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String Theory, Strongly Correlated Systems, and Duality Twists

by

Nesty Ray Torres Chicon

A dissertation submitted in partial satisfaction of the

requirements for the degree of

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in

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of the

University of California, Berkeley

Committee in charge:

Professor Ori J. Ganor, Chair

Professor Petr Hořava

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Nesty Ray Torres Chicon

Abstract

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Professor Ori J. Ganor, Chair

In the first part of this dissertation (Chapter 1), I present a construction of a six dimensional $(2, 0)$ -theory model that describes the dynamics of the Fractional Quantum Hall Effect (FQHE). The FQHE appears as part of the low energy description of the Coulomb branch of the A_1 $(2, 0)$ -theory formulated on a geometry $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_k$. At low-energy, the configuration is described in terms of a $4 + 1$ D supersymmetric Yang-Mills (SYM) theory on a cone $(\mathbb{R}^2/\mathbb{Z}_k)$ with additional $2 + 1$ D degrees of freedom at the tip of the cone that include fractionally charged particles. These fractionally charged “quasi-particles” are BPS strings of the $(2, 0)$ -theory wrapped on short cycles. In this framework, a W-boson can be modeled as a bound state of k quasi-particles, which can be used to understand the dynamics of the FQHE.

In the second part of this dissertation (Chapters 2-3), I investigate the $\mathcal{N} = 4$ SYM theory compactified on a circle, with a varying coupling constant (*Janus configuration*) and an S-duality twist. I relate this setup to a three dimensional topological theory and to a dual string theory. The equality of these descriptions is exhibited by matching the operator algebra, and the dimensions of the Hilbert space. Additionally, this dissertation addresses a classic result in number theory, called *quadratic reciprocity*, using string theory language. I present a proof that quadratic reciprocity is a direct consequence of T-duality of Type-II string theory. This is demonstrated by analyzing a partition function of abelian $\mathcal{N} = 4$ SYM theory on a certain supersymmetry-preserving four-manifold with variable coupling constant and a $SL(2, \mathbb{Z})$ -duality twist.

I dedicate this dissertation to my dad, mom, brother, and the love of my life, Mana.

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Chapter 1

Q-balls of Quasi-particles

A toy model of the fractional quantum Hall effect appears as part of the low-energy description of the Coulomb branch of the A_1 $(2, 0)$ -theory formulated on $(S^1 \times \mathbb{R}^2)/\mathbb{Z}_k$, where the generator of \mathbb{Z}_k acts as a combination of translation on S^1 and rotation by $2\pi/k$ on \mathbb{R}^2 . At low energy the configuration is described in terms of a 4+1D Super-Yang-Mills theory on a cone $(\mathbb{R}^2/\mathbb{Z}_k)$ with additional 2+1D degrees of freedom at the tip of the cone that include fractionally charged particles. These fractionally charged “quasi-particles” are BPS strings of the $(2, 0)$ -theory wrapped on short cycles. We analyze the large k limit, where a smooth cigar-geometry provides an alternative description. In this framework a W-boson can be modeled as a bound state of k quasi-particles. The W-boson becomes a Q-ball, and it can be described as a soliton solution of Bogomolnyi monopole equations on a certain auxiliary curved space. We show that axisymmetric solutions of these equations correspond to singular maps from AdS_3 to AdS_2 , and we present some numerical results and an asymptotic expansion.

1.1 Introduction

The fractional quantum Hall effect (FQHE) with filling-factor $1/\mathbf{k}$ ($\mathbf{k} \in \mathbb{Z}$) appears in 2+1D condensed matter systems whose low-energy effective degrees of freedom can be described by the Chern-Simons action

$$I = \frac{\mathbf{k}}{4\pi} \int \mathbf{A}_{\text{int}} \wedge d\mathbf{A}_{\text{int}} + \frac{1}{2\pi} \int \mathbf{A} \wedge d\mathbf{A}_{\text{int}}. \quad (1.1)$$

Here, \mathbf{A} is the electromagnetic gauge field, and \mathbf{A}_{int} is a 2+1D $U(1)$ gauge field that describes the low-energy internal degrees of freedom of the system. It is related to the electromagnetic current by $j = *d\mathbf{A}_{\text{int}}$. Excited states of the system may include quasi-particle excitations that are charged under the gauge symmetry associated with \mathbf{A}_{int} . Such quasi-particles with one unit of \mathbf{A}_{int} -charge will have $1/\mathbf{k}$ electromagnetic charge.

The goal of this chapter is to construct an integrally charged particle as a bound state of quasi-particles using a particularly intuitive string-theoretic toy model of the FQHE. Over

the past two decades several realizations of the integer and fractional quantum Hall effects in string theory have been constructed [1]-[10]. Generally speaking, these constructions engineer the Chern-Simons action (1.1) as a low-energy effective description of a $(d+2)$ -dimensional brane compactified on a d -dimensional space, possibly in the presence of suitable fluxes, to yield the requisite 2 + 1D effective description. In the present chapter, we will begin by constructing an FQHE model by compactifying the 5+1D $(2,0)$ -theory. Our system is a special case of a general class of 2 + 1D theories obtained from the $(2,0)$ -theory by taking three of the dimensions to be a nontrivial manifold. (We note that a beautiful framework for understanding such compactifications has been developed in [11]-[14].) We will focus on a particular aspect of the system which is the dynamics of the quasi-particles that in the condensed-matter system can arise from impurities. As we will see, the quasi-particles and their relationship to the integrally charged particles have a simple geometrical interpretation in terms of the $(2,0)$ theory, as follows. In our construction, the geometry of the extra dimensions will have long 1-cycles and short 1-cycles, the short ones being $1/\mathbf{k}$ the size of the long ones. The quasi-particles will be realized as BPS strings of the $(2,0)$ theory wound around short 1-cycles, while the integrally charged particles will be realized as strings wound around long 1-cycles.

We are especially interested in the limit $\mathbf{k} \gg 1$, where the filling fraction becomes extremely small. This is the strong-coupling limit of the condensed-matter system, and as we will see, our model has a dual description where quasi-particles are elementary and the integrally charged particles can be described as classical solitons, or rather Q-balls, in terms of the fundamental quasi-particle fields. We will show that solutions to the equations of motion describing these solitons correspond to certain singular harmonic maps from AdS_3 to AdS_2 .

The chapter is organized as follows. In §1.2 we describe the $(2,0)$ theory setting for our model. In §1.3 we study the quasi-particles, which are BPS strings, and we calculate their quantum numbers. In §1.4 we study the large \mathbf{k} limit and write down the semiclassical action of the system. In §1.5 we develop the differential equations that describe the integrally charged particles as solitons of the fundamental quasi-particle fields in the large \mathbf{k} limit. We show that they can be mapped to the equations describing a magnetic monopole on a 3D space with metric $ds^2 = x_3^2(dx_1^2 + dx_2^2 + dx_3^2)$. In §1.6 we analyze the soliton equations in more detail and show the connection to harmonic maps from AdS_3 to AdS_2 . The equations are not integrable in the standard sense, and we were unable to solve them in closed form, but we were able to make several additional observations: (i) we present an expansion up to second order in the inverse of the distance from the “center” of the solution to the origin; (ii) using a rather complicated transformation we can recast the equations in terms of a single “potential” function; and (iii) we plot an example of a numerical solution. Points (ii)-(iii) are explored in Appendices A.1-A.2.

1.2 The $(2, 0)$ theory on $(\mathbb{R}^2 \times S^1)/\mathbb{Z}_k$

Our starting point is the 5+1D $A_1(2, 0)$ -theory on $\mathbb{R}^{2,1} \times \mathbf{M}_3$, where $\mathbb{R}^{2,1}$ is 2+1D Minkowski space and $\mathbf{M}_3 \simeq (\mathbb{R}^2 \times S^1)/\mathbb{Z}_k$ is the flat, noncompact, smooth three-dimensional manifold defined as the quotient of $\mathbb{R}^2 \times S^1$ by the isometry that acts as a simultaneous rotation of \mathbb{R}^2 by an angle $2\pi/k$, and a translation of S^1 by $1/k$ of its circumference. The $A_1(2, 0)$ -theory is the low-energy limit of either type-IIB on $\mathbb{R}^4/\mathbb{Z}_2$ [15] or of 2 M5-branes [16] (after decoupling of the center of mass variables). We are interested in the low-energy description of the Coulomb branch of the theory, and in particular in the low-energy degrees of freedom that are localized near the origin of \mathbb{R}^2 . The fractional quantum Hall effect, as we shall see, naturally appears in this context. We will now expand on the details. (See [17] for a related study of M-theory and type-II string theory in this geometry and [18]-[26] for the study of effects on other kinds of branes in a similar geometry.)

1.2.1 The geometry

The space \mathbf{M}_3 can be constructed as a quotient of \mathbb{R}^3 as follows. We parameterize \mathbb{R}^3 by x_3, x_4, x_5 and set $\mathbf{z} \equiv x_4 + ix_5$. Then, \mathbf{M}_3 is defined by the equivalence relation

$$(x_3, \mathbf{z}) \sim (x_3 + 2\pi R, \mathbf{z}e^{-2\pi i/k}), \quad [\text{defining relation of } \mathbf{M}_3] \quad (1.2)$$

where R is a constant parameter that sets the scale, and $k > 1$ is an integer. The Euclidean metric on \mathbf{M}_3 is given by

$$ds^2 = dx_3^2 + dx_4^2 + dx_5^2 = dx_3^2 + |d\mathbf{z}|^2.$$

For future reference we define the $(2k)^{\text{th}}$ root of unity:

$$\omega \equiv e^{\pi i/k}. \quad (1.3)$$

We also set

$$\mathbf{z} = re^{i\theta},$$

so that (1.2) can be written as

$$(x_3, r, \theta) \sim (x_3 + 2\pi R, r, \theta - \frac{2\pi}{k}). \quad (1.4)$$

The $\mathbf{z} = 0$ locus [i.e., the set of points $(x_3, 0)$ with arbitrary x_3] forms an S^1 of radius R that we will call *the minicircle* and denote by S_m^1 . The space $\mathbf{M}_3 \setminus S_m^1$ (which is \mathbf{M}_3 with the minicircle excluded) is a circle-bundle over a cone (with the origin $\{0\}$ excluded):

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathbf{M}_3 \setminus S_m^1 \\ & & \downarrow \\ & & \mathbb{C}/\mathbb{Z}_k \setminus \{0\} \end{array} \quad (1.5)$$

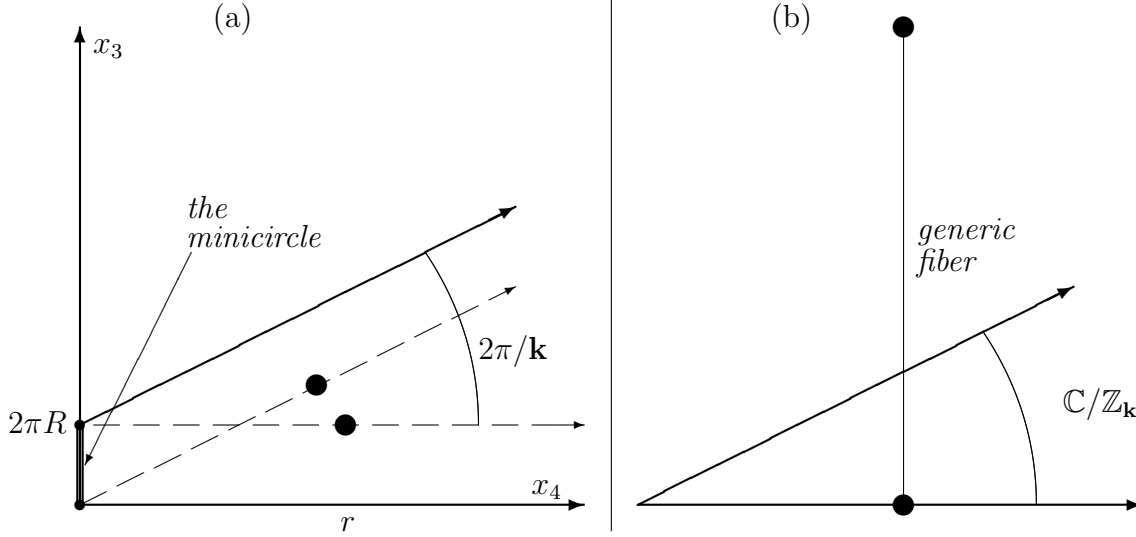


Figure 1.1: (a) The geometry of $\mathbf{M}_3 \simeq (\mathbb{C} \times S^1)/\mathbb{Z}_k$: in the coordinate system $(x_4 + ix_5, x_3)$, the point $(r, 0)$ is identified with $(re^{-2\pi i/k}, 2\pi R)$ and $(r, 2\pi kR)$; The large dots indicate equivalent points; (b) The fibration $\mathbf{M}_3 \rightarrow \mathbb{C}/\mathbb{Z}_k$ with the generic fiber that is of radius kR .

The cone \mathbb{C}/\mathbb{Z}_k is parameterized by \mathbf{z} , subject to the equivalence relation $\mathbf{z} \sim \omega^2 \mathbf{z}$. In polar coordinates the cone is parameterized by (r, θ) with $0 < r < \infty$ and $0 \leq \theta < 2\pi/k$. (θ is understood to have period $2\pi/k$ when describing the cone.) The projection $\mathbf{M}_3 \rightarrow \mathbb{C}/\mathbb{Z}_k$ is given by $(x_3, \mathbf{z}) \mapsto \mathbf{z}$. For a given $\mathbf{z} \neq 0$, the fiber S^1 of the fibration (1.5) over $\mathbf{z} \simeq \omega^2 \mathbf{z}$ is given by all points (x_3, \mathbf{z}) with $0 \leq x_3 < 2\pi kR$. The equivalence (1.2) then implies $(x_3 + 2\pi kR, \mathbf{z}) \sim (x_3, \mathbf{z})$, and so this S^1 has radius kR .

In order to preserve half of the 16 supersymmetries we augment (1.2) by an appropriate R-symmetry twist as follows. Let $\text{Spin}(5) \simeq Sp(2)$ be the R-symmetry of the $(2, 0)$ -theory. In the M5-brane realization of the $(2, 0)$ -theory [16], $\text{Spin}(5)$ is the group of rotations (acting on spinors) in the five directions transverse to the M5-branes, which we take to be $6, \dots, 10$. We now split them into the subsets $6, 7$ and $8, 9, 10$. This corresponds to the rotation subgroup $[\text{Spin}(3) \times \text{Spin}(2)]/\mathbb{Z}_2 \subset \text{Spin}(5)$. Let $\gamma \in \text{Spin}(5)$ correspond to a $2\pi/k$ rotation in the $6, 7$ plane. We then augment the RHS of the geometrical identification (1.2) by an R-symmetry transformation γ . The setting now preserves 8 supersymmetries.

We now go to the Coulomb branch of the $(2, 0)$ -theory by separating the two M5-branes of §1.2.1 in the M-theory direction x_{10} . This breaks $\text{Spin}(3)$ to an $SO(2)$ subgroup (corresponding to rotations in directions $8, 9$) which we denote by $SO(2)_r$. On the Coulomb branch of the $(2, 0)$ -theory there is a BPS string whose tension we denote by \tilde{V} .

At energies $E \ll 1/kR$, sufficiently far away from S_m^1 , the dynamics of the $(2, 0)$ -theory on $\mathbb{R}^{2,1} \times \mathbf{M}_3$ reduces to $SU(2)$ 4+1D Super-Yang-Mills theory on $\mathbb{R}^{2,1} \times (\mathbb{C}/\mathbb{Z}_k)$. The coupling

constant is given by

$$\frac{4\pi^2}{g_{\text{ym}}^2} = \frac{1}{\mathbf{k}R}. \quad (1.6)$$

All fields are functions of the coordinates $(x_0, x_1, x_2, r, \theta)$, but the periodicity $\theta \sim \theta + 2\pi/\mathbf{k}$ is modified in two ways:

- The shift by $2\pi R$ in x_3 , expressed in (1.4), implies that as we cross the $\theta = 2\pi/\mathbf{k}$ ray a translation by $2\pi R$ in x_3 is needed in order to patch smoothly with the $\theta = 0$ ray. Since x_3 -momentum corresponds to conserved instanton charge in the low-energy SYM, we find that we have to add to the standard SYM action an additional term

$$\frac{1}{16\mathbf{k}\pi} \int_{\theta=0} \text{tr}(F \wedge F), \quad (1.7)$$

where the integral is performed on the ray at $\theta = 0$.

- the R-symmetry twist γ introduces phases in the relation between values of fields at $\theta = 0$ and at $\theta = 2\pi/\mathbf{k}$. Of the five (gauge group adjoint-valued) scalar fields Φ^6, \dots, Φ^{10} (corresponding to M5-brane fluctuations in directions 6, \dots , 10) the last three $\Phi^8, \Phi^9, \Phi^{10}$ are neutral under γ and hence periodic in θ , while the combination $Z \equiv \Phi^6 + i\Phi^7$ satisfies

$$Z(x_0, x_1, x_2, r, \theta + \frac{2\pi}{\mathbf{k}}) = \omega^2 \Omega^{-1} Z(x_0, x_1, x_2, r, \theta) \Omega. \quad (1.8)$$

where we have included an arbitrary gauge transformation $\Omega(x_0, x_1, x_2, r) \in SU(2)$. The gluinos have similar boundary conditions with appropriate $\exp(\pm\pi/\mathbf{k})$ phases.

At the origin, $\mathbf{z} = 0$, which is the tip of the cone \mathbb{C}/\mathbb{Z}_k , boundary conditions need to be specified and additional 2+1D degrees of freedom need to be added. These degrees of freedom and their interactions with the bulk SYM fields are the main focus of this chapter and will be discussed in §1.2.4. But at this point we can make a quick observation. When a BPS string of the $(2, 0)$ -theory wraps the S^1 of (1.5) we get the W -boson of the effective 4+1D SYM. The circle has radius $\mathbf{k}R$ and so the mass of the W -boson is $2\pi\mathbf{k}R\tilde{V}$. On the other hand, the BPS string can also wrap the minicircle S_m^1 whose radius is only R . (A similar effect has been pointed out in [17] in the context of type-IIA string theory on this same geometry.) The resulting particle in 2+1D has mass $2\pi R\tilde{V}$ which is $1/\mathbf{k}$ of the mass of the W -boson. Its charge is also $1/\mathbf{k}$ of the charge of the W -boson. This is our first hint that we are dealing with a system that exhibits a fractional quantum Hall effect (FQHE). We will soon see that indeed a BPS string that wraps S_m^1 can be identified with a *quasiparticle* of FQHE.

1.2.2 Symmetries

Now, let us discuss the symmetries of the theory at a generic point on the Coulomb branch. The continuous isometries of \mathbf{M}_3 are generated by translations of x_3 and rotations of the

\mathbf{z} -plane. We denote the latter by $SO(2)_{\mathbf{z}}$ and normalize the respective charge so that the differential $d\mathbf{z}$ has charge +1. The isometry group of \mathbf{M}_3 also contains a discrete \mathbb{Z}_2 factor generated by the orientation-preserving isometry

$$(x_3, \mathbf{z}) \mapsto (-x_3, \bar{\mathbf{z}}).$$

This by itself does not preserve our setting because it converts the R-symmetry twist γ to γ^{-1} . To cure this problem, we introduce an extra reflection $x_7 \rightarrow -x_7$ in the plane on which γ acts, and finally, in order to preserve parity we also introduce one more reflection in a transverse direction, say, $x_{10} \rightarrow -x_{10}$. Altogether, we define the discrete symmetry \mathbb{Z}'_2 to be generated by

$$(x_0, x_1, x_2, x_3, \mathbf{z}, x_6, x_7, x_8, x_9, x_{10}) \mapsto (x_0, x_1, x_2, -x_3, \bar{\mathbf{z}}, x_6, -x_7, x_8, x_9, -x_{10}). \quad [\mathbb{Z}'_2] \quad (1.9)$$

The $SO(2)$ subgroup of the R-symmetry that corresponds to rotations in the 6 – 7 plane will be referred to as $SO(2)_{\gamma}$ and normalized so that $\Phi^6 + i\Phi^7$ has charge +1. The $SU(2) = \text{Spin}(3)$ subgroup of the R-symmetry that corresponds to rotations in the 8, 9, 10 directions will be referred to as $SU(2)_R$. For future reference we also denote the $SO(2)$ subgroup of rotations in the 8, 9 plane by $SO(2)_r$.

The parity symmetry of M-theory [27], which acts as reflection on an odd number of dimensions combined with a reversal of the 3-form gauge field ($C_3 \rightarrow -C_3$) can also be used to construct a symmetry of our background. We define \mathbb{Z}''_2 as the discrete symmetry generated by the reflection that acts as

$$x_{10} \rightarrow -x_{10}, \quad C_3 \rightarrow -C_3. \quad [\mathbb{Z}''_2]$$

This symmetry preserves the M5-brane configuration and the twist. We have summarized the symmetries in the table below.

$SO(2)_{\mathbf{z}}$	rotations of the \mathbf{z} ($x_4 - x_5$) plane
$SO(2)_{\gamma}$	rotations of the $x_6 - x_7$ plane
$SU(2)_R$	rotations of the x_8, x_9, x_{10} plane
$SO(2)_r$	rotations of the x_8, x_9 plane
\mathbb{Z}'_2	reflection in directions x_3, x_5, x_7, x_{10}
\mathbb{Z}''_2	reflection in direction x_{10} (and $C_3 \rightarrow -C_3$)

Table 1.1: Symmetries of the theory.

We denote the conserved charges associated with $SO(2)_{\mathbf{z}}$, $SO(2)_{\gamma}$, and $SO(2)_r$ by $q_{\mathbf{z}}$, q_{γ} , and q_r , respectively. These are the spins in the 4 – 5, 6 – 7, and 8 – 9 planes. The supersymmetry generators are also charged under these groups, and the background preserves those supercharges for which $q_{\mathbf{z}} + q_{\gamma} = 0$. These observations will become useful in §1.3, where we will study the quantum numbers of the quasi-particles.

1.2.3 Relation to D3-(p, q)5-brane systems

As we have seen in §1.2.1, following dimensional reduction on the S^1 fiber of (1.5), we get a low-energy description in terms of 4+1D SYM on the cone $\mathbb{C}/\mathbb{Z}_{\mathbf{k}}$, interacting with additional (as yet unknown, but to be described below) degrees of freedom at the tip of the cone (at $x_4 = x_5 = 0$). These additional degrees of freedom are three-dimensional and can be expressed in terms of $SU(2)$ Chern-Simons theory coupled to the IR limit of a $U(1)$ gauge theory with two charged hypermultiplets (with $\mathcal{N} = 4$ supersymmetry in 2+1D). The latter is the self-mirror theory introduced in [28], and named $T(SU(2))$ by Gaiotto and Witten [7]. The arguments leading to the identification of the degrees of freedom at the tip of the cone were presented, in a somewhat different but related context, in [29]. The idea is to relate the local degrees of freedom of M-theory on the geometry of §1.2.1 to those of a (p, q) 5-brane of type-IIB, as originally done in [17], and then map our two M5-branes to two D3-branes, to obtain the problem of two D3-branes ending on a (p, q) 5-brane. This problem was solved in [7] in terms of $T(SU(2))$ (and see also [30] for previous work on this subject, and [31] for generalizations with less supersymmetry). The Gaiotto-Witten solution thus also furnishes the solution to our problem. On the Coulomb branch, the gauge part of the system reduces to $U(1)$ Chern-Simons theory interacting with $T(U(1))$, which reproduces (1.1). Although the details of the argument will not be needed for the rest of this chapter, we will review them below for completeness. More details can be found in [29].

Our geometry in directions 3, \dots , 7 is of the form $(S^1 \times \mathbb{C}^2)/\mathbb{Z}_{\mathbf{k}}$, and leads to a $(1, \mathbf{k})$ 5-brane according to [17]. This was demonstrated in [17] by replacing \mathbb{C}^2 with a Taub-NUT space, whose metric can be written as

$$ds^2 = \tilde{R}^2 \left(1 + \frac{\tilde{R}}{2\tilde{r}}\right)^{-1} (dy + \sin^2(\frac{\tilde{\theta}}{2}) d\tilde{\phi})^2 + \left(1 + \frac{\tilde{R}}{2\tilde{r}}\right) [d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2)], \quad (1.10)$$

where y is a periodic coordinate with range $0 \leq y < 2\pi$. We then introduce the S^1 , parameterized by x_3 as in (1.2). The plane \mathbb{C} that appears in (1.2) is now embedded in the \mathbb{C}^2 tangent space of the Taub-NUT space at the origin $\tilde{r} = 0$, and is recovered in the limit $\tilde{R} \rightarrow \infty$. In that limit, and with a change of variables $\tilde{r} = r^2/\tilde{R}$, we can identify the \mathbb{C} plane of (1.2) as a plane at constant $(\tilde{\theta}, \tilde{\phi})$ (say $\tilde{\theta} = \pi/2$ and $\tilde{\phi} = 0$), and the $\mathbf{z} \equiv x_4 + ix_5$ coordinate of (1.2) is identified with

$$\mathbf{z} = r e^{iy} = \sqrt{\tilde{R}\tilde{r}} e^{iy}.$$

In this limit ($\tilde{R} \rightarrow \infty$), the x_6, x_7 plane is identified with a plane transverse to the \mathbf{z} -plane, which we can take to be given by $\tilde{\theta} = \pi/2$ and $\tilde{\phi} = \pi$. We now return to the finite \tilde{R} geometry, and impose the $\mathbb{Z}_{\mathbf{k}}$ equivalence of (1.2) by setting

$$(x_3, y, \tilde{r}, \tilde{\theta}, \tilde{\phi}) \sim (x_3 + 2\pi R, y - \frac{2\pi}{\mathbf{k}}, \tilde{r}, \tilde{\theta}, \tilde{\phi}).$$

We then wrap two M5-branes on the $(\tilde{\theta} = \frac{\pi}{2}, \tilde{\phi} = 0)$ subspace of this 5-dimensional geometry. In the limit $\tilde{R} \rightarrow \infty$ this reproduces the setting of §1.2.1.

The technique that Witten employed in [17] is to convert the Taub-NUT geometry to a D6-brane by reduction on the y -circle from M-theory to type-IIA, and then apply T-duality on the x_3 -circle to get type-IIB with a complex string coupling constant of the form

$$\tau_{\text{IIB}} = \frac{2\pi i}{g_{\text{IIB}}} - \frac{1}{\mathbf{k}}.$$

This turns out to be strongly coupled ($g_{\text{IIB}} \rightarrow \infty$) in the limit $\tilde{R} \rightarrow \infty$, but it can, in turn, be converted to weak coupling with an $\text{SL}(2, \mathbb{Z})$ transformation

$$\tau_{\text{IIB}} \rightarrow \tau'_{\text{IIB}} = \frac{\tau_{\text{IIB}}}{\mathbf{k}\tau_{\text{IIB}} + 1} = \frac{1}{\mathbf{k}} + \frac{ig_{\text{IIB}}}{2\pi\mathbf{k}^2} \rightarrow \frac{1}{\mathbf{k}} + i\infty.$$

As explained in [17], the combined transformations convert the Taub-NUT geometry to a 5-brane of (p, q) -type $(1, \mathbf{k})$ (where \mathbf{k} is the NS5-charge and 1 is the D5-charge). It also converts the M5-branes to D3-branes. The boundary degrees of freedom where the two D3-branes end on the $(1, \mathbf{k})$ 5-brane were found in [7] as follows. Let A denote the boundary 2+1D value of the $SU(2)$ gauge field of the D3-branes (with the superpartners left implicit). Using the identity

$$\begin{pmatrix} 1 \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & (-\mathbf{k}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we see that we can obtain a $(1, \mathbf{k})$ 5-brane from an NS5-brane by applying an $\text{SL}(2, \mathbb{Z})$ transformation that acts as $\tau \rightarrow \tau - \mathbf{k}$, followed by another transformation that acts as $\tau \rightarrow -1/\tau$. Each transformation can be implemented on the boundary conditions. The $\tau \rightarrow \tau - \mathbf{k}$ transformation introduces a level- \mathbf{k} Chern-Simons theory expressed in terms of an ancillary $SU(2)$ gauge field that we denote by A' , and the $\tau \rightarrow -1/\tau$ (S-duality) transformation introduces 2+1D degrees of freedom, named $T(SU(2))$ by Gaiotto and Witten, that couple to both the A and A' gauge fields. $T(SU(2))$ was identified with the Intriligator-Seiberg theory [28] that is defined as the low-energy limit of $\mathcal{N} = 4$ $U(1)$ gauge theory coupled to two hypermultiplets. The theory has a classical $SU(2)$ flavor symmetry (which will ultimately couple to, say, the gauge field A), and it also has a $U(1)$ global symmetry under which only magnetic operators are charged, and this symmetry is enhanced to $SU(2)$ in the (strongly coupled) low-energy limit. This $SU(2)$ is then coupled to A' . It is also not hard to check that A is the $r \rightarrow 0$ limit of the 4+1D gauge field on the cone. To see this, consider the T^2 formed by varying (x_3, y) for fixed $\tilde{r}, \tilde{\theta}$, and $\tilde{\phi}$. The $\text{SL}(2, \mathbb{Z})$ transformation $\begin{pmatrix} 1 & 0 \\ \mathbf{k} & 1 \end{pmatrix}$ converts 1-cycle from $(0, 0)$ to $(2\pi R, -2\pi/\mathbf{k})$ into the 1-cycle from $(0, 0)$ to $(2\pi\mathbf{k}R, 0)$, and this is precisely the 1-cycle used in the reduction from the $(2, 0)$ -theory to 4+1D SYM.

1.2.4 Appearance of the fractional quantum Hall effect

On the Coulomb branch the $SU(2)$ gauge group of 4+1D SYM is broken to $U(1)$. At energies below the breaking scale, the $SU(2)$ gauge fields A and A' reduce to $U(1)$ gauge fields which

we denote by \mathbf{A} and \mathbf{A}_{int} . The theory $T(SU(2))$ reduces to $T(U(1))$ which is described by the action [7] $(1/2\pi) \int \mathbf{A} \wedge d\mathbf{A}_{\text{int}}$. The total gauge part of the action at the tip of the cone is therefore given by (1.1). As we have already seen, the BPS strings that wrap the minicircle S_m^1 have fractional charge $1/\mathbf{k}$ under the bulk \mathbf{A} , which we have now identified as the unbroken $U(1)$ gauge field of the bulk 4+1D SYM. If we slowly move such a string away from the tip, we get a string that, in the (x_3, y) coordinates of §1.2.3, wraps the 1-cycle from $(0, 0)$ to $(2\pi R, -2\pi/\mathbf{k})$. This implies that it has one unit of charge under \mathbf{A}_{int} , which lends credence to the proposal of identifying such a string with a quasi-particle of FQHE. The quasi-particle is confined to $\mathbb{R}^{2,1}$, because everywhere else a wound string is longer than the BPS bound $2\pi R$.

Following the breaking of $SU(2)$ to $U(1)$, the bulk 4+1D W -boson gets a mass. The W -boson corresponds to a $(2, 0)$ -string wound around the S^1 fiber of (1.5), and the homotopy class of the bulk S^1 fiber is \mathbf{k} times the homotopy class of the minicircle S_m^1 . It is therefore clear that, in principle, we should be able to design a process in which a bulk W -boson reaches the tip of the cone and breaks-up into \mathbf{k} strings that wrap the minicircle:

$$W \longrightarrow \mathbf{k} \text{ quasi-particles.} \quad (1.11)$$

Alternatively, it should be possible to describe the W -boson as a bound state of \mathbf{k} quasi-particles. In §1.4-§1.6, we will show how this works in the limit of large \mathbf{k} . Before we proceed to the analysis, which is the main focus of our chapter, let us compute the spin quantum numbers of the quasi-particles.

1.3 Quasi-particles

The quasi-particle is obtained by wrapping the $(2, 0)$ BPS string on the minicircle S_m^1 . Its quantum numbers can be deduced by quantizing the zero-modes of the low-energy fermions that live on the BPS string. Let us begin by reviewing the low-energy fermionic degrees of freedom on a BPS string. We assume that the M5-branes are in directions $0, \dots, 5$, separated in direction 10, and the BPS string is in direction x_3 . We first ignore the equivalence (1.2) and the R-symmetry twist. For simplicity we will now refer to rotation groups as $SO(m)$ instead of $\text{Spin}(m)$. Thus, the VEV breaks the R-symmetry to $SO(4)_R \subset SO(5)_R$, and the presence of the string breaks the Lorentz group down to $SO(1, 1) \times SO(4)$. We will denote the last factor by $SO(4)_T$, and we will describe representations of $SO(1, 1) \times SO(4)_T \times SO(4)_R$ as $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)_s$, where $(\mathbf{r}_1, \mathbf{r}_2)$ is a representation of $SO(4)_T \sim SU(2) \times SU(2)$, $(\mathbf{r}_3, \mathbf{r}_4)$ is a representation of $SO(4)_R \sim SU(2) \times SU(2)$, and s is an $SO(1, 1)$ charge (spin). The representation of the unbroken supersymmetry charges is the same as the supersymmetry that is preserved by an M2-brane ending on an M5-brane. If the M2-brane is in directions $0, 3, 10$ and the M5-brane is in directions $0, 1, 2, 3, 4, 5$ then a preserved SUSY parameter ϵ satisfies

$$\epsilon = \Gamma^{031} \epsilon = \Gamma^{012345} \epsilon, \quad (1.12)$$

where we denote $\natural \equiv 10$, to avoid ambiguity. The SUSY parameter therefore transforms as

$$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})_{+\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_{-\frac{1}{2}}.$$

On the worldsheet of the BPS string there are 4 scalars X^A ($A = 1, 2, 4, 5$) that correspond to translations of the string in transverse directions. These are in the representation $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_0$. In addition, there are fermions in

$$(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})_{+\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})_{-\frac{1}{2}}. \quad (1.13)$$

Now, consider this theory on $\mathbb{R}^{2,1} \times \mathbf{M}_3$ and let the BPS string be at rest at $x_1 = x_2 = 0$. It thus breaks the Lorentz group $SO(2, 1)$ to the rotation group $SO(2)$ in the $x_1 - x_2$ plane, which we denote by $SO(2)_J$. The representations appearing in the brackets of (1.13) refer to $SO(4)_T \times SO(4)_R$, but in our setting, according to the discussion above, we have to reduce $SO(4)_T \rightarrow SO(2)_J \times SO(2)_z$ and $SO(4)_R \rightarrow SO(2)_\gamma \times SO(2)_r$. Thus, denoting representations as

$$(q_J, q_z, q_\gamma, q_r)_s, \quad (1.14)$$

we decompose the left-moving spinors of (1.13) as

$$\left(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)_{+\frac{1}{2}} \oplus \left(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right)_{+\frac{1}{2}} \oplus \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)_{+\frac{1}{2}} \oplus \left(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right)_{+\frac{1}{2}} \quad (1.15)$$

and the right-movers as

$$\left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)_{-\frac{1}{2}} \oplus \left(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)_{-\frac{1}{2}} \oplus \left(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)_{-\frac{1}{2}} \oplus \left(-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)_{-\frac{1}{2}} \quad (1.16)$$

These modes can be described by fermionic fields on the string worldsheet, which are functions of (x_0, x_3) . To get the quantum numbers of the lowest-energy multiplet we need to find the zero-modes of these fermionic fields. For that, we need to know the boundary conditions of these fields in the x_3 direction. Due to the rotation by $2\pi/\mathbf{k}$ in the $x_4 - x_5$ and $x_6 - x_7$ planes that were introduced in §1.2.1, there are nontrivial phases in the boundary conditions of some of the fields that appear in (1.15)-(1.16). The boundary conditions on a field $\psi(x_0, x_3)$ with charges q_z and q_γ are

$$\psi(x_0, x_3 + 2\pi R) = \omega^{2(q_z + q_\gamma)} \psi(x_0, x_3). \quad (1.17)$$

The only zero modes are therefore of those modes with $q_z + q_\gamma = 0$. These have quantum numbers

$$\left(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)_{+\frac{1}{2}} \oplus \left(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right)_{+\frac{1}{2}} \oplus \left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)_{-\frac{1}{2}} \oplus \left(-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)_{-\frac{1}{2}} \quad (1.18)$$

Quantizing these modes gives a multiplet with quantum numbers

$$\begin{pmatrix} q_J^{(0)} - \frac{1}{2}, & q_z^{(0)}, & q_\gamma^{(0)}, & q_r^{(0)} - \frac{1}{2}, & (q_J^{(0)}, & q_z^{(0)} + \frac{1}{2}, & q_\gamma^{(0)} - \frac{1}{2}, & q_r^{(0)}), \\ (q_J^{(0)}, & q_z^{(0)} - \frac{1}{2}, & q_\gamma^{(0)} + \frac{1}{2}, & q_r^{(0)}), & (q_J^{(0)} + \frac{1}{2}, & q_z^{(0)}, & q_\gamma^{(0)}, & q_r^{(0)} + \frac{1}{2}), \end{pmatrix} \quad (1.19)$$

where the charges $q_J^{(0)}$, $q_{\mathbf{z}}^{(0)}$, $q_\gamma^{(0)}$, $q_r^{(0)}$ still need to be determined. To determine them, consider the discrete symmetry \mathbb{Z}'_2 , defined in §1.2.2. It preserves the setting and the BPS particle but does not commute with all the charges q_J , $q_{\mathbf{z}}$, q_γ , q_r . It acts on the charges as follows:

$$q_J \rightarrow q_J, \quad q_{\mathbf{z}} \rightarrow -q_{\mathbf{z}}, \quad q_\gamma \rightarrow -q_\gamma, \quad q_r \rightarrow q_r. \quad [\text{generator of } \mathbb{Z}'_2]$$

The constants $q_J^{(0)}$, $q_{\mathbf{z}}^{(0)}$, $q_\gamma^{(0)}$, $q_r^{(0)}$ must therefore be chosen so that the charges (1.19) will be invariant, as a set, under \mathbb{Z}'_2 . In other words, \mathbb{Z}'_2 is allowed to permute the states in (1.19), but must convert an allowed state to an allowed state. This is only possible if both $q_{\mathbf{z}}^{(0)}$ and $q_\gamma^{(0)}$ vanish. The BPS states are therefore in a multiplet with quantum numbers given by:

$$(q_J^{(0)} - \frac{1}{2}, 0, 0, q_r^{(0)} - \frac{1}{2}) \oplus (q_J^{(0)}, +\frac{1}{2}, -\frac{1}{2}, q_r^{(0)}) \oplus (q_J^{(0)}, -\frac{1}{2}, +\frac{1}{2}, q_r^{(0)}) \oplus (q_J^{(0)} + \frac{1}{2}, 0, 0, q_r^{(0)} + \frac{1}{2}).$$

Note that the setting of (1.4) can be defined for any value of \mathbf{k} , not necessarily an integer (as suggested in [17]). We can then easily determine $q_J^{(0)}$ and $q_r^{(0)}$ in the limit $\mathbf{k} \rightarrow \infty$ at which the multiplet must become part of the multiplet of the wrapped string of the (2, 0)-theory. This determines the charges up to an overall sign (which can be determined arbitrarily and flipped with a parity transformation). So we pick $q_J^{(0)} = -q_r^{(0)} = \frac{1}{2}$ and find the following multiplet structure:

$$(0, 0, 0, -1) \oplus (+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \oplus (+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}) \oplus (1, 0, 0, 0), \quad (\mathbf{k} \rightarrow \infty) \quad (1.20)$$

This is as far as we can go with an analysis of the quantum mechanics of the zero modes. We can do better by considering the full 1+1D low-energy effective action on a string wrapped on the minicircle whose worldsheet is in directions (x_0, x_3) . This is a 1+1D CFT of 4 free bosons together with 4 free left-moving and 4 free right-moving fermions in the representations given by (1.15)-(1.16). Half of the fermionic fields have twisted boundary conditions with nontrivial phases, according to (1.17), and the other half have periodic boundary conditions, whose zero modes we quantized above. The CFT of the 4 fermionic fields (2 left-moving and 2 right-moving) whose boundary conditions include nontrivial phases has a unique ground state, but quantum corrections lead to corrections to the q_J and q_r quantum numbers of this ground state. That, in turn, leads to $\frac{1}{\mathbf{k}}$ corrections to the q_J and q_r charges, as we will now explain.¹ We recall from basic 1+1D conformal field theory that a free complex left-moving fermion satisfying the boundary condition $\psi(x_0 + x_3 + 2\pi R) = e^{2\pi i\nu}\psi(x_0 + x_3)$ with $0 < \nu < 1$, and charged under a global $U(1)$ symmetry such that ψ has charge q and $\bar{\psi}$ has charge $-q$, has a unique ground state with charge $(\frac{1}{2} - \nu)q$. For a right-moving fermion with boundary condition $\psi(x_0 - x_3 - 2\pi R) = e^{2\pi i\nu}\psi(x_0 - x_3)$ the ground state charge is $(\nu - \frac{1}{2})q$. For $\nu = 0$ (periodic Ramond-Ramond boundary conditions) there are two ground states with charge $\pm\frac{1}{2}q$. The charge assignments of the fermions were calculated in (1.15)-(1.16). We set $q = q_r$ or $q = q_J$ and according to (1.17), we need to set $\nu = \frac{1}{\mathbf{k}}(q_{\mathbf{z}} + q_\gamma)$. The bosonic fields with

¹ The $1/\mathbf{k}$ correction to the spin discussed below was missed in an earlier version of this chapter. We corrected this part of §1.3 following a related observation in [32].

twisted boundary conditions have neither q_r nor q_J charge, and so do not contribute to the ground state charge. Combining the modes in (1.15)-(1.16), we find that the left-moving sector of the CFT has ground states of q_J charge $\pm\frac{1}{4} + \frac{1}{2}(\frac{1}{2} - \frac{1}{\mathbf{k}})$ and the right-moving sector has ground states of q_J charge $\pm\frac{1}{4} - \frac{1}{2}(\frac{1}{\mathbf{k}} - \frac{1}{2})$. For q_r we find that the left-moving sector of the CFT has ground states of charge $\pm\frac{1}{4} - \frac{1}{2}(\frac{1}{2} - \frac{1}{\mathbf{k}})$ and the right-moving sector has ground states of charge $\pm\frac{1}{4} + \frac{1}{2}(\frac{1}{\mathbf{k}} - \frac{1}{2})$. Altogether, we find the quantum-corrected quasi-particle quantum numbers:

$$\left(-\frac{1}{\mathbf{k}}, 0, 0, -1 + \frac{1}{\mathbf{k}}\right) \oplus \left(+\frac{1}{2} - \frac{1}{\mathbf{k}}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} + \frac{1}{\mathbf{k}}\right) \oplus \left(+\frac{1}{2} - \frac{1}{\mathbf{k}}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2} + \frac{1}{\mathbf{k}}\right) \oplus \left(1 - \frac{1}{\mathbf{k}}, 0, 0, \frac{1}{\mathbf{k}}\right), \quad (1.21)$$

As a corollary, we can immediately restrict the types of processes described in (1.11). Let us write down the q_J , q_z , q_γ , and q_r quantum numbers of the W -boson supermultiplet. The bosons (vectors and scalars) are in

$$\left(\pm 1, 0, 0, 0\right) \oplus \left(0, \pm 1, 0, 0\right) \oplus \left(0, 0, \pm 1, 0\right) \oplus \left(0, 0, 0, \pm 1\right). \quad (1.22)$$

and the gluinos are in

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right). \quad [\text{even number of } (-\frac{1}{2})\text{'s}] \quad (1.23)$$

Starting with, say, a W -boson with charges $(-1, 0, 0, 0)$, consider a process such as

$$W\text{-boson} \longrightarrow \mathbf{k} \text{ quasi-particles}. \quad (1.24)$$

By examining q_r charge conservation, we see that out of the \mathbf{k} quasi-particles either (i) $(\mathbf{k} - 1)$ quasi-particles are of charge $(1 - \frac{1}{\mathbf{k}}, 0, 0, \frac{1}{\mathbf{k}})$, and one is of charge $(-\frac{1}{\mathbf{k}}, 0, 0, -1 + \frac{1}{\mathbf{k}})$, or (ii) $(\mathbf{k} - 2)$ are of charge $(1 - \frac{1}{\mathbf{k}}, 0, 0, \frac{1}{\mathbf{k}})$, one is of charge $(+\frac{1}{2} - \frac{1}{\mathbf{k}}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} + \frac{1}{\mathbf{k}})$, and one is of charge $(+\frac{1}{2} - \frac{1}{\mathbf{k}}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2} + \frac{1}{\mathbf{k}})$. Therefore, examining the q_J charge, we see that $(\mathbf{k} - 1)$ units of orbital angular momentum need to convert into spin. We therefore expect that if the typical product quasi-particle's velocity u in the $x_1 - x_2$ plane is small, the amplitude will be suppressed by a factor of $u^{\mathbf{k}-1}$.

The process (1.24) also suggests that the W boson can be viewed as a bound state of \mathbf{k} quasi-particles. This is similar to the well-known result in FQHE theory that in some contexts the electron can be regarded as a bound state of \mathbf{k} fractionally charged edge-states. The edge-states are the low-energy excitations of the Chern-Simons theory that reside on the boundary, or on impurities in the bulk. In this analogy, our quasi-particles correspond to external impurities that couple to the Chern-Simons theory gauge field. The fractional corrections of $\frac{1}{\mathbf{k}}$ that we found for the spin of the quasi-particles are consistent with the well-known anyonic properties of quasi-particles of the FQHE.

Our goal is to develop a concrete description of the W -boson as a composite of \mathbf{k} quasi-particles. For this purpose we will first need to switch to a dual formulation of the low-energy theory whereby the quasi-particles are fundamental.

1.4 The large \mathbf{k} limit

A weakly-coupled dual formulation of our system can be constructed in the limit $\mathbf{k} \rightarrow \infty$. In FQHE terminology, this is the *small filling fraction* regime which in ordinary systems corresponds to very strong interactions. More insight can be gained in this limit by choosing a different fibration structure for \mathbf{M}_3 than the one represented in (1.5). While (1.5) is convenient to work with, because the fibers are of constant size and are geodesics, the fibration is singular at the origin $\mathbf{z} = 0$ — indeed the tip of the cone is singular, and the fiber over $\mathbf{z} = 0$ is smaller by a factor of \mathbf{k} from the generic one.

Instead, in this section we will represent \mathbf{M}_3 as a smooth fibration in another way. The base is the well-known cigar geometry and the fiber corresponds to a loop at constant $|\mathbf{z}|$. (See also [33, 34] for other uses of this technique.) We will then reduce the (2, 0)-theory to 4 + 1D SYM along this fiber. The fiber’s size varies and the base’s geometry is curved, but nevertheless this representation is very useful, as we shall see momentarily. (See for example [35, 36] for recent discussions of dimensional reductions of this type.)

1.4.1 Cigar geometry

To arrive at the alternative fibration we change variables on \mathbf{M}_3 from (x_3, \mathbf{z}) to x_3 and

$$\tilde{\mathbf{z}} \equiv \exp\left(\frac{ix_3}{\mathbf{k}R}\right) \mathbf{z} \equiv r e^{i\bar{\theta}}. \quad (1.25)$$

We then write the metric as

$$ds^2 = dx_3^2 + |d\mathbf{z}|^2 = \tilde{\alpha}(dx_3 - \frac{r^2}{\mathbf{k}R\tilde{\alpha}}d\bar{\theta})^2 + dr^2 + \tilde{\alpha}^{-1}r^2d\bar{\theta}^2, \quad (\tilde{\alpha} \equiv 1 + \frac{r^2}{\mathbf{k}^2R^2}) \quad (1.26)$$

This metric describes a circle fibration over a cigar-like base with metric

$$ds_B^2 = dr^2 + \tilde{\alpha}^{-1}r^2d\bar{\theta}^2 = dr^2 + \left(\frac{\mathbf{k}^2R^2r^2}{\mathbf{k}^2R^2 + r^2}\right)d\bar{\theta}^2. \quad (1.27)$$

We denote the cigar space by Υ . Note that the cigar-metric is smooth everywhere and for $r \gg \mathbf{k}R$ it behaves like a cylinder $\mathbb{R}_+ \times S^1$, where S^1 has radius $\mathbf{k}R$. The “global angular form” of the circle fibration is

$$\chi \equiv dx_3 - \frac{r^2}{\mathbf{k}R\tilde{\alpha}}d\bar{\theta} \equiv dx_3 - R\mathbf{a}, \quad (1.28)$$

where we have defined the 1-form

$$\mathbf{a} \equiv \frac{r^2}{\mathbf{k}R^2\tilde{\alpha}}d\bar{\theta} = \left(\frac{\mathbf{k}r^2}{\mathbf{k}^2R^2 + r^2}\right)d\bar{\theta}. \quad (1.29)$$

In this context, \mathbf{a} is a $U(1)$ gauge field on the cigar with associated field-strength

$$d\mathbf{a} = -\frac{1}{R}d\chi = \frac{2\mathbf{k}^3R^2r}{(\mathbf{k}^2R^2 + r^2)^2}dr \wedge d\bar{\theta}.$$

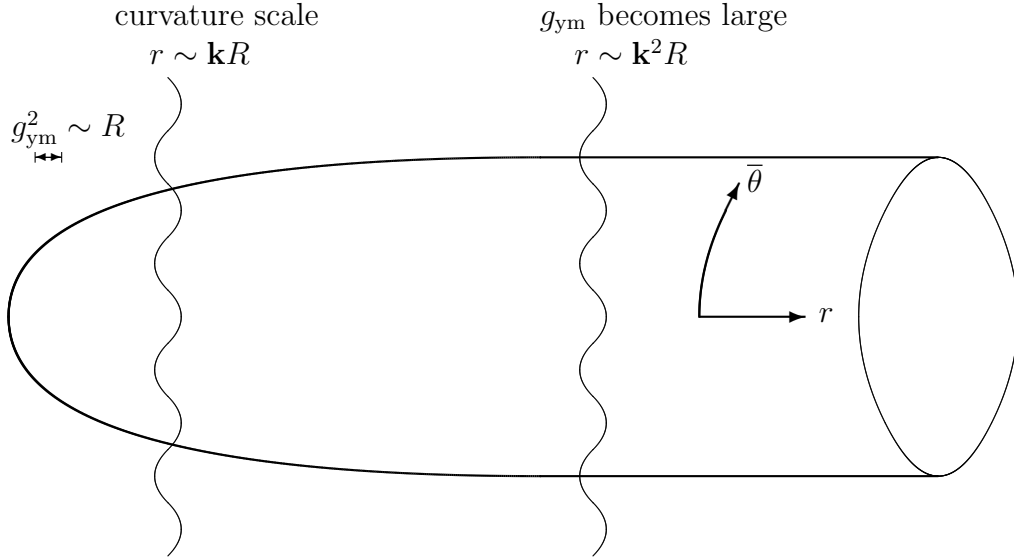


Figure 1.2: The cigar geometry with the typical scales indicated. The curvature of the cigar sets the length scale $\mathbf{k}R$, and the 4 + 1D SYM coupling constant sets the length scale g_{ym}^2 .

The total magnetic flux of the gauge field \mathbf{a} is $\int_B d\mathbf{a} = 2\pi\mathbf{k}$.

An anti-self-dual field $H = -*H$ on $\mathbf{M}_3 \times \mathbb{R}^{2,1}$ can be reduced along the fibers of the circle fibration (1.26) to obtain a 4 + 1D gauge field strength f on $\Upsilon \times \mathbb{R}^{2,1}$ as follows:

$$H = \left(dx_3 - \frac{r^2}{\mathbf{k}R\tilde{\alpha}}d\bar{\theta}\right) \wedge f - \tilde{\alpha}^{-\frac{1}{2}}(*f). \quad (1.30)$$

Here $*f$ is the 4+1D Hodge dual of the 2-form f on $\Upsilon \times \mathbb{R}^{2,1}$. The coupling constant of the effective 4 + 1D super Yang-Mills theory for f is

$$g_{\text{ym}}^2 = (2\pi)^2 \tilde{\alpha}^{1/2} R = (2\pi)^2 \left(1 + \frac{r^2}{\mathbf{k}^2 R^2}\right)^{\frac{1}{2}} R. \quad (1.31)$$

The coupling constant g_{ym}^2 has dimensions of length and can be compared to the length scale set by the order of magnitude of the curvature of the cigar metric at the origin – this length-scale is $\mathbf{k}R$. For $r \sim \mathbf{k}R$ we find $g_{\text{ym}}^2 \ll \mathbf{k}R$ (in the large \mathbf{k} limit), and so the Yang-Mills theory is weakly coupled on length scales of the order of the curvature. The Yang-Mills theory becomes strongly coupled only when the two scales become comparable, which happens for $r \sim \mathbf{k}^2 R$, and therefore for large \mathbf{k} our low-energy semi-classical 4 + 1D SYM approximation is valid, because the strongly coupled region $r \gg \mathbf{k}^2 R$ is pushed to $r \rightarrow \infty$. The various length scales are depicted in Figure 1.2.

1.4.2 Equations of motion

The bosonic fields of our maximally supersymmetric 4+1D SYM are the $SU(2)$ gauge field and 5 adjoint-valued scalars. The scalars correspond to the relative motion in directions x_6, \dots, x_{10} of the M5-branes (which become D4-branes after dimensional reduction on direction x_3). We will be interested in supersymmetric solutions where only the scalar corresponding to direction x_{10} can be nonzero. We will therefore ignore the remaining 4 scalars, as well as the fermions, and we will denote the scalar associated with direction x_{10} by Φ . The boundary conditions at infinity are

$$\Phi \rightarrow \begin{pmatrix} \frac{1}{2}v & 0 \\ 0 & -\frac{1}{2}v \end{pmatrix} \quad (\text{up to a gauge transformation}),$$

where $v \equiv 2\pi R\tilde{V}$, and \tilde{V} is the tension of the BPS string defined in §1.2.1.

We convert to polar coordinates in the $x_1 - x_2$ plane by

$$\rho \equiv \sqrt{x_1^2 + x_2^2}, \quad x_1 + ix_2 = \rho e^{i\varphi}. \quad (1.32)$$

The 4+1D SYM theory is therefore formulated on a space with 4+1D metric

$$ds^2 = -dt^2 + dr^2 + \tilde{\alpha}^{-1}r^2 d\bar{\theta}^2 + d\rho^2 + \rho^2 d\varphi^2.$$

The action contains three terms,

$$I_{\text{bosonic}} = I_{\Phi} + I_{\text{YM}} + I_{\theta}, \quad (1.33)$$

where I_{Φ} is the action of the scalar field, I_{YM} is the standard Yang-Mills action with variable coupling constant, and I_{θ} is the 4+1D θ -term that arises due to the nonzero connection \mathfrak{a} [see (1.29)]. We will only consider $\bar{\theta}$ -independent field configurations. For such configurations the explicit expressions for the terms in the action are

$$I_{\Phi} = \frac{1}{8\pi^2 R} \text{tr} \int [(D_0\Phi)^2 - (D_{\rho}\Phi)^2 - \frac{1}{\rho^2}(D_{\varphi}\Phi)^2 - (D_r\Phi)^2] r \rho dr d\rho d\varphi dt, \quad (1.34)$$

$$I_{\text{YM}} = \frac{1}{8\pi^2 R} \text{tr} \int \frac{1}{\tilde{\alpha}} (F_{0r}^2 + F_{0\rho}^2 + \frac{1}{\rho^2} F_{0\varphi}^2 - F_{r\rho}^2 - \frac{1}{\rho^2} F_{r\varphi}^2 - \frac{1}{\rho^2} F_{\rho\varphi}^2) r \rho dr d\rho d\varphi dt, \quad (1.35)$$

$$I_{\theta} = \frac{1}{4\pi^2 R} \text{tr} \int \frac{r^2}{\mathbf{k}R\tilde{\alpha}} (F_{0r}F_{\rho\varphi} - F_{0\rho}F_{r\varphi} + F_{0\varphi}F_{r\rho}) dr d\rho d\varphi dt, \quad (1.36)$$

where $D_{\mu}\Phi = \partial_{\mu}\Phi + i[A_{\mu}, \Phi]$ is the covariant derivative of an adjoint-valued field. The equations of motion are

$$0 = D^{\beta}F_{\alpha\beta} + D_r F_{\alpha r} - \frac{1}{r}F_{\alpha r} - i\tilde{\alpha}[D_{\alpha}\Phi, \Phi] - (\frac{\tilde{\alpha}'}{\tilde{\alpha}})F_{\alpha r} - \frac{\tilde{\alpha}}{2r}(\frac{r^2}{\mathbf{k}R\tilde{\alpha}})' \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}, \quad (1.37)$$

$$0 = D^{\beta}F_{r\beta} - i\tilde{\alpha}[D_r\Phi, \Phi], \quad (1.38)$$

$$0 = D^{\alpha}D_{\alpha}\Phi + D_r D_r\Phi + \frac{1}{r}D_r\Phi, \quad (1.39)$$

where $\alpha, \beta = 0, 1, 2$ are lowered and raised with the Minkowski metric $ds^2 = -dt^2 + dx_1^2 + dx_2^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2$, the notation $(\dots)'$ denotes a derivative with respect to r , and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor.

We note that the term $-i\tilde{\alpha}[D_\alpha\Phi, \Phi]$ in (1.37) leads to a quadratic potential in the r direction for A_α , when Φ gets a nonzero VEV. The ground states of this “harmonic-oscillator” are the $(\pm 1, 0, 0, 0)$ -charged states in (1.22), which have spin ± 1 in the $x_1 - x_2$ plane. The next term in (1.22), with charges $(0, \pm 1, 0, 0)$, describes states with $x_4 - x_5$ spin and corresponds to the ground states of the $(A_r, A_{\bar{\theta}})$ field components. Note that A_r gets an r -dependent potential by a similar mechanism through (1.38). The $A_{\bar{\theta}}$ component was set to zero in our analysis, so its equation of motion does not appear in (1.37)-(1.39). The remaining terms in (1.22) correspond to excitations of scalar field components that we also set to zero.

1.5 Integrally charged particles as bound states of quasi-particles

We now have two alternative descriptions of the low-energy limit in terms of 4+1D SYM. In the first description, studied in §1.2, the 4+1D SYM theory is formulated on a cone, with extra degrees of freedom at the tip. In the second description, studied in §1.4, the 4+1D SYM theory is formulated on a cigar geometry. The latter description is most suitable in the large \mathbf{k} limit, as we have seen at the end of §1.4.1. The quasi-particles that we studied in §1.3 are the fundamental fields of 4+1D SYM in the cigar-setting. We have seen that \mathbf{k} quasi-particles can form a bound state that is free to move into the bulk of the cone. Let us now identify this state in the cigar-setting.

From the perspective of the $(2, 0)$ -theory, the bound state is a string wrapped on the fiber of (1.5). Let us consider such a wrapped string at the cone base point given by coordinates $r = a$ and $\theta = x_1 = x_2 = 0$, with variable x_3 . In the cigar variables, this reduces to a string at fixed $r = a$ and $x_1 = x_2 = 0$ but variable $\bar{\theta}$. Recall that on the Coulomb branch of $SU(2)$ 4+1D SYM, the monopole is a 1+1D object – a monopole-string. The bound state of \mathbf{k} quasi-particles is therefore associated with a monopole-string wrapped around the $\bar{\theta}$ -circle of the cigar at $r = a$, as depicted in Figure 1.3. Thanks to the θ -term (1.36), the monopole-string gains \mathbf{k} units of charge, as required.

In flat space, a monopole-string is described by the Prasad-Sommerfield solution [37]. In our case, the Prasad-Sommerfield solution is a good approximation if the thickness of the monopole is small compared to the typical scale $\mathbf{k}R$ over which the coupling constant varies, and also small compared to a . In this case, setting $\mathbf{w} \equiv \sqrt{(r-a)^2 + \rho^2}$, we find the gauge invariant magnitude of the scalar field near the core $r = a$ to be given by [37]:

$$|\Phi| \equiv \sqrt{2 \operatorname{tr}(\Phi^2)} = \tilde{v} \coth(\tilde{v}\mathbf{w}) - \frac{1}{\mathbf{w}}, \quad (1.40)$$

where $\tilde{v} \equiv (1 + \frac{a^2}{\mathbf{k}^2 R^2})^{1/2} v$ is the effective VEV of the normalized scalar field $\tilde{\alpha}^{1/2}\Phi$ at the core ($r = a$) of the monopole. The “thickness” of the Prasad-Sommerfield solution is of the order

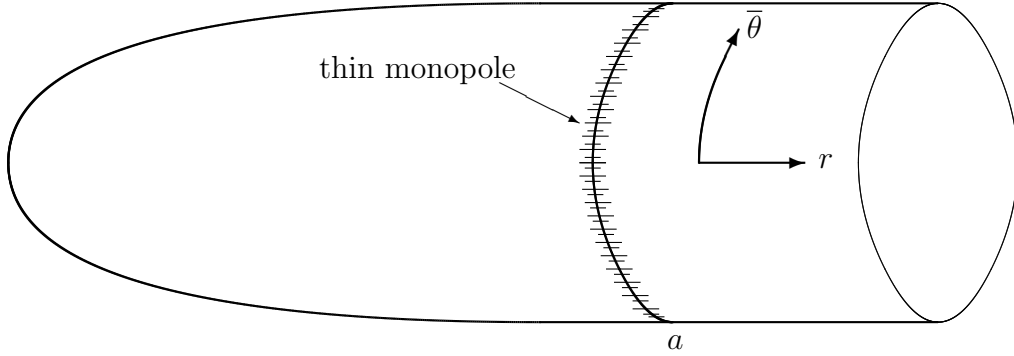


Figure 1.3: In the limit $va^2 \gg 1$ the soliton is approximately described by the Prasad-Sommerfield solution (of width $1/va$) near $r = a$ and $x_1 = x_2 = 0$. Note that directions x_1, x_2 are not drawn since they are perpendicular to the r, θ directions.

of $1/\tilde{v}$, and the condition that the monopole should be “thin” becomes $a \gg 1/v$. If this condition is not met, the Prasad-Sommerfield solution does not provide a good approximation for the particle that corresponds to a $(2, 0)$ -string wrapped on the generic fiber (of size $\mathbf{k}R$) of (1.5). Nevertheless, this is a BPS state with charge \mathbf{k} , which can be described in the large \mathbf{k} limit by a soliton solution to the equations of motion (1.37)-(1.39). The solution describes a Q-ball, and we expect the position a to be a free parameter. In the next subsection we present the BPS equations that this soliton satisfies.

1.5.1 BPS equations

As we will derive in §1.5.3, the BPS equations that describe stationary solutions that preserve the same amount of supersymmetry as a $(2, 0)$ -string wrapped on a fiber of (1.5) are

$$D_r \Phi = \frac{\mathbf{k}R}{r} F_{12} = F_{0r}, \quad D_1 \Phi = \frac{\mathbf{k}R}{r} F_{2r} = F_{01}, \quad D_2 \Phi = -\frac{\mathbf{k}R}{r} F_{1r} = F_{02}. \quad (1.41)$$

Assuming that A_r, A_1, A_2 are time independent, we find $D_\mu \Phi = F_{0\mu} = -D_\mu A_0$ (for $\mu = 1, 2, r$), which is solved by $\Phi = -A_0$. So the equations are reduced to

$$D_r \Phi = \frac{\mathbf{k}R}{r} F_{12}, \quad D_1 \Phi = \frac{\mathbf{k}R}{r} F_{2r}, \quad D_2 \Phi = -\frac{\mathbf{k}R}{r} F_{1r}, \quad \Phi = -A_0. \quad (1.42)$$

These equations imply the equations of motion (1.37)-(1.39). In fact, for a stationary configuration (all fields are t -independent), using the Bianchi identity for the gauge field, we can

rewrite the action (1.33) as:

$$\begin{aligned}
 I_\Phi + I_{\text{YM}} + I_\theta = & \\
 & \frac{1}{8\pi^2 R} \text{tr} \int \left\{ [A_0, \Phi]^2 - \left[\frac{1}{\rho^2} \left(\frac{\mathbf{k}R\rho}{r} F_{r\rho} - D_\varphi \Phi \right)^2 + \left(\frac{\mathbf{k}R}{r\rho} F_{r\varphi} + D_\rho \Phi \right)^2 + \left(\frac{\mathbf{k}R}{r\rho} F_{\rho\varphi} - D_r \Phi \right)^2 \right] \right. \\
 & + \frac{1}{\tilde{\alpha}} \left[\frac{1}{\rho^2} \left(\frac{\mathbf{k}R\rho}{r} F_{r\rho} - F_{0\varphi} \right)^2 + \left(\frac{\mathbf{k}R}{r\rho} F_{r\varphi} + F_{0\rho} \right)^2 + \left(\frac{\mathbf{k}R}{r\rho} F_{\rho\varphi} - F_{0r} \right)^2 \right] \left. \right\} r \rho dr d\rho d\varphi \\
 & + \frac{\mathbf{k}}{4\pi^2} \int \left\{ \partial_\rho \text{tr} [F_{r\varphi}(\Phi + A_0)] + \partial_\varphi \text{tr} [F_{\rho r}(\Phi + A_0)] + \partial_r \text{tr} [F_{\varphi\rho}(\Phi + A_0)] \right\} dr d\rho d\varphi.
 \end{aligned} \tag{1.43}$$

The expressions of the form $(\dots)^2$ on the 2nd and 3rd lines of (1.43) are squares of combinations that vanish if (1.42) holds, while the 4th line is a total derivative, so a configuration that satisfies (1.42) is therefore a saddle point of the action.

The nonzero A_0 in the solution (1.42) is consistent with the configuration being a Q-ball [38]. A_0 can be gauged away at the expense of creating time-varying phases for the other fields, but we will not do so.

We can rewrite the first three equations of (1.42) as the Prasad-Sommerfield [37] equations

$$D_i \tilde{\Phi} = B_i, \tag{1.44}$$

where

$$\tilde{\Phi} \equiv \frac{1}{\mathbf{k}R} \Phi, \quad B_i \equiv \frac{1}{2\sqrt{g}} g_{ij} \epsilon^{jkl} F_{kl}, \tag{1.45}$$

are defined on a 3D auxiliary space \mathcal{W} parameterized by x_1, x_2, r , with metric g_{ij} given by

$$ds^2 = g_{ij} dx^i dx^j = r^2 (dr^2 + dx_1^2 + dx_2^2) = r^2 (dr^2 + d\rho^2 + \rho^2 d\varphi^2). \tag{1.46}$$

In §1.6.3 we will show that the problem of finding an axisymmetric (φ -independent) BPS soliton can be converted to the problem of finding a harmonic map from the AdS_3 space with metric

$$ds^2 = \frac{1}{r^2} (dr^2 + d\rho^2 + \rho^2 d\varphi^2)$$

to AdS_2 , with a certain singular behavior along a Dirac-like string at $\rho = 0$ and $0 < r < a$.

1.5.2 Energy

The energy of a general solution of the equations of motion [not necessarily stationary and not necessarily obeying (1.42)] is given by

$$\begin{aligned}
 \mathcal{E} = & \frac{1}{8\pi^2 R} \text{tr} \int \left[(D_0 \Phi)^2 + (D_r \Phi)^2 + (D_\rho \Phi)^2 + \frac{1}{\rho^2} (D_\varphi \Phi)^2 \right] r \rho dr d\rho d\varphi \\
 & + \frac{1}{8\pi^2 R} \text{tr} \int \tilde{\alpha}^{-1} \left[F_{0r}^2 + F_{0\rho}^2 + \frac{1}{\rho^2} F_{0\varphi}^2 + F_{r\rho}^2 + \frac{1}{\rho^2} F_{r\varphi}^2 + \frac{1}{\rho^2} F_{\rho\varphi}^2 \right] r \rho dr d\rho d\varphi.
 \end{aligned} \tag{1.47}$$

Using the equations of motion (1.37)-(1.39), it is not hard to check that if Π_Φ , Π_{A_r} , Π_{A_ρ} , and Π_{A_φ} are the canonical momenta dual to the fields Φ , A_r , A_ρ , A_φ , then the Hamiltonian is related to \mathcal{E} by a total derivative:

$$\begin{aligned} H &\equiv \text{tr} \int (\Pi_\Phi \partial_0 \Phi + \Pi_{A_r} \partial_0 A_r + \Pi_{A_\rho} \partial_0 A_\rho + \Pi_{A_\varphi} \partial_0 A_\varphi) dr d\rho d\varphi - I_\Phi - I_{\text{YM}} - I_\theta \\ &= \mathcal{E} + \frac{1}{4\pi^2 R} \text{tr} \int \left\{ \partial_\rho \left[\frac{r\rho}{\alpha} A_0 (F_{0\rho} - \frac{r}{\rho \mathbf{k}R} F_{r\varphi}) \right] + \partial_r \left[\frac{r\rho}{\alpha} A_0 (F_{0r} + \frac{r}{\rho \mathbf{k}R} F_{\rho\varphi}) \right] \right. \\ &\quad \left. + \partial_\varphi \left[\frac{r}{\rho\alpha} A_0 (F_{0\varphi} + \frac{r\rho}{\mathbf{k}R} F_{r\rho}) \right] \right\} dr d\rho d\varphi. \end{aligned} \quad (1.48)$$

For a stationary configuration that satisfies the equations of motion and also satisfies $A_0 = -\Phi$, the energy can be written as a sum of squares of the BPS equations plus total derivatives:

$$\begin{aligned} \mathcal{E}_{\text{stat}} &= \frac{1}{8\pi^2} \text{tr} \int \frac{1}{\alpha} \left[(F_{r\rho} - \frac{r}{\mathbf{k}R\rho} D_\varphi \Phi)^2 + \frac{1}{\rho^2} (F_{r\varphi} + \frac{r\rho}{\mathbf{k}R} D_\rho \Phi)^2 + \frac{1}{\rho^2} (F_{\rho\varphi} - \frac{r\rho}{\mathbf{k}R} D_r \Phi)^2 \right] r \rho dr d\rho d\varphi \\ &+ \frac{1}{4\pi^2 R} \text{tr} \int \left\{ \partial_\rho \left(\frac{r\rho}{\alpha} \Phi F_{0\rho} \right) + \partial_r \left(\frac{r\rho}{\alpha} \Phi F_{0r} \right) + \partial_\varphi \left(\frac{r}{\rho\alpha} \Phi F_{0\varphi} \right) \right\} dr d\rho d\varphi \\ &+ \frac{1}{4\pi^2 \mathbf{k}R^2} \text{tr} \int \left\{ \partial_\rho \left(\frac{r^2}{\alpha} \Phi F_{\varphi r} \right) + \partial_r \left(\frac{r^2}{\alpha} \Phi F_{\rho\varphi} \right) + \partial_\varphi \left(\frac{r^2}{\alpha} \Phi F_{r\rho} \right) \right\} dr d\rho d\varphi \end{aligned} \quad (1.49)$$

Equation (1.49) assumes (1.37)-(1.39), but not (1.42) (other than $A_0 = -\Phi$). The term on the RHS of the first line vanishes when the BPS equations (1.42) are satisfied. Substituting (1.42) into (1.49), we find

$$\mathcal{E}_{\text{BPS}} = \frac{1}{4\pi^2 R} \text{tr} \int \left\{ \partial_\rho (\Phi F_{\varphi r}) + \partial_r (\Phi F_{\rho\varphi}) + \partial_\varphi (\Phi F_{r\rho}) \right\} dr d\rho d\varphi, \quad (1.50)$$

which depends only on the behavior of the fields at infinity and reduces to the VEV v times the magnetic charge of the soliton [regarded as a monopole in the metric (1.46)].

We note that (1.42) also lead to another set of 2^{nd} order differential equations:

$$0 = D_n F_{mn} + D_r F_{mr} - \frac{1}{r} F_{\alpha r} - i \frac{r^2}{\mathbf{k}^2 R^2} [D_m \Phi, \Phi], \quad (1.51)$$

$$0 = D_n F_{rn} - i \frac{r^2}{\mathbf{k}^2 R^2} [D_r \Phi, \Phi], \quad (1.52)$$

$$0 = D_n D_n \Phi + D_r D_r \Phi + \frac{1}{r} D_r \Phi, \quad (1.53)$$

where $m, n = 1, 2$. Equations (1.51)-(1.53) are the stationary equations for a Yang-Mills field A , minimally coupled to an adjoint scalar Φ , on a space with metric (1.46). These equations presumably have additional solutions that do not solve (1.37)-(1.39).

1.5.3 Derivation of the BPS equations

In this subsection we explain how (1.41) was derived. (The rest of the chapter does not rely on this subsection, and it may be skipped safely.) We wish to find the equations that

describe the “W-boson” that appeared in (1.24) in terms of the low-energy fields of 4+1D SYM on $\Upsilon \times \mathbb{R}^{2,1}$, where $\mathbb{R}^{2,1}$ corresponds to directions 0, 1, 2, and Υ is the “cigar” defined in §1.4.1. That “W-boson” is *not* the W-boson of the 4+1D SYM on Υ , but rather the W-boson of a dual 4+1D SYM on the $\mathbb{R}^{2,1} \times (\mathbb{C}/\mathbb{Z}_{\mathbf{k}})$ background that appeared in (1.5). But, anyway, to derive the BPS equations it is convenient to start in six dimensions.

Let us first discuss the equations on the Coulomb branch of the (2, 0)-theory. The contents of the low-energy theory is a free tensor multiplet with 2-form field B (and anti-self-dual field strength $H = dB = -*H$), five scalar fields Φ^6, \dots, Φ^{10} , and chiral fermions ψ in the spinor representation $\mathbf{4} \otimes \mathbf{4}$ of $SO(5, 1) \times SO(5)$. We assume

$$\Phi^6 = \Phi^7 = \Phi^8 = \Phi^9 = 0$$

and only allow $\Phi^{10} \equiv \phi$ to be nonzero. The BPS equations are derived from the SUSY transformation of the fermions. Let ϵ be a constant SUSY parameter, which we represent as a 32-component spinor on which the 10+1D Dirac matrices Γ^I ($I = 0, \dots, 10$) can act. The BPS conditions on ϵ are:

- Invariance of ϵ under simultaneous rotations by $2\pi/\mathbf{k}$ in the planes 4 – 5 and 6 – 7;
- Invariance of an M5-brane along directions 0, \dots , 5 under a SUSY transformation of 10+1D SUGRA with parameter ϵ ; and
- Invariance of an M2-brane along directions 0, 3, 10 under a SUSY transformation of 10+1D SUGRA with parameter ϵ .

Therefore, the equations are (we set $10 \equiv \natural$ in Dirac matrices):

$$\epsilon = \Gamma^{012345}\epsilon = \Gamma^{03\natural}\epsilon = \Gamma^{4567}\epsilon. \quad (1.54)$$

To get the BPS equations we require that the fermions ψ of the tensor multiplet of the (2, 0)-theory be invariant under any SUSY transformation with a parameter ϵ that satisfies (1.54):

$$0 = \delta\psi \equiv (H_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma} - \partial_\mu\phi\Gamma^{\mu\natural})\epsilon. \quad (1.55)$$

There are four linearly independent solutions to (1.54), and substituting these into (1.55) we find the equations:

$$H_{03\mu} = \partial_\mu\phi, \quad H_{0ij} = 0, \quad (i, j = 1, 2, 4, 5, \quad \mu = 0, \dots, 5). \quad (1.56)$$

The other components of H are determined by anti-self-duality $H = -*H$.

We now convert the 5+1D BPS equations (1.56) to 4+1D equations on $\Upsilon \times \mathbb{R}^{2,1}$ using (1.30) and the change of variables (1.25). To avoid ambiguity, we momentarily denote by x'_3 and θ' the coordinates before the change of variables, so that the change of variables is given by

$$x_3 = x'_3, \quad \bar{\theta} = \theta' - \frac{x'_3}{\mathbf{k}R}.$$

We then find:

$$0 = H_{03'r} - \partial_r \phi = H_{03'\theta'} - \partial_{\theta'} \phi = \partial_{3'} \phi = \partial_0 \phi, \quad 0 = H_{03'i} - \partial_i \phi, \quad (i = 1, 2), \quad (1.57)$$

and

$$0 = H_{012} = H_{0ir} = H_{0i\theta'} = H_{0i\bar{\theta}} = H_{0r\theta'} = H_{0r\bar{\theta}}, \quad (i = 1, 2).$$

The dual relations are

$$0 = H_{3'r\theta'} = H_{3'i\theta'} = H_{3'ir} = H_{3'12}, \quad (i = 1, 2),$$

which become in $(x_3, \bar{\theta})$ coordinates:

$$0 = H_{3r\bar{\theta}} = H_{3i\bar{\theta}} = H_{3ir} - \frac{1}{\mathbf{k}R} H_{\bar{\theta}ir} = H_{312} - \frac{1}{\mathbf{k}R} H_{\bar{\theta}12}, \quad (i = 1, 2). \quad (1.58)$$

Next we use the anti-self-duality conditions

$$H_{03'r} = \frac{1}{r} H_{\theta'12} = \frac{1}{r} H_{\bar{\theta}12}, \quad H_{03'1} = \frac{1}{r} H_{r\theta'2} = \frac{1}{r} H_{r\bar{\theta}2}, \quad H_{03'2} = -\frac{1}{r} H_{r\theta'1} = -\frac{1}{r} H_{r\bar{\theta}1},$$

and the relations (1.58) to write

$$H_{03'r} = \frac{1}{r} H_{\bar{\theta}12} = \frac{\mathbf{k}R}{r} H_{312}, \quad H_{03'1} = \frac{1}{r} H_{r\bar{\theta}2} = \frac{\mathbf{k}R}{r} H_{32r}, \quad H_{03'2} = -\frac{1}{r} H_{r\bar{\theta}1} = -\frac{\mathbf{k}R}{r} H_{31r}. \quad (1.59)$$

Combining with (1.57), we end up with the BPS equations

$$\partial_r \phi = H_{03'r} = \frac{\mathbf{k}R}{r} H_{312}, \quad \partial_1 \phi = H_{03'1} = \frac{\mathbf{k}R}{r} H_{32r}, \quad \partial_2 \phi = H_{03'2} = -\frac{\mathbf{k}R}{r} H_{31r}, \quad (1.60)$$

and further combining with (1.30) we have

$$\partial_r \phi = \frac{\mathbf{k}R}{r} f_{12}, \quad \partial_1 \phi = \frac{\mathbf{k}R}{r} f_{2r}, \quad \partial_2 \phi = -\frac{\mathbf{k}R}{r} f_{1r}. \quad (1.61)$$

Altogether, we have

$$\partial_r \phi = \frac{\mathbf{k}R}{r} f_{12} = f_{0r}, \quad \partial_1 \phi = \frac{\mathbf{k}R}{r} f_{2r} = f_{01}, \quad \partial_2 \phi = -\frac{\mathbf{k}R}{r} f_{1r} = f_{02}. \quad (1.62)$$

The equations (1.41) are the nonabelian extension of (1.62), and the fact that they imply the equations of motion (1.37)-(1.39) shows that no additional terms are needed.

1.5.4 The moduli space

We are interested in solutions to (1.44) that correspond to a monopole on the space with metric (1.46) with m units of monopole charge. We focus on $m = 1$, but the comments we make in this section apply to any number m of monopole charge. Recall that the moduli space of m BPS $SU(2)$ monopoles on \mathbb{R}^3 is hyper-Kähler and can be described as the space of

solutions to Nahm's equations [39], written in terms of three $m \times m$ anti-hermitian matrices T^i which depend on a parameters s :

$$\frac{dT^i}{ds} = \frac{1}{2}\epsilon_{ijk}[T^i, T^j], \quad -1 \leq s \leq 1, \quad i, j, k = 1, 2, 3, \quad T^i(s) \in \mathfrak{u}(m), \quad (1.63)$$

with prescribed boundary conditions (Nahm poles) at $s = \pm 1$, and a reality condition $T(s)^* = T(-s)$. It was given a nice string-theoretic interpretation in [40] (using previous results on the moduli space of instantons [41, 42]), was related to the moduli space of 2+1D gauge theories with 8 supercharges in [43], and was further generalized to singular monopoles in [44]-[46].

Our setting has only 4 supercharges – 16 are preserved by the $(2, 0)$ -theory, half are broken by the geometry, and another half is broken by the Q-ball. Our moduli space of solutions is therefore only Kähler and not hyper-Kähler. We can show this explicitly using an adaptation of the Hamiltonian (Marsden-Weinstein) reduction technique of [47].

We start with the space of all possible $SU(2)$ gauge field and scalar field configurations $(A_1, A_2, A_r, \tilde{\Phi})$ on the $r \geq 0$ portion of space, subject to the boundary conditions

$$|\tilde{\Phi}| \rightarrow v \quad \text{at } x_1^2 + x_2^2 + r^2 \rightarrow \infty. \quad (1.64)$$

At $r = 0$ we note that (1.44) implies $F_{12} = 0$ [see the left-most equation of (1.42)], and so $A_1 dx_1 + A_2 dx_2$ reduces to a flat connection on the $r = 0$ plane. We can therefore pick a gauge so that $A_1 = A_2 = 0$ at $r = 0$. We still have the freedom to perform a gauge transformation with a gauge parameter λ that approaches a constant (independent of x_1, x_2) at $r = 0$ but with a possibly nonconstant $\partial_r \lambda$. We use this gauge freedom to set $A_r = 0$ at $r = 0$ as well. We therefore require:

$$A_1 = A_2 = A_r = 0 \quad \text{at } r = 0. \quad (1.65)$$

We denote the space of $(A_1, A_2, A_r, \tilde{\Phi})$ configurations with the boundary conditions (1.64)-(1.65) by \mathcal{N} . The infinite dimensional space \mathcal{N} is Kähler with a complex structure defined so that $A_1 + iA_2$ and $A_r + ir\tilde{\Phi}$ (evaluated at any point x_1, x_2, r) are holomorphic, and with a symplectic Kähler form given by

$$\omega = \text{tr} \int \left(\frac{1}{r} \delta A_1 \wedge \delta A_2 + \delta \tilde{\Phi} \wedge \delta A_r \right) dx_1 dx_2 dr. \quad (1.66)$$

The associated Kähler metric is

$$\text{tr} \int \left[\frac{1}{r} (\delta A_1^2 + \delta A_2^2 + \delta A_r^2) + r \delta \tilde{\Phi}^2 \right] dx_1 dx_2 dr. \quad (1.67)$$

The combination $A_r + ir\tilde{\Phi}$ was chosen so that the two middle equations of (1.42) will be the real and imaginary parts of a holomorphic equation $(D_1 + iD_2)\tilde{\Phi} = -\frac{i}{r}(F_{1r} + iF_{2r})$.

We are interested in the moduli space \mathcal{M}_m of solutions to (1.44) with the boundary conditions (1.64)-(1.65), modulo gauge transformations with gauge parameter λ that approaches a constant at $r = 0$ and at $x_1^2 + x_2^2 + r^2 \rightarrow \infty$, and such that

$$vm = \text{tr} \int \sqrt{g} g^{ij} B_i D_j \tilde{\Phi} d^3x = \text{tr} \int \left[F_{12} D_r \tilde{\Phi} + F_{2r} D_1 \tilde{\Phi} - F_{1r} D_2 \tilde{\Phi} \right] dr dx_1 dx_2.$$

We note that the metric (1.67) does not lead to the physical metric on the moduli space \mathcal{M}_m (that is, the metric determined from the energy of a slowly time-varying configuration that corresponds to motion on \mathcal{M}_m), but rather to the metric that would result from the action of minimally coupled scalar and gauge fields, leading to the equations of motion (1.51)-(1.53). This metric is more directly related to the derived problem of 3D monopoles on the space with metric (1.46).

For any Lie-algebra valued gauge parameter λ (that is a constant at $r = 0$) we define the “moment-map”:

$$\mu_\lambda = \text{tr} \int \lambda \left(D_r \tilde{\Phi} - \frac{1}{r} F_{12} \right) dx_1 dx_2 dr. \quad (1.68)$$

When μ_λ is set to the Hamiltonian on the (infinite dimensional) symplectic manifold with symplectic form ω , the generated flow (“time evolution”) corresponds to gauge transformations with gauge parameter λ . The moduli space \mathcal{M}_m is then equivalent to the Hamiltonian reduction of \mathcal{N} by these moment-maps (for all allowed λ 's). It is the subset of \mathcal{N} for which $\mu_\lambda = 0$ for all admissible λ , modulo the equivalence relations corresponding to the gauge transformations generated by all the λ 's. Since the gauge transformations preserve the complex structure (acting in an affine-linear way on the complex variables $A_1 + iA_2$ and $A_r + ir\tilde{\Phi}$) and the symplectic form, the arguments of [47] show that the resulting (finite dimensional) moduli space \mathcal{M}_m is Kähler.

One can shed more light on the form of the metric (1.67) as follows.² One can derive (1.44) by reducing to \mathcal{W} the instanton equations on $\mathbb{R} \times \mathcal{W}$ that are invariant under translations in \mathbb{R} [where \mathcal{W} was defined as the 3D space with metric (1.46)]. The metric on $\mathbb{R} \times \mathcal{W}$ is taken to be $ds^2 = dx_4^2 + r^2(dx_1^2 + dx_2^2 + dr^2)$, but since instanton equations are conformally invariant, we can replace this metric with the conformally equivalent metric $\frac{1}{r^2} |d(x_4 + \frac{i}{2}r^2)|^2 + |d(x_1 + ix_2)|^2$. The latter is clearly a Kähler manifold, as it describes the product of a 2D surface, parameterized by complex coordinate $x_4 + \frac{i}{2}r^2$ and a copy of \mathbb{C} , parameterized by $x_1 + ix_2$, and so the instanton moduli space is Kähler. Requiring invariance under translations in \mathbb{R} is a holomorphic constraint, and so the space of \mathbb{R} -invariant solutions is also Kähler.

The metric on \mathcal{M}_m is induced from the metric (1.67) on \mathcal{N} as follows. Let $(A, \tilde{\Phi})$ be a solution of (1.44), and let $(\delta A, \delta \tilde{\Phi})$ be a deformation to a nearby solution. We need to fix the right gauge so that (1.67) will be minimal among gauge equivalent deformations. This is equivalent to the gauge condition

$$0 = r^2[\tilde{\Phi}, \delta \tilde{\Phi}] + D_1 \delta A_1 + D_2 \delta A_2 + D_r \delta A_r - \frac{1}{r} \delta A_r. \quad (1.69)$$

²The reasoning presented in this paragraph was pointed out to us by Sergey Cherkis.

Now take a constant $r_0 \gg 1/\sqrt{v}$ and consider a portion of the moduli space comprising of solutions whose bulk of the energy is concentrated in the vicinity of r_0 , allowing a spread of $O(1/vr_0)$ away from r_0 . Then $(A, r_0\tilde{\Phi})$ is an approximate solution of the flat space monopole equations, and if we approximate the explicit r and $1/r$ factors in (1.66)-(1.68) by r_0 and $1/r_0$, we get the corresponding Kähler form, metric, and moment map of [47], in one of the complex structures of the corresponding hyper-Kähler moduli space. Set $\tilde{\Phi}_0 \equiv r_0\tilde{\Phi}$. Then $(A_1, A_2, A_3, \tilde{\Phi}_0)$ approximately solve the BPS problem on \mathbb{R}^3 , which we will refer to as the “hyper-Kähler problem”. In this context the \mathbb{R}^3 coordinates are taken to be x_1, x_2 and $x'_3 \equiv r - r_0$.

Now consider the case $m = 1$. There are three moduli corresponding to the “position” of the monopole (a_1, a_2, a_3) , with $a_3 \equiv a - r_0$. (Note that this “position” is not necessarily the maximum of energy density for finite a , but it is so in the limit $a \rightarrow \infty$.) The combination $a_1 + ia_2$ is holomorphic in the complex structure of \mathcal{M}_1 , and the “missing” modulus θ that combines with a_3 to form a holomorphic $a_3 + i\theta$ can be recovered as follows. First recall that for the hyper-Kähler problem, if we perform a large gauge transformation with gauge parameter $\Lambda = \exp(i\theta\tilde{\Phi}_0/r_0v)$, where $0 \leq \theta \leq \pi$, we obtain a different solution that still satisfies the correct boundary conditions at infinity of \mathbb{R}^3 . The infinitesimal version $\lambda = (\delta\theta)\tilde{\Phi}_0/r_0v$ solves the hyper-Kähler gauge condition, which we can recover from (1.69) by dropping the last term on the RHS, as $r_0 \rightarrow \infty$. Plugging the corresponding deformations $\delta A_i = D_i\lambda$, $\delta A_r = D_r\lambda$ and $\delta\tilde{\Phi} = 0$ into (1.67), we find that the metric on the θ direction behaves as $(\delta\theta)^2/r_0^2v$. In our case, we also expect a modulus that corresponds to a large gauge transformation, but setting λ to be proportional to $\tilde{\Phi}$, say $\lambda = \epsilon\tilde{\Phi}$, would not work, because: (i) $\tilde{\Phi}$ does not vanish at $r = 0$, and (ii) the gauge condition (1.69) requires

$$0 = -r^2[\tilde{\Phi}, [\tilde{\Phi}, \lambda]] + D_1^2\lambda + D_2^2\lambda + D_r^2\lambda - \frac{1}{r}D_r\lambda, \quad (1.70)$$

but $\lambda = \epsilon\tilde{\Phi}$ does not satisfy (1.70). The sign of the rightmost term of (1.70) is in conflict with what the equation of motion (1.39) requires it to be. Instead, we need to look for a solution to (1.70) with $\lambda = \epsilon\Psi$ such that Ψ approaches a constant, say σ^3 , at $r = 0$ and approaches $\tilde{\Phi}/v$ at infinity. In addition, Ψ should map the boundary of the $r \geq 0$ space (the $x_1 - x_2$ plane at $r = 0$ together with a hemisphere at infinity) to S^2 in such a way as to have winding number $m = 1$. Gauge transformations by $\Lambda = \exp(i\theta\Psi)$ then correspond to a circular direction $0 \leq \theta < \pi$ in moduli space. The corresponding deformations are

$$\delta A_1 = \delta\theta D_1\Psi, \quad \delta A_2 = \delta\theta D_2\Psi, \quad \delta A_r = \delta\theta D_r\Psi, \quad \delta\tilde{\Phi} = -i\delta\theta[\tilde{\Phi}, \Psi].$$

The metric on this direction is given by

$$(\delta\theta)^2 \text{tr} \int \left\{ \frac{1}{r} [(D_1\Psi)^2 + (D_2\Psi)^2 + (D_r\Psi)^2] - r[\tilde{\Phi}, \Psi]^2 \right\} dx_1 dx_2 dr,$$

which can be integrated by parts, using (1.70) (for $\lambda = \epsilon\Psi$), to give a surface integral on the boundary of the $r \geq 0$ space:

$$(\delta\theta)^2 \int \frac{1}{r} [\partial_1 \text{tr}(\Psi^2) dx_2 dr + \partial_2 \text{tr}(\Psi^2) dx_1 dr + \partial_r \text{tr}(\Psi^2) dx_1 dx_2].$$

This integral depends on the subleading terms in the behavior of Ψ^2 near the boundary, which, unfortunately, we do not know.

Now, consider the mode associated with translations. In the hyper-Kähler limit the associated deformation that satisfies the gauge condition (1.69) is

$$\delta A_1 = (\delta a)F_{31}, \quad \delta A_2 = (\delta a)F_{32}, \quad \delta A_3 = 0, \quad \delta \tilde{\Phi}_0 = (\delta a)D_3 \tilde{\Phi}_0,$$

where we have augmented the translation by δa in the x_3 direction by a gauge transformation with gauge parameter $(\delta a)A_3$. Plugging into (1.67) we get a metric $(\delta a)^2 v$. Rescaling by v , so far we have the approximate metric

$$ds^2 \sim da^2 + \frac{d\theta^2}{v^2 a^2}. \quad (1.71)$$

In general, the modulus a is defined from the boundary conditions of the solution $(A, \tilde{\Phi})$. Like the hyper-Kähler counterpart, for $r \rightarrow \infty$ the solution to (1.44) reduces, up to a gauge transformation, to the field of an abelian monopole centered at, say, $(0, 0, a)$. We will discuss the abelian solution and present its exact form in §1.6.2, but for now suffice it to say that the modulus a can be read off from the asymptotic form. The metric that we found above in (1.71) would be consistent with a Kähler manifold if $\frac{1}{2}va^2 + i\theta$ is a holomorphic coordinate. From the discussion above, we find the asymptotic form of the metric on moduli space as

$$ds^2 \sim da_1^2 + da_2^2 + da^2 + \frac{d\theta^2}{v^2 a^2}, \quad a \rightarrow \infty.$$

and the asymptotic Kähler form is

$$k \sim da_1 \wedge da_2 + \frac{da}{va} \wedge d\theta, \quad a \rightarrow \infty.$$

Beyond these observations, we do not have a simple description of the moduli space \mathcal{M}_1 , and as we have seen, unlike the moduli space of \mathbb{R}^3 BPS monopoles, in our case the x_3 coordinate of the “center” (corresponding to $\text{tr} T^3$ in Nahm’s equations) does not decouple. Moreover, the Bogomolnyi equations that describe monopoles on \mathbb{R}^3 can be obtained as a limit of (1.44) (see §1.6.5 for more details) when $r \rightarrow \infty$. Thus, we expect to recover the moduli space of BPS monopoles with fixed center of mass at the boundary $r \rightarrow \infty$ of the moduli space of (1.44).

1.6 Analysis of the BPS equations

In this section we will present several observations regarding the solution of the BPS equations (1.42). It is convenient to regard the BPS equations as Bogomolnyi monopole equations (1.44) on a curved space with metric (1.46). We are looking for a solution of unit monopole charge. We also require axial symmetry (i.e., independence of φ), since we can assume that

the string of the $(2, 0)$ -theory, which the solution describes, sits at the origin of the $x_1 - x_2$ plane. The fields are therefore functions of two variables, r and ρ , only. The Bogomolnyi monopole equations on \mathbb{R}^3 have the renowned Prasad-Sommerfield solution [37] for one $SU(2)$ monopole, and the general solution was given by Nahm [39]. It was given a string-theoretic interpretation in [40]. The extension to hyperbolic space is also known [48], but we are unaware of an extension of Nahm's technique to the space given by the metric (1.46), and standard techniques that exploit the integrability of the \mathbb{R}^3 problem do not work in our case. We were unable to find an exact solution, but we can make a few observations. In §1.6.1 we will reduce the number of independent fields from twelve to two by adapting a method developed in [49, 50] for finding axially symmetric (generally multi-monopole) solutions of the Bogomolnyi equations on \mathbb{R}^3 . We will then present the asymptotic form of the solution far away from the origin. In this region the solution reduces to a $U(1)$ monopole whose fields we write down explicitly. We then show that the solution can be encoded in a harmonic map from AdS_3 to AdS_2 . We conclude in §1.6.5 with an expansion up to second order in inverse VEV.

1.6.1 Manton gauge

We adopt an ansatz proposed in [50] for axially symmetric solutions. Adapted from \mathbb{R}^3 to our metric (1.46) we look for a solution in the form:

$$\Phi = \frac{1}{2}(\Phi_1\sigma_1 + \Phi_2\sigma_2), \quad A = -[(\eta_1\sigma_1 + \eta_2\sigma_2)d\varphi + W_2\sigma_3d\rho + W_1\sigma_3dr], \quad (1.72)$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices, and $\Phi_1, \Phi_2, \eta_1, \eta_2, W_1, W_2$ are scalar fields. The BPS equations then reduce to

$$\partial_\rho\Phi_1 - W_2\Phi_2 = -\frac{1}{r\rho}(\partial_r\eta_1 - W_1\eta_2), \quad (1.73)$$

$$\partial_\rho\Phi_2 + W_2\Phi_1 = -\frac{1}{r\rho}(\partial_r\eta_2 + W_1\eta_1), \quad (1.74)$$

$$\eta_2\Phi_1 - \eta_1\Phi_2 = \frac{\rho}{r}(\partial_\rho W_1 - \partial_r W_2), \quad (1.75)$$

$$\partial_r\Phi_1 - W_1\Phi_2 = \frac{1}{r\rho}(\partial_\rho\eta_1 - W_2\eta_2), \quad (1.76)$$

$$\partial_r\Phi_2 + W_1\Phi_1 = \frac{1}{r\rho}(\partial_\rho\eta_2 + W_2\eta_1), \quad (1.77)$$

Next, we adapt to our metric the technique developed in [49], solving (1.73)-(1.75) by setting

$$\Phi_1 = -\frac{1}{r}f^{-1}\partial_r\chi, \quad \Phi_2 = \frac{1}{r}f^{-1}\partial_r f, \quad \eta_1 = \rho f^{-1}\partial_\rho\chi, \quad \eta_2 = -\rho f^{-1}\partial_\rho f, \quad (1.78)$$

and

$$W_1 = -f^{-1}\partial_r\chi, \quad W_2 = -f^{-1}\partial_\rho\chi, \quad (1.79)$$

where f and χ are as yet undetermined real functions of r and ρ . We plug the ansatz (1.78)-(1.79) into (1.76)-(1.77) and get:

$$0 = f\chi_{rr} + f\chi_{\rho\rho} - 2f_r\chi_r - 2f_\rho\chi_\rho + \frac{1}{\rho}f\chi_\rho - \frac{1}{r}f\chi_r, \quad (1.80)$$

$$0 = f_r^2 + f_\rho^2 - \chi_r^2 - \chi_\rho^2 - ff_{rr} - ff_{\rho\rho} + \frac{1}{r}ff_r - \frac{1}{\rho}ff_\rho, \quad (1.81)$$

where subscripts $(\dots)_r$ and $(\dots)_\rho$ denote derivatives with respect to r and ρ , respectively.

1.6.2 The abelian solution

We can trivially solve (1.80) by setting $\chi = 0$. The remaining equation (1.81) then states that $\log f$ is a harmonic function on AdS_3 . Alternatively, the solution describes a $U(1)$ monopole on the (x_1, x_2, r) space with metric (1.46). It is easiest to construct the solution starting from 5+1D. Let us take the center of the monopole to be $(0, 0, a)$, which will then have to be a singular point for f . In the abelian limit, the fields of the $(2, 0)$ theory that are relevant to our problem reduce to a free anti-self-dual 3-form field $H = -*H$ and a free scalar field ϕ . We start by solving (1.56) on $\mathbb{R}^{5,1}$, which in particular implies that ϕ is harmonic. Consider a solution that describes the H and ϕ fields that emanate from a $(2, 0)$ -string centered at $(x_1, x_2, x_4, x_5) = (0, 0, a \cos \theta, a \sin \theta)$. The scalar field is given by

$$\phi = v + \frac{1}{x_1^2 + x_2^2 + (x_4 - a \cos \theta)^2 + (x_5 - a \sin \theta)^2}. \quad (1.82)$$

But the solution that we need must be independent of θ , so we “smear” (1.82) to obtain the requisite field:

$$\phi(x_1, x_2, r) = v + \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\rho^2 + (r \cos \theta - a)^2 + r^2 \sin^2 \theta} = v + \frac{1}{\sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2 r^2}}. \quad (1.83)$$

The $U(1)$ gauge field is now easy to calculate from (1.42) and we find

$$A = \left(\frac{\rho^2 + a^2 - r^2}{2\sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2 r^2}} - 1 \right) \frac{x_2 dx_1 - x_1 dx_2}{\rho^2}, \quad (1.84)$$

where we picked a gauge for which $A_r = 0$. It is easy to find the associated (f, χ) fields. We have $\chi = 0$ and

$$f = \exp \int \phi(r, \rho) r dr = e^{-\frac{1}{2} v r^2} \left(\rho^2 + r^2 - a^2 + \sqrt{(\rho^2 + r^2 + a^2)^2 - 4a^2 r^2} \right). \quad (1.85)$$

Equation (1.84) exhibits a Dirac string singularity that extends from $r = a$ to $r = \infty$ at $x_1 = x_2 = 0$. The abelian solution must describe the asymptotic behavior of the nonabelian solution when either $r \rightarrow \infty$ or $\rho \rightarrow \infty$ (or both).

1.6.3 Relation to harmonic maps from AdS_3 to AdS_2

The equations (1.80)-(1.81) can be derived from the action

$$I = \int \frac{\rho}{r f^2} (f_r^2 + f_\rho^2 + \chi_r^2 + \chi_\rho^2) dr d\rho. \quad (1.86)$$

We can therefore give a simple geometrical meaning to the equations of motion (1.80)-(1.81) by considering an auxiliary AdS_3 space parameterized by (r, ρ, φ) with metric

$$ds^2 = \frac{1}{r^2}(dr^2 + d\rho^2 + \rho^2 d\varphi^2)$$

and interpreting the functions $f(r, \rho)$ and $\chi(r, \rho)$ as describing an axisymmetric map from AdS_3 to the two-dimensional (f, χ) “target-space.” If we further endow this target-space with the AdS_2 metric

$$ds^2 = \frac{1}{f^2}(df^2 + d\chi^2), \quad (1.87)$$

it is easy to see that the equations of motion derived from (1.86) describe harmonic maps

$$(f, \chi) : AdS_3 \mapsto AdS_2. \quad (1.88)$$

The connection between AdS_2 (the “pseudosphere”) and axisymmetric solutions to monopole equations on \mathbb{R}^3 was first noted in [49]. The harmonic map (1.88) is required to have a singularity along a Dirac-like string, as we saw in §1.6.2.

To reproduce the abelian solution of §1.6.2, we set $\chi = 0$ and find that $\log f$ is a harmonic function on AdS_3 , as stated at the beginning of §1.6.2. To present its Dirac string more clearly, it is convenient to use instead of the Poincaré coordinates on AdS_3 , a coordinate system with the point $r = a$ at the origin. The change from (r, ρ, φ) to the new coordinate system (μ, α, φ) is given by:

$$\frac{\rho}{r} = \sinh \mu \sin \alpha, \quad \frac{\rho^2 + r^2 - a^2}{2ar} = \sinh \mu \cos \alpha,$$

and the coordinates are defined in the range

$$0 \leq \mu < \infty, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \varphi < 2\pi.$$

The metric in terms of (μ, α, φ) is

$$ds^2 = d\mu^2 + \sinh^2 \mu (d\alpha^2 + \sin^2 \alpha d\varphi^2),$$

and the inverse coordinate transformations are:

$$r = a \frac{\cosh \mu + \sinh \mu \cos \alpha}{1 + \sinh^2 \mu \sin^2 \alpha}, \quad \rho = a \frac{\cosh \mu + \sinh \mu \cos \alpha}{1 + \sinh^2 \mu \sin^2 \alpha} \sinh \mu \sin \alpha.$$

In (μ, α, φ) coordinates we have, up to an unimportant constant,

$$\begin{aligned} \log f &= -\frac{1}{2}va^2 \left(\frac{\cosh \mu + \sinh \mu \cos \alpha}{1 + \sinh^2 \mu \sin^2 \alpha} \right)^2 + \log \left(\frac{\cosh \mu + \sinh \mu \cos \alpha}{1 + \sinh^2 \mu \sin^2 \alpha} \right) \\ &\quad + \log \sinh \mu + \log(1 + \cos \alpha). \end{aligned} \quad (1.89)$$

The singularity in the last term at $\alpha = \pi$ represents the Dirac string.

1.6.4 Comments on (lack of) integrability

The classic Bogomolnyi equations for monopoles on \mathbb{R}^3 admit the well-known Nahm solutions [39], which also have a nice string-theoretic interpretation [40]. The rich properties of these solutions essentially stem from an underlying integrable structure. One way to describe the structure is to map a solution of the Bogomolnyi equations to a holomorphic vector bundle over minitwistor space [51, 52]. (Minitwistor space is the space of oriented straight lines on \mathbb{R}^3 , and it has a complex structure.) The Bogomolnyi equations arise as the integrability condition for an auxiliary set of differential equations for an auxiliary 2-component field ψ , that require ψ 's gauge-covariant derivative along a line in \mathbb{R}^3 to be related to multiplication by the scalar field $\tilde{\Phi}$, and also require ψ to be holomorphic in the directions transverse to the line. This technique can be extended to other metrics, such as AdS_3 (whose corresponding minitwistor space also possesses a complex structure and is equivalent to $\mathbb{C}P^1 \times \mathbb{C}P^1$). But this technique fails for the metric (1.46), whose space of geodesics is not complex, and the monopole equations (1.44) cannot be expressed as the integrability condition for an auxiliary system of linear differential equations, at least not in an obvious way.

Another way to see where integrability fails is to focus on axially-symmetric solutions as in [49]. Defining the symmetric $SL(2, \mathbb{R})$ matrix

$$\mathcal{G} \equiv \frac{1}{f} \begin{pmatrix} 1 & -\chi \\ -\chi & (f^2 + \chi^2) \end{pmatrix},$$

the equations of motion (1.80)-(1.81) can then be recast as

$$0 = \nabla^\alpha (\nabla_\alpha \mathcal{G} \mathcal{G}^{-1}), \quad (1.90)$$

where the covariant derivatives are with respect to another auxiliary metric,

$$ds^2 = dr^2 + d\rho^2 + \left(\frac{\rho^2}{r^2}\right) d\varphi^2, \quad (1.91)$$

and $\mathcal{G}(r, \rho)$ is, of course, assumed to be independent of φ . It is possible [49] to recast axially symmetric solutions of the Bogomolnyi equations on \mathbb{R}^3 in the form (1.90) – the metric in that case would be the Euclidean metric

$$ds^2 = dr^2 + d\rho^2 + \rho^2 d\varphi^2,$$

and the connection with the σ -model (1.90) leads to an integrable structure. To describe the integrable structure we switch to complex coordinates,

$$\xi \equiv r + i\rho, \quad \bar{\xi} \equiv r - i\rho,$$

and write (1.90) as the integrability condition for a system of first order linear differential equations for a two-component field $\Psi(\xi, \bar{\xi})$:

$$\Psi_\xi = \frac{1}{1 + \gamma} \mathcal{G}_\xi \mathcal{G}^{-1} \Psi, \quad \Psi_{\bar{\xi}} = \frac{1}{1 - \gamma} \mathcal{G}_{\bar{\xi}} \mathcal{G}^{-1} \Psi,$$

where $(\dots)_\xi$ and $(\dots)_{\bar{\xi}}$ are derivatives with respect to ξ and $\bar{\xi}$, and the function $\gamma(\xi, \bar{\xi})$ has to be suitably chosen (so that the integrability condition $(\Psi_\xi)_{\bar{\xi}} = (\Psi_{\bar{\xi}})_\xi$ will be automatically satisfied). There are, in fact, infinitely many choices for the function γ , but it has to be a solution of

$$\gamma_\xi = \frac{\gamma}{\xi - \bar{\xi}} \left(\frac{1 + \gamma}{1 - \gamma} \right), \quad \gamma_{\bar{\xi}} = -\frac{\gamma}{\xi - \bar{\xi}} \left(\frac{1 - \gamma}{1 + \gamma} \right),$$

which are compatible (see [53] for review). This construction is easy to extend to any metric of the form

$$ds^2 = dr^2 + d\rho^2 + \Lambda(r, \rho)^2 d\varphi^2,$$

as long as $\Lambda(r, \rho)$ is harmonic (in the metric $dr^2 + d\rho^2$). In our case $\Lambda = \rho/r$ is not harmonic, so the standard integrability structure is not present.

One can also attempt to extend the technique of [40], to “probe” the solution with a string that extends in an extra dimension, say x_8 . It is not hard to construct BPS string solutions that preserve some supersymmetry, compatible with that of the M5-branes and the twist. For example, in the M-theory variables we can take an M2-brane along a holomorphic curve given by $x_4 + ix_5 = C_0 e^{\frac{i}{kR}(x_3 + ix_8)}$, where C_0 is a constant. This would translate in type-IIA to a string whose x_8 coordinate varies logarithmically with r . However, this string does not preserve any common SUSY with the soliton. We were unable to find an exact solution to (1.44), and in fact, the appearance of polylogarithms in the expansion at large VEV (see §1.6.5) suggests that even if a closed form exists, it is very complicated. We therefore resorted to an asymptotic expansion for large VEV, described below, and to numerical analysis, which we describe in Appendix A.2.

1.6.5 Large VEV expansion

In this section we will discuss the asymptotic expansion of the solution to the BPS equations (1.42) for large VEV v . Since the only dimensionless combination that governs the behavior of the solution is va^2 , we can just as well discuss fixed v and large a , which means that the core of the monopole solution is far from the tip. Let us set³ $x_3 \equiv r - a$ and rescale $\phi = a\Phi/kR$, so that equations (1.42) can be rewritten as

$$(1 + \frac{x_3}{a}) D_i \phi = \frac{1}{2} \epsilon_{ijk} F_{jk}, \quad (1.92)$$

where in this section $i, j, k = 1, 2, 3$ refer to x_1, x_2, x_3 with Euclidean metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

In the limit $a \rightarrow \infty$, the equations (1.92) reduce to Bogomolnyi’s equations, and the one-monopole solution is [37]:

$$A_i^{a(0)} = \epsilon_{iaj} x_j K(u), \quad \phi^{a(0)} = x_a H(u), \quad (1.93)$$

³We hope that no confusion will arise with the coordinate x_3 that was used in §1.2.1. That coordinate plays no role here, and the only coordinates relevant for this section are x_1, x_2 and $r = a + x_3$.

where

$$H \equiv \frac{1}{u} \coth u - \frac{1}{u^2}, \quad K \equiv \frac{1}{u \sinh u} - \frac{1}{u^2}, \quad (1.94)$$

and here $u \equiv \sqrt{\sum_{i=1}^3 x_i^2}$. We set

$$b \equiv \frac{1}{a}, \quad \vec{\ell} \equiv (0, 0, b),$$

so that $\frac{x_3}{a} = \vec{\ell} \cdot \vec{x}$, and (1.92) can be written as:

$$(1 + \vec{\ell} \cdot \vec{x}) D_i \phi = \frac{1}{2} \epsilon_{ijk} F_{jk}. \quad (1.95)$$

We can now expand around the Prasad-Sommerfield solution:

$$A_i = A_i^{(0)} + b A_i^{(1)} + b^2 A_i^{(2)} + \dots, \quad \phi = \phi^{(0)} + b \phi^{(1)} + b^2 \phi^{(2)} + \dots,$$

where we set the 0th order terms to the Prasad-Sommerfield solution (1.93).

At order $O(b)$ we write all possible terms that are allowed by spherical symmetry and we keep only the terms that are also invariant under the parity symmetry

$$\phi^a(\vec{x}, \vec{\ell}) \rightarrow -\phi^a(-\vec{x}, -\vec{\ell}), \quad A_i^a(\vec{x}, \vec{\ell}) \rightarrow -A_i^a(-\vec{x}, -\vec{\ell}). \quad (1.96)$$

The general expression is then

$$b \phi^{a(1)} = \ell_a \mathbf{f}_{1,1}(u) + x_a (\ell_k x_k) \mathbf{f}_{1,2}(u), \quad (1.97)$$

$$b A_i^{a(1)} = x_a \epsilon_{ijk} x_j \ell_k \mathbf{f}_{1,3}(u) + x_i \epsilon_{ajk} x_j \ell_k \mathbf{f}_{1,4}(u) + \epsilon_{aij} \ell_j \mathbf{f}_{1,5}(u), \quad (1.98)$$

and we note the identity

$$x_{[i} \epsilon_{a]jk} x_j \ell_k = \frac{1}{2} \epsilon_{aij} \ell_j u^2 - \frac{1}{2} (\ell_k x_k) \epsilon_{aij} x_j, \quad (1.99)$$

which is the reason why we did not include a term of the form $(\ell_k x_k) \epsilon_{aij} x_j \mathbf{f}_{1,6}$ in (1.98). The coefficients $\mathbf{f}_{1,1}, \dots, \mathbf{f}_{1,5}$ are unknown functions of u .

We also have the freedom to apply an infinitesimal $O(b)$ gauge transformation which takes the form

$$\delta \phi^a = \epsilon_{abc} \lambda^b \phi^c, \quad \delta A_i^a = \partial_i \lambda^a - \epsilon_{abc} A_i^b \lambda^c$$

with

$$\lambda^a = \epsilon_{abc} x_b \ell_c \mathbf{g}_{1,1}(u).$$

This gives

$$\delta \phi^a = -x_a \ell_k x_k \mathbf{g}_{1,1} H + \ell_a u^2 \mathbf{g}_{1,1} H, \quad (1.100)$$

$$\delta A_i^a = -\epsilon_{iab} \ell_b \mathbf{g}_{1,1} + \frac{1}{u} x_i \epsilon_{abc} x_b \ell_c \mathbf{g}'_{1,1} + x_a \epsilon_{ibc} x_b \ell_c \mathbf{g}_{1,1} K. \quad (1.101)$$

Using this gauge transformation we can set one of the parameters in (1.97)-(1.98) to zero. We choose to set

$$\mathbf{f}_{1,5} = 0. \quad (1.102)$$

We end up with the general form of the $O(b)$ correction:

$$b\phi^{a(1)} = \ell_a \mathbf{f}_{1,1}(u) + x_a (\ell_k x_k) \mathbf{f}_{1,2}(u), \quad (1.103)$$

$$bA_i^{a(1)} = x_a \epsilon_{ijk} x_j \ell_k \mathbf{f}_{1,3}(u) + x_i \epsilon_{ajk} x_j \ell_k \mathbf{f}_{1,4}(u). \quad (1.104)$$

Plugging (1.93) and (1.103)-(1.104) into (1.95) and comparing terms of order $O(b)$ we get:

$$HK - \frac{1}{u}H' = \frac{1}{u}(\mathbf{f}'_{1,2} + \mathbf{f}'_{1,3}) - K\mathbf{f}_{1,2} - K\mathbf{f}_{1,3} + (K - H)\mathbf{f}_{1,4}, \quad (1.105)$$

$$0 = \frac{1}{u}\mathbf{f}'_{1,1} + (1 + u^2K)\mathbf{f}_{1,3} + u^2H\mathbf{f}_{1,4}, \quad (1.106)$$

$$0 = u\mathbf{f}'_{1,3} + K\mathbf{f}_{1,1} - \mathbf{f}_{1,2} + 3\mathbf{f}_{1,3} + (1 + u^2K)\mathbf{f}_{1,4} \quad (1.107)$$

$$-H(1 + u^2K) = K\mathbf{f}_{1,1} + (1 + u^2K)\mathbf{f}_{1,2} + \mathbf{f}_{1,4}, \quad (1.108)$$

These are ordinary inhomogeneous linear differential equations in $\mathbf{f}_{1,1}, \dots, \mathbf{f}_{1,4}$. Note that $\mathbf{f}_{1,4}$ can be eliminated from (1.108), so the general solution is given by an arbitrary solution of the full equations (1.105)-(1.108) plus a linear combination of three linearly independent solutions of the homogeneous equations:

$$0 = \frac{1}{u}(\mathbf{f}'_{1,2} + \mathbf{f}'_{1,3}) - K\mathbf{f}_{1,2} - K\mathbf{f}_{1,3} + (K - H)\mathbf{f}_{1,4}, \quad (1.109)$$

$$0 = \frac{1}{u}\mathbf{f}'_{1,1} + (1 + u^2K)\mathbf{f}_{1,3} + u^2H\mathbf{f}_{1,4}, \quad (1.110)$$

$$0 = u\mathbf{f}'_{1,3} + K\mathbf{f}_{1,1} - \mathbf{f}_{1,2} + 3\mathbf{f}_{1,3} + (1 + u^2K)\mathbf{f}_{1,4} \quad (1.111)$$

$$0 = K\mathbf{f}_{1,1} + (1 + u^2K)\mathbf{f}_{1,2} + \mathbf{f}_{1,4}, \quad (1.112)$$

The general solution to (1.105)-(1.108) that is nonsingular at $u = 0$ is

$$\mathbf{f}_{1,1} = -\frac{u}{2\sinh u} + c_1\left(\frac{u \cosh^2 u}{\sinh^3 u} - \frac{\cosh u}{\sinh^2 u}\right) + c_2\left(\frac{3u}{\sinh u} - \frac{3u^2 \cosh u}{\sinh^2 u} + \frac{u^3 \cosh^2 u}{\sinh^3 u}\right), \quad (1.113)$$

$$\mathbf{f}_{1,2} = \frac{1}{2u^2} + \frac{1-2\cosh u}{2u \sinh u} + c_1\left(\frac{1}{u^4} - \frac{\cosh^2 u}{u \sinh^3 u} + \frac{\cosh u - 1}{u^2 \sinh^2 u}\right) + c_2\left(-\frac{u \cosh^2 u}{\sinh^3 u} + \frac{(2\cosh u - 3)}{u \sinh u} + \frac{(3\cosh u - 1)}{\sinh^2 u}\right), \quad (1.114)$$

$$\mathbf{f}_{1,3} = \frac{1}{2u^2} - \frac{1}{2u} \coth u + c_1\left(-\frac{1}{u^4} + \frac{\cosh u}{u \sinh^3 u}\right) + c_2\left(\frac{u \cosh u}{\sinh^3 u} + \frac{\cosh u}{u \sinh u} - \frac{2}{\sinh u}\right), \quad (1.115)$$

$$\mathbf{f}_{1,4} = c_1\left(-\frac{1}{u^3 \sinh u} - \frac{\cosh u}{u^2 \sinh^2 u} + \frac{1 + \cosh^2 u}{u \sinh^3 u}\right) + c_2\left(\frac{u(1 + \cosh^2 u)}{\sinh^3 u} + \frac{3}{u \sinh u} - \frac{5 \cosh u}{\sinh^2 u}\right), \quad (1.116)$$

where c_1, c_2 are undetermined constants. Note that the functions (1.113)-(1.116) have a regular power series expansion at $u = 0$ with nonnegative even powers of u only. We note that there is another homogeneous solution that we discarded because it is singular at $u = 0$:

$$\begin{aligned} \mathbf{f}_{1,1} &= c_3 \left(\frac{\cosh^2 u}{\sinh^3 u} \right), & \mathbf{f}_{1,2} &= -c_3 \left(\frac{\coth u}{u^4} + \frac{1}{u^3 \sinh^2 u} + \frac{\cosh^2 u}{u^2 \sinh^3 u} \right), \\ \mathbf{f}_{1,3} &= c_3 \left(\frac{\coth u}{u^4} + \frac{1}{u^3 \sinh^2 u} + \frac{\cosh u}{u^2 \sinh^3 u} \right), & \mathbf{f}_{1,4} &= c_3 \left(\frac{\cosh u}{u^3 \sinh^2 u} + \frac{1 + \cosh^2 u}{u^2 \sinh^3 u} \right). \end{aligned} \quad (1.117)$$

We are now left with two unknown parameters c_1, c_2 , but one can be adjusted to zero by a shift of the center of the zeroth order solution, $\vec{x} \rightarrow \vec{x} + c_0 \vec{\ell}$, followed by a suitable gauge transformation to fix back the $\mathbf{f}_{1,6} = 0$ gauge. This allows us to set $c_1 = 0$. The parameter c_2 is undetermined at this point, since it depends on the proper boundary conditions at $u = \infty$ and at $x_3 = -1/b$.

Now we move on to order $O(b^2)$. The general ansatz at this order is:

$$b^2 \phi^{a(2)} = \ell^2 x_a \mathbf{f}_{2,1}(u) + [\ell_a (\ell_k x_k) - \frac{1}{3} \ell^2 x_a] \mathbf{f}_{2,3}(u) + x_a [(\ell_k x_k) (\ell_m x_m) - \frac{1}{3} \ell^2 u^2] \mathbf{f}_{2,4}(u), \quad (1.118)$$

$$b A_i^{a(2)} = \ell^2 \epsilon_{iak} x_k \mathbf{f}_{2,2}(u) + x_a \epsilon_{ijk} x_j \ell_k (\ell_m x_m) \mathbf{f}_{2,5}(u) + x_i \epsilon_{ajk} x_j \ell_k (\ell_m x_m) \mathbf{f}_{2,6}(u) + \epsilon_{aij} [\ell_j (\ell_m x_m) - \frac{1}{3} \ell^2 x_j] \mathbf{f}_{2,7}(u) + (\ell_i \epsilon_{ajk} x_j \ell_k - \frac{1}{3} \ell^2 \epsilon_{aji} x_j) \mathbf{f}_{2,8}(u), \quad (1.119)$$

where we have separated the different terms according to whether they can be expressed in terms of the spin-0 combination $\ell^2 \equiv \ell_k \ell_k$ or the spin-2 combination $\ell_k \ell_m - \frac{1}{3} \ell^2 \delta_{km}$. We again used the identity (1.99) to eliminate the term $\epsilon_{aij} x_j (\vec{\ell} \cdot \vec{x})^2$, and we also note the identity $\ell_{[i} \epsilon_{a]jk} x_j \ell_k = \frac{1}{2} \epsilon_{aij} \ell_j (\ell_k x_k) - \frac{1}{2} \ell^2 \epsilon_{aij} x_j$, which we used to eliminate a term of the form $\ell_a \epsilon_{ijk} x_j \ell_k \mathbf{f}_{2,9}$. At order $O(b^2)$ the possible gauge parameters are of the form

$$\lambda^a = \epsilon_{abc} x_b \ell_c (\ell_k x_k) \mathbf{g}_{2,1}(u),$$

and we use the corresponding gauge transformation to gauge fix $\mathbf{f}_{2,8} = 0$.

Our parameters $\mathbf{f}_{2,1}, \mathbf{f}_{2,2}$ correspond to spin-0 terms, while $\mathbf{f}_{2,3}, \dots, \mathbf{f}_{2,7}$ correspond to spin-2 terms. The spin-2 equations are:

$$0 = \frac{1}{u} \mathbf{f}'_{2,4} - \frac{1}{u} \mathbf{f}'_{2,5} - K \mathbf{f}_{2,4} + K \mathbf{f}_{2,5} + (H - K) \mathbf{f}_{2,6} + \frac{1}{u} \mathbf{f}'_{1,2} - K \mathbf{f}_{1,2} - H \mathbf{f}_{1,4} - \mathbf{f}_{1,2} \mathbf{f}_{1,4} - \mathbf{f}_{1,3} \mathbf{f}_{1,4}, \quad (1.120)$$

$$0 = \frac{1}{u} \mathbf{f}'_{2,7} + H \mathbf{f}_{2,7} + K \mathbf{f}_{2,3} + (1 + u^2 K) \mathbf{f}_{2,4} - 2 \mathbf{f}_{2,6} + K \mathbf{f}_{1,1} + \mathbf{f}_{1,2} + u^2 K \mathbf{f}_{1,2} - \mathbf{f}_{1,1} \mathbf{f}_{1,3}, \quad (1.121)$$

$$0 = \frac{1}{u} \mathbf{f}'_{2,3} - (1 + u^2 K) \mathbf{f}_{2,5} + (1 - u^2 H) \mathbf{f}_{2,6} + (K - H) \mathbf{f}_{2,7} + \frac{1}{u} \mathbf{f}'_{1,1} + u^2 H \mathbf{f}_{1,4} + \mathbf{f}_{1,1} \mathbf{f}_{1,3} + \mathbf{f}_{1,1} \mathbf{f}_{1,4} + u^2 \mathbf{f}_{1,2} \mathbf{f}_{1,4} + u^2 \mathbf{f}_{1,3} \mathbf{f}_{1,4}, \quad (1.122)$$

$$0 = u \mathbf{f}'_{2,5} - \frac{1}{u} \mathbf{f}'_{2,7} - K \mathbf{f}_{2,3} + 2 \mathbf{f}_{2,4} + 4 \mathbf{f}_{2,5} + (2 + u^2 K) \mathbf{f}_{2,6} + K \mathbf{f}_{2,7} - K \mathbf{f}_{1,1} + \mathbf{f}_{1,2} + \mathbf{f}_{1,1} \mathbf{f}_{1,3} + u^2 \mathbf{f}_{1,3} \mathbf{f}_{1,4}, \quad (1.123)$$

$$0 = \mathbf{f}_{2,3} - u^2 \mathbf{f}_{2,6} - \mathbf{f}_{2,7} - u^2 \mathbf{f}_{1,1} \mathbf{f}_{1,3} - u^4 \mathbf{f}_{1,3} \mathbf{f}_{1,4}. \quad (1.124)$$

The spin-0 equations are:

$$0 = \mathbf{f}'_{2,1} + \frac{1}{u} \mathbf{f}_{2,1} + \frac{2}{u} (1 + u^2 K) \mathbf{f}_{2,2} + \frac{1}{3} \mathbf{f}'_{1,1} + \frac{1}{3} u^2 \mathbf{f}'_{1,2} + \frac{2}{3} u \mathbf{f}_{1,2} - \frac{2}{3} u \mathbf{f}_{1,1} \mathbf{f}_{1,4}, \quad (1.125)$$

$$0 = \mathbf{f}'_{2,2} + \left(\frac{2}{u} + uH\right) \mathbf{f}_{2,2} + \frac{1}{u} (1 + u^2 K) \mathbf{f}_{2,1} + \frac{1}{3} u K \mathbf{f}_{1,1} + \frac{1}{3} u (1 + u^2 K) \mathbf{f}_{1,2} + \frac{1}{2} u \mathbf{f}_{1,1} \mathbf{f}_{1,3} - \frac{1}{3} u^3 \mathbf{f}_{1,3} \mathbf{f}_{1,4}. \quad (1.126)$$

We first solve the spin-0 equations. The general solution is given by:

$$\mathbf{f}_{2,1} = \frac{u^2}{36 \sinh^2 u} + \frac{1}{6} u \coth u + c_4 \left(\frac{1}{\sinh^2 u} - \frac{1}{u} \coth u \right) + c_5 \left(\frac{1}{u \sinh^2 u} \right), \quad (1.127)$$

$$\mathbf{f}_{2,2} = \frac{u^2 \cosh u}{36 \sinh^2 u} - \frac{u}{8 \sinh u} + c_4 \left(\frac{\cosh u}{\sinh^2 u} - \frac{1}{u \sinh u} \right) + c_5 \left(\frac{\cosh u}{u \sinh^2 u} \right). \quad (1.128)$$

Since c_5 multiplies an u -odd and singular solution, we set $c_5 = 0$. The unknown c_4 needs to be determined by the boundary conditions at $u = \infty$ and $x_3 = -a$.

Now, we move on to the spin-2 equations. First we look for a solution of the homogeneous spin-2 part:

$$0 = \frac{1}{u} \mathbf{f}'_{2,4} - \frac{1}{u} \mathbf{f}'_{2,5} - K \mathbf{f}_{2,4} + K \mathbf{f}_{2,5} + (H - K) \mathbf{f}_{2,6}, \quad (1.129)$$

$$0 = \frac{1}{u} \mathbf{f}'_{2,7} + H \mathbf{f}_{2,7} + K \mathbf{f}_{2,3} + (1 + u^2 K) \mathbf{f}_{2,4} - 2 \mathbf{f}_{2,6}, \quad (1.130)$$

$$0 = \frac{1}{u} \mathbf{f}'_{2,3} - (1 + u^2 K) \mathbf{f}_{2,5} + (1 - u^2 H) \mathbf{f}_{2,6} + (K - H) \mathbf{f}_{2,7}, \quad (1.131)$$

$$0 = u \mathbf{f}'_{2,5} - \frac{1}{u} \mathbf{f}'_{2,7} - K \mathbf{f}_{2,3} + 2 \mathbf{f}_{2,4} + 4 \mathbf{f}_{2,5} + (2 + u^2 K) \mathbf{f}_{2,6} + K \mathbf{f}_{2,7}, \quad (1.132)$$

$$0 = \mathbf{f}_{2,3} - u^2 \mathbf{f}_{2,6} - \mathbf{f}_{2,7}. \quad (1.133)$$

The general solution that is well behaved as $u \rightarrow \infty$ is:

$$\mathbf{f}_{2,3}^{(homog)} = c_6 \left\{ \frac{4u}{\sinh u} \right\} + c_7 \left\{ \frac{4}{u^4 \sinh u} \right\}, \quad (1.134)$$

$$\mathbf{f}_{2,4}^{(homog)} = c_6 \left\{ \frac{6 \cosh u - 4}{u \sinh^2 u} - \frac{2}{\sinh^2 u} \right\} + c_7 \left\{ -\frac{4(\cosh u + 1)}{u^6 \sinh u} - \frac{2}{u^5 \sinh^2 u} \right\}, \quad (1.135)$$

$$\mathbf{f}_{2,5}^{(homog)} = c_6 \left\{ -\frac{2 \cosh u}{\sinh^2 u} - \frac{4 \cosh u - 6}{u \sinh u} \right\} + c_7 \left\{ -\frac{4(\cosh u + 1)}{u^6 \sinh u} - \frac{2 \cosh u}{u^5 \sinh^2 u} \right\}, \quad (1.136)$$

$$\mathbf{f}_{2,6}^{(homog)} = c_6 \left\{ -\frac{2}{u \sinh u} + \frac{2 \cosh u}{\sinh^2 u} \right\} + c_7 \left\{ \frac{8}{u^6 \sinh u} + \frac{2 \cosh u}{u^5 \sinh^2 u} \right\}, \quad (1.137)$$

$$\mathbf{f}_{2,7}^{(homog)} = c_6 \left\{ \frac{6u}{\sinh u} - \frac{2u^2 \cosh u}{\sinh^2 u} \right\} + c_7 \left\{ -\frac{4}{u^4 \sinh u} - \frac{2 \cosh u}{u^3 \sinh^2 u} \right\}. \quad (1.138)$$

Additionally, there are two more linearly independent solutions that grow exponentially as $u \rightarrow \infty$. They are given by:

$$\begin{aligned} \mathbf{f}_{2,3}^{(homog)} &= c_8 \left\{ -\frac{2 \cosh^2 u}{u^2 \sinh u} + \frac{6 \cosh u}{u^3} - \frac{6 \sinh u}{u^4} \right\} \\ &\quad + c_9 \left\{ -\frac{6 \cosh u}{u^4} - \frac{2 \cosh u}{u^2} + \frac{6 \cosh^2 u}{u^3 \sinh u} \right\}, \end{aligned} \quad (1.139)$$

$$\begin{aligned} \mathbf{f}_{2,4}^{(homog)} &= c_8 \left\{ \frac{6 \sinh u}{u^6} - \frac{3(1+2 \cosh u)}{u^5} + \frac{2 \cosh^2 u}{u^4 \sinh u} + \frac{1}{u^3 \sinh^2 u} \right\} \\ &\quad + c_9 \left\{ \frac{6(1+\cosh u)}{u^6} - \frac{3(\cosh u + 2 \cosh^2 u)}{u^3 \sinh u} + \frac{2 \cosh u}{u^4} - \frac{3}{u^4 \sinh^2 u} \right\}, \end{aligned} \quad (1.140)$$

$$\begin{aligned} \mathbf{f}_{2,5}^{(homog)} &= c_8 \left\{ \frac{6 \sinh u}{u^6} - \frac{3(2+\cosh u)}{u^5} + \frac{2 \coth u}{u^4} + \frac{\cosh u}{u^3 \sinh^2 u} \right\} \\ &\quad + c_9 \left\{ \frac{6 \cosh u}{u^6} + \frac{6}{u^6} - \frac{3 \cosh^2 u}{u^5 \sinh u} - \frac{6 \coth u}{u^5} + \frac{2}{u^4} - \frac{3 \cosh u}{u^4 \sinh^2 u} \right\}, \end{aligned} \quad (1.141)$$

$$\begin{aligned} \mathbf{f}_{2,6}^{(homog)} &= c_8 \left\{ -\frac{12 \sinh u}{u^6} + \frac{9 \cosh u}{u^5} - \frac{2 \cosh^2 u}{u^4 \sinh u} - \frac{\cosh u}{u^3 \sinh^2 u} \right\} \\ &\quad + c_9 \left\{ -\frac{12 \cosh u}{u^6} + \frac{9 \cosh^2 u}{u^5 \sinh u} + \frac{3 \cosh u}{u^4 \sinh^2 u} - \frac{2 \cosh u}{u^4} \right\}, \end{aligned} \quad (1.142)$$

$$\begin{aligned} \mathbf{f}_{2,7}^{(homog)} &= c_8 \left\{ \frac{6 \sinh u}{u^4} - \frac{3 \cosh u}{u^3} + \frac{\cosh u}{u \sinh^2 u} \right\} \\ &\quad + c_9 \left\{ \frac{6 \cosh u}{u^4} - \frac{3 \cosh^2 u}{u^3 \sinh u} - \frac{3 \cosh u}{u^2 \sinh^2 u} \right\}. \end{aligned} \quad (1.143)$$

Once we have a complete linearly independent set of solutions to the homogeneous problem, we can find the solution to the inhomogeneous problem by integration. When we perform the integrals we obtain complicated expressions that contain polylogarithms $\text{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n}$. For example, if we set $c_2 = 0$ in (1.113)-(1.116), we get:

$$\begin{aligned} \mathbf{f}_{2,3}^{(inhomog)} &= -\frac{9}{2u^4 \sinh u} \text{Li}_4(e^{-2u}) + \left[\frac{3}{u^3} (\sinh u - \frac{2}{\sinh u}) - (\frac{3}{u^4} + \frac{1}{u^2}) \cosh u \right] \text{Li}_3(e^{-2u}) \\ &\quad - \left[\frac{3}{u^2 \sinh u} + (\frac{6}{u^3} + \frac{2}{u}) \cosh u - \frac{6 \sinh u}{u^2} \right] \text{Li}_2(e^{-2u}) \\ &\quad + \left[(\frac{6}{u^2} + 2) \cosh u - \frac{6}{u} \sinh u \right] \log(1 - e^{-2u}) - \frac{1}{\sinh u} \left(\frac{1}{2} + \frac{45}{2u^3} + \frac{2}{u} + \frac{59}{120} u \right) \\ &\quad + \left(\frac{u^2}{8 \sinh^2 u} + \frac{45}{2u^4} + \frac{15}{2u^2} + \frac{2}{u} + 2 + \frac{2}{3} u \right) \cosh u - \left(\frac{45}{2u^3} + \frac{2}{u} + 2 + \frac{2}{3} u \right) \sinh u, \end{aligned} \quad (1.144)$$

$$\begin{aligned} \mathbf{f}_{2,4}^{(inhomog)} &= \left(\frac{9}{2u^6 \sinh u} + \frac{9}{2u^6} \coth u + \frac{9}{4u^5 \sinh^2 u} \right) \text{Li}_4(e^{-2u}) \\ &\quad + \left[\frac{3}{u^6} + (\frac{3}{u^6} + \frac{1}{u^4}) \cosh u + \frac{15}{2u^5} \coth u + \frac{6}{u^5 \sinh u} + \frac{3}{u^4 \sinh^2 u} - \frac{3}{u^5} \sinh u \right] \text{Li}_3(e^{-2u}) \\ &\quad + \left[\frac{6}{u^5} + \frac{3}{2u^3 \sinh^2 u} + (\frac{6}{u^5} + \frac{2}{u^3}) \cosh u + \frac{6}{u^4} \coth u + \frac{3}{u^4 \sinh u} - \frac{6}{u^4} \sinh u \right] \text{Li}_2(e^{-2u}) \\ &\quad + \left[\frac{6}{u^3} \sinh u - \frac{3}{u^3} \coth u - (\frac{6}{u^4} + \frac{2}{u^2}) \cosh u - \frac{6}{u^4} \right] \log(1 - e^{-2u}) \\ &\quad + \left(\frac{45}{2u^5} + \frac{2}{u^3} + \frac{2}{u^2} + \frac{2}{3u} \right) \sinh u + \left(\frac{45}{4u^4} + \frac{1}{u^2} + \frac{1}{4u} + \frac{37}{120} \right) \frac{1}{\sinh^2 u} - \frac{\cosh u}{8 \sinh^2 u} \\ &\quad + \left(\frac{45}{2u^5} + \frac{2}{u^3} + \frac{1}{2u^2} + \frac{59}{120u} \right) \frac{1}{\sinh u} + \left(\frac{45}{4u^5} + \frac{2}{u^3} - \frac{1}{2u^2} + \frac{1}{5u} \right) \coth u \\ &\quad - \left(\frac{45}{2u^6} + \frac{15}{2u^4} + \frac{2}{u^3} + \frac{2}{u^2} + \frac{2}{3u} \right) \cosh u - \frac{45}{2u^6} - \frac{2}{u^3} + \frac{1}{8u^2}, \end{aligned} \quad (1.145)$$

$$\begin{aligned} \mathbf{f}_{2,5}^{(inhomog)} &= \left(\frac{9}{2u^6 \sinh u} + \frac{9}{2u^6} \coth u + \frac{9 \cosh u}{4u^5 \sinh^2 u} \right) \text{Li}_4(e^{-2u}) \\ &\quad + \left(\frac{3}{u^6} + \frac{1}{u^4} + \frac{3}{u^6} \cosh u - \frac{3}{2u^5} \sinh u + \frac{6}{u^5} \coth u + \frac{15}{2u^5 \sinh u} + \frac{3 \cosh u}{u^4 \sinh^2 u} \right) \text{Li}_3(e^{-2u}) \\ &\quad + \left(\frac{6}{u^5} + \frac{2}{u^3} + \frac{6}{u^5} \cosh u - \frac{3}{u^4} \sinh u + \frac{3}{u^4} \coth u + \frac{6}{u^4 \sinh u} + \frac{3 \cosh u}{2u^3 \sinh^2 u} \right) \text{Li}_2(e^{-2u}) \\ &\quad - \left[\frac{6}{u^4} + \frac{2}{u^2} + \frac{6}{u^4} \cosh u - \frac{3}{u^3} \sinh u + \frac{3}{u^3 \sinh u} \right] \log(1 - e^{-2u}) \\ &\quad - \frac{45}{2u^6} - \frac{15}{2u^4} - \frac{2}{u^3} - \frac{3}{8u^2} - \frac{2}{3u} - \left(\frac{45}{2u^6} + \frac{2}{u^3} + \frac{1}{u^2} \right) \cosh u \\ &\quad + \left(\frac{45}{4u^5} + \frac{2}{u^3} + \frac{1}{u^2} \right) \sinh u + \left(\frac{45}{2u^5} + \frac{2}{u^3} + \frac{1}{2u^2} + \frac{13}{15u} \right) \coth u \\ &\quad + \left(\frac{45}{4u^5} + \frac{2}{u^3} - \frac{1}{2u^2} - \frac{27}{40u} \right) \frac{1}{\sinh u} + \left(\frac{45}{4u^4} + \frac{1}{u^2} + \frac{1}{4u} + \frac{37}{120} \right) \frac{\cosh u}{\sinh^2 u} - \frac{1}{8 \sinh^2 u}, \end{aligned} \quad (1.146)$$

$$\begin{aligned} \mathbf{f}_{2,6}^{(inhomog)} &= -\left(\frac{9}{u^6 \sinh u} + \frac{9 \cosh u}{4u^5 \sinh^2 u} \right) \text{Li}_4(e^{-2u}) \\ &\quad - \left[\left(\frac{6}{u^6} + \frac{1}{u^4} \right) \cosh u - \frac{9}{2u^5} \sinh u + \frac{27}{2u^5 \sinh u} + \frac{3 \cosh u}{u^4 \sinh^2 u} \right] \text{Li}_3(e^{-2u}) \\ &\quad - \left[\left(\frac{12}{u^5} + \frac{2}{u^3} \right) \cosh u - \frac{9}{u^4} \sinh u + \frac{9}{u^4 \sinh u} + \frac{3 \cosh u}{2u^3 \sinh^2 u} \right] \text{Li}_2(e^{-2u}) \\ &\quad + \left[\left(\frac{12}{u^4} + \frac{2}{u^2} \right) \cosh u - \frac{9}{u^3} \sinh u + \frac{3}{u^3 \sinh u} \right] \log(1 - e^{-2u}) \\ &\quad - \left(\frac{45}{4u^4} + \frac{1}{u^2} + \frac{1}{4u} + \frac{13}{30} \right) \frac{\cosh u}{\sinh^2 u} + \left(\frac{45}{u^6} + \frac{15}{2u^4} + \frac{4}{u^3} + \frac{3}{u^2} + \frac{2}{3u} \right) \cosh u \\ &\quad - \left(\frac{135}{4u^5} + \frac{4}{u^3} + \frac{3}{u^2} + \frac{2}{3u} \right) \sinh u - \left(\frac{135}{4u^5} + \frac{4}{u^3} - \frac{13}{30u} \right) \frac{1}{\sinh u}, \end{aligned} \quad (1.147)$$

$$\mathbf{f}_{2,7}^{(inhomog)} = \mathbf{f}_{2,3}^{(inhomog)} - u^2 \mathbf{f}_{2,6}^{(inhomog)} - u^2 \mathbf{f}_{1,1} \mathbf{f}_{1,3} - u^4 \mathbf{f}_{1,3} \mathbf{f}_{1,4}. \quad (1.148)$$

We note that the combinations of polylogarithms that appear here are the results of the integrals

$$\int u^3 \coth u \, du = -\frac{3}{4} \text{Li}_4(e^{-2u}) - \frac{3}{2} u \text{Li}_3(e^{-2u}) - \frac{3}{2} u^2 \text{Li}_2(e^{-2u}) + u^3 \log(1 - e^{-2u}) + \frac{1}{4} u^4,$$

and

$$\int u^2 \coth u \, du = -\frac{1}{2} \text{Li}_3(e^{-2u}) - u \text{Li}_2(e^{-2u}) + u^2 \log(1 - e^{-2u}) + \frac{1}{3} u^3.$$

When we turn on $c_2 \neq 0$ we get additional terms, but they can be expressed as rational functions of e^u and u and do not cancel the polylogarithms. In any case, this demonstrates that a simple solution to the BPS equations (1.42), involving only basic functions, does not exist.

1.7 Discussion

We have studied a 2+1D system constructed from the compactification of the $(2, 0)$ -theory on $(\mathbb{R}^2 \times S^1)/\mathbb{Z}_k$. In the large k limit, we have reduced it to 4+1D SYM on the ‘‘cigar’’ geometry, and we have developed the BPS equations that describe Q-ball solitons. In terms of the effective FQHE low-energy action, these solitons are bound states of k quasi-particles (each of $1/k$ charge). We mapped the BPS equations to the Bogomolnyi equations $D\Phi = *F$ on a manifold with metric

$$ds^2 = x_3^2(dx_1^2 + dx_2^2 + dx_3^2), \quad (1.149)$$

and we described a relation between axisymmetric solutions (in particular, the 1-monopole solution) and harmonic maps $\varphi : AdS_3 \rightarrow AdS_2$. It would be interesting to explore this system further. We note that other interesting extensions of the classic Bogomolnyi equations were discovered in [23], in the context of D3-brane probes of a Melvin space (which is in fact T-dual to the orbifold background in our work), where the D3-branes are oriented in such a way that noncommutative geometry with a variable parameter is generated.

Our problem is reminiscent of the problem of monopoles on AdS_3 [if x_3^2 is replaced with $1/x_3^2$ in (1.149)]. The latter is integrable, with known solutions, and in particular the one-monopole solution is not difficult to construct [54]. Like the case of monopoles on AdS_3 , the monopole solutions on the space (1.149) contain as a limit the classic Prasad-Sommerfield solutions (by going to the outskirts $x_3 \rightarrow \infty$). Indeed, in §1.6.5 we outlined an expansion around the Prasad-Sommerfield solution, up to second order in $1/x_3$, albeit with a few undetermined coefficients.

Monopole equations on a three-dimensional space can be recast as the dimensional reduction of instanton equations on a four-dimensional space, which can provide new insights. For example, instanton equations on Taub-NUT spaces can be reduced to Bogomolnyi’s equations on \mathbb{R}^3 (with singularities) [55], which recently led to the discovery of new explicit solutions [56, 57], using the techniques developed in [58, 59] for solving instanton equations on Taub-NUT spaces. It might therefore be interesting to explore instanton equations on

circle fibrations over (1.149) and look for their applications in string theory. More recently, a set of partial differential equations on G_2 -manifolds was discovered [60], which can be reduced in special cases to Bogomolnyi's equations on \mathbb{R}^3 . It would be interesting to explore whether the system studied in this chapter and the related Bogomolnyi equations on (1.149) have an interesting 7-dimensional origin.

In this chapter we focused on the case of a single monopole, corresponding to a $(2, 0)$ -string wound once. It would be interesting to generalize the discussion to the case of multiple $(2, 0)$ -strings, which corresponds to monopole charge higher than 1 in the effective metric (1.149). Techniques for analyzing the low-energy description of multiple $(2, 0)$ -strings have recently been developed in [61]-[62].

Chapter 2

Janus configurations with $SL(2, \mathbb{Z})$ -duality twists

We develop an equivalence between two Hilbert spaces: (i) the space of states of $U(1)^n$ Chern-Simons theory with a certain class of tridiagonal matrices of coupling constants (with corners) on T^2 ; and (ii) the space of ground states of strings on an associated mapping torus with T^2 fiber. The equivalence is deduced by studying the space of ground states of $SL(2, \mathbb{Z})$ -twisted circle compactifications of $U(1)$ gauge theory, connected with a Janus configuration, and further compactified on T^2 . The equality of dimensions of the two Hilbert spaces (i) and (ii) is equivalent to a known identity on determinants of tridiagonal matrices with corners. The equivalence of operator algebras acting on the two Hilbert spaces follows from a relation between the Smith normal form of the Chern-Simons coupling constant matrix and the isometry group of the mapping torus, as well as the torsion part of its first homology group.

2.1 Introduction and summary of results

Our goal is to develop tools for studying circle compactifications of $\mathcal{N} = 4$ Super-Yang-Mills theory on S^1 with a general $SL(2, \mathbb{Z})$ -duality twist (also known as a “duality wall”) inserted at a point on S^1 . The low-energy limit of such compactifications encodes information about the operator that realizes the $SL(2, \mathbb{Z})$ -duality, and can potentially teach us new facts about S-duality itself. Some previous works on duality walls and related compactifications include [7, 11, 29, 64, 65, 66, 67, 68, 69, 70].

In this chapter we consider only the abelian gauge group $G = U(1)$, leaving the extension to nonabelian groups for a separate publication [71]. We focus on the Hilbert space of ground states of the system and study it in two equivalent ways: (i) directly in field theory; and (ii) via a dual type-IIA string theory system (extending the techniques developed in [29]). As we will show, the equivalence of these two descriptions implies the equivalence of:

(i) the Hilbert space of ground states of $U(1)^n$ Chern-Simons theory with action

$$L = \frac{1}{4\pi} \sum_{i=1}^n \mathbf{k}_i A_i \wedge dA_i - \frac{1}{2\pi} \sum_{i=1}^{n-1} A_i \wedge dA_{i+1} - \frac{1}{2\pi} A_1 \wedge dA_n,$$

on T^2 , and

(ii) the Hilbert space of ground states of strings of winding number $w = 1$ on a certain target space that contains the *mapping torus* with T^2 fiber:

$$\mathbf{M}_3 \equiv \frac{I \times T^2}{(0, v) \sim (1, f(v))}, \quad (v \in T^2),$$

where $I = [0, 1]$ is the unit interval, and f is a large diffeomorphism of T^2 corresponding to the $SL(2, \mathbb{Z})$ matrix

$$W \equiv \begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.1)$$

We will explain the construction of these Hilbert spaces in detail below.

An immediate consequence of the proposed equivalence of Hilbert spaces (i) and (ii) is the identity

$$\det \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ -1 & & 0 & -1 & \mathbf{k}_n \end{pmatrix} = \text{tr} \left[\begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix} \right] - 2. \quad (2.2)$$

which follows from the equality of dimensions of the Hilbert spaces above. This is a known identity (see for instance [76]), and we will present a proof in Appendix B.1, for completeness.¹

Moreover, equivalence of the operator algebras of the systems associated with (i) and (ii) allows us to make a stronger statement. The operator algebra of (i) is generated by Wilson loops along two fundamental cycles of T^2 , and keeping only one of these cycles gives a maximal finite abelian subgroup. Let $\Lambda \subseteq \mathbb{Z}^n$ be the sublattice of \mathbb{Z}^n generated by the columns of the Chern-Simons coupling constant matrix, which appears on the LHS of (2.2). Then, the abelian group generated by the maximal commuting set of Wilson loops

¹ The continuum limit of (2.2) with $n \rightarrow \infty$ and $\mathbf{k}_i \rightarrow 2 + \frac{1}{n^2} V(\frac{i}{n})$ might be more familiar. It leads to a variant of the Gelfand-Yaglom theorem [82] with a periodic potential: $\det[-d^2/dx^2 + V(x)] = \text{tr} \left[P \exp \oint \begin{pmatrix} \sqrt{V} + \frac{V'}{2V} & \frac{V'}{2V} \\ -\sqrt{V} & -\sqrt{V} \end{pmatrix} dx \right] - 2$ (up to a renormalization-dependent multiplicative constant).

is isomorphic to \mathbb{Z}^n/Λ . The operator algebra of (ii), on the other hand, is constructed by combining the isometry group of \mathbf{M}_3 with the group of operators that measure the various components of string winding number in \mathbf{M}_3 . The latter is captured algebraically by the Pontryagin dual ${}^\vee(\dots)$ of the torsion part Tor of the homology group $H_1(\mathbf{M}_3, \mathbb{Z})$. (The terms will be explained in more detail in §2.4.3.) Thus, ${}^\vee \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ as well as the isometry group are both equivalent to \mathbb{Z}^n/Λ . Together, ${}^\vee \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ and $\text{Isom}(\mathbf{M}_3)$ generate a noncommutative (but reducible) group that is equivalent to the operator algebra of the Wilson loops of the Chern-Simons system in (i). The subgroup ${}^\vee \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ corresponds to the group generated by the Wilson loops along one fixed cycle of T^2 (let us call it “the α -cycle”) and $\text{Isom}(\mathbf{M}_3)$ corresponds to the group generated by the Wilson loops along another cycle (call it “the β -cycle”), where α and β generate $H_1(T^2, \mathbb{Z})$. The situation is summarized in the diagram below.

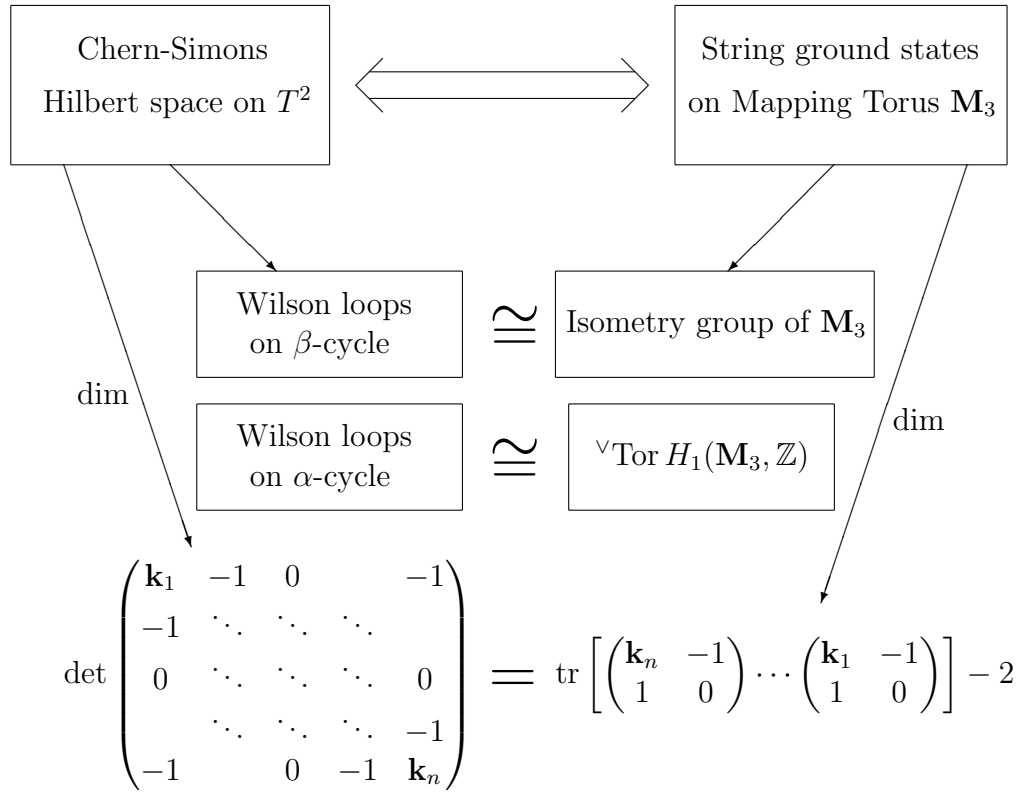


Figure 2.1: Equivalence between our two Hilbert spaces. The operator algebra and the dimension of both Hilbert spaces and their relationship is presented in this figure.

We will now present a detailed account of the statements made above. In §2.2 we construct the $SL(2, \mathbb{Z})$ -twist from the QFT perspective, and in §2.3 we take its low-energy limit and make connection with $U(1)^n$ Chern-Simons theory, leading to Hilbert space (i). In §2.4

we describe the dual construction of type-IIA strings on \mathbf{M}_3 . In §2.5 we develop the “dictionary” that translates between the states and operators of (i) and (ii). We conclude in §2.6 with a brief summary of what we have found so far and a preview of the nonabelian case.

2.2 The $SL(2, \mathbb{Z})$ -twist

Our starting point is a free 3+1D $U(1)$ gauge theory with action

$$I = \frac{1}{4g_{\text{ym}}^2} \int F \wedge^* F + \frac{\theta}{2\pi} \int F \wedge F,$$

where $F = dA$ is the field strength. As usual, we define the complex coupling constant

$$\tau \equiv \frac{4\pi i}{g_{\text{ym}}^2} + \frac{\theta}{2\pi} \equiv \tau_1 + i\tau_2.$$

The $SL(2, \mathbb{Z})$ group of dualities is generated by S and T that act as $\tau \rightarrow -1/\tau$ and $\tau \rightarrow \tau + 1$, respectively.

Let the space-time coordinates be x_0, \dots, x_3 . We wish to compactify direction x_3 on a circle (so that $0 \leq x_3 \leq 2\pi$ is a periodic coordinate), but allow τ to vary as a function of x_3 in such a way that

$$\tau(0) = \frac{\mathbf{a}\tau(2\pi) + \mathbf{b}}{\mathbf{c}\tau(2\pi) + \mathbf{d}},$$

where $W \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in SL(2, \mathbb{Z})$ defines an electric/magnetic duality transformation. Such a compactification contains two ingredients:

- The variable coupling constant τ ; and
- The “duality-twist” at $x_3 = 0 \sim 2\pi$.

We will discuss the ingredients separately, starting from the duality-twist.

The duality-twist can be described concretely in terms of an abelian Chern-Simons theory as follows. Represent the $SL(2, \mathbb{Z})$ matrix in terms of the generators S and T (nonuniquely) as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = T^{k_1} S T^{k_2} S \dots T^{k_n} S, \quad (2.3)$$

where $\mathbf{k}_1, \dots, \mathbf{k}_n$ are integers, some of which may be zero. To understand how each of the operators T and S act separately, we pretend that x_3 is a time-direction and impose the temporal gauge condition $A_3 = 0$. At any given x_3 the wave-function is formally $\Psi(\mathcal{A})$, where \mathcal{A} is the gauge field 1-form on the three-dimensional space parameterized by x_0, x_1, x_2 .

The action of the generators S and T on the wave-functions is then given by (see for instance [83, 84]):

$$S : \Psi(\mathcal{A}) \rightarrow \int e^{-\frac{i}{2\pi} \int \mathcal{A} \wedge d\mathcal{A}'} \Psi(\mathcal{A}') \mathcal{D}\mathcal{A}', \quad T : \Psi(\mathcal{A}) \rightarrow e^{\frac{i}{4\pi} \int \mathcal{A} \wedge d\mathcal{A}} \Psi(\mathcal{A}).$$

It is now clear how to incorporate the duality twist by combining these two elements to realize the $SL(2, \mathbb{Z})$ transformation (2.3). We have to add to the action a Chern-Simons term at $x_3 = 0$ with additional auxiliary fields A_1, \dots, A_{n+1} and with action

$$I_{CS} = \frac{1}{4\pi} \sum_{i=1}^n k_i A_i \wedge dA_i - \frac{1}{2\pi} \sum_{i=1}^n A_i \wedge dA_{i+1}, \quad (2.4)$$

and then set

$$A_1 = A|_{x_3=0}, \quad A_{n+1} = A|_{x_3=2\pi}.$$

The second ingredient is the varying coupling constant $\tau(x_3)$. Systems with such a varying τ are known as *Janus configurations* [85]. They have supersymmetric extensions [5, 86, 87] where the Lagrangian of $N = 4$ Super-Yang-Mills with variable τ is modified so as to preserve 8 supercharges. In such configurations the function $\tau(x_3)$ traces a geodesic in the hyperbolic upper-half τ -plane, namely, a half-circle centered on the real axis [5]. In this model, the surviving supersymmetry is described by parameters that vary as a function of x_3 , so that in general the supercharges at $x_3 = 0$ are not equal to those at $x_3 = 2\pi$. This might have been a problem for us, since we need to continuously connect $x_3 = 0$ to $x_3 = 2\pi$ to form a consistent supersymmetric theory, but luckily, we also have the $SL(2, \mathbb{Z})$ -twist, and as shown in [88], in $N = 4$ Super-Yang-Mills (with a fixed coupling constant τ), the $SL(2, \mathbb{Z})$ duality transformations do not commute with the supercharges. Following the action of duality, the SUSY generators pick up a known phase. But as it turns out, this phase exactly matches the phase difference from 0 to 2π in the Janus configuration. Therefore, we can combine the two separate ingredients and close the supersymmetric Janus configuration on the segment $[0, 2\pi]$ with an $SL(2, \mathbb{Z})$ duality twist that connects 0 to 2π . We describe this construction in more detail in Appendix B.2.

The details of the supersymmetric action, however, will not play an important role in what follows, so we will just assume supersymmetry and proceed. Thanks to mass terms that appear in the Janus configuration (which are needed to close the SUSY algebra [5]), at low-energy the superpartners of the gauge fields are all massive (see Appendix B.2), with masses of the order of the Kaluza-Klein scale, and we can ignore them. We will therefore proceed with a discussion of only the free $U(1)$ gauge fields.

2.3 The Low-energy limit and Chern-Simons theory

At low-energy we have to set $A_1 = A_{n+1}$ in (2.4), since the dependence of A on x_3 is suppressed. Then, the low-energy system is described by a 2+1D Chern-Simons action with

gauge group $U(1)^n$ and action

$$I = \frac{1}{4\pi} \sum_{i,j=1}^n K_{ij} A_i \wedge dA_j,$$

with coupling-constant matrix that is given by

$$K \equiv \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ -1 & & 0 & -1 & \mathbf{k}_n \end{pmatrix}. \quad (2.5)$$

We now make directions x_1, x_2 periodic, so that the theory is compactified on T^2 , leaving only time uncompactified. The dimension of the resulting Hilbert space of states of this compactified Chern-Simons theory is $|\det K|$.

Next, we pick two fundamental cycles whose equivalence classes generate $H_1(T^2, \mathbb{Z})$. Let α be the cycle along a straight line from $(0, 0)$ to $(1, 0)$, and let β be a similar cycle from $(0, 0)$ to $(0, 1)$, in (x_1, x_2) coordinates. We define $2n$ Wilson loop operators:

$$U_j \equiv \exp \left(i \oint_{\alpha} A_j \right), \quad V_j \equiv \exp \left(i \oint_{\beta} A_j \right), \quad j = 1, \dots, n.$$

They are unitary operators with commutation relations given by

$$U_i U_j = U_j U_i, \quad V_i V_j = V_j V_i, \quad U_i V_j = e^{2\pi i (K^{-1})_{ij}} V_j U_i.$$

[$(K^{-1})_{ij}$ is the i, j element of the matrix K^{-1} .] In particular, for any $j = 1, \dots, n$ the operator $\prod_{i=1}^n U_i^{K_{ij}}$ commutes with all $2n$ operators, and hence is a central element. In an irreducible representation, it can be set to the identity. The U_i 's therefore generate a finite abelian group, which we denote by \mathcal{G}_α . Similarly, we denote by \mathcal{G}_β the finite abelian group generated by the V_i 's. Both groups are isomorphic and can be described as follows. Let $\Lambda \subseteq \mathbb{Z}^n$ be the sublattice of \mathbb{Z}^n generated by the columns of the matrix K . Then, \mathbb{Z}^n/Λ is a finite abelian group and $\mathcal{G}_\alpha \cong \mathcal{G}_\beta \cong \mathbb{Z}^n/\Lambda$, since an element of \mathbb{Z}^n represents the powers of a monomial in the U_i 's (or V_i 's), and an element in Λ corresponds to a monomial that is a central element. We therefore map

$$\mathcal{G}_\alpha \ni \prod_{i=1}^n U_i^{N_i} \mapsto (N_1, N_2, \dots, N_n) \in \mathbb{Z}^n \pmod{\Lambda}, \quad (2.6)$$

and similarly

$$\mathcal{G}_\beta \ni \prod_{i=1}^n V_i^{M_i} \mapsto (M_1, M_2, \dots, M_n) \in \mathbb{Z}^n \pmod{\Lambda}. \quad (2.7)$$

We denote the operator in \mathcal{G}_α that corresponds to $\mathbf{v} \in \mathbb{Z}^n/\Lambda$ by $\mathcal{O}_\alpha(\mathbf{v})$, and similarly we define $\mathcal{O}_\beta(\mathbf{v}) \in \mathcal{G}_\beta$ to be the operator in \mathcal{G}_β that corresponds to \mathbf{v} . For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n/\Lambda$ we define

$$\chi(\mathbf{u}, \mathbf{v}) \equiv e^{2\pi i \sum_{i,j} (K^{-1})_{ij} N_i M_j}, \quad (\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n/\Lambda). \quad (2.8)$$

The definition is independent of the particular representatives (N_1, \dots, N_n) or (M_1, \dots, M_n) in \mathbb{Z}^n/Λ . The commutation relations can then be written as

$$\mathcal{O}_\alpha(\mathbf{u})\mathcal{O}_\beta(\mathbf{v}) = \chi(\mathbf{u}, \mathbf{v})\mathcal{O}_\beta(\mathbf{v})\mathcal{O}_\alpha(\mathbf{u}). \quad (2.9)$$

We recall that for any nonsingular matrix of integers $K \in \mathrm{GL}(n, \mathbb{Z})$, one can find matrices $P, Q \in \mathrm{SL}(n, \mathbb{Z})$ such that

$$PKQ = \mathrm{diag}(d_1, d_2, \dots, d_n) \quad (2.10)$$

is a diagonal matrix, d_1, \dots, d_n are positive integers, and d_i divides d_{i+1} for $i = 1, \dots, n-1$. The integers d_1, \dots, d_n are unique, and we have

$$\mathbb{Z}^n/\Lambda \cong \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_n},$$

where \mathbb{Z}_d is the cyclic group of d elements. The matrix on the RHS of (2.10) is known as the *Smith normal form* of K . The integer d_j is the greatest common divisor of all $j \times j$ minors of K . For K of the form (2.5), the minor that is made of rows $2, \dots, n-1$ and columns $1, \dots, n-2$ is $(-1)^{n-2}$, so it follows that $d_{n-2} = 1$ and therefore also $d_1 = \dots = d_{n-2} = 1$. We conclude that

$$\mathcal{G}_\alpha \cong \mathcal{G}_\beta \cong \mathbb{Z}_{d_{n-1}} \oplus \mathbb{Z}_{d_n}.$$

2.4 Strings on a mapping torus

The system we studied in §2.2 has a dual description as the Hilbert space of ground states of strings of winding number $w = 1$ (around a 1-cycle to be defined below) on a certain type-IIA background. We will begin by describing the background geometry and then explain in §2.5 why its space of ground states is isomorphic to the space of ground states of the $\mathrm{SL}(2, \mathbb{Z})$ -twisted compactification of §2.2.

Set

$$W = \begin{pmatrix} \mathbf{k}_n & -1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} \mathbf{k}_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_1 & -1 \\ 1 & 0 \end{pmatrix} = T^{\mathbf{k}_n} S \dots T^{\mathbf{k}_2} S T^{\mathbf{k}_1} S \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \quad (2.11)$$

We will assume that $|\mathrm{tr} W| > 2$ so that W is a hyperbolic element of $\mathrm{SL}(2, \mathbb{Z})$. (The case of elliptic elements with $|\mathrm{tr} W| < 2$ was covered in [29], and parabolic elements with $|\mathrm{tr} W| = 2$ are conjugate to $\pm T^{\mathbf{k}}$ for some $\mathbf{k} \neq 0$, and since they do not involve S , they are elementary.)

Let $0 \leq \eta \leq 2\pi$ denote the coordinate on the interval $I = [0, 2\pi]$ and let (ξ_1, ξ_2) denote the coordinates of a point on T^2 . The coordinates ξ_1 and ξ_2 take values in \mathbb{R}/\mathbb{Z} (so they are periodic with period 1). We impose the identification

$$(\xi_1, \xi_2, \eta) \sim (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2, \eta + 2\pi). \quad (2.12)$$

The metric is

$$ds^2 = R^2 d\eta^2 + \left(\frac{4\pi^2 \rho^2}{\tau_2}\right) |d\xi_1 + \tau(\eta) d\xi_2|^2$$

where R and ρ are constants, and $\tau = \tau_1 + i\tau_2$ is a function of η (with real and imaginary parts denoted by τ_1 and τ_2) such that

$$\tau(\eta - 2\pi) = \frac{\mathbf{a}\tau(\eta) + \mathbf{b}}{\mathbf{c}\tau(\eta) + \mathbf{d}},$$

thus allowing for a continuous metric.

2.4.1 The number of fixed points

We will need the number of fixed points of the $SL(2, \mathbb{Z})$ action on T^2 , i.e., the number of solutions to:

$$(\xi_1, \xi_2) = (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2) \pmod{\mathbb{Z}^2}.$$

Let $f : T^2 \rightarrow T^2$ be the map given by

$$f : (\xi_1, \xi_2) \rightarrow (\mathbf{d}\xi_1 + \mathbf{b}\xi_2, \mathbf{c}\xi_1 + \mathbf{a}\xi_2). \tag{2.13}$$

The Lefschetz fixed-point formula states that

$$\sum_{\text{fixed point } p} i(p) = \sum_{j=0}^2 (-1)^j \text{tr}(f_* | H_j(T^2, \mathbb{Z})) = 2 - \text{tr } W = 2 - \mathbf{a} - \mathbf{d}.$$

The index $i(p)$ of a fixed point is given by [89]:

$$i(p) = \text{sgn det}(\mathcal{J}(p) - \mathbf{I}) = \text{sgn det}(W - \mathbf{I}),$$

where $\mathcal{J}(p)$ is the Jacobian matrix of the map f at p . In our case, $i(p)$ is either $+1$ or -1 for all p , and therefore the number of fixed points is

$$|2 - \text{tr } W| = |\text{det}(W - \mathbf{I})| = |2 - \mathbf{a} - \mathbf{d}|.$$

2.4.2 Isometries

Let $v_1, v_2 \in \mathbb{R}/\mathbb{Z}$ be constants and consider the map

$$(\xi_1, \xi_2, \eta) \mapsto (\xi_1 + v_1, \xi_2 + v_2, \eta). \tag{2.14}$$

It defines an isometry of \mathbf{M}_3 if

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \equiv \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \pmod{\mathbb{Z}}. \tag{2.15}$$

Set

$$H \equiv W^T - \mathbf{I} = \begin{pmatrix} \mathbf{a} - 1 & \mathbf{c} \\ \mathbf{b} & \mathbf{d} - 1 \end{pmatrix}, \quad \mathbf{v} \equiv \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}. \quad (2.16)$$

Then, the isometries are given by $\mathbf{v} = H^{-1} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$ for some $n_1, n_2 \in \mathbb{Z}$. The set of vectors \mathbf{v} that give rise to isometries therefore live on a lattice $\tilde{\Lambda}$ generated by the columns of H^{-1} . Since $H \in GL(2, \mathbb{Z})$ we have $\mathbb{Z}^2 \subseteq \tilde{\Lambda}$, and since the isometries that correspond to $\mathbf{v} \in \mathbb{Z}^2$ are trivial, the group of isometries of type (2.14) is isomorphic to $\tilde{\Lambda}/\mathbb{Z}^2$. Changing basis to $\mathbf{u} \equiv H\mathbf{v}$, we can replace $\mathbf{v} \in \tilde{\Lambda}/\mathbb{Z}^2$ with $\mathbf{u} \in \mathbb{Z}^2/\Lambda'$, where $\Lambda' \subseteq \mathbb{Z}^2$ is the sublattice generated by the columns of H , and the group \mathcal{G}_{iso} of isometries of type (2.14) is therefore

$$\mathcal{G}_{\text{iso}} \cong \tilde{\Lambda}/\mathbb{Z}^2 \cong \mathbb{Z}^2/\Lambda'. \quad (2.17)$$

Its order is

$$|\mathcal{G}_{\text{iso}}| = |\det H| = |2 - \mathbf{a} - \mathbf{d}|. \quad (2.18)$$

2.4.3 Homology quantum numbers

To proceed we also need the homology group $H_1(\mathbf{M}_3, \mathbb{Z})$. Let γ be the cycle defined by a straight line from $(0, 0, 0)$ to $(0, 0, 2\pi)$, in terms of (ξ_1, ξ_2, η) coordinates. Let α' be the cycle from $(0, 0, 0)$ to $(1, 0, 0)$ and let β' be the cycle from $(0, 0, 0)$ to $(0, 1, 0)$. The homology group $H_1(\mathbf{M}_3, \mathbb{Z})$ is generated by the equivalence classes $[\alpha']$, $[\beta']$ and $[\gamma]$, subject to the relations

$$[\alpha'] = \mathbf{d}[\alpha'] + \mathbf{c}[\beta'], \quad [\beta'] = \mathbf{b}[\alpha'] + \mathbf{a}[\beta']. \quad (2.19)$$

Now suppose that $(c_1 \ c_2)$ is a linear combination of the columns of H [defined in (2.16)] with integer coefficients. Then the relations (2.19) imply that $c_1[\alpha'] + c_2[\beta']$ is zero in $H_1(\mathbf{M}_3, \mathbb{Z})$. With $\Lambda' \subset \mathbb{Z}^2$ being the sublattice generated by the columns of H , as defined in §2.4.2, it follows that

$$H_1(\mathbf{M}_3, \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}^2/\Lambda'), \quad (2.20)$$

where the \mathbb{Z} factor is generated by $[\gamma]$ and (\mathbb{Z}^2/Λ') is generated by $[\alpha']$ and $[\beta']$. In particular, the torsion part is

$$\text{Tor } H_1(\mathbf{M}_3, \mathbb{Z}) \cong \mathbb{Z}^2/\Lambda'. \quad (2.21)$$

Denote the Smith normal form [see (2.10)] of the matrix H by $\begin{pmatrix} d'_1 & \\ & d'_2 \end{pmatrix}$. We prove in Appendix B.1 that $d_{n-1} = d'_1$ and $d_n = d'_2$, where d_{n-1} and d_n were defined in (2.10). Thus, combining (2.17) and (2.20) we have

$$\mathbb{Z}^2/\Lambda' \cong \mathcal{G}_{\text{iso}} \cong \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z}) \cong \mathbb{Z}_{d_{n-1}} \oplus \mathbb{Z}_{d_n}.$$

The physical meaning of these results will become clear soon.

2.4.4 The Hilbert space of states

As we have seen in §2.4.2, the Hilbert space of string ground states has a basis of states of the form $|\mathbf{v}'\rangle$ with $\mathbf{v}' \in \tilde{\Lambda}/\mathbb{Z}^2$. In this state, the string is at (ξ_1, ξ_2) coordinates given by \mathbf{v}' . According to (2.17), an element $\mathbf{v} \in \tilde{\Lambda}/\mathbb{Z}^2$ defines an isometry, which we denote by $\mathcal{Y}(\mathbf{v})$, that acts as

$$\mathcal{Y}(\mathbf{v})|\mathbf{v}'\rangle = |\mathbf{v} + \mathbf{v}'\rangle, \quad \mathbf{v}, \mathbf{v}' \in \tilde{\Lambda}/\mathbb{Z}^2.$$

Given the string state $|\mathbf{v}'\rangle$, we can ask what is the element in $H_1(\mathbf{M}_3, \mathbb{Z})$ that represents the corresponding 1-cycle. The answer is $[\gamma] + N'_1[\alpha'] + N'_2[\beta']$, where the torsion part $N'_1[\alpha'] + N'_2[\beta']$ is mapped under (2.21) to \mathbf{v}' . To see this, note that for $0 \leq t \leq 1$, the loops C_t in \mathbf{M}_3 that are given by

$$\begin{cases} (4\pi s, tv'_1, tv'_2) & \text{for } 0 \leq s \leq \frac{1}{2} \\ (2\pi, tv'_1 + (2s-1)t[(\mathbf{d}-1)v'_1 + \mathbf{b}v'_2], tv'_2 + (2s-1)t[\mathbf{c}v'_1 + (\mathbf{a}-1)v'_2]) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

[which go along direction η at a constant (ξ_1, ξ_2) given by $t\mathbf{v}'$, and then connect $t\mathbf{v}'$ to its $SL(2, \mathbb{Z})$ image $tW\mathbf{v}'$] are homotopic to the loop corresponding to string state $|0\rangle$. Setting $t = 1$ we find that C_1 breaks into two closed loops, one corresponding to string state $|\mathbf{v}'\rangle$, and the other is a closed loop in the T^2 fiber above $\eta = 0$, which corresponds to the homology element

$$((\mathbf{d}-1)v'_1 + \mathbf{b}v'_2)[\alpha'] + (\mathbf{c}v'_1 + (\mathbf{a}-1)v'_2)[\beta'],$$

and this is precisely the element corresponding to $H\mathbf{v}' \in \mathbb{Z}^2/\Lambda' \cong \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$, as defined in §2.4.3.

We now wish to use the torsion part of the homology to define a unitary operator $\mathcal{R}(\tilde{\mathbf{u}})$ for every $\tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'$. This operator will measure a component of the charge associated with the homology class of the string. For this purpose we need to construct the Pontryagin dual group ${}^\vee \text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$, which is defined as the group of characters of $\text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ (i.e., homomorphisms from $\text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ to \mathbb{R}/\mathbb{Z}). The dual group is isomorphic to \mathbb{Z}^2/Λ' , but not canonically. In our construction $\tilde{\mathbf{u}}$ is naturally an element of the dual group and not the group itself. We define $\mathcal{R}(\tilde{\mathbf{u}})$ as follows. For

$$\tilde{\mathbf{u}} = (M'_1, M'_2) \in \mathbb{Z}^2/\Lambda', \quad \mathbf{v} = (N'_1, N'_2) \in \mathbb{Z}^2/\Lambda',$$

we define the phase

$$\varphi(\tilde{\mathbf{u}}, \mathbf{v}) \equiv e^{2\pi i (H^{-1})_{ij} N'_i M'_j}, \quad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda', \quad \mathbf{v} \in \mathbb{Z}^2/\Lambda'. \quad (2.22)$$

This definition is independent of the representatives (N'_1, N'_2) and (M'_1, M'_2) of \mathbf{v} and $\tilde{\mathbf{u}}$, and it corresponds to the character of $\text{Tor } H_1(\mathbf{M}_3, \mathbb{Z})$ associated with $\tilde{\mathbf{u}}$. We then define the operator $\mathcal{R}(\tilde{\mathbf{u}})$ to be diagonal in the basis $|\mathbf{v}\rangle$ and act as:

$$\mathcal{R}(\tilde{\mathbf{u}})|\mathbf{v}\rangle = \varphi(\tilde{\mathbf{u}}, \mathbf{v})|\mathbf{v}\rangle, \quad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda', \quad \mathbf{v} \in \mathbb{Z}^2/\Lambda'.$$

From the discussion above about the homology of the string state, and from the linearity of the phase of $\varphi(\tilde{\mathbf{u}}, \mathbf{v})$ in $\tilde{\mathbf{u}}$ and \mathbf{v} , it follows that

$$\mathcal{R}(\tilde{\mathbf{u}})\mathcal{Y}(\mathbf{v}) = \varphi(\tilde{\mathbf{u}}, \mathbf{v})\mathcal{Y}(\mathbf{v})\mathcal{R}(\tilde{\mathbf{u}}). \quad (2.23)$$

2.5 Duality between strings on \mathbf{M}_3 and compactified $SL(2, \mathbb{Z})$ -twisted $U(1)$ gauge theory

We can now connect the string theory model of §2.4 with the field theory model of §2.3. We claim that the Hilbert space of ground states of a compactification of a $U(1)$ gauge theory on S^1 with an $SL(2, \mathbb{Z})$ twist and string ground states on \mathbf{M}_3 are dual. This is demonstrated along the same lines as in [29]. We realize the (supersymmetric extension of the) $U(1)$ gauge theory on a D3-brane along directions x_1, x_2, x_3 . We compactify direction x_3 on a circle with a Janus-like configuration and $SL(2, \mathbb{Z})$ -twisted boundary conditions. We assume that the Janus configuration can be lifted to type-IIB, perhaps with additional fluxes, but we will not worry about the details of the lift. We then compactify (x_1, x_2) on T^2 and perform T-duality on direction 1, followed by a lift from type-IIA to M-theory (producing a new circle along direction 10), followed by reduction to type-IIA along direction 2. This combined U-duality transformation transforms the $SL(2, \mathbb{Z})$ -twist to the geometrical transformation (2.12). It also transforms some of the charges of the type-IIB system to the following charges of the type-IIA system:

$$D3_{123} \rightarrow F1_3, \quad F1_1 \rightarrow P_1, \quad F1_2 \rightarrow F1_{10}, \quad D1_1 \rightarrow F1_1, \quad D1_2 \rightarrow P_{10}. \quad (2.24)$$

where P_j is Kaluza-Klein momentum along direction j , $Dp_{j_1 \dots j_r}$ is a Dp -brane wrapped along directions j_1, \dots, j_r , and $F1_j$ is a fundamental string along direction j . A summary of the U-duality transformation is provided in the table below.

Brane	1	2	y	\dots	10	Next step
D3	✓	✓	✓		N/A	T_1 -duality
D2		✓	✓		N/A	lift to M-theory
M2		✓	✓			red. to IIA on x_2
F1		N/A	✓			
	x_1		y		x_2	\Leftarrow New coords.

Table 2.1: U-Duality transformation summary.

Now suppose we take the limit that all directions of \mathbf{M}_3 are large. The dual geometry has a Hilbert space of ground states which corresponds to classical configurations of strings of minimal length that wind once around the x_3 circle. This means that the projection of their $H_1(\mathbf{M}_3, \mathbb{Z})$ homology class on the \mathbb{Z} factor of (2.20) is required to be the generator $[\gamma]$. The torsion part of their homology is unrestricted. The string configurations of minimal length must have constant (x_1, x_2) which in particular means that (x_1, x_2) is invariant under the $SL(2, \mathbb{Z})$ twist, i.e.,

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \equiv \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \pmod{\mathbb{Z}}.$$

But this is precisely the same equation as (2.15), and indeed when the isometry that corresponds to a vector $\mathbf{v} \in \tilde{\Lambda}/\mathbb{Z}^2$ acts on the solution with $(x_1, x_2) = (0, 0)$ it converts it to the solution with $(x_1, x_2) = (v_1, v_2)$. The dimension of the Hilbert space of ground states of the type-IIA system is therefore the order of \mathcal{G}_{iso} , which is given by (2.18). This is also the number of fixed points of the W action on T^2 , as we have seen in §2.4.1. Since the number of ground states of the Chern-Simons theory is $|\det K|$, we conclude from the duality of the Chern-Simons theory and string theory that

$$|\det K| = |\mathcal{G}_{\text{iso}}| = |2 - \mathbf{a} - \mathbf{b}|.$$

This is the physical explanation we are giving to (2.2).

2.5.1 Isomorphism of operator algebras

Going one step beyond the equality of dimensions of the Hilbert spaces, we would like to match the operator algebras of the string and field theory systems. Starting with the field theory side, realized on a D3-brane in type-IIB, consider a process whereby a fundamental string that winds once around the β -cycle of T^2 is absorbed by the D3-brane at some time t . This process is described in the field theory by inserting a Wilson loop operator V_1 at time t into the matrix element that calculates the amplitude. On the type-IIA string side, the charge $F1_2$ that was absorbed is mapped by (2.24) to winding number along the α' cycle (denoted by $F1_{10}$). The operator that corresponds to V_1 on the string side must therefore increase the homology class of the string state by $[\alpha']$. Since the state $|\mathbf{v}\rangle$, for $\mathbf{v} = (N'_1, N'_2)$, has homology class $[\gamma] + N'_1[\alpha'] + N'_2[\beta']$, it follows that the isometry operator $\mathcal{Y}(\mathbf{v}')$ with $\mathbf{v}' = (1, 0)$ does what we want. We therefore propose to identify

$$V_1 \rightarrow \mathcal{Y}(\mathbf{v}'), \quad \text{for } \mathbf{v}' = (1, 0).$$

By extension, we propose to map the abelian subgroup \mathcal{G}_β generated by the Wilson loops V_1, \dots, V_n with the isometry group generated by $\mathcal{Y}(\mathbf{v}')$ for $\mathbf{v}' \in \mathbb{Z}^2/\Lambda'$.

Next, on the type-IIB side, consider a process whereby a fundamental string that winds once around the α -cycle of T^2 is absorbed by the D3-brane. This process is described in the field theory by inserting a Wilson loop operator U_1 into the matrix element that calculates the amplitude. On the type-IIA string side, the charge $F1_1$ that was absorbed is mapped by (2.24) to momentum along the β' cycle (denoted by $P1_1$). The operator that corresponds to U_1 on the string side must therefore increase the momentum along the $[\alpha']$ cycle by one unit. We claim that this operator is $\mathcal{R}(\tilde{\mathbf{u}})$ for $\tilde{\mathbf{u}} = (1, 0)$. To see this we note that, by definition of “momentum”, an operator \mathcal{X} that increases the momentum by M'_1 units along the $[\alpha']$ cycle and M'_2 units along the $[\beta']$ cycle must have the following commutation relations with the translational isometries $\mathcal{Y}(\mathbf{v}')$:

$$\mathcal{Y}(\mathbf{v}')^{-1} \mathcal{X} \mathcal{Y}(\mathbf{v}') = \varphi(\tilde{\mathbf{u}}, \mathbf{v}') \mathcal{X}, \quad \tilde{\mathbf{u}} = (M'_1, M'_2) \in \mathbb{Z}^2/\Lambda'.$$

But given (2.23), this means that up to an unimportant central element, we can identify $\mathcal{X} = \mathcal{R}(\tilde{\mathbf{u}})$, as claimed. So, we have

$$U_1 \rightarrow \mathcal{Y}(\tilde{\mathbf{u}}), \quad \text{for } \tilde{\mathbf{u}} = (1, 0),$$

and by extension, we propose to map the abelian subgroup \mathcal{G}_α generated by the Wilson loops U_1, \dots, U_n with the subgroup generated by $\mathcal{R}(\tilde{\mathbf{u}})$ for $\tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'$.

In particular, $\mathcal{G}_\alpha \cong \mathcal{G}_\beta \cong \mathbb{Z}^n/\Lambda$ implies that $(\mathbb{Z}^2/\Lambda') \cong (\mathbb{Z}^n/\Lambda)$. This is equivalent to requiring that the Smith normal form of H is

$$P'HQ' = \text{diag}(d_{n-1}, d_n)$$

where d_{n-1} and d_n are the same last two entries in the Smith normal form of K . We provide an elementary proof of this fact in Appendix B.1.

Since the Smith normal forms of H and K are equal, the abelian groups \mathbb{Z}^n/Λ and \mathbb{Z}^2/Λ' are equivalent, and it is also not hard to see that under this equivalence χ that was defined in (2.8) is mapped to φ defined in (2.22). We have the mapping

$$\mathcal{O}_\alpha(\mathbf{v}) \rightarrow \mathcal{Y}(\mathbf{v}'), \quad \mathbf{v} \in \mathbb{Z}^n/\Lambda, \quad \mathbf{v}' \in \mathbb{Z}^2/\Lambda'$$

and

$$\mathcal{O}_\beta(\mathbf{u}) \rightarrow \mathcal{R}(\tilde{\mathbf{u}}), \quad \mathbf{u} \in \mathbb{Z}^n/\Lambda, \quad \tilde{\mathbf{u}} \in \mathbb{Z}^2/\Lambda'.$$

The commutation relations (2.9) are then mapped to (2.23).

2.6 Discussion

We have argued that a duality between $U(1)^n$ Chern-Simons theory on T^2 with coupling constant matrix (2.5) and string configurations on a mapping torus provide a geometrical realization to the algebra of Wilson loop operators in the Chern-Simons theory. Wilson loop operators along one cycle of T^2 correspond to isometries that act as translations along the fiber of the mapping torus, while Wilson loop operators along the other cycle correspond to discrete charges that can be constructed from the homology class of the string.

These ideas have an obvious extension to the case of $U(N)$ gauge group with $N > 1$, where $SL(2, \mathbb{Z})$ -duality is poorly understood. The techniques presented in this chapter can be extended to construct the algebra of Wilson loop operators. The Hilbert space on the string theory side is constructed from string configurations on a mapping torus whose $H_1(\mathbf{M}_3, \mathbb{Z})$ class maps to N under the projection map $\mathbf{M}_3 \rightarrow S^1$. In other words, the homology class projects to $N[\gamma]$ when the torsion part is ignored. Such configurations could be either a single-particle string state wound N times, or a multi-particle string state. A string state with r strings with winding numbers N_1, \dots, N_r is described by a partition $N = N_1 + \dots + N_r$, and the j^{th} single-particle string state is described by an unordered set of N_j points on T^2 that is invariant, as a set, under the action of f in (2.13). The counterparts of the Wilson

loops on the string theory side can then be constructed from operations on these sets. A more complete account of the nonabelian case will be reported elsewhere [71].

It is interesting to note that some similar ingredients to the ones that appear in this work also appeared in [13] in the study of vacua of compactifications of the free $(2, 0)$ theory on Lens spaces. More specifically, a Chern-Simons theory with a tridiagonal coupling constant matrix and the torsion part of the first homology group played a role there as well. It would be interesting to further explore the connection between these two problems.

Chapter 3

Quadratic Reciprocity, Janus Configurations, and String Duality Twists

Quadratic reciprocity is a classic result in Number Theory that relates the question “does the equation $x^2 \equiv q$ modulo p , for given odd prime numbers p and q , have an integer solution x ?” to a similar question with the roles of p and q interchanged. The number of solutions x is encoded in the *quadratic-residue*, which is related to a *quadratic Gauss sum*. In this chapter, quadratic reciprocity is shown to be a direct consequence of T-duality of type-II string theory. This is demonstrated by recasting the quadratic Gauss sum as the partition function of abelian $\mathcal{N} = 4$ Super-Yang-Mills theory on a certain supersymmetry-preserving four-manifold with variable coupling constant and θ -angle. The manifold is a (three-dimensional) mapping-torus times a circle, with an Olive-Montonen $SL(2, \mathbb{Z})$ duality twist along the circle (creating a discontinuity in the Yang-Mills coupling constant and θ -angle). The recently discovered supersymmetric Janus configuration plays a crucial role in the construction. The geometry of the mapping-torus depends on p and the $SL(2, \mathbb{Z})$ duality twist depends on q . String theory dualities act by exchanging p and q , leading to a relation between quadratic Gauss-like sums known as the Landsberg-Schaar relation, from which quadratic reciprocity follows.

3.1 Quadratic reciprocity and the Landsberg-Schaar relation

This section is a review of basic facts from Number Theory, and more details can be found in the references [72, 73]. The identity [73, 74, 75]

$$\frac{e^{\pi i/4}}{\sqrt{2p}} \sum_{n=0}^{2p-1} e^{-\pi i n^2 q/2p} = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} e^{2\pi i n^2 p/q}, \quad (3.1)$$

where p and q are positive integers, is known as the *Landsberg-Schaar relation*. It can be proved using the modular transformation properties of the Jacobi theta-function $\theta(0; \tau) = \sum_{n=-\infty}^{\infty} \exp(2\pi i \tau n^2)$ and its asymptotic behavior as the argument τ approaches the real axis. However, (3.1) is an identity of finite sums, and a proof (for general p and q) that doesn't involve taking a limit is not known at the moment. We are interested in constructing a physical system with a finite number of quantum states that reproduces (3.1) directly.

We note that the identity (3.1) is related to an elementary duality in Number Theory known as *Quadratic Reciprocity*. If p is an odd prime number and a is an integer, then a is called a *quadratic residue mod p* if $x^2 \equiv a \pmod{p}$ has integer solutions x . The *Legendre symbol* $\lambda(a, p)$ is defined to have the value 1 if a is a quadratic residue mod p , to have the value 0 if p divides a , and to have the value -1 otherwise. [In the math literature, it is often denoted by the rather confusing symbol (a/p) .] The *Law of Quadratic Reciprocity* states that if p and q are odd primes then

$$\lambda(p, q) = (-1)^{((p-1)/2)((q-1)/2)} \lambda(q, p). \quad (3.2)$$

It is a nontrivial statement that relates the existence or nonexistence of solutions to a quadratic equation modulo p to the existence or nonexistence of solutions to a completely different quadratic equation modulo q .

A *Quadratic Gauss Sum* is, by definition, the discrete Fourier transform of the Legendre symbol:

$$\chi_p(a) = \sum_{b=0}^{p-1} e^{2\pi i ab/p} \lambda(b, p) = \sum_{n=0}^{p-1} e^{2\pi i an^2/p}, \quad (3.3)$$

where the last equality follows from the identity $\sum_{b=0}^{p-1} e^{2\pi i ab/p} = 0$ for $a \not\equiv 0 \pmod{p}$. It is not hard to prove that $\chi_p(a) = \lambda(a, p) \chi_p(1)$, so the quadratic Gauss sum is proportional to the quadratic residue. It can also be shown that $\chi_p(1) = \sqrt{p}$ if $p \equiv 1 \pmod{4}$ and $\chi_p(1) = i\sqrt{p}$ if $p \equiv 3 \pmod{4}$ (see [72]). Quadratic reciprocity is then a statement about the relation between quadratic Gauss sums. For example, if both p and q are $1 \pmod{4}$ then $\chi_p(q)/\sqrt{p} = \chi_q(p)/\sqrt{q}$. Variants of quadratic Gauss sums are provided in Appendix C.1

For prime and odd p and q , the Landsberg-Schaar identity is a slightly modified version of quadratic reciprocity. To see this, define

$$\varrho_p(a) = \sum_{n=0}^{2p-1} e^{\pi i an^2/2p}, \quad (3.4)$$

which is periodic in a with period $2p$. It is elementary to check (by splitting the sum over n into odd and even numbers) that

$$\varrho_p(a) = (1 + i^{pa}) \chi_p(a). \quad (3.5)$$

It follows by simple algebra that if $p \neq q$ are odd primes then

$$\frac{1}{\sqrt{2p}} \varrho_p(q) = \frac{1}{\sqrt{q}} e^{\frac{1}{4}(2q-1)i\pi} \chi_q(p). \quad (3.6)$$

Taking the complex conjugate and noting the elementary result that $\lambda(-1, q) = (-1)^{(q-1)/2}$, the Landsberg-Schaar identity (3.1) follows in the form

$$\frac{1}{\sqrt{2p}} \varrho_p(-q) = \frac{1}{\sqrt{q}} e^{-\frac{1}{4}i\pi} \chi_q(p). \quad (3.7)$$

We will now construct a (p, q) -dependent type-II string theory setting whose partition function can be calculated in two T-dual ways. One gives an expression proportional to $\chi_q(p)$ while the other gives an expression proportional to $\varrho_p(-q)$. While the argument can be made directly in type-II string theory, it is more instructive to proceed through an intermediate step that is $U(1)$ Chern-Simons theory.

3.2 Chern-Simons partition function

Consider Chern-Simons theory at level \mathbf{k} (which we assume is a positive integer) with $U(1)$ gauge group, formulated on $T^2 \times \mathbb{R}$ where \mathbb{R} is the (Euclidean) time direction and T^2 is a torus parameterized by periodic coordinates $0 \leq x_1, x_2 < 1$. The action is $I = \frac{\mathbf{k}}{2\pi} \int A \wedge dA$, where A is the gauge field. It is well known [77] that the Hilbert space of the theory has exactly \mathbf{k} states, which we will denote by $|a\rangle$ (with $a = 0, \dots, \mathbf{k} - 1$). Let α and β be two fundamental 1-cycles of T^2 , where α corresponds to a loop at constant x_2 , with x_1 varying from 0 to 1, and β corresponds to a similar loop at constant x_1 with x_2 varying from 0 to 1. Consider the Wilson loop operators $W_1 \equiv \exp(i \oint_{\alpha} A)$ and $W_2 \equiv \exp(i \oint_{\beta} A)$. Their action on the states is given by the clock and shift matrices:

$$W_1|a\rangle = e^{2\pi ia/\mathbf{k}}|a\rangle, \quad W_2|a\rangle = |a+1\rangle. \quad (3.8)$$

We now assume that \mathbf{k} is an even integer and set $\mathbf{k} = 2p$. We will need the action of large coordinate transformations on the Hilbert space. This action only depends on the topological nature of the transformation, i.e., on the representative of the coordinate transformation in the mapping class group $SL(2, \mathbb{Z})$. The action of a general transformation can be calculated from the action of the two generators $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. They act on the Hilbert space as

$$\hat{S}|a\rangle = \frac{1}{\sqrt{\mathbf{k}}} \sum_{b=0}^{\mathbf{k}-1} e^{2\pi i ab/\mathbf{k}} |b\rangle, \quad \hat{T}|a\rangle = e^{-i\pi/12} e^{i\pi a^2/\mathbf{k}} |a\rangle. \quad (3.9)$$

Up to the phase, this can be checked by making sure that the commutation relations $\hat{S}^{-1}W_i\hat{S}$ and $\hat{T}^{-1}W_i\hat{T}$ are as they should be (for $i = 1, 2$), given the geometrical interpretation of \hat{T} and \hat{S} . The phase $e^{-i\pi/12}$ is restricted by requiring $(\hat{S}\hat{T})^3 = \hat{S}^2$. It can be derived more systematically by writing explicit wavefunctions (as a function of holonomies of the gauge fields) or by recalling the connection between $U(1)$ Chern-Simons theory and the 2D CFT

of a free chiral boson. The states can be associated with characters of primary states and \hat{S} and \hat{T} act by modular transformations [78]. The factor $e^{-i\pi/12}$ is related to the shift in energy by $-c/24$ where $c = 1$ is the central charge. Note also that the equation for \hat{T} is ill-defined for odd \mathbf{k} . In that case only even powers of \hat{T} are well-defined. The quadratic sum appearing on the LHS of the Landsberg-Schaar relation (3.1) can now be written as

$$\begin{aligned} \varrho_p(-q)/\sqrt{2p} &= e^{-\pi i(q+2)/12} \sum_{n=0}^{2p-1} \langle n | \hat{S} \hat{T}^{-q-2} | n \rangle \\ &= e^{-\pi i(q+2)/12} \text{tr}(\hat{S} \hat{T}^{-q-2}). \end{aligned} \quad (3.10)$$

The trace $\text{tr}(\hat{S} \hat{T}^{-q-2})$, in turn, can be recast as a partition function of Chern-Simons theory on a 3-manifold obtained by including time (parameterized by x_0) and identifying $x_0 = 0$ with $x_0 = 1$ up to a diffeomorphism of T^2 that corresponds to the $SL(2, \mathbb{Z})$ transformation $\hat{S} \hat{T}^{-q-2}$. In other words, the 3-manifold is given by $T^2 \times [0, 1]$ with the identification

$$(x_0, x_1, x_2) \sim (x_0 + 1, x_2, (2 + q)x_2 - x_1).$$

This is a special case of what is known as a *mapping torus* – a manifold formed by fibering a T^2 (directions x_1, x_2) over S^1 (direction x_0).

3.3 Type-II string on a mapping torus

The next step is to realize the Hilbert space of $U(1)$ Chern-Simons on T^2 at level \mathbf{k} as the Hilbert space of ground states of a type-IIA string configuration on a different mapping torus, as was done in [77]. The series of steps that take us from Chern-Simons to type-IIA string theory will be briefly recalled for completeness below, but let us start by describing the resulting type-IIA background. We take as coordinates (y_0, \dots, y_9) and take (y_1, y_2) to describe a T^2 , where y_1 and y_2 are periodic with period 1. We let the shape (complex structure) of the torus vary as a function of y_3 so that the metric at $y_4 = y_5 = \dots = y_9 = 0$ is

$$ds^2 = \frac{A}{\text{Im } \tau(y_3)} |dy_1 + \tau(y_3) dy_2|^2 + R^2 dy_3^2,$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure parameter, A is the area of T^2 that (for the time being) is kept constant, and R is a constant as well. Directions y_4, \dots, y_9 are irrelevant to our story (except for their role in preserving supersymmetry), and so we will ignore them. We then connect the T^2 at $y_3 = 0$ with the T^2 at $y_3 = 1$ via a linear $SL(2, \mathbb{Z})$ transformation as follows. Let $\mathbf{M} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ be a fixed matrix in $SL(2, \mathbb{Z})$. We require that $\tau(1) = \frac{\mathbf{a}\tau(0) + \mathbf{b}}{\mathbf{c}\tau(0) + \mathbf{d}}$ and impose the extra identification

$$(y_1, y_2, y_3) \sim (\mathbf{d}y_1 + \mathbf{b}y_2, \mathbf{c}y_1 + \mathbf{a}y_2, y_3 + 1). \quad (3.11)$$

We add a fundamental string to this background and require it to wind once around the direction of y_3 . Ground states of this string will be configurations of minimal length. Such configurations have constant (y_1, y_2) for which both $(\mathbf{d} - 1)y_1 + \mathbf{b}y_2$ and $\mathbf{c}y_1 + (\mathbf{a} - 1)y_2$ are integers. It is not hard to check that the number of ground states is $|\mathbf{a} + \mathbf{d} - 2|$. (See [77] for details.) We will now restrict to the case $\mathbf{M} = \mathbf{T}^{\mathbf{k}-2}\mathbf{S}$. We can then identify the ground states with the states of Chern-Simons theory of the previous section. Note that the operator W_1 of (3.8) increases the momentum in the y_1 direction by one unit and decreases the momentum in the y_2 direction by one unit, but due to the identification (3.11), momentum is only defined modulo \mathbf{k} . On the other hand, the operator W_2 increases the string winding number in direction y_1 by one unit. To see this, note that the loops associated with the string states $|a\rangle$ are topologically different for different a 's. The first Homology group of the target space is $\mathbb{Z} \oplus \mathbb{Z}_{\mathbf{k}}$ and the label a corresponds to the element in $\mathbb{Z}_{\mathbf{k}}$ that describes the homology class of the string. It is not hard to check that attaching a loop of a string wound around the y_1 direction to the state $|a\rangle$ results in a loop that is equivalent in homology to the state $|a + 1\rangle$. In other words, the string states $|a + 1\rangle$ and $|a\rangle$ differ by one unit of winding number along y_1 . Similarly, it is not hard to check that W_2 decreases the winding number in direction y_2 by one unit. (See [77] for more details.) These winding numbers are again defined up to the identifications implied by (3.11).

Next, we ask what is the interpretation in the string theory setting of the operators \hat{T} and \hat{S} defined in (3.9). To describe them, we first need to define the complex Kähler modulus $\rho = \frac{1}{2\pi}B + i\alpha'A$, where B is the integral of the NS-NS 2-form field on T^2 and α' is the string tension. In Chern-Simons theory \hat{S} and \hat{T} act as diffeomorphisms. For example, $\hat{S}^{-1}W_1\hat{S} = W_2$. In the string-on-mapping-torus setting, the operator that converts W_1 to W_2 must exchange momentum with winding number. It therefore must be identified with T -duality on the torus, which acts as $\rho \rightarrow -1/\rho$. Similarly, \hat{T} acts as $\rho \rightarrow \rho + 1$. Set $\hat{\mathbf{R}} = \hat{S}\hat{T}^{-q-2}$. Then, $\hat{\mathbf{R}}$ corresponds to an $SL(2, \mathbb{Z})$ transformation $\mathbf{R} = \begin{pmatrix} \mathbf{a}' & \mathbf{b}' \\ \mathbf{c}' & \mathbf{d}' \end{pmatrix}$ that preserves τ and acts on ρ as $\rho \rightarrow \frac{\mathbf{a}'\rho + \mathbf{b}'}{\mathbf{c}'\rho + \mathbf{d}'}$. For $\mathbf{k} = 2p$, the operator $\hat{\mathbf{R}}$ acts on the ground states according to (3.9) as

$$\hat{\mathbf{R}}|a\rangle = \frac{1}{\sqrt{2p}} e^{-\pi i(q+2)/12} \sum_{b=0}^{2p-1} e^{\pi i[2ab + (-q-2)b^2]/2p} |b\rangle.$$

The formal argument for why \hat{S} is to be identified with T-duality on T^2 can be made more precise by following the chain of dualities described in [77] that leads from Chern-Simons theory to the string-on-mapping-torus setting. The first step is to realize Chern-Simons theory as the low-energy limit of a compactification of $N = 4$ supersymmetric $U(1)$ Yang-Mills theory on S^1 with boundary conditions that include an S-duality twist. At this point τ is identified with the complex Yang-Mills coupling constant $\frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$, which is allowed to vary along direction x_3 in a supersymmetric way known as a *Janus configuration* [5, 79, 80]. The coupling constants at $x_3 = 0$ and $x_3 = 1$ are not equal, and to close the configuration

smoothly we have to insert an unconventional boundary condition that connects a Yang-Mills configuration at $x_3 = 1$ with an Olive-Montonen S-dual configuration at $x_3 = 0$, with the particular duality transformation taken to be $\mathbf{M} \in SL(2, \mathbb{Z})$. An R-symmetry twist is also usually necessary to preserve supersymmetry. The proof that the low energy limit of this configuration is indeed Chern-Simons theory at level $\mathbf{k} = 2p$, as well as other details, can be found in [77]. Next, we realize the Super-Yang-Mills theory as the low-energy limit of a D3-brane in type-IIB, and we compactify directions x_1, x_2 on T^2 . We proceed by performing T-duality on direction x_1 to get a D2-brane in type-IIA, and we refer to the T-dual direction 1 as “ y_1 ”. We follow by a lift to M-theory to get an M2-brane wrapping the M-theory direction, which we refer to as “ y_2 ”, and we finish with a reduction to type-IIA by eliminating direction x_2 . The result is type-IIA with the background geometry of the mapping torus, and a fundamental string wrapping direction $x_3 = y_3$. Following the known action of these dualities, which combined amount to a U-duality transformation, it is not hard to see that \mathbf{M} is realized as T-duality, as stated.

3.4 Quadratic Reciprocity is T-duality

We can now see how to realize the quadratic Gauss sum (3.10) in terms of the string-on-mapping-torus setting. Start with a compactification of type-IIA on T^2 (directions y_1 and y_2) and a string with worldsheet in directions (y_0, y_3) . It has an $SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\rho$ duality group, with $SL(2, \mathbb{Z})_\tau$ acting on the complex structure modulus τ , and $SL(2, \mathbb{Z})_\rho$ acting on the Kähler modulus ρ . If we now compactify direction y_3 on S^1 with an $\mathbf{M} \in SL(2, \mathbb{Z})_\tau$ twist and direction y_0 on another S^1 with an $\hat{\mathbf{R}} \in SL(2, \mathbb{Z})_\rho$ twist, the partition function of the string is, up to a phase, going to be $\varrho_p(-q)/\sqrt{2p}$, according to (3.10). (The phase is important and will be discussed separately below.) But type-IIA string theory has another T-duality symmetry that exchanges $SL(2, \mathbb{Z})_\rho \leftrightarrow SL(2, \mathbb{Z})_\tau$. In this T-dual perspective the role of \mathbf{M} and $\hat{\mathbf{R}}$ is interchanged. We now have type-IIB strings on a mapping torus defined by $\mathbf{R} \in SL(2, \mathbb{Z})$ which has q ground states. Following the same kind of arguments that led to (3.10), we see that with the role of \mathbf{M} and \mathbf{R} reversed, the partition function will be, up to a phase, given by $\chi_q(p)/\sqrt{q}$. We therefore see that, if we can explain the phase, the Landsberg-Schaar relation is a direct consequence of $\rho \leftrightarrow \tau$ duality.

3.5 The phase

We have seen that T-duality $\rho \leftrightarrow \tau$ is the natural framework for understanding the Landsberg-Schaar equality (3.1), which in essence expresses the phenomenon of quadratic reciprocity. But to complete the argument it is necessary to understand the emergence of the phase $e^{\pi i/4}$ on the LHS of (3.1). For this purpose we have to describe in more detail the supersymmetric construction. We will see that the phase emerges as a difference in Berry phases when

either ρ or τ are allowed to vary as a function of time (a role played by either y_0 or y_3) in a Janus-like configuration.

We begin by reviewing in more detail the supersymmetric Janus configuration. As shown in [5], to preserve half the supersymmetry the function $\tau(y_3)$ has to trace a semicircle centered at the origin. For $\tau(0)$ and $\tau(1)$ to be connected via \mathbf{M} , we find [77]:

$$\tau = \left(\frac{\mathbf{a} - \mathbf{d}}{2\mathbf{c}} \right) + \left(\frac{\sqrt{(\mathbf{a} + \mathbf{d})^2 - 4}}{2|\mathbf{c}|} \right) e^{2i\psi},$$

where $\psi(y_3)$ is an arbitrary real function. One can also check that if $\tau(0) = (\mathbf{a}\tau(1) + \mathbf{b})/(\mathbf{c}\tau(1) + \mathbf{d})$ then

$$\frac{|\mathbf{c}\tau(1) + \mathbf{d}|}{\mathbf{c}\tau(1) + \mathbf{d}} = e^{i(\psi(0) - \psi(1))}.$$

If τ is understood as the complex structure of T^2 , then the j^{th} ($j = 0, \dots, q-1$) wavefunction of level- q Chern-Simons theory can be expressed as a wavefunction of the holonomies $0 \leq \xi_1, \xi_2 < 2\pi$ as

$$\Psi_{j,q} = \frac{(2q\tau_2)^{\frac{1}{4}}}{2\pi} e^{\frac{iq\xi_1\xi_2}{4\pi}} \sum_{n=-\infty}^{\infty} e^{i(qn+j)\xi_1 + \pi iq\tau(\frac{\xi_2}{2\pi} + n + \frac{j}{q})^2} \quad (3.12)$$

This is, in fact, a wavefunction of the lowest Landau level of a nonrelativistic charged particle on a torus labeled by ξ_1, ξ_2 with q units of magnetic flux and gauge field $q(\xi_1 d\xi_2 - \xi_2 d\xi_1)/2\pi$. Now let τ vary adiabatically. The corresponding Berry connection is

$$\mathcal{A}_\tau = i\langle \Psi_{j,q} | \partial_\tau | \Psi_{l,q} \rangle = -\frac{1}{8\tau_2} \delta_{lj}, \quad \mathcal{A}_{\bar{\tau}} = \mathcal{A}_\tau^*,$$

and the Berry phase acquired from $\tau(0)$ to $\tau(1)$ is

$$e^{i\gamma_b} = e^{i\int \mathcal{A}} = e^{\frac{i}{2}(\psi(0) - \psi(1))} = \left(\frac{|\mathbf{c}\tau + \mathbf{d}|}{\mathbf{c}\tau + \mathbf{d}} \right)^{1/2}. \quad (3.13)$$

For $\mathbf{M} = \mathbf{T}^{2(p-1)}\mathbf{S}$ we set $\mathbf{c} = 1$ and $\mathbf{d} = 0$. Next we apply the $\hat{\mathbf{M}}$ operator to convert $\tau(1)$ to $\tau(0)$. The modular properties of $\Psi_{j,q}$ are as follows

$$\Psi_{j,q}(\xi_2, -\xi_1; -\frac{1}{\tau}) = \left(\frac{\tau}{|\tau|} \right)^{\frac{1}{2}} \frac{e^{-\frac{\pi i}{4}}}{\sqrt{q}} \sum_{l=0}^{q-1} e^{\frac{2\pi i}{q}jl} \Psi_{l,q}(\xi_1, \xi_2; \tau)$$

and

$$\Psi_{j,q}(\xi_1 - 2\xi_2, \xi_2; \tau + 2) = e^{\frac{2\pi ij^2}{q}} \Psi_{j,q}(\xi_1, \xi_2; \tau)$$

Thus, $\mathbf{M} = \mathbf{T}^{2(p-1)}\mathbf{S}$ acts as

$$\Psi_{j,q} \rightarrow \left(\frac{\tau}{|\tau|} \right)^{\frac{1}{2}} \frac{e^{-\frac{\pi i}{4}}}{\sqrt{q}} e^{\frac{2\pi i(p-1)j^2}{q}} \sum_{l=0}^{q-1} e^{\frac{2\pi i}{q}jl} \Psi_{l,q}$$

Including the Berry phase (3.13), we calculate the partition function as

$$Z = e^{i\gamma b} \operatorname{tr}(\hat{\mathbf{M}}) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{q}} \chi_q(p).$$

Note that the precise semicircle graph of $\tau(y_3)$ found by Gaiotto-Witten is crucial to have a $\tau(0)$ -independent partition function. Switching the role of $\mathbf{k} = 2p$ and $-q$ we similarly get

$$Z' = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2p}} \varrho_p(-q).$$

So, to connect with (3.7) we only need to show that $Z = e^{\frac{\pi i}{4}} Z'$. We note that the above calculation of the phase is most natural when \mathbf{M} is the $SL(2, \mathbb{Z})_\rho$ twist, because the system reduces to geometric quantization of a torus with wavefunctions naturally mapped to the lowest Landau level [66, 81]. So, in that case both Z and Z' would be type-IIB partition functions but the $\rho \leftrightarrow \tau$ duality converts type-IIA to type-IIB. Because the only difference is in the fermion sector, and the fermions are massive, this shouldn't make much of a difference. Even if there was a phase difference $e^{i\phi}$ coming from the fermion sector, it couldn't be the explanation, because we could convert either Z or Z' to type-IIA in which case the phase would be either $e^{i\phi}$ or $e^{-i\phi}$. The explanation of the $e^{\frac{\pi i}{4}}$ phase mismatch must be elsewhere!).

3.6 Review of $U(1)$

In this section, let us review the abelian $U(1)$ case of our theory. We want to understand this example thoroughly in order to attempt to extend the partition function to nonabelian gauge theories, specifically, the $U(2)$ case. We define the theory as a Gaiotto-Witten configuration compactified with an $SL(2, \mathbb{Z})$ -twist and then twisted in 3D. Consider the $U(1)$ case with an $M \in SL(2, \mathbb{Z})$ twist. Set

$$M = ST^{\mathbf{k}_1} \dots ST^{\mathbf{k}_r}.$$

There are two equivalent ways of thinking about the Hilbert space of ground states [77]:

1. as a space of minimal-length string states in a three dimensional mapping torus;
2. as a space of ground states of Chern-Simons theory on T^2 with gauge group $U(1)^r$.

In the following two subsections, let us study both of these equivalent ways.

3.6.1 Mapping Torus Description

First, let us start by studying the string states of our three dimensional mapping torus. The mapping torus is described by a T^2 fibered over S^1 . Let $\theta \in \mathbb{R}$ be a periodic coordinate on

the S^1 base, and let $\mathbf{x} \in \mathbb{R}^2$ be a periodic set of coordinates on the T^2 fiber. The mapping torus is defined as the set of points (θ, \mathbf{x}) with identification

$$(\theta, \mathbf{x}) \sim (\theta, \mathbf{x} + \mathcal{N}) \sim (\theta + 1, M^{-1}\mathbf{x}), \quad \forall \mathcal{N} \in \mathbb{Z}^2. \quad (3.14)$$

A minimal-length winding-number 1 string configuration corresponds to a point on the fiber (described by coordinates $v \in \mathbb{R}^2$ modulo the lattice \mathbb{Z}^2) that is invariant modulo \mathbb{Z}^2 under the M -twist, that is $v - Mv \in \mathbb{Z}^2$.

This has rational solutions $v \in \mathbb{Q}^2$, and we define a Hilbert space of states with basis $\{|v\rangle\}$ comprising of states $|v\rangle$ such that

$$(M - \mathbb{I})v \in \mathbb{Z}^2 \quad (3.15)$$

with $|v\rangle = |u\rangle$ if $v - u \in \mathbb{Z}^2$.

Now define the lattice:

$$\Xi \simeq (M - \mathbb{I})^{-1}(\mathbb{Z}^2) \equiv \{v \in \mathbb{Q}^2 : (M - \mathbb{I})v \in \mathbb{Z}^2\} \supset \mathbb{Z}^2. \quad (3.16)$$

Then a solution to (3.15) with the identification “ $v \sim u$ whenever $v - u \in \mathbb{Z}^2$ ” defines an element of the coset Ξ/\mathbb{Z}^2 . This coset is a finite abelian group which can be identified with isometries on a mapping torus defined by the $\text{SL}(2, \mathbb{Z})$ -twist given by M . (See [77] for more details.) It is easy to check that the number of states is

$$|\Xi/\mathbb{Z}^2| = |\det(M - \mathbb{I})| = |\text{tr } M - 2|.$$

Define the antisymmetric matrix

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $M \in \text{SL}(2, \mathbb{Z})$ satisfies the identity

$$M^t \epsilon = \epsilon M^{-1}, \quad \epsilon M = (M^{-1})^t \epsilon. \quad (3.17)$$

We need to know the action of the T-duality group of the T^2 fiber. This is an $\text{SL}(2, \mathbb{Z})$ group generated by \tilde{T} and \tilde{S} that act as:

$$\tilde{T}^{\mathbf{k}}|v\rangle = e^{\mathbf{k}\pi i v^t \epsilon M v}|v\rangle, \quad \tilde{S}|v\rangle = \frac{1}{\sqrt{|\Xi/\mathbb{Z}^2|}} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M - \mathbb{I})v}|u\rangle. \quad (3.18)$$

Technically, $\tilde{T}^{\mathbf{k}}$ might be ill-defined for odd \mathbf{k} (unless M satisfies additional restrictions to be discussed later), so for now we will assume that $\mathbf{k} \in 2\mathbb{Z}$. Note that these definitions are independent of the representatives v and u , because $\mathcal{N} \in \mathbb{Z}^2$ we have $\mathcal{N}^t \epsilon (M - \mathbb{I})v \in \mathbb{Z}$ when $(M - \mathbb{I})v \in \mathbb{Z}^2$, and also

$$u^t \epsilon (M - \mathbb{I})\mathcal{N} = u^t (M^{-1})^t \epsilon \mathcal{N} - u^t \epsilon \mathcal{N} = [(\mathbb{I} - M)u]^t (M^{-1})^t \epsilon \mathcal{N} \in \mathbb{Z}.$$

Note also that since ϵ is antisymmetric we have

$$v^t \epsilon M v = v^t \epsilon (M - \mathbb{I}) v.$$

The phase $\exp(\mathbf{k}\pi i v^t \epsilon M v)$ has a nice geometrical interpretation. The expression $\frac{1}{2} v^t \epsilon M v$ is the area of a triangle in \mathbb{R}^2 with sides given by the vectors v and Mv . To see how this is related to \tilde{T} consider a string worldsheet that interpolates between the states $|0\rangle$ (a string at $v = 0$) and $|v\rangle$ (for $v \neq 0$). We can do that constructing a section of the mapping torus with $\mathbf{x} = \zeta v$ and let $\zeta \in [0, 1]$ and $\theta \in (0, 1)$ be the coordinates of the worldsheet. If we attach to this surface the triangle with vertices $\{0, v, Mv\}$ we obtain a surface whose boundary is the union of three loops: the loop corresponding to string state $|0\rangle$, the loop corresponding to string state $|v\rangle$, and the loop from $(0, v)$ to $(0, Mv) \sim (1, v)$ at constant $\theta = 0$, which is a closed loop thanks to (3.14) and (3.15). If we now consider the scattering amplitude of an inelastic scattering process with two strings states going into two string states:

$$|0\rangle \otimes |0\rangle \rightarrow |v\rangle \otimes |-v\rangle$$

then it is calculated in string theory by a path integral over worldsheets Σ with boundaries corresponding to the four string states $|v\rangle, |-v\rangle$ and $|0\rangle$ (wrapped twice with opposite orientation). Then the duality operation $\tilde{T}^{\mathbf{k}}$ acts on the Kalb-Ramond field B as $B \rightarrow B + \pi \mathbf{k} d\mathbf{x}^t \wedge \epsilon d\mathbf{x}$ and multiplies the scattering amplitude by the phase $\exp(i \int_{\Sigma} B)$. The construction above shows that this phase is $4\pi \mathbf{k}$ times the area of the triangle with vertices $\{0, v, Mv\}$, which corresponds to a wavefunction normalization of each of the $|\pm v\rangle$ states by $\exp(\mathbf{k}\pi i v^t \epsilon M v)$, as required by (3.18). (See [29] for a similar argument.)

Now let us check that \tilde{S}^2 acts as $-\mathbb{I}$. To see this, we first calculate

$$u^t \epsilon (M - \mathbb{I}) v = -v^t (\mathbb{I} - M^t) \epsilon u = -v^t \epsilon u + v^t M^t \epsilon u = -v^t \epsilon u + v^t \epsilon M u = v^t \epsilon (M - \mathbb{I}) u$$

So

$$\tilde{S}|v\rangle = \frac{1}{\sqrt{|\Xi/\mathbb{Z}^2|}} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M - \mathbb{I}) v} |u\rangle = \frac{1}{\sqrt{|\Xi/\mathbb{Z}^2|}} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i v^t \epsilon (M - \mathbb{I}) u} |u\rangle,$$

and

$$\tilde{S}^2|v\rangle = \frac{1}{|\Xi/\mathbb{Z}^2|} \sum_{u \in \Xi/\mathbb{Z}^2} \sum_{v' \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M - \mathbb{I})(v + v')} |v'\rangle$$

Now, it is not hard to check that

$$\frac{1}{|\Xi/\mathbb{Z}^2|} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M - \mathbb{I})(v + v')} = \delta(v + v')$$

where $\delta(v + v')$ is the δ -function in the abelian group Ξ/\mathbb{Z}^2 (that is, $v + v'$ needs to be zero not in Ξ but in Ξ/\mathbb{Z}^2). To see this, note that we can change the summation variable u to $u + u'$ for any constant $u' \in \Xi/\mathbb{Z}^2$, so

$$\frac{1}{|\Xi/\mathbb{Z}^2|} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M - \mathbb{I})(v + v')} = \frac{1}{|\Xi/\mathbb{Z}^2|} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i (u' + u)^t \epsilon (M - \mathbb{I})(v + v')}$$

and the latter expression can be written as

$$e^{-2\pi i u^t \epsilon(M-\mathbb{I})(v+v')} \frac{1}{|\Xi/\mathbb{Z}^2|} \sum_{u \in \Xi/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon(M-\mathbb{I})(v+v')}$$

So the sum can only be nonzero if

$$u^t \epsilon(M-\mathbb{I})(v+v') \in \mathbb{Z} \quad \forall u' \in \Xi/\mathbb{Z}^2.$$

But this can be written as

$$u^t \epsilon(M-\mathbb{I})(v+v') = (v+v')^t \epsilon(M-\mathbb{I})u'$$

and $(M-\mathbb{I})u'$ can be taken to be an arbitrary vector in \mathbb{Z}^2 . So, $v+v' \in \mathbb{Z}^2$. So, we conclude that

$$\tilde{S}^2|v\rangle = |-v\rangle.$$

We also need the relation

$$\tilde{S}^{-1}|v\rangle = \tilde{S}|-v\rangle = \frac{1}{\sqrt{|\Xi/\mathbb{Z}^2|}} \sum_{u \in \Xi/\mathbb{Z}^2} e^{2\pi i u^t \epsilon(M-\mathbb{I})v} |u\rangle. \quad (3.19)$$

3.6.2 Chern-Simons Theory Description

Now, let us analyze the other side of the equivalence established in [77], that is, the space of ground states of Chern-Simons theory on T^2 with gauge group $U(1)^{\mathbf{r}}$. The $U(1)^{\mathbf{r}}$ Chern-Simons theory description of our system has a coupling constant matrix

$$K \equiv \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & & -1 \\ -1 & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ -1 & & 0 & -1 & \mathbf{k}_r \end{pmatrix}.$$

The Hilbert space of $U(1)^{\mathbf{r}}$ Chern-Simons theory on T^2 with coupling constant matrix K has a basis of states $|\tilde{v}\rangle$ parameterized by $\tilde{v} \in \mathbb{Z}^{\mathbf{r}}$ such that $|\tilde{v}\rangle = |\tilde{u}\rangle$ if $\tilde{v} - \tilde{u} = K\mathcal{N}$ for some $\mathcal{N} \in \mathbb{Z}^{\mathbf{r}}$. Define the lattice

$$\Lambda \simeq K(\mathbb{Z}^{\mathbf{r}}) \equiv \{K\tilde{w} : \tilde{w} \in \mathbb{Z}^{\mathbf{r}}\} \subset \mathbb{Z}^{\mathbf{r}}.$$

So Λ is the sublattice of $\mathbb{Z}^{\mathbf{r}}$ that is generated by the columns of the matrix K . The coset $\mathbb{Z}^{\mathbf{r}}/\Lambda$ is a finite abelian group. The Hilbert space of $U(1)^{\mathbf{r}}$ Chern-Simons theory on T^2 with coupling constant matrix K has a basis of states which can be identified with elements of $\mathbb{Z}^{\mathbf{r}}/\Lambda$. Pick a nontrivial generator of $\pi_1(T^2)$, and consider the corresponding \mathbf{r} Wilson loops

acting on the Hilbert space. They form a commuting abelian group which can be identified with \mathbb{Z}^r/Λ .

Next, we define the action of another $\mathrm{SL}(2, \mathbb{Z})$ on the Hilbert space. From the Chern-Simons perspective, this is the mapping class group of T^2 . The generators act on states as

$$\tilde{S}|\tilde{v}\rangle = \frac{1}{\sqrt{|\mathbb{Z}^r/\Lambda|}} \sum_{\tilde{u} \in \mathbb{Z}^r/\Lambda} e^{-2\pi i \tilde{u}^t K^{-1} \tilde{v}} |\tilde{v}\rangle, \quad \tilde{T}^{\mathbf{k}}|\tilde{v}\rangle = e^{\mathbf{k} \pi i \tilde{v}^t K^{-1} \tilde{v}} |\tilde{v}\rangle$$

Note that $\tilde{T}^{\mathbf{k}}$ could be ill-defined if \mathbf{k} is odd.

3.7 Generalized Landsberg-Schaar relation for $U(1)$

Now that we have reviewed the details of our theory in the latter section, let us generalize our calculations. We will study a basic version of $U(1)$ below, and then we will explore a generalization extension.

3.7.1 Basic version $M = ST^{p+2}$ and $\tilde{M} = \tilde{S}\tilde{T}^{q+2}$

We obtain the original Landsberg-Schaar relation by taking

$$M = ST^{p+2} = \begin{pmatrix} (p+2) & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{M} = \tilde{S}\tilde{T}^{q+2} = \begin{pmatrix} (q+2) & -1 \\ 1 & 0 \end{pmatrix}.$$

Solutions to (3.15) can then be represented by

$$v = \begin{pmatrix} \frac{n}{p} \\ \frac{n}{p} \end{pmatrix},$$

so that

$$(M - \mathbb{I})v = \begin{pmatrix} (p+1) & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{n}{p} \\ \frac{n}{p} \end{pmatrix} = \begin{pmatrix} n \\ 0 \end{pmatrix} \in \mathbb{Z}^2, \quad n = 0, \dots, p-1.$$

So, instead of writing $|v\rangle$ we can write $|n\rangle$ ($n = 0, \dots, p-1$). Then, using (3.18) we can calculate

$$\tilde{T}^{q+2}|n\rangle = e^{(q+2)\pi i n^2/p} |n\rangle, \quad \tilde{S}|n\rangle = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} e^{-2\pi i m n/p} |m\rangle. \quad (3.20)$$

where we used the relations

$$v^t \epsilon M v = \begin{pmatrix} \frac{n}{p} & \frac{n}{p} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (p+2) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{n}{p} \\ \frac{n}{p} \end{pmatrix} = \frac{n^2}{p}$$

and for

$$u = \begin{pmatrix} \frac{m}{p} \\ \frac{m}{p} \\ p \end{pmatrix},$$

we calculate

$$u^t \epsilon(M - \mathbb{I})v = \begin{pmatrix} \frac{m}{p} & \frac{m}{p} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (p+1) & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{n}{p} \\ \frac{n}{p} \\ p \end{pmatrix} = \frac{mn}{p}$$

Define

$$\mathrm{tr}_p^{(1)} = \text{trace in } U(1) \text{ Hilbert space for strings on mapping torus with twist } M.$$

Then,

$$\mathrm{tr}_p^{(1)}(\widetilde{S}\widetilde{T}^{q+2}) = \sum_{n=0}^{p-1} e^{\frac{\pi i q n^2}{p}} \quad (3.21)$$

It is well-defined if at least one of p and q is even. (Otherwise, replacing n by $n + p$ in the expression gives an opposite sign.) The Landsberg-Schaar relation states that

$$\mathrm{tr}_p^{(1)}(\widetilde{S}\widetilde{T}^{q+2}) = e^{\frac{\pi i}{4}} \mathrm{tr}_q^{(1)}((\widetilde{S}\widetilde{T}^{p+2})^{-1}), \quad pq \in 2\mathbb{Z}.$$

3.7.2 Generalization I: $M = ST^{p_1+2}ST^{p_2+2}$ and $\widetilde{M} = \widetilde{S}\widetilde{T}^{q+2}$

For our first generalization we take

$$M = ST^{p_1+2}ST^{p_2+2} = \begin{pmatrix} [(p_1+2)(p_2+2)-1] & -2(p_1+2) \\ (p_2+2) & -1 \end{pmatrix}$$

We have

$$\det(\mathbb{I} - M) = p_1 p_2 + 2(p_1 + p_2).$$

Let us first assume that both p_1 and p_2 are even and set

$$p_1 = 2s_1, \quad p_2 = 2s_2.$$

The Smith normal form of $\mathbb{I} - M$ is then given by

$$\mathbb{I} - M = \begin{pmatrix} 1 & 0 \\ 1+s_2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2(s_1 s_2 + s_1 + s_2) \end{pmatrix} \begin{pmatrix} -1 & -(1+s_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2(1+s_2) & 1 \end{pmatrix}.$$

Solutions to (3.15) can then be represented by

$$v = \begin{pmatrix} 1 & 0 \\ -2(1+s_2) & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & -(1+s_1) \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a/2 \\ n/[2(s_1 + s_2 + s_1 s_2)] \end{pmatrix},$$

with

$$a = 0, 1, \quad n = 0, \dots, 2(s_1 s_2 + s_1 + s_2) - 1.$$

So

$$v = \begin{pmatrix} 1 & 0 \\ 2(1+s_2) & 1 \end{pmatrix} \begin{pmatrix} -1 & -(1+s_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a/2 \\ n/[2(s_1 + s_2 + s_1 s_2)] \end{pmatrix}, \quad (3.22)$$

and it is easy to check that

$$(\mathbb{I} - M)v = \begin{pmatrix} a \\ n + a(1 + s_2) \end{pmatrix}$$

and

$$v^t \epsilon M v = -\frac{(1+s_1)n^2}{2(s_1 + s_2 + s_1 s_2)} + \frac{(1+s_2)a^2}{2}$$

and if we set

$$u = \begin{pmatrix} 1 & 0 \\ 2(1+s_2) & 1 \end{pmatrix} \begin{pmatrix} -1 & -(1+s_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b/2 \\ m/[2(s_1 + s_2 + s_1 s_2)] \end{pmatrix},$$

then

$$u^t \epsilon (M - \mathbb{I}) v = -\frac{(1+s_1)mn}{2(s_1 + s_2 + s_1 s_2)} + \frac{ma - nb + (1+s_2)ab}{2}$$

Define

$$\mathrm{tr}_{(p_1, p_2)}^{(1)} = \text{trace in } U(1) \text{ Hilbert space with twist } M.$$

Then,

$$\mathrm{tr}_{(p_1, p_2)}^{(1)}(\tilde{S}\tilde{T}^{q+2}) = \frac{1}{2(s_1 + s_2 + s_1 s_2)^{1/2}} \sum_{a=0}^1 \sum_{n=0}^{2(s_1 + s_2 + s_1 s_2) - 1} \exp\left(-\frac{\pi i q(1+s_1)n^2}{2(s_1 + s_2 + s_1 s_2)} + \frac{\pi i q(1+s_2)a^2}{2}\right)$$

This can be written as

$$\mathrm{tr}_{(p_1, p_2)}^{(1)}(\tilde{S}\tilde{T}^{q+2}) = \left(\frac{1 + \exp\left(\frac{\pi i q(1+s_2)}{2}\right)}{2\sqrt{s_1 + s_2 + s_1 s_2}}\right)^{2(s_1 + s_2 + s_1 s_2) - 1} \sum_{n=0} \exp\left(-\frac{\pi i q(1+s_1)n^2}{2(s_1 + s_2 + s_1 s_2)}\right)$$

or as

$$\mathrm{tr}_{(p_1, p_2)}^{(1)}(\tilde{S}\tilde{T}^{q+2}) = \left(\frac{1 + i^q(1+s_2)}{2\sqrt{s_1 + s_2 + s_1 s_2}}\right)^{2(s_1 + s_2 + s_1 s_2) - 1} \sum_{n=0} \exp\left(-\frac{\pi i q(1+s_1)n^2}{2(s_1 + s_2 + s_1 s_2)}\right) \quad (3.23)$$

On the other side of the generalized Landsberg-Schaar relation we should have

$$\mathrm{tr}_q^{(1)}((\tilde{S}\tilde{T}^{2s_1+2}\tilde{S}\tilde{T}^{2s_2+2})^{-1}) = \mathrm{tr}_q^{(1)}(\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1})$$

Using (3.19) and (3.20) we calculate (replacing $p \rightarrow q$, of course):

$$\tilde{T}^{-2(s+1)}|n\rangle = e^{-2\pi i(s+1)n^2/q}|n\rangle, \quad \tilde{S}^{-1}|n\rangle = \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} e^{2\pi imn/q}|m\rangle.$$

So,

$$\begin{aligned} & \text{tr}_q^{(1)}(\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1}) \\ &= \sum_{n=0}^{q-1} \langle n|\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1}|n\rangle \\ &= \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} e^{2\pi imn/q} \langle n|\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}|m\rangle \\ &= \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} e^{2\pi imn/q} e^{-2\pi i(s_1+1)m^2/q} \langle n|\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}|m\rangle \\ &= \frac{1}{q} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} \sum_{l=0}^{q-1} e^{2\pi imn/q} e^{-2\pi i(s_1+1)m^2/q} e^{2\pi ilm/q} \langle n|\tilde{T}^{-2(s_2+2)}|l\rangle \\ &= \frac{1}{q} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} \sum_{l=0}^{q-1} e^{2\pi imn/q} e^{-2\pi i(s_1+1)m^2/q} e^{2\pi ilm/q} e^{-2\pi i(s_2+1)l^2/q} \delta_{nl} \end{aligned}$$

Altogether, we get

$$\begin{aligned} & \text{tr}_q^{(1)}(\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1}) = \\ & \frac{1}{q} \sum_{n=0}^{q-1} \sum_{m=0}^{q-1} e^{2\pi imn/q} e^{-2\pi i(s_1+1)m^2/q} e^{2\pi imn/q} e^{-2\pi i(s_2+1)n^2/q} \end{aligned}$$

So,

$$\text{tr}_q^{(1)}(\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1}) = \frac{1}{q} \sum_{m,n=0}^{q-1} e^{\frac{2\pi i}{q}(2mn-(s_1+1)m^2-(s_2+1)n^2)} \quad (3.24)$$

It can be checked that for even q we indeed have:

$$\text{tr}_{(p_1,p_2)}^{(1)}(\tilde{S}\tilde{T}^{q+2}) = -i \text{tr}_q^{(1)}(\tilde{T}^{-2(s_2+2)}\tilde{S}^{-1}\tilde{T}^{-2(s_1+1)}\tilde{S}^{-1})^*$$

Note that we need the phase $e^{-\pi i/2}$ the complex conjugate on the right. The phase of $e^{-\pi i/2}$ makes perfect sense – it is $(e^{-\pi i/4})^2$, where each $e^{-\pi i/4}$ comes from one \tilde{S} .

Setting $a = 1 + s_1$ and $b = 1 + s_2$, we can write the identity as:

$$-\frac{i}{q} \sum_{m,n=0}^{q-1} e^{-\frac{2\pi i}{q}(2mn-am^2-bn^2)} = \left(\frac{1+i^{qb}}{2\sqrt{ab-1}} \right) \sum_{n=0}^{2ab-3} \exp\left(-\frac{\pi i q a n^2}{2(ab-1)}\right), \quad (3.25)$$

for

$$q \in 2\mathbb{Z}_+, \quad a, b \in \mathbb{Z}, \quad ab > 1.$$

We prove this identity in Appendix C.2.

3.8 The states for $U(2)$

The Hilbert space of strings with winding number n on the mapping torus decomposes into a direct sum of Hilbert spaces – each subspace corresponding to a partition of n . For $n = 2$ we have only two subspaces which we denote by $\mathcal{H}_{(2)}$ and $\mathcal{H}_{(1+1)}$. The Hilbert space $\mathcal{H}_{(2)}$ corresponds to a single string with winding number 2, and $\mathcal{H}_{(1+1)}$ corresponds to two (identical bosons) strings, each with winding number 1.

Define the lattices

$$\Xi_1 = (M - \mathbb{I})^{-1}(\mathbb{Z}^2) \equiv \{v \in \mathbb{Q}^2 : Mv - v \in \mathbb{Z}^2\} \supset \mathbb{Z}^2, \quad (3.26)$$

$$\Xi_2 = (M^2 - \mathbb{I})^{-1}(\mathbb{Z}^2) \equiv \{v \in \mathbb{Q}^2 : M^2v - v \in \mathbb{Z}^2\} \supset \Xi_1. \quad (3.27)$$

Note that if $v \in \Xi_2$ then also $Mv \in \Xi_2$. In Ξ_2 we define an equivalence relation \mathcal{R} by $v \sim Mv$. Then,

$$\mathcal{H}_{(1+1)} \simeq \langle S^2(\Xi_1/\mathbb{Z}^2) \rangle, \quad \mathcal{H}_{(2)} \simeq \langle (\Xi_2/\mathbb{Z}^2)/\mathcal{R} \rangle,$$

where $\langle (\dots) \rangle$ denotes the vector space generated by a basis set (\dots) , and $S^2(\dots)$ denotes the symmetric product. In other words, $\mathcal{H}_{(1+1)}$ is the space of two-particle states with a basis:

$$|v, v\rangle, \quad v \in \Xi/\mathbb{Z}^2, \quad \text{and} \quad \frac{1}{\sqrt{2}}(|v_1, v_2\rangle + |v_2, v_1\rangle), \quad v_1, v_2 \in \Xi_1/\mathbb{Z}^2, \quad v_1 \neq v_2.$$

Basis elements of $\mathcal{H}_{(2)}$ can be taken as

$$|v\rangle, \quad v \in \Xi_1/\mathbb{Z}^2, \quad \text{and} \quad \frac{1}{\sqrt{2}}(|v\rangle + |Mv\rangle), \quad v \in \Xi_2/\mathbb{Z}^2, \quad v \neq Mv.$$

The $U(1)$ Hilbert space defined in §3.6.1 is simply $\langle \Xi_1/\mathbb{Z}^2 \rangle$ and we define the action of $\text{SL}(2, \mathbb{Z})$ on $\mathcal{H}_{(1+1)}$ by extending the formulas (3.18) to act on the symmetric product:

$$\begin{aligned} \tilde{T}^{\mathbf{k}}|v_1, v_2\rangle &= e^{\mathbf{k}\pi i(v_1^t \epsilon M v_1 + v_2^t \epsilon M v_2)}|v_1, v_2\rangle, \\ \tilde{S}|v_1, v_2\rangle &= \frac{1}{|\Xi_1/\mathbb{Z}^2|} \sum_{u_1, u_2 \in \Xi_1/\mathbb{Z}^2} e^{-2\pi i u_1^t \epsilon (M - \mathbb{I}) v_1 - 2\pi i u_2^t \epsilon (M - \mathbb{I}) v_2} |u_1, u_2\rangle. \end{aligned}$$

(Note again that $\tilde{T}^{\mathbf{k}}$ could be ill-defined if \mathbf{k} is odd.)

We define the action of $\text{SL}(2, \mathbb{Z})$ on $\mathcal{H}_{(2)}$ by replacing $M \rightarrow M^2$ in (3.18):

$$\tilde{T}^{\mathbf{k}}|v\rangle = e^{\mathbf{k}\pi i v^t \epsilon M^2 v}|v\rangle, \quad \tilde{S}|v\rangle = \frac{1}{\sqrt{|\Xi_2/\mathbb{Z}^2|}} \sum_{u \in \Xi_2/\mathbb{Z}^2} e^{-2\pi i u^t \epsilon (M^2 - \mathbb{I}) v}|u\rangle. \quad (3.28)$$

Note that this definition is consistent with the equivalence relation $|v\rangle = |Mv\rangle$. To see this, we use (3.17) to calculate

$$u^t M^t \epsilon(M^2 - \mathbb{I}) M v = u^t \epsilon M^{-1} (M^2 - \mathbb{I}) M v = u^t \epsilon (M^2 - \mathbb{I}) v.$$

We can easily calculate the dimensions of $\mathcal{H}_{(2)}$ and $\mathcal{H}_{(1+1)}$ as follows.

$$\dim \mathcal{H}_{(1+1)} = \frac{1}{2} |\Xi_1/\mathbb{Z}^2| (|\Xi_1/\mathbb{Z}^2| + 1) = \frac{1}{2} |\operatorname{tr}(M) - 2| (1 + |\operatorname{tr}(M) - 2|).$$

$$\dim \mathcal{H}_{(2)} = \frac{1}{2} (|\Xi_2/\mathbb{Z}^2| - |\Xi_1/\mathbb{Z}^2|) + |\Xi_1/\mathbb{Z}^2| = \frac{1}{2} |\operatorname{tr}(M) - 2| (1 + |\operatorname{tr}(M) + 2|)$$

For

$$M = \begin{pmatrix} (p+2) & -1 \\ 1 & 0 \end{pmatrix}$$

with $p > 0$, we have $|\Xi_1/\mathbb{Z}^2| = p$ and

$$\dim \mathcal{H}_{(1+1)} = \frac{1}{2} p(p+1), \quad \dim \mathcal{H}_{(2)} = \frac{1}{2} p(p+3)$$

We can now explore extensions of the Landsberg-Schaar relation. Set

$$\tilde{M} = \begin{pmatrix} (q+2) & -1 \\ 1 & 0 \end{pmatrix} = \tilde{S} \tilde{T}^{q+2}.$$

Define

$$\operatorname{tr}_p^{(1+1)} = \text{trace in } \mathcal{H}_{(1+1)} \text{ for strings on mapping torus with twist } M.$$

and

$$\operatorname{tr}_p^{(2)} = \text{trace in } \mathcal{H}_{(2)} \text{ for strings on mapping torus with twist } M.$$

We now define

$$\mathcal{X}_1(p, q) \equiv \operatorname{tr}_p^{(1+1)}(\tilde{S} \tilde{T}^{q+2}), \quad \mathcal{X}_2(p, q) \equiv \operatorname{tr}_p^{(2)}(\tilde{S} \tilde{T}^{q+2}).$$

The generalized relation we are looking for is a relation involving

$$\mathcal{X}_1(p, q), \quad \mathcal{X}_2(p, q), \quad \mathcal{X}_1(q, p), \quad \mathcal{X}_2(q, p).$$

We would then like to compare to the Landsberg-Schaar relation for $U(1)$ and M and the generalized Landsberg-Schaar relation for $U(1)$ and M^2 .

3.8.1 The calculation

It is not hard to check that for $\hat{\mathcal{O}}_2$ a product of single-particle operators which we denote by $\hat{\mathcal{O}}$ we have

$$\operatorname{tr}_p^{(1+1)}(\hat{\mathcal{O}}_2) = \frac{1}{2} \left(\sum_{v \in \Xi_1/\mathbb{Z}^2} \langle v | \hat{\mathcal{O}} | v \rangle \right)^2 + \frac{1}{2} \sum_{v \in \Xi_1/\mathbb{Z}^2} \langle v | \hat{\mathcal{O}}^2 | v \rangle.$$

For $\hat{\mathcal{O}} = \tilde{S}\tilde{T}^{q+2}$ we use the results of §3.7.1, and in particular (3.21), to write

$$\sum_{v \in \Xi_1/\mathbb{Z}^2} \langle v | \tilde{S}\tilde{T}^{q+2} | v \rangle = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} e^{\frac{\pi i q n^2}{p}}.$$

From (3.24) (or its complex conjugate) we get

$$\sum_{v \in \Xi_1/\mathbb{Z}^2} \langle v | (\tilde{S}\tilde{T}^{q+2})^2 | v \rangle = \frac{1}{p} \sum_{m,n=0}^{p-1} e^{\frac{\pi i}{p}((q+2)m^2 + (q+2)n^2 - 4mn)}.$$

So,

$$\mathrm{tr}_p^{(1+1)}(\tilde{S}\tilde{T}^{2t+2}) = \frac{1}{2p} \left(\sum_{n=0}^{p-1} e^{\frac{\pi i q n^2}{p}} \right)^2 + \frac{1}{2p} \sum_{m,n=0}^{p-1} e^{\frac{\pi i}{p}[(q+2)(m^2+n^2) - 4mn]}.$$

It is also not hard to check that for $\hat{\mathcal{O}}$ that commutes with M we have

$$\mathrm{tr}_p^{(2)}(\hat{\mathcal{O}}) = \frac{1}{2} \sum_{v \in \Xi_2/\mathbb{Z}^2} (\langle v | \hat{\mathcal{O}} | v \rangle + \langle v | \hat{\mathcal{O}} | Mv \rangle).$$

We now assume that p is even and set

$$p = 2s.$$

For $\hat{\mathcal{O}} = \tilde{S}\tilde{T}^{q+2}$ we calculate using (3.23) (with $s_1 = s_2 = s$):

$$\sum_{v \in \Xi_2/\mathbb{Z}^2} \langle v | \tilde{S}\tilde{T}^{q+2} | v \rangle = \left(\frac{1 + i^{q(1+s)}}{2\sqrt{s(s+2)}} \right)^{2s(s+2)-1} \sum_{n=0}^{2s(s+2)-1} \exp\left(-\frac{\pi i q(1+s)n^2}{2s(s+2)}\right)$$

Before we proceed, we note that

$$\tilde{T}^{q+2} | v \rangle = e^{(q+2)\pi i v^t \epsilon M^2 v} | v \rangle$$

and

$$\tilde{T}^{q+2} | Mv \rangle = e^{(q+2)\pi i v^t M^t \epsilon M^3 v} | Mv \rangle = e^{(q+2)\pi i v^t \epsilon M^{-1} M^3 v} | Mv \rangle = e^{(q+2)\pi i v^t \epsilon M^2 v} | Mv \rangle,$$

and

$$|\Xi_2/\mathbb{Z}^2| = \det(M^2 - \mathbb{I}) = p(p+4)$$

Now we calculate (using (3.18))

$$\begin{aligned} \sum_{v \in \Xi_2/\mathbb{Z}^2} \langle v | \tilde{S}\tilde{T}^{q+2} | Mv \rangle &= \sum_{v \in \Xi_2/\mathbb{Z}^2} e^{(q+2)\pi i v^t \epsilon M^2 v} \langle v | \tilde{S} | Mv \rangle \\ &= \frac{1}{\sqrt{p(p+4)}} \sum_{v \in \Xi_2/\mathbb{Z}^2} \sum_{u \in \Xi_2/\mathbb{Z}^2} e^{(q+2)\pi i v^t \epsilon M^2 v} e^{-2\pi i u^t \epsilon (M^2 - \mathbb{I}) M v} \langle v | u \rangle \\ &= \frac{1}{\sqrt{p(p+4)}} \sum_{v \in \Xi_2/\mathbb{Z}^2} e^{(q+2)\pi i v^t \epsilon M^2 v} e^{-2\pi i v^t \epsilon (M^2 - \mathbb{I}) M v} \end{aligned}$$

Now we use (3.22) to write

$$v = \begin{pmatrix} 1 & 0 \\ 2(1+s) & 1 \end{pmatrix} \begin{pmatrix} -1 & -(1+s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a/2 \\ n/[2s(s+2)] \end{pmatrix},$$

We calculate:

$$v^t \epsilon M^2 v = \frac{(1+s)(n^2 - s(s+2)a^2)}{2s(s+2)},$$

and

$$\begin{aligned} v^t \epsilon (M^2 - \mathbb{1}) M v &= \frac{(1+4s+2s^2)(n^2 - s(s+2)a^2)}{2s(s+2)} \\ &= \frac{(n^2 - s(s+2)a^2)}{2s(s+2)} + n^2 - s(s+2)a^2 \end{aligned} \quad (3.29)$$

So,

$$e^{-2\pi i v^t \epsilon (M^2 - \mathbb{1}) M v} = \exp\left(-\frac{\pi i (n^2 - s(s+2)a^2)}{s(s+2)}\right)$$

So,

$$\begin{aligned} &\sum_{v \in \Xi_2/\mathbb{Z}^2} \langle v | \tilde{S} \tilde{T}^{q+2} | M v \rangle \\ &= \frac{1}{\sqrt{p(p+4)}} \sum_{a=0}^1 \sum_{n=0}^{2s(s+2)-1} \exp\left(\frac{\pi i (q+2)(1+s)(n^2 - s(s+2)a^2)}{2s(s+2)} - \frac{2\pi i (n^2 - s(s+2)a^2)}{2s(s+2)}\right) \\ &= \frac{1}{\sqrt{p(p+4)}} \sum_{a=0}^1 \sum_{n=0}^{2s(s+2)-1} \exp\left(\frac{\pi i [s(q+2) + q](n^2 - s(s+2)a^2)}{2s(s+2)}\right) \\ &= \frac{1}{2\sqrt{s(s+2)}} [1 + (-i)^{s(q+2)+q}] \sum_{n=0}^{2s(s+2)-1} \exp\left(\frac{\pi i [s(q+2) + q]n^2}{2s(s+2)}\right) \\ &= \left(\frac{1 + (-i)^{q(1+s)+2s}}{2\sqrt{s(s+2)}}\right) \sum_{n=0}^{2s(s+2)-1} \exp\left(\frac{\pi i [q(1+s) + 2s]n^2}{2s(s+2)}\right) \end{aligned}$$

This is how you do the calculation for the $U(2)$ case. For other nonabelian groups, the calculation is much harder given the fact that there is a difficulty writing the operator expression for \tilde{S}, \tilde{T} . Work related to this extension is left for future research.

3.9 Discussion

In this chapter, we have successfully shown how quadratic reciprocity is a direct consequence of T-duality of type-II string theory. The entire analysis was done for gauge group $U(1)$, and

sketched for the non-abelian gauge group $U(2)$. It would be very interesting to extend this analysis to other more complex non-abelian groups in order to see what number theoretic identities we find. The extension to gauge group $U(n)$, for $n > 2$ is left for future research.

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Appendix A

Q-balls of Quasi-particles

A.1 Recasting the BPS equations in terms of a single potential

The action (1.86) is invariant under dilatations that act as

$$f(r, \rho) \rightarrow f(\lambda r, \lambda \rho), \quad \chi(r, \rho) \rightarrow \chi(\lambda r, \lambda \rho).$$

The components of the associated Noether current are given by

$$\begin{aligned} J^r &= \frac{\rho f_r^2}{2f^2} + \frac{\rho^2 f_r f_\rho}{r f^2} - \frac{\rho f_\rho^2}{2f^2} + \frac{\rho \chi_r^2}{2f^2} + \frac{\rho^2 \chi_r \chi_\rho}{r f^2} - \frac{\rho \chi_\rho^2}{2f^2}, \\ J^\rho &= \frac{\rho^2 f_\rho^2}{2r f^2} + \frac{\rho f_r f_\rho}{f^2} - \frac{\rho^2 f_r^2}{2r f^2} + \frac{\rho^2 \chi_\rho^2}{2r f^2} + \frac{\rho \chi_r \chi_\rho}{f^2} - \frac{\rho^2 \chi_r^2}{2r f^2}. \end{aligned}$$

The equations of motion (1.80)-(1.81) imply the conservation equation¹

$$(J^r)_r + (J^\rho)_\rho = 0,$$

which implies that there exists a potential function Φ such that

$$J^\rho = \Phi_r, \quad J^r = -\Phi_\rho. \tag{A.1}$$

To proceed, we think of the functions f and χ as defining a change of coordinates from (f, χ) to (r, ρ) [similar to (1.88), except with the ϕ coordinate absent]. In (r, ρ) coordinates, the AdS_2 metric (1.87) becomes:

$$ds^2 = \mathfrak{G}_{rr} dr^2 + 2\mathfrak{G}_{r\rho} dr d\rho + \mathfrak{G}_{\rho\rho} d\rho^2, \tag{A.2}$$

¹But note that (J^r, J^ρ) are not directly related to the stress-energy tensor derived from the original (“physical”) action in the original fields A_i and Φ . The “physical” conserved currents associated with dilatations generally vanishes on BPS configurations [63].

where the metric \mathfrak{G} can be expressed, using (A.1), as:

$$\begin{aligned}\mathfrak{G}_{rr} &= -\frac{r^2}{r^2+\rho^2}(\Phi_{\rho\rho} + \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{\rho}\Phi_\rho), \\ \mathfrak{G}_{\rho\rho} &= -\frac{r^2}{r^2+\rho^2}(\Phi_{\rho\rho} + \Phi_{rr} - \frac{1}{r}\Phi_r - \frac{1}{\rho}\Phi_\rho), \\ \mathfrak{G}_{r\rho} &= \frac{r}{\rho(r^2+\rho^2)}(r\Phi_r - \rho\Phi_\rho).\end{aligned}$$

Φ then satisfies a nonlinear differential equation that states that the Ricci scalar of (A.2) is $R = -2$. In order to incorporate the Dirac string for $r < a$, the function Φ must diverge like $\log \rho$ as $\rho \rightarrow 0$ and $r < a$. Define \mathbf{Z} and \mathbf{R} by:

$$\mathbf{Z} \equiv \frac{1}{2a}(\rho^2 + r^2 - a^2), \quad \mathbf{R} \equiv \sqrt{\rho^2 + \mathbf{Z}^2} = \frac{1}{2a}\sqrt{(\rho^2 + r^2 - a^2)^2 + 4a^2\rho^2}. \quad (\text{A.3})$$

For large a , the solution to f and χ is given by adapting the Prasad-Sommerfield solution as given by [49]:

$$f = \frac{\rho \sinh \mathbf{R}}{\mathbf{R} + \mathbf{R} \cosh \mathbf{R} \cosh \mathbf{Z} - \mathbf{Z} \sinh \mathbf{Z} \sinh \mathbf{R}}, \quad \chi = \frac{\mathbf{Z} \cosh \mathbf{Z} \sinh \mathbf{R} - \mathbf{R} \sinh \mathbf{Z} \cosh \mathbf{R}}{\mathbf{R} + \mathbf{R} \cosh \mathbf{R} \cosh \mathbf{Z} - \mathbf{Z} \sinh \mathbf{Z} \sinh \mathbf{R}}, \quad (\text{A.4})$$

where we have set the VEV to $v = 1$, and we have used \mathbf{R} as a substitute for the distance from the core of the monopole. From this we find, in the large a limit,

$$\Phi \rightarrow -\frac{1}{4}\rho^2 + \frac{1}{2}\log \rho - \log \mathbf{R} + \log \sinh \mathbf{R}. \quad (\text{A.5})$$

We also note that the abelian solution

$$f = \left(\frac{\mathbf{R} - \mathbf{Z}}{2a}\right) e^{-\frac{1}{2}vr^2}, \quad \chi = 0,$$

can be derived from the potential

$$\Phi = \frac{1}{4}v^2r^2\rho^2 + \frac{1}{2}v(2a\mathbf{R} + r^2 - \rho^2) + \log \left[\frac{2a\mathbf{R}}{(\mathbf{R} - \mathbf{Z})(a + \mathbf{R} + \mathbf{Z})} \right].$$

Finally, we note that a change of variables,

$$r + i\rho = ae^\xi, \quad r - i\rho = ae^{\bar{\xi}},$$

converts the metric (A.2) to the more compact form:

$$ds^2 = -4 \cosh^2\left(\frac{\xi - \bar{\xi}}{2}\right) \Phi_{\xi\bar{\xi}} d\xi d\bar{\xi} + \coth\left(\frac{\xi - \bar{\xi}}{2}\right) (\Phi_\xi d\xi^2 - \Phi_{\bar{\xi}} d\bar{\xi}^2), \quad (\text{A.6})$$

where $\Phi_\xi \equiv \partial\Phi/\partial\xi$, $\Phi_{\bar{\xi}} \equiv \partial\Phi/\partial\bar{\xi}$, and $\Phi_{\xi\bar{\xi}} \equiv \partial^2\Phi/\partial\xi\partial\bar{\xi}$. The equation to solve is again $R = -2$, where R is the Ricci scalar calculated from the metric (A.6), and the result is a rather length nonlinear partial differential equation for the single field Φ , which we will not present here.

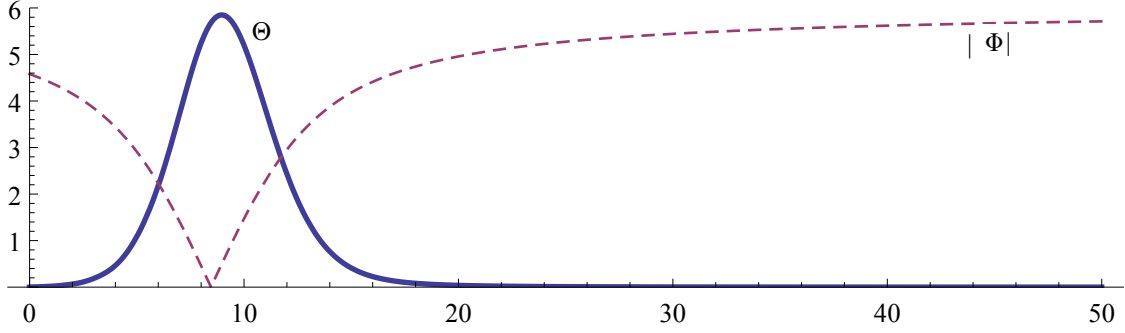


Figure A.1: Results of a numerical analysis with parameters $b = 2.60$ and $N = 22$. The graphs show the energy density $\Theta \equiv \mathcal{U}/\mathbf{V}$ (solid line) and the gauge invariant absolute value of the scalar field $|\tilde{\Phi}| \equiv (\tilde{\Phi}^a \tilde{\Phi}^a)^{1/2}$ (dashed line) for VEV $v = 1$ and soliton center at $a = 1$. The graphs are on the axis $\mathbf{U} = 0$ and the horizontal axis is \mathbf{V} . The vertical axis refers to Θ , and the asymptotic value of $|\tilde{\Phi}|$ is 1. At $\mathbf{V} = 0$ the value of Θ is 1.5×10^{-3} and the value of $|\tilde{\Phi}|$ is 0.76. The value of the excess energy \mathcal{E} for this configuration is less than 2×10^{-5} of \mathcal{E}_{BPS} .

A.2 Numerical results

As a first step towards a numerical analysis of the solution to the BPS equations (1.44) we find it convenient to recast the equations in a different gauge from the one we used in the main text. We begin by parameterizing the scalar field components as:

$$\phi^\alpha = x_\alpha(\mathcal{P} + \mathcal{T}), \quad \phi^3 = \mathcal{S}, \quad (\text{A.7})$$

and the gauge field components as:

$$A_\beta^\alpha = x_{(\beta} \epsilon_{\alpha)\gamma} x^\gamma \mathcal{M} + \frac{1}{2} \epsilon_{\alpha\beta} \mathcal{K}, \quad A_r^\alpha = -r \epsilon_{\alpha\gamma} x_\gamma (\mathcal{P} - \mathcal{T}), \quad A_\beta^3 = \epsilon_{\beta\gamma} x^\gamma \mathcal{W}, \quad A_r^3 = 0. \quad (\text{A.8})$$

with $\alpha, \beta, \gamma = 1, 2$, $\epsilon_{\alpha\beta}$ the anti-symmetric Levi-Civita symbol, and \mathcal{P} , \mathcal{S} , \mathcal{T} , \mathcal{M} , \mathcal{K} , and \mathcal{W} functions of (r, ρ) only. Next, we fix the gauge by setting $\mathcal{M} = 0$. Defining

$$\mathbf{U} \equiv \rho^2, \quad \mathbf{V} \equiv r^2,$$

the BPS equations (1.44) reduce (after rescaling ϕ by $\mathbf{k}R$) to:

$$0 = \mathcal{T}\mathcal{W} - 2\frac{\partial\mathcal{T}}{\partial\mathbf{U}}, \quad (\text{A.9})$$

$$0 = 2\mathbf{U}\frac{\partial\mathcal{P}}{\partial\mathbf{U}} + \mathbf{U}\mathcal{W}\mathcal{P} + 2\mathcal{P} + \frac{\partial\mathcal{K}}{\partial\mathbf{V}} + \frac{1}{2}\mathcal{S}\mathcal{K}, \quad (\text{A.10})$$

$$0 = \mathbf{V}(\mathcal{T} - \mathcal{P})\mathcal{S} + \frac{1}{2}\mathcal{K}\mathcal{W} + 2\mathbf{V}\frac{\partial\mathcal{P}}{\partial\mathbf{V}} + 2\mathbf{V}\frac{\partial\mathcal{T}}{\partial\mathbf{V}} - \frac{\partial\mathcal{K}}{\partial\mathbf{U}}, \quad (\text{A.11})$$

$$0 = \frac{\partial\mathcal{W}}{\partial\mathbf{V}} - \frac{\partial\mathcal{S}}{\partial\mathbf{U}} + \frac{1}{2}\mathcal{T}\mathcal{K}, \quad (\text{A.12})$$

$$0 = \mathbf{U}\mathbf{V}(\mathcal{P}^2 - \mathcal{T}^2) + \frac{1}{4}\mathcal{K}^2 + 2\mathcal{W} + 2\mathbf{V}\frac{\partial\mathcal{S}}{\partial\mathbf{V}} + 2\mathbf{U}\frac{\partial\mathcal{W}}{\partial\mathbf{U}}. \quad (\text{A.13})$$

Let us also set

$$\mathbf{Z} \equiv \frac{1}{2a}(\rho^2 + r^2 - a^2), \quad \mathbf{R} \equiv \sqrt{\rho^2 + \mathbf{Z}^2} = \frac{1}{2a}\sqrt{(\rho^2 + r^2 - a^2)^2 + 4a^2\rho^2}, \quad (\text{A.14})$$

as in (A.3). The advantage of the ansatz (A.7)-(A.8) is that the abelian solution (1.83)-(1.84) can be written in the form:

$$\mathcal{P} = \frac{v}{2\mathbf{R}} - \frac{1}{a\mathbf{R}^2}, \quad \mathcal{S} = \frac{v\mathbf{Z}}{\mathbf{R}} - \frac{\mathbf{Z}}{a\mathbf{R}^2}, \quad \mathcal{T} = \frac{v}{2\mathbf{R}}, \quad \mathcal{K} = \frac{a^2 + \mathbf{U} - \mathbf{V}}{a\mathbf{R}^2}, \quad \mathcal{W} = -\frac{1}{\mathbf{R}^2} - \frac{\mathbf{Z}}{a\mathbf{R}^2}. \quad (\text{A.15})$$

which has no singularities except at $r = a$ (and in particular no Dirac string).

We now require that at either limit $r \rightarrow \infty$ or $\rho \rightarrow \infty$ the full solution should reduce to the abelian solution. At the tip $r = 0$ the solution is required to be regular. This allows us to determine \mathcal{K} , \mathcal{T} , and \mathcal{W} at the tip as follows. Setting $\mathbf{V} = 0$ in (A.9), (A.11), and (A.13), we get the ordinary differential equations

$$\mathcal{T}\mathcal{W} - 2\frac{\partial\mathcal{T}}{\partial\mathbf{U}} = \frac{1}{2}\mathcal{K}\mathcal{W} - \frac{\partial\mathcal{K}}{\partial\mathbf{U}} = \frac{1}{4}\mathcal{K}^2 + 2\mathcal{W} + 2\mathbf{U}\frac{\partial\mathcal{W}}{\partial\mathbf{U}} = 0, \quad (\mathbf{V} = 0) \quad (\text{A.16})$$

which we can solve uniquely, given the known boundary conditions at $\mathbf{U} \rightarrow \infty$. This is easily done by expressing \mathcal{K} and \mathcal{T} in terms of the function $(1 + \mathbf{U}\mathcal{W})$ and its derivatives, and changing variables to $\log \mathbf{U}$. The result is that unique solution to (A.16) that satisfies the boundary conditions at $\mathbf{U} \rightarrow \infty$ is

$$\mathcal{K} = \frac{4a}{\mathbf{U} + a^2}, \quad \mathcal{W} = -\frac{2}{\mathbf{U} + a^2}, \quad \mathcal{T} = \frac{va}{\mathbf{U} + a^2}, \quad (\mathbf{V} = 0), \quad (\text{A.17})$$

which is none other than the abelian solution (A.15) at $\mathbf{V} = 0$.

We cannot determine \mathcal{P} and \mathcal{S} at $\mathbf{V} = 0$ so easily, and our strategy will be to find an approximate solution to (A.9)-(A.13) by the variational method, minimizing the energy of the field configuration within a certain class of trial functions of (\mathbf{U}, \mathbf{V}) . For the energy we take the expression for the excess energy above the BPS bound for a stationary configuration of gauge field and minimally coupled adjoint scalar on a manifold given by the three dimensional metric (1.46), that is,

$$\begin{aligned} \mathcal{E} &\equiv \frac{1}{2} \text{tr} \int \sqrt{g} g^{ij} (D_i \tilde{\Phi} - B_i)(D_j \tilde{\Phi} - B_j) d^3x \\ &= \frac{1}{2} \text{tr} \int \left[(rD_r \tilde{\Phi} - F_{12})^2 + (rD_1 \tilde{\Phi} - F_{2r})^2 + (rD_2 \tilde{\Phi} - F_{r1})^2 \right] \rho d\rho \left(\frac{dr}{r} \right), \end{aligned} \quad (\text{A.18})$$

where B_i and $\tilde{\Phi}$ were defined in (1.45), and the ‘‘tr’’ is in the fundamental representation. Note that \mathcal{E} is different from the physical energy (1.49). The integrand in (A.18) is $\tilde{\alpha}/r^2$ bigger than the integrand in the first term on the RHS of (1.49), but they are both minimized on the BPS configurations, and (A.18) gives more weight to the vicinity of $r = 0$. We can

rewrite \mathcal{E} in terms of the right-hand-sides of (A.9)-(A.13) as follows. Setting

$$\mathcal{X}_1 = \mathcal{T}\mathcal{W} - 2\frac{\partial\mathcal{T}}{\partial\mathbf{U}}, \quad (\text{A.19})$$

$$\mathcal{X}_2 = 2\mathbf{U}\frac{\partial\mathcal{P}}{\partial\mathbf{U}} + \mathbf{U}\mathcal{W}\mathcal{P} + 2\mathcal{P} + \frac{\partial\mathcal{K}}{\partial\mathbf{V}} + \frac{1}{2}\mathcal{S}\mathcal{K}, \quad (\text{A.20})$$

$$\mathcal{X}_3 = \mathbf{V}(\mathcal{T} - \mathcal{P})\mathcal{S} + \frac{1}{2}\mathcal{K}\mathcal{W} + 2\mathbf{V}\frac{\partial\mathcal{P}}{\partial\mathbf{V}} + 2\mathbf{V}\frac{\partial\mathcal{T}}{\partial\mathbf{V}} - \frac{\partial\mathcal{K}}{\partial\mathbf{U}}, \quad (\text{A.21})$$

$$\mathcal{X}_4 = \frac{\partial\mathcal{W}}{\partial\mathbf{V}} - \frac{\partial\mathcal{S}}{\partial\mathbf{U}} + \frac{1}{2}\mathcal{T}\mathcal{K}, \quad (\text{A.22})$$

$$\mathcal{X}_5 = \mathbf{U}\mathbf{V}(\mathcal{P}^2 - \mathcal{T}^2) + \frac{1}{4}\mathcal{K}^2 + 2\mathcal{W} + 2\mathbf{V}\frac{\partial\mathcal{S}}{\partial\mathbf{V}} + 2\mathbf{U}\frac{\partial\mathcal{W}}{\partial\mathbf{U}}, \quad (\text{A.23})$$

we get (A.18) in the form

$$\mathcal{E} = \int \left(\frac{1}{8}\mathbf{U}^2\mathcal{X}_1^2 + \frac{1}{8}\mathcal{X}_2^2 + \frac{\mathcal{X}_3^2}{16\mathbf{V}} + \frac{1}{4}\mathbf{U}\mathcal{X}_4^2 + \frac{\mathcal{X}_5^2}{16\mathbf{V}} \right) d\mathbf{U}d\mathbf{V}. \quad (\text{A.24})$$

We also note that the BPS bound on energy is given by

$$\begin{aligned} \mathcal{E}_{\text{BPS}} &= \text{tr} \int \sqrt{g}g^{ij}(B_j D_i \tilde{\Phi})d^3x \\ &= \text{tr} \int \left[F_{12}D_r \tilde{\Phi} + F_{2r}D_1 \tilde{\Phi} + F_{r1}D_2 \tilde{\Phi} \right] \rho d\rho dr = \int d\lambda, \end{aligned} \quad (\text{A.25})$$

where the 1-form λ is defined by

$$\begin{aligned} \lambda &\equiv \left[\frac{1}{8}\mathbf{U}\mathcal{K}\mathcal{W}(\mathcal{P} + \mathcal{T}) + \frac{1}{16}\mathcal{K}^2\mathcal{S} + \frac{1}{2}\mathcal{W}\mathcal{S} + \frac{1}{2}\mathbf{U}\mathcal{S}\frac{\partial\mathcal{W}}{\partial\mathbf{U}} - \frac{1}{4}\mathbf{U}(\mathcal{P} + \mathcal{T})\frac{\partial\mathcal{K}}{\partial\mathbf{U}} \right] d\mathbf{U} \\ &+ \left[\frac{1}{8}\mathbf{U}\mathcal{S}\mathcal{K}(\mathcal{T} - \mathcal{P}) + \frac{1}{4}\mathbf{U}(\mathcal{T}^2 - \mathcal{P}^2)(1 + \mathbf{U}\mathcal{W}) - \frac{1}{4}\mathbf{U}(\mathcal{P} + \mathcal{T})\frac{\partial\mathcal{K}}{\partial\mathbf{V}} + \frac{1}{2}\mathbf{U}\mathcal{S}\frac{\partial\mathcal{W}}{\partial\mathbf{V}} \right] d\mathbf{V}. \end{aligned}$$

Requiring the asymptotic behavior for large \mathbf{U} and \mathbf{V} to be as in (A.15), we find

$$\mathcal{E}_{\text{BPS}} = 2v.$$

We construct our trial functions by modifying the abelian solution (A.15). But first we need to smooth out the singularity of that solution at $\mathbf{V} = a^2$ while preserving the asymptotic behavior at large \mathbf{U} and \mathbf{V} , as well as the behavior (A.17) at $\mathbf{V} = 0$. For this purpose we define:

$$\mathcal{R} \equiv \sqrt{\mathbf{U} + \mathbf{V} + a^2} = \sqrt{r^2 + \rho^2 + a^2} \quad (\text{A.26})$$

and then define smoothed versions of \mathcal{P} , \mathcal{S} , \mathcal{T} , \mathcal{K} , \mathcal{W} :

$$\begin{aligned} \tilde{\mathcal{P}} &\equiv \frac{av}{\mathcal{R}^2} + \frac{2a(va^2 - 2)}{\mathcal{R}^4} - \frac{2a^3v\mathbf{U}}{\mathcal{R}^6}, \\ \tilde{\mathcal{S}} &\equiv v - \frac{2}{\mathcal{R}^2} - \frac{2va^2\mathbf{U}}{\mathcal{R}^4}, \\ \tilde{\mathcal{T}} &\equiv v \left(\frac{a}{\mathcal{R}^2} + \frac{2a^3}{\mathcal{R}^4} - \frac{2a^3\mathbf{U}}{\mathcal{R}^6} - \frac{2a^5(a^2 + \mathbf{U})}{\mathcal{R}^8} \right), \\ \tilde{\mathcal{K}} &\equiv \frac{4a}{\mathcal{R}^2} - \frac{8a\mathbf{V}}{\mathcal{R}^4}, \\ \tilde{\mathcal{W}} &\equiv -\frac{2}{\mathcal{R}^2} - \frac{8a^2}{\mathcal{R}^4} + \frac{8a^2(a^2 + \mathbf{U})}{\mathcal{R}^6}, \end{aligned}$$

so that for $\mathbf{V} \rightarrow \infty$ at fixed \mathbf{U} we have

$$\begin{aligned}\tilde{\mathcal{P}} &= \frac{v}{2\mathbf{R}} - \frac{1}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^4}\right), \\ \tilde{\mathcal{S}} &= \frac{v\mathbf{Z}}{\mathbf{R}} - \frac{\mathbf{Z}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^3}\right), \\ \tilde{\mathcal{T}} &= \frac{v}{2\mathbf{R}} + O\left(\frac{1}{\mathbf{V}^4}\right), \\ \tilde{\mathcal{K}} &= \frac{a^2 + \mathbf{U} - \mathbf{V}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^3}\right), \\ \tilde{\mathcal{W}} &= -\frac{1}{\mathbf{R}^2} - \frac{\mathbf{Z}}{a\mathbf{R}^2} + O\left(\frac{1}{\mathbf{V}^4}\right),\end{aligned}$$

and $\tilde{\mathcal{P}}, \tilde{\mathcal{S}}, \tilde{\mathcal{T}}, \tilde{\mathcal{W}}, \tilde{\mathcal{K}}$ are smooth everywhere. We also define

$$\mathcal{R}_b \equiv \sqrt{\mathbf{U} + \mathbf{V} + b^2} = \sqrt{r^2 + \rho^2 + b^2},$$

where b is a parameter to be determined dynamically by the variational principle. We now pick a sufficiently large integer N (we chose $N = 20$ below), and construct trial functions in the form:

$$\begin{aligned}\mathcal{P} &= \tilde{\mathcal{P}} + \frac{1}{\mathcal{R}_b^{5+2N}} \sum_{\substack{n+m \leq N \\ n, m \geq 0}} \mathcal{P}_{m,n} \mathbf{U}^m \mathbf{V}^n, \\ \mathcal{S} &= \tilde{\mathcal{S}} + \frac{1}{\mathcal{R}_b^{4+2N}} \sum_{\substack{n+m \leq N \\ n, m \geq 0}} \mathcal{S}_{m,n} \mathbf{U}^m \mathbf{V}^n, \\ \mathcal{T} &= \tilde{\mathcal{T}} + \frac{\mathbf{V}}{\mathcal{R}_b^{5+2N}} \sum_{\substack{n+m \leq N-1 \\ n, m \geq 0}} \mathcal{T}_{m,n} \mathbf{U}^m \mathbf{V}^n, \\ \mathcal{K} &= \tilde{\mathcal{K}} + \frac{\mathbf{V}}{\mathcal{R}_b^{4+2N}} \sum_{\substack{n+m \leq N-1 \\ n, m \geq 0}} \mathcal{K}_{m,n} \mathbf{U}^m \mathbf{V}^n, \\ \mathcal{W} &= \tilde{\mathcal{W}} + \frac{\mathbf{V}}{\mathcal{R}_b^{4+2N}} \sum_{\substack{n+m \leq N-1 \\ n, m \geq 0}} \mathcal{W}_{m,n} \mathbf{U}^m \mathbf{V}^n,\end{aligned}$$

where $\mathcal{P}_{m,n}, \mathcal{S}_{m,n}, \mathcal{T}_{m,n}, \mathcal{K}_{m,n}, \mathcal{W}_{m,n}$ are constant coefficients to be determined. These expressions are designed to preserve the boundary condition (A.17), as well as the asymptotic behavior for large \mathbf{U} and \mathbf{V} . We then find the coefficients $\mathcal{P}_{m,n}, \mathcal{S}_{m,n}, \mathcal{T}_{m,n}, \mathcal{K}_{m,n}, \mathcal{W}_{m,n}$ that minimize \mathcal{E} , using the Newton-Raphson method for given b , and finally we optimize b . For example, we find for the dimensionless coefficient $va^2 = 1$ and $N = 22$ that the optimal b is $2.6a$. We define the energy density

$$\mathcal{U} \equiv \frac{1}{2} \text{tr} \left[(D_r \tilde{\Phi})^2 + (D_1 \tilde{\Phi})^2 + (D_2 \tilde{\Phi})^2 \right] + \frac{1}{2} r^2 \text{tr} \left[F_{12}^2 + F_{1r}^2 + F_{2r}^2 \right], \quad (\text{A.27})$$

for the exact solution we have

$$\mathcal{U} = \mathcal{U}_{\text{BPS}} \equiv r \operatorname{tr} \left[F_{12} D_r \tilde{\Phi} + F_{2r} D_1 \tilde{\Phi} + F_{r1} D_2 \tilde{\Phi} \right]. \quad (\text{A.28})$$

The total energy is then

$$\mathcal{E}_{\text{BPS}} = \frac{1}{4} \int \frac{1}{\mathbf{V}} \mathcal{U}_{\text{BPS}} d\mathbf{V} d\mathbf{U}.$$

We present in Figure A.1 our² numerical results for $\Theta \equiv \mathcal{U}/\mathbf{V}$ as well as for the gauge invariant absolute value of the scalar field

$$|\tilde{\Phi}| \equiv (\tilde{\Phi}^a \tilde{\Phi}^a)^{1/2} = \sqrt{\mathbf{U}(\mathcal{P} + \mathcal{T})^2 + \mathcal{S}^2}.$$

The results are for $va^2 = 1$, and it is interesting to note that for such a relatively small value of va^2 , the core of the soliton (where $|\tilde{\Phi}| = 0$) is at $r \approx 2.9$ ($\mathbf{V} = 8.46$ in the graph of Figure A.1), which is far from $a = 1$.

²The graph was drawn by *Mathematica*, Version 9.0, (Wolfram Research, Inc.).

Appendix B

Janus Configurations with SL(2, \mathbb{Z})-duality twists

B.1 A proof of the determinant identity and the Smith normal form of the coupling constant matrix

Molinari gave an elegant proof [76] to a generalization of (2.2) using only polynomial analysis. Here we present an alternative basic linear-algebra proof for (2.2). At the same time we also demonstrate that the Smith normal form of the coupling constant matrix K defined in (2.5),

$$K = \begin{pmatrix} \mathbf{k}_1 & -1 & 0 & \ddots & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & \ddots & 0 & -1 & \mathbf{k}_n \end{pmatrix},$$

is identical to the Smith normal form of

$$H = W - \mathbf{I} = \begin{pmatrix} \mathbf{a} - 1 & \mathbf{b} \\ \mathbf{c} & \mathbf{d} - 1 \end{pmatrix},$$

where W was defined in (2.11).

We begin by moving the first row of K to the end, to get K'_1 . We have

$$\det K = (-1)^n \det K'_1$$

- Add $[H'_m]_{21}$ times the m^{st} row to the n^{th} row;
- For $j = m + 1, \dots, n$, add $[K'_m]_{mj}$ times the m^{th} column to the j^{th} column.

It is not hard to see that these operations produce a matrix that fits the general form (B.1) with $m \rightarrow m + 1$. Tracking how the bottom two rows transform, we find that for $m < n - 2$,

$$H'_{m+1} = \begin{pmatrix} [H'_{m+1}]_{11} & [H'_{m+1}]_{12} \\ [H'_{m+1}]_{21} & [H'_{m+1}]_{22} \end{pmatrix} = \begin{pmatrix} [H'_m]_{12} + \mathbf{k}_{m+1}[H'_m]_{11} & -[H'_m]_{11} \\ [H'_m]_{22} + \mathbf{k}_{m+1}[H'_m]_{21} & -[H'_m]_{21} \end{pmatrix} = H'_m \begin{pmatrix} \mathbf{k}_{m+1} & 1 \\ -1 & 0 \end{pmatrix}.$$

Since, by definition, $H'_1 = \begin{pmatrix} -1 & 0 \\ \mathbf{k}_1 & -1 \end{pmatrix}$, it follows that

$$H'_{n-2} = \begin{pmatrix} -1 & 0 \\ \mathbf{k}_1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{k}_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathbf{k}_{n-2} & 1 \\ -1 & 0 \end{pmatrix}.$$

It can then be easily checked that the last two steps yield:

$$H'_n = H'_{n-2} \begin{pmatrix} \mathbf{k}_{n-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{k}_n & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

B.2 Compatibility of the supersymmetric Janus configuration and the duality twist

In this section we describe the details of the supersymmetric Lagrangian. As explained in §2.2, the system is composed of two ingredients: (i) the supersymmetric Janus configuration; and (ii) an $SL(2, \mathbb{Z})$ duality twist. We will now review the details of both ingredients and demonstrate that their combination preserves supersymmetry.

B.2.1 Supersymmetric Janus

Extending the work of [85]-[87], Gaiotto and Witten [5] have constructed a supersymmetric deformation of $\mathcal{N} = 4$ Super-Yang-Mills theory with a complex coupling constant τ that varies along one direction, which we denote by x_3 . We will now review this construction, using the same notation as in [5]. First, the real and imaginary parts of the coupling constant are defined as

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e^2}, \quad (\text{B.2})$$

It is taken to vary along a semi-circle on the upper half τ -plane, centered on the real axis:

$$\tau = a + 4\pi D e^{2i\psi}, \quad (\text{B.3})$$

where $\psi(x_3)$ is an arbitrary function.

The action is defined as

$$I = I_{N=4} + I' + I'' + I'''$$

where $I_{N=4}$ is the standard $\mathcal{N} = 4$ action, modified only by making τ a function of x_3 , and I' , I'' , and I''' are correction terms listed below. We will list the actions for a general gauge group, as derived by Gaiotto and Witten, although the application in this chapter is for a $U(1)$ gauge group, and so several terms drop out. The bosonic fields are: a gauge field A_μ ($\mu = 0, 1, 2, 3$), 3 adjoint-valued scalar fields X^a ($a = 1, 2, 3$) and 3 adjoint-valued scalar fields Y^p ($p = 1, 2, 3$). In the $U(1)$ case, X^a and Y^p are real scalar fields. In the type-IIB realization on D3-branes, the D3-brane is in directions 0, 1, 2, 3, X^a corresponds to fluctuations in directions 4, 5, 6, and Y^p corresponds to directions 7, 8, 9. The fermionic fields are encoded in a 16-dimensional Majorana-Weyl spinor Ψ on which even products of the 9+1D Dirac matrices $\Gamma_0, \dots, \Gamma_9$ act. Products of pairs from the list $\Gamma_0, \dots, \Gamma_3$ correspond to generators of the Lorentz group $SO(1, 3)$, while products of pairs from the list $\Gamma_4, \Gamma_5, \Gamma_6$ correspond to generators of the R-symmetry subgroup $SO(3)_X$ acting on X^1, X^2, X^3 , and products of pairs from the list $\Gamma_7, \Gamma_8, \Gamma_9$ correspond to generators of the R-symmetry subgroup $SO(3)_Y$ acting on Y^1, Y^2, Y^3 . We have the identity $\Gamma_{0123456789} = 1$.

The additional terms are

$$\begin{aligned} I' &= \frac{i}{e^2} \int d^4x \operatorname{Tr} \bar{\Psi} (\alpha \Gamma_{012} + \beta \Gamma_{456} + \gamma \Gamma_{789}) \Psi, \\ I'' &= \frac{1}{e^2} \int d^4x \operatorname{Tr} \left(\mathbf{u} \epsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda) + \frac{\mathbf{v}}{3} \epsilon^{abc} X_a [X_b, X_c] + \frac{\mathbf{w}}{3} \epsilon^{pqr} Y_p [Y_q, Y_r] \right), \\ I''' &= \frac{1}{2e^2} \int d^4x \operatorname{Tr} (\mathbf{r} X_a X^a + \tilde{\mathbf{r}} Y_p Y^p), \end{aligned}$$

where

$$-\frac{1}{4} \mathbf{u} = \alpha = -\frac{1}{2} \psi', \quad -\frac{1}{4} \mathbf{v} = \beta = -\frac{\psi'}{2 \cos \psi}, \quad -\frac{1}{4} \mathbf{w} = \gamma = \frac{\psi'}{2 \sin \psi}, \quad (\text{B.4})$$

$$\mathbf{r} = 2(\psi' \tan \psi)' + 2(\psi')^2, \quad \tilde{\mathbf{r}} = -2(\psi' \cot \psi)' + 2(\psi')^2. \quad (\text{B.5})$$

As we are working with a $U(1)$ gauge group, we will not need the cubic terms in I'' . They are nevertheless listed here for reference, and they will become relevant for extensions to a nonabelian gauge group.

To describe the preserved supersymmetry we follow Gaiotto-Witten and work in a spinor representation where

$$\Gamma_{0123} = -\Gamma_{456789} = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad \Gamma_{3456} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad \Gamma_{3789} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix},$$

where \mathbf{I} is an 8×8 identity matrix. The surviving supersymmetries are those parameterized by a 16-component ε_{16} which takes the form

$$\varepsilon_{16} = \begin{pmatrix} \cos(\frac{\psi}{2}) \varepsilon_8 \\ \sin(\frac{\psi}{2}) \varepsilon_8 \end{pmatrix}, \quad (\text{B.6})$$

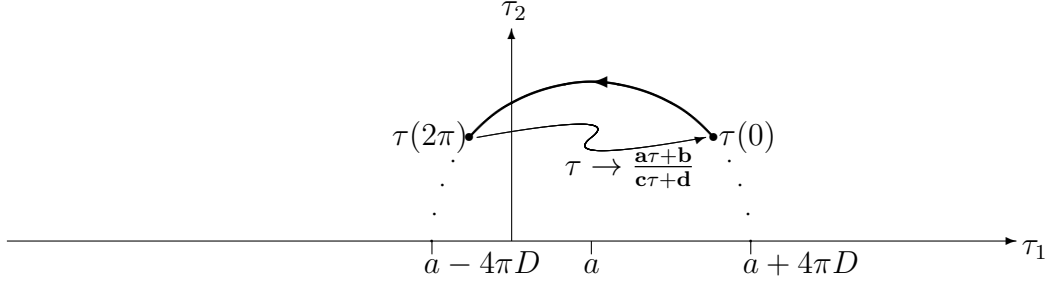


Figure B.1: In the Janus configuration the coupling constant τ traces a portion of a semi-circle of radius $4\pi D$ in the upper-half plane, whose center a is on the real axis. We augment it with an $SL(2, \mathbb{Z})$ duality twist that glues $x_3 = 2\pi$ to $x_3 = 0$.

where ε_8 is an arbitrary constant 8-component spinor.

B.2.2 Introducing an $SL(2, \mathbb{Z})$ -twist

Here ψ is a function of x_3 such that $\tau(x_3)$ traces a geodesic on τ -plane with metric $|d\tau|^2/\tau_2^2$. We pick the parameters a and D so that the semi-circle (B.3) will be invariant under

$$\tau \rightarrow \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}.$$

This amounts to solving the two equations

$$(a - 4\pi D) = \frac{\mathbf{a}(a - 4\pi D) + \mathbf{b}}{\mathbf{c}(a - 4\pi D) + \mathbf{d}}, \quad (a + 4\pi D) = \frac{\mathbf{a}(a + 4\pi D) + \mathbf{b}}{\mathbf{c}(a + 4\pi D) + \mathbf{d}}.$$

The solution is:

$$a = \frac{\mathbf{a} - \mathbf{d}}{2\mathbf{c}}, \quad 4\pi D = \frac{\sqrt{(\mathbf{a} + \mathbf{d})^2 - 4}}{2|\mathbf{c}|},$$

and is real for a hyperbolic element of $SL(2, \mathbb{Z})$ (with $|\mathbf{a} + \mathbf{d}| > 2$). Note that it is important to have both $(a \pm 4\pi D)$ as fixed-points of the $SL(2, \mathbb{Z})$ transformation, so as not to reverse the orientation of the $\tau(x_3)$ curve, and not create a discontinuity in $\tau'(x_3)$. So, given \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} , our configuration is constructed by first calculating a and D , and then picking an arbitrary $\psi(2\pi)$ with a corresponding $\tau(2\pi) = a + 4\pi D e^{2i\psi(2\pi)}$. Next, we calculate the $SL(2, \mathbb{Z})$ dual $\tau(0) = (\mathbf{a}\tau(2\pi) + \mathbf{b})/(\mathbf{c}\tau(2\pi) + \mathbf{d})$ and match it to a point on the semicircle according to $\tau(0) = a + 4\pi D e^{2i\psi(0)}$. The function $\psi(x_3)$ can then be chosen arbitrarily, as long as it connects $\psi(0)$ to $\psi(2\pi)$. It can then be checked that \mathbf{r} and $\tilde{\mathbf{r}}$ are continuous at $x_3 = 2\pi$.

At low-energy, the mass parameters \mathbf{r} and $\tilde{\mathbf{r}}$ in I''' make the scalar fields (X^a and Y^p) massive. Note that in principle, the parameters can be locally negative [although this can be averted by choosing $\psi(x_3)$ so that $\psi'' = 0$], but the effective 2+1D masses, [obtained by solving for the spectrum of the operators $-d^2/dx_3^2 + \mathbf{r}(x_3)$, and $-d^2/dx_3^2 + \tilde{\mathbf{r}}(x_3)$] have to

be positive, since the configuration is supersymmetric and the BPS bound prevents us from having a profile of $X^a(x_3)$ or $Y^p(x_3)$ with negative energy. Similar statements hold for the fermionic masses in I' .

B.2.3 The supersymmetry parameter

As explained in [88], the $SL(2, \mathbb{Z})$ duality transformation acts nontrivially on the SUSY generators. Define the phase φ by

$$e^{i\varphi} = \frac{|\mathbf{c}\tau + \mathbf{d}|}{\mathbf{c}\tau + \mathbf{d}}.$$

Then, the SUSY transformations act on the supersymmetry parameter as

$$\varepsilon \rightarrow e^{\frac{1}{2}\varphi\Gamma_{0123}}\varepsilon.$$

(See equation (2.25) of [88].)

We can now check that

$$\frac{|\mathbf{c}\tau + \mathbf{d}|}{\mathbf{c}\tau + \mathbf{d}} = e^{i(\tilde{\psi} - \psi)}, \quad (\text{B.7})$$

where $\tilde{\psi}$ is defined by

$$\tilde{\tau} \equiv \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}} \equiv a + 4\pi D e^{2i\tilde{\psi}}.$$

It follows from (B.7) that the Gaiotto-Witten phase that is picked up by the supersymmetry parameter as it traverses the Janus configuration from $\eta = 0$ (corresponding to angular variable ψ) to $\eta = 2\pi$ (corresponding to $\tilde{\psi}$) is precisely canceled by the Kapustin-Witten phase of the $SL(2, \mathbb{Z})$ -duality twist. The entire ‘‘Janus plus twist’’ configuration is therefore supersymmetric.

B.2.4 Extending to a type-IIA supersymmetric background

In section §2.4 we assumed that there is a lift of the gauge theory construction to type-IIB string theory and, following a series of dualities, we obtained a type-IIA background with NSNS fields turned on. Here we would like to outline how such a lift might be constructed. We start with the well-known $AdS_3 \times S^3 \times T^4$ type-IIB background, and perform S-duality (if necessary) to get the 3-form flux to be NSNS. Then, take AdS_3 to be of Euclidean signature and replace T^4 with \mathbb{R}^4 , which we then Wick rotate to $\mathbb{R}^{1,3}$. We take the AdS_3 metric in the form

$$ds^2 = \frac{r^2}{r_1 r_5} (-dt^2 + dx_5^2) + \frac{r_1}{r_5} \sum_{i=6}^9 dx_i^2 + \frac{r_1 r_5}{r^2} dr^2 + r_1 r_5 d\Omega_3^2$$

$$H^{(RR)} = \frac{2r_5^2}{g} (\epsilon_3 + {}^*_6\epsilon_3), \quad e^\phi = \frac{gr_1}{r_5}$$

where ϵ_3 is the volume form on the unit sphere, and $*$ is the Hodge dual in the six dimensions x_0, \dots, x_5 (of $AdS_3 \times S^3$), and where r_1, r_5 are constants. (We follow the notation of [90].)

We need to change variables $r \rightarrow x_3$, $t \rightarrow ix_1$ and $x_9 \rightarrow ix_0$, and perform S-duality (where the RHS of arrows are the variables of §2.5). We then compactify directions x_1 and x_2 so that $0 \leq x_i < 2\pi L_i$ ($i = 1, 2$). As a function of x_3 , we define the Kähler modulus of the $x_1 - x_2$ torus to be

$$\rho = i \frac{4\pi^2 r_1^2 L_1 L_2}{x_3^2}$$

Finally, we perform T-duality on direction x_5 to replace ρ with the complex structure τ of the resulting T^2 . In an appropriate limit, this gives a solution where τ goes along a straight perpendicular line in the upper half plane. We can convert it to a semi-circle with an $SL(2, \mathbb{R})$ transformation.

Appendix C

Quadratic Reciprocity, Janus Configurations, and String Duality Twists

C.1 Variants of Quadratic Gauss Sums

Let $p \neq 2$ be a prime and a an integer. Define

$$\chi_p(a) = \sum_{n=0}^{p-1} e^{2\pi i a n^2 / p}, \quad \varrho_p(a) = \sum_{n=0}^{2p-1} e^{\pi i a n^2 / 2p}.$$

What is the relation between the two? We calculate

$$\begin{aligned} \varrho_p(a) &= \sum_{n=0}^{2p-1} e^{\pi i a n^2 / 2p} = \sum_{m=0}^{p-1} e^{\pi i a (2m)^2 / 2p} + \sum_{m=0}^{p-1} e^{\pi i a (2m+1)^2 / 2p} \\ &= \sum_{m=0}^{p-1} e^{2\pi i a m^2 / 2p} + e^{\pi i a / 2p} \sum_{m=0}^{p-1} e^{2\pi i a m(m+1) / p} = \chi_p(a) + e^{\pi i a / 2p} \sum_{m=0}^{p-1} e^{2\pi i a m(m+1) / p}. \end{aligned}$$

To evaluate the second sum we define

$$t_p = \begin{cases} (1-p)/4 & \text{if } p \equiv 1 \pmod{4} \\ (1+p)/4 & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

so that in all cases $4t_p \equiv 1 \pmod{4}$. We can then evaluate

$$\begin{aligned} \sum_{m=0}^{p-1} e^{2\pi i a m(m+1) / p} &= \sum_{m=0}^{p-1} e^{2\pi i (4t_p) a m(m+1) / p} \\ &= e^{2\pi i t_p a / p} \sum_{m=0}^{p-1} e^{2\pi i t_p a [(2m+1)^2 - 1] / p} = e^{-2\pi i t_p a / p} \sum_{m=0}^{p-1} e^{2\pi i t_p a (2m+1)^2 / p} \end{aligned}$$

We now note that as $m = 0, \dots, p-1$ runs over all p values (mod p), the set of numbers $(2m+1)$ also runs over all p values (mod p). So

$$\sum_{m=0}^{p-1} e^{2\pi i t_p a (2m+1)^2 / p} = \sum_{n=0}^{p-1} e^{2\pi i t_p a n^2 / p} = \chi_p(a).$$

So

$$\begin{aligned} \varrho_p(a) &= \chi_p(a) + e^{\pi i a / 2p} \sum_{m=0}^{p-1} e^{2\pi i a m(m+1) / p} = \chi_p(a) + e^{\pi i a / 2p} e^{-2\pi i t_p a / p} \chi_p(a) \\ &= \chi_p(a) \left(1 + e^{\pi i (\frac{1}{2} - 2t_p) a / p} \right) \\ &= \begin{cases} \chi_p(a) \left(1 + e^{\pi i (\frac{1}{2} + \frac{p-1}{2}) a / p} \right) = \chi_p(a) (1 + e^{\pi i a / 2}) & \text{if } p \equiv 1 \pmod{4} \\ \chi_p(a) \left(1 + e^{\pi i (\frac{1}{2} - \frac{p+1}{2}) a / p} \right) = \chi_p(a) (1 + e^{-\pi i a / 2}) & \text{if } p \equiv 3 \pmod{4} \end{cases} \end{aligned}$$

So, we can write

$$\varrho_p(a) = \chi_p(a) (1 + i^{pa}).$$

If a is odd, we can further write this as

$$\varrho_p(a) = \chi_p(a) (1 + i^{pa}) = \sqrt{2} e^{(-1)^{(pa-1)/2} \pi i / 4} \chi_p(a).$$

We also have

$$\chi_p(a) = \left(\frac{a}{p} \right) \chi_p(1)$$

and

$$\chi_p(1) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod{4} \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

Quadratic reciprocity states that

$$\left(\frac{q}{p} \right) = (-1)^{\binom{p-1}{2} \binom{q-1}{2}} \left(\frac{p}{q} \right),$$

or more explicitly,

$$\left(\frac{q}{p} \right) / \left(\frac{p}{q} \right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ and } q \equiv 1 \pmod{4} \\ 1 & \text{if } p \equiv 1 \text{ and } q \equiv 3 \pmod{4} \\ 1 & \text{if } p \equiv 3 \text{ and } q \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \text{ and } q \equiv 3 \pmod{4} \end{cases}$$

In terms of quadratic Gauss sums we get,

$$\frac{\sqrt{q} \chi_p(q)}{\sqrt{p} \chi_q(p)} = \begin{cases} 1 & \text{if } p \equiv 1 \text{ and } q \equiv 1 \pmod{4} \\ -i & \text{if } p \equiv 1 \text{ and } q \equiv 3 \pmod{4} \\ i & \text{if } p \equiv 3 \text{ and } q \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \text{ and } q \equiv 3 \pmod{4} \end{cases}$$

We also have

$$\frac{\varrho_p(q)}{\sqrt{2}\chi_p(q)} = \begin{cases} e^{i\pi/4} & \text{if } p \equiv 1 \text{ and } q \equiv 1 \pmod{4} \\ e^{-i\pi/4} & \text{if } p \equiv 1 \text{ and } q \equiv 3 \pmod{4} \\ e^{-i\pi/4} & \text{if } p \equiv 3 \text{ and } q \equiv 1 \pmod{4} \\ e^{i\pi/4} & \text{if } p \equiv 3 \text{ and } q \equiv 3 \pmod{4} \end{cases}$$

So, altogether we have

$$\frac{\sqrt{q}\varrho_p(q)}{\sqrt{2p}\chi_q(p)} = \begin{cases} e^{i\pi/4} & \text{if } p \equiv 1 \text{ and } q \equiv 1 \pmod{4} \\ -e^{i\pi/4} & \text{if } p \equiv 1 \text{ and } q \equiv 3 \pmod{4} \\ e^{i\pi/4} & \text{if } p \equiv 3 \text{ and } q \equiv 1 \pmod{4} \\ -e^{i\pi/4} & \text{if } p \equiv 3 \text{ and } q \equiv 3 \pmod{4} \end{cases}$$

So,

$$\sqrt{q}\varrho_p(q) = \sqrt{2p}\chi_q(p)e^{i\pi/4}(-1)^{(q-1)/2}$$

The identity

$$\frac{e^{\pi i/4}}{\sqrt{2p}} \sum_{n=0}^{2p-1} e^{-\pi i n^2 q/2p} = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} e^{2\pi i n^2 p/q},$$

is known as the Landsberg-Schaar relation, and holds for all positive odd P and q . We can write it as

$$\sqrt{q}\varrho_p(-q) = \sqrt{2p}e^{-\pi i/4}\chi_q(p)$$

or, taking the complex conjugate,

$$\sqrt{q}\varrho_p(q) = \sqrt{2p}e^{\pi i/4}\chi_q(-p) = \sqrt{2p}e^{\pi i/4}\left(\frac{-1}{p}\right)\chi_q(p) = \sqrt{2p}e^{\pi i/4}(-1)^{(q-1)/2}\chi_q(p)$$

which is the same as the identity above:

$$\sqrt{q}\varrho_p(q) = \sqrt{2p}e^{(2q-1)\pi i/4}\chi_q(p).$$

C.2 Proof of Identity in Equation 3.25

Setting $a = 1 + s_1$ and $b = 1 + s_2$, the identity to be proved is:

$$-\frac{i}{q} \sum_{m,n=0}^{q-1} e^{-\frac{2\pi i}{q}(2mn-am^2-bn^2)} = \left(\frac{1+i^q}{2\sqrt{ab-1}}\right) \sum_{n=0}^{2ab-3} \exp\left(-\frac{\pi i q a n^2}{2(ab-1)}\right), \quad (\text{C.1})$$

for

$$q \in 2\mathbb{Z}_+, \quad a, b \in \mathbb{Z}, \quad ab > 1.$$

we will prove this by discussing two situations, in which $q = 4r$ and $q = 4r + 2$, respectively, where r is an odd integer.

C.2.1 Situation where $q = 4r$

To prove this identity we start with Poisson resummation in two variables:

$$\sum_{m,n \in \mathbb{Z}} f(m, n) = \sum_{x,y \in \mathbb{Z}} \hat{f}(x, y)$$

where the Fourier transform is defined as:

$$\hat{f}(x, y) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(mx+ny)} f(u, v) du dv$$

Take

$$f(u, v) = e^{2\pi i\tau(au^2+bv^2-2uv)}, \quad \text{Im } \tau > 0, \quad a, b > 1.$$

We calculate

$$\begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}^{-1} = \frac{1}{ab-1} \begin{pmatrix} b & 1 \\ 1 & a \end{pmatrix}.$$

Now, set

$$x_0 \equiv \frac{bu+v}{2(ab-1)\tau}, \quad y_0 \equiv \frac{av+u}{2(ab-1)\tau}$$

Then,

$$a\tau x^2 + b\tau y^2 - 2\tau xy - ux - vy = a\tau(x-x_0)^2 + b\tau(y-y_0)^2 - 2\tau(x-x_0)(y-y_0) - \frac{bu^2 + av^2 + 2uv}{4(ab-1)\tau}.$$

So

$$\begin{aligned} \hat{f}(x, y) &= \iint e^{2\pi i(a\tau x^2 + b\tau y^2 - 2\tau xy - ux - vy)} dx dy \\ &= \iint \exp\left[2\pi i\left(a\tau(x-x_0)^2 + b\tau(y-y_0)^2 - 2\tau(x-x_0)(y-y_0) - \frac{bu^2 + av^2 + 2uv}{4(ab-1)\tau}\right)\right] dx dy \\ &= \frac{1}{2\sqrt{-ia\tau}\sqrt{-\frac{i(ab-1)\tau}{a}}} \exp\left[-\frac{\pi i(bu^2 + av^2 + 2uv)}{2(ab-1)\tau}\right] \\ &= -\frac{1}{2i\sqrt{ab-1}} \exp\left[-\frac{\pi i(bu^2 + av^2 + 2uv)}{2(ab-1)\tau}\right] \end{aligned}$$

So, we have

$$-\frac{1}{2i\sqrt{ab-1}} \sum_{m,n \in \mathbb{Z}} \exp\left[-\frac{\pi i(bm^2 + an^2 + 2mn)}{2(ab-1)\tau}\right] = \sum_{m,n \in \mathbb{Z}} e^{2\pi i\tau(am^2 + bn^2 - 2mn)}. \quad (\text{C.2})$$

by changing variables u and v into m and n , respectively.

Now, set

$$\tau = \frac{1}{q} + i\epsilon$$

and take the limit $\epsilon \rightarrow 0$, so that

$$-\frac{1}{\tau} = -q + iq^2\epsilon + O(\epsilon^2).$$

The expression on the RHS of (C.2) can be expanded by setting

$$m = m_1q + m_0, \quad n = n_1q + n_0, \quad m_0, n_0 = 0, \dots, q-1, \quad m_1, n_1 \in \mathbb{Z},$$

so that

$$\exp[2\pi i\tau(am^2 + bn^2 - 2mn)] \approx \exp\left[\frac{2\pi i}{q}(am_0^2 + bn_0^2 - 2m_0n_0)\right] e^{-2\pi q^2\epsilon(am_1^2 + bn_1^2 - 2m_1n_1)}$$

where we ignore terms $2am_0m_1q$, bm_0^2 , $2bn_0n_1q$, bn_0^2 , $-2m_1n_0q$, $-2m_0n_1q$, and $-2m_0n_0$, which are proportional to q and 1, in the exponent of the second factor, because both m_0 and n_0 are of order q at most.

Then

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i\tau(am^2 + bn^2 - 2mn)} = \left(\sum_{m,n=0}^{q-1} e^{\frac{2\pi i}{q}(am^2 + bn^2 - 2mn)} \right) \sum_{m,n \in \mathbb{Z}} e^{-2\pi q^2\epsilon(am^2 + bn^2 - 2mn)}$$

In the limit $\epsilon \rightarrow 0$, the leftmost sum can be evaluated by converting into an integral with $u = m\sqrt{\epsilon}$ and $v = n\sqrt{\epsilon}$:

$$\lim_{\epsilon \rightarrow 0} \sum_{m,n \in \mathbb{Z}} e^{-2\pi q^2\epsilon(am^2 + bn^2 - 2mn)} \approx \frac{1}{\epsilon} \iint e^{-2\pi q^2\epsilon(am^2 + bn^2 - 2mn)} dudv = \frac{1}{2\epsilon q^2 \sqrt{ab-1}}$$

So, we have

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i\tau(am^2 + bn^2 - 2mn)} \approx \frac{1}{2\epsilon q^2 \sqrt{ab-1}} \left(\sum_{m,n=0}^{q-1} e^{\frac{2\pi i}{q}(am^2 + bn^2 - 2mn)} \right)$$

To approximate the LHS of (C.2), we need to perform a similar manipulation to the double-sum

$$\sum_{m,n \in \mathbb{Z}} \exp\left[-\frac{\pi i(bm^2 + an^2 + 2mn)}{2(ab-1)\tau}\right].$$

It would help to know the Smith Normal Form of the matrix $\begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}$. This matrix is related to the inverse of the quadratic form in the exponent by

$$\begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix} = (ab-1) \begin{pmatrix} b & 1 \\ 1 & a \end{pmatrix}^{-1}.$$

The Smith Normal Form is:

$$\begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ab-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}$$

We want to convert the sum over $(m, n) \in \mathbb{Z}^2$ into a sum over the finite points in the fundamental cell generated by the columns of $\begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}$, times a sum over the lattice points generated by these columns. So, we want to write (m, n) as

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m_0 \\ n_0 \end{pmatrix} + \begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} m_0 + am_1 - n_1 \\ n_0 + bn_1 - m_1 \end{pmatrix},$$

where (m_0, n_0) have $(ab - 1)$ possible values. Replacing

$$\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$$

we can write

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m_0 \\ n_0 \end{pmatrix} + \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ab-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}.$$

Setting

$$\begin{pmatrix} m_0 \\ n_0 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j \\ 0 \end{pmatrix}, \quad j = 0, \dots, ab - 2,$$

we find

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ab-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} j + m_1(ab-1) - n_1a \\ n_1 \end{pmatrix}.$$

So, we set $n = n_1$ and

$$m = j + (ab - 1)k - na, \quad j = 0, \dots, ab - 2$$

Then,

$$bm^2 + an^2 + 2mn = bj^2 + (ab - 1)[a(bk - n)^2 - b(k - j)^2 + 2(k - j)n + bj^2].$$

Also,

$$\begin{aligned} & \sum_{m, n \in \mathbb{Z}} \exp \left[-\frac{\pi i (bm^2 + an^2 + 2mn)}{2(ab-1)\tau} \right] \\ &= \sum_{j=0}^{ab-2} \left(e^{-\frac{\pi i bj^2}{2(ab-1)\tau}} \sum_{n, k \in \mathbb{Z}} \exp \left[-\frac{\pi i [a(bk - n)^2 - b(k - j)^2 + 2(k - j)n + bj^2]}{2\tau} \right] \right) \end{aligned}$$

In the limit $\tau = \frac{1}{q} + i\epsilon$ with $\epsilon \rightarrow 0$, we can approximate

$$\begin{aligned} & -\frac{\pi i}{2\tau} [a(bk - n)^2 - b(k - j)^2 + 2(k - j)n + bj^2] \\ & \approx -\frac{\pi i}{2} q [a(bk - n)^2 - b(k - j)^2 + 2(k - j)n + bj^2] - \frac{\pi}{2} q^2 \epsilon [a(bk - n)^2 - bk^2 + 2kn] \end{aligned} \quad (\text{C.3})$$

Because $4|q$, the first term on the last line is an integer multiple of $2\pi i$ and can be dropped. And so, we are left with

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} \exp \left[-\frac{\pi i (bm^2 + an^2 + 2mn)}{2(ab-1)\tau} \right] \\ & = \left(\sum_{j=0}^{ab-2} e^{-\frac{\pi i q b j^2}{2(ab-1)}} \right) \lim_{\epsilon \rightarrow 0} \sum_{n,k \in \mathbb{Z}} e^{-\frac{\pi}{2} q^2 \epsilon [a(bk-n)^2 - bk^2 + 2kn]} \end{aligned}$$

The limit $\epsilon \rightarrow 0$ can be evaluated by converting to an integral with $u = k\sqrt{\epsilon}$ and $v = n\sqrt{\epsilon}$ we get

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \sum_{n,k \in \mathbb{Z}} e^{-\frac{\pi}{2} q^2 \epsilon [a(bk-n)^2 - bk^2 + 2kn]} \\ & = \frac{1}{\epsilon} \iint e^{-\frac{\pi}{2} q^2 [a(bu-v)^2 - bu^2 + 2uv]} du dv = \frac{2}{q^2(ab-1)\epsilon} \end{aligned}$$

Finally, we need to show that

$$\sum_{j=0}^{ab-2} e^{-\frac{\pi i q b j^2}{2(ab-1)}} = \sum_{j=ab-1}^{2ab-3} e^{-\frac{\pi i q b j^2}{2(ab-1)}}$$

We consider the pairings between exponents with forms

$$-\frac{q b j^2}{ab-1} \frac{\pi i}{2} \quad \text{and} \quad -\frac{q b (ab-1+j)^2}{ab-1} \frac{\pi i}{2}, \quad j = 0, \dots, ab-2$$

then the difference between j^2 and $(ab-1+j)^2$ is $a^2b^2 - 2ab + 1 + 2j(ab-1) = (ab-1)^2 + 2j(ab-1)$, so the difference between two exponents is $(ab-1+2j)\frac{q b}{2}\pi i$, which is an integer multiple of $2\pi i$ because $4|q$. So the summands can be paired by the 1-to-1 correspondence between j and $ab-1+j$. At this point we have:

$$\frac{1}{-2i\sqrt{ab-1}} \sum_{m,n \in \mathbb{Z}} \exp \left[-\frac{\pi i (bm^2 + an^2 + 2mn)}{2(ab-1)\tau} \right] = \sum_{m,n \in \mathbb{Z}} e^{2\pi i \tau (am^2 + bn^2 - 2mn)}.$$

and

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i \tau (am^2 + bn^2 - 2mn)} \rightarrow \frac{1}{2\epsilon q^2 \sqrt{ab-1}} \left(\sum_{m,n=0}^{q-1} e^{\frac{2\pi i}{q} (am^2 + bn^2 - 2mn)} \right)$$

and

$$\sum_{m,n \in \mathbb{Z}} \exp \left[-\frac{\pi i (bm^2 + an^2 + 2mn)}{2(ab-1)\tau} \right] \rightarrow \frac{2}{q^2(ab-1)\epsilon} \sum_{j=0}^{ab-2} e^{-\frac{\pi i q b j^2}{2(ab-1)}} = \frac{1}{q^2(ab-1)\epsilon} \sum_{j=0}^{2ab-3} e^{-\frac{\pi i q b j^2}{2(ab-1)}}$$

By inspection, we can see that (C.1) holds if $4|q$.

C.2.2 Situation where $q = 4r + 2 = 2s$

C.2.2.1 If b is odd

In this case, the RHS of (C.1) is obviously 0. Hence we need to show that the LHS of (C.1) also vanishes. After trying two cases: $(q, a, b) = (6, 2, 3)$ and $(6, 2, 5)$, we hypothesize that for fixed q and m ,

$$\sum_{n=0}^{q-1} e^{-\frac{2\pi i}{q} (2mn - am^2 - bn^2)} = 0.$$

We need to show that with m fixed, for every n , there is an $n' = n + t$ such that $am^2 + bn^2 - 2mn$ and $am^2 + b(n+t)^2 - 2m(n+t)$ differ by an odd multiple of s , hence $e^{\frac{\pi i}{s} (am^2 + bn^2 - 2mn)} + e^{\frac{\pi i}{s} (am^2 + bn'^2 - 2mn')} = 0$.

Calculate $am^2 + b(n+t)^2 - 2m(n+t) - (am^2 + bn^2 - 2mn) = bt^2 + 2bnt - 2mt$, and set it to be $\pm \delta s$. Then we have a quadratic equation $bt^2 + (2bn - 2m)t \pm \delta s = 0$, and its solutions are $-n + \frac{m \pm \sqrt{(bn-m)^2 \pm \delta s}}{b}$. Let the discriminant Δ be equal to x^2 , then one particular solution admits

$$(bn - m)^2 - \delta bs = (bn - m + x)(bn - m - x) = \delta bs. \quad (\text{C.4})$$

We also denote $\frac{m \pm \sqrt{\Delta}}{b} = y$, leading to $m = by \pm x$. Then (C.4) becomes

$$(n - y)(bn - by - 2x) = \delta s$$

or

$$(n - y)(bn - by + 2x) = \delta s$$

In order to make the pairing between n and n' stable for all n , we need to impose $n - y$, a solution to the above mentioned quadratic equation, to be $s = q/2$ (diagonally, with respect to the polygon whose vertices are $e^{-\frac{2\pi i}{q} (2mn - am^2 - bn^2)}$). So

$$bn - by - 2x = bs - 2x = \delta$$

or

$$bn - by + 2x = bs + 2x = \delta$$

Since δ , b and s are all odd, b has to be odd. Hence, for $q = 4r + 2$ and odd b , both sides of (C.1) are zero.

C.2.2.2 If b is even and a is odd

In this case, the first term in (C.3) expands as

$$s\pi i[ab^2k^2 - 2abnk + an^2 - bk^2 + 2bjk + 2(k - j)n]$$

Since all terms in the square brackets, except for an^2 , are even, the overall contributing exponent provided by (C.3) is

$$-\frac{\pi}{2}q^2\epsilon[a(bk - n)^2 - bk^2 + 2kn] - asn^2\pi i$$

So the sum concerning n and k following (C.3) becomes

$$\lim_{\epsilon \rightarrow 0} \sum_{n, k \in \mathbb{Z}} e^{-\frac{\pi}{2}q^2\epsilon[a(bk - n)^2 - bk^2 + 2kn]} e^{-asn^2\pi i}$$

Notice that the second factor is just $(-1)^n$. We evaluate the first factor by converting into an integral by change of variable $u = k\sqrt{\epsilon}$ and $v = n\sqrt{\epsilon}$:

$$\frac{1}{\epsilon} \iint e^{-\frac{\pi}{2}q^2[a(bu - v)^2 - bu^2 + 2uv]} dudv = \frac{1}{q} \sqrt{\frac{2}{b(ab - 1)\epsilon}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi q^2 \epsilon n^2}{2b}} \quad (\text{C.5})$$

Then we consider the sum:

$$\sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{\pi q^2 \epsilon n^2}{2b}}$$

This should be zero because this is an alternating Riemann sum, and the decrease of the exponential is $\sim \epsilon$, while the denominator of the prefactor in (C.5) goes as $\sim \frac{1}{\epsilon}$. Notice that this result agrees with the symmetry between a and b which is manifest on the LHS of (C.1), since both sides are zero if b is odd as shown above.

C.2.2.3 If both a and b are even

Then again the first term in (C.3) is an integer multiple of $2\pi i$, and the remaining argument coincide with that in the previous section.