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Mathematics

Banach–Tarski paradox using pieces with the property of Baire

(equidecomposable sets/paradoxical decomposition)

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ABSTRACT In 1924 Banach and Tarski, using ideas of Hausdorff, proved that there is a partition of the unit sphere S^2 into sets $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ and a collection of isometries $\{\sigma_1, \ldots, \sigma_k, \rho_1, \ldots, \rho_\ell\}$ so that $\{\sigma_1 A_1, \ldots, \sigma_k A_k\}$ and $\{\rho_1 B_1, \ldots, \rho_\ell B_\ell\}$ both are partitions of S^2. The sets in these partitions are constructed by using the axiom of choice and cannot all be Lebesgue measurable. In this note we solve a problem of Marczewski from 1930 by showing that there is a partition of S^2 into sets $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ with a different strong regularity property, the *Property of Baire*. We also prove a version of the Banach–Tarski paradox that involves only *open* sets and does not use the axiom of choice.

In 1924, Banach and Tarski (1), using ideas of Hausdorff (2) proved a very striking theorem:

THEOREM (THE BANACH–TARSKI PARADOX). *For* $n \geq 2$, *there is a partition* $\{A_1, \ldots, A_k, B_1, \ldots, B_\ell\}$ *of the unit sphere* S^n *and a collection of isometries* $\{\sigma_1, \ldots, \sigma_k, \rho_1, \ldots, \rho_\ell\}$ *of* S^n, *so that* $\{\sigma_1 A_1, \ldots, \sigma_k A_k\}$ *and* $\{\rho_1 B_1, \ldots, \rho_\ell B_\ell\}$ *are both partitions of* S^n.

As a corollary:

COROLLARY. *If* $n \geq 3$ *and* A *and* B *are bounded subsets of* $\mathbb{R}^n$ *with nonempty interior, then there is a partition of* A *into* $\{A_1, \ldots, A_k\}$ *and a collection of isometries of* $\mathbb{R}^n$, $\{\rho_1, \ldots, \rho_k\}$, *so that* $\{\rho_1 A_1, \ldots, \rho_k A_k\}$ *partitions* B.

Colloquially, this states that one can "cut up a pea and rearrange the pieces to get the sun."

Because A and B typically do not have the same volume, the pieces in the partition must not have a well-defined volume; in particular, they are not all Lebesgue measurable. In ref. 3 (assuming the consistency of "ZFC+ there is an inaccessible cardinal"), Solovay shows that it is consistent with Zermelo–Fraenkel set theory with the countable axiom of choice that *every* subset of $\mathbb{R}^n$ is Lebesgue measurable. Hence the controversial (uncountable) axiom of choice is required to construct the partition used in the corollary and thus the sets must be extremely wild in this sense.

There is, however, another well-behaved countably complete boolean algebra (σ-algebra) of subsets of $\mathbb{R}^n$, the sets with the *property of Baire*. We now define this notion in a somewhat abstract setting.

A topological space X is *Polish* if the topology admits a complete separable metric. A closed set $K \subseteq X$ is *nowhere dense* iff it contains no nonempty open set. A set $B \subseteq X$ is *meager* if it is included in a countable union of closed nowhere-dense sets. If B is meager, then $X \backslash B$ is *comeager*. A set $Y \subseteq X$ has the property of Baire iff it belongs to the smallest σ-algebra containing the Borel sets and the meager sets. Note that in $\mathbb{R}^n$ this is analogous to the algebra of Lebesgue-measurable sets—i.e., the smallest σ-algebra containing the Borel sets and the Lebesgue measure zero sets. The *Baire category theorem* implies that no nonempty Polish space is meager. A standard fact (4) is that for each $Y \subseteq X$ with the property of Baire, there is an open set $O \subset X$ so that the symmetric difference $Y \triangle O$ is meager. If $A, B \in \mathcal{B} \subseteq \mathcal{P}(X)$, and G is a group acting on X, we say that A is *equidecomposable with* B *using pieces in* $\mathcal{B}$ *and elements of* G ($A \approx_{\mathcal{B}} B$ with respect to G) iff there is a partition of A into $\{A_1, \ldots, A_n\}$ with $A_i \in \mathcal{B}$ and $\{\gamma_1, \ldots, \gamma_n\} \subseteq G$ so that $\{\gamma_1 A_1, \ldots, \gamma_n A_n\}$ is a partition of B. We omit $\mathcal{B}$ if $\mathcal{B} = \mathcal{P}(X)$ and we omit G if it is clear from context. We say $A \approx_{\mathcal{B}} 2B$ iff there is a partition of A, $\{A_1, A_2\}$ with each $A_i \approx_{\mathcal{B}} B$. A is *paradoxical* iff $A \approx 2A$. A group G acting on a set X is said to act *freely* on $Y \subseteq X$ iff whenever $\gamma \in G$ and $y \in Y$, $\gamma y = y$ implies $\gamma = e$.

Our main result is that the Banach–Tarski paradox can be performed using pieces with the property of Baire:

THEOREM 1. *Let* $n \geq 2$. *Then* $S^n \approx 2S^n$ *using isometries and pieces with the property of Baire*.

This solves a problem posed by E. Marczewski in 1930 (see ref. 5).

COROLLARY 1. *Let* $n \geq 3$ *and* A *and* B *be bounded subsets of* $\mathbb{R}^n$ *with nonempty interior. Then* $A \approx B$ *using sets with the property of Baire and isometries of* $\mathbb{R}^n$.

COROLLARY 2. *Let* $n \geq 3$, *and* A *and* B *be nonempty bounded open subsets of* $\mathbb{R}^n$. *Then there is a pairwise disjoint collection* $\{A_1, \ldots, A_k\}$ *of* open *subsets of* A *whose union is dense in* A *and a collection* $\{\rho_1, \ldots, \rho_k\}$ *of isometries of* $\mathbb{R}^n$ *so that* $\{\rho_1 A_1, \ldots, \rho_k A_k\}$ *is a pairwise disjoint collection of open subsets of* B *whose union is dense in* B.

COROLLARY 3. *Let* $n \geq 2$. *There is no rotation-invariant finitely additive probability measure on the Borel subsets of* S^n *giving meager sets measure zero*.

We note that *Corollary 2* is proven entirely constructively, with no axiom of choice. In the "pea–sun" metaphor, it says that there is a collection $O_1, \ldots, O_n$ of disjoint open subsets of the sun that fill the sun (in the sense that there are no "holes" of positive radius) and that can be rearranged by rigid motions to remain disjoint and fit inside a pea. Our other results are as follows:

THEOREM 2. *If* $n \geq 2$ *and* $A, B \subset S^n$ *have nonempty interior and the property of Baire, then* $A \approx B$ *using sets with the property of Baire and elements of* $SO(n + 1)$.

THEOREM 3. *Let* $n \geq 2$. *For each* $N \geq 3$, *there is a partition* $\{A_1, \ldots, A_N\}$ *of* S^n *into congruent pieces with the property of Baire. Further, if* $1 \leq i < j < k \leq N$, *then there are rotations* ρ_1, ρ_2, ρ_3 *so that* $\{\rho_1 A_i, \rho_2 A_j, \rho_3 A_k\}$ *partitions* S^n.

THEOREM 4. *The sphere* S^2 *and the unit ball in* $\mathbb{R}^3$ *each have paradoxical decompositions using 6 pieces with the property of Baire*. [Wehrung (personal communication) previously showed that one needs at least 6 pieces.]

These results will appear in a forthcoming paper of the authors.

MAIN LEMMA

We now state our main lemma, prove the Banach–Tarski paradox, and use the proof to reduce *Theorem 1* to the *Main Lemma*:

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MAIN LEMMA. *Suppose* X *is a separable metric space and* G *is a countable group of homeomorphisms of* X *acting freely on* X. *Suppose that* $\{\rho_i, \gamma_i: 1 \leq i \leq 3\} \subseteq G$ *generate a free subgroup of* G *of rank 6. Then there are disjoint open sets* $\{R_i, G_i: 1 \leq i \leq 3\}$ *so that* $\cup_{1\leq i\leq 3}\rho_i R_i$ *and* $\cup_{1\leq i\leq 3}\gamma_i G_i$ *are dense open subsets of* X.

Let G be a group acting on X and $\mathscr{B} \subseteq \mathscr{P}(X)$. For $A, B \in \mathscr{B}$ define $A \leq_{\mathscr{B}} B$ (with respect to G) iff there is $B' \subseteq B$ with $A \approx_{\mathscr{B}} B'$. Banach noticed that the usual Schröder–Bernstein proof works in this context (see ref. 5, pp. 25–26 and 116): Let $\mathscr{B} = \mathscr{P}(X)$ or (if X is a Polish space and G is a group of homeomorphisms) $\mathscr{B} = \{Y \subseteq X: Y$ has the property of Baire$\}$ and suppose that $A \leq_{\mathscr{B}} B$ and $B \leq_{\mathscr{B}} A$, then $A \approx_{\mathscr{B}} B$.

Our first step in reducing *Theorem 1* (for $n = 2$) to the *Main Lemma* is to note that if $\mathscr{F} \subseteq S^2$ is countable, then $S^2 \approx_{\mathscr{B}} S^2 \backslash \mathscr{F}$ for any rotation-invariant σ-algebra $\mathscr{B}$ containing all singletons. To see this, choose a line ℓ through the origin, whose intersection with S^2 doesn't meet $\mathscr{F}$. Choose a rotation ρ about ℓ so that for all natural numbers $m > 0$, $\rho^m\mathscr{F} \cap \mathscr{F} = \varnothing$. (Since $\mathscr{F}$ is countable the collection of such rotations ρ about ℓ is cocountable.) Then: $S^2 = (S^2\backslash \cup_{m\geq 0}\rho^m\mathscr{F}) \cup (\cup_{m\geq 0}\rho^m\mathscr{F}) \approx_{\mathscr{B}} (S^2\backslash\cup_{m\geq 0}\rho^m\mathscr{F}) \cup \rho(\cup_{m\geq 0}\rho^m\mathscr{F}) = (S^2\backslash\cup_{m\geq 0}\rho^m\mathscr{F}) \cup \cup_{m\geq 1}\rho^m\mathscr{F} = S^2\backslash\mathscr{F}$.

Now choose rotations ϕ, ψ of S^2 that generate a free group on two generators. For $v \in \{\phi, \phi^{-1}, \psi, \psi^{-1}\}$, let $W_v = \{w \in \langle\phi, \psi\rangle$: w is a reduced work beginning with $v\}$. Then for $v \neq v'$, $W_v \cap W_{v'} = \varnothing$ and $\langle\phi, \psi\rangle = W_\phi \cup \phi W_{\phi^{-1}} = W_\psi \cup \psi W_{\psi^{-1}}$. Let $F = \{s \in S^2$: s is fixed by some element of $\langle\phi, \psi\rangle\}$, and let $Y \subset S^2\backslash F$ be $\langle\phi, \psi\rangle$-invariant. For each $\langle\phi, \psi\rangle$-orbit $\mathscr{O}$ of Y, choose $x_{\mathscr{O}} \in \mathscr{O}$. For $v \in \{\phi, \phi^{-1}, \psi, \psi^{-1}\}$, let $Y_v = \cup\{W_v x_{\mathscr{O}}$: $\mathscr{O}$ is an orbit$\}$. Since $\langle\phi, \psi\rangle$ acts freely on $S^2\backslash F$, for $v \neq v'$ we have $Y_v \cap Y_{v'} = \phi$ and $Y = Y_\phi \cup \phi Y_{\phi^{-1}} = Y_\psi \cup \psi Y_{\psi^{-1}}$. Hence $Y \approx Y_\phi \cup Y_{\phi^{-1}} \leq Y\backslash(Y_\psi \cup Y_{\psi^{-1}}) \leq Y$, and thus the Banach–Schröder–Bernstein theorem implies that $Y \approx Y_\psi \cup Y_{\psi^{-1}} \approx Y\backslash(Y_\psi \cup Y_{\psi^{-1}})$, or $Y \approx 2Y$. In particular, if $Y = S^2\backslash F$, we have $S^2 \approx Y \approx 2Y \approx 2S^2$ (the Banach–Tarski paradox).

To prove *Theorem 1* from the *Main Lemma*, let $X = S^2\backslash F$ with the inherited topology. Then X is a G_δ subset of S^2 and hence is a Polish space with the subspace topology (6). Further, $\langle\phi, \psi\rangle$ acts freely on X by homeomorphisms. Let $\{\rho_i, \gamma_i: 1 \leq i \leq 3\}$ be free generators of a free subgroup of $\langle\phi, \psi\rangle$ of rank 6. Applying the *Main Lemma*, we get pairwise disjoint open sets $\{R_i, G_i: 1 \leq i \leq 3\}$. By "shrinking" the R_is and G_is we can assume that both $\{\rho_1R_1, \rho_2R_2, \rho_3R_3\}$ and $\{\gamma_1G_1, \gamma_2G_2, \gamma_3G_3\}$ are pairwise disjoint collections of sets. Let $B = \cup_{1\leq i\leq 3}\rho_iR_i \cap \cup_{1\leq i\leq 3}\gamma_iG_i$ and $D = \cap_{g\in\langle\phi,\psi\rangle} g(B)$. Then D is a $\langle\phi, \psi\rangle$-invariant comeager set. Letting $R_i' = R_i \cap D$ and $G_i' = G_i \cap D$, we find $D = \cup_{1\leq i\leq 3}\rho_iR_i' = \cup_{1\leq i\leq 3}\gamma_iG_i'$. Let $\mathscr{B}$ be the collection of sets with the property of Baire. Then $D \approx_{\mathscr{B}} \cup_{1\leq i\leq 3}R_i' \leq_{\mathscr{B}} (D\backslash\cup_{1\leq i\leq 3}G_i') \leq_{\mathscr{B}} D$, so by an application of the Banach–Schröder–Bernstein theorem, $(D\backslash\cup_{1\leq i\leq 3}G_i') \approx_{\mathscr{B}} D \approx_{\mathscr{B}} \cup_{1\leq i\leq 3}G_i'$. Hence $D \approx_{\mathscr{B}} 2D$. On the other hand, $Y = S^2\backslash(D \cup F)$ is $\langle\phi, \psi\rangle$-invariant and meager. Hence $Y \approx 2Y$ and, since Y is meager, the pieces used trivially have the property of Baire. Putting all of this together, we see

$$S^2 \approx_{\mathscr{B}} S^2\backslash F = D \cup S^2\backslash(D \cup F) \approx_{\mathscr{B}} 2D \cup 2(S^2\backslash(D \cup F)) = 2(S^2\backslash F) \approx_{\mathscr{B}} 2S^2.$$

This proves *Theorem 1*.

PROOF OF *MAIN LEMMA*

We now prove the *Main Lemma*. Fix a countable dense G-invariant subset D in X, and endow it with the subspace topology. Then G acts on D by homeomorphisms. It suffices to build pairwise disjoint open subsets of D, $\{R_i, G_i: 1 \leq i \leq 3\}$ so that $\cup_{1\leq i\leq 3}\rho_iR_i$ and $\cup_{1\leq i\leq 3}\gamma_iG_i$ are dense in D. (Then, if we let R_i' be the interior of the closure of R_i in X and G_i' be the interior of the closure of G_i in X, $\{R_i', G_i': 1 \leq i \leq 3\}$ witness the lemma for X.)

Consider the subgroup H of G generated by $P = \{\rho_i^{-1} \circ \rho_j: 1 \leq i \neq j \leq 3\} \cup \{\gamma_i^{-1} \circ \gamma_j: 1 \leq i \neq j \leq 3\}$. On each H-orbit of D, put the undirected Cayley diagram of H with respect to the presentation P. So, for x and y in D, x is connected to y iff for some $i \neq j$, $\rho_i^{-1} \circ \rho_j(x) = y$ or $\gamma_i^{-1} \circ \gamma_j(x) = y$. A triple (x, y, z) is an (i, j, k)-red triangle iff $\rho_j^{-1} \circ \rho_i(x) = y$ and $\rho_k^{-1} \circ \rho_i(x) = z$. Define an (i, j, k)-green triangle similarly, using γ_i, γ_j, and γ_k. Since H acts freely on D and is a free group, each connected component of the Cayley graph on D consists of red and green triangles linked at the vertices and no simple cycles except these triangles. In particular, if x and y are in the same connected component then there is a unique path from x to y of minimal length (which we call the *graph distance* from x to y) and for each w and each triangle (x, y, z) in the component of w there is a unique vertex of that triangle closest to w. Note that each path determines a word ω in the letters P and the graph distance from x to y is the length of the unique shortest word ω (written in the letters P) so that $\omega(x) = y$. (We will call these minimal words "reduced"—they are the words in P that are reduced in the group $\langle\rho_i, \gamma_i: 1 \leq i \leq 3\rangle$.)

LEMMA 2. *Let* $x_0 \in D$ *and* $m \in \mathbb{N}$. *Suppose that* $A_1, \ldots, A_n$ *is a partition of* D *into open sets. Then there is a neighborhood* O *of* x_0 *so that for all* $y \in O$, *all* $i \leq n$, *and all words* $\omega \in H$ *(expressed in the alphabet* P*) of length less than or equal to* m, $\omega x_0 \in A_i$ *iff* $\omega y \in A_i$.

Proof: Let $O = \cap\{\gamma^{-1}(A_i)$: length $\gamma \leq m$ and $\gamma(x_0) \in A_i\}$.

To prove the *Main Lemma*, we must construct R_i, G_i so that they can be rearranged into a "red" open dense set and a "green" open dense set. So the main task is to arrange density and maintain the disjointness of $\{R_i, G_i: 1 \leq i \leq 3\}$. Fix $\langle O_n: n \in \mathbb{N}\rangle$ a basis for the topology on D. We now "color" the points in D into open colors $\{R_i, G_i: 1 \leq i \leq 3\}$ so that for each n, $\cup\rho_iR_i \cap O_n \neq \varnothing$ and $\cup\gamma_iG_i \cap O_n \neq \varnothing$. Viewing each O_n as two "tasks," we build sequences of open sets $R_i^0 \subseteq R_i^1 \subseteq \cdots \subseteq R_i^n \subseteq R_i^{n+1} \subseteq \ldots$ and $G_i^0 \subseteq G_i^1 \subseteq \cdots \subseteq G_i^n \subseteq G_i^{n+1} \subseteq \ldots$ so that $\{R_i^{n+1}, G_i^{n+1}: 1 \leq i \leq 3\}$ accomplishes the nth task; i.e., $\cup\rho_iR_i^{2n+1} \cap O_n \neq \varnothing$ and $\cup\gamma_iG^{2n+2} \cap O_n \neq \varnothing$. (Setting $R_i = \cup_nR_i^n$ and $G_i = \cup_nG_i^n$ then suffices.) There may be an obstacle, however: trying to arrange that $O \cap \cup_{1\leq i\leq 3}\rho_iR_i^{n+1} \neq \varnothing$, it may occur that for each i, $\rho_i^{-1}(O) \subseteq \cup_{i'\neq i}R_{i'}^n \cup \cup_iG_i^n$. It would then be impossible to add an open subset of any $\rho_i^{-1}(O)$ to R_i^n and maintain disjointness. We thus have an induction hypothesis preventing this. Note that if $w \in O$ and $x = \rho_i^{-1}(w)$, $y = \rho_j^{-1}(w)$ and $z = \rho_k^{-1}(w)$, then (x, y, z) is an (i, j, k)-red triangle. If O were the obstacle described then $x \in \cup_{i'\neq i}R_{i'}^n \cup \cup_iG_i^n$, $y \in \cup_{i'\neq j}R_{i'}^n \cup \cup_iG_i^n$, and $z \in \cup_{i'\neq k}R_{i'}^n \cup \cup_iG_i^n$. This is called a *bad* (i, j, k)-red triangle. [Bad (i, j, k)-green triangles are defined similarly with γs replacing ρs and the Gs and Rs interchanged.] Our induction hypothesis ensures that there are no bad triangles. In fact, we will demand the stronger requirement that if $x \in \cup_{i'\neq i}R_{i'}^n \cup \cup_iG_i^n$ and $y \in \cup_{i'\neq j}R_{i'}^n \cup \cup_iG_i^n$, then $z \in R_k^n$, and of course, similarly for green triangles. We call triangles that violate this stronger hypothesis "delinquent."

Presented with a "red" task O at stage n, if there is no delinquent triangle, it is possible to add some x_0 to some R_i^n with $x_0 \in \rho_i^{-1}(O)$. But adding this to R_i^n (to be an element of R_i^{n+1}) threatens to create new delinquent triangles, so we must add further points to the $R_{i'}$s and G_is adjacent to x in the Cayley graph. This creates further complications. We must show that no contradictory requirements are created (by induction on graph distance) and that we can "blow up" x_0 to an open neighborhood which exactly mimics the behavior of x_0 and hence can be added to R_i^n with the resulting consequences.

The induction hypotheses on the sets $\{R_i^n, G_i^n: 1 \leq i \leq 3\}$ are:

Hypothesis 1: For each $i, n \geq 1$, R_i^n and G_i^n are open subsets of D with no boundary, and $\{R_i^n, G_i^n: 1 \leq i \leq 3\}$ is pairwise disjoint, and $R_i^{n-1} \subseteq R_i^n$, $G_i^{n-1} \subseteq G_i^n$.

Hypothesis 2: For $n = 2j$ (resp. $n = 2j + 1$), $\cup_i \rho_i R_i^{n+1} \cap O_j \neq \emptyset$ (resp. $\cup_i \gamma_i G_i^{n+1} \cap O_j \neq \emptyset$).

Hypothesis 3: Let $R_i^* = \cup_{i' \neq i} R_{i'}^n \cup \cup_i G_i^n$ and $G_i^* = \cup_{i' \neq i} G_{i'}^n \cup \cup_i R_i^n$. If (x, y, z) is an (i, j, k)-red triangle and $x \in R_i^*$, $y \in R_j^*$, then $z \in R_k^n$, and similarly for the Gs and (i, j, k)-green triangles.

Define the *live* Cayley graph at stage n to be the edge-subgraph consisting of all edges adjacent to a vertex in $\cup_{1 \leq i \leq 3} R_i^n \cup G_i^n$.

Hypothesis 4: The *live* Cayley graph at stage n has finite connected components.

The following clearly suffices to prove the *Main Lemma*:

Main Claim. There is a sequences of open sets $\{R_i^n, G_i^n: n \in \mathbb{N}, 1 \leq i \leq 3\}$ so that for each n, *Hypotheses 1–4* hold.

We construct the sequence by induction on n. For $n = 0$, let $R_i^0 = G_i^0 = \emptyset$.

Suppose we have $\{R_i^n, G_i^n: 1 \leq i \leq 3\}$ satisfying *Hypotheses 1–4*. By symmetry we may assume $n = 2j$. Choose $w \in O_j$. The triple $(\rho_1^{-1}(w), \rho_2^{-1}(w), \rho_3^{-1}(w))$ is a (1,2,3)-red triangle, so by *Hypothesis 3*, for some $\bar{\imath}$, $x_0 = \rho_{\bar{\imath}}^{-1}(w) \notin R_{\bar{\imath}}^*$. Let C be the (finite) connected component of x_0 in the live Cayley graph. We define by induction on graph distance, a pairwise disjoint collection of subsets of C, $\{\hat{R}_i, \hat{G}_i: 1 \leq i \leq 3\}$.

Let $x_0 \in \hat{R}_{\bar{\imath}}$. Suppose $m \geq 1$ and we have defined $\hat{R}_i$ and $\hat{G}_i$ restricted to points of distance $\leq m - 1$ from x_0. Each point z in C of distance m from x_0 belongs to a unique triangle $\{x, y, z\}$ with vertex x of distance $m - 1$ from x_0. If (x, y, z) is an (i, j, k)-red triangle and $x \in \cup_{i' \neq i} \hat{R}_{i'} \cup \cup_i \hat{G}_i$ and $y \in R_j^*$, then we put z into $\hat{R}_k$. [If (x, y, z) is an (i, j, k)-green triangle we do analogous things with the role of the Rs and Gs reversed.] Since C is finite this process terminates. An induction on graph distance shows that at stage k in the construction of $\{\hat{R}_i, \hat{G}_i: 1 \leq i \leq 3\}$, the $\hat{R}_i$s and $\hat{G}_i$s are pairwise disjoint, there are no bad triangles, and every triangle of whose vertices of distance at most k from x_0 is not delinquent.

Let C' be the union of all the connected components of the live Cayley graph at stage n that have a vertex adjacent or equal to some element of $\cup_i(\hat{R}_i \cup \hat{G}_i)$. Then C' is finite, since each $\hat{R}_i$ and $\hat{G}_i$ is. Let $m = |C'| + 1$ and $U = D \backslash (\cup_i R_i^n \cup \mathrm{U} G_i^n)$. By *Lemma 2*, there is an open neighborhood O of x_0 so that for all $y \in O$ and all words ω (in the letters P) of length less than or equal to m, $\omega\, x_0 \in R_i^n$ (resp. G_i^n, U) iff $\omega\, y \in R_i^n$ (resp. G_i^n, U). This implies that if we "label" the elements of the Cayley graph of distance less than or equal to m from $y \in O$ with the labels R_i^n and G_i^n and U according to which sets they belong to we get a graph isomorphic to the labeled graph of the elements of distance less than or equal to m from x_0. By shrinking O further we may assume that O has no boundary and for all $y \in O$ and all reduced words $\omega \neq \omega'$ (in the letters P) of length less than or equal to m, $\omega O \cap \omega' O = \emptyset$.

Let $R_i^{n+1} = \cup\{\omega O: \omega x_0 \in \hat{R}_i\} \cup R_i^n$ and $G_i^{n+1} = \cup\{\omega O: \omega\, x_0 \in \hat{G}_i\} \cup G_i^n$. Then *Hypotheses 1–3* are easily seen to hold. *Hypothesis 4* holds because C' is the connected component of x_0 in the live Cayley graph at stage $n + 1$. Hence for all $y \in O$, the connected component at stage $n + 1$ is isomorphic to C'. Thus all y' connected to some y in O have finite connected components in the live graph at stage $n + 1$. For all other z, the live component at stage $n + 1$ is unchanged from the live component at stage n.

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