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UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture VII

Roy Haddock

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University of California
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FOURIER ANALYSIS

1. Introduction

Fourier in 1822 was the first to assert that an arbitrary function defined in the interval $(-\pi, \pi)$ could be expressed as a trigonometrical series of the form

$$a_o + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots .$$

where

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad ; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \quad ; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx .$$

He rigorously proved that the expansion was possible for certain simple functions he needed in the problems of heat conduction. About the same time D'Alembert, Euler, D. Bernoulli found that the solution to the vibrating string problem could be expressed as the sum or the integral of a sum of terms similar to Fourier's series. From this beginning the theory of Fourier analysis has grown. Its major application is still in the solution of boundary value problems.

Dirichlet in 1829-1837 proved that for the class of functions which are piece-wise continuous and bounded that .

$$\frac{1}{2}\left[f(x_o + 0) + f(x_o - 0)\right] = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx_o + b_n \sin nx_o$$

for all points in the open interval $-\pi$ to π and $\frac{1}{2}\left[f(-\pi + 0) + f(\pi - 0)\right]$ at $x = \pm\pi$. $f(x_o + 0) = \lim_{\lambda \to 0} f(x_o + \lambda)$ is the right hand limit at a point in the interval. $f(x_o - 0) = \lim_{\lambda \to 0} f(x_o - \lambda)$ is the left hand limit at a point in the interval. Notice that if x_o is a point at which $f(x)$ is continuous then $\frac{1}{2}\left[f(x_o + 0) + f(x_o - 0)\right] = f(x)$ and if x_o is a point at which $f(x)$ is piece-wise continuous then $\frac{1}{2}\left[f(x_o + 0) + f(x_o - 0)\right]$ is the arithmetic mean of the left and right hand limits at the point.

2. The Fourier Expansion Theorem.

The Fourier expansion theorem was proved using the following lemmas.

2.1 Riemann-Lebesgue Theorem. Let $f(x)$ be bounded and integrable in (a, b), or, if $f(x)$ is unbounded, let $\int_a^b f(x)dx$ be absolutely convergent. Then

$$\lim_{n \to \infty} \int_a^b f(x) \begin{matrix} \sin \\ \cos \end{matrix} nx \, dx = 0 .$$

2.2 If $f(x)$ is bounded and piece-wise continuous in $(0, a)$ where $0 < a$ and if $f(+0)$ exists, then

$$\lim_{k \to \infty} \int_0^a f(x) \frac{\sin kx}{x} dx = \frac{\pi}{2} f(+0) .$$

2.3 If $f(x)$ is bounded and piece-wise continuous in (a, b) and has a right hand and left hand derivative at $x = x_o$ where $a < x_o < b$, then

$$\lim_{k\to\infty} \int_a^b f(x) \frac{\sin k(x - x_o)}{x - x_o}\, dx = \frac{\pi}{2} \left[f(x_o + 0) + f(x_o - 0) \right].$$

The right hand derivative $= f'(x_o + 0) = \lim_{\lambda\to 0} \frac{f(x_o + \lambda) - f(x_o + 0)}{\lambda}$

and the left hand derivative $= f'(x_o - 0) = \lim_{\lambda\to 0} \frac{f(x_o - 0) - f(x_o - \lambda)}{\lambda}$.

Now consider

$$S_n(x) = \frac{a_o}{2} + \sum_{v=1}^{n} (a_v \cos vx + b_v \sin vx)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi)\, d\xi + \frac{1}{\pi} \sum_{v=1}^{n} \left\{ \cos vx \int_{-\pi}^{\pi} f(\xi) \cos v\xi \, d\xi + \sin vx \int_{-\pi}^{\pi} f(\xi) \sin v\xi \, d\xi \right\}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \left\{ \tfrac{1}{2} + \sum_{v=1}^{n} \cos v(\xi - x) \right\} d\xi .$$

By Lagrange's trigonometric identity,

$$\sum_{v=1}^{n} \cos v(\xi - x) = -\tfrac{1}{2} + \tfrac{1}{2} \frac{\sin(n + \frac{1}{2})(\xi - x)}{\sin\left(\frac{\xi - x}{2}\right)}$$

thus

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \left\{ \tfrac{1}{2} \frac{\sin (n + \frac{1}{2})(\xi - x)}{\sin\left(\frac{\xi - x}{2}\right)} \right\} d\xi$$

and

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\, \frac{\frac{\xi - x}{2}}{\sin\left(\frac{\xi - x}{2}\right)}\; \frac{\sin\left[(n+\frac{1}{2})(\xi - x)\right]}{\xi - x}\, d\xi \,.$$

Now if $f(\xi)$ is bounded and piece-wise continuous in $(-\pi, \pi)$, then so is

$$f(\xi)\left\{\frac{\frac{\xi - x}{2}}{\sin\left(\frac{\xi - x}{2}\right)}\right\} = F(\xi) \,.$$

Thus by lemma 2.3,

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} F(\xi)\, \frac{\sin\left[(n+\frac{1}{2})(\xi - x)\right]}{\xi - x}\, d\xi$$

$$= \tfrac{1}{2}\left[F(x+0) + F(x-0)\right]$$

$$= \tfrac{1}{2}\left[f(x+0) + f(x-0)\right] .$$

Thus if $f(x)$ is bounded and piece-wise continuous in $(-\pi, \pi)$, then

$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges to the value

$\frac{1}{2}\left[f(x+0) + f(x-0)\right]$ at every point where $f(x)$ has a right and left hand derivative.

3. Special Case of Fourier Series.

3.1 Odd functions $(f(x) = -f(-x))$.

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin n x \, dx$$

thus

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

, or $f(x)$ is represented by a sine series.

3.2 Even functions $(f(x) = f(-x))$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad \text{and} \quad b_n = 0$$

thus

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$, or $f(x)$ is represented by a

cosine series.

3.3 Partially-defined functions.

If $f(x)$ is defined only on part of the interval $(-\pi, \pi)$, say $(0, \pi)$, then $f(x)$ may be defined as either an odd or even function and considered as either a sine or cosine series.

3.4 Functions defined on arbitrary intervals.

If $f(x)$ is defined in the interval $(-L, L)$, then by a change of variables, $y = \frac{\pi}{L}x$,

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\right)$$

where

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)dx \quad , \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos \frac{n\pi}{L}x\,dx \quad ,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin \frac{n\pi}{L}x\,dx \quad .$$

4. Properties of Fourier Coefficients.

4.1 Bounds on Fourier coefficients.

4.11 Suppose $f(x)$ is piece-wise continuous in $\langle -\pi, \pi \rangle$ and has a finite number of maxima and minima. Divide the interval into subintervals in which $f(x)$ is monotonic. Then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$ can be expressed as the sum of a finite number of integrals of the form $\int_a^b f(x)\cos nx\, dx$. Consider $\int_a^b f(x)\cos nt\, dt$. By the second mean value theorem this is equal to

$$f(a+)\int_a^{\xi} \cos nx\, dx + f(b-)\int_{\xi}^{b} \cos nx\, dx \quad \text{where } a \leq \xi \leq b .$$

Integrating,

$$\int_a^b f(x)\cos nx\, dx = f\frac{(a+)}{n}\left[\sin n\xi - \sin na\right] + \frac{f(b-)}{n}\left[\sin nb - \sin n\xi\right].$$

Since $f(x)$ is piece-wise continuous and bounded and $(\sin a - \sin b) \leq 2$ and independent of n, then $\int_a^b f(x)\cos nx\, dx < \frac{M_i}{n}$. Thus,

$$|a_n| = \sum_{i=1}^{k} \frac{M_i}{n}, \quad M_i \text{ some real finite number, (k finite). Similarly}$$

$$|b_n| = \sum_{j=1}^{m} \frac{M_j}{n}, \quad M_j \text{ some real number, (m finite). Let}$$

$M = \max(k\,M_i, k\,M_j)$ for all i and j. Then $|a_n|$ and $|b_n| < \frac{M}{n}$ and a_n and b_n are of order $\frac{1}{n}$, i.e., a_n and $b_n \sim O(\frac{1}{n})$.

4.12 Suppose $f(x)$ is continuous and $f'(x)$ piece-wise continuous in $\langle -\pi, \pi \rangle$, $f'(x)$ having a finite number of maxima or minima.

Then
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -\frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$$

when integrated by parts. But $f'(x)$ can be expressed as a Fourier series with coefficients of order $\frac{1}{n}$ from case 4.11 so $a_n \sim O(\frac{1}{n^2})$ and correspondingly $b_n \sim O(\frac{1}{n^2})$.

4.2 General Theorem.

Theorem: If $f(x)$ and its first $(k-1)$ derivatives are continuous in $\langle -\pi, \pi \rangle$ and the k'th derivative is piece-wise continuous with a finite number of maxima and minima, then the Fourier coefficients are of order $\frac{1}{n^{k+1}}$.

Convergence criteria:

$$\left| f(x) \right| = \left| \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right|$$
$$\leq \left| \frac{a_o}{2} \right| + \sum_{n=1}^{\infty} \left| a_n \cos nx + b_n \sin nx \right|$$
$$< \left| a_o \right| + \sum_{n=1}^{\infty} \frac{M}{n^{k+1}} .$$

The series of constants converges for $k > 0$. Therefore, the Fourier series converges uniformly for $k=1, 2, 3 \ldots$. We can say then:

(1) Due to uniform convergence for $k = 1, 2, 3 \ldots$ (k denotes which of the derivatives of $f(x)$ is piece-wise continuous with a finite number of maxima and minima) one can integrate Fourier series term-wise and the result will converge to $\int f(x)dx$.

If $k=0$, i.e., the function itself is piece-wise continuous, we can also integrate term by term and the result will converge to $\int f(x)dx$.

(2) For $k = 1, 2, 3 \ldots$ we can differentiate term-wise and the series will converge to $f'(x)$. If $k = 0$ differentiation is impossible.

5. Approximation of Functions by Linear Interpolation.

A function tabulated in arbitrary intervals can be approximated by using linear interpolation and Fourier analysis.

The tabulated points are connected by straight lines and then the Fourier series is obtained for this function.

For example, suppose the tabulated points are (x_i, y_i), $i = 1, \ldots, n$, and assume the true function to have period 2ℓ. It may also be assumed that $x_n - x_1 = 2\ell$ and $x_1 = -\ell$ and $x_n = +\ell$. Then

$$f(x) = \begin{cases} f_1(x) & x_1 \leq x \leq x_2 \\ f_2(x) & x_2 \leq x \leq x_3 \\ \vdots & \\ f_{n-1}(x) & x_{n-1} \leq x \leq x_n \end{cases}$$

where

$$f_k(x) = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x - x_k) + y_k .$$

Then

$$f(x) = \sum_{m=0}^{\infty} (a_m \cos mx + b_m \sin mx)$$

where

$$a_o = \frac{1}{\ell}\left[\int_{x_1}^{x_2} f_1(x)dx + \int_{x_2}^{x_3} f_2(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f_{n-1}(x)\, dx\right.$$

$$a_m = \frac{1}{\ell}\left[\int_{x_1}^{x_2} f_1(x) \cos \frac{m\pi}{\ell} x\, dx + \ldots + \int_{x_{n-1}}^{x_n} f_{n-1}(x) \cos \frac{m\pi}{\ell} x\, dx\right]$$

$$f_m = \frac{1}{\ell}\left[\int_{x_1}^{x_2} f_1(x)\ \sin\frac{m\pi}{\ell}x\,dx + \ldots + \int_{x_{n-1}}^{x_n} f_{n-1}(x)\ \sin\frac{m\pi}{\ell}x\,dx\right]$$

6. Approximation of Functions Using a Finite Number of Trigonometric Terms.

A periodic function, tabulated in arbitrary intervals, may be approximated by a Fourier series involving a finite number of terms.

The coefficients are determined so as to minimize the sum of the squares of the differences obtained at the tabulated points. (This is an application of the method of least squares which leads to the Fourier coefficients in the infinite series expansion.)

If the function is tabulated in equidistant intervals, the calculations may be simplified.

Let f(x) be the tabulated function and its period may be assumed to be 2π. Let the interval $(-\pi, \pi)$ be divided into 2k equal parts and let the points of subdivision be denoted by x_j where $x_j = \frac{j\pi}{k}$, $j = -k, \ldots, -1, 0, 1, \ldots, k$. Then minimizing

$$\sum_{j=-k}^{k-1}\left[f(x_j) - \sum_{m=0}^{n}(a_m \cos mx_j + b_m \sin mx_j)\right]^2$$

by partially differentiating with respect to the a's and b's and equating to zero, the normal equations involving the coefficients are obtained.

$$\sum_{m=0}^{n}\left[a_m \sum_{j=-k}^{k-1} \cos mx_j \cos qx_j + b_m \sum_{j=-k}^{k-1} \sin mx_j \cos qx_j\right] = \sum_{j=-k}^{k-1} f(x_j)\cos qx_j$$

and

$$\sum_{m=0}^{n}\left[a_m \sum_{j=-k}^{k-1} \cos mx_j \sin qx_j + b_m \sum_{j=-k}^{k-1} \sin mx_j \sin qx_j\right] = \sum_{j=-k}^{k-1} f(x_j) \sin qx_j$$

where $q = 0, 1, \ldots, n$.

Now if $0 \leqq m \leqq k$ and $0 \leqq q \leqq k$, then

$$\sum_{j=-k}^{k-1} \sin mx_j \cos qx_j = 0$$

$$\sum_{j=-k}^{k-1} \cos mx_j \cos qx_j = \begin{cases} 0 & \text{if } m \neq q \\ k & \text{if } m = q \neq 0 \text{ and } \neq k \\ 2k & \text{if } m = q = 0 \text{ or } = k \end{cases} .$$

$$\sum_{j=-k}^{k-1} \sin mx_j \sin qx_j = \begin{cases} 0 & \text{if } m \neq q \\ k & \text{if } m = q \neq 0 \text{ and } \neq k \\ 0 & \text{if } m = q = 0 \text{ or } = k \end{cases} .$$

Using these relations, the coefficients may be obtained.

$$a_o = \frac{1}{2\,k} \sum_{j=-k}^{k-1} f(x_j) ,$$

$$a_q = \frac{1}{k} \sum_{j=-k}^{k-1} f(x_j) \cos qx_j , \qquad 0 < q < k$$

$$b_q = \frac{1}{k} \sum_{j=-k}^{k-1} f(x_j) \sin qx_j .$$

For modifications of procedure see Milne.

7. Bibliography

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7.2 Byerly, W. E., "Fourier Series and Spherical Harmonics".

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7.4 Scarborough, "Numerical Analysis".

7.5 Milne, "Numerical Calculus".