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Amalgamation Constructions And Recursive Model Theory

Author
Andrews, Uri

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Amalgamation Constructions And Recursive Model Theory

by

Uri Andrews

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Committee in charge:

Professor Thomas Scanlon, Chair
Professor Leo Harrington
Professor Roger Purves

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Abstract

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Uri Andrews

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Professor Thomas Scanlon, Chair

We employ the Hrushovski Amalgamation Construction to generate strongly minimal examples of interesting recursive model theoretic phenomena. We show that there exists a strongly minimal theory whose only recursively presentable models are prime or saturated. We show that there exists a strongly minimal theory in a language with finite signature whose only recursively presentable model is the saturated model. Similarly, we show that for every $k \in \omega + 1$ there exists a strongly minimal theory in a language with finite signature whose recursively presentable models are those with dimension less than k. Finally, we characterize the complexity of strongly minimal or $\aleph_0$-categorical theories that have only recursively presentable models by generating examples in every possible tt-degree.

This thesis is dedicated to my parents, Fred and Esther, who have raised me to be the person that I am today. You have always been there for me, supporting me, proud of me, encouraging me, and loving me. You gave me the confidence to succeed and the foundation to risk failure. I love you both very much.

Contents

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Chapter 1

Introduction & Background

Recursive mathematics explores the extent to which mathematical constructions can be carried out effectively. The particular additional perspective of recursive model theory is to analyze structures based on their theories. Model theorists classify the level of structure afforded to models by their theories. This allows for a hierarchy beginning with the most structured, the strongly minimal theories, extending to the ω-stable theories, then the super-stable, stable, and much work has been done in understanding even unstable theories. We will focus on the strongly minimal theories and despite the structural characterization of strongly minimal models, we will generate examples of interesting recursive model theoretic phenomena.

Hrushovski [8] developed the amalgamation method for generating strongly minimal theories. We will present new tools for using amalgamation constructions to partly answer the following three general questions regarding strongly minimal theories:

Question 1. *What relationships exist between the complexity of the theory and the complexity of its models?*

Question 2. *If some models of a theory are recursive, do we know that other models of the theory are recursive?*

Question 3. *If we restrict our sights to theories in languages with finite signatures, does that change the answer to the first two questions?*

1.1 Basic Definitions

Definition 1.

- *A subset S of the natural numbers is recursive if there exists a Turing machine which given input x outputs 1 if $x \in S$ and 0 otherwise.*

- *We will work with recursive languages. We say a set of formulae is recursive if the set of Gödel codes for formulae in the set is a recursive subset of the natural numbers.*
- *A set A is turing reducible to a set B, written $A \leq_T B$, if there exists a turing machine using B as an oracle determining membership in A. Two sets A and B are turing equivalent, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$. The equivalence class of A under $\equiv_T$ is the Turing degree of A.*
- *A set A is tt-reducible to a set B, written $A \leq_{tt} B$ if there exists a total turing machine using B as an oracle determining membership in A, where a total turing machine is one which halts on any input and any oracle. Two sets A and B are tt-equivalent, written $A \equiv_{tt} B$, if $A \leq_{tt} B$ and $B \leq_{tt} A$. The equivalence class of A under $\equiv_{tt}$ is the tt-degree of A.*
- *Unless otherwise stated, all languages are countable and recursive. Further, we will often conflate a language with its signature, saying for instance that L is finite to mean that L has finite signature.*

Definition 2.
- *A model M with universe ω is decidable if $\{\phi(a_1, \dots a_n) | \phi(x_1 \dots x_n) \in L \wedge_{i=1}^n a_i \in \omega \wedge M \models \phi(a_1, \dots a_n)\}$ is a recursive set.*
- *A model M is decidably presentable if M is isomorphic to a decidable model.*
- *A model M with universe ω is recursive if $\{\phi(a_1, \dots a_n) | \phi(x_1 \dots x_n) \in L \wedge_{i=1}^n a_i \in \omega \wedge M \models \phi(a_1, \dots a_n) \wedge \phi$ is quantifier-free$\}$ is a recursive set.*
- *A model M is recursively presentable if M is isomorphic to a recursive model.*
- *Let M be any L-structure, and suppose L' is a language with signature contained in the signature of L. Then the structure $M|_{L'}$ is the L'-structure where each relation symbol is interpreted as in M. We also think of $M|_{L'}$ as the L-structure where each symbol in $L \setminus L'$ is interpreted as the empty relation.*

Definition 3.
- *A subset of $\mathbb{N}$ is Σ_n it is defined in $(\mathbb{N}, +, \cdot)$ by a formula with $\leq n$ alternations of unbounded quantifiers starting with an existential quantifier. Such a formula is called a Σ_n formula. A subset of $\mathbb{N}$ is Σ_n in X if it is defined in $(\mathbb{N}, +, \cdot, X)$ by a Σ_n formula.*
- *A subset of $\mathbb{N}$ is Π_n it is defined in $(\mathbb{N}, +, \cdot)$ by a formula with $\leq n$ alternations of unbounded quantifiers starting with a universal quantifier. Such a formula is called a Π_n formula. A subset of $\mathbb{N}$ is Π_n in X if it is defined in $(\mathbb{N}, +, \cdot, X)$ by a Π_n formula.*
- 0^n *is the set of Gödel codes of true Σ_n formulas in the structure $(\mathbb{N}, +, \cdot)$.*

- $0^\omega = \{\langle i, n\rangle | n \in 0^i\}$, *where* $\langle \cdot, \cdot \rangle$ *is a recursive pairing function, ie: a bijection from* $\mathbb{N}^2$ *to* $\mathbb{N}$.

1.2 Model Theory of Strongly Minimal Theories

The content of this section is standard and can be found for instance in Marker [13].

Definition 4. *A theory T is strongly minimal if for all $M \models T$ and all $\phi(x) \in L_M$ (in one variable and allowing parameters from M), $\{x \in M | M \models \phi(x)\}$ is either finite or co-finite in M.*

A model M is strongly minimal if $\mathrm{Th}(M)$ *is strongly minimal.*

Strongly minimal theories form one of the nicest classes of structures. In one variable, all definable sets are definable from the language with just equality. The following are some classical examples of strongly minimal theories.

- Let L be the language generated by equality alone. Let T be the theory of an infinite set.
- Let L be the language generated by a single binary relation symbol. Let T be the theory of an infinite regular acyclic graph.
- Let L be the language generated by $\{+\} \cup \{c \cdot | c \in \mathbb{Q}\}$. Let T be the theory of a vector space over $\mathbb{Q}$ where $c \cdot x$ is interpreted as multiplying the vector x by the scalar c.
- Let L be the language generated by $\{+, \cdot, 0, 1\}$. Let T be the theory of any algebraically closed field.

Also note that any reduct of a strongly minimal theory to a smaller language is strongly minimal, and strongly minimal theories remain strongly minimal after naming constants. All of these example theories are very well behaved. In particular, we have notions of closure and dimension which characterize models up to isomorphism. In the first example, the closure of a set is the set itself and the dimension of a model is its cardinality. In the second example, the closure of a set is the union of the connected components of its elements and dimension is the number of connected components in the model. In the third example, the closure of a set is its span, and dimension is regular dimension of vector spaces. In the fourth case, closure is algebraic closure and dimension is transcendence degree. This is no coincidence. In fact, we will see that these are particular cases of a notion of closure and dimension which are defined for any strongly minimal theory.

Definition 5. *Suppose $A \subseteq M$ and M is strongly minimal. We write $acl_M(A)$ for the set $\{b \in M | \exists \phi(x) \in L_A \exists n \in \omega\, M \models ((\neg \exists^n y \phi(y)) \wedge \phi(b))\}$, ie: the set of elements contained in finite A-definable sets.*

Claim 6. *M along with the acl_M operator forms a pre-geometry, ie: it satisfies the following 4 conditions:*

1. *$A \subseteq acl_M(A)$*
2. *$acl_M(acl_M(A)) = acl_M(A)$*
3. *$acl_M(A) = \bigcup \{acl_M(A_0) | A_0$ a finite subset of $A\}$*
4. *If $a \in acl_M(A \cup \{b\}) \setminus acl_M(A)$ then $b \in acl_M(A \cup \{a\})$*

Proof. 1-3 are relatively easy to see, so let us prove 4. Suppose $\phi(x) \in L_A$, $M \models \phi(a, b)$, and there are only finitely many x realizing $\phi(x, b)$. Then let us examine the set $\{y | \phi(a, y)\}$. If this set is finite, then $b \in acl_M(A \cup \{a\})$. If this set is infinite then it is co-finite and since $a \notin acl_M(A)$, the set $S_x = \{y | \phi(x, y)\}$ is co-finite for all but finitely many x. But $b \in S_x$ for only finitely many x, say n such x. Thus $M \models \neg \exists^{n+1} x (b \in S_x)$, but there can only be finitely many elements satisfying this formula as the intersection of any $n + 1$ co-finite sets is co-finite. So $b \in acl_M(A)$. □

Definition 7. *Suppose $A \subseteq M$. We say A generates M if $acl_M(A) = M$. We say A is independent if for each $a \in A$, $a \notin acl_M(A \setminus \{a\})$. If A generates M and A is independent, then we say A is a basis for M.*

Claim 8. *Every pre-geometry has a basis. In particular, every strongly minimal model has a basis. Further, any two bases must have the same cardinality.*

Proof. This proof is similar to the analogous statement about vector spaces. The one difference is that we use the exchange property, property 4 of pre-geometries, where the proof for vector spaces multiplies by scalars and reorganizes variables within an equation. □

Definition 9. *Let M be a strongly minimal model. We set $dim(M)$, the dimension of M, to be the cardinality of a basis for M.*

Claim 10. *Let M and N be models of a strongly minimal theory T. If $dim(M) = dim(N)$ then $N \cong M$.*

Proof. We build an elementary map one element at a time. We start with ρ_0 mapping one basis to the other. To build $\rho_{\alpha+1}$, we take an element a of M. We set the formula ϕ_a to define the smallest $dom(\rho_\alpha)$-definable set containing a. There must be an element c satisfying $\rho(\phi)$ (the formula achieved by replacing parameters $\bar{b}$ by $\rho(\bar{b})$). The extended map $\rho_{\alpha+1} = \rho_\alpha \cup \{\langle a, c \rangle\}$ is still elementary. This defines a forth step. Reversing roles of M and N gives the back step. □

Claim 11. *Let M be a strongly minimal model. Suppose $N \subseteq M$ is an infinite substructure so that $\mathrm{acl}_M(N) = N$, ie: N is algebraically closed in M. Then $N \preceq M$.*

Proof. We use the Tarski-Vaught test to verify that $N \preceq M$: Fix a formula $\phi(x) \in L_N$, and suppose $M \models \exists x \phi(x)$. There are two cases to consider depending on whether the set $\{x \in M | M \models \phi(x)\}$ is finite or co-finite. Suppose this set is finite. Then the set is contained in $acl_M(N) = N$, so $M \models \exists x \in N \phi(x)$. Suppose the set is co-finite. Since N is infinite, $M \models \exists x \in N \phi(x)$. □

Claim 12. *Suppose M is strongly minimal and infinite, $dim(M) = k$, $k \in \omega$, and $k < n \in \omega + 1$. Then there is a model $N \models \mathrm{Th}(M)$ such that $dim(N) = n$. Further, there is such an N such that $M \preceq N$.*

Proof. For $n \in \omega$, it suffices to show this for $n = k + 1$. We will apply compactness to get a larger model than M, then we will take the algebraic closure of a generic $k + 1$ element set to form our model N. Fix $\bar{a}$ a basis for M. Let $p(x)$ be the 1-type over $\bar{a}$ defined by $\phi(x) \in p \leftrightarrow \{x \in M | M \models \phi(x)\}$ is infinite (equivalently co-finite). Let Γ be the set of sentences $Diag(M) \cup p(c)$, where $Diag(M)$ is the full elementary diagram of M and c is a new constant symbol. By compactness and the fact that the intersection of finitely many co-finite sets is non-empty, Γ is consistent. Let M' be a model of Γ. We have $M \preceq M'$, and $\{\bar{a}, c\}$ is independent in M'. Let $N = acl_{M'}(\{\bar{a}, c\})$. By property 2 of pre-geometries, $N = acl_{M'}(N)$, so $N \preceq M'$. Thus $N \models \mathrm{Th}(M)$ and it is clear by the definition of N that $\{\bar{a}, c\}$ forms a basis for N and thus $dim(N) = k + 1$.

Having the result for $n \in \omega$, we have a chain: $M_k \preceq M_{k+1} \preceq M_{k+2} \preceq \ldots \preceq M_\omega$, where $M_\omega = \bigcup_{i \geq k} M_i$. It is easy to see that M_ω cannot have a finite basis, thus has dimension ω. □

Baldwin and Lachlan [2] showed that any $\aleph_1$-categorical theory is 'controlled' by a strongly minimal part, and thus they extended the following result to apply to any $\aleph_1$-categorical theory.

Corollary 13. *(Baldwin-Lachlan Theorem for Strongly Minimal Theories) Let T be a strongly minimal non-$\aleph_0$-categorical theory with infinite models. Then the countable models of T form an elementary $\omega + 1$-chain: $M_0 \prec M_1 \prec M_2 \prec \ldots \prec M_\omega$.*

Proof. Let M_0 be the countable model of smallest dimension. This dimension is finite since otherwise each countable model would have dimension ω, and T would be $\aleph_0$-categorical by claim 10. Repeatedly applying claim 12 gives us the desired chain of models. As every model is characterized up to isomorphism by its dimension, every countable model of T appears in the chain. □

1.3 The Spectrum Problem

Definition 14. *Fix a non-$\aleph_0$-categorical strongly minimal theory T with infinite models. Let $\mathcal{M}_0 \prec \mathcal{M}_1 \ldots \prec \mathcal{M}_\omega$ be the elementary chain of countable models described by the Baldwin-Lachlan theorem. Then we say the Spectrum of Recursive Models of T (written $SRM(T)$) is the set $\{i|\mathcal{M}_i$ has a recursive presentation$\}$. We will often refer to this as simply the spectrum of T.*

Let S be a subset of $\omega + 1$. If there exists a strongly minimal theory T so that $SRM(T) = S$, then we say that S is a spectrum.

We now re-pose Question 2 in a more precise way.

Question (2'). *Which sets are spectra?*

There are relatively few known examples of spectra. The trivial examples are easy to construct:

Example 15. *The empty set is a spectrum. Let L be the language with signature $\{E\}$, a single binary relation symbol. Let S be a complete Π_1-set and let M be the model $\bigcup_{n \in S}$(one n-cycle). If $N \models \mathrm{Th}(M)$, $n \in S \leftrightarrow \exists x_1 \ldots x_n N \models$ "$x_1 \ldots x_n$ form an n-cycle". Thus, a complete Π_1-set is Σ_1 in the atomic diagram of N. Thus N cannot be recursive.*

Example 16. *$\omega + 1$ is a spectrum. Let T be the theory of $(\mathbb{Q}, +)$. The models of T are determined by the number of linearly independent elements. It is easy to see that $(\mathbb{Q}^n, +)$ is a recursively presentable structure for each $n \in \omega + 1$.*

Constructing other examples becomes more difficult. Due to the scarcity of known examples, included below is a complete list of known results and sketches of their proofs.

Theorem 17. *(Goncharov, 1978 [3]) The set $\{0\}$ is a spectrum.*

Proof. Let L be the language with signature $\{c_i | i \in \omega\} \cup \{R_j | j \in \omega\}$ where each c_i is a constant symbol and each R_j is a unary relation symbol. We define a theory by specifying a model. Let M be a model with universe ω where c_i is interpreted as the number i and $M \models R_j(c_i)$ if and only if $j \in K_i$, where K is a complete Σ_1 set and K_i is the part of K enumerated by the i^{th} step. Let T be the theory of M. Note that if $j \notin K$ then $T \models \forall x \neg R_j(x)$, and if $j \in K$, say first in K_i, then $T \models R_j(x) \leftrightarrow \bigwedge_{k=0}^{i-1} x \neq c_k$. This shows that T is strongly minimal as it is a definitional expansion of the theory of pure equality with countably many constants named. Further, any non-prime model has an element not named by a constant. Suppose $N \models T$ and $a \in N$ with $N \models a \neq c_i$ for all i. Then $j \in K$ if and only if $N \models R_j(a)$. This shows that N cannot be recursive. Thus only the prime model has a recursive presentation. □

Theorem 18. *(Kudaibergenov, 1980 [12]) The set $\{0, \ldots, n\}$ is a spectrum.*

Proof. Let L be the language with signature $\{c_i | i \in \omega\} \cup \{R_j | j \in \omega\}$ where each c_i is a constant symbol and each R_j is an $n+1$-ary relation symbol. We define the theory by specifying a model. Let M be a model with universe ω where c_i is interpreted as the number i and $M \models R_j(c_{i_1}, \ldots c_{i_{n+1}})$ if and only if the c_{i_k} are distinct and $j \in K_{min\{i_1, \ldots i_n\}}$. Again, each R_j is definable in terms of the constant symbols, so T is strongly minimal. Let N be a model with fewer than $n+1$ elements not named by constants. Then to determine whether R_j should hold on a distinct tuple of length $n+1$ involves only a finite stage approximation to K, since the tuple must include a constant. Thus N has a recursive presentation. Let N' be any model with at least $n+1$ elements not named by constants: $a_1, \ldots a_{n+1}$. Then $j \in K$ if and only if $N' \models R_j(a_1, \ldots a_{n+1})$. Thus N' cannot be recursive. □

Theorem 19. *(Khoussainov, Nies, Shore, 1997 [10]) The set ω is a spectrum.*

Proof. Let L be the language generated by $\{R_{k,s} | k, s \in \omega\} \cup \{c_i | i \in \omega\}$, where each $R_{k,s}$ is a k-ary relation. Fix a complete Π_2 set $S = \{k | \forall l \exists j \phi(k, l, j)\}$. Let M be a model with universe ω where the number n is named by the constant c_n, and $R_{k,s}(\bar{x})$ holds if and only if the x_i are distinct and $\forall n \leq s \exists j \leq B \phi(k, n, j)$ where $B = \min\{\bar{x}\}$.

Again, we see that a model of dimension k has exactly k elements not named by constants. Let M_ω be the model of dimension ω. Then we see that

$$k \in S \text{ if and only if } \exists \bar{y} (\forall s M_\omega \models R_{k,s}(\bar{y}))$$

But then if M_ω were recursive, S would be presented as a Σ_2 set, which is impossible.

To construct the m-dimensional model for m finite, we take non-constructively the finite set of information describing whether $k \in S$ for each $k \leq m$, and if $k \notin S$, what is the first s such that $\neg \forall n \leq s \exists j \phi(k, n, j)$. This is a finite amount of information, so we can take it as given in our construction. Using this information, we can recursively determine whether $R_{k,s}$ should hold for any tuple in the m-dimensional model. □

Theorem 20. *(Khoussainov, Nies, Shore, 1997 [10]) The set $\omega + 1 \setminus \{0\}$ is a spectrum.*

Proof. We fix the language L generated by $\{R_i | i \in \omega\}$ where each R_i is binary.

Definition 21. *The canonical n-cube is the L-structure with universe $(\mathbb{Z}_2)^n$ where $R_i(x, y)$ holds if $x = y + e_i$ where e_i is the vector with a single 1 in the i^{th} copy of $\mathbb{Z}_2$.*

An n-cube is an L-structure isomorphic to the canonical n-cube.

An ω-cube is the L-structure which is the direct limit of n-cubes (eg: it has universe $\oplus_{i \in \omega} \mathbb{Z}_2$ with $R_i(x, y)$ holding if $x + e_i = y$ where e_i is the element with a single 1 in the i^{th} copy of $\mathbb{Z}_2$).

Let S be any subset of ω. We define A_S to be the structure comprised of the disjoint union of n-cubes, one for each $n \in S$. We set T_S to be $\mathrm{Th}(A_S)$.

Claim 22. *If S is Σ_2, then $\omega + 1 \setminus \{0\} \subseteq SRM(T_S)$.*

Proof. For $i \geq 1$, the i-dimensional model is comprised of A_S along with i ω-cubes. As long as S is Σ_2, we have recursive guesses about whether $k \in S$, and $k \in S$ if and only if we guess $k \notin S$ only finitely many times. The construction proceeds by building an n-cube for each $n \in \omega$. At each stage when we guess $k \notin S$, we join the k-cube into one of the ω-cubes we are building and create a new k-cube. In the limit, the only finite cubes which exist are the ones which we only finitely often join with an ω-cube. This is precisely the set S. □

Definition 23. *A function $g(x) : \omega \to \omega$ is a limitwise monotonic function if there is a total recursive function $f(s, x)$ so that $f(s, x) \leq f(s + 1, x)$ for all s, x and $lim_s f(s, x)$ always exists and equals $g(x)$.*

Claim 24. $0 \in SRM(T_S)$ *implies that S is the range of a limitwise monotonic function.*

Proof. Let N be any recursive model of dimension 0. In particular, N has universe ω. Then N is isomorphic to A_S. At stage s, we conduct a search in the model N of the first s elements and the first s relations. Define $f(s, x)$ to be the largest cube x is seen to be in during the stage s search in N. Then $lim_s f(s, x)$ is n if x is in an n-cube, but x is not in an $n+1$-cube. Thus S is the range of this limitwise monotonic function. □

We need to pick S to be any Σ_2 set which is not the range of a limitwise monotonic function. Such an S can be constructed via a finite injury construction. □

Theorem 25. *(Nies, 1999 [14]) The set* $\{1\}$ *is a spectrum.*

Proof. The proof, which will not appear in full here, proceeds by a refinement of the proof in the previous theorem. The idea is to add predicates P_k so that if (x, y) forms a sufficiently generic pair then $P_k(x, y) \leftrightarrow k \in K$. A similar refinement produces theories T_α so that $SRM(T) = [1, \alpha)$ for any $\alpha \in \omega + 1$ (Hirschfeldt, Nies). □

Theorem 26. *(Hirschfeldt, Khoussainov, Semukhin, 2006 [6]) The set* $\{\omega\}$ *is a spectrum.*

Proof. The proof is a slick refinement of the two previous examples. What follows is a sketch of the idea of the proof. First, generalize the notion of n-cube to A-cube for A any finite subset of ω. Then, we decide that each finite cube we build will have two edges R_k, R_l not appearing in any other cube. Upon observing the recursion-theoretic trigger for that cube, we change our mind and declare that all big-enough cubes should have an R_k edge, and we decide that our cube is actually much larger than it first appeared. In doing this, we add all the recently declared non-special edges. We add a new edge that only this cube should contain. By repeating this, there are 2 cases for the outcome of this cube in our construction. Either it turns into a copy of the infinite generic cube or at some point we stop observing triggers. The trigger for a cube is that a particular recursive structure contains an isomorphic copy of that cube. The two cases correspond to either the particular recursive model building an infinite generic cube (pushing its dimension up) or it does not copy this cube, in which case, the cube being algebraic ensures that the model is not of the same theory as the model we construct. By doing this, we end up building infinitely many copies of the infinite generic cube, and we ensure that every other recursive model of the theory also has infinitely many of the infinite generic cube. □

Finally, completing the list of known spectra, the following theorem is one of the main results of chapter 3 and negatively answers the question of whether all spectra are intervals in $\omega + 1$.

Theorem 27. *The set* $\{0, \omega\}$ *is a spectrum.*

1.3.1 The Finite Language Spectrum Problem

One of the main tools of coding information into models that we have seen in the examples above is as follows: We take a strongly minimal theory T (such as equality alone with constants named) and we append infinitely many relation symbols to the language. We decide that each relation symbol will be definable in terms of T in one of two or more ways. Which way we use codes one bit of a set (such as K). Thus, by scrolling through the sequence of all relation symbols, we code the full set K. This pattern is especially apparent in the examples of $SRM(T) = \{0\}$, $SRM(T) = \{0, \dots, n\}$ and $SRM(T) = \omega$. This particular tool of coding shows no relationship between the model theory involved in the theory and its spectrum, since this tool adds no definable sets to the theory. In view of this, one way to focus on the general relationships between the geometry of a theory and the recursion theoretic patterns in the models of a theory is to focus on theories in finite languages. In view of this, we pose the following version of Question 2:

Question $(2'')$**.** *Which sets are spectra of strongly minimal theories in finite languages?*

Note that Example 15 and Example 16 use finite languages, so $\emptyset$ and $\omega + 1$ are spectra of strongly minimal theories in finite languages. The following theorem provided the first interesting example.

Theorem 28. *(Herwig, Lempp, Ziegler, 1999 [5]) The set* $\{0\}$ *is the spectrum of a strongly minimal theory in a finite language.*

The following theorem is one of the main results of chapter 3 and the one after that summarizes the results in chapter 5.

Theorem 29. *The set $\{\omega\}$ is a spectrum of a strongly minimal theory in a finite language.*

Theorem 30. *For any $n \in \omega$, the set $\{0, \dots n\}$ as well as ω are spectra of strongly minimal theories in a finite language.*

1.4 The Complexity of Theories with Recursive Models

It is well known that if a theory T is recursive then the Henkin construction effectivizes to yield that T has a decidable model. Also, if T is recursive and $\aleph_1$-categorical, then all of its countable models are decidable [4], since the generic n-type as well as the procedure of taking algebraic closures are recursive. This yields a one-directional answer to Question 1.

On the other hand, if T has a recursive model (a model whose quantifier-free diagram is recursive), then in general we can only say that T is *tt*-reducible to 0^ω, $(\mathbb{N}, +, \cdot)$ being an example. Naturally, one would like to know whether this bound can be improved upon for 'tame' theories. Two natural classes of 'tame' theories are the $\aleph_0$-categorical and $\aleph_1$-categorical theories. Within the $\aleph_1$-categorical theories, our focus is primarily on the strongly minimal theories.

Goncharov and Khoussainov in [3] showed that for each n there is an $\aleph_1$-categorical, non-$\aleph_0$-categorical non-strongly minimal theory turing equivalent to $0^{(n)}$ all of whose countable models have recursive presentations. They also showed that for each n there is an $\aleph_0$-categorical theory turing equivalent to $0^{(n)}$ with a recursive countable model. Goncharov and Khoussainov conclude by asking the following two questions, which clarify the remaining direction of Question 1:

Question $(1')$**.** *Does there exist a theory turing equivalent to 0^ω which is $\aleph_1$-categorical and all of its countable models are recursive?*

Question $(1'')$**.** *Does there exist is an $\aleph_0$-categorical theory turing equivalent to 0^ω with a recursive model.*

The latter question was settled by Montalban and Khoussainov [9] in the affirmative. They generalize the construction of the random graph to allow the theory to code true arithmetic. The following theorem settles the first question also in the affirmative. Further, the theory used is strongly minimal and in a finite language.

Theorem 31. *There is a strongly minimal theory T in a finite language all of whose models are recursive such that $T \equiv_T 0^\omega$.*

In fact, in chapter 4 we show that every *tt*-degrees below 0^ω contains a strongly minimal theory with recursive models.

Theorem 32. *Let* $\mathbf{d}$ *be a tt-degree below* 0^ω*. Then there exists a strongly minimal theory* T *in a finite language such that* $T \in \mathbf{d}$ *and each countable model of* T *is recursively presentable.*

We also give the following analogous result for $\aleph_0$-categorical theories.

Theorem 33. *Let* $\mathbf{d}$ *be a tt-degree below* 0^ω*. Then there exists an* $\aleph_0$*-categorical theory* T *in a finite language such that* $T \in \mathbf{d}$ *and the countable model of* T *is recursively presentable.*

As an immediate consequence we have the following improvement of the result of Montalban and Khoussainov.

Theorem 34. *There exists an* $\aleph_0$*-categorical theory* T *in a finite language with a recursive model such that* $T \equiv_T 0^\omega$*.*

Chapter 2

Amalgamation Constructions

The content of this chapter derives from an alteration to the construction of Hrushovski [8] and is presented here as we will use this and an analog in further chapters as a major tool to build strongly minimal theories. As in any amalgamation construction, the aim is to define a class of structures with a well-behaved amalgamation property. At first, we work in generality so that the definitions and Lemma 42 can be used in future chapters without repetition. We will then switch gears to choose a δ and make some choices for the class $\mathcal{C}$ to demonstrate the Hrushovski amalgamation method in an infinite language.

2.1 General Definitions

Definition 35. *Let L be any fixed relational language. Let $\{B_i\}_{i\in I}$ be a collection of finite L-structures whose pairwise intersection is A. We say $\bigcup_{i\in I} B_i$ is a free-join over A if whenever $R(\bar{a})$ holds for any relation symbol R in L and $\bar{a} \subseteq \bigcup_{i\in I} B_i$, then $\bar{a} \subseteq B_i$ for some i.*

The core idea in Hrushovski's amalgamation construction for building strongly minimal sets is to use a pre-dimension function to give a coherent notion of what algebraicity should be in the constructed theory.

Definition 36. *A pre-dimension function is a function δ from finite L-structures to $\mathbb{Z} \cup \{-\infty\}$ with the following properties.*

1. *For any finite L-structures A and B, $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$*

2. *For M any finite L-structure and $B_1, B_2 \subseteq M$, $\delta(B_1 \cup B_2) = \delta(B_1) + \delta(B_2) - \delta(B_1 \cap B_2)$ if and only if $B_1 \cup B_2$ is the free-join of B_1 and B_2 over $B_1 \cap B_2$ in M.*

Note that we define pre-dimension functions to have range in $\mathbb{Z} \cup \{-\infty\}$, though this construction can be adapted to pre-dimension functions with other ranges, such as $\mathbb{R}$. Our choice of $\mathbb{Z} \cup \{-\infty\}$ is motivated by our focus on constructions of strongly minimal models. From a pre-dimension function, we will define the ideas of dimension relative to a set, dimension in a set, and precisely identify which extensions we want to limit in our amalgamation class. The fundamental idea is that having $\delta \leq 0$ should mean algebraicity. These notions are defined for any pre-dimension function, and in what follows we will specify a particular pre-dimension function δ and use these corresponding definitions for δ.

Definition 37. *For any finite L-structures A and B and infinite L-structure D, we define:*

- $\delta(B/A) = \delta(A \cup B) - \delta(A)$. *This is the relative dimension of B over A.*
- *If $A \subseteq B$, we set $\delta(A, B) = \min\{\delta(C) | A \subseteq C \subseteq B\}$. This is the dimension of A in B.*
- *If $A \subseteq B$, we say A is strong in B or $A \leq B$ if $\delta(A) = \delta(A, B)$.*
  We say A is strong in D if $A \subseteq D$ and A is strong in C for each finite $A \subseteq C \subseteq D$.
- *We say B is simply algebraic over A if $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset B' of B such that $\delta(B'/A) = 0$.*
- *We say that B is minimally simply algebraic over A if B is simply algebraic over A and there is no proper subset A' of A such that B is simply algebraic over A'.*

We verify that strongness forms a transitive reflexive relation, justifying the use of the symbol $\leq$. Also, we verify that relative dimension acts as we expect.

Lemma 38. *Let $A \subseteq N$ be L-structures. Suppose $A \leq N$*

1. $\delta(X \cap A) \leq \delta(X)$ *whenever $X \subseteq N$.*
2. $\delta(A', A) = \delta(A', N)$ *whenever $A' \subseteq A$.*
3. *In particular, if $A' \leq A \leq N$, then $A' \leq N$*

Proof. 3 is immediate from 2, which in turn is immediate from 1, so we will only prove 1.
$\delta(X \cup A) \leq \delta(X) + \delta(A) - \delta(X \cap A)$. So, $0 \leq \delta(X \cup A) - \delta(A) \leq \delta(X) - \delta(X \cap A)$. □

Lemma 39. *If X, A, and B are finite L-structures such that $A \subseteq B$, then $\delta(X/A \cup (X \cap B)) \geq \delta(X/B)$. In particular, if $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$.*

Proof. $\delta((X \cup A) \cup B) \leq \delta(X \cup A) + \delta(B) - \delta((X \cup A) \cap B)$, which simplifies to $\delta(X \cup B) - \delta(B) \leq \delta(X \cup A) - \delta(A \cup (X \cap B))$, as needed. □

Lemma 40. *Let M be a finite L-structure. Let $A \subseteq M$ and suppose B_j are simply algebraic over A and $A \leq (A \cup \bigcup_j B_j), (j \in J)$. Then:*

1. *The B_j are pairwise equal or disjoint.*
2. *$A \cup \bigcup_j B_j$ is a free join of the B_j over A.*
3. *Suppose $A \subseteq A' \subseteq M$, $A' \leq A' \cup B_j$, and B_j is not a subset of A' (j=1,2). Then any isomorphism of B_1 with B_2 over A extends to an isomorphism over A'. In fact, $A' \cup B_j$ is a free join of A' and B_j over A.*

Proof.

1. We need to show that $B_1 \cap B_2 = \emptyset$ assuming $B_1 \neq B_2$.
   $\delta(A) \leq \delta(A \cup B_1 \cup B_2) \leq \delta(A \cup B_1) + \delta(A \cup B_2) - \delta(A \cup (B_1 \cap B_2))$.
   So, $\delta(A) \leq 2\delta(A) - \delta(A \cup (B_1 \cap B_2))$. Hence, $\delta(A \cup (B_1 \cap B_2)) \leq \delta(A)$. By strongness of A, these are equal. But B_1 and B_2 are simply algebraic over A, so by the minimality condition in the definition of simply algebraic, $B_1 \cap B_2$ is empty or else equal to both B_1 and B_2.

2. As we saw above:
   $\delta(A \cup B_1 \cup B_2) = \delta(A \cup B_1) + \delta(A \cup B_2) - \delta(A)$ and $A = (A \cup B_1) \cap (A \cup B_2)$.
   Hence $A \cup B_1$ and $A \cup B_2$ are freely joined over A. Inductively repeating this argument shows that $A \cup \bigcup_j B_j$ is a free join of the B_j over A.

3. $0 \leq \delta(B_1/A') = \delta(B_1 \cup A/A') \leq \delta(B_1 \cup A/A' \cap (B_1 \cup A)) = \delta(B_1 \cup A/A \cup (B_1 \cap A')) = \delta(B_1 \cup A) - \delta(A \cup (A' \cap B_1)) = \delta(A) - \delta(A \cup (A' \cap B_1)) \leq 0$, where the last inequality is because $A \leq A \cup B_1$. So, $\delta(A \cup (A' \cap B_1)) = \delta(A)$, but by the fact that B_1 is simply algebraic over A, $A' \cap B_1 = \emptyset$ or B_1. By assumption, $A \cap B_1$ must be $\emptyset$. Similarly for B_2.
   As $\delta(B_1 \cup A/A') = \delta(B_1 \cup A/A' \cap (B_1 \cup A))$, we see that
   $\delta(A' \cup B_1) = \delta(A') + \delta(A \cup B_1) - \delta(A \cup (A' \cap B_1)) = \delta(A') + \delta(A \cup B_1) - \delta(A)$.
   So, we have that $A' \cup B_1$ is a free join over A.

□

Definition 41. *Let $\mathcal{C}_0$ be the class of finite L-structures C such that if $A \subseteq C \in \mathcal{C}_0$, then $\delta(A) \geq 0$.*

We have worked in generality by not specifying a particular pre-dimension function or a specific class $\mathcal{C} \subseteq \mathcal{C}_0$ so that the technical details of the following combinatorial lemma will be widely applicable.

Lemma 42. *Suppose $A, B_1, B_2 \in \mathcal{C}_0$, $A = B_1 \cap B_2$, and $A \leq B_1$. Let E be the free-join of B_1 with B_2 over A. Suppose $C^1, \ldots C^r, F$ are disjoint substructures of E such that each C^i is minimally simply algebraic over F and the structures C^i and C^j are isomorphic over F for each $1 \leq i, j \leq r$. Then one of the following holds:*

1. *One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$.*

2. *Either $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_2 or $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_1 and one of the C^i is contained in $B_1 \setminus A$.*

3. *$r \leq \delta(F)$*

4. *For one C^i, setting $X = (F \cap A) \cup (C^i \cap B_2)$, $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$ by Lemma 38).*

Proof. Let $C_0^j = C^j \cap A$, $C_1^j = C^j \cap B_1$, and $C_2^j = C^j \cap B_2$, and define F_0, F_1, F_2 similarly. Renumber the C^j so that $\delta(C_1^j/F) < \delta(C_0^j/F)$ (ie: $\delta(C_1^j \setminus C_0^j/C_0^j \cup F) < 0$) if and only if $j \leq r_0$, and for $j > r_0$, $C^j = C_2^j$ if and only if $j \leq r_1$ ($r_1 \geq r_0$).

Claim 1: $r_0 \leq \delta(F_1/A)$

Proof. Take $j \leq r_0$.
$\delta(C_1^j \cup A \cup F) \leq \delta(A \cup F) + \delta(C_1^j \cup F) - \delta(C_0^j \cup F)$.
So, $\delta(C_1^j - C_0^j/A \cup F) \leq \delta(C_1^j - C_0^j/C_0^j \cup F) < 0$.

As $A \cup F$ and $C_1^j \cup A \cup F_1$ are freely joined over $A \cup F_1$, we get $\delta(C_1^j - C_0^j/A \cup F_1) = \delta(C_1^j - C_0^j/A \cup F) \leq \delta(C_1^j - C_0^j/C_0^j \cup F) \leq -1$.

We set $C^* = \bigcup_{j \leq r_0} C_1^j$. We will inductively show that $\delta(C^*/A \cup F_1) \leq r_0 \cdot (-1) = -r_0$. We have shown the base case. Now, suppose $\delta(\bigcup_{j<m} C_1^j/A \cup F_1) \leq m(-1)$, and $m \leq r_0$. We see that $\delta(\bigcup_{j \leq m} C_1^j/A \cup F_1) = \delta(\bigcup_{j<m} C_1^j \cup C_1^m \cup A \cup F_1) - \delta(A \cup F_1) \leq \delta(\bigcup_{j<m} C_1^j \cup A \cup F_1) + \delta(C_1^m \cup A \cup F_1) - \delta(A \cup F_1) - \delta(A \cup F_1) = \delta(\bigcup_{j<m} C_1^j/A \cup F_1) + \delta(C_1^m/A \cup F_1) \leq -m - 1$ as needed.

But $A \leq B_1$, so $A \leq (A \cup C^* \cup F_1)$. Thus $0 \leq \delta(C^* \cup F_1/A) = \delta(C^*/F_1 \cup A) + \delta(F_1/A)$, showing that $\delta(F_1/A) \geq r_0$. □

Claim 2: For each j, $\delta(C_1^j/F) + \delta(C_2^j/F) - \delta(C_0^j/F) = 0$

Proof. $\delta(C_1^j/F) + \delta(C_2^j/F) - \delta(C_0^j/F) = \delta(C_2^j \cup F) + \delta(C_1^j \cup F) - \delta(C_0^j \cup F) - \delta(F)$. $(C_2^j \cup F)$ and $(C_1^j \cup F)$ are freely joined over $C_0^j \cup F$, so $\delta(C_2^j \cup F) + \delta(C_1^j \cup F) - \delta(C_0^j \cup F) - \delta(F) = \delta(C_2^j \cup C_1^j \cup F) - \delta(F) = \delta(C^j \cup F) - \delta(F) = 0$. □

Claim 3: If $j > r_1$, then $C_2^j = \emptyset$

Proof. $\delta(C_2^j/F) = -\delta(C_1^j/F) + \delta(C_0^j/F) \leq 0$, as $j > r_0$. But $F \leq F \cup C_2^j$. So, $\delta(C_2^j/F) = 0$. Thus, by minimality of C^j, $C_2^j = C^j$ or $\emptyset$. The first case is ruled out since $j > r_1$. □

Case 1: $F \subseteq B_2$

Proof. Then $F_1 \subseteq A$. By Claim 1, $r_0 = 0$. By Claim 3, $C_2^j = \emptyset$ or $C_2^j = C^j$ for every j. If $C_2^j = C^j$ for each j, then conclusion 2 of our lemma holds. We may assume $C_2^j = \emptyset$ for one j. Since $C^j \subseteq B_1 \setminus A$, we see that $F \subseteq A$. (To see that $F \subseteq A$, note that $\delta(C^j \cup F) = \delta(C^j \cup F_0) + \delta(F_2) - \delta(F_0)$. So, $0 = \delta(C^j \cup F) - \delta(F) = \delta(C^j \cup F_0) - \delta(F_0)$, and by the fact that C^j is minimally simply algebraic over F, we see that $F = F_0$.) This shows that conclusion 1 of our lemma holds. From here on, we may assume that $F \neq F_2$. □

Claim 4: $r_1 - r_0 \leq \delta(F_1/F_0) - \delta(F_1/A)$

Proof. Take any j so that $r_0 < j \leq r_1$, ie: $C^j = C_2^j$. We will show that F_1 is not freely joined with C^j over F_0. Suppose for a contradiction that F_1 was freely joined with C^j over F_0. Then we see that F_1 is freely joined with $C^j \cup F_2$ over F_0. Thus, $0 = \delta(C^j/F) = \delta(C^j \cup F) - \delta(F) = \delta(F_1) + \delta(C^j \cup F_2) - \delta(F_0) - \delta(F) = (\delta(F_1) + \delta(F_2) - \delta(F_0)) - \delta(F) + \delta(C^j \cup F_2) - \delta(F_2) = \delta(C^j/F_2)$ Thus C^j is simply algebraic over F_2, showing that $F = F_2$ contrary to assumption. We conclude that F_1 is not freely joined with C^j over F_0. Thus F_1 is not freely joined with C_0^j over F_0, and $\delta(C_0^j \cup F_1) < \delta(C_0^j \cup F_0) + \delta(F_1) - \delta(F_0)$

Combining these bounds as in Claim 1, we see that $\delta(\bigcup_{r_0<j\leq r_1} C_0^j \cup F_1) \leq \delta(\bigcup_{r_0<j\leq r_1} C_0^j \cup F_0) + \delta(F_1) - \delta(F_0) - (r_1 - r_0)$. Set $C^* = \bigcup_{r_0<j\leq r_1} C_0^j$ and we have that $\delta(C^* \cup F_1) - \delta(C^* \cup F_0) \leq \delta(F_1) - \delta(F_0) - (r_1 - r_0)$. Thus, $\delta(F_1/A) \leq \delta(F_1/(C^* \cup F_0)) \leq \delta(F_1/F_0) - (r_1 - r_0)$ yielding the claim by rearranging terms. □

Case 2: $r > r_1$, ie: $C_2^r = \emptyset$.

As we saw above, this implies that $F = F_1$. If all the C^j are contained in B_1, then conclusion 2 of our lemma holds (C^r witnessing the second part of the conclusion). So, we may assume that $C^j \neq C_1^j$ for some j. If it were that $C^j \subseteq B_2 - A$, then F would equal F_2, and we have already dealt with that case, so we may assume also that $C_0^j \neq \emptyset$.

$\delta(C_2^j - C_0^j / C_0^j \cup F) = \delta(C_2^j/F) - \delta(C_0^j/F) = -\delta(C_1^j/F) \leq 0$. This last inequality is true because $F \leq F \cup C^j$.

Case 2a: $-\delta(C_1^j/F) = 0$. Then by minimality of C^j, either $C_1^j = C^j$ or $C_1^j = \emptyset$. Either way, this contradicts one assumption.

Case 2b: $\delta(C_2^j - C_0^j / C_0^j \cup F) = -\delta(C_1^j/F) < 0$. Then, letting $X = F_0 \cup C_2^j$, we have $\delta(X/X \cap A) = \delta(C_2^j/C_0^j \cup F_0) = \delta(C_2^j \cup F_0) - \delta(C_0^j \cup F_0) = \delta(C_2^j \cup F) - \delta(C_0^j \cup F) = \delta(C_2^j/F) - \delta(C_0^j/F) = -\delta(C_1^j/F) < 0$, showing that the last conclusion of our lemma holds. The equality replacing F_0 by F in the previous chain holds because F and C^j are freely joined over $F_0 \cup C_0^j$ since $F \subseteq B_1$ and $C^j \subseteq B_2$.

Case 3: Not Case 1 or Case 2.

$r = r_1 = (r_1 - r_0) + r_0 \leq (\delta(F_1/F_0) - \delta(F_1/A)) + \delta(F_1/A) = \delta(F_1/F_0) = \delta(F) - \delta(F_2) \leq \delta(F)$ showing that the third condition of our lemma holds. □

2.2 The Amalgamation Class

Now we switch gears by declaring a particular δ-function and a particular amalgamation class to demonstrate the construction of a strongly minimal theory.

Definition 43.
- *Let L be a countable relational language with signature $\{R_i | i \in I\}$ Let k_i be the arity of R_i.*
- *Throughout the construction, we enforce that each relation symbol is symmetric and holds only on distinct tuples.*
- *For R a relation symbol in L, we write $|R(A)|$ for the number of subsets of A on which R holds.*
- *Let $\delta(A) = |A| - \sum_{i \in I} |R_i(A)|$.*
- *For any disjoint L-structures $\bar{a}, \bar{b} \subseteq C$, we write $tp_{r.q.f.}(\bar{b}/\bar{a})$ for the set $\{R_i(\bar{x}_i, \bar{y}_i) | (\bar{b}_i \bar{a}_i) \subseteq (\bar{b} \cup \bar{a})^{k_i} \smallsetminus \bar{a}^{k_i}, i \in \omega,$ and $R_i(\bar{b}_i, \bar{a}_i)$ holds$\}$. We call this set the relative quantifier-free type of $\bar{b}$ over $\bar{a}$. We say two relative quantifier free types are the same if they are equal after a re-ordering of $\bar{b}$ and a re-ordering of $\bar{a}$. Thus we can talk about the relative quantifier-free type of the set B over A, and we write $tp_{r.q.f}(B/A)$.*
- *Let $\mu(B, A)$ be a function from pairs of L-structures B, A with B minimally simply algebraic over A to $\mathbb{N}$ so that μ depends only on the isomorphism type of the pair (B, A) and $\mu(B, A) \geq |A|$. We further suppose that if Γ is a relative quantifier-free type, then there exists a sub-language L' with a finite subsignature of L so that if $tp_{r.q.f.}(B/A) = \Gamma = tp_{r.q.f.}(B'/A')$ and $\mathrm{tp}_{q.f.}(A)|_{L'} = \mathrm{tp}_{q.f.}(A')|_{L'}$ then $\mu(B, A) = \mu(B', A')$.*

On first introduction to amalgamation constructions, one should take $L = \{R\}$, R a single ternary relation symbol, and $\mu(B, A) = |A|$.

Definition 44. *Let Y and X be finite L-structures such that Y is minimally simply algebraic over X. Let $L_{Y/X}$ be the language generated by $\{R_i | \exists \bar{x} \subseteq (B \cup A)^{k_i} \smallsetminus A^{k_i}(R_i(\bar{x}))\}$, ie: the language occurring in $tp_{r.q.f.}(Y/X)$. Suppose B and A are finite L-structures such that $tp_{r.q.f.}(B/A)|_{L_{Y/X}} = tp_{r.q.f.}(Y/X)$ and $tp_{qf}(X) = tp_{qf}(A)$. Then we say the extension B over A is of the form of Y over X.*

Definition 45. *Let b be in ω and μ a function as described above. Then we define $\mathcal{C}_{b,\mu}$ to be the class of finite L-structures C such that:*

- *If $A \subseteq C$, then $\delta(A) \geq min(|A|, b)$*

- *Suppose Y over X is some minimally simply algebraic extension. If $B_1, \ldots, B_n$, A are disjoint subsets of C such that each B_i over A is an extension of the form of Y over X. Then $n \le \mu(Y, X)$.*

Though $\mathcal{C}_{b,\mu}$ is defined only relative to b and μ, we will prove the results in this section for any such $\mathcal{C}_{b,\mu}$ and will simply refer to the class as $\mathcal{C}$. The following two theorems verify that $\mathcal{C}$ behaves as expected when we pass to smaller languages.

Lemma 46. *Suppose $C \in \mathcal{C}$ and L' is any language whose signature is a subset of $\{R_i | i \in \omega\}$. Then $C|_{L'} \in \mathcal{C}_0$.*

Proof. Let A' be any subset of $C|_{L'}$. Let A be the corresponding subset of C. Then $\delta(A') \ge \delta(A) \ge 0$. □

Lemma 47. *Suppose A and B are any finite L-structures such that $A \le B$ and L' is a language whose signature is a subset of $\{R_i | i \in \omega\}$. Let $A' = A|_{L'}$ and $B' = B|_{L'}$. Then $A' \le B'$*

Proof. Fix any C' such that $A' \subseteq C' \subseteq B'$. Letting C be the corresponding subset of B, we see that $0 \le \delta(C/A) \le \delta(C'/A')$. Thus $\delta(C') \ge \delta(A')$ showing that $A' \le B'$. □

We wish to amalgamate the class $\mathcal{C}$ to form a model. Though $\mathcal{C}$ does not satisfy the standard amalgamation property (as if $\delta(B/A) = \delta(C/A) = -\delta(A)$, then B and C cannot be amalgamated over A without $\delta(B \cup C) < 0$), as the next lemmas show, $\mathcal{C}$ still has a form of amalgamation property for strong substructures.

Lemma 48. *(Algebraic Amalgamation Lemma) Suppose $A, B_1, B_2 \in \mathcal{C}$, $A = B_1 \cap B_2$, and $B_1 \setminus A$ is simply algebraic over A. Let E be the free-join of B_1 with B_2 over A. Then $E \in \mathcal{C}$ unless one of the following hold:*

1. *$B_1 \setminus A$ is minimally simply algebraic over some $F \subseteq A$, and B_2 contains $\mu(B_1 \setminus A, F)$ disjoint extensions over F of the form of $B_1 \setminus A$ over F.*

2. *There exists a set $Z \subseteq B_2$, and a subset $\hat{L}$ of $L_{B_1/A}$ such that $(A \cap Z)|_{\hat{L}} \not\le Z|_{\hat{L}}$. Further, $B_1|_{\hat{L}}$ contains an isomorphic copy of $Z|_{\hat{L}}$.*

Proof. We will show that E satisfies the first condition of being in $\mathcal{C}$. Let X be any subset of E. Then $\delta(X) = \delta(X \cap B_1) + \delta(X \cap B_2) + \delta(X \cap A)$. Note that from this, we see $\delta(X) \ge \delta(X \cap B_2)$. If $|X \cap B_2| \ge b$, then $\delta(X) \ge b$. So, we may suppose $|X \cap B_2| = \delta(X \cap B_2) < b$. Thus no relations hold on $X \cap B_2$. Thus $\delta(X \cap A) = |X \cap A|$ as well. Finally, we have $\delta(X) = \delta(X \cap B_1) + |X \cap (B_2 \setminus A)| \ge \min(b, |X \cap B_1|) + |X \cap (B_2 \setminus A)| \ge \min(b + |X \cap (B_2 \setminus A)|, |X \cap B_1| + |X \cap (B_2 \setminus A)|) \ge \min(b, |X|)$.

Suppose Y is minimally simply algebraic over X and E contains disjoint sets $C^1, \ldots, C^r$, F where each of the C^j over F are of the form of Y over X. We will look at the structure $E|_{L_{Y/X}}$. Call this structure E'. Our focus will be on using the structure E' to count the C^js, so we abuse notation and write C^j also for the structure induced by the corresponding set in E'. Here, each of the C^j over F are minimally simply algebraic extensions. Note further that E' is the free-join of $B_1|_{L_{Y/X}}$ with $B_2|_{L_{Y/X}}$ over $A|_{L_{Y/X}}$ and $B_1|_{L_{Y/X}}$ and $B_2|_{L_{Y/X}}$ are members of $\mathcal{C}_0$ by Lemma 46. Each of the C^j are minimally simply algebraic over F and $A|_{L_{Y/X}} \leq B_1|_{L_{Y/X}}$ by Lemma 47. By Lemma 42, there are 4 cases:

1. One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$. Since $B_1 \setminus A$ is simply algebraic over A in E, $C^i = B_1 \setminus A$. As the C^j and F are disjoint and one is $B_1 \setminus A$, each of the other C^j and F are contained in B_2. If $r > \mu(Y, X)$ then there must be $\mu(Y, X)$ of them contained in B_2 putting us in the case of the first exception to this lemma.

2. $F \cup \bigcup_{i=1}^{r} C^i$ is entirely contained in either B_1 or B_2. Then $r \leq \mu(Y, X)$ as $B_1, B_2 \in \mathcal{C}$

3. $r \leq \delta(F)$. $\delta(F) \leq |F| = |X| \leq \mu(Y, X)$

4. For one C^i, setting $X = (F \cap A) \cup (C^i \cap B_2)$, we see that $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. Using this set as Z and using $L_{Y/X}$ as $\hat{L}$ yields the second exception in our lemma.

□

Lemma 49. *(Strong Amalgamation Lemma) Suppose $A, B, C \in \mathcal{C}$, $A \leq B$, $A \leq C$. Then there exists $D \in \mathcal{C}$ so that $C \leq D$, and an $g : B \to D$ an embedding so that $g(B) \leq D$ and $g(A) = id|_A$.*

Proof. This follows from the Algebraic Amalgamation Lemma by induction on $|B - A| + |C - A|$. We may assume there is no $A \subsetneq B' \subsetneq B$ such that $A \leq B' \leq B$. Otherwise, using the inductive hypothesis we can amalgamate B' with C over A and then B can be amalgamated with $B' \cup C$ over B', and we are done. We have two cases remaining to consider.

Case 1: B is comprised of A along with a single element unrelated to A. In this case, the free-join of B with C over A suffices.

Case 2: B is simply algebraic over A, say minimally simply algebraic over $F \subseteq A$. If neither of the conditions of the Algebraic Amalgamation Lemma hold, then the free-join of B with C over A suffices. Since $A \leq C$, if $Z \subseteq C$ then $Z \cap A \leq Z$. Thus by Lemma 47 for any language $\hat{L}$, $(Z \cap A)|_{\hat{L}} \leq Z|_{\hat{L}}$, so the second condition of the Algebraic Amalgamation Lemma can not hold.

Suppose the first condition of the Algebraic Amalgamation Lemma holds. C contains $\mu(B \setminus A, F)$ disjoint extensions of the form of $B \setminus A$ over F.

Claim 50. *Let $K \subseteq C$ be such that K over F is of the form of $B \setminus A$ over F. Then $K \subseteq A$ or $K \cap A = \emptyset$.*

Proof. Suppose otherwise, that $K \cap A$ is a proper subset of K. As $C \in \mathcal{C}$ and $A \leq C$, we have that $A' = A|_{L_{Y/X}} \leq C|_{L_{Y/X}}$ by Lemma 47. Also, since K over F is of the form of Y over X, $K' = K|_{L_{Y/X}}$ is minimally simply algebraic over $F' = F|_{L_{Y/X}}$. See that $0 = \delta(K'/F') = \delta(K'/(K' \cap A') \cup F') + \delta((K' \cap A')/F')$. But $\delta(K'/(K' \cap A') \cup F') \geq 0$ as $A' \leq C'$ and $\delta((K' \cap A')/F') > 0$ as K' is minimally simply algebraic over F'. This yields the contradiction. □

At most $\mu(B \setminus A, F) - 1$ disjoint extensions of the form of $B \setminus A$ over F can occur inside A since $B \in \mathcal{C}$. Thus there is an $X \subseteq C \setminus A$ such that X over F is of the form of $B \setminus A$ over F. Since $A \leq C$, we see that there can be no additional relations holding on X, ie: $tp_{r.q.f.}(X/F) = tp_{r.q.f.}(B \setminus A/F)$. By Lemma 40, $tp_{r.q.f}(X/Q) = tp_{r.q.f}(B \setminus A/A)$, and identifying $B \setminus A$ with X suffices. □

Using the Strong Amalgamation Lemma, we can build a generic model $\mathcal{M}$ whose age is $\mathcal{C}$.

2.3 The Theory of $\mathcal{M}$

Using the Strong Amalgamation Lemma, we get a model $\mathcal{M}$ which satisfies the following 3 properties:

1. $\mathcal{M}$ is countable
2. Every finite substructure of $\mathcal{M}$ is an element of $\mathcal{C}$
3. Suppose $B \leq \mathcal{M}$, $B \leq C$, and $C \in \mathcal{C}$. Then there exists an embedding $f : C \to \mathcal{M}$ such that $f|_B = id_B$ and $f(C) \leq \mathcal{M}$.

Note that for any $A \subseteq \mathcal{M}$, there exists a finite $B \subseteq \mathcal{M}$ such that $A \subseteq B \leq \mathcal{M}$ because $\mathbb{N}$ is well-ordered. By a standard back-and-forth argument using finite strong substructures, (1,2,3) defines $\mathcal{M}$ up to isomorphism.

We would like to show that $\mathcal{M}$ is saturated by showing that any countable elementary supermodel of $\mathcal{M}$ must also have properties (1,2,3) and thus would have to be isomorphic to $\mathcal{M}$. The problem is that 3 is not a first-order property. To make the argument work, we replace 3 by $3'$ and $3''$:

$3'$. M contains an infinite set I such that no relations hold on any tuples from I and $A \leq M$ for each finite $A \subset I$.

$3''$. Suppose $B \subseteq \mathcal{M}$, $B \leq C$, $C \in \mathcal{C}$, and $C \setminus B$ is simply algebraic over B, say minimally simply algebraic over $F \subseteq B$. Suppose also that for any subset $\hat{L}$ of $L_{C/B}$ and any $X \subseteq C$, there is no set $X' \subseteq \mathcal{M}$ such that $X|_{\hat{L}} \cong X'|_{\hat{L}}$ and $(B \cap X')|_{\hat{L}} \not\leq X'|_{\hat{L}}$. Then there are $\mu(C \setminus B, F)$ disjoint extensions over F of the form of $C \setminus B$ over F in $\mathcal{M}$.

Note that $3''$ is first order since $L_{C/B}$ has finite signature and $\mu(C \setminus B, F)$ is determined by the type of F in a finite signature sublanguage of L.

Claim 51. *$(1,2,3)$ and $(1,2,3',3'')$ are equivalent.*

Proof. $\rightarrow$: $3'$ follows from 3 directly and $3''$ follows from the Algebraic Amalgamation Lemma.

$\leftarrow$: 3 follows as in the proof of the Strong Amalgamation Lemma. □

Corollary 52. *$\mathcal{M}$ is saturated.*

Proof. Since $3'$ and $3''$ are first order conditions, any countable elementary extension of $\mathcal{M}$ satisfies $(1,2,3',3'')$, hence is isomorphic to $\mathcal{M}$. It follows that there are only countably many types realized in elementary extensions of $\mathcal{M}$. Hence, there is a saturated countable elementary extension of $\mathcal{M}$, which $\mathcal{M}$ must be isomorphic to. □

We want to explain what algebraicity amounts to in $\mathcal{M}$. We define $d(A) = \min\{\delta(C)|A \subseteq C \subseteq \mathcal{M},\ C \text{ finite}\}$. Clearly for any A and x, either $d(xA) = d(A)$ or $d(xA) = d(A) + 1$.

Lemma 53. *If $d(xA) = d(A) + 1$ and $d(yA) = d(A) + 1$, then $(\mathcal{M}, Ax) \cong (\mathcal{M}, Ay)$.*

Proof. Let B be such that $A \subseteq B$, $\delta(B) = d(A)$. Then $B \leq \mathcal{M}$. $d(xB) = d(xA) = d(A) + 1$. Thus, $xB \leq \mathcal{M}$, and similarly $yB \leq \mathcal{M}$. Using property 3 and a standard back-and forth, we see that $(\mathcal{M}, xB)$ and $(\mathcal{M}, yB)$ are isomorphic. □

We have shown that there is a unique type over A of an element x such that $d(xA) > d(A)$. Next we show that $d(xA) = d(A)$ implies that $x \in acl_{\mathcal{M}}(A)$.

Lemma 54. *If $d(xA) = d(A)$, then $x \in acl_{\mathcal{M}}(A)$.*

Proof. Suppose $d(xA) = d(A)$. First, let B be a minimal set such that $A \subseteq B$ and $\delta(B) = d(A)$. We show that B is algebraic over A in $\mathcal{M}$. Suppose there were two realizations of the positive atomic type of B over A. Call the second realization B'. Then $\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') \leq d(A)$. The strict inequality is due to B being a minimal set with the properties that $A \subseteq B$ and $\delta(B) = d(A)$. This inequality contradicts the definition of $d(A)$.

Fix E to be a set such that $xA \subseteq E$ and $\delta(E) = d(A)$. Then $\delta(E \cup B) \leq \delta(E) + \delta(B) - \delta(E \cap B)$. If E does not contain B, then $\delta(E \cap B) > d(A)$ by minimality of B. Then $\delta(E \cup B) \leq d(A) + d(A) - \delta(E \cap B) < d(A)$, again a contradiction. Thus, E contains B and $d(xB) = d(B)$.

Take a sequence of extensions $B_0, B_1, B_2, \ldots B_n$ such that $B_0 = B$, $B_n = E$, and B_{i+1} is a minimal set such that $B_i \subsetneq B_{i+1} \subseteq E$ and $\delta(B_{i+1}) = d(A)$. Then $B_{i+1} \setminus B_i$ is simply algebraic over B_i, say minimally simply algebraic over F_i. Since $B_i \leq \mathcal{M}$, any two realizations of the positive atomic type of $B_{i+1} \setminus B_i$ over B_i must be disjoint by Lemma 40 and there can be no more than $\mu(B_{i+1} \setminus B_i, F_i)$ many of these. Thus B_{i+1} is algebraic over B_i. We conclude that E is algebraic over A. In particular, $x \in acl_{\mathcal{M}}(A)$. □

Corollary 55. *$Th(\mathcal{M})$ is strongly minimal.*

Proof. In the previous lemma, we showed that over any set there is a unique non-algebraic type realized in $\mathcal{M}$. Since $\mathcal{M}$ is saturated, we see that $Th(\mathcal{M})$ is strongly minimal. □

In two of the following chapters, this construction will be used explicitly via naming a μ and b to apply this construction to. In chapter 5, we will need an alteration of this method and will have to verify the validity of that construction.

Chapter 3

A New Spectrum

In this chapter, we will use the Hrushovski Amalgamation method as well as a diagonalization argument to produce strongly minimal theories as in the following theorems.

Theorem 56. *There exists a strongly minimal theory T such that $SRM(T) = \{0, \omega\}$.*

Theorem 57. *There exists a strongly minimal theory T in a finite language such that $SRM(T) = \{\omega\}$*

Both main theorems will proceed via use of the Hrushovski construction presented in Chapter 2 by specifying an integer b and function μ.

3.1 The Amalgamation Class

In this section, we will describe the construction of a model relative to any given set $S \subseteq \omega$. In section 3.3, we will fix a particular set S to yield the desired theorems. We view S as a set of pairs of natural numbers $\langle j, k\rangle$ by using a standard pairing function (a recursive bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$). We refer to $\{m \in S | \exists k \ (m = \langle j, k\rangle)\}$ as the j^{th} column of S and will write $S^{[j]}$ to denote this set. From the set S, we define the set Q to consist of the first two elements of each column not contained in S, ie: $Q = \{\langle j, k\rangle | \langle j, k\rangle \notin S \text{ and } \neg\exists^2 k'(k' < k \wedge \langle j, k'\rangle \notin S)\}$.

We define L to be the language with signature $\{R\} \cup \{R_i | i \in \omega\}$ where each relation symbol is ternary. Throughout the construction, we enforce that each relation is symmetric and holds only on distinct triples. We use the construction from section 2.2, using the same δ function ($\delta(A) = |A| - |R(A)| - \sum_{i \in omega} |R_i(A)|$). We need only specify a function μ and an integer b.

We enumerate recursively the relative quantifier-free types of all minimally simply algebraic extensions over a 3 element set involving only the relation R. We refer to the i^{th} enumerated relative quantifier-free types as Λ_i.

Definition 58. *We say B/A is a Λ_i-extension if $tp_{r.q.f}(B/A)|_R = \Lambda_i$.*

Definition 59.

$$\mu(Y,X) = \begin{cases} 4 & \textit{if } Y/X \textit{ is a } \Lambda_{\langle j,k\rangle}\textit{-extension}, \langle j,k\rangle \in Q, \textit{ and } R_j(X) \textit{ holds} \\ 4 & \textit{if } Y/X \textit{ is a } \Lambda_{\langle j,k\rangle}\textit{-extension and } \langle j,k\rangle \in S \\ |X| & \textit{otherwise} \end{cases}$$

Note that in the first two cases $|X| = 3$ as the Λ's are relative quantifier-free types over 3 element sets. Any integer greater than 3 could be used in the place of 4 in the above definition.

Now we use μ to bound the number of extensions allowed of the form of a given minimally simply algebraic extension and define our amalgamation class, $\mathcal{C}_{2,\mu}$. Recall the following definition of $\mathcal{C}_{2,\mu}$.

Definition 60. *Let $\mathcal{C}$ be the class of finite L-structures C such that the following hold:*

- *If $A \subseteq C$ then $\delta(A) \geq min(|A|, 2)$*
- *Let Y/X be a minimally simply algebraic extension. Let B_i, $i = 1, \ldots, n$, and A be disjoint subsets of C such that B_i/A is an extension of the form Y/X for each i. Then $n \leq \mu(Y,X)$.*

From the results of section 2.3, we know that we get a saturated amalgam $\mathcal{M}$ of the class $\mathcal{C}$ and that $\mathrm{Th}(M)$ is strongly minimal.

Lemma 61. *Suppose S is a Σ_1 set. Then $\mathcal{M}$ is a recursively presentable structure.*

Proof. If S is finite, then μ is a recursive function, $\mathcal{C}$ is a recursive set of L-structures and repeatedly applying the strong amalgamation lemma lets us recursively build $\mathcal{M}$. We may assume S is infinite. We fix uniformly recursive approximations S_i to S such that $S_i \subseteq S_{i+1}$ and $|S_i| = i$. We will use these recursive approximations to S to build recursive approximations to the amalgamation class and will be able to amalgamate to build $\mathcal{M}$. We define Q_i to be the first 2 elements of the first i columns not contained in S_i.
At stage i, define

$$\mu_i(Y,X) = \begin{cases} 4 & \text{if } Y/X \text{ is of the form of } \Lambda_{\langle j,k\rangle}, \langle j,k\rangle \in Q_i, \text{ and } R_j(X) \text{ holds} \\ 4 & \text{if } Y/X \text{ is of the form of } \Lambda_{\langle j,k\rangle} \text{ and } \langle j,k\rangle \in S_i \\ |X| & \text{otherwise} \end{cases}$$

We define $\mathcal{C}_i$, the amalgamation class allowed at the i^{th} stage, from μ_i.

Let $\mathcal{C}_i = \mathcal{C}_{2,\mu_i}$ be the class of finite L-structures C such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq min(|A|, 2)$.
- Let Y/X be a minimally simply algebraic extension. Let B_j, $j = 1, \ldots, n$, and A be disjoint subsets of C such that B_j/A is an extension of the form Y/X for each j. Then $n \leq \mu_i(Y, X)$.

Since $\mu_i(Y, X) \leq \mu_{i+1}(Y, X)$, we see that $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$. As $\lim_i \mu_i = \mu$, we see that $\mathcal{C} = \bigcup_i \mathcal{C}_i$. To construct $\mathcal{M}$, we work in stages. At the i^{th} stage, we amalgamate the first i possible amalgamations allowed in $\mathcal{C}_i$. As $\mathcal{C} = \bigcup_i \mathcal{C}_i$, every possible amalgamation in $\mathcal{C}$ is amalgamated at a finite stage, and since $\mathcal{C}_i \subseteq \mathcal{C}$, we never leave the amalgamation class $\mathcal{C}$. This constructs a generic model for $\mathcal{C}$ which is therefore isomorphic to $\mathcal{M}$. □

From here forward we assume S is a Σ_1 set, and thus the result of the lemma holds. We fix a recursive presentation of $\mathcal{M}$, and we refer to this particular presentation as $\mathcal{M}$ from here on.

3.2 The Restricted Language

To obscure the recursion theoretic content of the construction from the presentation of the model, we will restrict to the language generated by the single relation symbol R. Also, to force the prime model to be recursive in Theorem 56, we will name constants which will identify the prime model.

We fix a non-algebraic pair of elements x and y from $\mathcal{M}$. By the characterization of algebraic closure in section 2.3, $acl_{\mathcal{M}}(\{x, y\})$ is a Σ_1 set (ie: $z \in acl_{\mathcal{M}}(\{x, y\})$ if and only if $d(\{z, x, y\}) = 2$ if and only if $\exists A \supseteq \{x, y, z\}(\delta(A) = 2)$, which is a Σ_1 condition). Using this observation, we fix a recursive enumeration of $acl_{\mathcal{M}}(\{x, y\})$, $i \mapsto z_i$.

Definition 62. *Let $\mathcal{M}'$ be the model obtained by restricting $\mathcal{M}$ to the language generated by $\{R\}$.*
Let $\mathcal{M}''$ be the model constructed by adding constant symbols $\{c_i | i \in \omega\}$ to $\mathcal{M}'$ where c_i names the element z_i.

Our next goal is to understand algebraicity in the model $\mathcal{M}'$. In particular, we'll see that the relations that 'count' are R and the R_i such that $S^{[i]} \neq \omega^{[i]}$. From here forth, we call the language generated by $\{R\} \cup \{R_i | S^{[i]} \neq \omega^{[i]}\}$ by the name L'. Recall Λ_j is a relative quantifier-free type of an extension of a 3-element set. In the context of a first order formula, we write $\Lambda_j(\bar{y}, \bar{x})$ to represent the formula which states that $\bar{y}$ is a Λ_j-extension of $\bar{x}$ (ie: $tp_{r.q.f}(\bar{y}/\bar{x})|_R = \Lambda_j$). Note that $\Lambda_j(\bar{y}, \bar{x})$ is a formula involving only the relation R.

Lemma 63. *Let i be an integer such that $S^{[i]} \neq \omega^{[i]}$. Then $Q^{[i]} \neq \emptyset$. Let $\langle i, k \rangle$ be an element of $Q^{[i]}$. Then $\mathcal{M} \models \forall \bar{x}(R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i,k \rangle}(\bar{y}, \bar{x}))$ (here we read $\exists^4$ as there exist 4 disjoint tuples $\bar{y}$ satisfying the condition).*

Proof. $\leftarrow$: If there are 4 disjoint Λ_j-extensions over $\bar{x}$ and $R_i(\bar{x})$ does not hold, then taking the finite set A comprised of the 4 extensions and $\bar{x}$, we see that $A \notin \mathcal{C}$ contradicting property (2) of $\mathcal{M}$.

$\rightarrow$: Suppose $R_i(\bar{x})$ holds. Then $\delta(\bar{x}) = 2$, which shows $\bar{x} \leq \mathcal{M}$. By $(3'')$, we see that there are 4 disjoint Λ_j-extensions over $\bar{x}$. □

Since each of the relations $R_i \in L'$ are definable in $\mathcal{M}'$, we will abuse notation and say $R_i(\bar{x})$ holds in $\mathcal{M}'$ to mean that the equivalent statement involving only R holds in $\mathcal{M}'$. Similarly for $\mathcal{M}''$. This shows that these relations still 'count' in the reduct $\mathcal{M}'$. The next lemma shows that these are the only relations that still count.

Definition 64.
- *For finite $A \subseteq M'$, let $\delta'(A) = \delta(A|_{L'}) = |A| - |R(A)| - \sum_{R_j \in L'} |R_j(A)|$.*
- *Let $d'(A) = \min\{\delta'(B) | A \subseteq B \subseteq \mathcal{M}', B \text{ finite}\}$*
- *The class $\mathcal{C}_{L'}$ is the set of L'-structures in $\mathcal{C}$.*

The following lemma holds as well for any sub-language of L containing L'.

Lemma 65. *$\mathcal{M}|_{L'}$ is generic for the class $\mathcal{C}_{L'}$.*

Proof. We need to show that $\mathcal{M}|_{L'}$ satisfies the conditions to be a generic model of $\mathcal{C}_{L'}$. We use the versions of $(1, 2, 3', 3'')$ for $\mathcal{C}_{L'}$:
1: $\mathcal{M}|_{L'}$ is countable
2: For any finite $A \subseteq \mathcal{M}|_{L'}$, $A \in \mathcal{C}_{L'}$
$3'$: $\mathcal{M}|_{L'}$ contains an infinite set I such that there are no relations holding on I, and for any finite $A \subseteq I$, $d'(A) = |A|$.
$3''$: Suppose $B \subseteq \mathcal{M}|_{L'}$, $B \leq C$, $C \in \mathcal{C}_{L'}$, and $C \setminus B$ is simply algebraic over B, say minimally simply algebraic over $F \subseteq B$. Suppose also that for any subset $\hat{L}$ of $L_{C/B}$ and any $X \subseteq C$, there is no set X' such that $X|_{\hat{L}} \cong X'|_{\hat{L}}$ and $(B \cap X')|_{\hat{L}} \not\leq X'|_{\hat{L}}$. Then there are $\mu(C \setminus B, F)$ many copies of $C \setminus B$ over F in $\mathcal{M}|_{L'}$.

1 is equivalent to the 1 above. 2 follows from the fact that for any $A \in \mathcal{C}$, $A|_{L'} \in \mathcal{C}_{L'}$, which follows from the identity $\mu(Y, X) = \mu(Y, X|_{L'})$. $3'$ is formally weaker than the $3'$ above. Since $\mu(Y, X|_{L'}) = \mu(Y, X)$, $3''$ follows from the $3''$ above. □

Lemma 66. *$x \in acl_{\mathcal{M}'}(A)$ if and only if $d'(xA) = d'(A)$.*

Proof. Above we showed that for $\mathcal{M}$ the generic model of $\mathcal{C}$, algebraicity meant $d(xA) = d(A)$. By the analogous argument for $\mathcal{M}|_{L'}$, we see that algebraicity here means $d'(xA) = d'(A)$. Since $\mathcal{M}|_{L'}$ is a definitional expansion of $\mathcal{M}'$, algebraicity is the same for $\mathcal{M}'$. □

Lemma 67. *$\mathcal{M}'$ and $\mathcal{M}''$ are both recursive, saturated, and strongly minimal.*

Proof. $\mathcal{M}'$ is recursive, saturated, and strongly minimal, as it is a reduct to a recursive language of a model with all of these properties.

$\mathcal{M}''$ is recursive since the assignment of the constants is recursive. It is strongly minimal, as adding constants to a strongly minimal theory retains strong minimality. Take I an infinite algebraically independent sequence in $\mathcal{M}$ beginning with $\{x, y\}$. $I - \{x, y\}$ is algebraically independent over the algebraic closure of $\{x, y\}$ in $\mathcal{M}$. Thus $I - \{x, y\}$ is algebraically independent in $\mathcal{M}''$. This shows that $\mathcal{M}''$ has infinite algebraic dimension, thus is saturated. □

3.3 Defining S

Thus far we have constructed the two recursive models $\mathcal{M}'$ and $\mathcal{M}''$ relative to any given Σ_1 set S. We aim for a construction where $SRM(Th(\mathcal{M}')) = \{\omega\}$ and $SRM(Th(\mathcal{M}'')) = \{0, \omega\}$. To ensure this, we need to diagonalize against the possible finite-dimensional models of each theory. In this section, we construct the Σ_1 set S to ensure these results.

We want to ensure that finite dimensional models are not recursive. There is no 0-dimensional model of $Th(\mathcal{M}')$ (ie: $acl(\emptyset) = \emptyset$), so we will diagonalize only against positive dimensional models. We fix a recursive enumeration of all pairs (f, U) where f is a partial recursive function from the set of quantifier-free formulas in the language $\{R\} \cup \{c_i | i \in \omega\} \cup \mathbb{N}$ to {true, false} and U is a non-empty finite subset of $\mathbb{N}$. This is to be interpreted as f giving the quantifier-free diagram of a model N with universe $\mathbb{N}$ and U representing a basis of the model.

We will describe a routine for enumerating S. For the i^{th} pair (f, U), we will have an i^{th} subroutine $Routine_i$ whose job it is to ensure that this pair does not represent a model N with a basis U satisfying either of the theories of $\mathcal{M}'$ or $\mathcal{M}''$.

Given a pair (f, U), at stages we read off information about the model it describes from f_j the computation of f at stage j. We let N_0 be the empty model, and N_j be comprised of all $n \leq j$ such that for each $m < n$, $f_j(n = m) \downarrow=$ 'false'. In N_j, we say $R(\bar{x})$ holds if $f_j(R(\bar{x})) \downarrow=$ 'true'. We say for $R_i \in L$ with $i < j$, $R_i(\bar{x})$ holds if there is a $\langle i, k\rangle \in T_j^{[i]}$ such that $N_j \models \exists^4 \bar{y}\ \Lambda_{\langle i,k\rangle}(\bar{y}, \bar{x})$, where $\Lambda_k(\bar{y}, \bar{x})$ has already been defined in N_j as a conjunction of R-statements. For a set A of natural numbers, we write $\delta_j(A)$ for $\delta(A)$ as A is seen in the structure N_j. Finally, we set $K_j \subseteq N_j$ to be the set of elements $x \in N_j$ such that $f_j(x = c_i) \downarrow=$ 'true' for some $i \leq j$.

$Routine_i$ is the only part of our program allowed to enumerate anything into $S^{[i]}$. When $Routine_i$ is initialized, $S^{[i]} = \emptyset$. The routine runs in parts as follows:

Part 1) Wait until a stage j when there is some set $X \subseteq N_j$ and a set $K \subseteq K_j$ such that $(X \cup U \cup K)|_{R_i}$ is a minimally simply algebraic extension over $(U \cup K)|_{R_i}$. Once found, for the duration of its run $Routine_i$ refers to these sets as X and K.

Part 2) The first thing $Routine_i$ does when it gets to part 2 is to define the set of obstructions to moving to part 3. A set $Y \subseteq N_j$ is an obstruction to moving to part 3 if $\delta_j(Y/K_j) < |U|$, and $U \subseteq Y$. Suppose we first got to part 2 on stage j and defined the sets Y as such. If during a stage $s > j$ an element is enumerated into $S^{[k]}$, then we say R_k is removed. If at a stage s enough R_k are removed so that "counting only the non-removed R_k", $\delta_s(Y/K_j) \geq |U|$, then we say the obstruction Y has been removed. That is: if

$$\left[|Y \cup K_t| - |R(Y \cup K_t)| - \sum_{\substack{R_j \text{ not} \\ \text{removed}}} |R_j(Y \cup K_t)| \right] - \left[|K_t| - |R(K_t)| - \sum_{\substack{R_j \text{ not} \\ \text{removed}}} |R_j(K_t)| \right] \geq |U|,$$

then the obstruction Y is removed.

If for each tuple $\bar{x} \in X \cup U \cup K$, $N_j \models \exists^4 \bar{y} \Lambda_l(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_m(\bar{y}, \bar{x})$ where $\{l, m\} = T_j^{[i]}$, then we say $Routine_i$ is ready for part 3. If $Routine_i$ is ready for part 3 and all obstructions have been removed, $Routine_i$ moves to part 3.

Part 3) Take the least element of $\omega^{[i]}$ which has not yet been enumerated into S, and enumerate it into S. Now, $Routine_i$ goes back to part 2.

The possible outcomes of a run of $Routine_i$ are that it gets stuck in part 1, it gets stuck in part 2, or it cycles between part 2 and part 3 infinitely often. In the first two cases, $S^{[i]}$ is finite, and in the third case, $S^{[i]} = \omega^{[i]}$. In any case, we will show that either N does not satisfy the right theory or U is not its basis.

3.4 Verifying the Spectra

In the last section we defined a Σ_1 set S, and in the previous section we gave a construction of two models $\mathcal{M}'$ and $\mathcal{M}''$ from any fixed Σ_1 set. We fix $\mathcal{M}'$ and $\mathcal{M}''$ to be those models obtained by applying the construction to the set S defined in the last section.

It is clear that $\omega \in SRM(Th(\mathcal{M}'))$ and $0, \omega \in SRM(Th(\mathcal{M}''))$. The first being because $\mathcal{M}'$ has a recursive presentation and is saturated. The second is because $\mathcal{M}''$ has a recursive presentation and is saturated and the set of constants in $\mathcal{M}''$ is algebraically closed and infinite, hence also a model of the same theory. Since the set of constants is Σ_1 in the recursive presentation of $\mathcal{M}''$, they form a recursive prime model. It remains to show that for any other $n \in \omega + 1$, n is not in $SRM(Th(\mathcal{M}'))$ or $SRM(Th(\mathcal{M}''))$

Theorem 68. $SRM(Th(\mathcal{M}')) = \{\omega\}$

Proof. Suppose N is a recursive model of $Th(\mathcal{M}')$, and N has a finite basis U. Let i be the index of the pair (f, U) where f is the recursive function describing the quantifier-free diagram of N.

Case 1: $Routine_i$ gets stuck in part 1.
$R_i \in L'$, as $Routine_i$ is never in stage 3. $\mathcal{M}'$ has minimally simply algebraic extensions involving only the relation R_i. Thus N does not satisfy $Th(\mathcal{M}')$.

Case 2: $Routine_i$ gets stuck in part 2.

Case 2a: $Routine_i$ gets stuck in part 2 because it is never ready for part 3.
This means that for some $\bar{x} \in X \cup U$ and $l, m \in Q_j^{[i]}$, $N \not\models \exists^4 \bar{y} \Lambda_l(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_m(\bar{y}, \bar{x})$. Since $Routine_i$ never gets to part 3 again, $Q^{[i]} = Q_j^{[i]}$. By Lemma 63, $\mathcal{M}' \models \exists^4 \bar{y} \Lambda_l(\bar{y}, \bar{x}) \leftrightarrow R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_m(\bar{y}, \bar{x})$. Thus $N \not\models Th(M')$.

Case 2b: There is an obstruction Y which is never removed.
As N is a model of $Th(\mathcal{M}')$, there are no constants in N. Thus when counting the non-removed relations, $\delta_j(Y) = \delta_j(Y/K_j) < |U|$. Since the obstruction is never removed, $\delta'(Y) < |U|$, contradicting U being an independent set in N.

Case 3: $Routine_i$ loops through part 2 and part 3 infinitely often.
By assumption, U is a basis for N. Thus X is algebraic over U, which means that there is a set Y such that $\delta'(Y) = |U|$, and $X \cup U \subseteq Y$. Let s be a stage when $Routine_i$ enters Part 2 and s is large enough that $Y \subseteq N_s$ and for each relation R_j in L' occurring on Y, $S_s^{[j]} = S^{[j]}$. We will show that $Routine_i$ never enters part 3 after stage s, leading to a contradiction.

As $Routine_i$ is ready for part 3 each time it leaves part 2, we see that for all j, N_j realizes occurrences of R_i on X. Let $t > s$ be a stage when $Routine_i$ is in part 2. Then $\delta_t(Y) < \delta'(Y)$ since R_i occurs on X but does not count in δ'. Then R_i is a relation which has not been removed since entering part 2 and neither has any of the relations counted in δ', so $\delta_t(Y) < |U|$. Thus, Y is an obstruction which is never removed after stage s, contradicting our being in case 3.

In any case, we get a contradiction to the assumption that N is a recursive model of $Th(M')$ with finite basis U. □

Lemma 69. *Let U be a finite subset of $\mathcal{M}''$. Then $x \in acl_{\mathcal{M}''}(U)$ if and only if there is a finite set K' of elements named by constants and $x \in acl_{\mathcal{M}'}(U \cup K')$.*

Proof. The left direction is trivial. To prove the rightward direction, take an algebraic formula $\phi(x, U, K')$ defining x over U involving constants K'. See that ϕ is an algebraic formula over $U \cup K'$ in $\mathcal{M}'$. □

Theorem 70. $SRM(Th(\mathcal{M}'')) = \{0, \omega\}$

Proof. Suppose N is a recursive model of $Th(\mathcal{M}'')$ and N has a finite basis U. Let i be the index of the pair (f, U) where f is the function describing the quantifier-free diagram of N.

Case 1: $Routine_i$ gets stuck in part 1.
$R_i \in L'$, as $Routine_i$ is never in stage 3. $\mathcal{M}''$ has minimally simply algebraic extensions involving only the relation R_i. Thus N does not satisfy $Th(\mathcal{M}'')$.

Case 2: $Routine_i$ gets stuck in part 2.

Case 2a: $Routine_i$ gets stuck in part 2 because it is never ready for part 3.
This means that for some $\bar{x} \in X \cup U \cup K$, $l, m \in Q_j^{[i]}$, $N \not\models \exists^4 \bar{y} \Lambda_l(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_m(\bar{y}, \bar{x})$. But since $Routine_i$ never gets to part 3 again, $Q^{[i]} = Q_j^{[i]}$. By Lemma 63, $\mathcal{M}'' \models \exists^4 \bar{y} \Lambda_l(\bar{y}, \bar{x}) \leftrightarrow R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_m(\bar{y}, \bar{x})$. Thus $N \not\models Th(M'')$.

Case 2b: There is an obstruction Y that is never removed.
There is a finite set of constants C in N such that counting only the non-removed relations on Y, $\delta_j(Y/C) < |U|$. As the obstruction is never removed, $\delta'(Y/C) < |U|$, implying that U is not an M'-independent set over the constants C. Hence U is not algebraically independent over $\emptyset$.

Case 3: $Routine_i$ loops through part 2 and part 3 infinitely often.
By assumption, U is a basis for N. Thus X is algebraic over U, which means that there is a finite set Y and a finite set of constants C such that $\delta'(Y/C) = |U|$ and $X \cup U \subseteq Y$. Let s be a stage when $Routine_i$ enters part 2 and s is large enough that $Y \cup C \subseteq N_s$ and for each relation R_j in L' occurring on $Y \cup C$, $S_s^{[j]} = S^{[j]}$. We will show that $Routine_i$ never enters part 3 after stage s, leading to a contradiction.

As $Routine_i$ is ready for part 3 each time it leaves part 2, we see that for all j, N_j realizes occurrences of R_i on X. Let $t > s$ be a stage when $Routine_i$ is in part 2. Then $\delta_t(Y/C) < \delta'(Y/C)$ since R_i occurs on X but does not count in δ'. Then R_i is a relation which has not been removed since entering part 2 and neither has any of the relations counted in δ', so $\delta_t(Y/C) < |U|$. Thus, Y is an obstruction which is never removed after stage s, contradicting our being in case 3.

In any case, we get a contradiction to the assumption of N being a recursive model of $Th(\mathcal{M}'')$ with finite basis U. □

Chapter 4

On Degrees of Strongly Minimal Theories with Recursive Models

We will use a new application of the Hrushovski amalgamation method to answer the following questions of Goncharov and Khoussainov.

Question ($1'$)**.** *Does there exist an $\aleph_1$-categorical theory T turing equivalent to 0^ω such that all of the countable models of T are recursively presentable?*

Question ($1''$)**.** *Does there exist an $\aleph_0$-categorical theory turing equivalent to 0^ω with a recursive model.*

We answer $1'$ via the following theorem.

Theorem 71. *There exists a strongly minimal theory T in a finite language such that $T \equiv_T 0^\omega$ and each countable model of T is recursively presentable.*

In fact, we prove the following somewhat stronger theorem.

Theorem 72. *Let $\mathbf{d}$ be a tt-degree below 0^ω. Then there exists a strongly minimal theory T in a finite language such that $T \in \mathbf{d}$ and each countable model of T is recursively presentable.*

The proof of this theorem uses the Hrushovski Amalgamation construction to produce a strongly minimal theory as well as the Ash-Knight Meta-theorem, which will be introduced in Section 4.2, to manage the recursion theoretic needs of a construction managing 0^ω-level information.

In section 4.4, we provide the following refinement of the result of Khoussainov and Montalban which answered question $1''$ affirmatively.

Theorem 73. *There exists an $\aleph_0$-categorical theory T in a finite language such that $T \equiv_T 0^\omega$ and the countable model of T is recursively presentable.*

In fact, we prove the following somewhat stronger theorem.

Theorem 74. *Let* $\mathbf{d}$ *be a tt-degree below* 0^ω*. Then there exists an* $\aleph_0$*-categorical theory* T *in a finite language such that* $T \in \mathbf{d}$ *and the countable model of* T *is recursively presentable.*

4.1 The Theory

We will define a theory relative to any set $S \subseteq \mathbb{N}$ such that whether $n \in S$ is uniformly recursive in 0^n. We will construct the theory to be tt-equivalent to S and we will show that any tt-degree below 0^ω contains such a set. We begin with an infinite language L, construct a model $\mathcal{M}$ in the language L, and our theory will be a reduct of $\mathrm{Th}(\mathcal{M})$ to a sublanguage with finite signature. We define the language $L = \{R_i | i \in \omega\}$ where each R_i is ternary. Recalling the conventions from Chapter 2, we ensure that the relations defined by each of the R_i are symmetric and hold only on distinct tuples. We write $|R_i(A)|$ for the number of sets $\bar{x}$ in A so that $R_i(\bar{x})$ holds, and we set $\delta(A) = |A| - \sum_{i \in \omega} |R_i(A)|$.

We will follow the construction for section 2.2, and will specify an integer b and function μ. We fix a particular minimally simply algebraic extension H over G, G a three element set such that $tp_{r.q.f}(H/G)$ involves a single ternary relation symbol. We fix $k = |G \cup H|$. For $i \geq 1$, we write Γ_i for the relative quantifier-free type received by replacing the relation symbol in $tp_{r.q.f.}(H/G)$ by R_{i-1}.

Definition 75. *Let* B *be minimally simply algebraic over* A*.*

$$\mu(B, A) = \begin{cases} |A| + 2^k & \text{if } \forall i \; tp_{r.q.f.}(B/A) \neq \Gamma_i \\ |A| + 2^k + 2 & \text{if } tp_{r.q.f.}(B/A) = \Gamma_i \text{ and } \neg R_i(A) \\ |A| + 2^k + 1 & \text{if } tp_{r.q.f.}(B/A) = \Gamma_i \text{ and } R_i(A) \text{ and } i \in S \\ |A| + 2^k & \text{if } tp_{qf}(B/A) = \Gamma_i \text{ and } R_i(A) \text{ and } i \notin S \end{cases}$$

Definition 76. *If* B *and* A *are finite* L*-structures such that* $tp_{r.q.f.}(B/A)|_{R_{i-1}} = \Gamma_i$*, then we say the extension* B *over* A *is a* Γ_i*-extension.*

We work with the amalgamation class $\mathcal{C} = \mathcal{C}_{0,\mu}$. Recall the following definition of the amalgamation class $\mathcal{C}$.

Definition 77. *Let* $\mathcal{C}$ *be the class of finite* L*-structures* C *such that*

- $\delta(A) \geq 0$ *for all* $A \subseteq C$*.*
- *If* $B_1, \ldots, B_n$*,* A *are disjoint subsets of* C *such that each* B_i *over* A *is an extension of the form of* Y *over* X*. Then* $n \leq \mu(Y, X)$*.*

From the amalgamation of this class we generate a model and a theory. In section 2.3, we showed that the amalgamated structure is saturated and the theory is strongly minimal.

Th($\mathcal{M}$) encodes S. We want to restrict to a finite language while still encoding this set. To do so, we will show that each R_i is definable in terms of R_{i-1}, so we can restrict to the language generated by the single relation symbol R_0. To show this, we need a stronger version of the Algebraic Amalgamation Lemma specifically for the Γ_i. The following lemma should explain the occurrences of 2^k in the definition of our chosen μ function.

Lemma 78. *(Algebraic Amalgamation Lemma for Γ_i) Suppose $A, B_1, B_2 \in \mathcal{C}$, $A = B_1 \cap B_2$, $B_1 \setminus A$ is simply algebraic over A. Suppose further that $B_1 \setminus A$ is minimally simply algebraic over $A' \subseteq A$, and $tp_{r.q.f.}(B_1 \setminus A/A') = \Gamma_i$. Then one of the following two conditions holds:*

- *The free-join of B_1 and B_2 over A is in $\mathcal{C}$.*
- *B_2 contains $\mu(B_1 \setminus A, A')$ disjoint Γ_i-extensions of A' (ie: extensions of the form of $B_1 \setminus A$ over A').*

Proof. Let E be the free-join of B_1 with B_2 over A. Suppose $E \notin \mathcal{C}$. This means that there are disjoint $C^1, \dots C^n$, F contained in E and a pair (Y, X) such that each C^j over F is of the form of Y over X and $n > \mu(Y, X)$. Restricting E to the language $L_{Y/X}$, we see each of the C^j are minimally simply algebraic over F in the same way. Here we have the same set-up as in Lemma 42. Claims 1-3 and case 1 of Lemma 42 hold exactly as proved there. We need only count the number of C^j which are entirely contained in $B_1 \setminus A$. There are certainly fewer than $2^{|(B_1 \setminus A) \cup A'|} = 2^k$ such C^js. So $n \leq |F| + 2^k = |X| + 2^k \leq \mu(Y, X)$. □

We will write $\Gamma_i(\bar{y}, \bar{x})$ to denote the first order formula designating that $\bar{y}$ over $\bar{x}$ is an Γ_i-extension.

Lemma 79. *Fix $i \geq 1$. $\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg\exists^{5+2^k}\bar{y}(\Gamma_i(\bar{y}, \bar{x}))$. (Note: we write $\exists^m \bar{y}$ to say that there exists m disjoint tuples $\bar{y}$ satisfying the property.)*

Proof. The rightward direction follows from the fact that any finite substructure of $\mathcal{M}$ is an element of $\mathcal{C}$. If the rightward direction did not hold, then we would be explicitly violating the μ-bound.

The leftward direction follows from the previous lemma. Suppose $\neg R_i(\bar{x})$ holds. Let A be such that $\bar{x} \subseteq A \leq \mathcal{M}$. Repeated application of the previous lemma shows that there is a $B \in \mathcal{C}$ such that $A \leq B$ and B contains $5 + 2^k$ disjoint Γ_i-extensions over $\bar{x}$. Property 3 of $\mathcal{M}$ guarantees that this B embeds in $\mathcal{M}$ over A. □

Since the first order formula $\Gamma_i(\bar{y}, \bar{x})$ is defined using only the relation R_{i-1}, we see that each of the R_i are definable via the relation R_0.

Definition 80. *Let* $T = \mathrm{Th}(\mathcal{M})|_{R_0}$.

By Lemma 79, $\mathrm{Th}(\mathcal{M})$ is a definitional expansion of T. This T is going to be the strongly minimal theory of the theorem. The following lemma shows that $\mathrm{Th}(\mathcal{M})$, and thus T, computes S.

Lemma 81. *Fix* $i \in \omega$. *Then* $i \in S$ *if and only if* $\mathcal{M} \models \forall \bar{x} \exists^{4+2^k} \bar{y}(\Gamma_i(\bar{y}, \bar{x}))$.

Proof. There are as many disjoint Γ_i-extensions in $\mathcal{M}$ over $\bar{x}$ as μ allows by the same argument as in Lemma 79. If $i \in S$, then μ always allows at least $4 + 2^k$ extensions. If $i \notin S$ and $R_i(\bar{x})$, then μ allows only $3 + 2^k$ Γ_i-extensions over $\bar{x}$, witnessing that $\mathcal{M} \models \neg\forall \bar{x} \exists^{4+2^k} \bar{y}(\Gamma_i(\bar{y}, \bar{x}))$. □

This shows that $S \leq_{tt} \mathrm{Th}(\mathcal{M}) \equiv_{tt} T$. As this construction works for any S, we see that the recursive function computing T from S is total, so $T \equiv_{tt} \mathrm{Th}(\mathcal{M}) \equiv_{tt} S$. In Section 4.3 we will complete the proof of the main theorem by showing that all of the countable models of T are recursive.

Here we recall the construction of a countable model of T, ignoring the recursion-theoretic obstructions. As follows from the proof of $\mathcal{M}$ being strongly minimal, the set of formulae which describe algebraicity are those that describe the various ways that $d(xA)$ could be the same as $d(A)$. Thus, looking at the structure of a model with a basis U of size k, we see that for any x, $d(xU) = d(U) = k$. Taking unions, we see that $d(A) = k$ for any finite set containing U and $d(A) \leq k$ for any finite set. Further, any countable model of T satisfies 1,2, and $3''$. In fact, these properties characterize the k-dimensional model. Thus, we present a way to construct the k-dimensional model of T:

At each stage s, we have a finite L-structure M_s built by stage s.

Stage 0: start with M_0 being a k-element set with no relation symbols holding on it.

Stage s: Ensure $3''$ holds for the first s possible minimally simply algebraic extensions on some list of all minimally simply algebraic extensions. Do this by repeated use of the algebraic amalgamation lemma over M_{s-1}. Doing so ensures that $M_{s-1} \leq M_s$ and $\delta(M_s) = k$.

Any finite set $A \subseteq \bigcup_s M_s$ is contained in some M_s, so $d(A) \leq \delta(M_s) = k$. Also $M_0 \leq M$ and $d(M_0) = |M_0| = k$, so M_0 is a basis for $\bigcup_s M_s$. Thus this procedure produces the k-dimensional model of T.

This is the construction we will employ, but we need to do so recursively. The obstruction to doing this is that μ is determined by S. The saving grace is that the μ-bound is on occurrences of Γ_i, which take many quantifiers to describe in T. We can employ a worker construction via the metatheorem to deal with the non-recursive information.

4.2 The Metatheorem

This section entirely follows (Ash-Knight [1], see pg. 235), with the notational exception that the set which we call V is there referred to as L. To maintain as much of the Ash-Knight notation as possible, we still refer to elements of V as l's. The elements of an ω-system should be understood as follows: V will be the set of finite L-structures we might build in our construction, U is the set of estimates to S we may have in our construction, $\hat{l}$ will be the structure we start our construction from, P will be the tree of partial constructions pairing finite pieces of S with the finite structure we build given that information, E enumerates the content of the construction that we commit to, and $\leq_n$ represents the potential layers of injury to be handled in the construction. Finally, q will provide 'free' information about the true value of S, and the Meta-theorem will allow us to handle enough injury that we can carry out a construction along a path through P agreeing with q.

Let V and U be recursively enumerable sets, E be a partial recursive enumeration function on V, and let P be a recursively enumerable alternating tree on V and U made up of non-empty finite sequences which all start with the same $\hat{l} \in V$. Let $(\leq_n)_{n\in\omega}$ be uniformly recursively enumerable binary relations on V.

We define the structure $(V, U, \hat{l}, P, E, (\leq_n)_{n<\omega})$ to be an ω-system if it satisfies the following properties:

1. $\leq_n$ is reflexive and transitive for all $n < \omega$.
2. $l \leq_n l' \Rightarrow l \leq_m l'$ for $m < n < \omega$.
3. If $l \leq_0 l'$, then $E(l) \subseteq E(l')$.
4. If $\sigma u \in P$, where σ has length 2n+1 ending in $l^0 \in V$ and
$$l^0 \leq_{n_0} l^1 \leq_{n_1} \ldots \leq_{n_{t-1}} l^t$$
for $n > n_0 > \ldots > n_t$, then there exists l^* such that $\sigma u l^* \in P$ and $l^i \leq_{n_i} l^*$ for each $0 \leq i \leq t$.

Theorem 82. *(Ash-Knight Metatheorem)*
Let $(V, U, \hat{l}, P, E, (\leq_n)_{n<\omega})$ be an ω-system, and let q be a uniformly 0^n instruction function for P (ie: uniformly in n, q computes u_n on input $\hat{l}u_0l_0u_1l_1\ldots l_{n-1}$ using oracle 0^n). Then there is a path $\pi = \hat{l}u_0l_0u_1l_1\ldots$ through P which agrees with the instruction function q such that $E(\pi)$ is recursively enumerable.

We will use the metatheorem to get a structure whose atomic diagram is recursively enumerable, ie: a recursive structure. The method of building the structure, the ω-system, will be explained in the next section. Note that the proof that a particular ω-system satisfies property 4 necessarily contains all the details as to how injury is handled in the construction.

4.3 Constructing the Countable Models of T

We are going to use the Ash-Knight Metatheorem to construct the k-dimensional model of T by defining an ω-system $(V, U, \hat{l}, P, E, (\leq_n)_{n\in\omega})$. Throughout the construction, we will be working with various estimates to the set $\{i | i \in 0^i\}$. These estimates will be represented by elements of $2^{<\omega}$. Given the estimate τ, we define μ_τ and $\mathcal{C}_\tau$, the corresponding approximations to μ and $\mathcal{C}$.

Definition 83. *Let τ be an element of $2^{<\omega}$ where length$(\tau) = n + 1$ ($\tau(0)$ is never referenced, so this index is off by one).*

We define $L_\tau = L_n$ to be the language generated by the relation symbols $\{R_i | i < n\}$
For B a minimally simply algebraic extension over A, let

$$\mu_\tau(B, A) = \begin{cases} |A| + 2^k & \text{if } B \text{ over } A \text{ is not a } \Gamma_i\text{-extension for any } i \\ |A| + 2^k + 2 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension and } \neg R_i(A) \\ |A| + 2^k + 1 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A)\text{, and } \tau(i) = 1 \\ |A| + 2^k & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A)\text{, and } \tau(i) = 0 \end{cases}$$

Let $\mathcal{C}_\tau$ be the class of finite L_τ structures C such that the following conditions hold:

- *$\delta(A) \geq 0$ for all $A \subseteq C$.*
- *Let Y over X be a minimally simply algebraic extension. Suppose $B_1, \ldots, B_n$, A are disjoint subsets of C such that each B_i over A is an extension of the form of Y over X. Then $n \leq \mu_\tau(Y, X)$.*

Definition 84. *Fix $k \in \omega$. We define S_k to be the following system:*

- *V is the set of pairs (M, σ), where M is a finite L-structure whose universe is an initial segment of ω and $\sigma \in 2^{<\omega}$ such that $M \in \mathcal{C}_\sigma$. We write $l = (M_l, \sigma_l)$.*
- *U is $2^{<\omega}$.*
- *$\hat{l}$ is the pair (M, σ) where M is the structure with k elements and no relations and σ is the trivial string of length 0.*
- *$E(l)$ is the set of primitive statements true about M_l in the language generated by the single relation symbol R_0*
- *$l \leq_n l'$ if the following conditions hold:*
  1. *$\sigma_l|_n = \sigma_{l'}|_n$ (ie: $\sigma_l(i) = \sigma_{l'}(i)$ for $i \leq n$).*
  2. *The universe of M_l is a subset of the universe of $M_{l'}$.*

3. *$M_l|_{L_n} \leq M_{l'}|_{L_n}$. By this, we mean that as L_n-structures $M_l|_{L_n} \subseteq M_{l'}|_{L_n}$ and it is a strong substructure.*

- *P is the tree defined by $\hat{l}, u_0, l_0, \ldots, u_t, l_t \in P$ if*

  1. *For each i, $\sigma_{l_i} = u_i$.*
  2. *For each i, $M_{l_i} \leq M_{l_{i+1}}$.*
  3. *For each i, $\delta(M_{l_i}) = k$.*
  4. *Take a universal list of all minimally simply algebraic extensions in L along with sets the extension could be over, call it List. For the first i entries on List, if the extension is in $\mathcal{C}_{\sigma_{l_i}}$, then $3''$ holds for that extension and M_{l_i}.*

(the last item says that all the allowed copies of each of the first i extensions occur already in M_{l^i})

Theorem 85. *S_k is an ω-system.*

Proof. We focus on the difficult condition.

Suppose $\tau u \in P$, $length(\tau)$=2n+1, τ ends in l^0, and

$$l^0 \leq_{n_0} l^1 \leq_{n_1} \ldots \leq_{n_{t-1}} l^t$$

for $n > n_0 > n_1 > \ldots n_{t-1} > n_t$. Without loss of generality, we assume that $n_0 = n-1$. We need to show that there exists an l^* such that $\tau u l^* \in P$, and for each i, $l^i \leq_{n_i} l^*$. First we will define an auxiliary structure $l^\#$ which will handle the injury occurring in this sequence of l's. Then we will extend $l^\#$ to an l^* which has the right dimension and contains amalgamations of the required structures from $\mathcal{C}_u$. To avoid notation such as M_{l^j}, we write $l^j = (M_j, \sigma_j)$.

Let $l^\#$ be the pair $(\mathcal{N}, \sigma)$, defined as follows. $\sigma = u$ and $\mathcal{N}$ has the same universe as M_t. Let $\bar{x}$ be a tuple in $\mathcal{N}$. We will describe whether or not R_i holds on $\bar{x}$. Let m be least such that $\bar{x} \subseteq M_m$. Then R_i holds on $\bar{x}$ in $\mathcal{N}$ if and only if R_i is in L_{n_m} and holds on $\bar{x}$ in M_m.

We will write $\mathcal{N}_i$ for the substructure of $\mathcal{N}$ with the same universe as M_i, and we will write Ł$_i$ for L_{n_i}.

Claim 86. *$M_i|_{Ł_i} \subseteq \mathcal{N}|_{Ł_i}$, ie: for each relation in Ł$_i$ and every tuple in M_i, the relation holds in M_i if and only if it holds in $\mathcal{N}$.*

Proof. For any $j \leq i$, $l^j \leq_{n_{i-1}} l^i$, in particular, $l^j \leq_{n_i} l^i$. Let $\bar{x}$ be any tuple in M_i and let m be minimal such that $\bar{x} \subseteq M_m$. Then $m \leq i$, so $l^m \leq_{n_i} l^i$. Thus for $R \in$ Ł$_i$, $R(\bar{x})$ holds in M_m if and only if it holds in $\mathcal{N}$ (by definition of $\mathcal{N}$) and the first condition is equivalent to $R(\bar{x})$ holding in M_i as $l^m \leq_{n_i} l^i$. □

In particular, since $n_0 = n - 1$, $M_0 = \mathcal{N}_0$.

Lemma 87. $l^{\#} \in V$

Proof. we need to verify that $\mathcal{N} \in \mathcal{C}_\sigma$. We verify this by verifying each condition in the definition of $\mathcal{C}_\sigma$.

1. $\delta(A) \geq 0$ for all $A \subseteq \mathcal{N}$

   Proof. Let A be a subset of $\mathcal{N}$. Let $A_i = A \cap \mathcal{N}_i$. We need to show that $\delta(A_t) \geq 0$. We achieve this by showing that $\delta(A_{i+1}/A_i) \geq 0$ for each i. This suffices since $\delta(A_t) = \delta(A_t/A_{t-1}) + \delta(A_{t-1}/A_{t-2}) + \ldots + \delta(A_1/A_0) + \delta(A_0)$ and A_0 is a subset of $\mathcal{N}_0 = M_0$, hence has non-negative dimension.

   $\delta(A_{i+1}/A_i)$ is $|A_{i+1} \setminus A_i|$ − (the number of relations holding in A_{i+1} involving at least one element in $A_{i+1} \setminus A_i$). Consider B the subset of M_{i+1} with the same underlying set as A_{i+1}. Since $M_i \leq_{n_i} M_{i+1}$, $\delta(B|_{Ł_i}/(B \cap M_i)|_{Ł_i}) \geq 0$, but $\delta(B|_{Ł_i}/(B \cap M_i)|_{L_i}) \leq \delta(A_{i+1}/A_i)$, as every relation counting on the right counts on the left as well. Thus, each summand is non-negative and $\delta(A) \geq 0$. □

2. If $C^1, \ldots, C^n$, F are disjoint subsets of $\mathcal{N}$, and each C^j over F is of the form of Y over X (for Y over X a minimally simply algebraic extension), then $n \leq \mu_\sigma(Y, X)$.

   Proof. We proceed by induction to show that the condition holds for each $\mathcal{N}_i$. The condition holds on $\mathcal{N}_0$, as this is just M_0. Suppose the condition holds for $\mathcal{N}_{s-1}$. We will show that the condition holds on $\mathcal{N}_s$ as well.

   The proof follows via Lemma 42. Suppose $C^1, \ldots, C^n$, F are disjoint subsets of $\mathcal{N}_s$, and each C^j over F is of the form of Y over X (for Y over X a minimally simply algebraic extension). Since $\mathcal{N}_{s-1} \leq \mathcal{N}_s$, we apply Lemma 42 with $B_1 = \mathcal{N}_s|_{L_{Y/X}}$, $A = B_2 = \mathcal{N}_{s-1}|_{L_{Y/X}}$. There are 4 cases to consider. In one case, $r \leq |X| < \mu_\sigma(Y, X)$. In each of the other cases, one C^j is entirely contained in $\mathcal{N}_s \setminus \mathcal{N}_{s-1}$.

   In this case, $tp_{r.q.f.}(Y/X)$ only includes relations from the language $Ł_s$. There are a number of possibilities to consider:

   - Y over X is not a Γ_i-extension for any i. Since $M_s|_{Ł_s} \subseteq \mathcal{N}_s|_{Ł_s}$, we see that each of the C^j over F, looked at as subsets of M_s, are of the form of Y over X. Since for non-Γ_i-extensions, μ_τ does not depend on τ, and since M_s satisfies the property for μ_{σ_s}, $n \leq \mu_{\sigma_s}(Y, X) = \mu_\sigma(Y, X)$.
   - Y over X is a Γ_i-extension and $\neg R_i(X)$. In this case, we look at the C^j and F as subsets of M_s. The C^j are each Γ_i-extensions over F. If $R_i(F)$ in M_s, then the number of C^j is bounded by $\mu_{\sigma_s}(Y, X')$ (where X' is the same as X but with $R_i(X)$ holding) which is even less than $\mu_{\sigma_s}(Y, X)$. If

$\neg R_i(F)$ in M_s, then the number of C^j is bounded by $\mu_{\sigma_s}(Y,X)$. Since $\sigma_s|_{n_{s-1}} = \sigma|_{n_{s-1}}$, $\mu_{\sigma_s}(Y,X) = \mu_\sigma(Y,X)$.

- Y over X is a Γ_i-extension and $R_i(X)$. If $R_i \in Ł_s$, then we have the corresponding fact in M_s, so we get the μ-bound from the fact that M_s satisfies the property for μ_{σ_s} and $\sigma_s|_{n_{s-1}} = \sigma|_{n_{s-1}}$. We may assume $R_i \notin Ł_s$. Since Y over X is a Γ_i-extension, $R_{i-1} \in Ł_s$. So $R_i \in Ł_{s-1}$. Clearly, $R_i(F)$ implies that $F \subseteq \mathcal{N}_{s-1}$. But then $R_i(F)$ holds in M_{s-1}, and $l^{s-1} \leq_{n_{s-1}} l^s$, so $R_i(F)$ holds in M_s as well. Again, we get the μ-bound from M_s.

This concludes the inductive step, showing that $\mathcal{N} = \mathcal{N}_t$ satisfies the condition. □

□

Claim 88. *For each i, $l^i \leq_{n_i} l^\#$.*

Proof. We verify the two properties. First we verify $\sigma_i|_{n_i} = \sigma|_{n_i}$. We know that $l^0 \leq_{n_{i-1}} l^i$. So, $\sigma|_{n_i} = \sigma_0|_{n_i} = \sigma_i|_{n_i}$.

Second we verify that $M_i|_{Ł_i} \leq \mathcal{N}|_{Ł_i}$. Claim 86 gives us that $M_i|_{Ł_i} \subset \mathcal{N}|_{Ł_i}$.

Now, let X be a subset of $\mathcal{N}|_{Ł_i}$. We use the same argument as before (when we showed that $\delta(A) \geq 0$ for all $A \subseteq \mathcal{N}$). We need to show that $\delta(X/\mathcal{N}_i|_{Ł_j}) \geq 0$. For $i \leq j \leq t$, we write $X_j = ((X \cap \mathcal{N}_j) \cup \mathcal{N}_i)|_{Ł_i}$. Then we write $\delta(X/\mathcal{N}_i|_{Ł_i}) = \delta(X_t/X_{t-1}) + \delta(X_{t-1}/X_{t-2}) + \ldots + \delta(X_{i+1}/X_i)$. As in the previous argument, each summand is non-negative, so $\delta(X/\mathcal{N}_i|_{Ł_i}) \geq 0$. □

The only obstructions to $l^\#$ being what we need for l^* is that it might not contain the first n minimally simply algebraic extensions and perhaps $\delta(\mathcal{N}) > k$. Extend $\mathcal{N}$ using only the relation symbol R_{n-1} to $\mathcal{N}'$ so that $\delta(\mathcal{N}') = k$, $M_0 \leq \mathcal{N}'$, and $\mathcal{N}' \in \mathcal{C}_\sigma$. Then proceed to extend $\mathcal{N}'$ to $\mathcal{N}^*$ by amalgamating to ensure that the first n minimally simply algebraic extensions occur inside $\mathcal{N}^*$ if they are allowed in $\mathcal{C}_\sigma$. We set l^* to be $(\mathcal{N}^*, u)$. By construction, $l^i \leq_{n_i} l^\# \leq_{n_i} l^*$ and $\tau u l^* \in P$ as $l^* \in V$, $M_0 \leq \mathcal{N}^*$, $\sigma_{l^*} = u$, and $\delta(\mathcal{N}^*) = k$. Having found this l^*, we have shown that $(V, U, \hat{l}, E, P, (\leq_n)_{n\in\omega})$ is an ω-system. □

We have a uniformly 0^n instruction function for u, namely $u_n = S|_n$, the string in $2^{<\omega}$ describing membership in S for integers $\leq n$. Thus, the metatheorem gives us a run, $\pi = \hat{l} u_0, l_0, u_1, l_1, \ldots$ such that $E(\pi)$ is recursively enumerable. $E(\pi)$ gives us the R_0-atomic diagram of $\bigcup_i M_{l_i}$.

Theorem 89. *The k-dimensional model of T is recursively presentable.*

Proof. All that remains to be shown is that this model $M = \bigcup_i M_{l_i}$ is isomorphic to the k-dimensional model of T.

To see this, we see by construction that M satisfies properties $1, 2$, and $3''$, thus $M \models T$. Furthermore, $\hat{l} \leq M$, so the dimension is at least k. For any finite $A \subseteq M$, there is some M_{l_i} such that $A \subseteq M_{l_i}$ and $\delta(M_{l_i}) = k$, so $d(A) \leq k$. Thus the dimension of M is k. □

Theorem 90. *The saturated model of T is recursively presentable.*

Proof. We use a similar ω-system which is identical except that instead of insisting that $\delta(M_{l_i}) = k$ in the definition of P, we insist that $\delta(M_{l_i}) \geq i$. To get $\mathcal{N}'$ from $\mathcal{N}$ we add a single element not related to anything else. In the final model, we constructed it so that $1, 2, 3', 3''$ all hold so the constructed model is the saturated model of T. □

We conclude the desired theorem.

Theorem 91. *Let $\mathbf{d}$ be a tt-degree below 0^ω. Then there exists a strongly minimal theory T in a finite language such that $T \in \mathbf{d}$ and each countable model of T is recursively presentable.*

Proof. All that remains to show is that each tt-degree $\mathbf{d}$ below 0^ω contains a set S for which whether $n \in S$ is uniformly recursive in 0^n. Let S' be any member of $\mathbf{d}$, and let f be a recursive function such that $x \in S$ if and only if $f(x) \in 0^\omega$. Let S be the set $\{\langle x, f(x)\rangle | x \in S\}$. It is easy to see that $S \equiv_{tt} S'$ and $\langle x, f(x)\rangle \in S$ is uniformly recursive in $0^{\langle x, f(x)\rangle}$. □

4.4 The $\aleph_0$-categorical Case

Again, we fix a set S so that whether $n \in S$ is uniformly recursive in 0^n. We work at first in an infinite language to produce the theory, then we will take a reduct to a finite language. Let L be the language with signature $\{P, Q\} \cup \{R_i | i \geq 3\}$, where P and Q are binary relation symbols and each R_i is i-ary.

Definition 92. *Let $\mathcal{K}$ be the class of finite L-structures C which satisfy the following properties:*

- *Each relation symbol is symmetric and holds only on tuples of distinct elements*
- *If $i - 10 \notin S$ or $i < 10$, then C satisfies*

$$\neg\exists \bar{x}, y, z \left[R_i(\bar{x}) \land P(y, z) \land \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}| = i-2} (R_{i-1}(y, \bar{w}) \land R_{i-1}(z, \bar{w})) \right]$$

- *If $i-10 \in S$, then C satisfies*

$$\neg\exists \bar{x}, y, z \left[R_i(\bar{x}) \wedge Q(y,z) \wedge \bigwedge_{\bar{w}\subset\bar{x}, |\bar{w}|=i-2} (R_{i-1}(y,\bar{w}) \wedge R_{i-1}(z,\bar{w})) \right]$$

To carry out a Fraïssé construction using $\mathcal{K}$, we must verify the following lemma:

Lemma 93. *$\mathcal{K}$ has HP, JEP, and AP.*

Proof. Since $\mathcal{K}$ is defined via $\forall_1$ formulas, it automatically satisfies HP.
Given two disjoint structures $B, C \in \mathcal{K}$, see that the free-join of B with C over $\emptyset$ is in $\mathcal{K}$. So $\mathcal{K}$ satisfies the JEP.
Lastly, given three structures $A, B, C \in \mathcal{K}$ where $A = B \cap C$, we will show that the free-join of B with C over A is in $\mathcal{K}$. Suppose not, then there exists $i, \bar{x}, y, z$ witnessing this. We may assume $i - 10 \in S$. So, $\bar{x}, y, z$ satisfy:

$$R_i(\bar{x}) \wedge Q(y,z) \wedge \bigwedge_{\bar{w}\subset\bar{x}, |\bar{w}|=i-2} (R_{i-1}(y,\bar{w}) \wedge R_{i-1}(z,\bar{w}))$$

Since $R_i(\bar{x})$, we know that one of the following holds:
Case 1:$\bar{x} \subseteq A$. Here, since $Q(y,z)$, $\{y,z\} \subseteq B$ or $\{y,z\} \subseteq C$. Thus $\{\bar{x},y,z\} \subseteq B$ or $\{\bar{x},y,z\} \subseteq C$, witnessing that $B \notin \mathcal{K}$ or $C \notin \mathcal{K}$, either way yielding a contradiction.
Case 2:$\bar{x} \subseteq B$, $x_i \in \bar{x}$, $x_i \in B \smallsetminus A$. Take any $\bar{w} \subset \bar{x}$ so that $|\bar{w}| = i-2$ and $x_i \in \bar{w}$. Then $R_{i-1}(y,\bar{w})$ and $R_{i-1}(z,\bar{w})$ implies that y and z are both in B. Thus $\{\bar{x},y,z\} \subseteq B$, which shows $B \notin \mathcal{K}$, a contradiction.
Case 3:$\bar{x} \subseteq C$, $x_i \in \bar{x}$, $x_i \in C \smallsetminus A$. This is the same as case 2. □

Now we use Fraïssé's theorem ([7],6.1.2) which guarantees a countable ultra-homogeneous L-structure $\mathcal{M}$ with $Age(\mathcal{M}) = \mathcal{K}$. Further, as $\mathcal{M}$ is ultra-homogeneous, it admits quantifier elimination. Thus, the number of n-types is bounded by the number of possible configurations of the finitely many relations in L of arity less than n. Thus, Th($\mathcal{M}$) is $\aleph_0$-categorical. The following lemma allows the reduction to a finite sub-language.

Lemma 94. *Suppose $i > 3$:*
If $i - 10 \notin S$ or $i < 10$, then

$$\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg\exists y, z \left[P(y,z) \wedge \bigwedge_{\bar{w}\subset\bar{x}, |\bar{w}|=i-2} (R_{i-1}(y,\bar{w}) \wedge R_{i-1}(z,\bar{w})) \right]$$

Similarly, if $i - 10 \in S$, then

$$\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg\exists y, z \left[Q(y,z) \wedge \bigwedge_{\bar{w}\subset\bar{x}, |\bar{w}|=i-2} (R_{i-1}(y,\bar{w}) \wedge R_{i-1}(z,\bar{w})) \right]$$

Proof. The rightward direction follows via the fact that $Age(M) = \mathcal{K}$.
To show the leftward direction, take a tuple $\bar{x}$ such that $\mathcal{M} \models \neg R_i(\bar{x})$. By ultrahomogeneity of $\mathcal{M}$, it suffices to show that $\bar{x}$ embeds into an element of $\mathcal{K}$ where such a y and z exist. Consider the structure $A = \bar{x} \cup \{a, b\}$, where each of a and b are R_{i-1}-related to every $i-2$-element subset of $\bar{x}$, a and b are P-related (Q-related in the case of the second equivalence above), and no other relations hold involving a or b. It is easy to verify that this structure is in $\mathcal{K}$. □

Definition 95. *Let* $T = \mathrm{Th}(\mathcal{M})|_{\{P,Q,R_3\}}$

By the previous lemma, $\mathrm{Th}(\mathcal{M})$ is a definitional expansion of T. Thus, T is also $\aleph_0$-categorical. Also, from T we can in a tt way determine which definition of R_i is correct and recover $\mathrm{Th}(\mathcal{M})$, and thus $S \equiv_{tt} \mathcal{K} \equiv_t t\mathrm{Th}(\mathcal{M}) \equiv_{tt} T$. Thus it remains only to prove that the countable model of T is recursively presentable.

Lemma 96. *Uniformly in n, $T \cap \exists_n$ is computable in 0^{n-7}.*

Proof. We first show that any $\exists_n$ formula in T is equivalent to a quantifier-free formula in relations $\{P, Q, R_i\}_{i \leq 3+n}$. The proof proceeds by setting up the appropriate Ehrenfuecht-Fraïssé game and seeing that '∃loise' has a winning strategy. The game is the standard Ehrenfuecht-Fraïssé game of length n where we start with tuples $\bar{a}$ and $\bar{b}$ which have the same $\{P, Q, R_i\}_{i \leq 3+n}$-quantifier-free type. Then whichever tuple $\bar{c}$ '∀belard' chooses, ∃loise can choose a tuple $\bar{d}$ so that $\bar{a}\bar{c}$ and $\bar{b}\bar{d}$ satisfy the same $\{P, Q, R_i\}_{i \leq 3+n-1}$ type. Proceeding as such, ∃loise wins the game of length n.
This shows that any $\exists_n$ formula depends only on the relations $\{P, Q, R_i\}_{i \leq 3+n}$ ([13], Lemma 2.4.9). Thus, the $\exists_n$ formula $\exists\forall \ldots \phi(\bar{x})$ is equivalent to

$$\bigvee_{(\text{configurations in } \{P,Q,R_i\}_{i \leq 3+n} \text{ in } \mathcal{K})} \bigwedge_{(\text{configurations in } \{P,Q,R_i\}_{i \leq 3+n-1} \text{ in } \mathcal{K})} \ldots \phi(\bar{x})$$

To verify whether this statement is true, we need only to be able to parse "in $\mathcal{K}$" for configurations in the language $\{P, Q, R_i\}_{i \leq 3+n}$. The conditions of being in $\mathcal{K}$ is then described recursively in 0^{3+n-10}. □

We use the following case of a theorem of Knight [11] to show that the countable model of T is recursively presentable.

Theorem 97. *(Knight) Let T be an $\aleph_0$-categorical theory. If $T \cap \exists_{n+1}$ is Σ^0_n uniformly in n, then T has a recursive model.*

Since T is $\aleph_0$-categorical, Lemma 96 shows that T satisfies the conditions of this theorem. Thus we conclude the promised theorem.

Theorem 98. *Let* $\mathbf{d}$ *be a tt-degree below* 0^ω*. Then there exists an $\aleph_0$-categorical theory T in a finite language such that $T \in \mathbf{d}$ and the countable model of T is recursively presentable.*

Proof. Let $S \in \mathbf{d}$ be a set with the property that whether $n \in S$ is uniformly recursive in 0^n. Let T be the theory attained by applying this construction to S. $T \equiv_{tt} S$ and the countable model of T is recursively presentable. $\square$

Chapter 5

New Spectra in Finite Languages

In this chapter, we present an alteration of the Hrushovski method which produces strongly minimal theories as in the following theorems.

Theorem 99. *For each $n \in \omega$, there exists a strongly minimal theory T in a finite language so that $SRM(T) = \{0, \dots, n\}$*

Theorem 100. *There exists a strongly minimal theory T in a finite language so that $SRM(T) = \omega$*

Recall the methods of coding used in Theorems 18 and 19. We want to employ similar encodings of non-recursive content into the generic (n+1)-type or ω-type, but use a finite language to do so.

5.1 Altering the Hrushovski Construction

We will use the definitions and results from section 2.1 and we will specify a new amalgamation class. We fix the language L generated by the single ternary relation symbol R. We write $R(A)$ for the set of tuples from A on which R holds. We define a function $\delta : \{\text{finite L-structures}\} \to \mathbb{Z}$ by $\delta(A) = |A| - |R(A)|$. Note that δ is a pre-dimension function. From δ, we use the definitions of relative dimension, dimension in, strong substructure, simply algebraic extension, and minimally simply algebraic extension as in section 2.1.

Unlike the standard Hrushovski construction of a strongly minimal set, we provide the definition:

Definition 101. *For $A \subseteq B$ L-structures, A finite, we define $f_B(A) = \min\{|C| | A \subseteq C \subseteq B, \delta(C) < |A|\}$, where we say the* min *of an empty set is ∞.*

Lemma 102. *If $A \subseteq B \subseteq C$ and $B \leq C$, then $f_B(A) = f_C(A)$*

Proof. Take $X \subseteq C$ of minimal size with $\delta(X) < |A|$. Then $\delta(X \cap B) \leq \delta(X) < |A|$. Thus $X \subseteq B$ by minimality. □

One can think of $f_B(A)$ as a measure of how much A looks independent to the set B. We define the μ function similarly to its analog in chapter 2, but we want to incorporate f into our definition.

Definition 103. *Let $\mu(A, B, n)$ be a function from quantifier free types of finite L-structures A, B and an $n \in \omega \cup \{\infty\}$ to ω so that for all but finitely many $n \in \omega$, $\mu(A, B, n) = \mu(A, B, \infty)$. Furthermore, we suppose $\mu(A, B, n) \geq \delta(A)$ for all triples A, B, n.*

Given a pair A, B of finite L-structures, set $h(A, B)$ to be the least $n \geq |A|$ so that $\mu(A, B, m)$ is constant for all $m \geq n$. For $k \in \omega$, we set $g(k) = max\{h(A, B)||A|, |B| \leq k\}$.

From any such μ function, we define the following amalgamation class:

Definition 104. *Let $\mathcal{C}$ be the class of finite L-structures C such that the following hold:*

1. $\delta(A) \geq 0$ *for all* $A \subseteq C$
2. *Suppose X_i, $i = 1, \ldots, n$, Y are disjoint subsets of C so that the X_i are minimally simply algebraic over Y, and the X_i are isomorphic over Y. Then $n \leq \mu(Y, X_1, f_C(Y))$*

Note that unlike the original construction, μ depends on $f_C(Y)$, which means that it is possible that $A \subset C \in \mathcal{C}$, but $A \notin \mathcal{C}$. Despite this, we will show that $\mathcal{C}$ leads us to a strongly minimal amalgam.

Definition 105. *Let $A \subseteq B$ be L-structures. We say A is n-strong in B if $\delta(A \cup X) \geq \delta(A)$ for all $X \subseteq B$ with $|X| \leq n$.*

Lemma 106. *If $B \leq C \in \mathcal{C}$, then $B \in \mathcal{C}$. In fact, if B is $g(|B|)$-strong in C, then $B \in \mathcal{C}$.*

Proof. The first condition holds as any subset A of B is a subset of C. Suppose X_i, $i = 1, \ldots, n$, Y are disjoint subsets of B so that the X_i are minimally simply algebraic over Y, and the X_i are isomorphic over Y. Then $n \leq \mu(Y, X_1, f_C(Y))$. Since $f_B(Y) \geq f_C(Y)$, if $f_C(Y) > h(Y, X_1)$, then $\mu(Y, X_1, f_C(Y)) = \mu(Y, X_1, f_B(Y))$. So, we may assume there exists a Z of minimal size so that $Y \subseteq Z$ and $\delta(Z) < |Y|$ with $|Z| < h(Y, X_1)$. Since $B \leq B \cup Z$ by assumption, $\delta(B \cap Z) \leq \delta(Z)$ showing that $Z \subseteq B$ and $f_C(Y) = f_B(Y)$. □

Lemma 107. *(Algebraic Amalgamation Lemma) Suppose $A = B_1 \cap B_2$, $A, B_1, B_2 \in \mathcal{C}$, and $B_1 \setminus A$ is simply algebraic over A. Let E be the free-join of B_1 with B_2 over A. Then $E \in \mathcal{C}$ unless one of the following holds:*

- *$B_1 \setminus A$ is minimally simply algebraic over $F \subseteq A$, and there are $\mu(F, B_1 \setminus A, f_{B_2}(F))$ disjoint copies of $B_1 \setminus A$ in B_2.*
- *There is a set $X \subseteq B_2$ such that $X \cap A \not\leq X$, and X satisfies a type realized by a subset of B_1.*
- *There is a set $F \subseteq B_1$ and $C \subseteq B_1$ minimally simply algebraic over F so that $\mu(F, C, B_1) > \mu(F, C, E)$.*

Proof. If $X \subseteq E$, then $\delta(X) = \delta(X \cap B_1) + \delta(X \cap B_2) - \delta(X \cap A) \geq \delta(X \cap B_2) \geq 0$. If there are disjoint $C^i, F \subseteq E$ so that each of the C^i are minimally simply algebraic over F and each (C^i, F) is isomorphic, then by Lemma 42, we need consider only four cases:

- One of the C^i is $B_1 \setminus A$. As the C^j and F are disjoint, each of the other C^j and F are contained in B_2. If $r > \mu(Y, X, f_E(F))$ then there must be $\mu(Y, X, f_E(F))$ of them contained in B_2. Since $B_2 \leq E$, $f_E(F) = f_{B_2}(F)$, showing that the first exception in this lemma holds.
- $F \cup \bigcup_{i=1}^{r} C^i$ is entirely contained in either B_1 or B_2. Here, $r \leq \mu(C^1, F, f_{B_j}(F))$ as $B_1, B_2 \in \mathcal{C}$. Since $B_2 \leq E$, if $F, C^i \subseteq B_2$, then $r \leq \mu(C^1, F, f_E(F))$ as $f_E(F) = f_{B_2}(F)$. So, we need only consider the case where $F, C^i \subseteq B_1$ and $\mu(F, C^1, f_E(F)) < \mu(F, C^1, f_{B_1}(F))$. In this case, the third exception of this lemma holds.
- $r \leq \delta(F)$. In this case $r \leq \delta(F) \leq \mu(C^1, F, f_E(F))$.
- For one C^j, setting $X = (F \cap A) \cup (C^j \cap B_2)$, we see that $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. This yields the second exception in this lemma.

□

Lemma 108. *(Strong Amalgamation Lemma) Suppose $A, B_1, B_2 \in \mathcal{C}$, $A \leq B_i$ Then there exists $D \in \mathcal{C}$ so that $B_2 \leq D$, and an $g : B_1 \to D$ an embedding so that $g(B_1) \leq D$ and $g(A) = id|_A$.*

Proof. We may assume there is no B' such that $A \leq B' \leq B_1$. Thus, $B_1 = A \cup \{x\}$ where x is unrelated to A by R, or $B_1 \setminus A$ is simply algebraic over A. In the first case, the free-join suffices. In the second case, the free-join fails only if one of the conditions of the last lemma holds. The second and third conditions cannot hold, as $A \leq B_2$. Let

$F \subseteq A$ be so that $B_1 \setminus A$ is minimally simply algebraic over F. As $A \leq B_1$ and $A \leq B_2$, $f_{B_1}(F) = f_A(F) = f_{B_2}(F)$. If condition 1 holds, then we have $\mu(F, B_1 \setminus A, f_{B_2}(F))$ many copies of $B_1 \setminus A$ in B_2. There must be no more than $\mu(F, B_1 \setminus A, f_{B_2}(F)) - 1$ many contained in A, as $B_1 \in \mathcal{C}$ and $f_{B_2}(F) = f_A(F) = f_{B_1}(F)$. As no copy of $B_1 \setminus A$ in B_2 can be partially in A (as $A \leq B_2$), we have one contained in $B_2 \setminus A$ with which to identify $B_1 \setminus A$. This gives us the required amalgamation. □

The above lemma guarantees that there is a generic amalgamation of the class $\mathcal{C}$, which we call $\mathcal{M}$. $\mathcal{M}$ is characterized by three properties:

1. $\mathcal{M}$ is countable.

2. For any finite $A \leq \mathcal{M}$, $A \in \mathcal{C}$.

3. Suppose $A \leq \mathcal{M}$, $A \leq B$, and $B \in \mathcal{C}$. Then there is an embedding $g : B \to \mathcal{M}$ so that $g|_A = id_A$ and $g(B) \leq M$.

By a standard back-and-forth on strong substructures, and that each A is a subset of a finite B such that $B \leq M$, we see that these 3 properties fully characterize $\mathcal{M}$ up to isomorphism. Showing that $\mathcal{M}$ is strongly minimal will be analogous to the proof in section 2.3 with the exception of the difference of the new μ appearing in $3''$ and the change to $2'$.

As in the proof that the construction of section 2.2 yields a saturated model, we would like to show that $\mathcal{M}$ is saturated by showing that any elementary extension of $\mathcal{M}$ satisfies properties $(1, 2, 3)$, but properties 2 and 3 are not first order. So, we replace them by $2'$, $3'$,and $3''$:
$2'$: For any finite $A \subseteq \mathcal{M}$, if A is $h(|A|)$-strong in $\mathcal{M}$, then $A \in \mathcal{C}$.
$3'$: There is an infinite set I with R not holding on any tuple in I such that for all finite $A \subset I$, $A \leq \mathcal{M}$.
$3''$: Suppose $A \subset \mathcal{M}$, $A \leq B$, and $B \setminus A$ is minimally simply algebraic over $F \subseteq A$. Further, suppose that A is $g(|B|)$-strong in $\mathcal{M}$. Then there are $\mu(F, B_1 \setminus A, f_{\mathcal{M}}(F))$ many distinct realizations of $tp_{r.q.f.}(B/F)$ over F in $\mathcal{M}$.
Note that $2'$, $3'$, $3''$ are first order conditions. Note that if A is $g(|B|)$-strong in $\mathcal{M}$, then $\mu(F, B_1 \setminus A, f_{\mathcal{M}}(F)) = \mu(F, B_1 \setminus A, f_A(F))$ as in the proof of Lemma 106.

Claim 109. *The conditions* $(1, 2, 3)$ *are equivalent to the conditions* $(1, 2', 3', 3'')$.

Proof. Assume $(1, 2, 3)$. To see $2'$ from 2, let B be least so that $A \subseteq B \leq \mathcal{M}$. Apply lemma 106 to the pair (A, B). $3'$ follows trivially from 3. $3''$ is a consequence of the algebraic amalgamation lemma employed for any A, B, and set C so that $A \subset C \leq \mathcal{M}$. If the free-join of C with B over A is in $\mathcal{C}$, then 3 implies that we can amalgamate E into $\mathcal{M}$ over C. Otherwise, one of the conditions in the algebraic amalgamation lemma holds. Since A is $g(|B|)$-strong in $\mathcal{M}$, the second and third conditions cannot hold, and if the first condition holds, then there are already $\mu(B \setminus A, F, f_{\mathcal{M}}(F))$ many copies of $B \setminus A$ over F in C.

Assume $(1, 2', 3', 3'')$. 2 is formally weaker than $2'$, so it follows immediately. We show 3: Suppose $A \leq \mathcal{M}$, $A \leq B$. We may assume that there is no B' such that $A \leq B' \leq B$. Thus, B is either simply algebraic over A, or $B = A \cup \{x\}$ a singleton unrelated to A. In the latter case, $3'$ gives us an infinite independent sequence from which to choose an embedding of B over A. In the former case, $3''$ guarantees that there is an embedding of B over A exactly as in the strong amalgamation lemma. □

Corollary 110. *$\mathcal{M}$ is saturated.*

Proof. Let N be any countable model elementarily containing $\mathcal{M}$. Then since N satisfies $(1, 2, 3', 3'')$ and hence $(1, 2, 3)$, N is isomorphic to $\mathcal{M}$. Thus there are only countably many types realized in elementary extensions of $\mathcal{M}$, so there is a countable saturated model elementarily containing $\mathcal{M}$, which $\mathcal{M}$ must be isomorphic to. □

We define $d(A) = \min\{\delta(C) | A \subseteq C \subseteq \mathcal{M},\ C \text{ finite}\}$. Clearly for any A and x, either $d(xA) = d(A)$ or $d(xA) = d(A) + 1$.

Lemma 111. *If $d(xA) = d(A) + 1$ and $d(yA) = d(A) + 1$, then $(\mathcal{M}, Ax) \cong (\mathcal{M}, Ay)$.*

Proof. Let B be such that $A \subseteq B$, $\delta(B) = d(A)$. Then $B \leq \mathcal{M}$. $d(xB) = d(xA) = d(A) + 1$. Thus, $xB \leq \mathcal{M}$, and similarly $yB \leq \mathcal{M}$. Using property 3 and a standard back-and forth, we see that $(\mathcal{M}, xB)$ and $(\mathcal{M}, yB)$ are isomorphic. □

Lemma 112. *If $d(xA) = d(A)$ then $x \in \operatorname{acl}_{\mathcal{M}}(A)$.*

Proof. Suppose $d(xA) = d(A)$. First, let B be a minimal set such that $A \subseteq B$ and $\delta(B) = d(A)$. We show that B is algebraic over A in $\mathcal{M}$. Suppose there were two realizations of the positive quantifier-free type of B over A. Call the second realization B'. Then $\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') = d(A)$. The strict inequality is due to B being a minimal set with the properties that $A \subseteq B$ and $\delta(B) = d(A)$. This inequality contradicts the definition of $d(A)$.

Fix E to be a set such that $xA \subseteq E$ and $\delta(E) = d(A)$. Then $\delta(E \cup B) \leq \delta(E) + \delta(B) - \delta(E \cap B)$. If E does not contain B, then $\delta(E \cap B) > d(A)$ by minimality of B. Then $\delta(E \cup B) \leq d(A) + d(A) - \delta(E \cap B) < d(A)$, again a contradiction. Thus, E contains B and $d(xB) = d(B)$.

Take a sequence of extensions $B_0, B_1, B_2, \ldots B_n$ such that $B_0 = B$, $B_n = E$, and B_{i+1} is a minimal set such that $B_i \subseteq B_{i+1} \subseteq E$ and $\delta(B_{i+1}) = d(A)$. Then B_{i+1} is simply algebraic over B_i, say minimally simply algebraic over F_i. Thus B_{i+1} is algebraic over B_i (any two extensions of B_i satisfying the positive atomic type of $B_{i+1} \setminus B_i$ over B_i must be disjoint and isomorphic to $B_{i+1} \setminus B_i$ over B_i since $B_i \leq \mathcal{M}$, so we explicitly forced there to be no more than $\mu(F_i, B_{i+1} \setminus B_i, f_{\mathcal{M}}(B_i)$ of these). We conclude that E is algebraic over A. In particular, $x \in acl_{\mathcal{M}}(A)$. □

Corollary 113. *$\mathcal{M}$ is strongly minimal.*

Proof. In the previous lemma, we showed that over any set there is a unique non-algebraic type realized in $\mathcal{M}$. Since $\mathcal{M}$ is saturated, we see that $Th(\mathcal{M})$ is strongly minimal. □

5.2 $SRM(T) = \{0, \ldots, m\}$

Fix an integer m. We define K to be the standard complete Σ_1 set, ie: the halting problem. We set K_s to be the part of K enumerated by stage s (ie: the first s programs to halt by stage s). We set $K_\infty = K$. We will construct a theory T via the construction from the previous section so that $SRM(T) = \{0, \ldots, m\}$. We need only define the μ function and the previous section will give a corresponding strongly minimal theory. Fix a recursive enumeration of all the relative quantifier-free types in our language L of minimally simply algebraic extensions over a set of size $m + 1$. We will refer to these as Λ_i, and will say $\Lambda_i(A, B)$ to mean that B is a minimally simply algebraic extension of A of relative type enumerated as Λ_i.

Definition 114.

$$\mu(A, B, k) = \begin{cases} |A| + 1 & \text{if for all } i,\ \neg\Lambda_i(A, B) \text{(ie: } |A| \neq m + 1\text{)} \\ |A| + 1 & \text{if } \Lambda_i(A, B),\ \text{and } i \in K_k \\ |A| + 2 & \text{if } \Lambda_i(A, B),\ \text{and } i \notin K_k \end{cases}$$

We employ the previous section, and we get a generic model $\mathcal{M}$, which is saturated and strongly minimal. Let $T = Th(\mathcal{M})$. We need only to verify that $SRM(T)$ is as promised.

Claim 115. $k > m \to k \notin SRM(T)$

Proof. Let N be any model of dimension $> m$. Let $\bar{x}$ be any algebraically independent tuple of size $m + 1$ in N. Then $i \in K \leftrightarrow \neg\exists^{m+3}\bar{y}\ N \models \Lambda_i(\bar{x}, \bar{y})$. Thus, a complete Σ_1 set can be represented as a Π_1 set using an oracle for quantifier-free statements true about N. Therefore, N cannot be recursive. □

Claim 116. $k \leq m \to k \in SRM(T)$.

Proof. If X is a finite L-structure and $\delta(X) \leq m$, then whether $X \in \mathcal{C}$ is a recursive question. This is simply because $f_X(Y)$ is finite for any $m+1$ element set $Y \subseteq X$, so we can compute $\mu(A, B, f_X(A))$ for any $A, B \subseteq X$. To construct the k-dimensional model, we start with M_0 as k elements unrelated by R. At stage s, we start with M_{s-1} and we list off the first s possible simply algebraic extensions over subsets of M_{s-1}. Then we check if the associated free-join keeps us in $\mathcal{C}$. If it does, we pass to the free-join. After doing this for these s possible extensions, we call the result M_s. This yields a model where we have amalgamated every simply algebraic extension possible, in particular we amalgamate B over A for any strong enough A. Thus we get a model of $1, 2', 3''$. By compactness, there is an elementary superstructure satisfying $2', 3', 3''$, and by downward Lowenheim-Skolem, there is an elementary superstructure satisfying $1, 2', 3', 3''$. Thus we get a model of T. Since $\delta(M_i) = k$ for each M_i and $M_0 \leq M_i$ for each M_i, we have built the k-dimensional model. □

Thus we have proved the promised theorem.

Theorem 117. *There exists a strongly minimal theory in a language with a single ternary relation symbol such that* $SRM(T) = \{0, \ldots, m\}$.

5.3 $SRM(T) = \omega$

We will be employing the same construction as above, so we need only define a new μ function. In order to work with the more complicated recursion theoretic necessities of this proof, we will be using a complete Π_2 set. We fix one now: $S = \{k | \forall l \exists j \phi(k, l, j)\}$. Fix a recursive enumeration of all relative quantifier-free types of minimally simply algebraic extensions $\Lambda_{k,s}$, so that the extension $\Lambda_{k,s}$ is over a set of size k. Now we can define the bounding function μ:

Definition 118.

$$\mu(A, B, n) = \begin{cases} |A| + 1 & \text{if } \Lambda_{k,s}(A, B), \text{ and } \forall l \leq s \exists j \leq n \phi(k, l, j) \\ |A| + 2 & \text{if } \Lambda_{k,s}(A, B), \text{ and } \neg\forall l \leq s \exists j \leq n \phi(k, l, j) \end{cases}$$

Note that μ satisfies the required property that all but finitely many integers agree with the value outputted at ∞.

We employ the construction above, and we thus get a generic model $\mathcal{M}$ which is saturated and strongly minimal. Let $T = Th(\mathcal{M})$. Now, we verify that $SRM(T)$ is as promised.

Claim 119. $\omega \notin SRM(T)$.

Proof. Let N be any particular presentation of the saturated model. For any k,

$$k \in S \leftrightarrow \exists \bar{x} \left((\forall s \neg \exists^{k+2} \bar{y} N \models \Lambda_{k,s}(\bar{x}, \bar{y})) \wedge (\bar{x} \text{ is strong in } N) \right)$$

Then, we see that a complete Π_2 set is Σ_2 (being strong in N is a Π_1-condition) in a presentation of the quantifier-free diagram of N. Thus N has no recursive presentation. □

Claim 120. $n \in \omega \rightarrow n \in SRM(T)$.

Proof. Fix $n \in \omega$.

Claim 121. *The set of finite L-structures X such that $\delta(X) \leq n$ and $X \in \mathcal{C}$ is a recursive set.*

Proof. Non-uniformly, fix a finite set of information detailing for each $i \leq n$, whether $i \in S$, and if not, which is the first s so that $\neg\exists j \phi(i, s, j)$.

Given any $A \subseteq X$, either $|A| \leq n$ or $f_X(A)$ is finite. In the latter case, computing μ is recursive, since all the quantifiers are bound. In the former case, the information we specified tells us how to compute μ when $f_X(A) = \infty$. □

To construct the n-dimensional model, we start with M_0 as n elements unrelated by R. At stage s, we start with M_{s-1} and we list off the first s possible simply algebraic extensions over subsets of M_{s-1}. Then we check if the associated free-join keeps us in $\mathcal{C}$. If it does, we pass to the free-join. After doing this for these s possible extensions, we call the result M_s. This yields a model where we have amalgamated every simply algebraic extension possible, in particular we amalgamate B over A for any strong enough A. Thus we get a model of $1, 2', 3''$. By compactness and Lowenheim-Skolem, there is an elementary superstructure satisfying $1, 2', 3', 3''$, so we get a model of T. Since $\delta(M_i) = n$ for each M_i and $M_0 \leq M_i$ for each M_i, we have built the n-dimensional model. □

Thus we have proved the promised theorem.

Theorem 122. *There exists a strongly minimal theory in a language with a single ternary relation symbol such that $SRM(T) = \omega$.*

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