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I. Two dimensional homogenization

by

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On Cohesive Micro-crack Damage Theory. I. Two dimensional homogenizations

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Abstract

A novel continuum damage theory is proposed to model the overall damage effects on material's properties due to distributed cohesive micro-crack growth and coalescence. A new class of continuum damage models is constructed based on homogenizations of Dugdale-Bilby-Cottrell-Swinden (Dugdale-BCS) type micro-cracks in an elastic elastic representative volume element (RVE) of two dimensional space. The new theory proposed rest upon two postulates on the statistical closure:

- *the maximum average octahedral elastic strain of an RVE, and*
- *the maximum distortional strain energy density of an RVE.*

The newly proposed damage models are distinctly different from the existing damage models such as the Gurson model. Instead of considering void growth in a perfectly plastic medium, the new damage models are derived from homogenizations of cohesive crack growth in a linear elastic RVE; they mimic the realistic interactions among various bond forces at micro-scale. Therefore a statistical average of such interaction can effectively represent the overall damage in a material due to bond breakings, or surface separations. In this part of work, damage models are derived based on homogenizations of analytical solutions of Dugdale-BCS cracks in two-dimensional space.

Keywords:

Cohesive model, Damage, Fracture, Homogenization, Micro-cracks, Micromechanics, Plasticity

Contents

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1. Introduction

Micromechanics-based constitutive modeling of materials that contain distributed defects is an important subject in reliability analysis, in order to predict material failure and degradation. The popular Gurson model (Gurson [1975,1977]; Tvergaard [1981,1982]) is such an example. In this model, the material's failure mechanism at micro-level is postulated to be void growth, and the macro-level constitutive relation obtained from the homogenization is a form of pressure sensitive plasticity, which depends on a damage indicator — the volume fraction of the void in a representative volume element (RVE). The most distinguishing feature of the Gurson model is that its constitutive relation at macro-level, a form of pressure sensitive plasticity with damage-softening, differs from the constitutive relation at the micro-level, the perfect plasticity. This feature is absent in early developments in micro-elasticity, in which homogenization does not produce any new forms of constitutive relations. At both micro and macro-levels, constitutive equations are the same: the generalized Hooke's law, except having different material constants (e.g. Eshelby [1961], and Hill [1963,1965ab,1967]). The *motif* of contemporary micromechanics is far more ambitious. It aims to discovering of unknown but vital constitutive information by homogenizing simple micromechanics object in a massive ensemble.

In principle, a more accurate micromechanics model should lead to a more accurate constitutive relation at macro-level, provided that a feasible homogenization can be carried out. For the Gurson model, the void growth mechanism is supported by many experimental observations on failures of ductile materials (see McClintock [1968], Rice and Tracy [1969]). On the other hand, in most brittle or quasi-brittle materials such as concrete, rocks, ceramics, and some metals, the damage failure mechanism is usually attributed to nucleation and coalescence of micro-cracks. In fact, if a dislocation pile-up may be viewed as a micro-crack physically, not just a fundamental solution in mathematical sense, its nucleation and coalescence may be attributed to the failure of ductile materials as well (e.g. Rice [1992], Rice and Thomson [1974]), which is the essence of Bilby-Cottrell-Swinden theory [1963].

Although several micro-crack based damage models have been proposed to describe the brittle failure process (e.g. Budiansky and O'Connell [1976], Hoenig [1979], Hutchinson [1982], Horii and Nemat-Nesser [1983,1986], Fleck [1992], Gudmundson [1994], Kachanov [1985,1994], Krajcinovic [1996]). Few micro-crack damage models are available for both ductile and quasi-brittle materials.

Furthermore, the Gurson model is hardly a genuine micro-mechanics model in physical sense, its basic assumption on perfectly plastic medium inside an RVE excludes any first principle based damage mechanisms, such as decohesion, or surface separation.

It is generally believed that at micro-level, or even at meso-level, physics-based constitutive laws should be adopted to model real material behavior. The cohesive model has been long regarded as a sensible approximation to model fracture, fatigue, and other failure phenomena of solids. Since ultimately, this approximation may be better justified on physical ground because the separation of two solid surfaces may attributed to any types of damage.

Since Barenblatt [1959,1962] and Dugdale's pioneer contribution [1960], the cohesive models have been studied extensively and the concept has become the very foundation for both theoretical and experimental fracture mechanics. Notable contributions have been made by Bilby, Cottrell, and Swinden [1963,1964], Keer [1964], Keer and Mura [1966], Goodier [1968], Rice [1968ab], Kanninen [1964,1967], Becker and Gross [1987ab,1988ab,1989], Lu and Chow [1992], Weertman [1984ab,1996], Zhang and Gross [1998], Feng and Gross [2000], and many others.

In reality, the cohesive zone has a very small length scale, and how to assess the overall effect of cohesive zone degradation is important for study brittle/ductile fracture in macro-level. Recently, the Barenblatt-type model has been implemented in finite element analysis based numerical computations, e.g. Xu et al [1994s] and Ortiz et al [1999]. In their approach, no homogenization procedure has been taken into consideration, and the cohesive force only exists between finite element edges. Other cohesive models with homogenization features have also been proposed in literature, e.g. Gao and Klein's *Internal Virtual Bonds* model [1998]. Most of these homogenized cohesive models follow a numerical homogenization procedure, i.e. a numerical averaging procedure.

In this paper, an analytical homogenization procedure is developed to homogenize an elastic solid with randomly distributed cohesive cracks — Dugdale-Barenblatt cracks. The homogenization leads to a new damage evolution law at macro-level. New pressure sensitive elasto-plastic constitutive relations are obtained, which reflect the accumulated damaged effect due to the distribution of micro-cracks.

2. Damage models based on model III Dugdale-BCS crack solution

As modeling a multi-dimensional damage process is a complex task, it is valuable to illustrate the philosophy and procedures of what is to follow with an one-dimensional damage model based on homogenization of an anti-plane Dugdale-Bilby-Cottrell-Swinden (Dugdale-BCS) problem.

(a) *The mode III Dugdale-BCS crack solution*

The mode III (anti-plane) Dugdale-BCS crack problem and its solution (Dugdale [1960], Bilby et al [1963]) has now become a classical example in standard references (e.g. Kanninen and Popelar [1985], Mura [1987], Broberg [1999]). For convenience, the main results of the mode III Dugdale-BCS crack solution are outlined in the following.

Consider a single Dugdale-BCS crack at the center of an RVE as shown in Fig. 1. The displacement fields are

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = w(x_1, x_2)$$

The stress fields inside the RVE are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = 0 \; ; \tag{2.1}$$

$$\sigma_{13} = \mu^* \frac{\partial w}{\partial x_1}, \quad \sigma_{23} = \mu^* \frac{\partial w}{\partial x_2}. \tag{2.2}$$

where the shear modulus μ^* may be either μ, the virgin shear modulus, or $\bar{\mu}$, the shear modulus of the damaged material, depending on which homogenization procedure is chosen at a later point in the analysis, e.g. the method based on the assumption of dilute crack distribution or the self-consistent scheme,

External traction is present on the remote boundary Γ_∞ (the boundary of the RVE) such that

$$\sigma_{13} = 0 \, , \quad \sigma_{23} = \Sigma_\infty \, , \quad \forall \;\; \mathbf{x} \in \Gamma_\infty \tag{2.3}$$

On crack surfaces and inside the cohesive zones,

$$\sigma_{23}(x_1, 0) = 0 \qquad \forall \;\; |x_1| < a \tag{2.4}$$

$$\sigma_{23}(x_1, 0) = \sigma_0 \qquad \forall \;\; a < |x_1| < b \tag{2.5}$$

where $2a$ is the length of elastic crack and $2b$ is the total length of the crack including the cohesive zone.

The displacement solution along x_1-axis is given by Bilby et al [1963],

$$w(x_1, 0) = \frac{\sigma_0}{\pi \mu^*} \left\{ x_1 \ln \left| \frac{x_1\sqrt{b^2 - a^2} - a\sqrt{b^2 - x_1^2}}{x_1\sqrt{b^2 - a^2} + a\sqrt{b^2 - x_1^2}} \right| - a \ln \left| \frac{\sqrt{b^2 - a^2} - \sqrt{b^2 - x_1^2}}{\sqrt{b^2 - a^2} + \sqrt{b^2 - x_1^2}} \right| \right\} \tag{2.6}$$

where $b - a$ is the length of the cohesive zone, and

$$\frac{a}{b} = \cos\left(\frac{\pi \Sigma_\infty}{2\sigma_0}\right) \tag{2.7}$$

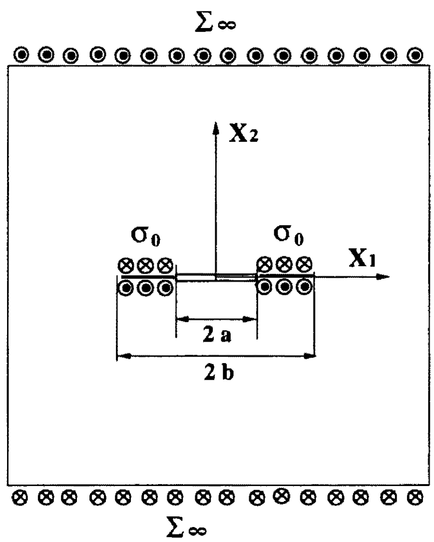


Figure 1. An anti-plane cohesive crack.

At the crack tip $x_1 = a$, the crack tip opening (slip) displacement is

$$\delta_t = w(a,0^+) - w(a,0^-) = \frac{4\sigma_0 a}{\pi\mu^*}\ln\frac{b}{a} \tag{2.8}$$

Note that under small-scale yielding condition $(a/b \approx 1)$,

$$w(a,0) = \frac{2\sigma_0 a}{\pi\mu^*}\ln\frac{b}{a} \approx \frac{2\sigma_0 a}{\pi\mu^*}\Big(\frac{b}{a}-1\Big) = \frac{2\sigma_0}{\pi\mu^*}(b-a) \tag{2.9}$$

Inside the cohesive zone,

$$\sigma_{13} = \mu^*\frac{\partial w}{\partial x_1} \approx \mu^*\frac{w(b,0)-w(a,0)}{b-a} = -\frac{2\sigma_0}{\pi} \tag{2.10}$$

$$\sigma_{23} = \sigma_0 \tag{2.11}$$

Assume that the plastic yielding at micro-level obeys the Huber-von Mises criterion. The cohesive stress σ_0 can then be related to the true yield stress by

$$\frac{\sigma_Y}{\sigma_0} = \frac{\sqrt{3(\pi^2+4)}}{\pi} \tag{2.12}$$

The crack opening (slip) volume for mode III Dugdale-BCS crack is:

$$\begin{aligned} V(a) &= \int_{-a}^{a}\Big[w(x_1,0^+) - w(x_1,0^-)\Big]dx_1 \\ &= \frac{2\sigma_0 a^2}{\pi\mu^*}\Big\{\Big(1-\frac{\Sigma_\infty}{\sigma_0}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big) - \frac{4}{\pi}\ln\Big[\cos\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big]\Big\} \end{aligned} \tag{2.13}$$

or

$$V(b) = \int_{-b}^{b}\Big[w(x_1,0^+) - w(x_1,0^-)\Big]dx = \frac{2\sigma_0 a^2}{\mu^*}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big) \tag{2.14}$$

Shown by Rice [1968a], the J-integral related energy release rate in this case is

$$\frac{\partial \mathcal{R}_1}{\partial \ell} = J = \sigma_0 \delta_t = \frac{4\sigma_0^2 a}{\pi \mu^*} \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{2.15}$$

where $\ell = 2a$ is the length of the elastic crack. Assume that during crack growth (quasi-static) the ratio Σ_∞/σ_0 remains constant. Integrating (2.15), one may find the energy release of a single Dugdale-BCS crack

$$\mathcal{R}_1 = \frac{4\sigma_0^2 a^2}{\pi \mu^*} \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{2.16}$$

The energy release expression given in Eq. (2.16) does not include plastic dissipation, nor is it the exact crack separation energy release (see: Kfouri and Rice [1978] and Kfouri [1979]). To seek a upper bound solution, one may assume that all the energy release is consumed in crack separation, and it is

$$\mathcal{R}_2 = \int_{-a}^{a} \Sigma_\infty [w](x_1) dx_1 + 2\int_a^b (\Sigma_\infty - \sigma_0)[w](x_1) dx_1 \tag{2.17}$$

where $[w](x) = w(x_1, 0^+) - w(x_1, 0^-)$. It is straightforward that

$$\mathcal{R}_2 = \Sigma_\infty V(b) - \sigma_0 (V(b) - V(a)) = \frac{8\sigma_0^2 a^2}{\pi \mu^*} \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{2.18}$$

One may notice that $\mathcal{R}_2 = 2\mathcal{R}_1$. Combining (2.16) and (2.18), one may write

$$\mathcal{R}_\omega = \frac{4\omega\sigma_0^2 a^2}{\pi \mu^*} \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big]\ , \quad \omega = 1, 2 \tag{2.19}$$

(b) Averaging theorem for aligned cohesive crack distribution

For elastic solids containing cohesive cracks, there is no averaging theorem available. An extension of averaging theorem to the present case of cohesive cracks will provide sound theoretical footing.

Define macro stress and macro elastic strain tensors as

$$\Sigma_{ij} := \ < \sigma_{ij} > = \frac{1}{V}\int_V \sigma_{ij} dV \tag{2.20}$$

$$\mathcal{E}_{ij} := \frac{\partial \bar{W}^c}{\partial \Sigma_{ij}} = \bar{D}_{ijk\ell} \Sigma_{k\ell} \tag{2.21}$$

where $\bar{W}^c$ is the average complementary energy density of an RVE, and $\bar{D}_{ijk\ell}$ is the overall (average) elastic compliance tensor. Note that first Eq. (2.21) is a nonlinear relationship in general since the overall elastic compliance tensor $\bar{D}_{ijk\ell}$ may depend on macro stress Σ_{ij}; second the average elastic strain is not the average strain, i.e. $\mathcal{E}_{ij} \neq E_{ij} := < \epsilon_{ij} >$.

The deviatoric counterparts and the corresponding second invariants of Σ_{ij} and $\mathcal{E}_{ij}$ are defined as

$$\begin{aligned} \Sigma'_{ij} &= \Sigma_{ij} - \frac{1}{3}\Sigma_{kk}\delta_{ij}, \quad J_2 = \frac{1}{2}\Sigma'_{ij}\Sigma'_{ij} \\ \mathcal{E}'_{ij} &= \mathcal{E}_{ij} - \frac{1}{3}\mathcal{E}_{kk}\delta_{ij}, \quad I_2 = \frac{1}{2}\mathcal{E}'_{ij}\mathcal{E}'_{ij} \end{aligned} \tag{2.22}$$

First consider a single antiplane crack in the center of an RVE (see Fig. 1). On the remote boundary of RVE Γ_∞, the prescribed traction is generated by a constant stress tensor Σ^∞_{ij}; on elastic crack surfaces, ∂V_{ec}, the traction is zero; and cohesive crack surfaces, ∂V_{pz}, the traction is constant. By the divergence theorem and considering $\partial V = \Gamma_\infty \cup \partial V_{ec} \cup \partial V_{pz}$, it may be found that

$$\begin{aligned}
\Sigma_{ij} &= \frac{1}{V}\int_V \sigma_{ij} dV = \frac{1}{V}\int_V \Big(\sigma_{mj} x_i\Big)_{,m} dV \\
&= \frac{1}{V}\left\{\int_V \Sigma^\infty_{mj}\delta_{im} dV - \int_{\partial V_{ec}} 0\cdot x_i n_m dS - \int_{\partial V_{pz}} \sigma_{mj} x_i n_m dS\right\} \\
&= \Sigma^\infty_{ij} - \frac{1}{V}\int_{\partial V_{pz}} \sigma_{mj} x_i n_m dS = \Sigma^\infty_{ij} - \frac{1}{V} t^0_j \int_{\partial V_{pz}} x_i dS
\end{aligned} \tag{2.23}$$

where $t^0_j = \sigma^0_{kj} n_k$. The last term in Eq. (2.23) is

$$\frac{1}{V}\sigma_0 \int_{\partial V_{pz}} x_2 dS = 0 \tag{2.24}$$

since the crack lies on x_1 axis. Now assuming that there are N cohesive cracks inside the RVE, and they are aligned in the direction of x_1 axis. Therefore,

$$\Sigma_{ij} = \Sigma^\infty_{ij} - \frac{1}{V}\sum_{k=1}^{N} \sigma_0 \int_{\partial V_{k-pz}} x_2 dx_1 \tag{2.25}$$

Assume that the center of the k-th crack is located at $(x_{k1}, x_{k2}, 0)$ and introduce a local coordinate x'_i such that $x_i = x_{ki} + x'_i$. Hence Eq. (2.25) becomes

$$\begin{aligned}
\Sigma_{ij} &= \Sigma^\infty_{ij} - \frac{1}{V}\sum_{k=1}^{N} \sigma_0 (x_{k2}|\partial V_{k-pz}| + \int_{\partial V_{k-pz}} x'_2 dx_1) \\
&= \Sigma^\infty_{ij} - \frac{1}{V}\sum_{k=1}^{N} \sigma_0 \Big(x_{k-2}|\partial V_{k-pz}|\Big)
\end{aligned} \tag{2.26}$$

because in each local coordinate $\int_{\partial V_{k-pz}} x'_2 dx_1 = 0$. If the crack distribution is symmetric about x_1 axis (aligned and uniform), it would require that the weighted average

$$\sum_{k=1}^{N} x_{k2} |\partial V_{k-pz}| = 0 \tag{2.27}$$

To this end, we recover the averaging theorem for solids with cohesive defects: $< \sigma_{ij} >= \Sigma^\infty_{ij}$ in the case of aligned cohesive crack distribution.

Define

$$\Sigma_{eq} := \sqrt{3J_2} \tag{2.28}$$

$$\Sigma_m := \frac{1}{3}\Sigma_{kk} \tag{2.29}$$

Considering the remote boundary conditions (2.3), one has

$$\Sigma_{eq} = \sqrt{3}\Sigma_\infty \tag{2.30}$$
$$\Sigma_m = 0\,. \tag{2.31}$$

(c) *Additional strain formulas*

The key technical step in deriving continuum damage models is finding the damaged elastic compliances. There are several means by which to accomplish this: (1) use Hill-Kachanov additional strain formula suitable for traction-free defects (Hill [1967] and Kachanov [1985]), (2) use energy methods.

(i) *Overall elastic compliance via Hill-Kachanov additional strain formula*

The additional strain caused by a crack Ω may be estimated by the Hill-Kachanov formula (Hill [1967], Kachanov [1985]),

$$\boldsymbol{\epsilon}^{(add)} = \frac{1}{2V}\int_{\partial\Omega}\Big(\mathbf{n}\otimes[\mathbf{u}] + [\mathbf{u}]\otimes\mathbf{n}\Big)dS \tag{2.32}$$

where the superscript *add* stands for *additional* strain, and $\partial\Omega$ is the upper part of the crack surface.

Strictly speaking, Eq. (2.32) is only applicable to solids with traction-free defects; it may not be valid in homogenizations of cohesive cracks. Becker and Gross [1987a] showed that the Hill-Kachanov formula is still be valid in homogenization of cohesive defects, which, the present authors believe, is true only under certain approximations.

For cohesive cracks, there are two choices for crack upper surface: $\partial\Omega = [-a, a]$ or $\partial\Omega = [-b, b]$. Consequently,

$$\begin{aligned}
\epsilon_{23}^{(add)} &= \frac{1}{2V}\int_{\partial\Omega} n_2[u_3]dS = \frac{1}{2V}\int_{\partial\Omega}[u_3]dx_1 = \frac{1}{2V}\begin{cases} V(a) \\ V(b)\end{cases} \\
&= \frac{\sigma_0}{\pi\mu^*}\Big(\frac{\pi a^2}{V}\Big)\begin{cases} \Big(1-\dfrac{\Sigma_\infty}{\sigma_0}\Big)\tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big) - \dfrac{4}{\pi}\ln\Big[\cos\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] \\ \tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big)\end{cases}
\end{aligned} \tag{2.33}$$

From $\mathcal{E}_{23} = \epsilon_{23}^{(0)} + \epsilon_{23}^{(add)}$, one may find

$$\frac{\Sigma_\infty}{2\bar{\mu}} = \frac{\Sigma_\infty}{2\mu} + \frac{\Sigma_\infty}{\pi\mu^*}\Big(\frac{\pi a^2}{V}\Big)\Big(\frac{\sigma_0}{\Sigma_\infty}\Big)\begin{cases} \Big(1-\dfrac{\Sigma_\infty}{\sigma_0}\Big)\tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big) - \dfrac{4}{\pi}\ln\cos\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big) \\ \tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big)\end{cases} \tag{2.34}$$

Suppose that there are N randomly distributed cohesive cracks inside the RVE, and each of them with the half length a_ℓ, $\ell = 1, 2, \cdots, N$. Define the crack opening volume fraction

$$f := \sum_{\ell=1}^{N} \frac{\pi a_\ell^2}{V} \tag{2.35}$$

and use the self-consistent scheme (e.g. Hill [1965]), Budiansky and O'Connell [1976]) such that $\mu^* = \bar{\mu}$. Then one can find the damaged shear modulus

$$\frac{\bar{\mu}}{\mu} = 1 - \frac{2f}{\pi} B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big) \tag{2.36}$$

where

$$B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big) := \Big(\frac{\sigma_0}{\Sigma_\infty}\Big) \begin{cases} \Big(1 - \dfrac{\Sigma_\infty}{\sigma_0}\Big) \tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big) - \dfrac{4}{\pi} \ln\cos\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big), & (a); \\ \tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big), & (b). \end{cases} \tag{2.37}$$

On the other hand, if the crack distribution is dilute, one may find

$$\frac{\mu}{\bar{\mu}} = 1 + \frac{2f}{\pi} B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big) \tag{2.38}$$

Eqs. (2.36) and (2.38) reveal the dependence of the overall shear modulus, $\bar{\mu}$, on remote stress Σ_∞.

(ii) *Overall compliance via energy methods*

If the energy release contribution to the damage process can be determined, one can derive the overall elastic compliance with ease.

In this 1-D model, the complementary strain energy density of the virgin material is

$$W^c = \frac{1}{2\mu} \Sigma_\infty^2 \tag{2.39}$$

Consider strain energy balance

$$\bar{W} = \Big(< \sigma_{ij}\epsilon_{ij} > - W^c\Big) - \frac{\mathcal{R}_\omega}{V} \tag{2.40}$$

where $\bar{W}$ is the overall strain energy density. The overall complementary energy density can be obtained via Legendre transformation

$$\bar{W}^c = W^c + \frac{\mathcal{R}_\omega}{V}\ , \tag{2.41}$$

because

$$\bar{W} = \Big(< \sigma_{ij}\epsilon_{ij} > - W^c\Big) - \frac{\mathcal{R}_\omega}{V} \tag{2.42}$$

It is then straightforward that

$$\begin{aligned}\mathcal{E}_{23} &= \frac{\partial \bar{W}^c}{\partial \Sigma_{23}} = \frac{\partial W^c}{\partial < \sigma_{23} >} + \frac{\partial}{\partial < \sigma_{23} >}\Big(\frac{\mathcal{R}_\omega}{V}\Big) \\ &= \epsilon_{23}^{(0)} + \frac{2\omega\sigma_0 f}{\pi\mu^*}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\end{aligned} \tag{2.43}$$

where

$$\frac{\mathcal{R}_\omega}{V} = \frac{4\omega\sigma_0^2 f}{\pi^2\mu^*}\ln\Big[\cos\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{2.44}$$

Consider the self-consistent scheme ($\mu^* = \bar{\mu}$). One may find the ratio between the damaged shear modulus and the initial shear modulus

$$\frac{\bar{\mu}}{\mu} = 1 - \omega f\Big(\frac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\ , \quad \omega = 1,2 \tag{2.45}$$

It is interesting to note that when $\omega = 1$, (2.45) is exactly the same as the result obtained from Hill-Kachanov formula in Eqs. (2.36) and (2.37 b).

If a crack distribution is dilute, it can be shown that

$$\frac{\mu}{\bar{\mu}} = 1 + \omega f\Big(\frac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big), \quad \omega = 1,2 \tag{2.46}$$

Obviously, the energy method is a more theoretically sound approach.

(d) Damage models

Without proper statistical closure, averaging alone may not be able to provide sensible results, and often leads to frustration.

In this paper, two hypotheses on statistical closure are postulated for isotropic materials:

Hypothesis 2.1[Ensemble Averaging Closure]

1. *The macroscopic yielding of an RVE begins when the average elastic octahedral strain reaches a threshold, i.e.*

$$\mathcal{E}_{oct} := \frac{2\sqrt{2}}{\sqrt{3}}\sqrt{I_2} = \epsilon_{cr} \tag{2.47}$$

where $I_2 = \frac{1}{2}\mathcal{E}'_{ij}\mathcal{E}'_{ij}$ *and* $\mathcal{E}'_{ij}$ *are calculated based on elastic unloading, i.e.*

$$\mathcal{E}'_{ij} = \frac{1}{2\bar{\mu}}\Sigma'_{ij} \tag{2.48}$$

2. *The macroscopic yielding of an RVE begins when elastic distortional energy density of the RVE,*

$$U_d := \int_0^{\mathcal{E}'_{ij}} \tilde{\Sigma}'_{ij} d\tilde{\mathcal{E}}'_{ij} \tag{2.49}$$

reaches a threshold. In other words, the maximum elastic distortional energy of an RVE is a material constant.

$$U_d \leq U_d^{(cr)} \ . \tag{2.50}$$

♣

Remark 2.1

1. *The above statistical closures are postulates, and they are not based on micromechanics principles. In other words, these conditions are pre-requisite properties assigned to all the RVEs in a material under consideration.*

2. *Since the relationship (2.48) is nonlinear, in general,*

$$U_d \neq \frac{1}{2\bar{\mu}} J_2 \tag{2.51}$$

   Interestingly, if the overall shear modulus only depends on Σ_m*, i.e.* $\bar{\mu} = \bar{\mu}(\Sigma_m)$*, it can be shown that*

$$\begin{aligned} U_d &= \int_0^{\mathcal{E}'_{ij}} \tilde{\Sigma}'_{ij} d\tilde{\mathcal{E}}'_{ij} = \int_0^{\mathcal{E}'_{ij}} 2\bar{\mu}(\Sigma_m)\tilde{\mathcal{E}}'_{ij} d\tilde{\mathcal{E}}'_{ij} = \int_0^{I_2} 2\bar{\mu}(\Sigma_m) dI_2 \\ &= 2\bar{\mu}(\Sigma_m) I_2 = \frac{1}{2\bar{\mu}} J_2 \end{aligned} \tag{2.52}$$

3. *Eq. (2.49) is reminiscent of Hencky's maximum distortional energy principle in classical plasticity (Hencky [1924]). According to this principle, the threshold of yielding for a material point can be measured by its ability to absorb certain amount of elastic distortional energy. However, the elastic distortional energy density of an RVE does not equal to the average elastic distortional energy density, i.e.*

$$U_d = \int_0^{\mathcal{E}'_{ij}} \tilde{\Sigma}'_{ij} d\tilde{\mathcal{E}}'_{ij} \neq \frac{1}{V} \int_V \int_0^{\epsilon'_{ij}} \tilde{\sigma}'_{ij} d\tilde{\epsilon}'_{ij} dV \tag{2.53}$$

   In other words

$$\int_0^{\mathcal{E}'_{ij}} < \sigma_{ij} >' d < \epsilon^e_{ij} >' \neq \frac{1}{2} < \sigma'_{ij} \epsilon'_{ij} > \tag{2.54}$$

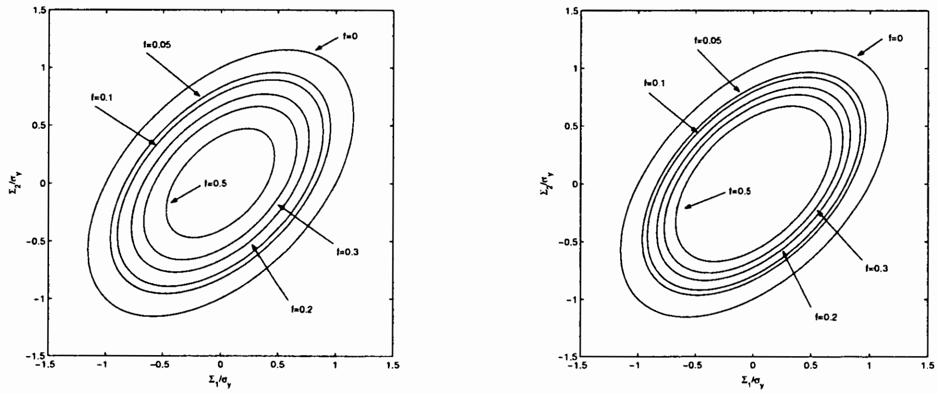
4. *The two criteria can be calibrated with uniaxial tension tests of virgin materials*

$$\mathcal{E}_{oct} = \frac{1}{2\mu}\Sigma_{oct} = \frac{1}{3\sqrt{2}\mu}\sigma_Y \tag{2.55}$$

$$U_d = \frac{1}{6\mu}\Sigma^2_{eq} = \frac{1}{6\mu}\sigma^2_Y \tag{2.56}$$

where $\Sigma_{oct} := \sqrt{\frac{2}{3}J_2}$ *and* $\Sigma_{eq} := \sqrt{3J_2}$.

In the damage evolution process, the two criteria take the following forms



(a) (b)

Figure 2. One dimensional damage models (a) Model 1a (self-consistent method); (b) Model 1b (dilute crack distribution) .

1. The maximum average elastic octahedral strain criterion:

$$\mathcal{E}_{oct} = \frac{1}{3\sqrt{2}\bar{\mu}}\Sigma_{eq} \leq \frac{1}{3\sqrt{2}\mu}\sigma_Y \quad \Rightarrow \quad \frac{\Sigma_{eq}}{\sigma_Y} = \frac{\bar{\mu}}{\mu} \tag{2.57}$$

2. The criterion of the maximum distortional strain energy density in an RVE

$$U_d = \frac{\Sigma_{eq}^2}{6\bar{\mu}} \leq \frac{1}{6\mu}\sigma_Y^2 \qquad \Rightarrow \quad \frac{\Sigma_{eq}^2}{\sigma_Y^2} = \frac{\bar{\mu}}{\mu} \tag{2.58}$$

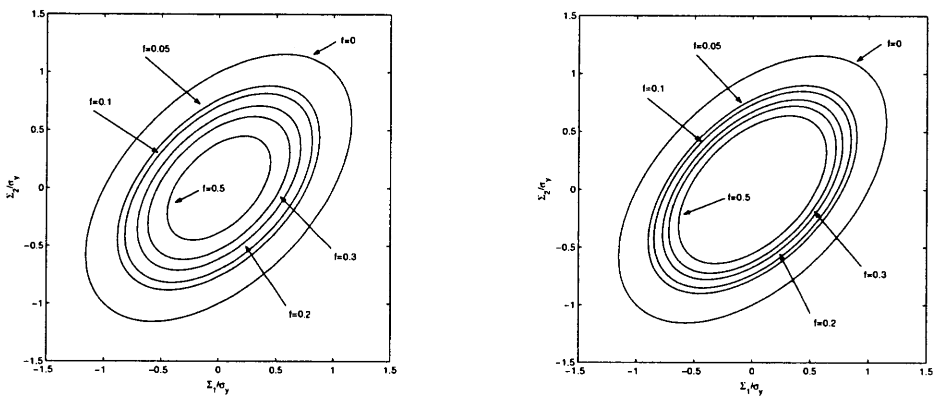
In the 1D model, $\bar{\mu} = \bar{\mu}(\Sigma_{eq})$, and the criterion of maximum distortional energy density in an RVE may not be cast into a convenient form. Use the maximum average elastic octahedral strain criterion and substitute (2.36) and (2.37a,b) into (2.57), and combine (2.12) and (2.30)

$$\frac{\Sigma_\infty}{\sigma_0} = \sqrt{1 + \frac{4}{\pi^2}\Big(\frac{\Sigma_{eq}}{\sigma_Y}\Big)} \tag{2.59}$$

The following damage models may be derived

Damage Model 1a (self − consistent) :

$$\boxed{\begin{aligned}&\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2f}{\sqrt{\pi^2+4}}\Bigg\{\Big(1 - \sqrt{1 + \frac{4}{\pi^2}\Big(\frac{\Sigma_{eq}}{\sigma_Y}\Big)}\Big)\tan\Big(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\Big)\\ &-\frac{4}{\pi}\ln\Big[\cos\Big(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\Big)\Big]\Bigg\} - \frac{\Sigma_{eq}}{\sigma_Y} = 0\end{aligned}} \tag{2.60}$$



(a) (b)

Figure 3. One dimensional damage models (a) Model 2a (self-consistent method); (b) Model 2b (dilute crack distribution) .

Damage Model 1b (dilute crack distribution) :

$$\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2f\Sigma_{eq}}{\sigma_Y\sqrt{\pi^2+4}}\left\{\left(1-\sqrt{1+\frac{4}{\pi^2}}\left(\frac{\Sigma_{eq}}{\sigma_Y}\right)\right)\tan\left(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\right) -\frac{4}{\pi}\ln\left[\cos\left(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\right)\right]\right\} - \frac{\Sigma_{eq}}{\sigma_Y} = 0 \qquad (2.61)$$

Damage Model 2a (self − consistent) :

$$\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2\omega f}{\sqrt{\pi^2+4}}\tan\left(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\right) - \frac{\Sigma_{eq}}{\sigma_Y} = 0\,, \quad \omega = 1,2 \qquad (2.62)$$

Damage Model 2b (dilute crack distribution) :

$$\frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{2\omega f\Sigma_{eq}}{\sigma_Y\sqrt{\pi^2+4}}\tan\left(\frac{\sqrt{\pi^2+4}\Sigma_{eq}}{2\sigma_Y}\right) - \frac{\Sigma_{eq}}{\sigma_Y} = 0\,, \quad \omega = 1,2 \qquad (2.63)$$

In Figs. 2 and 3, the damaged yield surfaces are plotted in the two-dimensional stress space, which are a set of coaxial elliptics.

3. Two-dimensional Dugdale-BCS crack in an RVE

(a) *Mode I Dugdale-BCS crack under uniform biaxial tension*

Before proceeding to construct damage models, we first briefly outline the mode I Dugdale-BCS crack solution in a representative volume element (RVE). Consider a two-dimensional RVE with a Dugdale-BCS crack in the center. Uniform biaxial tension is applied on the remote boundary of the RVE, Γ_∞.

$$\Sigma_{11} = \Sigma_{22} = \Sigma_\infty \, , \quad \forall \; \mathbf{x} \in \Gamma_\infty \tag{3.1}$$

On macro-level, the remote stress Σ_∞ may be related with the spherical stress Σ_m of the RVE

$$\Sigma_m = \phi \Sigma_\infty = \begin{cases} \dfrac{2}{3}\Sigma_\infty, & \text{plane stress} \\[2ex] \dfrac{2}{3}(1+\nu^*)\Sigma_\infty, & \text{plane strain} \end{cases} \tag{3.2}$$

The Dugdale-BCS mode I crack solution can be obtained via superposition.

(i) *Trivial solution*

Consider an RVE without cracks. A trivial solution of uniform stress state is: $\forall \, \mathbf{x} \in V$,

$$\begin{aligned} \sigma_{11}^{(0)} &= \Sigma_\infty \\ \sigma_{22}^{(0)} &= \Sigma_\infty \\ \sigma_{12}^{(0)} &= 0 \end{aligned}$$

and

$$\begin{bmatrix} \epsilon_{11}^{(0)} \\[2ex] \epsilon_{22}^{(0)} \\[2ex] 2\epsilon_{12}^{(0)} \end{bmatrix} = \frac{1}{\mu^*} \begin{bmatrix} \dfrac{\kappa^*+1}{8} & \dfrac{\kappa^*-3}{8} & 0 \\ \dfrac{\kappa^*-3}{8} & \dfrac{\kappa^*+1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma_\infty \\[2ex] \Sigma_\infty \\[2ex] 0 \end{bmatrix} \tag{3.3}$$

where the constants μ^*, κ^* not only depend on material properties but also depend on ensuing homogenization procedures, and κ^* is the Kolosov's constant

$$\kappa^* := \begin{cases} \dfrac{(3-\nu^*)}{(1+\nu^*)}, & \text{plane stress ;} \\[2ex] 3 - 4\nu^*, & \text{plane strain.} \end{cases} \tag{3.4}$$

By inspection, one may find the displacement fields of the trivial solution

$$u_1^{(0)} = \frac{\kappa^*+1}{8\mu^*}\Sigma_\infty x_1 + \frac{\kappa^*-3}{8\mu^*}\Sigma_\infty x_2 \tag{3.5}$$

$$u_2^{(0)} = \frac{\kappa^*-3}{8\mu^*}\Sigma_\infty x_1 + \frac{\kappa^*+1}{8\mu^*}\Sigma_\infty x_2 \tag{3.6}$$

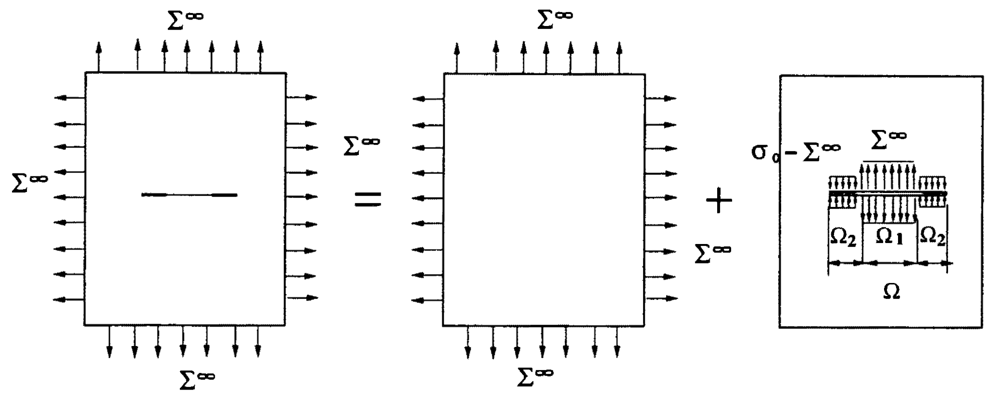


Figure 4. Illustration of superposition of cohesive crack problem

(ii) *Crack solution*

The crack solution has to satisfy the remote boundary conditions

$$\sigma_{11}^{(c)} = \sigma_{22}^{(c)} = \sigma_{12}^{(c)} = 0, \qquad r = \sqrt{x_1^2 + x_2^2} \to \infty \tag{3.7}$$

and crack surface traction boundary conditions and symmetric condition

$$\sigma_{22}^{(c)} = -\Sigma_\infty, \qquad \forall\, x_2 = 0, \;\; |x_1| < a \tag{3.8}$$

$$\sigma_{22}^{(c)} = \sigma_0 - \Sigma_\infty, \qquad \forall\, x_2 = 0, \;\; a \le |x_1| < b \tag{3.9}$$

$$u_2^{(c)} = 0, \qquad \forall\, x_2 = 0, \;\; |x_1| > b \tag{3.10}$$

The stress field solution on x_1 axis ($x_2 = 0$) is well known (e.g. Mura [1987] pages 280-285)

$$\sigma_{11}^{(c)}(x_1, 0) = \sigma_{22}^{(c)}(x_1, 0) = -\frac{d}{dx_1}\left\{ \int_0^b \frac{q(t) H(x_1 - t)}{\sqrt{x_1^2 - t^2}} dt \right\} \tag{3.11}$$

where $H(\cdot)$ is the Heaviside function, and

$$q(t) = \begin{cases} \Sigma_\infty t\ , & t < a \\ \Sigma_\infty t - \dfrac{2}{\pi}\sigma_0 \cos^{-1}(\dfrac{a}{t}) t\ , & a \le t < b \end{cases} \tag{3.12}$$

The stress distribution along x_1 axis are:

1. $(0 < |x_1| < a)$

$$\sigma_{11}^{(c)}(x_1, 0) = \sigma_{22}^{(c)}(x_1, 0) = -\Sigma_\infty\ , \qquad \sigma_{12}^{(c)} = 0;$$

2. $(a \le |x_1| < b)$

$$\sigma_{11}^{(c)}(x_1, 0) = \sigma_{22}^{(c)}(x_1, 0) = -\Sigma_\infty + \sigma_0\ , \qquad \sigma_{12}^{(c)} = 0\ ;$$

3. $b < |x_1| < \infty$

$$\begin{aligned}
\sigma_{11}^{(c)}(x_1,0) &= \sigma_{22}^{(c)}(x_1,0) = -\Sigma_\infty + \sigma_0 \\
&+ \frac{\sigma_0 a}{\pi}\frac{d}{dx_1}\left\{\frac{x_1}{a}\sin^{-1}\left[\frac{x_1^2(b^2-2a^2)+a^2b^2}{b^2(x_1^2-a^2)}\right] + \sin^{-1}\left[\frac{x_1^2+a^2-2b^2}{x_1^2-a^2}\right]\right\} \\
\sigma_{12}^{(c)} &= 0\,.
\end{aligned} \tag{3.13}$$

Inside the cohesive zone ($a < |x_1| < b$),

$$\sigma_{11}^{(t)} = \sigma_{11}^{(0)} + \sigma_{11}^{(c)} = \sigma_0 \tag{3.14}$$

$$\sigma_{22}^{(t)} = \sigma_{22}^{(0)} + \sigma_{22}^{(c)} = \sigma_0 \tag{3.15}$$

$$\sigma_{12}^{(t)} = 0\,. \tag{3.16}$$

It is assumed that the microscopic yielding of the material is governed by the Huber-von Mises criterion. Since the shear stresses are all zero inside the cohesive zone,

$$\sigma_{eq}^{micro} = \sqrt{\frac{1}{2}\left((\sigma_{11}^{(t)}-\sigma_{22}^{(t)})^2 + (\sigma_{22}^{(t)}-\sigma_{33}^{(t)})^2 + (\sigma_{33}^{(t)}-\sigma_{11}^{(t)})^2\right)} \le \sigma_Y \tag{3.17}$$

which links the cohesive stress with the uniaxial yield stress of the virgin material,

$$\sigma_Y = \chi\sigma_0 = \begin{cases} \sigma_0\,, & \text{plane stress;} \\ (1-2\nu^*)\sigma_0\,, & \text{plane strain.} \end{cases} \tag{3.18}$$

For crack solution, the displacement fields along the x_1 axis are

$$u_1^{(c)}(x_1,\pm 0) = \frac{(1-\kappa^*)}{4\mu^*}\Sigma_\infty x_1 + \frac{(\kappa^*-1)}{4\mu^*}\sigma_0 \begin{cases} (x_1+a), & \forall \; -b < x_1 \le -a \\ 0, & \forall \; -a < x_1 < a \\ (x_1-a), & \forall \; a \le x_1 < b \end{cases} \tag{3.19}$$

and

$$u_2^{(c)}(x_1,\pm 0) = \pm\frac{(1+\kappa^*)}{4\pi\mu^*}\sigma_0 \cdot \left\{x_1 \ln\left|\frac{x_1\sqrt{b^2-a^2} - a\sqrt{b^2-x_1^2}}{x_1\sqrt{b^2-a^2} + a\sqrt{b^2-x_1^2}}\right| - a\ln\left|\frac{\sqrt{b^2-a^2}-\sqrt{b^2-x_1^2}}{\sqrt{b^2-a^2}+\sqrt{b^2-x_1^2}}\right|\right\} \tag{3.20}$$

Therefore, in the cohesive zone ($a < |x_1| < b$),

$$u_1^{(t)}(x_1,0) = \frac{(\kappa^*-1)}{4\mu^*}\sigma_0 \begin{cases} (x_1+a), & \forall \; -b < x_1 \le -a \\ 0, & \forall \; -a < x_1 \le a \\ (x_1-a), & \forall \; a \le x_1 < b \end{cases} \tag{3.21}$$

and $u_2^{(t)}(x_1,0) = u_2^{(c)}(x_1,0)$. The elastic crack opening volume can be expressed as

$$V(a) = \int_{-a}^{a}[u_2^{(t)}]dx_1 = \left(\frac{1+\kappa^*}{2\mu^*}\right)a^2\sigma_0\left\{\left(1-\frac{\Sigma_\infty}{\sigma_0}\right)\tan\left(\frac{\pi\Sigma_\infty}{2\sigma_0}\right) - \frac{4}{\pi}\ln\left[\cos\left(\frac{\pi\Sigma_\infty}{2\sigma_0}\right)\right]\right\}, \tag{3.22}$$

and the total crack opening volume as

$$V(b) = \int_{-b}^{b} [u_2^{(t)}] dx_1 = \frac{(1+\kappa^*)}{2\mu^*} \sigma_0 a^2 \tan\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big) \tag{3.23}$$

The crack tip opening displacement is given by Rice [1968a],

$$\delta_t = u^{(t)}(a,+0) - u^{(t)}(a,-0) = \frac{(1+\kappa^*)\sigma_0 a}{\pi \mu^*} \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{3.24}$$

Rice [1968a,b] also showed that the J-integral for mode I Dugdale-BCS crack is

$$J = \sigma_0 \delta_t = \frac{(1+\kappa^*)}{\pi \mu^*} \sigma_0^2 a \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{3.25}$$

Since J is related to energy release, assume $\dfrac{\partial}{\partial \ell}\mathcal{R}_1 = J$, where $\ell = 2a$ is the total length of the crack. It may be found that

$$\mathcal{R}_1 = \frac{(1+\kappa^*)}{\pi \mu^*} \sigma_0^2 a^2 \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{3.26}$$

Note that in nonlinear fracture mechanics, J may not be the exact surface separation energy release rate (Wnuk [1972],[1990]).

The total energy release can be calculated as

$$\mathcal{R}_2 = \Sigma_\infty V(b) - \sigma_0\Big(V(b) - V(a)\Big) = \frac{2(1+\kappa^*)}{\pi \mu^*} \sigma_0^2 a^2 \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{3.27}$$

Considering the fact that $\mathcal{R}_2 = 2\mathcal{R}_1$, one can then combine the two into a single form

$$\mathcal{R}_\omega = \frac{\omega(1+\kappa^*)}{\pi \mu^*} \sigma_0^2 a^2 \ln\Big[\sec\Big(\frac{\pi \Sigma_\infty}{2\sigma_0}\Big)\Big], \quad \omega = 1, 2 \tag{3.28}$$

4. Additional Strain Formulas

Since the cohesive crack is not a traction-free defect, the usual averaging theorem used for solids containing traction-free defects may not be applicable for homogenization of cohesive defects. Before proceeding to homogenization, it may be necessary to examine the ensemble averaging technique first.

(a) *Averaging theorem*

Theorem 4.1. *Suppose*

1. *A 2D elastic representative volume element contains N Dugdale-BCS cracks;*
2. *The orientation of the cohesive crack distribution is isotropic and there are no shear components of the cohesive tractions on the surfaces of the cohesive zone;*
3. *The tractions on the remote boundary of the RVE are generated by a constant stress tensor, i.e,* $t^{\infty}_{\alpha} = n_{\beta}\Sigma^{\infty}_{\beta\alpha}$, *and* $\Sigma^{\infty}_{\alpha\beta} = const.$

Then average stress of the RVE equals to the remote constant stress, i.e.

$$\Sigma_{\alpha\beta} = < \sigma_{\alpha\beta} > = \Sigma^{\infty}_{\alpha\beta} \tag{4.1}$$

where all Greek letters range from 1 to 2.

Proof:
As shown previously,

$$< \sigma_{\alpha\beta} > = \Sigma^{\infty}_{\alpha\beta} - \frac{1}{V}\sum_{k=1}^{N}\int_{\partial V_{k-pz}} t^{(k)0}_{\beta} x_{\alpha} dS \tag{4.2}$$

Let $x_{\alpha} = x_{k\alpha} + x'_{k\alpha}$, where $x_{k\alpha}$ is the coordinate for the center of the k-th cohesive zone. By symmetry,

$$\int_{\partial V_{k-pz}} x'_{k\alpha} dS = 0 \tag{4.3}$$

Therefore

$$< \sigma_{ij} > = \Sigma^{\infty}_{ij} - \frac{1}{V}\sum_{k=1}^{N} t^{(\alpha)0}_{j} x_{k\alpha} |\partial V_{k-pz}| \tag{4.4}$$

This is valid for any crack orientation. In other words, the distribution of the crack orientation is isotropic for any given spatial point (see Fig. 6). The second term of Eq. (4.4) may be rewritten in terms of the probability of the crack orientation

$$\begin{aligned} < \sigma_{\alpha\beta} > &= \Sigma^{\infty}_{\alpha\beta} - \frac{1}{V}\sum_{k=1}^{N}\frac{|\partial V_{k-pz}|}{4\pi}\int_{S_k} t^{(k)0}_{\beta} x_{k\alpha} dS \\ &= \Sigma^{\infty}_{\alpha\beta} - \frac{1}{V}\sum_{k=1}^{N}\frac{x_{k\alpha}}{2\pi}|\partial V_{k-pz}|\int_{0}^{2\pi} t^{(k)0}_{\beta} d\theta \end{aligned} \tag{4.5}$$

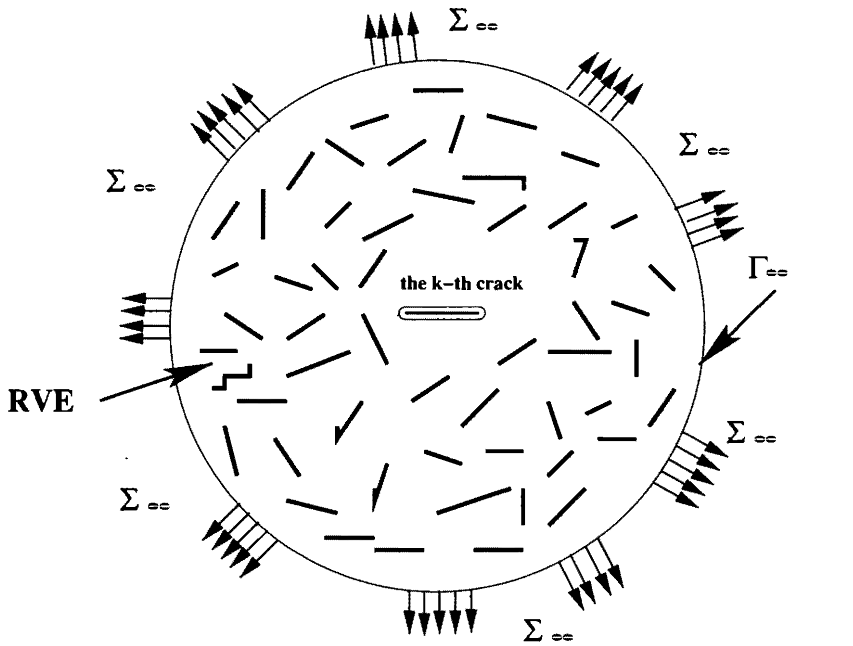


Figure 5. A two-dimensional RVE with randomly distributed cracks

Here S_k is a unit sphere surrounding the k-th cohesive zone, at center $x_{k\alpha}$.

Let the outward normal to crack surface be $\mathbf{n}$. And $\mathbf{t} = \sigma_0 \mathbf{n}$, where σ_0 is the cohesive stress. By assumption that there is no shear cohesive force, one may write the components of $\mathbf{t}^{(k)0}$ as

$$t_1^{(k)0} = \sigma_0 \cos\theta \tag{4.6}$$

$$t_2^{(k)0} = \sigma_0 \sin\theta \tag{4.7}$$

It is trivial to show

$$\frac{1}{2\pi}\int_0^{2\pi} t_\beta^{(k)0} d\theta = 0, \qquad \beta = 1, 2 \tag{4.8}$$

Thereby,

$$\Sigma_{\alpha\beta} = < \sigma_{\alpha\beta} > = \Sigma^{\infty}_{\alpha\beta} \tag{4.9}$$

♣

To construct continuum damage models, we first apply additional strain formulas to evaluate overall compliance tensor of a damaged solid. In the following, two types of additional strain formulas are used: (1) the Hill-Kachanov formula; (2) the formulas based on energy methods.

(b) *Additional strain formula for traction-free defects*

In the first approach, the Hill-Kachanov formula is taken as an acceptable approximation to estimate additional strain in solids containing cohesive defects. Assume that there is a single crack with radius a_k in a local coordinate $\{x^k_\alpha\}$. The

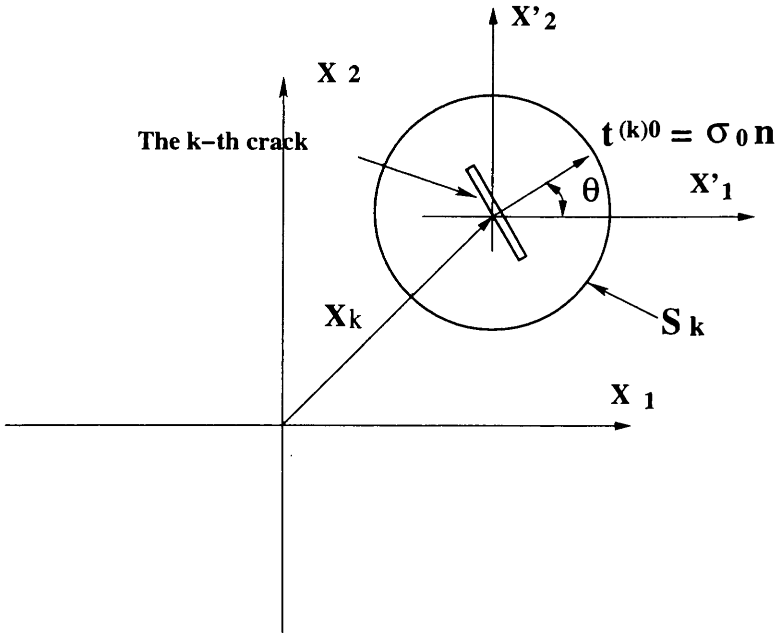


Figure 6. Homogenization of cohesive cracks with random orientations

outward normal to the crack surface is pointing to x_2^k direction. Then the only non-zero additional strain component is

$$\epsilon_{22}^{(add)} = \frac{1}{V}\int_{\partial\Omega_k} [u_2] dx_1^k = \frac{(1+\kappa^*)}{2\pi\mu^*}\Big(\frac{\pi a_k^2}{V}\Big) B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big)\Sigma_\infty \tag{4.10}$$

where function $B\Big(\Sigma_\infty/\sigma_0\Big)$ is defined in Eq. (2.37).

Let $\kappa^* = \kappa$ and $\mu^* = \mu$. The additional strain tensor may be expressed as

$$\boldsymbol{\epsilon}_k^{(add)} = \Big(\frac{\pi a_k^2}{V}\Big)\mathbf{H}^k : \boldsymbol{\Sigma}^\infty \tag{4.11}$$

where $\mathbf{H}^k = H^k_{\alpha\beta\zeta\eta}\mathbf{e}^k_\alpha \otimes \mathbf{e}^k_\beta \otimes \mathbf{e}^k_\zeta \otimes \mathbf{e}^k_\eta$, and $H^k_{\alpha\beta\zeta\eta} = 0, \;\; \forall\; \alpha,\beta,\zeta,\eta = 1,2$ except

$$H^k_{2222} = \frac{(1+\kappa)}{2\pi\mu} B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big)^2 \tag{4.12}$$

Note that $\mathbf{H}^k$ is anisotropic, due to the presence of the crack.

Assume that there exists a crack distribution density function $w(a,\theta) = w_r(a)w_0(\theta)$. The total additional strain introduced by crack distribution from crack size $2a_m$ to crack size $2a_M$ is

$$\boldsymbol{\epsilon}^{(add)} = \frac{1}{2\pi}\int_{a_m}^{a_M}\int_0^{2\pi} \boldsymbol{\epsilon}_k^{(add)} w(a,\theta) da d\theta = \mathbf{H} : \boldsymbol{\Sigma}^\infty \tag{4.13}$$

$$= \frac{1}{2\pi}\int_{a_m}^{a_M}\int_0^{2\pi} \Big(\frac{\pi a^2}{V}\Big)\mathbf{H}^k : \boldsymbol{\Sigma}^\infty w(a,\theta) da d\theta \tag{4.14}$$

The global **H** tensor can then be determined as

$$\begin{aligned}\mathbf{H} &= \frac{1}{2\pi}\int_{a_m}^{a_M}\int_0^{2\pi}\Bigl(\frac{\pi a^2}{V}\Bigr)\mathbf{H}^k(\theta)w(a,\theta)dad\theta \\ &= \Bigl\{\frac{1}{V}\int_{a_m}^{a_M} a^2 w_r(a)da\Bigr\}\cdot\Bigl\{\frac{1}{2\pi}\int_0^{2\pi} H^k_{\alpha\beta\zeta\eta}(\theta)\mathbf{e}^k_\alpha\otimes\mathbf{e}^k_\beta\otimes\mathbf{e}^k_\zeta\otimes\mathbf{e}^k_\eta w_0(\theta)d\theta\Bigr\}\end{aligned} \tag{4.15}$$

where V is the total volume of the RVE.

Define the crack opening volume fraction as

$$f := \frac{1}{V}\int_{a_m}^{a_M}\pi a^2 w_r(a)da \tag{4.16}$$

Eq. (4.15) becomes

$$\mathbf{H} = \frac{f}{2\pi}\int_0^{2\pi}\Bigl(Q^k_{\alpha\lambda}Q^k_{\beta\mu}Q^k_{\zeta\nu}Q^k_{\eta\xi}H^k_{\lambda\mu\nu\xi}(\theta)w_0(\theta)\Bigr)\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\otimes\mathbf{e}_\zeta\otimes\mathbf{e}_\eta d\theta \tag{4.17}$$

Assume that **H** is isotropic, i.e.

$$H_{\alpha\beta\zeta\eta} = \frac{h_1}{2}\delta_{\alpha\beta}\delta_{\zeta\eta} + \frac{h_2}{2}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}) \tag{4.18}$$

Consider the identity

$$Q^k_{\alpha\beta}Q^k_{\zeta\beta} = Q^k_{\beta\alpha}Q^k_{\beta\zeta} = \delta_{\alpha\zeta} \tag{4.19}$$

and the normalization

$$\frac{1}{2\pi}\int_0^{2\pi} w_0(\theta)d\theta = 1 \tag{4.20}$$

One may find the global **H** tensor by solving the following algebraic equations

$$2h_1 + 2h_2 = fH^k_{\lambda\lambda\mu\mu} = fH^k_{2222} \tag{4.21}$$

$$h_1 + 3h_2 = fH^k_{\lambda\mu\lambda\mu} = fH^k_{2222} \tag{4.22}$$

which yield the solution

$$h_1 = \frac{1}{4}\frac{(1+\kappa)}{2\pi\mu}fB\Bigl(\frac{\Sigma_\infty}{\sigma_0}\Bigr) \tag{4.23}$$

$$h_2 = \frac{1}{4}\frac{(1+\kappa)}{2\pi\mu}fB\Bigl(\frac{\Sigma_\infty}{\sigma_0}\Bigr) \tag{4.24}$$

The elastic compliance tensor for two dimensional isotropic materials has the form

$$D_{\alpha\beta\zeta\eta} = \frac{\kappa-3}{8\mu}\delta_{\alpha\beta}\delta_{\zeta\eta} + \frac{1}{4\mu}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}) \tag{4.25}$$

$$\bar{D}_{\alpha\beta\zeta\eta} = \frac{\bar{\kappa}-3}{8\bar{\mu}}\delta_{\alpha\beta}\delta_{\zeta\eta} + \frac{1}{4\bar{\mu}}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta}) \tag{4.26}$$

From the homogenization scheme

$$\bar{D}_{\alpha\beta\zeta\eta} = D_{\alpha\beta\zeta\eta} + H_{\alpha\beta\zeta\eta} \quad \Rightarrow \quad \frac{1}{4\bar{\mu}} = \frac{1}{4\mu} + \frac{h_2}{2} \tag{4.27}$$

It then yields

$$\frac{\mu}{\bar{\mu}} = 1 + \frac{(1+\kappa)}{4\pi} f B\Big(\frac{\Sigma_\infty}{\sigma_0}\Big) \tag{4.28}$$

(c) Energy method

A more rigorous procedure of deriving the additional strain formula for solids containing cohesive defects is to use energy method. By the balance of strain energy density of the damage process, one may be able to find the average potential energy density and hence the average complementary energy density, which are assumed to be potential functions of average strain and average stress. Consider an RVE with N randomly distributed cohesive cracks. The overall potential energy density in an RVE can be calculated by taking into account the average energy release density (see Fig. 4)

$$\bar{W} = < \sigma_{\alpha\beta}\epsilon_{\alpha\beta} > - W^c - \frac{\mathcal{R}_\omega}{V}, \qquad \omega = 1, 2 \tag{4.29}$$

Via Legender transform, one has

$$\bar{W}^c \;=\; W^c + \frac{\mathcal{R}_\omega}{V} = \frac{1}{2} D_{\alpha\beta\zeta\eta}\Sigma^\infty_{\alpha\beta}\Sigma^\infty_{\zeta\eta} + \Big(\sum_{k=1}^{N}\frac{\pi a_k^2}{V}\Big)\frac{\omega\sigma_0^2(1+\kappa^*)}{\pi^2\mu^*}\ln\Big[\sec\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{4.30}$$

where $\Sigma^\infty_{ij} = \Sigma_\infty \delta_{ij}$. Define the crack opening volume fraction

$$f := \sum_{k=1}^{N} \frac{\pi a_k^2}{V} \tag{4.31}$$

Since $\bar{W}^c$ is a potential function of $\Sigma_{\alpha\beta}$ and by the averaging Theorem 4.1 ($\Sigma_{\alpha\beta} = \Sigma^\infty_{\alpha\beta}$), one may find

$$\begin{aligned}
\mathcal{E}_{\alpha\beta} &= \frac{\partial \bar{W}^c}{\partial \Sigma_{\alpha\beta}} = \frac{\partial \bar{W}^c}{\partial \Sigma^\infty_{\alpha\beta}} \\
&= D_{\alpha\beta\zeta\eta}\Sigma^\infty_{\zeta\eta} + \frac{\partial}{\partial\Sigma_\infty}\left(\frac{\omega\sigma_0^2(1+\kappa^*)f}{\pi^2\mu^*}\ln\Big[\sec\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big]\right)\frac{\partial\Sigma_\infty}{\partial\Sigma^\infty_{\alpha\beta}} \\
&= \epsilon^{(0)}_{\alpha\beta} + \epsilon^{(add)}_{\alpha\beta}
\end{aligned} \tag{4.32}$$

where $\epsilon^0_{\alpha\beta} := D_{\alpha\beta\zeta\eta}\Sigma^\infty_{\zeta\eta}$, $\mathcal{E}_{\alpha\beta} = \bar{D}_{\alpha\beta\zeta\eta}\Sigma^\infty_{\alpha\beta}$, and

$$\epsilon^{(add)}_{\alpha\beta} = \frac{\omega(1+\kappa^*)}{4\mu^*} f\Big[\frac{2\sigma_0}{\pi\Sigma_\infty}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big]\Sigma_\infty\delta_{\alpha\beta}, \quad \omega = 1, 2 \tag{4.33}$$

One may wish to put above expression in a general expression,

$$\epsilon^{(add)} = \mathbf{H} : \mathbf{\Sigma}^{\infty} \tag{4.34}$$

where the tensor $\mathbf{H} = \dfrac{h_1}{2}\mathbf{1}^2 \otimes \mathbf{1}^2 + h_2\mathbf{1}^{(4s)}$ is a global, isotropic tensor, such that $\bar{\mathbf{D}} = \mathbf{D} + \mathbf{H}$.

Apparently, the information carried in (4.33) is not sufficient to determine the $\mathbf{H}$ tensor, because the degradation on the shear modulus is hidden. To evaluate $\mathbf{H}$, additional provisions on homogenization may be needed. In the following, two approaches are adopted.

(i) *Dilute crack distribution*

Assume that the crack distribution is dilute and let $\mu^* = \mu$, $\nu^* = \nu$. Consider a single crack embedded in an RVE. Eq. (4.33) take the form

$$\epsilon^{(add)}_{\alpha\beta} = \Big(\frac{\pi a^2}{V}\Big)\frac{\omega(1+\kappa)}{4\mu}\Big[\frac{2\sigma_0}{\pi\Sigma_\infty}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big]\Sigma_\infty\delta_{\alpha\beta}, \quad \omega = 1,2 \tag{4.35}$$

One may write the above expression in a tensor form

$$\epsilon^{(add)}_k = \Big(\frac{\pi a^2}{V}\Big)\mathbf{H}^k : \mathbf{\Sigma}^{\infty} \tag{4.36}$$

where the local tensor $\mathbf{H}^k = H^k_{\alpha\beta\zeta\eta}\mathbf{e}^k_\alpha \otimes \mathbf{e}^k_\beta \otimes \mathbf{e}^k_\zeta \otimes \mathbf{e}^k_\eta$ is again anisotropic. Among its components

$$H^k_{1111} = H^2_{2222} = \frac{\omega(1+\kappa)}{4\mu}\Big(\frac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\ , \tag{4.37}$$

and $H^k_{\alpha\beta\zeta\eta} = 0$ otherwise.

The global isotropic tensor $\mathbf{H}$ can then be obtained by spatial averaging

$$\begin{aligned}\mathbf{H} &= \frac{1}{2\pi}\int_{a_m}^{a_M}\int_0^{2\pi}\Big(\frac{\pi a^2}{V}\Big)\mathbf{H}^k w(a,\theta)da d\theta \\ &= \frac{f}{2\pi}\int_0^{2\pi} H^k_{\alpha\beta\zeta\eta}\mathbf{e}^k_\alpha \otimes \mathbf{e}^k_\beta \otimes \mathbf{e}^k_\zeta \otimes \mathbf{e}^k_\eta w_0(\theta)d\theta \end{aligned}\tag{4.38}$$

where f is defined in Eq. (4.16). Use the procedure outlined before, one has

$$2h_1 + 2h_2 = fH^k_{\lambda\lambda\mu\mu} = f(H^k_{1111} + H^k_{2222}) \tag{4.39}$$

$$h_1 + 3h_2 = fH^k_{\lambda\mu\lambda\mu} = f(H^k_{1111} + H^k_{2222}) \tag{4.40}$$

The solution of Eq. (4.40) is

$$h_1 = \frac{1}{4}\frac{\omega f(1+\kappa)}{2\mu}\Big(\frac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\ , \tag{4.41}$$

$$h_2 = \frac{1}{4}\frac{\omega f(1+\kappa)}{2\mu}\Big(\frac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\ . \tag{4.42}$$

From $\bar{\mathbf{D}} = \mathbf{D} + \mathbf{H}$, it then can be shown that

$$\frac{\mu}{\bar{\mu}} = 1 + \frac{\omega(1+\kappa)f}{4}\Big[\frac{2\sigma_0}{\pi\Sigma_\infty}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big], \quad \omega = 1,2 \tag{4.43}$$

(ii) *Self-consistent method*

The essence of self-consistent method is considering the effect of micro-crack interaction (Hill [1965b], Budiansky and O'Connell [1976]). Let $\mu^* = \bar{\mu}, \nu^* = \bar{\nu}$ and assume that inside RVE the damaged medium is microscopically isotropic and that there exists a global isotropic tensor $\mathbf{H} = \frac{h_1}{2}\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2)} + h_2 \mathbf{1}^{(4s)}$, such that

$$\boldsymbol{\epsilon}^{(add)} = \mathbf{H} : \boldsymbol{\Sigma}^{\infty} \ , \tag{4.44}$$

which may lead to the determination of the overall compliance tensor, $\bar{\mathbf{D}} = \mathbf{D} + \mathbf{H}$.

Let

$$\mathbf{D} = \frac{1}{3K}\mathbf{E}^1 + \frac{1}{2\mu}\mathbf{E}^2 \tag{4.45}$$

$$\bar{\mathbf{D}} = \frac{1}{3\bar{K}}\mathbf{E}^1 + \frac{1}{2\bar{\mu}}\mathbf{E}^2 \tag{4.46}$$

$$\mathbf{H} = (h_1 + h_2)\mathbf{E}^1 + h_2\mathbf{E}^2 \tag{4.47}$$

where $\mu = \dfrac{E}{2(1+\nu)}$ and

$$\frac{1}{3K} = \begin{cases} \dfrac{(1-\nu)}{E}, & \text{plane stress} \\[2ex] \dfrac{(1-\nu-2\nu^2)}{E}, & \text{plane strain} \end{cases} \tag{4.48}$$

and similar expressions may be held for $\bar{\mu}$ and $\bar{K}$. Note that

$$\mathbf{E}^1 := \frac{1}{2}\delta_{\alpha\beta}\delta_{\zeta\eta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\zeta \otimes \mathbf{e}_\eta \tag{4.49}$$

$$\mathbf{E}^2 := \frac{1}{2}\Big(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} - \delta_{\alpha\beta}\delta_{\zeta\eta}\Big)\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\zeta \otimes \mathbf{e}_\eta \tag{4.50}$$

Nevertheless, Eq. (4.44) is only valid when $\boldsymbol{\Sigma}^{\infty} = \Sigma_\infty \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$. It only admits one equation

$$\bar{\mathbf{D}} : \Big(\Sigma_\infty\delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\Big) = (\mathbf{D} + \mathbf{H}) : \Big(\Sigma_\infty\delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\Big) \tag{4.51}$$

Combining Eqs. (4.33) and (4.51), one may find that

$$\frac{1}{3\bar{K}} = \frac{1}{3K} + (h_1 + h_2) = \frac{1}{3K} + \Big(\frac{1}{3\bar{K}} + \frac{1}{2\bar{\mu}}\Big)\omega f\Big[\frac{2\sigma_0}{\pi\Sigma_\infty}\tan\Big(\frac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] \tag{4.52}$$

by virtue of identities $\mathbf{E}^1 : \mathbf{1}^{(2)} = \mathbf{1}^{(2)}$ and $\mathbf{E}^2 : \mathbf{1}^{(2)} = \mathbf{0}$.

In Eq. (4.52), there are two unkowns, $\bar{K}$ and $\bar{\mu}$, or equivalently h_1 and h_2. An additional condition is needed to uniquely determine $\bar{\mathbf{D}}$ or $\mathbf{H}$. Impose the restriction

$$\frac{\bar{K}}{K} = \frac{\bar{\mu}}{\mu} \tag{4.53}$$

This implies that the relative reduction of the bulk modulus is the same as that of the shear modulus. This restriction will guarantee the positive definiteness of the overall strain energy, and it is reasonable for uniform biaxial loading condition.

Remark 4.1. *There may be some other possibilities for additional restriction. For example, $h_1 = 0$. However, in this particular problem, the restriction, $h_1 = 0$, may not guarantee the positive definiteness of overall strain energy.*

A direct consequence of (4.53) is

$$\bar{\nu} = \nu \tag{4.54}$$

Then for plane stress problems, one may find

$$\frac{\bar{K}}{K} = \frac{\bar{\mu}}{\mu} = \begin{cases} 1 - \dfrac{2\omega f}{(1-\nu)}\Big[\Big(\dfrac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] & \text{plane stress;} \quad (a) \\ 1 - \dfrac{2(1-\nu^2)\omega f}{(1-\nu-2\nu^2)}\Big[\Big(\dfrac{2\sigma_0}{\pi\Sigma_\infty}\Big)\tan\Big(\dfrac{\pi\Sigma_\infty}{2\sigma_0}\Big)\Big] & \text{plane strain;} \quad (b) \end{cases} \tag{4.55}$$

5. Micro-cohesive-crack damage models

Recall

$$\frac{\Sigma_\infty}{\sigma_0} = \frac{\chi \Sigma_m}{\phi \sigma_Y} = \begin{cases} \dfrac{3}{2}\dfrac{\Sigma_m}{\sigma_Y}, & \text{plane stress;} \\ \dfrac{3(1-2\nu^*)}{2(1+\nu^*)}\dfrac{\Sigma_m}{\sigma_Y}, & \text{plane strain .} \end{cases} \tag{5.1}$$

The overall shear moduli derived above only depend on spherical stress Σ_m. Therefore, one may be able to derive continuum damage models based the maximum distortional energy density closure. By substituting (4.28) into criterion (2.58), the following plastic yield function may be derived,

$$\boxed{\frac{\Sigma_{eq}^2}{\sigma_Y^2} = \frac{\bar{\mu}}{\mu} = \frac{1}{1 + \dfrac{(1+\kappa)f}{4\pi} B\Big(\dfrac{\Sigma_\infty}{\sigma_0}\Big)}} \tag{5.2}$$

where $B(\Sigma_\infty/\sigma_0)$ is defined in Eq. (2.37). It can be rewritten as

$$\boxed{\text{Model 1}: \quad \Psi_1(\Sigma_{eq}, \Sigma_m, \mathbf{q}) = \frac{\Sigma_{eq}^2}{\sigma_Y^2} - \frac{1}{1 + \dfrac{(1+\kappa)}{4\pi} f B\Big(\dfrac{\chi \Sigma_m}{\phi \sigma_Y}\Big)} = 0} \tag{5.3}$$

where $\mathbf{q}$ represents other possible internal variables.

Since the model is only valid when $f << 1$, a first order approximation may be taken

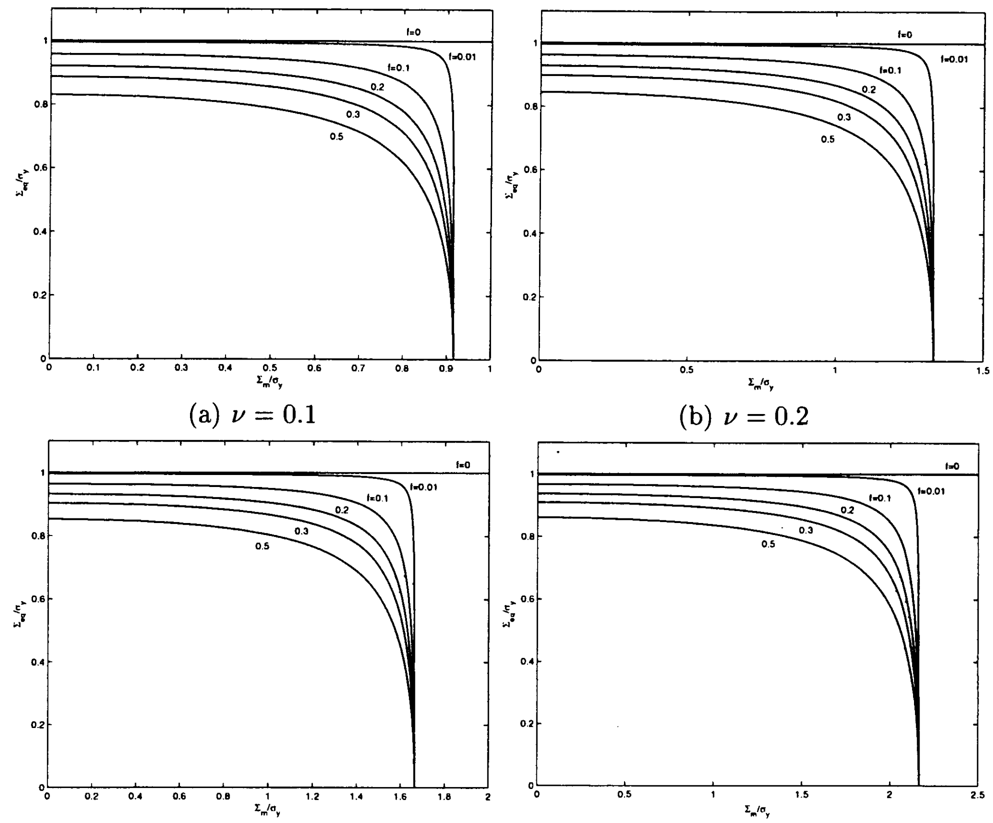
$$\Psi_1(\Sigma_{eq}, \Sigma_m, \mathbf{q}) = \frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{(1+\kappa)}{4\pi} f B\Big(\frac{\chi \Sigma_m}{\phi \sigma_Y}\Big) - 1 + \mathcal{O}(f^2) = 0 \tag{5.4}$$

Similarly, by substituting homogenization results obtained via energy methods (4.43) into (2.58), the following damage model may be derived,

$$\boxed{\text{Model 2}: \quad \Psi_2(\Sigma_{eq}, \Sigma_m, \mathbf{q}) = \frac{\Sigma_{eq}^2}{\sigma_Y^2} - \frac{1}{1 + \dfrac{\omega(1+\kappa)f}{4}\Big[\Big(\dfrac{2\phi\sigma_Y}{\pi\chi\Sigma_m}\Big)\tan\Big(\dfrac{\pi\chi\Sigma_m}{2\phi\sigma_Y}\Big)\Big]} = 0 \,.} \tag{5.5}$$

The first order approximation is

$$\Psi_2(\Sigma_{eq}, \Sigma_m, \mathbf{q}) = \frac{\Sigma_{eq}^2}{\sigma_Y^2} + \frac{\omega(1+\kappa)f}{4}\Big[\Big(\frac{2\phi\sigma_Y}{\pi\chi\Sigma_m}\Big)\tan\Big(\frac{\pi\chi\Sigma_m}{2\phi\sigma_Y}\Big)\Big] - 1 + \mathcal{O}(f^2) = 0 \,. \tag{5.6}$$



(a) $\nu = 0.1$ (b) $\nu = 0.2$

(c) $\nu = 0.25$ (d) $\nu = 0.30$

Figure 7. The cohesive micro-crack damage model: Ψ_1.

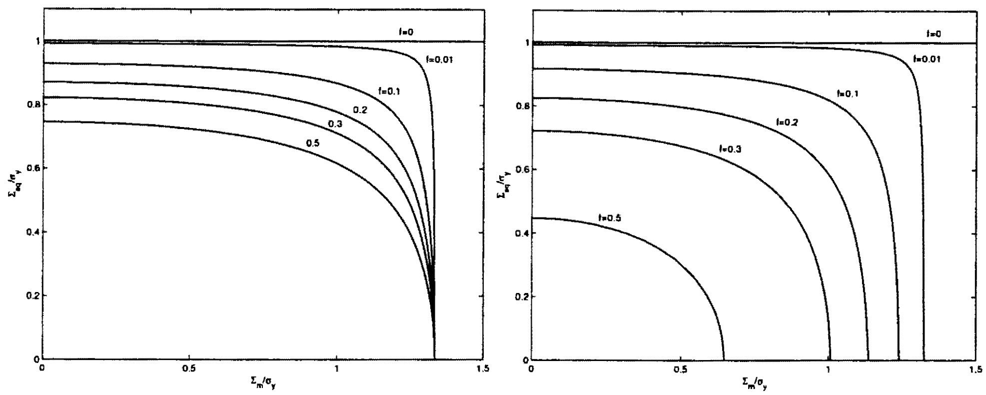
Substitute the corresponding homogenization results based on self-consistent method into Eq. (2.58). For plane stress problems, Eq. (4.55 (a)) leads to

$$\text{Model 3a}: \quad \Psi_{3a}(\Sigma_{eq}, \Sigma_m, \mathbf{q}) = \frac{\Sigma_{eq}^2}{\sigma_Y^2} + \Big(\frac{8\omega f \sigma_Y}{3\pi(1-\nu)\Sigma_m}\Big)\tan\Big(\frac{3\pi\Sigma_m}{4\sigma_Y}\Big) - 1 = 0\ . \tag{5.7}$$

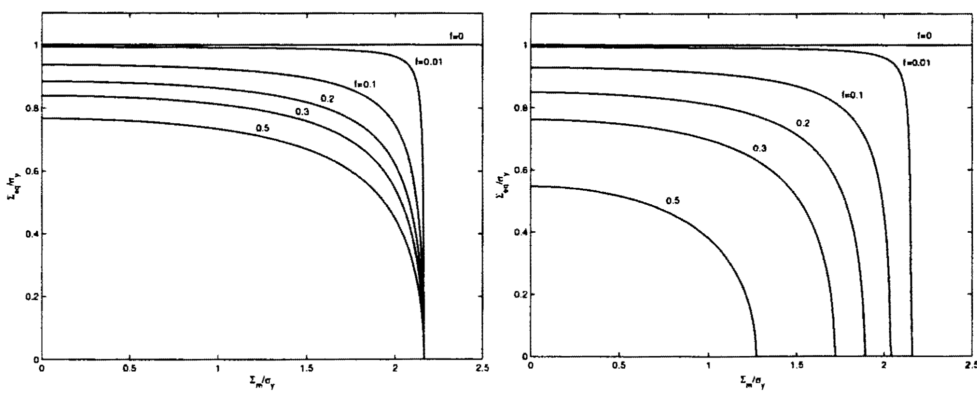
For plane strain problems, Eq. (4.55 (b)) leads to

$$\begin{aligned}\text{Model 3b}: \quad \Psi_{3b}(\Sigma_{eq}, \Sigma_m, \mathbf{q}) &= \frac{\Sigma_{eq}^2}{\sigma_Y^2} \\ &+ \Big(\frac{8(1-\nu^2)\omega f \sigma_Y}{3\pi(1-2\nu)^2\Sigma_m}\Big)\tan\Big(\frac{3\pi(1-2\nu)\Sigma_m}{4(1+\nu)\sigma_Y}\Big) - 1 = 0\ .\end{aligned} \tag{5.8}$$

Choose the overall yield function Ψ_{3a} as the paradigm for macro potential function of plastic flows. The macro plastic flow direction may be obtained by the



(a) The exact formula. (b) The first order approximation.



(c) The exact formula. (d) The first order approximation.

Figure 8. The cohesive crack damage model, Ψ_2, with different Poisson's ratios: (a), (b) $\nu = 0.2$; (c) (d) $\nu = 0.3$.

associative rule

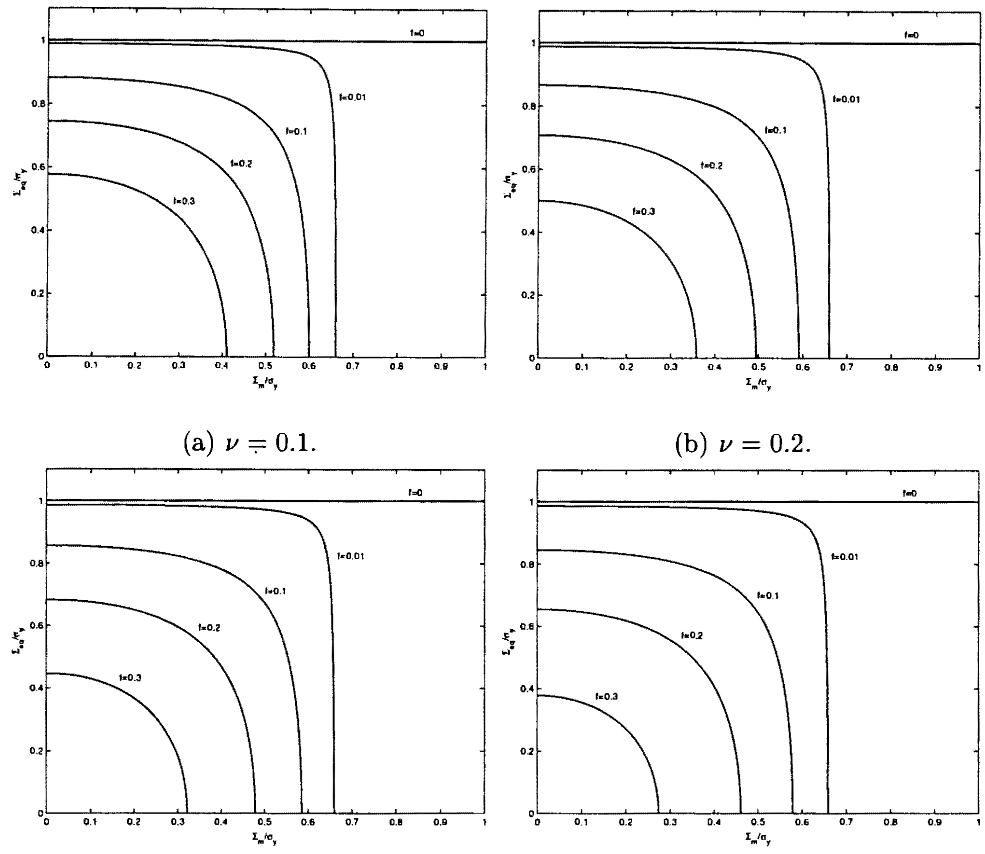
$$\mathbf{D}^p = \dot{\lambda}\mathbf{n} \tag{5.9}$$

where $\mathbf{D}^p = \dfrac{1}{2}\Big(\dot{u}^p_{i,j} + \dot{u}^p_{j,i}\Big)\mathbf{e}_i \otimes \mathbf{e}_j$, and

$$\begin{aligned} \mathbf{n} \;&=\; \frac{\partial \Psi_{3a}}{\partial \mathbf{\Sigma}} = \frac{3}{\sigma_Y^2}\mathbf{\Sigma}' + \frac{8\omega f \sigma_Y}{9\pi(1-\nu)\Sigma_m^2}\left\{\Big(\frac{3\pi\Sigma_m}{4\sigma_Y}\Big)\sec^2\Big(\frac{3\pi\Sigma_m}{4\sigma_Y}\Big)\right. \\ &\left. - \tan\Big(\frac{3\pi\Sigma_m}{4\sigma_Y}\Big)\right\}\mathbf{1} \end{aligned} \tag{5.10}$$

The scalar plastic flow rate can be determined by consistency condition as usual,

$$\dot{\lambda} = \frac{\langle \mathbf{r} : \mathbf{C} : \mathbf{D} \rangle}{-\Psi_{\mathbf{q}} \cdot \mathbf{h} + \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n}} \tag{5.11}$$



(a) $\nu = 0.1$. (b) $\nu = 0.2$.

(c) $\nu = 0.25$. (d) $\nu = 0.3$.

Figure 9. The cohesive crack damage model, Ψ_{3a}, with different Poisson's ratios (plane stress).

where $\mathbf{C}$ is the elastic stiffness tensor, and $< \cdot >$ is the Macauley bracket, and $\mathbf{q}$ are internal variables whose evolutions are assumed to be governed by

$$\dot{\mathbf{q}} = \dot{\lambda}\mathbf{h}(\mathbf{\Sigma}, \mathbf{q}) \tag{5.12}$$

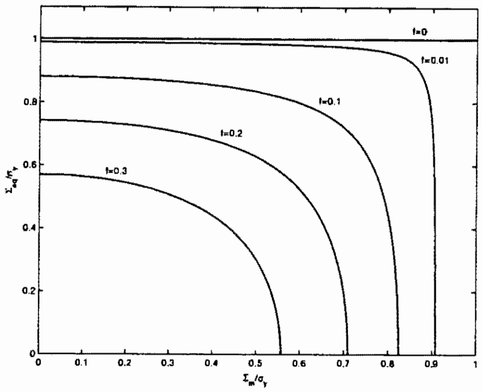
e.g. Lubliner [1990].

The damage evolution law for micro-cracks in a cohesive elastic environment may be significant different from that of voids in a perfectly viscoplastic environment. Let the volume of the RVE is denoted as $V = V_m + V_c$ where V_c is the crack opening volume and V_m is the volume of matrix. By assuming that the total rate change of the crack volume is proportional to *the volume rate change in a RVE*, i.e.

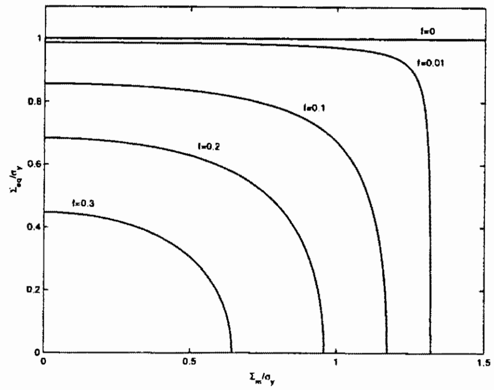
$$\dot{V}_c = \beta \dot{V} \tag{5.13}$$

where k is the proportionality constant. Then

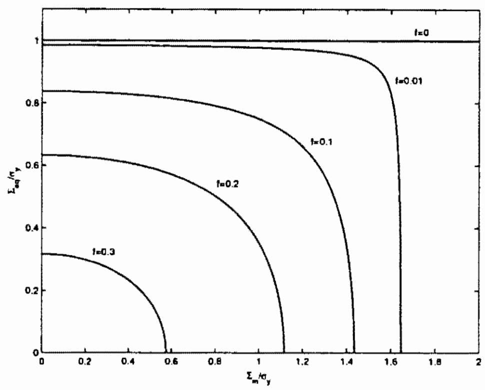
$$\dot{f}_{growth} = \frac{d}{dt}\Big(\frac{V_c}{V}\Big) = (\beta - f)\frac{\dot{V}}{V} = (\beta - f)trace(\mathbf{D}) \tag{5.14}$$



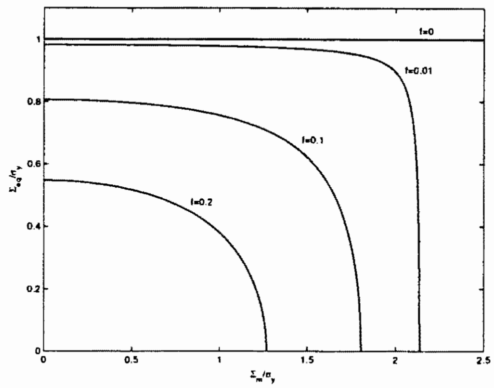
(a) $\nu = 0.1$.



(b) $\nu = 0.2$.



(c) $\nu = 0.25$.



(d) $\nu = 0.3$.

Figure 10. The cohesive crack damage model, Ψ_{3b}, with different Poisson's ratios (plane strain).

Neglecting elastic rate of deformation and assuming plastic rate of deformation is dominant, the conventional damage evolution law is recovered

$$\dot{f} = (1 - f) trace(\mathbf{D}^p) \tag{5.15}$$

On the other extreme, an argument can be made that $\dot{V}_e = \dot{V}_c$, since the volume fraction of cohesive cracks is obtained by integrating elastic crack opening displacement (COD),

$$\begin{aligned} \dot{f} &= (1 - \beta^{-1} f) trace(\mathbf{D}^e) = (1 - \beta^{-1} f) trace(\mathbf{D} - \mathbf{D}^p) \\ &= (1 - \beta^{-1} f) trace\left(\left(\mathbf{1} - \frac{(\mathbf{1} : \mathbf{n}) \otimes (\mathbf{n} : \mathbf{C})}{-\mathbf{\Psi_q} \cdot \mathbf{h} + \mathbf{n} : \mathbf{C} : \mathbf{n}} \right) : \mathbf{D} \right) \end{aligned} \tag{5.16}$$

6. Concluding remarks

The most distinguishing features of the present cohesive crack damage models are:

1. The homogenized macro-constitutive relations are different from the micro-constitutive relations: the reversible part of macro-constitutive relation is non-linear elastic versus the linear elastic behaviors at micro-level; the irreversible part of macro-constitutive relation is a form of pressure sensitive plasticity versus the Huber-von Mises plasticity or cohesive laws at micro-level.

2. When the ratio of spherical stress and the true yield reaches a finite value, i.e.

$$\frac{\Sigma^{\infty}}{\sigma_0} \to 1 \quad \Rightarrow \frac{\Sigma_m}{\sigma_Y} \to \begin{cases} \dfrac{2}{3}, & \text{plane stress} \\ \dfrac{2(1+\nu)}{3(1-2\nu)}, & \text{plane strain} \end{cases} \tag{6.1}$$

   the Dugdale-BCS cohesive crack model will predicts a complete failure in material with infinitesimal amount of initial damage, whereas under the same condition, the Gurson model, which is based on void growth, will not predict a complete material failure unless the spherical stress approaches to infinity;

3. In cohesive damage models, the overall yield surfaces as well as damage evolution equations depend on materials Poisson's ratio; whereas in the Gurson model, no such dependence is observed, because of the assumption of incompressible materials;

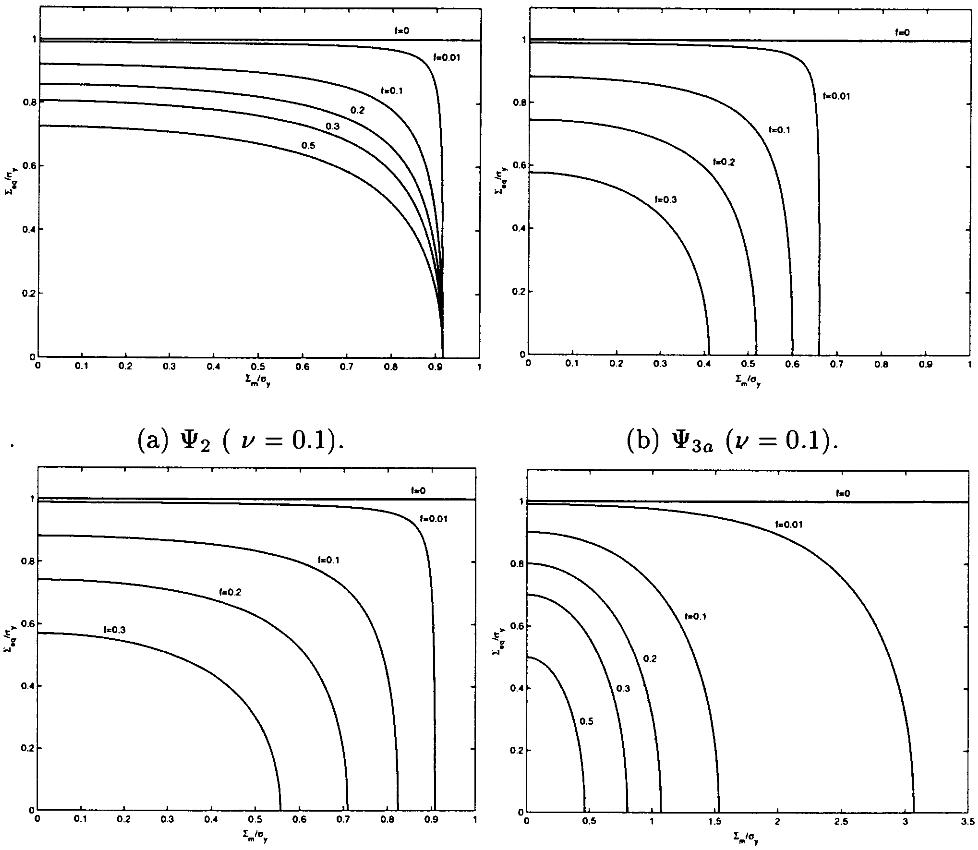
4. The rate of crack opening volume growth may depend on the rate of elastic deformation.

It may be worth noting that since cohesive damage models are obtained from the homogenization of the analytical solution of cohesive crack under uniform biaxial tension, it may break down for incompressible materials ($\mu = 0.5$). In that case, the dilatational energy density approaches zero. From this perspective, the cohesive damage model can not replace the Gurson model completely.

The damage model Ψ_1 and damage model Ψ_2 are derived based on the dilute crack distribution. Both cases are plotted for the case of plane strain (see Figs. 7 and 8). From Figs. 7 and 8(a) and (c), it may be observed that as Σ_m/σ_Y increases, all the yield loci converge to one point when $\Sigma_\infty/\sigma_0 \to 1$, or $\Sigma_{eq}/\sigma_Y \to 0$. An explanation for this unusual phenomenon is because when $\Sigma_\infty = \Sigma_m \to \sigma_0$, the remote stress reaches to the value of decohesion. At that point, the size of the cohesive zone will become infinite ($b \to \infty$), and the assumption of dilute crack distribution is no longer valid. Interestingly enough, this abnormality disappears when the first order approximation is adopted (see Figs. 8(b)(d)).

For the damaged yield loci obtained from self-consistent scheme, the abnormality does not appear (see Figs. 9 and 10), since the consideration of crack interactions.

Note that almost all the cohesive damage models derived from two-dimensional homogenization have the same functional format, except the model Ψ_1 based on 2.37(a). That is that damage due to hydrostatic stress state is controlled by a tangent function, contrasting to the hyperbolic cosine function for the Gurson model.



(a) Ψ_2 ($\nu = 0.1$).

(b) Ψ_{3a} ($\nu = 0.1$).

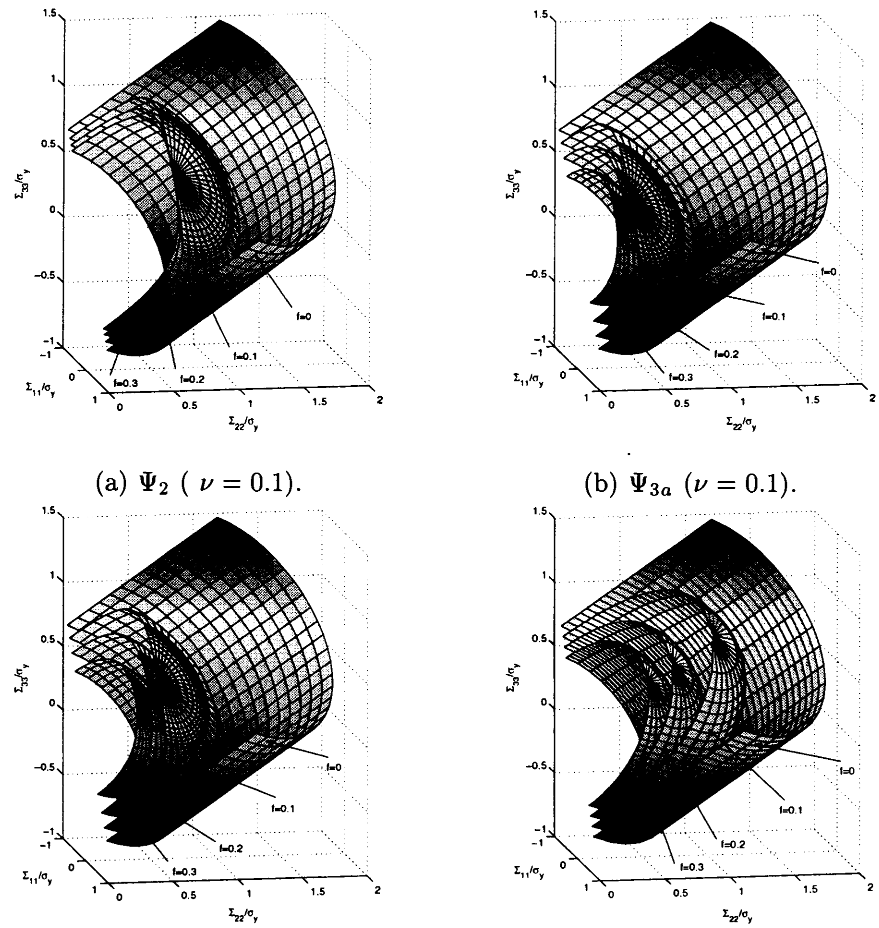
(c) Ψ_{3b} ($\nu = 0.1$).

(d) The Gurson model

Figure 11. Comparison of cohesive micro-crack damage models and the Gurson model.

All the damage models are juxtaposed and displayed in comparison with the Gurson model (see Figs. 11 and Figs. 12) It may be found that there is a major difference between these two types of damage models when the damaged volume fraction (either crack volume or void volume) is small. If $f << 1$, the Gurson model predicts a much higher tolerant hydrostatic stress level than cohesive damage model does. This is because that in cohesive crack solution once the remote stress reaches the decohesion level σ_∞, the material fails immediately, whereas for the Gurson model, the material only fails when hydrostatic stress reaches to infinite under the condition that f is infinitesimal. From this view point, it seems that the cohesive damage model makes more sense than the Gurson model does.

The key to accurately construct a cohesive damage model is to determine the energy release contribution to the material damage process. The energy release in nonlinear fracture mechanical process is consumed in several different dissipation processes, e.g. surface separation, heat generation, dislocation movement, and may be even phase transformation, etc. In fact, Wnuk [1972,1990], Kfouri and Rice [1978] , and Kfouri[1979] have studied energy release caused by crack extension of



(a) Ψ_2 ($\nu = 0.1$). (b) Ψ_{3a} ($\nu = 0.1$).

(c) Ψ_{3b} ($\nu = 0.1$). (d) The Gurson model

Figure 12. Comparison of cohesive micro-crack damage models and the Gurson model.

two-dimensional Dugdale-BCS cracks. An in-depth study may be needed to refine the damage models proposed here.

In fact, the damage due to interaction of cohesive micro-cracks has been attracted some attentions, e.g. recent work by Feng and Gross [2000]. It is speculated that in general by considering interaction induced coalescence among cohesive cracks, one may find a critical micro-crack opening volume fraction, f_c, based on Dugdale-BCS crack model, e.g. the solution of periodic distribution of cohesive crack by Bilby et al [1964].

A three-dimensional cohesive crack damage model is recently reported by Li et al [2001].

References

Barenblatt, G. I. [1959], "The formation of equilibrium cracks during brittle fracture," *J. Appl. Math. & Mechanics, [English translation PMM, Vol. 23, 434]*, **23**, 622

Barenblatt, G. I. [1962], "Mathematical theory of equilibrium cracks in brittle fracture," *Advances in Applied Mechanics*, **7**, 55-129, Academic Press.

Becker, W. & Gross, D. [1987a], "An one-dimensional micromechanical model of elastic-microplastic damage evolution," *Acta Mechanica*, **70**, 221-233.

Becker, W. & Gross, D. [1987b], "About the Mode II Dugdale crack solution," *International Journal of Fracture*, **34**, 65-70.

Becker, W. & Gross, D. [1988a], "About the Dugdale crack under mixed mode loading," *International Journal of Fracture*, **37**, 163-170.

Becker, W. & Gross, D. [1988b], "A two-dimensional micromechanical model of anisotropic elastic-microplastic damage evolution," *Ingenieur-Archiv*, **58**, 295-304.

Becker, W. & Gross, D. [1989], "About the penny-shaped Dugdale crack under shear and triaxial loading," *Proceedings of The Seventh International Conference on Fracture (ICF7)*, **3**, 2289-2299, Pergamon, Oxford, UK.

Bilby, B. A. & Cottrell, A. H. & Swinden, K. H. [1963], "The spread of plastic yield from a notch," *Proceedings of Royal Society of London, Series A*, **272**, 304-314.

Bilby, B. A. & Cottrell, A. H. & Smith, E. [1964]. "Plastic Yielding from Sharp Notches," *Proceedings of Royal Society of London, Series A*, **279**, 1-9.

Broberg, K. B. [1999], *Cracks and Fracture*, Academic Press, San Diego · London · Boston

Budiansky, B. [1965], "On the elastic moduli of some heterogeneous materials," *Journal of Mechanics and Physics of Solids*, **13**, 223-227

Budiansky, B. & Rice, J. R. [1973], "Conservation laws and energy release rates," *Journal of Applied Mechanics*, **26**, 201-203.

Budiansky, B. & O'Connell, R. J. [1976], "Elastic Moduli of a Cracked Solid," *International Journal of Solids and Structures*, **12**, 81-97.

Budiansky, B. & Huchinson. J. W. & Slutsky, S. [1982], "Void growth and collapse in viscous solids," In *Mechanics of Solids*, edited by Hopkins, H. G. & Sewell, M. J. Pergamon Press, 13-45.

Dugdale, D. S. [1960], "Yielding of Steel Sheets Containing Slits," *Journal of Mechanics and Physics of Solids*, **8**, 100-104.

Eshelby, J. D. [1957], " The Determination of the Elastic Field of An Ellipsoidal Inclusion, and Related Problems," *Proc. R. Soc. London, Ser. A*, Vol. 46, pp. 376-396

Eshelby, J. D. [1961], "Elastic inclusions and inhomogeneities," *Progress in Solid Mechanics*, **2**, edited by I. N. Sneddon, North-Holland, Amsterdam, 89-140.

Feng, X. Q. & Gross, D. [2000], "On the coalescence of collinear cracks in quasi-brittle materials," *Engineering Fracture Mechanics*, **65**, 511-524.

Fleck, N. [1991], "Brittle Fracture Due to an Array of Microcracks," *Proc. R. Soc. (London)*, **A432**, 55-76.

Gao, H. & Klein, P. [1999], "Numerical simulation of crack growth in an isotropic solid with randomized internal cohesive bonds," *Journal of Mechanics and Physics of Solids*, **46**, 187-218.

Gao, X. & Faleskog, J. & Shih, C. F. [1998], "Cell model for nonlinear fracture analysis — II. Fracture-process calibration," *International Journal of Fracture*, **89**, 374-386.

Goodier, J. N. [1968], "Mathematical Theory of Equilibrium Cracks," *Fracture: An Advanced Treatise*, edited by H. Liebowitz, **2**, 1-66, Academic Press, New York and London.

Gurson, A. L. [1975], "Plastic flow and fracture behavior of ductile materials incorporating void nucleation, growth and interaction," *Ph.D. Thesis, Brown University*, Rhode Island.

Gurson, A. L. [1977], "Continuum theory of ductile rupture by void nucleation and growth: Part I yield criteria and flow rules for porous ductile materials," *Journal of Engineering Materials and Technology*, **99**, 2-15.

Gudmundson, P. [1990], "Anisotropic microcrack nucleation in brittle materials," *Journal of the Mechanics and Physics of Solids*, **38**, 531-551.

Hencky, von H. [1924], "Zur Theorie Plastischer Deformationen und der Hierdurch im Material Hervorgerufen Nachspannungen," rnal = Zeits. ang. Math. Mech., **4**, 323-334.

Hill, R. [1963], "Elastic Properties of reinforced Solids: Some Theoretical Principles," *Journal of the Mechanics and Physics of Solids*, **11**, 357-372.

Hill, R. [1965a], "Theory of mechanical properties of fiber-strengthened materials-III: Self-consistent model," *Journal of the Mechanics and Physics of Solids*, **13**, 189-198.

Hill, R. [1965b], "A Self-consistent mechanics of composite materials," *Journal of the Mechanics and Physics of Solids*, **13**, 213-222.

Hill, R. [1967], "The essential structure of constitutive laws for metal composite and polycrystals," *Journal of the Mechanics and Physics of Solids*, **15**, 79-95.

Hoenig, A. [1979], "Elastic moduli of a non-randomly cracked body," *International Journal of Solids and Structures*, **15**, 137-154.

Horii, H. & Nemat-Nasser, S. [1983], "Overall moduli of solids with microcrack load-induced anisotropy," *Journal of the Mechanics and Physics of Solids*, **31**, 155-171.

Horii, H. & Nemat-Nasser, S. [1986], "Brittle fracture in compression: Splitting, faulting and brittle-ductile transition," *Phil. Trans. Roy. Soc. (London)* , **A 319**, 337-374.

Hutchinson, J. W. [1987], "Crack tip shielding by micro-cracking in brittle solids," *Acta Metallurgica*, **35**, 1605-1619.

Janson, J. [1977], "Dugdale crack in a material with continuous damage formation," *Engineering Fracture Mechanics*, **9**, 891-899.

Kachanov, M. [1985], "A simple technique of stress analysis in elastic solids with many cracks, *International Journal of Fracture*, **28**, R11-R19.

Kachanov, M. [1994], "Elastic solids with many cracks and related problems," *Advances in Applied Mechanics*, **32**, 259-445, edited by Hutchinson, J. W. & and Wu, T. Y. Academic Press, New York.

Kanninen, M. F. [1968], "An estimate of the limiting speed of a propagating ductile crack," *Journal of the Mechanics and Physics of Solids*, **16**, 215-128.

Kanninen, M. F. [1970], "A solution for a Dugdale crack subjected to a linearly varying tensile loading," *International Journal of Engineering Science*, **8**, 85-95.

Kanninen, M. F. & Popelar, C. H. [1985], *Advanced Fracture Mechanics*, Oxford University Press, New York · Oxford .

Keer, L. M. [1964], "Stress Distribution at the edge of an equilibrium crack," *Journal of the Mechanics and Physics of Solids*, **12**, 149-163.

Keer, L. M. & Mura, T. [1965], "Stationary crack and continuous distributions of dislocations," *Proceedings of The First International Conference on Fracture*, The Japanese Society for Strength and Fracture of Materials, **1**, 99-115.

Kfouri, A. P. & Rice, J. R. [1978], "Elastic/Plastic separation rate for crack advance in finite growth steps," *Advances in Research on the Strength and Fracture of Materials (Proceedings of Fourth International Conference on Fracture)*, **1**, 43-59, edited by D.M.R. Taplin, Pergamon Press.

Kfouri, A. P. [1979], "Crack separation energy-rates for the DBCS model under biaxial modes of loading," *Journal of Mechanics and Physics of Solids*, **27**, 135-150.

Krajcinovic, D. [1996], *Damage Mechanics*, Elsevier, Amsterdam · New York.

Li, S. & Wang, G. [2001], "A continuum damage model based on homogenization of distributed cohesive micro-cracks," *Technical Report: Department of Civil and Environmental Engineering, University of California*, **UCB/SEMM-2001/10**

Lu, T. J. & Chow, C. L. [1990], "A modified Dugdale model for crack tip plasticity and its related problems," *Engineering Fracture Mechanics*, **37**, 551-568.

Libliner, J. [1990], *Plasticity Theory*, Macmillan Publishing Company, New York

McClintock, F. A. [1968], "A criterion for ductile fracture by the growth of holes," *ASME Journal of Applied Mechanics*, **35**, 363-371.

Mou, Y. & Han, R. P. S. [1994], "Damage zones based on Dugdale model for materials," *International Journal of Fracture*, **68**, 245-259.

Mura, T. 1987 *Micromechanics of Defects in Solids*, Martinus Nijhoff Publishers, Dordrecht

Needleman, A. and V. Tvergaard, V. [1984], "An analysis of ductile rupture in notched bars," *Journal of the Mechanics and Physics*, **32**, 461-490.

Nemat-Nasser, S. & Hori, M. [1999] *Micromechanics: Overall Properties of Heterogeneous Materials*, Second Edition, North-Holland, Amsterdam

Ortiz, M. and Pandolfi, A. [1999], "Finite-deformation irreversible cohesive elements for three-dimensional crack-propagation analysis," *International Journal for Numerical Methods in Engineering*, **44**, 1267-1282.

Rice, J. R. [1968a], "A path independent integral and the approximate analysis of strain concentration by notches and cracks," *ASME, Journal of Applied Mechanics*, **35**, 379-386.

Rice, J. R. [1968b], "Mathematical analysis in the mechanics of fracture," *Fracture: An Advanced Treatise*, **2**, 191-311, Edited by Liebowitz, H. Academic Press.

Rice, J. R. & D. M. Tracey [1969], "On the ductile enlargement of voids in triaxial stress fields," *Journal of the Mechanics and Physics of Solids*, **17**, 201-217.

Rice, J. R. & Thomson, R. [1974], "Ductile versus brittle behavior of crystals," *Philosophical Magzine*, **29**, 73-97.

Rice, J. R. & Sorensen, E. P. [1976], "Continuing crack-tip deformation and fracture for plane-strain crack growth in elastic-plastic solids," *Journal of Mechanics and Physics of Solids*, **26**, 163-186.

Rice, J. R. [1992], "Dislocation nucleation from a crack tip: An analysis based on the Peierles concept," *Journal of Mechanics and Physics of Solids*, **40**, 239-271.

Tvergaard, V. [1981], "Influence of voids on shear band instabilities under plane strain conditions," *International Journal of Fracture*, **17**, 389-407.

Tvergaard, V. [1982], "On localization in ductile materials containing voids," *International Journal of Fracture*, **18**, 237-251.

Tvergaard, V. and Needleman, A. [1984], "Analysis of the cup-cone fracture in a round tensile bar," *Acta Metall.*, **32**, 157-169.

Tvergaard, V. [1990], "Material failure by void growth to coalescence," *Advances in Applied Mechanics*, **27**, 83-151, Edited by Hutchinson, J. W. & Wu, T. Y. **27**, Academic Press, New York.

Weertman, J. [1984a], "Crack growth for the double slip plane crack model. I. The R-curve," *Acta Metallurgica*, **32**, 563-573.

Weertman, J. [1984b], "Crack growth for the double slip plane and the modified DSP crack model. II. Fatigue crack growth with and without blunting," *Acta Metallurgica*, **32**, 575-584.

Weertman, J. [1996], *Dislocation Based Fracture Mechanics*, World Scientific, Singapore · New Jersey · Hong Kong

Wnuk, M. P. [1972], "Accelerating crack in a viscoelastic solid subject to subcritical stress intensity," *Proceedings of International Conference on Dynamic Crack Propagation*, edited by G. Sih, Lehigh University, 273-280.

Wnuk, M. P. [1990], "Mathematical modeling of nonlinear phenomena in fracture mechanics," *Nonlinear Fracture Mechanics*, 359-451, editor Enuk, M. P., Springer-Verlag, Wien-New York.

Xu, P. & Needleman, A. [1994], "Numerical simulations of fast crack growth in brittle solids," *Journal of the Mechanics and Physics of Solids*, **42**, 1397-1434.

Zhang, CH. & Gross, D. [1994], "Ductile crack analysis by a cohesive damage zone model," *Engineering Fracture Mechanics*, **47**, 237-248.