/2}_{(1-\lambda(n))(2^k-1),0} = ER(n)_{(1-\lambda(n))(2^k-1)}.$$

These map to $\hat{v}_k$ under the natural map

$$ER(n)_* \to E(n)_*$$

induced by inclusion of fixed points.

The element $\sigma^{2^{n+1}}$ is a unit and the $\mathbb{Z}/2$-action on it is trivial (since there are an even number of σ's), so it gives rise to an element in $\mathbb{ER}(n)^{h\mathbb{Z}/2}_{*,*}$, which we also denote by $\sigma^{2^{n+1}}$. Hence

$$v_n^{2^{n+1}}\sigma^{-(2^n-1)2^{n+1}} = (v_n\sigma^{-(2^n-1)})^{2^{n+1}} \in \mathbb{ER}(n)^{h\mathbb{Z}/2}_{2^{n+2}(2^n-1),0} = ER(n)_{2^{n+2}(2^n-1)} \tag{3.2.1}$$

is also a unit. This maps to $v_n^{2^{n+1}} \in E(n)_*$ and is the periodicity element for $ER(n)$.

In [KW06], Kitchloo and Wilson use the element $y(n)$ to construct a fibration of spectra

$$\Sigma^{\lambda(n)}\mathbb{ER}(n)^{h\mathbb{Z}/2}_V \xrightarrow{x(n)} \mathbb{ER}(n)^{h\mathbb{Z}/2}_V \xrightarrow{\iota} \mathbb{ER}(n)_V \tag{3.2.2}$$

where $x(n) = a \cdot y(n)$ and the subscript V indicates the spectrum obtained by fixing a $V \in RO(\mathbb{Z}/2)$. Fixing V to be the trivial representation, we obtain a fibration of spectra

$$\Sigma^{\lambda(n)}ER(n) \xrightarrow{x(n)} ER(n) \xrightarrow{\iota} E(n)$$

for $x(n) \in ER(n)_{\lambda(n)}$. This fibration gives rise to an exact couple

$$\begin{tikzcd} ER(n)^*(X) \arrow[rr, "x(n)"] & & ER(n)^*(X) \arrow[dl, "\iota"] \\ & E(n)^*(X) \arrow[ul, "\delta"] & \end{tikzcd}$$

and hence a Bockstein spectral sequence measuring $x(n)$-torsion in $ER(n)$ (see [KW], Thm. 4.2). The degree of d_r is $r\lambda(n) + 1$, and $E_{2^{n+1}-1} = 0$. Moreover, the spectral sequence measures $x(n)$-torsion in the following sense. Filter $ER(n)_*$ by F_j, the kernel of multiplication by $x(n)^j$, so that

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{2^{n+1}-1} = ER(n)_*.$$

Let $M = ER(n)_*/x(n)ER(n)_*$ and filter M by the image of F_j. Then

$$M/M_{r-1} \hookrightarrow E_r \quad \text{and} \quad M_r/M_{r-1} \cong \operatorname{im}(d_r).$$

Since $v_0 = 2$ is in the image of d_1, we have that E_r is entirely 2-torsion for $r \geq 2$. Note that the image of $ER(n)_* \to E(n)_*$ is the set of elements that are targets of differentials.

Now we will use this spectral sequence, together with the calculation of $\mathbb{ER}(n)_\star$, to compute $ER(n)_*$.

Theorem 3.2.1. *Let $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$ and set $x := x(n)$. We have*

$$ER(n)_* = \mathbb{Z}_{(2)}[x, \hat{v}_k(l), v_n^{\pm 2^{n+1}} \mid 0 \leq k < n, l \in \mathbb{Z}]/J$$

where J is the ideal generated by the relations

$$\hat{v}_0(0) = 2, \quad x^{2^{k+1}-1}\hat{v}_k(l) = 0, \quad \textit{and for } k \leq m,\ \hat{v}_m(l)\hat{v}_k(2^{m-k}s) = \hat{v}_m(l+s)\hat{v}_k(0).$$

The degrees of the generators are

$$|x| = \lambda(n) = 2^{2n+1} - 2^{n+2} + 1, \qquad |v_n^{2^{n+1}}| = 2^{n+2}(2^n - 1)^2,$$

$$|\hat{v}_k(l)| = 2(2^k - 1) + l2^{k+2}(2^n - 1)^2 - 2(2^k - 1)(2^n - 1)^2.$$

Remark 3.2.2. Though the degrees of these elements seem quite complicated, they are more easily remembered using the fact that

$$\hat{v}_k(l) \mapsto v_k v_n^{-(2^k-1)(2^n-1)+l2^{k+1}(2^n-1)}$$

under the natural map $ER(n)_* \to E(n)_*$ (which of course preserves the grading).

Remark 3.2.3. Note that we recover the homotopy of $KO_{(2)}$ as

$$\pi_*(ER(1)) = \mathbb{Z}_{(2)}[\hat{v}_0(1), \eta, v_1^{\pm 4}]/(2\eta = \eta^3 = \eta\hat{v}_0(1) = 0).$$

Proof of Theorem 3.2.1. We calculate the Bockstein spectral sequence for a point, using the identifications $E(n)_* = E(n)^{-*}$ and $ER(n)_* = ER(n)^{-*}$. The E_2-page is

$$E_2 = E(n)_* = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n^{\pm}].$$

Though there are potentially 2^{n+1} differentials, we will see that in fact only $n+1$ of them are nonzero, namely those of the form $d_{2^{j+1}-1}$ for $j = 0, \ldots, n$.

First, we show that for degree reasons, differentials can only originate on powers of v_n. This can also be seen using Kitchloo and Wilson's computation of $\mathbb{ER}(n)_\star$: for $k < n$, the $\hat{v}_k$ must be permanent cycles because they correspond to $a^{2^{k+1}-1}$-torsion elements in $\mathbb{ER}(n)_\star$. (They are therefore targets of differentials and hence permanent cycles.) However, we choose to include a short degree argument to demonstrate how much can be obtained from the Bockstein spectral sequence with very little input.

Write λ for $\lambda(n)$. We have

$$|\hat{v}_i| = (2^i - 1)(1 - \lambda) \text{ for } 0 \leq i < n$$

Consider differentials of the form $d_r(\hat{v}_i^m) = \hat{v}_j^k$. Such differentials must satisfy the degree equation

$$k|\hat{v}_j| = |\hat{v}_j^k| = |d_r| + |\hat{v}_i^m| = |d_r| + m|\hat{v}_i|. \tag{3.2.3}$$

If $0 \leq i < n$ and $0 \leq j < n$, the degree equation (3.2.3) becomes

$$k(2^j - 1)(1 - \lambda) = -r\lambda - 1 + m(2^i - 1)(1 - \lambda). \tag{3.2.4}$$

Since $1 - \lambda$ is divisible by 2^n, reducing modulo 2^n gives

$$r \equiv -1 \mod 2^n.$$

However, we know that $E_{2^{n+1}} = 0$, so the only possibilities are $r = 2^n - 1$ or $r = 2^{n+1} - 1$, both of which are be ruled out by plugging in to equation (3.2.3). An easy generalization of this argument shows that there are no differentials of the form

$$d_r\left(\prod_{i=0}^{n-1} a_i \hat{v}_i^{k_i}\right) = \prod_{i=0}^{n-1} b_i v_i^{m_i}$$

for $0 \leq i < n$ and $0 \leq j < n$, where $a_i, b_i \in \mathbb{Z}_{(2)}$ if $r = 1$ and $a_i, b_i \in \mathbb{Z}/2$ if $r > 1$.

Now consider differentials of the form $d_r(v_n^{m(2^n-1)}) = \hat{v}_j^k$ for $i, j \neq n$. If $i = n$ and $0 \leq j < n$, equation (3.2.3) becomes

$$k(2^j - 1)(1 - \lambda) = -r\lambda - 1 + m(\lambda + 1)$$

and reducing modulo λ gives

$$k(2^j - 1) - m + 1 \equiv 0 \mod \lambda.$$

We also have

$$2m \equiv r + 1 \mod (\lambda - 1).$$

Simple calculations and the fact that $r < 2^{n+1}$ show that

$$r = 2m - 1 \quad \text{and} \quad m = k(2^j - 1) + 1.$$

Since $0 < r < 2^{n+1}$, we have

$$0 < m \leq 2^n \quad \text{and} \quad 0 < k \leq \frac{2^n - 1}{2^j - 1}.$$

This is as much as we can say by looking at degrees alone. Notice that if $k = 1$, we see a possible differential of the form $d_{2^{j+1}-1}(v_n^{2^j(2^n-1)}) = \hat{v}_j$. It turns out that this differential does occur, though we need to invoke extra information to see it.

We now show that $\hat{v}_k$ is $x^{2^{k+1}-1}$-torsion so that the differential

$$d_{2^{j+1}-1}(v_n^{2^j(2^n-1)}) = \hat{v}_j \tag{3.2.5}$$

does indeed occur. Recall the definition

$$x = x(n) = a \cdot y(n).$$

By Kitchloo and Wilson's computation of $\mathbb{ER}(n)_\star$, we have that

$$a^{2^{k+1}-1} v_k(l) = 0 \tag{3.2.6}$$

in $\mathbb{ER}(n)_\star$. Now, since

$$v_k = v_k(0)\sigma^{1-2^k} \in \mathbb{ER}(n)_{2(2^k-1),0} = E(n)_{2(2^k-1)}$$

we have

$$\hat{v}_k = \underbrace{v_k v_n^{-(2^k-1)(2^n-1)}}_{\in E(n)_*} = \underbrace{v_k(0)\sigma^{1-2^k} v_n^{-(2^k-1)(2^n-1)} \sigma^{(-2^n+1)(1-2^k)(2^n-1)}}_{\in \mathbb{ER}(n)_\star}$$

and so

$$\begin{aligned} x^{2^{k+1}-1}\hat{v}_k &= (ay(n))^{2^{k+1}-1}(v_k(0)\sigma^{1-2^k} v_n^{-(2^k-1)(2^n-1)}\sigma^{(-2^n+1)(1-2^k)(2^n-1)}) \\ &= \underbrace{a^{2^{k+1}-1}v_k(0)}_{=0 \text{ by } (3.2.6)} y(n)^{2^{k+1}-1} v_n^{-(2^k-1)(2^n-1)}\sigma^{(-2^n+1)(1-2^k)^2(2^n-1)} = 0 \end{aligned}$$

Hence $\hat{v}_k$ is $x^{2^{k+1}-1}$-torsion.

So, we have proved that the only possible differentials originate on powers of v_n and that we have the differential (3.2.5), assuming that $v_n^{2^j(2^n-1)}$ survives long enough to support this differential (we'll see that it does below.) Hence we see the $x^{2^{k+1}-1}$-torsion elements in $\hat{v}_k(0) \in ER(n)_*$ that map to the $\hat{v}_k \in E(n)_*$. Now we complete the description of the differentials.

In [KW] the first differential is realized as

$$d_1(z) = v_n^{1-2^n}(1-c)(z)$$

where c denotes the action induced by complex conjugation. This agrees with our computation with the usual convention that $v_0 = 2$ (recall that $\hat{v}_0 = v_0$). This also implies that $d_1(v_n^{m(2^n-1)}) = 0$ for m even. Moreover, if m is odd, say $m = 2s+1$, then

$$\begin{aligned} d_1(v_n^{m(2^n-1)}) &= d_1(v_n^{(2s+1)(2^n-1)}) \\ &= d_1(v_n^{2s(2^n-1)})v_n^{2^n-1} + c(v_n^{2s(2^n-1)})d_1(v_n^{2^n-1}) = v_n^{2s(2^n-1)}\hat{v}_0 \end{aligned}$$

so odd powers of $v_n^{2^n-1}$ do not survive past E_1. Note that the same computation shows that odd powers of v_n also die on the E_1 page.

If m is a power of 2, say $m = 2^{j+1}$, and $r < 2^{j+1}-1$, then the product formula gives

$$\begin{aligned} d_r(v_n^{m(2^n-1)}) &= d_r(v_n^{2^j(2^n-1)})v_n^{2^j(2^n-1)} + c(v_n^{2^j(2^n-1)})d_r(v_n^{2^j(2^n-1)}) \\ &= 2v_n^{2^j(2^n-1)}d_r(v_n^{2^j(2^n-1)}). \end{aligned}$$

But since d_1 kills $\hat{v}_0 = 2$, each E_r is a vector space over $\mathbb{Z}/2$ when $r > 1$. Therefore $d_r(v_n^{m(2^n-1)})$ is zero and $v_n^{2^j(2^n-1)}$ does indeed survive long enough for the differential (3.2.5) to hold.

Now suppose that m is even but not a power of 2. Let $m = 2^{l_1} + \cdots + 2^{l_s}$ be the binary expansion of m, with $l_1 > l_2 > \cdots > l_s > 0$. For brevity, write $\check{v}_n$ for $v_n^{2^n-1}$. We compute

$$\begin{aligned}
d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_1}+\cdots+2^{l_s}}) &= \underbrace{d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_1}})}_{=0 \text{ since } l_1>l_s}\check{v}_n^{2^{l_2}+\cdots+2^{l_s}} + \underbrace{c(\check{v}_n^{2^{l_1}})}_{=\check{v}_n^{2^{l_1}}} d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_2}+\cdots+2^{l_s}}) \\
&= \check{v}_n^{2^{l_1}}(\underbrace{d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_2}})}_{=0 \text{ since } l_2>l_s}\check{v}_n^{2^{l_3}+\cdots+2^{l_s}} + \underbrace{c(\check{v}_n^{2^{l_2}})}_{=\check{v}_n^{2^{l_2}}} d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_3}+\cdots+2^{l_s}})) \\
&\vdots \\
&= \check{v}_n^{2^{l_1}+\cdots+2^{l_{s-2}}}(\underbrace{d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_{s-1}}})}_{=0 \text{ since } l_{s-1}>l_s}\check{v}_n^{2^{l_s}} + \underbrace{c(\check{v}_n^{2^{l_{s-1}}})}_{=\check{v}_n^{2^{l_{s-1}}}} d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_s}})) \\
&= \check{v}_n^{2^{l_1}+\cdots+2^{l_{s-1}}}\hat{v}_{l_s}
\end{aligned}$$

and so $\check{v}_n^m$ dies on the $(2^{l_s+1}-1)^{th}$ page. This is of course assuming that all the terms involved survive to the $(2^{l_s+1}-1)^{th}$ page, which is easily checked by induction. The key point is that $d_{2^{l_s+1}-1}(\check{v}_n^{2^{l_t}})$ will always vanish if $t < s$, because this implies that $l_s < l_t$. Since 2^{l_s} is the maximal power of 2 dividing m, we have proved: If $m = 2^j m'$ and 2 does not divide m', then $\check{v}_n^m$ dies on the $(2^{j+1}-1)^{th}$ page.

Thus if $j > 0$, an arbitrary monomial $a \in E_{2^{j+1}-1}$ is of the form

$$a = \prod_{i=j}^{n} a_i \hat{v}_i^{k_i}$$

where $a_i \in \mathbb{Z}/2$ and where the maximal power of 2 dividing k_n is larger than 2^{j-1}. That is,

$$E_{2^{j+1}-1} = \mathbb{Z}/2[\hat{v}_j, \ldots, \hat{v}_{n-1}, v_n^{\pm 2^j(2^n-1)}].$$

Let 2^l be the maximal power of 2 dividing k_n, so $l \geq j$. Writing the binary expansion for k_n as $k_n' + 2^l$, our previous calculation shows that

$$d_{2^{j+1}-1}(\check{v}_n^{k_n}) = d_{2^{j+1}-1}(\check{v}_n^{k_n'+2^l}) = \begin{cases} \check{v}_n^{k_n'}\hat{v}_j & \text{if } l = j \\ 0 & \text{else} \end{cases}$$

This and the product rule give

$$\begin{aligned}
d_{2^{j+1}-1}(a) &= d_{2^{j+1}-1}\left(\prod_{i=j}^{n} a_i \hat{v}_i^{k_i}\right) = d_{2^{j+1}-1}\left(\check{v}_n^{k_n} \cdot \prod_{i=j}^{n-1} a_i \hat{v}_i^{k_i}\right) \\
&= d_{2^{j+1}-1}(\check{v}_n^{k_n}) \cdot \prod_{i=j}^{n-1} a_i \hat{v}_i^{k_i} + c(\check{v}_n^{k_n}) \cdot \underbrace{d_{2^{j+1}-1}\left(\prod_{i=j}^{n-1} a_i \hat{v}_i^{k_i}\right)}_{=0} \\
&= d_{2^{j+1}-1}(\check{v}_n^{k_n}) \cdot \prod_{i=j}^{n-1} a_i \hat{v}_i^{k_i} \\
&= \begin{cases} \hat{v}_j \check{v}_n^{k_n'} \cdot \prod_{i=j}^{n-1} a_i \hat{v}_i^{k_i} = \hat{v}_j \check{v}_n^{-2^j} a & \text{if } l = j \\ 0 & \text{else} \end{cases}
\end{aligned}$$

In particular, for $l \in \mathbb{Z}$, we have

$$d_{2^{k+1}-1}(\check{v}_n^{(2^k-1)+l2^{k+1}}) = d_{2^{k+1}-1}(\check{v}_n^{2^k-1})\check{v}_n^{l2^{k+1}} + c(\check{v}_n^{2^k-1})\underbrace{d_{2^{k+1}-1}(\check{v}_n^{l2^{k+1}})}_{=0} = \hat{v}_k \check{v}_n^{l2^{k+1}}$$

so we see the elements $\hat{v}_k(l) \in ER(n)_*$ arising as the elements that map to

$$\hat{v}_k \check{v}_n^{l2^{k+1}} = v_k v_n^{-(2^k-1)(2^n-1)+l2^{k+1}(2^n-1)} \in E(n)_*.$$

This concludes the analysis of the possible differentials.

It remains to put these pieces together to determine the ring structure on $ER(n)$. To this end, recall that we have seen that

$$\hat{v}_k(l) \mapsto \hat{v}_k \check{v}_n^{l2^{k+1}} = v_k v_n^{-(2^k-1)(2^n-1)+l2^{k+1}(2^n-1)}$$

under map $ER(n)_* \to E(n)_*$. Since this is a ring homomorphism and

$$\hat{v}_m \check{v}_n^{l2^{m+1}} \hat{v}_k \check{v}_n^{(2^{m-k}s)2^{k+1}} = \hat{v}_k \check{v}_n \hat{v}_m \check{v}_n^{(l+s)2^{k+1}}$$

we obtain the relation

$$\hat{v}_m(l)\hat{v}_k(2^{m-k}s) = \hat{v}_k(0)\hat{v}_m(l+s)$$

As explained above, the Bocksetein spectral sequence measures x-torsion in the sense that the elements in the target of the k^{th} differential are x^k-torsion. Hence we see the relation

$$x^{2^{k+1}-1}\hat{v}_k(l) = 0.$$

Since we identified the periodicity element $v_n^{2^{n+1}}$ in equation (3.2.1), this completes the proof. □

3.3 $v_n^{-1}BPR_*$ and $v_n^{-1}MUR_*$

Let $\mathbb{BPR}$ denote Real Brown-Peterson theory, whose coefficients we recalled at the beginning of this chapter. We will need to know the coefficients of the theory $v_n^{-1}BPR$, where BPR is obtained from $\mathbb{BPR}$ in the same way as $ER(n)$ is obtained from $\mathbb{ER}(n)$, as explained in Section 2.7.

First consider the Real $\mathbb{MR}$-module $v_n(0)^{-1}\mathbb{BPR}$ obtained by inverting $v_n(0)$ in $\mathbb{BPR}$:

$$v_n(0)^{-1}\mathbb{BPR} := \operatorname{colim}(\mathbb{BPR} \xrightarrow{v_n(0)} \Sigma^{(2^n-1)(1+\alpha)}\mathbb{BPR} \xrightarrow{v_n(0)} \Sigma^{2(2^n-1)(1+\alpha)}\mathbb{BPR} \xrightarrow{v_n(0)} \cdots).$$

As is explained in [KW06], any Real spectrum derived from $\mathbb{MR}$ with $v_n(0)$ inverted has a fibration similar to the one for $\mathbb{ER}(n)$ given in (3.2.2). The proof is the same as the one given in [KW06] for $\mathbb{ER}(n)$. In particular, for $v_n(0)^{-1}\mathbb{BPR}$ we have

$$\Sigma^{\lambda(n)}\mathbb{BPR}_V^{h\mathbb{Z}/2} \xrightarrow{x(n)} \mathbb{BPR}_V^{h\mathbb{Z}/2} \xrightarrow{\iota} \mathbb{BPR}_V$$

where $x(n)$ is defined in exactly the same way as it was for $\mathbb{ER}(n)$. Hence we obtain a fibration

$$\Sigma^{\lambda(n)}v_n^{-1}BPR \xrightarrow{x} v_n^{-1}BPR \to v_n^{-1}BP$$

where $v_n^{-1}BPR$ is the homotopy fixed points of the naïve $\mathbb{Z}/2$-spectrum associated to the trivial representation, i.e.

$$v_n^{-1}BPR := \{v_n(0)^{-1}\mathbb{BPR}_{a,0}^{h\mathbb{Z}/2}\}_{a\in\mathbb{Z}}$$

The goal of this section is to prove the following proposition.

Proposition 3.3.1.

$$(v_n^{-1}BPR)_* = ER(n)_*[\hat{v}_{n+1}, \hat{v}_{n+2}, \ldots],$$

where $|\hat{v}_k| = (2^k-1)(1-\lambda)$ *for* $k > n$.

Proof. Just as in the calculation of $ER(n)_*$, we may set up the Bockstein spectral sequence measuring x-torsion. Working with the hat generators again (except for the periodicity element v_n), the E_2-term is

$$v_n^{-1}BP_* = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{n-1}, v_n, \hat{v}_{n+1}, \ldots][v_n^{-1}].$$

To prove the proposition, we will show that after inverting v_n, the element x becomes nilpotent with $x^{2^{n+1}} = 0$. Hence there are only the $n+1$ differentials that occurred in the spectral sequence for $ER(n)$, proving the result.

Recall the definition

$$x = x(n) = ay(n) = av_n^{2^n-1}\sigma^{-(2^{n-1}-1)2^{n+1}} \in v_n(0)^{-1}\mathbb{BPR}^{h\mathbb{Z}/2}_{\lambda(n),0} = v_n^{-1}BPR_{\lambda(n)}$$

By Theorem 3.1.1, we have the relation

$$a^{2^{k+1}-1}v_k(l) = 0$$

in the ring $\mathbb{BPR}_\star$. Setting $k = n$ and $l = 0$ and using the fact that $v_n(0)$ is a unit in $(v_n(0)^{-1}\mathbb{BPR})_\star$, we see that

$$a^{2^{n+1}-1} = 0$$

and hence the same holds for x. □

In Example 2.6.3, we remarked that there is a splitting

$$\mathbb{MR} = \bigvee_{m_i} \Sigma^{m_i(1+\alpha)}\mathbb{BPR}$$

of Real $\mathbb{MR}$-module spectra. Thus there is an analogous splitting of $v_n(0)^{-1}\mathbb{MR}$ in terms of $v_n(0)^{-1}\mathbb{BPR}$ and hence there is an equivariant splitting of $v_n^{-1}MU$ in terms of $v_n^{-1}BP$. We obtain a Bockstein spectral sequence converging to the homotopy of

$$v_n^{-1}MUR := v_n^{-1}MU^{h\mathbb{Z}/2}$$

in exactly the same way as we did for $v_n^{-1}BPR$. The equivariant splitting of $v_n^{-1}MU$ in terms of $v_n^{-1}BP$ gives a map of Bockstein spectral sequences from that of $v_n^{-1}MUR$ to that of

$$\bigvee_{m_i} \Sigma^{m_i(1+\alpha)}v_n^{-1}BPR$$

Since this map is an isomorphism on E_2, we have proved the following proposition.

Proposition 3.3.2. *$v_n^{-1}MUR_*$ is a free $v_n^{-1}BPR_*$-module on generators m_i ranging over the dimensions of additive free generators of a free symmetric algebra on generators of dimension $k \neq 2^i - 1$.*

3.4 $(\widehat{E(n)}^{h\mathbb{Z}/2})_*$

Here we calculate the coefficients of the fixed points of a completion of Johnson-Wilson theory. Consider the ideal

$$I_n = (v_0, \ldots, v_{n-1}) \subseteq E(n)_*$$

and let

$$\widehat{E(n)} := L^S_{K(n)} E(n)$$

as in Example 2.5.5. It is well known that this $K(n)$-localization induces completion at I_n on the level of coefficients (see, for example, [BJ02],[BH94], or [HS99]). Therefore, we have

$$\widehat{E(n)}_* = (E(n)_*)^{\wedge}_{I_n} = \widehat{\mathbb{Z}}_2[[v_1, \ldots, v_{n-1}]][v_n^{\pm}],$$

where $\widehat{\mathbb{Z}}_2$ denotes the 2-adics (the completion of $\mathbb{Z}_{(2)}$ at the its maximal ideal).

We'll see below (Remark 4.1.5) that Bousfield localization is functorial in the sense that a map $A \to B$ gives rise to a map $A_E \to B_E$ in a way that respects composition. Applying this to the S-module map

$$E(n) \to E(n)$$

given by complex conjugation gives rise to a map

$$\widehat{E(n)} \to \widehat{E(n)}$$

that squares to the identity. This map makes $\widehat{E(n)}$ into a naïve $\mathbb{Z}/2$-spectrum. By the method explained in Example 2.6.2, we obtain a Real spectrum $\widehat{\mathbb{ER}(n)}$ with naïve $\mathbb{Z}/2$-spectrum $\widehat{E(n)}$. Because $v_n(0)$ is a unit, we also have a fibration resulting in a Bockstein spectral sequence measuring x-torsion in $(\widehat{E(n)}^{h\mathbb{Z}/2})_*$.

Theorem 3.4.1. *Let* $\hat{I}_n = (\hat{v}_k(l) \mid 0 \le k < n, l \in \mathbb{Z}) \subseteq ER(n)_*$. *Then*

$$(\widehat{E(n)}^{h\mathbb{Z}/2})_* = (ER(n)_*)^{\wedge}_{\hat{I}_n}$$

Proof. This is another calculation with the Bockstein spectral sequence. The differentials are the same as in previous calculations, since the natural map $E(n) \to \widehat{E(n)}$ is equivariant. The E_2-term of the Bockstein spectral sequence is

$$E_2 = \widehat{E(n)}_* = \widehat{\mathbb{Z}}_2[[v_1, \dots, v_{n-1}]][v_n^{\pm}]$$

and again, we replace the v_k with the $\hat{v}_k$ for $k < n$ in order to facilitate computations. We then have

$$E_2 = \widehat{\mathbb{Z}}_2[[\hat{v}_1, \dots, \hat{v}_{n-1}]][v_n^{\pm}]$$

with the differentials as in the proof of Theorem 3.2.1. So again, if $j > 0$, an arbitrary monomial $a \in E_{2^{j+1}-1}$ is of the form

$$a = \prod_{i=j}^{n} a_i \hat{v}_i^{k_i}$$

where $a_i \in \mathbb{Z}/2$ and where the maximal power of 2 dividing k_n is larger than 2^{j-1}. Hence,

$$E_{2^{j+1}-1} = \mathbb{Z}/2[[\hat{v}_j, \dots, \hat{v}_{n-1}]][v_n^{\pm 2^j(2^n-1)}]$$

and again we see the $x^{2^{k+1}-1}$-torsion elements $\hat{v}_k(l)$ mapping to

$$\hat{v}_k \check{v}_n^{l2^{k+1}} = v_k v_n^{-(2^k-1)(2^n-1)+l2^{k+1}(2^n-1)} \in \widehat{E(n)}_*,$$

but now these generate a power series ring, rather than a polynomial ring. Thus, we have

$$(\widehat{E(n)}^{h\mathbb{Z}/2})_* = \widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \le k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

where J is the ideal generated by the relations

$$\hat{v}_0(0) = 2, \quad x^{2^{k+1}-1}\hat{v}_k(l) = 0, \quad \text{and for } k \le m,\ \hat{v}_m(l)\hat{v}_k(2^{m-k}s) = \hat{v}_m(l+s)\hat{v}_k(0)$$

as in Theorem 3.2.1. To complete the proof, we need only show that this is the same as the ring $ER(n)_*$ completed at the ideal $\hat{I}_n$. This is the result of the lemma below. $\square$

Lemma 3.4.2. *Let $\hat{I}_n$ be as above. Then*

$$(ER(n)_*)^{\wedge}_{\hat{I}_n} \cong \widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

where J is the ideal generated by the relations in Theorem 3.2.1.

Proof. First, let

$$\check{I}_n = (\hat{v}_0(0), \ldots, \hat{v}_{n-1}(0)) \subseteq ER(n)_*.$$

Recall ([Eis95], Lemma 7.14) that two ideals A and B of a ring R generate the same topology (i.e. $R^{\wedge}_A \cong R^{\wedge}_B$) if for every j there is an i such that $A^i \subseteq B^j$ and for every k there is an l such that $B^l \subseteq A^k$. Since $\check{I}_n \subseteq \hat{I}_n$ and $\hat{I}_n^2 \subseteq \check{I}_n$, we have

$$(ER(n)_*)^{\wedge}_{\hat{I}_n} \cong (ER(n)_*)^{\wedge}_{\check{I}_n}$$

so it suffices to prove that the latter ring is the one listed in the statement of the lemma.

Now, let $\hat{v}_k := v_k v_n^{-(2^k-1)(2^n-1)}$ for $0 \leq k < n$ and $\hat{v}_n := v_n^{2^{n+1}(2^n-1)}$. Consider the subring

$$L := \mathbb{Z}_{(2)}[\hat{v}_1, \ldots, \hat{v}_{n-1}, \hat{v}_n] \subseteq E(n)_*$$

of the coefficients of Johnson-Wilson theory. By the computations in the previous sections, the $\hat{v}_k$ are permanent cycles in the Bockstein spectral sequence converging to the coefficients of $ER(n)$, so the Bockstein spectral sequence is a spectral sequence of modules over L. Moreover, each page is a finitely generated L-module since there are only finitely many differentials.

Recall that completion at an ideal in a Noetherian ring preserves short exact sequences of finitely generated modules over that ring (see, for example, Prop. 10.12 of [AM69]). Moreover, the completion of such a module is simply the tensor product of the module with the completion of the ring. Let

$$\widetilde{I}_n := (\hat{v}_1, \ldots, \hat{v}_{n-1}) \subseteq L$$

and note that

$$L^{\wedge}_{\widetilde{I}_n} = \mathbb{Z}_{(2)}[[\hat{v}_1, \ldots, \hat{v}_{n-1}]][\hat{v}_n].$$

Thus each page of the Bockstein spectral sequence satisfies

$$(E_r)^{\wedge}_{\tilde{I}_n} \cong E_r \otimes_L L^{\wedge}_{\tilde{I}_n}.$$

Since the images of the $\hat{v}_k$ on the E_∞–page are exactly the $\hat{v}_k(0)$ (i.e. the module map $L \to E_\infty$ sends $\hat{v}_k \mapsto \hat{v}_k(0)$), we see that the E_∞-page is isomorphic to the desired ring. Since any nontrivial extension would be detected in $ER(n)_*$, this completes the proof. $\square$

3.5 $(v_n^{-1}\widehat{BP}^{h\mathbb{Z}/2})_*$ and $(v_n^{-1}\widehat{MU}^{h\mathbb{Z}/2})_*$

Let $v_n^{-1}\widehat{BP} := L_{K(n)}BP$ and note that

$$v_n^{-1}\widehat{BP}_* = (v_n^{-1}BP_*)^{\wedge}_{I_n}.$$

The proof of the following proposition is analogous to the $v_n^{-1}BP$ case.

Proposition 3.5.1.

$$(v_n^{-1}\widehat{BP}^{h\mathbb{Z}/2})_* = (\widehat{E(n)}^{h\mathbb{Z}/2})_*[\hat{v}_{n+1}, \hat{v}_{n+2}, \ldots]$$

Now, let $v_n^{-1}\widehat{MU} := L_{K(n)}MU$, as in Example 2.5.6. Again the complex conjugation action on MU gives rise to one on $v_n^{-1}\widehat{MU}$ and again there is a Bockstein spectral sequence converging to $(v_n^{-1}\widehat{MU}^{h\mathbb{Z}/2})_*$. As in Section 3.3, the equivariant splitting of $\mathbb{MR}$ as a wedge of suspensions of $\mathbb{BPR}$ implies the following.

Proposition 3.5.2. *$(v_n^{-1}\widehat{MU}^{h\mathbb{Z}/2})_*$ is a free $(v_n^{-1}\widehat{BP}^{h\mathbb{Z}/2})_*$-module on generators m_i ranging over the dimensions of additive free generators of a free symmetric algebra on generators of dimension $k \neq 2^i - 1$.*

Chapter 4

Bousfield localization

In this chapter, we construct a commutative MU-algebra $v_n^{-1}\widehat{MU}$ and an S-algebra map $\widehat{\sigma} : v_n^{-1}\widehat{MU} \to v_n^{-1}\widehat{MU}$ which squares to the identity $\mathrm{id}_{v_n^{-1}\widehat{MU}}$. The S-algebra $v_n^{-1}\widehat{MU}$ is constructed as a the Bousfield localization

$$v_n^{-1}\widehat{MU} := L^{MU}_{v_n^{-1}MU/I_n} MU.$$

As explained in Example 2.5.6, it has coefficients

$$\pi_*(v_n^{-1}\widehat{MU}) = (\pi_*(MU)[v_n^{-1}])^{\wedge}_{I_n}$$

where $I_n = (v_0, v_1, \ldots, v_{n-1})$.

4.1 Bousfield localization of R-algebras

We rely heavily on the following theorem from [EKMM97], which says that Bousfield localization preserves R-algebra structures.

Theorem 4.1.1 ([EKMM97], VIII.2.1)**.** *For a cell R-algebra A, the localization $\lambda : A \to A_E$ can be constructed as the inclusion of a subcomplex in a cell R-algebra A_E. Moreover, if $f : A \to B$ is a map of R-algebras into an E-local R-algebra B, then f lifts to a map of R-algebras $\widetilde{f} : A_E \to B$ such that $\widetilde{f}\lambda = f$, and $\widetilde{f}$ is unique up to homotopy through maps of R-algebras. If f is an E-equivalence, then $\widetilde{f}$ is a weak equivalence.*

4.1.2 Functoriality

To construct $\widehat{\sigma}$, we will need a few facts about the construction that are not made explicit in [EKMM97]. We show that the construction given in [EKMM97] is in fact functorial in the point-set category, though the universal property of the localization only holds in the homotopy category.

The construction of A_E

The R-algebra A_E is constructed as the colimit of a sequence of R-algebras[1]

$$A_0 \to A_1 \to A_2 \to \cdots$$

where $A_0 = A$ and A_{n+1} is constructed from A_n as follows. Let $\mathcal{T}$ be the set of E-acyclic inclusions $X \to Y$ of cell complexes of R-modules. For an R-algebra C, let $\mathcal{D}_C$ be the category of all diagrams of the form

$$Y \xleftarrow{i} X \xrightarrow{\alpha} C$$

where $i \in \mathcal{T}$ and α is a map of R-modules. Morphisms in this category are triples (f_y, f_x, f_c) of maps such that

$$\begin{array}{ccccc} Y & \xleftarrow{i} & X & \xrightarrow{\alpha} & C \\ {\scriptstyle f_y}\big\downarrow & & {\scriptstyle f_x}\big\downarrow & & {\scriptstyle f_c}\big\downarrow \\ Y' & \xleftarrow{i'} & X' & \xrightarrow{\alpha'} & C \end{array}$$

commutes and composition is given by stacking the diagrams in the obvious way. For any object of $\mathcal{D}_C$, we obtain a diagram of R-algebras

$$\mathbb{T}Y \xleftarrow{\mathbb{T}i} \mathbb{T}X \xrightarrow{\widetilde{\alpha}} C$$

where $\mathbb{T}$ is the free R-algebra functor and $\widetilde{\alpha}$ is the R-algebra adjoint to the R-module map α. Taking the sum over all elements of $\mathcal{T}$, we obtain a diagram

$$\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\alpha}} C.$$

Define A_{n+1} to be the pushout of this diagram for $C = A_n$.

[1]The construction actually uses a transfinite sequence, though for simplicity we ignore this technicality here. The construction and proof is easily modified to account for this.

Induced maps

In this section, we describe the induced maps between localizations and prove that their construction respects composition. For a map $f : A \to B$ of R-algebras, we will construct maps $f_n : A_n \to B_n$ inductively and then pass to colimits to obtain a map $f_E : A_E \to B_E$.

Suppose $f : A \to B$ is a map of R-algebras. Given an object

$$Y \xleftarrow{i} X \xrightarrow{\alpha} A$$

of $\mathcal{D}_A$, we obtain an object

$$Y \xleftarrow{i} X \xrightarrow{\alpha} A \xrightarrow{f} B$$

of $\mathcal{D}_B$. Applying $\mathbb{T}$ and taking coproducts as before, we obtain a diagram

$$\begin{tikzcd}
\coprod \mathbb{T}Y \arrow[d, "\mathrm{id}"'] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\tilde{\alpha}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "f"'] \\
\coprod \mathbb{T}Y \arrow[d, hook] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "f\circ\sum\tilde{\alpha}"] \arrow[d, hook] & B \arrow[d, "\mathrm{id}"'] \\
\coprod \mathbb{T}Y \arrow[dr] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\tilde{\gamma}"] \arrow[d] & B \arrow[dl] \\
& B' &
\end{tikzcd} \tag{4.1.1}$$

where $B' = \mathrm{colim}(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum\tilde{\gamma}} B)$ and the maps into B' are the natural ones. The coproducts in the middle row range over the objects

$$Y \xleftarrow{i} X \xrightarrow{\beta} B$$

of $\mathcal{D}_A$ such that $\beta = f \circ \alpha$ for some α in a object

$$Y \xleftarrow{i} X \xrightarrow{\alpha} A$$

of $\mathcal{D}_A$. The coproducts in the bottom row range over all objects of $\mathcal{D}_B$. Using the universal property of the colimit of the top row, we obtain a map

$$f' : \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\alpha}} A\right) \to \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\gamma}} B\right)$$

This construction respects composition. To see this, suppose $g : B \to C$. Consider the analogous diagrams for g and for the composition $gf : A \to C$

$$\begin{tikzcd}
\coprod \mathbb{T}Y \arrow[d, "\mathrm{id}"'] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\widetilde{\gamma}"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "g"'] \\
\coprod \mathbb{T}Y \arrow[d, hook] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "g\circ\sum\widetilde{\gamma}"] \arrow[d, hook] & C \arrow[d, "\mathrm{id}"'] \\
\coprod \mathbb{T}Y \arrow[dr] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\widetilde{\xi}"] \arrow[d] & C \arrow[dl] \\
& C' &
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod \mathbb{T}Y \arrow[d, "\mathrm{id}"'] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\widetilde{\alpha}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "f"'] \\
\coprod \mathbb{T}Y \arrow[d, "\mathrm{id}"'] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "f\circ\sum\widetilde{\alpha}"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "g"'] \\
\coprod \mathbb{T}Y \arrow[d, hook] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "g\circ f\circ\sum\widetilde{\alpha}"] \arrow[d, hook] & C \arrow[d, "\mathrm{id}"'] \\
\coprod \mathbb{T}Y \arrow[dr] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\widetilde{\xi}"] \arrow[d] & C \arrow[dl] \\
& C' &
\end{tikzcd}
\tag{4.1.2}$$

where $C' = \operatorname{colim}(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum\widetilde{\xi}} C)$. The diagram on the left gives rise to the map

$$g' : \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\gamma}} B\right) \to \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\xi}} C\right)$$

induced by g. In the diagram on the right, the coproducts in the first two rows range over the same sets as they did in (4.1.1). The coproducts in the third row range over all objects

$$Y \xleftarrow{i} X \xrightarrow{\xi} C$$

of $\mathcal{D}_C$ such that $\xi = g \circ f \circ \alpha$ for some α in an object

$$Y \xleftarrow{i} X \xrightarrow{\alpha} A$$

of $\mathcal{D}_A$ and the coproducts in the last row range over all objects of $\mathcal{D}_C$. The resulting map

$$(gf)' : \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\alpha}} A\right) \to \operatorname{colim}\left(\coprod \mathbb{T}Y \longleftarrow \coprod \mathbb{T}X \xrightarrow{\sum \widetilde{\xi}} C\right)$$

is the map induced by the composition gf. Now, the middle part of the diagram on the right in (4.1.2) factors as

$$\begin{tikzcd}
\coprod \mathbb{T}Y \arrow[d, hook] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "f\circ\sum\widetilde{\alpha}"] \arrow[d, hook] & B \arrow[d, "\mathrm{id}"] \\
\coprod \mathbb{T}Y \arrow[d, "\mathrm{id}"'] & \coprod \mathbb{T}X \arrow[l] \arrow[r, "\sum\widetilde{\gamma}"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "g"] \\
\coprod \mathbb{T}Y & \coprod \mathbb{T}X \arrow[l] \arrow[r, "g\circ\sum\widetilde{\gamma}"] & C
\end{tikzcd}$$

where the coproducts in the middle row range over all objects of $\mathcal{D}_B$. Therefore $(gf)'$ factors as $g'f'$. We have proved the following lemma.

Lemma 4.1.3. *The assignment $A \mapsto A', f \mapsto f'$ defines a functor from R-algebras to R-algebras.*

Returning to Bousfield localization, we iterate this construction to obtain a diagram

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "f"'] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & \cdots \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & \cdots
\end{tikzcd}$$

for each map $f : A \to B$. Passing to colimits, we obtain a map $f_E : A_E \to B_E$. We have $\lambda_B f = f_E \lambda_A$, where $\lambda_A : A \to A_E$ and $\lambda_B : B \to B_E$ are the maps into the colimits. The lemma ensures that $(gf)_E = g_E f_E$, so we have proved the following proposition.

Proposition 4.1.4. *Suppose $f : A \to B$ and $g : B \to C$ are maps of R-algebras and let $\lambda_X : X \to X_E$ be the localization maps for $X = A, B$, or C. Then there are maps $f_E : A_E \to B_E$ and $g_E : B_E \to C_E$ such that*

$$\lambda_B f = f_E \lambda_A, \quad \lambda_C g = g_E \lambda_B, \quad \textit{and} \quad (gf)_E = g_E f_E.$$

Remark 4.1.5. It was not necessary to restrict ourselves to R-algebras in the proof of the above proposition. The same argument applies to Bousfield localization of S- or R-modules.

4.2 Specialization to $v_n^{-1}MU$

As a special case of the discussion above, we obtain the following proposition.

Proposition 4.2.1. *There is an MU-algebra $v_n^{-1}\widehat{MU}$ and an MU-algebra map*

$$\widehat{\sigma} : v_n^{-1}\widehat{MU} \to v_n^{-1}\widehat{MU}$$

such that $\widehat{\sigma}^2 = \mathrm{id}_{v_n^{-1}\widehat{MU}}$ and $\lambda\sigma = \widehat{\sigma}\lambda$, where $\lambda : v_n^{-1}MU \to v_n^{-1}\widehat{MU}$.

Proof. Let $R = MU, A = MU$ and $E = v_n^{-1}MU/I_n$ in Theorem 4.1.1. Then

$$v_n^{-1}\widehat{MU} := L^{MU}_{v_n^{-1}MU/I_n}MU$$

is an MU-algebra. Applying the construction described above to the map

$$\sigma : MU \to MU$$

yields $\widehat{\sigma} := \sigma_{v_n^{-1}MU/I_n}$. Since $\sigma^2 = \mathrm{id}_{v_n^{-1}MU}$, functoriality ensures that

$$\widehat{\sigma}^2 = \mathrm{id}_{v_n^{-1}\widehat{MU}}\,.$$

$\square$

Chapter 5

Additional structure

Complex conjugation gives rise to an S-algebra map $\sigma : MU \to MU$ with $\sigma^2 = \mathrm{id}$. In Chapter 4, we extended this S-algebra $\mathbb{Z}/2$-action to an action $\widehat{\sigma}$ on $v_n^{-1}\widehat{MU}$. In this chapter, we prove some technical results that will allow us to compare this action with the Goerss-Hopkins-Miller action on $E_n(\mathrm{Gal})$. The key result is Theorem 5.5.8.

5.1 Background and notation

We recall a few constructions we will need. Fix a commutative S-algebra R and define the free S- and R-modules generated by a spectrum X as

$$\mathbb{F}_S X := S \wedge_{\mathcal{L}} \mathbb{L}X \quad \text{and} \quad \mathbb{F}_R X := R \wedge_R \mathbb{F}_S X$$

where $\mathcal{L}$ is the linear isometries operad and $\mathbb{L}$ is the $\mathbb{L}$-spectrum functor. Let

$$S_R^m = \mathbb{F}_R S_S^m = R \wedge_S \mathbb{F}_S S^m = R \wedge_S S \wedge_{\mathcal{L}} \mathbb{L}S^m = R \wedge_{\mathcal{L}} \mathbb{L}S^m$$

denote the m-sphere R-module (see [EKMM97], II.1.(1.7), V.1.(1.1)), so that

$$\pi_m(M) = h\mathcal{M}_S(S_S^m, M) = h\mathcal{M}_R(S_R^m, M)$$

where $\mathcal{M}_S$ and $\mathcal{M}_R$ are the categories of S-modules and R-modules, respectively.

Proposition 5.1.1 ([EKMM97], III.1.1.5)**.** *If X is a wedge of sphere spectra, then $\pi_*(\mathbb{F}_R X)$ is the free $\pi_*(R)$-module with one generator of degree m for each wedge summand S^m.*

For an R-module X, define

$$\mathbb{T}_R X := \bigvee_{j\geq 0} X^{\wedge_R^j} = R \vee X \vee (X \wedge_R X) \vee (X \wedge_R X \wedge_R X) \vee \cdots ,$$

the free R-algebra on the R-module X (see [EKMM97], VII.1.1.4). When clear from the context, we may drop the subscript R from the notation, writing $\mathbb{T}X$ for $\mathbb{T}_R X$.

For S-modules K and L, there is a natural isomorphism

$$(K \wedge_S R) \wedge_R (R \wedge_S L) \;\cong R \wedge_S (K \wedge_S L)$$

(see [EKMM97], III.4.3.6). Taking $K = L = S^m_S$, we have

$$S^m_R \wedge_R S^m_R = (S^m_S \wedge_S R) \wedge_R (S^m_S \wedge_S R) \;\cong R \wedge_S (S^m_S \wedge_S S^m_S)$$

and so $\mathbb{T}_R S^m_R = R \wedge_S \mathbb{T}_S S^m_S$.

Proposition 5.1.2. *$\pi_*(\mathbb{T}_R S^m_R) = \pi_*(R)[x]$, where $|x| = m$.*

Proof. Since $\mathbb{T}_R X = R \vee X \vee (X \wedge_R X) \vee (X \wedge_R X \wedge_R X) \vee \cdots$, we have

$$\begin{aligned} \pi_*(\mathbb{T}_R S^m_R) &= \pi_*(R \vee S^m_R \vee (S^m_R \wedge_R S^m_R) \vee (S^m_R \wedge_R S^m_R \wedge_R S^m_R) \vee \cdots) \\ &= \pi_*(R) \oplus \pi_*(S^m_R) \oplus \pi_*(S^m_R \wedge_R S^m_R) \oplus \pi_*(S^m_R \wedge_R S^m_R \wedge_R S^m_R) \oplus \cdots \\ &= \pi_*(R) \oplus \pi_*(\mathbb{F}_R S^m) \oplus \pi_*(\mathbb{F}_R S^m \wedge_R \mathbb{F}_R S^m) \\ &\qquad \oplus \pi_*(\mathbb{F}_R S^m \wedge_R \mathbb{F}_R S^m \wedge_R \mathbb{F}_R S^m) \oplus \cdots \end{aligned}$$

Since $\pi_*(\mathbb{F}_R S^m)$ is a free $\pi_*(R)$-module on one generator of degree m and $\pi_*(R)$ is a commutative ring, the Künneth theorem gives the result. □

Corollary 5.1.3. *Let $M = (\mathbb{T}_R S^{m_1}_R) \wedge_R \cdots \wedge_R (\mathbb{T}_R S^{m_k}_R)$. Then*

$$\pi_*(M) = \pi_*(R)[x_1, \ldots, x_k],$$

where $|x_i| = m_i$.

5.2 The S-algebra T

Now we will construct the S-algebra T, as promised so many pages ago in the introduction. Specializing the discussion of the previous section to the case $R = \widehat{E(n)}$ (which has a unique structure as a commutative S-algebra [Bak91]), we can construct a $\widehat{E(n)}$-module M with

$$\pi_*(M) = \pi_*(\widehat{E(n)})[x_i \mid i \in \Lambda]$$

for some index set Λ. Define an index set of integers

$$\Lambda := \{m \in \mathbb{Z} \mid m \neq 2^j - 1 \text{ for some } 0 \leq j \leq n\}.$$

Hence $2m$ is the degree of the polynomial generator x_m of $\pi_* MU$, and the set Λ encodes those which are not $v_1, \ldots, v_n$. In detail, we define

$$M := \bigwedge_{i \in \Lambda} \mathbb{T}_{\widehat{E(n)}} S^{|x_i|}_{\widehat{E(n)}} = \widehat{E(n)} \wedge_S \bigwedge_{i \in \Lambda} \mathbb{T}_S S^{|x_i|}_S$$

where the first Λ-indexed smash product is over $\widehat{E(n)}$ and the second is over S. Let

$$T := \bigwedge_{i \in \Lambda} \mathbb{T}_S S^{|x_i|}_S,$$

so that $M = \widehat{E(n)} \wedge_S T$. Notice that T is a commutative S-algebra. Moreover, since T is equivalent to a wedge of suspensions of spheres with exactly one 0-sphere S^0_S, there is a natural S-algebra map

$$t : T \to S_S. \tag{5.2.1}$$

Using this map, we can make any S-algebra X into a trivial T-algebra.

5.3 $v_n^{-1}\widehat{MU}$ is a T-algebra

In this section, we prove the following.

Proposition 5.3.1. *$v_n^{-1}\widehat{MU}$ is a T-algebra.*

It is enough to show that MU is a T-algebra, since $v_n^{-1}\widehat{MU}$ is a Bousfield localization of MU and Bousfield localization preserves such structures (see [EKMM97], VIII.2.2.1).

Just as in ordinary algebra, if T is an S-algebra and N is an S-algebra, a central map $T \to N$ of S-algebras makes N into a T-algebra. To show that MU is a T-algebra, we must write down a map $T \to MU$ of S-algebras (it is automatically central since MU is a commutative S-algebra).

Recall that $\pi_*(MU) = \mathbb{Z}[x_1, x_2, \ldots]$ where $|x_k| = 2k$. Since

$$\pi_m(MU) = h\mathcal{M}_S(S_S^m, MU),$$

we may choose S-module maps $\xi_k : S_S^{2k} \to MU$ representing the classes x_k. Since $\mathbb{T}_S S_S^m$ is the free S-algebra on the S-module S_S^m, we obtain S-algebra maps

$$\mathbb{T}_S S_S^{2k} \to MU$$

for each ξ_k. Slightly abusing notation, we will also call these maps ξ_k. Define the map $T \to MU$ by smashing together the ξ_k and then multiplying in MU. That is,

$$\varphi : T = \bigwedge_{k\in\Lambda} \mathbb{T}_S S_S^{|x_k|} \xrightarrow{\wedge\xi_k} \bigwedge_{k\in\Lambda} MU \xrightarrow{\mu} MU. \tag{5.3.1}$$

Proposition 5.3.2. *The map φ is a map of S-algebras, hence MU is a T-algebra.*

Proof. Each $\xi_k : \mathbb{T}_S S_S^{|x_k|} \to MU$ is an S-algebra map by construction, hence their product is also an S-algebra map. Since μ is an S-algebra map, the composition φ is an S-algebra map. □

5.4 Twisted maps and complex conjugation

We would like to say that the map $\hat{\sigma} : v_n^{-1}\widehat{MU} \to v_n^{-1}\widehat{MU}$ constructed in Chapter 4 is a map of T-algebras, but unfortunately complex conjugation (and hence $\hat{\sigma}$) does not preserve the maps used to give $v_n^{-1}\widehat{MU}$ its T-algebra structure. To remedy this, we carry the conjugation action along by replacing S^{2k} with $S(\mathbb{C}^k)$ throughout the construction of T. Now complex conjugation defines an automorphism

$$c : T \to T$$

with $c^2 = \mathrm{id}_T$. We can consider not only the space of T-algebra maps

$$F_{T-\mathrm{alg}}(A, B)$$

between T-algebras A and B, but also the space of *twisted* T-algebra maps

$$F_{cT-\mathrm{alg}}(A, B),$$

which are defined to be S-algebra maps $\varphi : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} T \wedge_S A \arrow[r] \arrow[d, "c\wedge\varphi"'] & A \arrow[d, "\varphi"] \\ T \wedge_S B \arrow[r] & B \end{tikzcd}$$

The following lemma says that our $\hat{\sigma}$ is a twisted T-algebra map.

Proposition 5.4.1. $\hat{\sigma} \in F_{cT-\mathrm{alg}}(v_n^{-1}\widehat{MU}, v_n^{-1}\widehat{MU})$

Proof. This follows directly from the T-module structure on $v_n^{-1}\widehat{MU}$. The S-algebra representatives

$$\xi_k : \mathbb{T}_S S_S^{2k} \to MU$$

of the generators $x_k \in \pi_{2k}(MU)$ are themselves twisted with respect to complex conjugation because they are odd desuspensions of the invariant generators in the homotopy of the Real spectrum $\mathbb{MR}$. □

5.5 Homotopy discrete mapping spaces

In this section, we prove that the space of T-algebra maps from $v_n^{-1}\widehat{MU}$ to a certain class of S-algebras is homotopy discrete. We need the following results of Lazarev [Laz03] and Elmendorf, Kriz, Mandell, and May [EKMM97]. The first result of Lazarev pertains to the class of strongly $K(n)$-complete S-algebras, which we do not define here (see [Laz03] for the definition). We simply note that $\widehat{E(n)}$ (and hence $E_n(\mathrm{Gal})$) belongs to this class.

Theorem 5.5.1 (Lazarev, [Laz03])**.** *For any strongly $K(n)$-complete S-algebra N, the space of maps $F_{S-\mathrm{alg}}(\widehat{E(n)}, N)$ is homotopy discrete.*

Theorem 5.5.2 (Lazarev, [Laz03])**.** *The S-algebra map $v_n^{-1}\widehat{MU} \to \widehat{E(n)}$ splits by a map of S-algebras.*

Proposition 5.5.3 ([EKMM97], VII.1.4)**.** *Let R be a commutative S-algebra and let Q be an S-algebra. Then $Q \wedge_S R$ is the free R-algebra generated by Q. That is, the functor $-\wedge_S R$ from S-algebras to R-algebras is left adjoint to the forgetful functor.*

Corollary 5.5.4. *$\widehat{E(n)} \wedge_S T$ is the free T-algebra generated by the S-algebra $\widehat{E(n)}$.*

Corollary 5.5.5. *For any T-algebra N, $F_{T-\mathrm{alg}}(\widehat{E(n)} \wedge_S T, N) \cong F_{S-\mathrm{alg}}(\widehat{E(n)}, N)$.*

Corollary 5.5.6. *For any strongly $K(n)$-complete S-algebra N with the trivial T-algebra structure, the space*

$$F_{T-\mathrm{alg}}(\widehat{E(n)} \wedge_S T, N)$$

is homotopy discrete.

Remark 5.5.7. Note that by freeness, Lazarev's splitting gives rise to a map

$$\psi : \widehat{E(n)} \wedge_S T \to v_n^{-1}\widehat{MU} \tag{5.5.1}$$

of T-algebras which, by construction, induces an isomorphism on homotopy and thus gives an isomorphism in the derived category.

Our goal for this section is to prove the following theorem.

Theorem 5.5.8. *$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, B)$ is homotopy discrete for any strongly $K(n)$-complete S-algebra B with the trivial T-algebra structure.*

We will prove this by computing the Bousfield-Kan spectral sequence and using the map (5.5.1) to compare the spectral sequence for maps out of $\widehat{E(n)} \wedge_S T$ with that of maps out of $v_n^{-1}\widehat{MU}$. The T-algebra map $\widehat{E(n)} \wedge_S T \to v_n^{-1}\widehat{MU}$ induces a map of spectral sequences and we will see that it is an isomorphism on the E_2-page because this page is entirely cohomological.

Proposition 5.5.9. *Let $X = v_n^{-1}\widehat{MU}$ or $\widehat{E(n)} \wedge_S T$. Let E be any strongly $K(n)$-complete S-algebra with trivial T-algebra structure. In the Bousfield-Kan spectral sequence computing the homotopy of $F_{T-\mathrm{alg}}(X, E)$, we have:*

(a) $E_2^{0,0} = \mathrm{Hom}_{E_*T-\mathrm{alg}}(E_*X, E_*)$

(b) $E_2^{s,t} = H^s(\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}^{\bullet+1}(E_*X), E_*[x]/x^2))$ *for $t - s \geq -1, t > 0$. Here $|x| = t$, $\mathbf{T}$ denotes the free E_*T-algebra functor, and the homomorphisms are those of E_*T-algebras augmented over E_*.*

Before proving the proposition, we describe the set up of the spectral sequence. For this and to compute the E_2-term, we follow Rezk [Rez98], though we work with the language of modules and algebras in the sense of [EKMM97] rather than in the operadic sense. Given a T-algebra X, we can construct an augmented simplicial T-algebra

$$X \leftarrow TX \leftleftarrows TTX \cdots \tag{5.5.2}$$

where TX is the free T-algebra on the S-algebra underlying X (i.e. $TX = X \wedge_S T$) and the iterated $T^n X$ are obtained considering $T^{n-1}X$ as an S-algebra (forgetting the T-algebra structure) and then applying T again.[1] Applying the functor $F_{T-\mathrm{alg}}(-, E)$ yields a cosimplicial space

$$Y^\bullet = F_{T-\mathrm{alg}}(T^{\bullet+1}X, E)$$

which under certain conditions gives rise to a Bousfield-Kan spectral sequence

$$E_2^{s,t} = \pi^s\pi_t(Y^\bullet, f) \quad\Longrightarrow\quad \pi_{t-s}(\mathrm{Tot}\, Y^\bullet, f)$$

[1] This is the simplicial resolution over the forget-free comonad. We suppress the notation for the forgetful functor here for simplicity. See below for the precise construction.

for some basepoint $f \in F_{T-\mathrm{alg}}(T^{\bullet+1}X, E)$ arising from a T-algebra map $X \to E$. In order for this spectral sequence to compute the homotopy of $F_{T-\mathrm{alg}}(X, E)$, we need that

$$\mathrm{Tot}(F_{T-\mathrm{alg}}(T^{\bullet+1}X, E)) \simeq F_{T-\mathrm{alg}}(X, E).$$

For this to be true, we must ensure that our resolution $T^{\bullet+1}X$ is Reedy cofibrant (see Chapter VII of [GJ99] and below). This requires a short detour and a few lemmas.

A comonad and Reedy cofibrancy

The augmented simplicial resolution (5.5.2) arises as the simplicial resolution over a forget-free comonad. The free T-algebra functor T from S-algebras to T-algebras is left adjoint to the forgetful functor U and hence defines a comonad $C := TU$ on $\mathcal{A}_T$ with structure maps

$$\mu : C \to C^2 \qquad \eta : C \to \mathrm{id}_{\mathcal{A}_T}$$

defined as follows. For a T-algebra X, define $\eta_X : TUX \to X$ to be the adjoint to the identity id_{UX}. Define $\mu_X : TUX \to TUTUX$ to be the image under the functor T of the adjoint $UX \to UTUX$ to the identity id_{TUX}. We obtain the simplicial resolution $T^{\bullet+1}X$ by putting $T^{k+1}X := C^{k+1}X$ and

$$\begin{aligned} d_i &:= C^i(\eta_{C^{k-i}X}) : T^kX \to T^{k-1}X \\ s_j &:= C^j(\mu_{C^{k-j}X}) : T^kX \to T^{k+1}X \end{aligned}$$

for $0 \leq i, j \leq n$. The simplicial identities are implied by the standard commuting diagrams for the structure maps η and μ. (See e.g. [Sch94], p. 65.)

The condition of Reedy cofibrancy ensures that the natural map $|T^{\bullet+1}X| \to X$ is a weak equivalence of T-algebras and that the resulting tower $F_{T-\mathrm{alg}}(T^{\bullet+1}X, E)$ is a tower of fibrations giving rise to the desired spectral sequence. A simplicial object $Y^\bullet$ in a model category is *Reedy cofibrant* if the natural map

$$L_k(Y^\bullet) \to Y^k$$

of the latching object is a cofibration. The latching object $L_k(Y^\bullet)$ is defined as

$$L_k(Y^\bullet) := \operatorname{colim}_{\varphi:[k]\to[m]} Y^m$$

where φ runs over nonidentity surjections in the ordinal number category.

Proposition 5.5.10. *The resolution $T^{\bullet+1}X$ is Reedy cofibrant.*

Proof. The latching object has an alternative description as the coequalizer

$$\coprod_{0\le i<j\le k} Y_{k-2} \rightrightarrows \coprod_{i=0}^{k-1} Y_{k-1} \to L_k(Y^\bullet)$$

where for $i<j$ the restrictions of the two left-hand maps are

$$Y_{k-2} \xrightarrow{s_i} Y_{k-1} \xrightarrow{ik_i} \coprod_{i=0}^{k-1} Y_{k-1}$$

$$Y_{k-2} \xrightarrow{s_{j-1}} Y_{k-1} \xrightarrow{ik_i} \coprod_{i=0}^{k-1} Y_{k-1}$$

The map $L_k(Y^\bullet) \to Y_k$ is induced by the degeneracies $s_i : Y_{k-1} \to Y_k$. (See Chapter VII of [GJ99].) In the case of a simplicial resolution over a comonad, the degeneracies arise from the structure map μ; Reedy cofibrancy will follow from the fact that μ_X is a cofibration.

The identity map

$$\mathrm{id}_{TUX} : TUX \to TUX$$

is a cofibration of T-algebras and its adjoint is

$$\iota \wedge \mathrm{id}_X : S \wedge_S X = UX \to T \wedge_S X = UTUX$$

where $\iota : S \to T$ is the unit of the S-algebra T. Since T is a cofibrant S-algebra, the unit map ι is a cofibration ([EKMM97], Thm. VII.6.2), so the adjoint to id_{TUX} is a cofibraiton. Applying T preserves cofibrations, so the map

$$\mu_X : TUX \to TUTUX$$

is a cofibration of T-algebras. Since

$$s_j := C^j(\mu_{C^{k-j}X}) : T^k X \to T^{k+1} X$$

and $C^i = (TU)^i$ preserves cofibrations ([EKMM97], Thm. VII.4.14), we see that the s_j are cofibrations. Since the map

$$L_k(Y^\bullet) \to Y_k$$

is induced by applying the universal property of coproducts to the s_i, it is also a cofibration, as desired. $\square$

We need one additional lemma before we can prove Proposition 5.5.9. Let $F_T(-,-)$ denote T-module maps.

Lemma 5.5.11. *For $Z = \widehat{E(n)} \wedge_S T$ or $v_n^{-1}\widehat{MU}$ and E any strongly $K(n)$-complete S-algebra, the evaluation map*

$$\pi_0 F_T(Z, E) \to \mathrm{Hom}_{E_*T}(E_*Z, E_*)$$

is an isomorphism.

Proof. First consider the case $Z = \widehat{E(n)} \wedge_S T$. In the diagram

$$\begin{array}{ccc} \pi_0 F_T(\widehat{E(n)} \wedge_S T, E) & \longrightarrow & \mathrm{Hom}_{E_*T}(E_*(\widehat{E(n)} \wedge_S T), E_*) \\ \downarrow & & \downarrow \\ \pi_0 F_S(\widehat{E(n)}, E) & \longrightarrow & \mathrm{Hom}_{E_*}(E_*\widehat{E(n)}, E_*) \end{array}$$

the vertical arrows are isomorphisms since $\widehat{E(n)} \wedge_S T$ is free. The bottom arrow is an isomorphism by Lemma 5.10 of Lazarev [Laz03], and hence the top arrow is also an isomorphism, as desired.

For $Z = v_n^{-1}\widehat{MU}$, consider the diagram

$$\begin{array}{ccc} \pi_0 F_T(v_n^{-1}\widehat{MU}, E) & \longrightarrow & \mathrm{Hom}_{E_*T}(E_*(v_n^{-1}\widehat{MU}), E_*) \\ \downarrow & & \downarrow \\ \pi_0 F_T(\widehat{E(n)} \wedge_S T, E) & \longrightarrow & \mathrm{Hom}_{E_*T}(E_*(\widehat{E(n)} \wedge_S T), E_*) \end{array}$$

where the vertical arrows are induced by the map ψ of (5.5.1). Since ψ is an isomorphism in the derived category of T-algebras, it induces an isomorphisms on these vertical arrows. Above we proved that the bottom arrow is an isomorphism, so the top arrow is as well. $\square$

Now we are ready to prove Proposition 5.5.9. Again, the argument here is the same as that of [Rez98], with minor adjustments made to account for working over T rather than S. The reader familiar with that paper may wish to skip these proofs.

Proof of Proposition 5.5.9(a). First, notice that the natural map

$$\pi_0 F_{T-\mathrm{alg}}(Z, E) \to \mathrm{Hom}_{E_*}(E_*Z, E_*)$$

obtained by sending $f : Z \to E$ to the composition

$$E_*Z \xrightarrow{f_*} E_*E \to E_*$$

actually lands in E_*T-algebra maps since the T-algebra structure $Z \wedge_T Z \to Z$ induces the desired E_*T-algebra structure

$$E_*Z \otimes_{E_*T} E_*Z \to E_*(Z \wedge_T Z) \to E_*Z$$

and a T-algebra map gives rise to a E_*T-algebra map.

Now, to compute $E_2^{0,0}$, we must compute the equalizer of the pair of maps

$$\pi_0 F_{T-\mathrm{alg}}(TX, E) \rightrightarrows \pi_0 F_{T-\mathrm{alg}}(T^2X, E).$$

We will reduce this to computing the equalizer of the pair of maps

$$\mathrm{Hom}_{E_*T-\mathrm{alg}}(\mathbf{T}(E_*X), E_*) \rightrightarrows \mathrm{Hom}_{E_*T-\mathrm{alg}}(\mathbf{T}^2(E_*X), E_*) \tag{5.5.3}$$

where $\mathbf{T}$ is the tensor algebra (over E_*T) on the underlying E_*T-module. Consider the diagram

$$\begin{tikzcd} \pi_0 F_{T-\mathrm{alg}}(TZ, E) \arrow[r] \arrow[d] & \mathrm{Hom}_{E_*T-\mathrm{alg}}(E_*(TZ), E_*) \arrow[d] \\ \pi_0 F_T(Z, E) \arrow[r] & \mathrm{Hom}_{E_*T}(E_*Z, E_*) \end{tikzcd}$$

The left vertical map is an isomorphism since TZ is the free T-algebra on Z. The inclusion $Z \to TZ$ induces a map $E_*Z \to E_*(TZ)$ that gives rise to the right hand vertical arrow. It also gives a map $\mathbf{T}(E_*Z) \to E_*(TZ)$. Since E_*Z is flat over E_* in the cases of interest (i.e. $Z = TX$ for $X = \widehat{E(n)} \wedge_S T$ or $X = v_n^{-1}\widehat{MU}$), this map is an isomorphism, so the right vertical map is an isomorpism. The bottom map is an isomorphism by Lemma 5.5.11, and so the top map is also an isomorphism. Taking $Z = TX$, we have reduced the original coequalizer to that of (5.5.3). But this is just the coequalizer of

$$\mathrm{Hom}_{E_*T}(E_*X, E_*) \rightrightarrows \mathrm{Hom}_{E_*T}(\mathbf{T}(E_*X), E_*)$$

where the top arrow sends an E_*T-module map $f : E_*X \to E_*$ to

$$\mathbf{T}(E_*X) \to E_*X \xrightarrow{f} E_*$$

and the bottom arrow sends it to

$$\mathbf{T}(E_*X) \xrightarrow{\mathbf{T}f} E_*X \to E_*$$

and so the coequalizer is $\mathrm{Hom}_{E_*T-\mathrm{alg}}(E_*X, E_*)$, as desired. □

Proof of Proposition 5.5.9(b). Following Rezk [Rez98], we will construct a complex whose cohomology gives the E_2-term. First, we identify

$$\pi_t(F_{T-\mathrm{alg}}(Z, E), f) \cong \pi_0 F_{T-\mathrm{alg}}(Z, F_{T-\mathrm{alg}}(S_T^t, E))$$

and then we construct a map from the latter to

$$\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(E_*Z, E_*[x]/x^2).$$

Taking $Z = T^sX$ for $s \geq 0$ gives a complex

$$\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}^\bullet(E_*X), E_*[x]/x^2)$$

and we'll see that

$$\pi_0 F_{T-\mathrm{alg}}(TZ, F_{T-\mathrm{alg}}(S_T^t, E)) \cong \mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}(E_*Z), E_*[x]/x^2)$$

for $Z = T^sX$. When $Z = T^sX$, the left side is $E_2^{s,t}$, so this will prove the result.

Fix a basepoint $f \in F_{T-\mathrm{alg}}(Z, E)$. The adjunction between $\wedge$ and $F(-,-)$ gives that the set $\pi_t(F_{T-\mathrm{alg}}(Z, E), f)$ is the same as the set of homotopy classes of T-algebra maps $\widetilde{\varphi} : Z \to F_{T-\mathrm{alg}}(S_T^t, E)$ such that $e\widetilde{\varphi} = f$, where $e : F_{T-\mathrm{alg}}(S_T^t, E) \to$

E is evaluation at the basepoint. In diagrams,

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "f"] \arrow[d] & F_{T-\mathrm{alg}}(Z,E) \\ S^t_T \arrow[ur, "\varphi"] & \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} & F_{T-\mathrm{alg}}(S^t_T,E) \arrow[d, "e"] \\ Z \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "f"] & E \end{tikzcd}$$

Thus by applying E_* to the diagram on the right and using the algebra structure of E_*, we obtain a diagram of E_*T-algebras

$$\begin{tikzcd} & E_*(F_{T-\mathrm{alg}}(S^t_T,E)) \arrow[r] \arrow[d, "e_*"] & E_*[x]/x^2 \arrow[d] \\ E_*Z \arrow[ur, "\widetilde{\varphi}_*"] \arrow[r, "f_*"] & E_*E \arrow[r] & E_* \end{tikzcd}$$

where $|x| = t$ and $E_*[x]/x^2 \cong \pi_* F_{T-\mathrm{alg}}(S^t_T, E)$. By sending φ to the top compsition in this diagram, we obtain a map

$$\pi_t(F_{T-\mathrm{alg}}(Z,E), f) \to \mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(E_*Z, E_*[x]/x^2)$$

where the right hand side is maps of E_*T-algebras augmented over E_*.

By an argument similar to that in the proof of part *(a)* above, the top map in the diagram

$$\begin{tikzcd} \pi_0 F_{T-\mathrm{alg}}(TZ, F_{T-\mathrm{alg}}(S^t_T,E)) \arrow[r] \arrow[d] & \mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}(E_*Z), E_*[x]/x^2) \arrow[d] \\ \pi_0 F_T(Z, F_{T-\mathrm{alg}}(S^t_T,E)) \arrow[r] & \mathrm{Hom}_{E_*T}(E_*Z, E_* \oplus E_{*+t}) \end{tikzcd}$$

is an isomorphism. Thus we may calculate the E_2-term as the cohomology

$$E_2^{s,t} \cong H^s(\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}^{\bullet+1}(E_*X), E_*[x]/x^2))$$

completing the proof. □

Proof of Theorem 5.5.8. Recall that we have a T-algebra map (5.5.1)

$$\psi : \widehat{E(n)} \wedge_S T \to v_n^{-1}\widehat{MU}$$

which is an isomorphism in the derived category. Since ψ is a T-algebra map, it induces a map of augmented simplicial T-algebras

$$T^{\bullet+1}(\widehat{E(n)} \wedge_S T) \to T^{\bullet+1}(v_n^{-1}\widehat{MU})$$

and thus a map of cosimplicial S-algebras

$$F_{T-\mathrm{alg}}(T^{\bullet+1}(v_n^{-1}\widehat{MU}), E) \to F_{T-\mathrm{alg}}(T^{\bullet+1}(\widehat{E(n)} \wedge_S T), E)$$

Under the identification of the E_2-page as given in Proposition 5.5.9, ψ yeilds a map

$$H^s(\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}^{\bullet+1}(E_*(v_n^{-1}\widehat{MU})), E_*[x]/x^2)) \to$$
$$H^s(\mathrm{Hom}_{E_*T-\mathrm{alg}\downarrow E_*}(\mathbf{T}^{\bullet+1}(E_*(\widehat{E(n)} \wedge_S T)), E_*[x]/x^2)).$$

Since the map ψ is an isomorphism in the derived category, it induces an isomorphism on E_* homology, and hence on the E_2-terms of these spectral sequences. Since the spectral sequence for $\widehat{E(n)} \wedge_S T$ collapses, so must that for $v_n^{-1}\widehat{MU}$. $\square$

Chapter 6

Comparing the actions

6.1 Relating $ER(n)$ and $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$

We have constructed an S-algebra $v_n^{-1}\widehat{MU}$ and an S-algebra map $\widehat{\sigma}$ that extends the complex conjugation action on MU. In this chapter, we compare this action to the action of the Goerss-Hopkins-Miller monoid corresponding to the subgroup generated by the formal inverse on $E_n(\mathrm{Gal})$. We will prove that $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ is a completion of $ER(n)$, in an appropriate sense, and calculate the homotopy of $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$. This final chapter ties together the results of the previous chapters to prove Theorem 6.2.4.

We will show that there is an equivariant map into $E_n(\mathrm{Gal})$ from an MU-module equivariantly equivalent to $v_n^{-1}\widehat{MU}$. We will see that the induced map on homotopy fixed points factors through $\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$ and that the resulting map

$$\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}} \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$$

is an equivalence. Composing with the natural map $ER(n) \to \widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$ will give a map

$$ER(n) \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$$

that induces the algebraic completion

$$ER(n)_* \to (ER(n)_*)^{\wedge}_{\hat{I}_n} \cong (E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}})_*$$

on the level of homotopy.

6.1.1 An equivariant map

In this section, we construct an equivariant map into $E_n(\mathrm{Gal})$ from an S-algebra equivalent to $v_n^{-1}\widehat{MU}$. This construction uses the S-algebra T from Chapter 5 and the fact that the space of T-algebra maps from $v_n^{-1}\widehat{MU}$ to $E_n(\mathrm{Gal})$ is homotopy discrete.

Write

$$G_1 \subseteq F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, v_n^{-1}\widehat{MU})$$

for the $\mathbb{Z}/2$ generated by $\widehat{\sigma}$ and

$$G_2 \subseteq F_{S-\mathrm{alg}}(E_n(\mathrm{Gal}), E_n(\mathrm{Gal}))$$

for the homotopy discrete monoid corresponding the subgroup of the Morava stabilizer group generated by the formal inverse, so that G_1 acts on $v_n^{-1}\widehat{MU}$ and G_2 acts on $E_n(\mathrm{Gal})$. Define $\widetilde{\mathbb{Z}/2}$ by the following pullback diagram

$$\begin{tikzcd} \widetilde{\mathbb{Z}/2} \arrow[r, "\widetilde{\Delta}"] \arrow[d] & G_1 \times G_2 \arrow[d, "\pi_0"] \\ \mathbb{Z}/2 \arrow[r, "\Delta"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \end{tikzcd}$$

where Δ is the diagonal map. We obtain an action of $\widetilde{\mathbb{Z}/2}$ on the space

$$F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$$

by conjugation. In detail, for $\varphi \in F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$ and $\psi \in \widetilde{\mathbb{Z}/2}$, we have

$$\varphi \mapsto \psi''\varphi(\psi')^{-1}$$

where $\psi \mapsto (\psi', \psi'')$ under the map $\widetilde{\Delta}$.

Give $E_n(\mathrm{Gal})$ the trivial T-algebra structure and let

$$\nu : v_n^{-1}\widehat{MU} \to E_n(\mathrm{Gal})$$

be a T-algebra representative for the map that induces the quotient map on coefficients. Write

$$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$$

for the component of ν. Notice that the composition

$$T \xrightarrow{\varphi} v_n^{-1}\widehat{MU} \xrightarrow{\nu} E_n(\mathrm{Gal}) \tag{6.1.1}$$

factors through the unit map $S \to E_n(\mathrm{Gal})$ via the projection $t : T \to S$ described in (5.2.1). Here φ is the T-algebra structure map of Prop. 5.3.2.

The map ν is homotopy equivariant with respect to the G_1 action on $v_n^{-1}\widehat{MU}$ and the G_2 action on $E_n(\mathrm{Gal})$, so the action of $\widetilde{\mathbb{Z}/2}$ on $F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$ descends to an action on the component of ν, which we denote by

$$F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu.$$

Lemma 6.1.2. *The $\widetilde{\mathbb{Z}/2}$ action on $F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$ descends to an action on the subspace*

$$F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \subseteq F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$$

of T-algebra maps.

Proof. Since ν is homotopy equivariant, it suffices to show that ν remains a T-algebra map after the action of $\widetilde{\mathbb{Z}/2}$. Consider the diagram

$$\begin{tikzcd}
v_n^{-1}\widehat{MU} \arrow[r, "\widehat{\sigma}"] & v_n^{-1}\widehat{MU} \arrow[r, "\nu"] & E_n(\mathrm{Gal}) \arrow[r, "f"] & E_n(\mathrm{Gal}) \\
T \arrow[u, "\varphi"] \arrow[r, "c"] & T \arrow[u, "\varphi"] \arrow[r, "t"] & S \arrow[u] \arrow[ur] &
\end{tikzcd}$$

for $f \in G_2$. The left square commutes because $\widehat{\sigma}$ is a twisted T-algebra map (Prop 5.4.1). The middle square commutes because of the factorization noted in (6.1.1). The right triangle commutes because f is an S-algebra map. Since $tc = t$, the commutativity of the outside of the diagram shows that the map across the top is a T-algebra map. Since the image of ν under the $\widetilde{\mathbb{Z}/2}$ action consists exactly of such maps, this completes the proof. $\square$

Because $F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$ is homotopy discrete, the space

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$$

is homotopy equivalent to $v_n^{-1}\widehat{MU}$. Moreover, this space admits a diagonal $\widetilde{\mathbb{Z}/2}$-action, where $\widetilde{\mathbb{Z}/2}$ acts on $v_n^{-1}\widehat{MU}$ via the map $\widetilde{\mathbb{Z}/2} \xrightarrow{\tilde{\Delta}} G_1 \times G_2 \to G_1$, which sends the identity component to the identity and the nonidentity component to $\widehat{\sigma}$.

Consider the evaluation map

$$\epsilon : v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal}) \tag{6.1.2}$$

and let $\widetilde{\mathbb{Z}/2}$ act on $E_n(\mathrm{Gal})$ via the map $\widetilde{\mathbb{Z}/2} \xrightarrow{\tilde{\Delta}} G_1 \times G_2 \to G_2$. The map ϵ is equivariant by the following lemma.

Lemma 6.1.3. *Let G be a group and let A and B be R-algebras with G-actions through R-algebra maps. Suppose $\varphi : A \to B$ is equivariant up to homotopy. Then the evaluation map*

$$\epsilon : A \wedge_R F_{R-\mathrm{alg}}(A, B)_\varphi \to B$$

is G-equivariant with respect to the conjugation action on $F_{R-\mathrm{alg}}(A, B)$.

Proof. The diagram

$$\begin{tikzcd} G \wedge_R A \wedge_R F_{R-\mathrm{alg}}(A,B)_\varphi \arrow[r] \arrow[d, "\mathrm{id} \wedge \epsilon"'] & A \wedge_R F_{R-\mathrm{alg}}(A,B)_\varphi \arrow[d, "\epsilon"] \\ G \wedge_R B \arrow[r] & B \end{tikzcd}$$

commutes since

$$g \cdot \epsilon(a, \psi) = g\psi(a) = g\psi((g^{-1}g)a) = g\psi(g^{-1}(ga)) = \epsilon(ga, g \cdot \psi)$$

for every $g \in G, a \in A, \psi \in F_{R-\mathrm{alg}}(A, B)_\varphi$. $\square$

Corollary 6.1.4. *The map*

$$\epsilon : v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal})$$

described in (6.1.2) *is equivariant and therefore induces a map*

$$(v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu)^{h\widetilde{\mathbb{Z}/2}} \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}. \tag{6.1.3}$$

6.2 The spectral sequence for $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$

Now that we have constructed an equivariant map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal})$$

we can calculate the coefficients of $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ using the Bockstein spectral sequence and the calculations of Chapter 3.

Theorem 6.2.1.

$$(E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}})_* \cong \widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

where J is the ideal generated by the relations in Theorem 3.2.1.

Proof. The natural map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to v_n^{-1}\widehat{MU}$$

given by projection onto the first component is equivariant and an equivalence, so it induces a weak equivalence on homotopy fixed points. In particular,

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu$$

and $v_n^{-1}\widehat{MU}$ have isomorphic Bockstein spectral sequences. Moreover, the equivariant map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal})$$

(6.1.2) induces a map of Bockstein spectral sequences. On the E_2-page, the map

$$v_n^{-1}\widehat{MU}_* \to E_n(\mathrm{Gal})_*$$

is the natural one, sending $x_i \mapsto 0$ for $i \neq 2(2^k - 1), 0 \leq k \leq n$, so it factors through $\widehat{E(n)}_*$. This factorization is a map of spectral sequences since the differentials on all three spectral sequences are the same. Therefore, the map

$$\widehat{E(n)}_* \to E_n(\mathrm{Gal})_*$$

is in fact a map of Bockstein spectral sequences. Since it is an isomorphism on E_2, it is an isomorphism of spectral sequences and thus $(E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}})_* \cong (\widehat{E(n)}^{h\mathbb{Z}/2})_*$. Since the latter was computed to be

$$\widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

in Section 3.4, this completes the proof. □

6.2.2 The map $ER(n) \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$

Recall that $(v_n^{-1}\widehat{MU}^{h\mathbb{Z}/2})_*$ is a free $(v_n^{-1}\widehat{BP}^{h\mathbb{Z}/2})_*$-module (see Section 3.5). Let A be the $MU^{h\mathbb{Z}/2}$-module spectrum constructed from $v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}}$ by taking successive cofibers in order to kill the free $(v_n^{-1}\widehat{BP})_*^{h\mathbb{Z}/2}$-module generators of $(v_n^{-1}\widehat{MU}^{h\mathbb{Z}/2})_*$. Let B be the $MU^{h\mathbb{Z}/2}$-module spectrum constructed from B by taking successive cofibers to kill the polynomial generators of $(v_n^{-1}\widehat{BP}^{h\mathbb{Z}/2})_*$ as a $(\widehat{E(n)}^{h\mathbb{Z}/2})_*$-module (again, see Section 3.5).

Proposition 6.2.3. *B is equivalent to $\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$.*

Proof. The map

$$v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}} \to \widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}},$$

arising from the natural map $v_n^{-1}\widehat{MU} \to \widehat{E(n)}$, factors through B. The resulting map $B \to \widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$ induces an isomorphism on coefficients, and is thus an equivalence. □

Since the equivariant map

$$v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to v_n^{-1}\widehat{MU}$$

is an equivalence of MU-modules, it induces an isomorphism

$$(v_n^{-1}\widehat{MU} \wedge_S F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu)^{h\widetilde{\mathbb{Z}/2}} \cong v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}}$$

in the derived category of $MU^{h\mathbb{Z}/2}$-modules. Composing this isomorphism with the map (6.1.3) gives a map

$$v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}} \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}.$$

Since this map induces the obvious quotient map on coefficients, it factors through B and hence through $\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$, yielding a diagram

$$\begin{tikzcd}
v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}} \arrow[r] \arrow[d] & E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}} \\
\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}} \arrow[ur, dashed] &
\end{tikzcd}$$

The dotted arrow is a homotopy isomorphism and hence an equivalence. Precomposing with the natural map

$$ER(n) \to \widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}}$$

gives the map

$$ER(n) \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$$

claimed in the introduction. We have proved the following theorem.

Theorem 6.2.4. *There is an equivalence* $\widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}} \simeq E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ *and a map*

$$ER(n) \to E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$$

that induces an algebraic completion on the level of coefficients. Let

$$\hat{I}_n = (\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}) \subseteq ER(n)_*.$$

The coefficients of $E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}}$ *are*

$$(E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}})_* = (ER(n)_*)^{\wedge}_{\hat{I}_n} = \widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

where J is the ideal generated by the relations

$$\hat{v}_0(0) = 2, \quad x^{2^{k+1}-1}\hat{v}_k(l) = 0, \quad \text{and for } k \leq m,\ \hat{v}_m(l)\hat{v}_k(2^{m-k}s) = \hat{v}_m(l+s)\hat{v}_k(0).$$

The degrees of the generators are

$$|x| = \lambda(n), \quad |v_n^{2^{n+1}}| = 2^{n+2}(2^n-1)^2,$$
$$|\hat{v}_k(l)| = 2(2^k-1) + l2^{k+2}(2^n-1)^2 - 2(2^k-1)(2^n-1)^2.$$

Corollary 6.2.5. *After completion, the $\mathbb{Z}/2$-action of complex conjugation on Johnson-Wilson theory $E(n)$ becomes E_∞, so* $\widehat{E(n)}^{h\mathbb{Z}/2}$ *is a commutative S-algebra.*

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