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DAMPING CHARACTERISTICS OF PRESTRESSED CONCRETE

BY

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DAMPING CHARACTERISTICS OF PRESTRESSED CONCRETE

by

Joseph Penzien

University of California

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TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

FOREWORD

The research described in this report, "Characteristics of Prestressed Concrete", was conducted under the supervision and technical responsibility of Joseph Penzien, Associate Professor of Civil Engineering, Division of Structural Engineering and Structural Mechanics, College of Engineering, University of California, Berkeley, California.

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ACKNOWLEDGMENT

NOTATION

DMF	Dynamic magnification factor
E	Modulus of elasticity of concrete
f'_c	Ultimate strength of concrete
f_t	Modulus of rupture of concrete
L	Length of beam
m	Mass per unit length of beam
M	Total mass of beam
M_1	Generalized mass of the first free-free vibration mode
p_o	Applied static load in free vibration test
$\bar{p}$	Load amplitude during steady state test
$\bar{x}_d$	Beam deflection amplitude during steady state test
$\bar{\dot{x}}$	Beam velocity amplitude during steady state test
$\bar{x}_s$	Static contribution to beam deflection
$x(0)$	Initial beam deflection - free vibration test
t	Time
z	Coordinate along beam axis
λ	Damping coefficient
ω	Circular frequency of harmonic input
ω_1	Circular frequency of first mode of vibration
ϕ	Phase angle between velocity and force,
	Mode shape function

I. INTRODUCTION

In recent years there has been an increase in the use of prestressed concrete structural elements which must be designed to withstand various types of dynamic loading conditions. These dynamic conditions may be transient in nature as produced by blast loading or earthquake loading or they may be steady state harmonic conditions as produced by heavy rotating equipment. In either case it is desirable in designing such elements that their damping characteristics be known.

Due to the lack of knowledge in this general area, the investigation as reported herein was undertaken to specifically study the damping characteristics of concrete under different types and intensities of prestress.

II. EXPERIMENTAL PROGRAM

A. Test Specimens

A total of 20 prestressed concrete beams and 4 standard reinforced concrete beams (no prestress) having the dimensions 6" x 6" x 90" were tested under dynamic conditions to determine their general damping characteristics. Table I summarizes the important features of each beam by giving the standard properties of the concrete, the location and size of reinforcing bars, and the prestress condition just prior to testing.

All concrete used in the program was manufactured using Santa Cruz Type I cement, Eliot sand, and Fair Oaks gravel. The gradation curves for the aggregates are shown in Figure 1.

Three different concrete mix designs were used in the investigation. The proportions by weight of Type I Santa Cruz Cement, Eliot sand, and Fair Oaks gravel used in each of these mixes were 1.00 : 2.88 : 3.60 (Mix No. 1), 1.00 : 2.31 : 3.29 (Mix No. 2), and 1.00 : 0.96 : 2.04 (Mix No. 3). Water cement ratios by weight were 0.56, 0.475, and 0.28, respectively, which produced ultimate strengths of approximately 4000, 5000, and 6000 pounds per square inch. All concrete when poured produced a slump in the range of 1 to 4 inches. The curing procedure was similar for all concrete specimens and consisted of 7 days in a fog room (100% humidity) followed by 21 days in an air dry condition. All specimens were tested at the age of 28 days.

Three standard control cylinders (6" x 12") and three control flexure beams (6" x 4-7/8" x 20") were poured along with each test beam. These control specimens were tested in accordance with ASTM specifications and gave average compressive strengths and moduli of rupture as presented in Table I. Strain measurements made on two of the three cylinders for each test beam gave the stress-strain curves shown in Figure 7 - 30. The modulus of elasticity for each cylinder tested is summarized in Table I.

Because of the desirability to restrict damping as much as possible to the concrete itself, high strength deformed bars were used as reinforcement rather than standard prestressing cables. Only two bar sizes, namely No. 6 and No. 9, were used in the investigation. While both size bars were of high strength steel, they were rolled from two different grades of steel having the stress-strain properties shown in Figure 2.

In order to maintain better control over the prestress, post-tensioning of the reinforcing bars was employed. This operation necessitated embedding flexible metal tubing (1-7/16" diameter) in the concrete through which the reinforcing bars could later be placed. Post-tensioning was accomplished in a standard manner using an hydraulic jack. Steel bearing plates 1-1/2 inches in thickness were used at the ends of each beam with a 1/4 inch plywood filler plate placed between the steel plate and the concrete to insure an even distribution of prestress. Howlett nuts and wedges were used to grip the bars. The photograph in Figure 3 shows all of the equipment used in the post-tensioning operation.

Post-tensioning of each concrete beam was started within a few days after removal from the fog room. The intensity of prestress applied at this early age was set as close to the design prestress as the strength of the concrete would permit without inflicting damage. Curves of ultimate strength versus age such as shown in Figure 5 for Mix No. 2 were used as a guide in establishing a safe limiting prestress at any age. The prestress was again increased or adjusted several times between the age of 7 and 21 days. At the age of 21 days, final adjustment of prestress was made.

Immediately following final adjustment of prestress, an expansive grouting mixture was pumped into the void space between the reinforcing bar and the flexible metal tubing from one end of the beam until it was forced out a small opening at the opposite end. The formula used to make the grouting mixture was the following: Type I Santa Cruz cement, 10.8 lbs.; Fly Ash, 1.2 lbs.; Interplast C, 55 gms.; 10% solution of plastiment, 6 m.l.; and water, 1944 m.l. An unrestrained

expansion test performed with this material produced the expansion versus age relationship as shown in Figure 6. Compressive strengths of approximately 2890 pounds per square inch at the age of 8 days were observed for this material.

The photograph in Figure 4 shows a reinforcing bar after removal from one of the test beams. In this photograph one will observe that the flexible tubing has been removed over a portion of the bar to expose the hardened grouting material and the grouting material has been removed over a short interval of length to expose the reinforcing bar. Bond between the reinforcing bar, the grout, and the flexible tubing was found to be very good; thus, one could reasonably assume no slippage took place in the immediate region of the reinforcing bars during dynamic testing. This condition was desirable to help restrict the internal damping to the concrete itself.

Several strain gauges were attached to each reinforcing bar prior to their being placed in the flexible tubing. Strains could thus be measured during the prestressing period to provide a check of the magnitude of prestress supplied by the calibrated hydraulic jack. Strains were also measured by these gauges just prior to testing each beam at the age of 28 days so that any loss of prestress, due to creep of the concrete after grouting (21 days), could be determined. However, due to the procedure of prestressing the concrete at as early an age as possible, very little loss of prestress was observed during the period between grouting and testing. These gauges were also used to detect any loss of prestress resulting from the dynamic tests. Because of the large initial strains in the steel bars, this loss was negligible up to the point of complete failure of the concrete.

B. Dynamic Testing Equipment

Each test beam was first subjected to a steady state harmonic excitation as a free-free beam. This free-free condition was obtained by hanging each beam from two relatively soft springs (k = 17.5 lb/in) which were attached to the beam at its nodal points. The harmonic excitation was obtained by attaching a Model C1-J, MB exciter to the lower side of the beam at its midpoint, as shown in Figure 31. The attachment of beam to exciter was made through a calibrated load cell which was strain gauged in such a manner that any bending in the load cell itself would not contribute to its output signal. Thus, the signal generated by this load cell was directly proportional to the concentrated load applied to the beam by the exciter driver coil. A velocity generator which is built into the above exciter provides a signal which is directly proportional to the vertical velocity of the driver coil. The above two signals, i.e. the signals from the load cell and the velocity generator, provide a measure of the steady state input and output, respectively.

The photograph in Figure 32 shows the associated electronic equipment used in the steady state testing program. The MB Model T112531 power supply unit for the exciter (Unit No. 1 in photograph) consists of separate power supplies for the driver and field coils, a power supply for the velocity generator, as well as an audio oscillator having a frequency range from 5 - 60,000 cps. The Ellis Model BA2 bridge and amplifier (Unit No. 2) shown in this photograph served an an intermediate element between the load cell and one channel of the Techtronic dual beam oscilloscope (Unit No. 3) shown in this same picture. The purpose of this unit is to balance the strain

gauge bridge on the load cell, amplify its signal before going to the
oscilloscope, and to serve as a calibration unit. Two other pieces of
equipment are shown in Figure 32, namely, a Hewlett Packard electronic
counter, Model 521A (Unit No. 4), which was used only to check the
audio oscillator frequency, and a power supply unit (Unit No. 5) for
the Techtronic oscilloscope. The output signal from the velocity
generator in the exciter is carried directly to one channel of the
Techtronic oscilloscope. Shielded cables were used for all connections
to eliminate the recording of any stray signals present.

After subjecting each beam to the steady state excitation as
previously described, they were supported at their ends as simple beams
and subjected to static concentrated loads of various magnitudes at
midspan which were then instantaneously released, thus producing a
free vibration condition.

This type of loading was accomplished by the system shown in the
photograph of Figure 33. As demonstrated in this photograph, the concrete
beam is bolted rigidly between two bearing plates at each end. The
bottom bearing plates are welded to flexible supporting plates which are
in turn welded to a very rigid steel supporting beam. The purpose of
these flexible supporting plates is to provide the concrete test beam
with supports which are very near to simple supports and which are
essentially free of damping. The restraining moments provided by these
plates as the ends of the test beam rotate are negligible. The rigid
steel supporting beam is clamped into the Baldwin hydraulic testing
machine shown in the picture with a force of approximately 15,000 pounds.

The static concentrated load at midspan is applied by a calibrated
hydraulic jack which sets on top of a railroad car coil spring which in

turn rests on the upper head of the Baldwin testing machine as shown
in Figure 33. The load from the jack is transmitted though a one inch
diameter high strength bolt which passed down through the cylinder of
the hydraulic jack and on down through the upper head of the Baldwin
testing machine with a one-half inch thick plate and a nut being
placed on its lower end just under the upper head of the testing
maching. To complete the connecting link between jack and concrete
test beam, a load **bar** approximately 9 inches long is threaded into the
lower end of the óne inch high strength bar. The lower end of this
load bar is threaded into a rigid bracket which is in turn bolted
rigidly to the test beam. As the hydraulic jack applies a load
through the load bar it must be reacted by the railroad car spring
($k = 8.7$ kips/in.). This spring compresses as the load is increased
to its desired magnitude and provides the necessary flexibility in the
system so that when the load bar is cut no appreciable loss of load
will be experienced due to local yielding of the load bar at the point
of fracture. Fracture of the load bar is initiated at the desired
time using bolt cutters. The one half inch thick plate mentioned
above served as a safety element which prevented, upon fracture of
the load bar, the sudden release of the strain energy stored in the
railroad car spring.

A Bourns linear motion potentiometer, powered with two 45 volt
dry cell batteries, was used to measure the center deflection of the
concrete test beam immediately following fracture of the load bar.
The output signal from this potentiometer was placed directly on one
channel of a dual beam Techtronic oscilloscope. Calibration of the

potentiometer was accomplished by measuring the static deflection at
the location of the potentiometer using a standard dial gauge.

C. Dynamic Testing Procedures

1. Steady state testing - As indicated in the previous section, each
concrete beam was first subjected to steady state harmonic oscil-
lation as a free-free beam. The testing procedure for this loading
condition consisted of setting the desired input frequency on the
audio oscillator located in the exciter power supply unit and then
adjusting the voltage gain on the exciter driver coil to produce
the desired force amplitude. This force amplitude was very easily
controlled by observing directly on an oscilloscope the signal
produced by the calibrated load cell. The response of the beam
to this known input was then measured by observing directly on
the oscilloscope the signal generated by the velocity generator
in the exciter. This signal gave a direct measure of the velocity
produced at the center of the beam.

Figure 34 shows two photographs taken directly from the
oscilloscope during steady state testing. Each photograph shows
two signals as received from the load cell and the signal generator
which gave force and velocity conditions, respectively. Photograph
A shows, for a particular test condition, a velocity which is
leading the force by approximately 80 degrees while photograph B
shows a test condition where this phase angle is nearly zero, thus
approaching a resonant condition. These example pictures illustrate
the method used to measure directly from the oscilloscope (1) the
input force amplitude, (2) the velocity amplitude, and (3) the

phase angle between force and velocity. The frequency is read directly off the audio oscillator for each test condition.

Two input parameters were varied during steady state testing, i.e. frequency and amplitude of the input force. The test procedure consisted of holding a constant force amplitude while varying the frequency of the input over a wide range of frequencies (50 - 500 cps). After measuring velocity amplitude and phase angle between velocity and force for many discrete frequencies in this range, the force amplitude was changed and the test repeated as described above.

2. Free vibration testing - The second method of testing each concrete beam, as previously noted, consisted of applying static concentrated loads of various magnitudes at the center of each beam and then suddenly removing these loads so that free vibration of each beam could be observed. In this phase of the program, each load was applied statically in increments up to some prescribed magnitude while noting the resulting static deflection at the center of the beam at each increment. Upon reaching the prescribed load, the load link was fractured, thus inducing free vibration from a known initial deflection.

This free vibration which damps out rapidly was recorded on one channel of a dual beam oscilloscope. The photograph in Figure 35 shows a typical deflection curve recorded during this period. To obtain such a picture it was necessary that the horizontal sweep on the oscilloscope be initiated at the instant of fracture of the load bar. This control was obtained by placing a metal foil along the load bar (but insulated from it), clamping

it to the bar at each end, and connecting it into the sweep circuit
of the oscilloscope in such a manner that when fracture of the load
bar occurred, the foil would also fracture thus opening a circuit
and causing the oscilloscope beam to sweep across the tube at a
preset rate.

To establish an accurate time scale for the dynamic response
as shown in Figure 35, a 60 cycle signal was placed on the second
channel of the dual beam oscilloscope and recorded simultaneously
with response during each period of free vibration.

III. EXPERIMENTAL RESULTS

A. Steady State Tests

The pertinent experimental results obtained in these series of
tests are shown in Figures 36 - 57. These figures show the rela-
tionship between the center deflection amplitude and frequency of
vibration over a fairly small range in the vicinity of the frequency
of the first mode of vibration. Each individual resonance curve in
these figures is for a constant force amplitude ($\bar{p}$) as indicated;
however, this force amplitude is the parameter which changes from
one curve to another in each figure. Dynamic magnification factors
(DMF) and damping factors (λ) as defined in a subsequent section
are given in each of these Figures.

As indicated previously these steady state tests were conducted
over the frequency range from 50 - 500 cps. This frequency range was
sufficiently high so that resonance of the third mode of vibration
(i.e. second symmetrical mode) was observed. However, due to the

small component of input forcing function present to excite this mode
it was impossible to get a reliable measure of the damping present in
this mode. Therefore, the experimental data obtained outside the
frequency ranges indicated in Figures 36 - 57 did not yield any
appreciable information regarding damping and as a result will not
be included in this report.

Phase angles between the force input and the velocity output were
measured throughout the frequency range 50 - 500 cps but again only
those data in the region of the frequencies shown in Figures 36 - 57
are considered of any importance. Since these phase angles were
observed to agree reasonably well with theory, they have not been
included in this report.

B. Free Vibration Tests

Figures 58 - 81 present the static load versus center deflection
relationship for each of the 24 beams tested. The continuous curves in
these Figures represent the deflections produced during the initial
loading while the dashed curves represent the deflections produced
during subsequent loadings. The arrows in Figure 58 illustrate the
manner in which loading was applied to Beam No. 1. All other load-
deflection curves should be interpreted in a similar manner.

Each beam was positioned on its supports with the cross section
oriented as shown in Table I and was loaded in an upward direction at
its center. Therefore, those beams having an unsymmetrical section
were loaded statically in their weak direction. This direction of
loading was considered desirable since any inelastic deformation
taking place during dynamic testing would be in that direction

regardless of the direction of load application. No attempt was made to measure a static load-deflection relation for negative loading (i.e. downward loading) for these unsymmetrical cases; however, from the geometry of the cross section and the positive loading results one can predict this relation reasonably well in range of loading applied during dynamic testing.

As previously described the free vibration tests were conducted by simply loading each beam statically in the positive or upward direction, suddenly releasing the load, and measuring the subsequent center deflection of the beam as it vibrates and finally comes to rest. Photographs taken directly from the oscilloscope which show this free vibration response are presented in Figures 82 - 89. The test number given on each of these Figures is also given on the static load-deflection curve for each corresponding beam (Figures 58 - 81) to show the static load-deflection condition of each beam when free vibration starts. This initial static load and its corresponding deflection are given for each test as presented in Figures 82 - 89. Effective viscous damping factors (λ) as defined in the subsequent section are also given for each of these tests.

IV. DISCUSSION OF EXPERIMENTAL RESULTS

A. Steady State Tests

The pertinent experimental data obtained during steady state testing consist of the resonance curves shown in Figures 36 - 57. As previously pointed out, each test beam was supported as a free-

free beam and subjected to a concentrated harmonic force of amplitude $\bar{p}$ at midspan. The resulting steady state vertical velocity amplitude $\bar{\dot{x}}$ at midspan was measured along with the phase angle ϕ as shown in Figure 34. By dividing this velocity amplitude by the known circular frequency ω of the input forcing function, the center deflection amplitude can be obtained.

Several standard methods can be used to measure the amount of damping as represented by a resonance curve. For this investigation the method considered to be most accurate was simply to base the amount of damping on the response amplitude at resonance. Thus, the dynamic magnification factors ($DMF = \bar{x}_d/\bar{x}_s$ = dynamic deflection/ static deflection) as shown in Figures 36 - 57 are based on resonance conditions. In calculating any particular dynamic magnification factor, the measured dynamic deflection $\bar{x}_d$ is taken directly from the resonance curve; however, the static deflection $\bar{x}_s$ is calculated using the equation

$$\bar{x}_s = \bar{p}/\omega_1^2 M_1 \tag{1}$$

where ω_1 is the resonant frequency of the first mode of vibration and M_1 is the generalized mass of the first mode of vibration. This generalized mass can be calculated using the well known equation

$$M_1 = \int_0^L m(z)\phi_1(z)^2 dz \tag{2}$$

where z is the coordinate along the beam axis, L is the length of the beam, $m(z)$ is the beam mass per unit length, and $\phi_1(z)$ is

the first mode shape normalized so that $\phi_1(\frac{L}{2}) = 1$. It has been assumed in calculating this generalized mass for each test beam that the first mode shape corresponds to that of a uniform free-free beam and is given by the equation

$$\phi_1(z) = -0.982 \left(\text{Sinh } 4.75 \frac{z}{L} + \text{Sin } 4.75 \frac{z}{L}\right)$$
$$+ \left(\text{Cosh } 4.75 \frac{z}{L} + \text{Cos } 4.75 \frac{z}{L}\right) \tag{3}$$

It has also been assumed in calculating the generalized mass that the mass per unit length $m(z)$ is constant and equal to M/L where M is the total known mass of the test beam. Substituting Equation (3) and M/L for $m(z)$ into Equation (2) will give

$$M_1 = 0.677M \tag{4}$$

It should be recognized that Equation (4) is approximate due to the presence of the steel end plates which change the mode shape $\phi_1(z)$ slightly from that given by Equation (3) and also influence the mass distribution term $m(z)$ appearing in Equation (2). The error introduced by this approximation is small, however, and therefore can be permitted.

In calculating the DMF's in the manner described above, it is assumed that only the first mode of vibration contributes to the dynamic displacement $\bar{x}_d$. This assumption is, of course, not strictly true; however, for low damped systems, the contributions from higher modes are negligible when first mode resonance is obtained.

The damping factors λ given in Figures 36 - 57 are based on the usual assumption of viscous damping and have been calculated

using the equation

$$\lambda = \frac{1}{2 \ \overline{\text{DMF}}} \qquad\qquad (5)$$

The above theoretical considerations are based on linear flexure theory and should be applied only to those beams showing linear response characteristics. As noted in Figures 58 - 81, the beams tested in this investigation are highly non-linear in character if the deflections produced are sufficiently high. However, in conducting the steady state tests all vibrations were limited to small amplitudes. Therefore, these beams generally responded in their initial linear range thus making the above theoretical considerations applicable.

To study the effect of various parameters on internal damping, consider first the results given in Figures 36 - 49 for Beams Nos. 1 - 16 (see Table I). Two important design parameters are involved, namely, magnitude and type of prestress.

It is apparent by comparing the results given in Figures 36 - 38 and also by comparing the results given in Figures 39 - 42, that for these tests magnitude of prestress did not greatly influence damping. In each of these cases, no tension cracks were developed due to the type of prestress and due to the small amplitude oscillations produced. Therefore, based on these results, it would appear that as long as the initial prestress keeps the entire cross section of the beam under compressive stresses during the vibration period, damping will not be affected greatly by the magnitude of the initial prestress. As noted on Figures 36 - 42, the amount of damping for these cases ranged from about 0.5 - 1.0 percent of critical.

Consider now the results presented in Figures 43 - 45 which represent the triangular initial prestress condition. Comparing these results with those discussed in the previous paragraph (Figs. 36 - 42), it is apparent an increased amount of damping is present. It would appear in this case that the tensile stresses produced by the vibration accounts for this increase. As noted, this increase in damping is accompanied by reduced resonant frequencies which indicate cracking of the concrete has taken place. For this triangular type of prestress, damping would seem to be reduced as the intensity of initial prestress is increased. This reduction is to be expected since the initial linear range of the beam is extended somewhat with increased intensity of prestress as indicated by the static load-deflection curves shown in Figures 66 - 69. The amount of damping given in Figures 43 - 45 for this prestress type ranges from approximately 0.75 - 1.50 percent of critical.

Now observe the results presented in Figures 46 - 49 for Beams Nos. 13 - 16. As shown in Table I, these beams have been prestressed so that tensile stresses are also produced by the initial prestress. However, as shown by Table I, the initial prestress for Beams Nos. 15 and 16 were sufficiently great to produce cracking. It is obvious from the resonance curves in Figures 46 - 49, that damping for this prestress type is greater than the damping observed for any of the other three prestress types previously discussed. This increase is due to the greater amount of cracking permitted. From the data presented for this prestress type, it is difficult to make any conclusions as to the effect which magnitude of initial prestress has on damping. Damping factors are observed for these cases in the approximate range of 1.0 - 2.0 percent of critical.

From the previous discussion of the results presented in Figures 36 - 49, it is generally concluded that the important parameter which affects the amount of internal damping in prestressed concrete is the degree to which tension cracks were allowed to develop in the concrete. The two design parameters involved in these cases, namely, magnitude and type of prestress, have an indirect influence on damping in that these parameters influence the amount of cracking which can take place.

It should be noted in all of the above cases that the damping factor λ increases with amplitude of oscillation. Even in those cases where the initial prestress keeps the entire cross section under compressive stresses during the vibration period, an increase in damping with amplitude of oscillation is observed. The rate of increase with amplitude is, of course, much more pronounced when the amplitudes increase sufficiently to develop tension cracks.

The resonance curves shown in Figures 50 - 53 were obtained from tests conducted on Beams Nos. 17 - 20 which were ordinary reinforced concrete beams (no prestress). Beam No. 17 obviously cracked during testing which resulted in damping factors of over two percent of critical and also resulted in a reduced frequency. Beam No. 18 apparently did not crack during testing as indicated by the low damping factors and by the resonant frequency of 90 cps which is about correct for an uncracked beam of this type. Beams Nos. 19 and 20 were reinforced on both top and bottom as shown in Table I and as a result no cracking took place during testing. Thus low damping factors, i.e. of the order of one percent of critical, were observed.

As pointed out in a previous section of this report, Beams Nos. 1 - 20 were manufactured using concrete Mix No. 1, Beams Nos. 21 and 22 were manufactured using Mix No. 2, and Beams Nos. 23 and 24 were manufactured using Mix No. 3. Therefore, it was the intent to check the effect which concrete ultimate strength has on damping by testing Beams Nos. 21 - 24. It is quite obvious, however, that the effect of this parameter is entirely overshadowed by the effect of cracking.

Beams Nos. 21 and 23 were prestressed and do not show evidence of cracking as shown by their resonance curves (Figs. 54 and 56), i.e. the damping factors are reasonably low and their frequencies are consistent with uncracked sections.

Beams Nos. 22 and 24, which were ordinary reinforced beams, produced the resonance curves shown in Figures 55 and 57. It is quite apparent that both of these beams cracked during the steady state tests as evidenced by the large increase in damping with amplitude and by the reduction of their resonant frequencies with amplitude.

As previously pointed out, all damping factors given on Figures 36 - 57 were based on the measured amplitude of response at resonance. However, the general shape of the resonance curves in these Figures also give an indication of the amount of damping as illustrated by comparing these resonance curves with those presented in Figure 90 for a single degree system.

In concluding this discussion on the steady state test results, it seems appropriate to re-emphasize that the important parameter

which appears to influence damping in prestress or ordinary reinforced
concrete elements is the degree to which cracking is permitted to develop.
Damping factors seem to be of the order of one percent or less for
uncracked members but increase with the degree of cracking permitted.
Maximum damping factors measured in these tests were of the order
of 2.5 percent of critical. For these extreme cases, cracks were
unquestionably produced; however, considering the small amplitudes of
of oscillation which produced them, it would be impossible to see
such cracks with the naked eye even if one could stop the motion of
the vibrating beam at its maximum amplitude of motion.

B. Free Vibration Tests

Due to the limited force output of the available excitor used
in the steady state test program, it was necessary that each test
beam be further tested using the free vibration method previously
described so that large amplitude oscillation could be achieved. The
results of this investigation are presented in Figures 82 - 89. The
applied static load p_o and resulting deflection $x(0)$ as indicated
for each dynamic test is represented by a point on the appropriate
static load-deflection curve in Figures 58 - 81.

Each damping coefficient λ shown in Figures 82 - 89 is based
on the assumption of viscous damping and has been obtained in the
usual way by measuring the decrement of damping directly from the
photographs of the decayed free vibration response. This decrement
was checked for successive cycles of oscillation for a number of
tests to determine if damping was a function of amplitude. While

some variations were observed, generally the damping decrement was reasonably constant over the earlier portion of the decayed response curve. Therefore, all λ's were calculated by considering the reduction in displacement amplitude over a period of four cycles of oscillation which results in the equation

$$\lambda = \frac{1}{2\pi} \left[1 - \sqrt[4]{\frac{x_{n+4}}{x_n}} \right]$$

where x_n is the displacement amplitude of the nth cycle of oscillation. Due to the accuracy of the amplitude measurements, it is very difficult to measure accurately the damping decrement towards the end of the decayed response curve. Therefore, damping for the relatively low amplitude oscillations could not be measured.

In attempting to correlate the damping factors in Figures 82 - 89 with the various parameters involved, considerably more difficulty arises than in the correlation of the steady state results previously discussed. This difficulty would seem to be due to the fact that in most cases the initial maximum displacements are sufficiently large to produce cracking in the concrete. Since the amount of this cracking can vary over a wide range and since each beam had been loaded only once or twice to the prescribed test load before the actual test was conducted, one might expect considerable differences in the damping factors obtained. This latter statement would imply that past history of loading is an important parameter. To illustrate this point, Test 22-2D (Fig. 88) was initiated at a load of 3.30 kips which was the first time Beam No. 22 had been loaded to this level. From the resulting decayed response curve it is quite clear that cracks were produced on the first cycle of oscillation; thus, a relatively large amount of the strain energy was immediately dissipated.

Obviously for this case any damping factor one might calculate would be meaningless. Test 22-3D is a rerun of Test 22-2D, which gave entirely different results. It can be assumed from these results that Test 22-3D did not produce any additional new cracks.

If one observes only those dynamic tests which were conducted strictly in the linear range (see Figs. 58 - 81), damping factors generally fall in the range 3 - 3 1/2 percent of critical. For those tests conducted in the non-linear range where large cracks develop, damping factors generally fall in the range 3 1/2 - 6 percent of critical. There is, however, one exception to this statement, i.e. for those tests involving Beams Nos. 20, 22, and 24 which were reinforced both top and bottom. For these beams damping seemed to show a decrease for the large amplitude oscillation.

In comparing the low amplitude free vibration results with the steady state results, greater damping factors are observed in the case of free vibration. While this difference is greater than one would expect, there are a number of factors which no doubt contribute to this difference. Some of these contributory factors are the following:

(1) In the steady state tests, many thousands of cycles of loading were accumulated before the basic test data were recorded, while in the free vibration case, each beam experienced only one or two previous cycles of load. It is expected that damping will decrease with number of load cycles to a given amplitude.

(2) The displacement amplitudes were higher in the free vibration tests than in the steady state tests.

(3) The frequency of oscillation was lower in the free vibration tests than in the steady state tests.

(4) Support damping was higher in the case of free vibration tests. As pointed out in numerous references, support damping can be very important in practical situations and should not be overlooked. To what extent support damping influenced the results presented in this report is unknown; however, considerable effort was made to minimize this effect. It is quite evident that further work is needed to study the effect of this very important parameter.

V. CONCLUSIONS AND RECOMMENDATIONS

Based on the experimental results obtained in this investigation, the following general conclusions have been deduced:

(1) Under steady state conditions, internal damping in prestressed concrete members may be less than one percent of critical if the initial prestress is sufficient to prevent tension cracks from developing. If tension cracks are allowed to develop, but on a miscroscopic scale, damping can be expected of the order of two percent of critical. If larger (visible) cracks are permitted to develop, one should expect higher damping.

(2) Under transient conditions, the amount of internal damping present in prestressed concrete members depends to a great extent on the past history of loading and on the amplitude

of displacements produced. For those cases where members have been dynamically loaded only a few times to a given stress level which produces considerable cracking, damping can be expected anywhere in the range of 3 - 6 percent of critical.

(3) Magnitude and type of prestress in concrete members have an indirect influence on internal damping only because these parameters control the amount of cracking which can take place.

There are numerous factors which affect the overall damping in a prestressed concrete structural system as pointed out in this report. Those factors which were not included as parameters in this investigation but which need further study are the following:

(1) Effect of history of loading on internal damping;

(2) Effect of support conditions on external damping; and

(3) Effect of frequency on overall damping.

VI. BIBLIOGRAPHY

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Table I

f_c^1 (28 day Compressive Strength), E (Modulus of Elasticity), f_t (Modulus of Rupture)
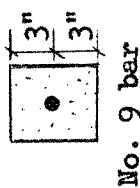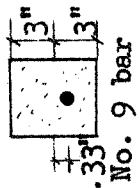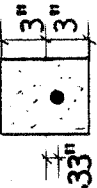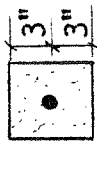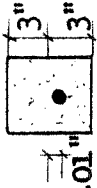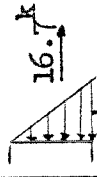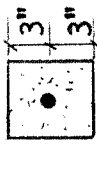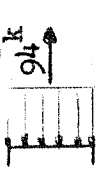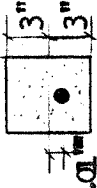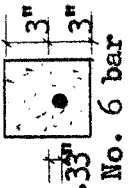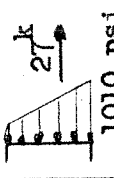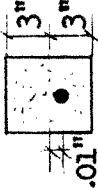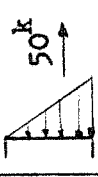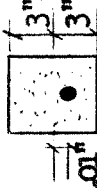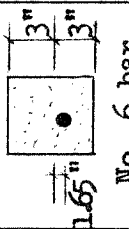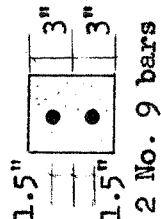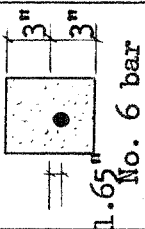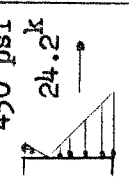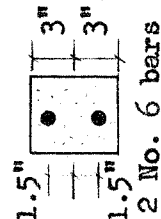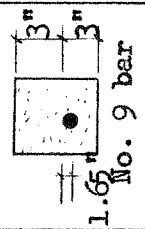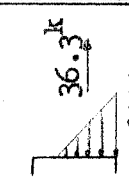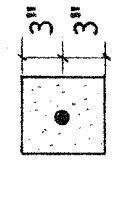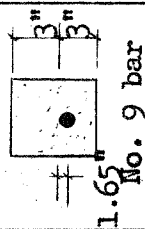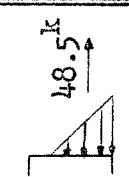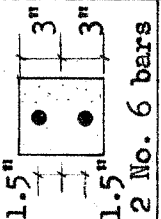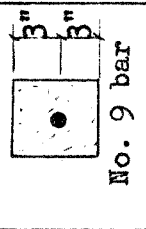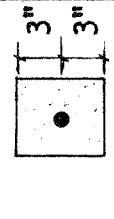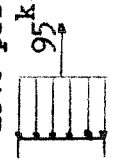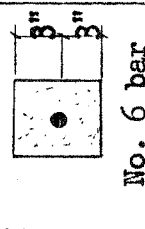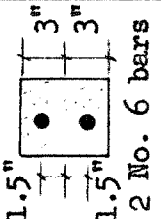
Beam No.	f_c^1 psi	E psi	f_t psi	Cross Section	Initial Prestress Condition
1	5040	3.1×10^6 / 4.0×10^6	571	No. 9 bar (3" / 3")	980 psi, 34.3^k
2	4920	4.7×10^6 / 4.7×10^6	550	No. 9 bar (3" / 3")	1950 psi, 68^k
3	4600	4.3×10^6 / 4.3×10^6	562	No. 9 bar (3" / 3")	2920 psi, 102^k
4	4755	3.6×10^6 / 3.6×10^6	599	No. 9 bar (3" / 3")	2680 psi, 94^k
5	5020	4.0×10^6 / 4.0×10^6	606	No. 6 bar (3" / 3" / 0.33")	505 psi, 27^k, 1010 psi
6	5170	4.2×10^6 / 4.6×10^6	654	No. 9 bar (3" / 3" / 0.33")	990 psi, 52^k, 1980 psi
7	5310	4.3×10^6 / 4.3×10^6	675	No. 9 bar (3" / 3" / 0.33")	1485 psi, 78^k, 2970 psi
8	4890	4.2×10^6 / 4.8×10^6	620	No. 9 bar (3" / 3" / 0.33")	2060 psi, 105^k, 4120 psi
9	5100	3.6×10^6 / 4.1×10^6	617	No. 6 bar (3" / 3" / 1.01")	16.7^k, 945 psi
10	4780	3.3×10^6 / 3.3×10^6	637	No. 9 bar (3" / 3" / 1.01")	33.3^k, 1910 psi
11	5010	3.6×10^6 / 3.6×10^6	610	No. 9 bar (3" / 3" / 1.01")	50^k, 2880 psi
12	4980	4.0×10^6 / 4.0×10^6	530	No. 9 bar (3" / 3" / 1.01")	70^k, 4050 psi

Table I (continued)

f_c^l (28 day Compressive Strength), E (Modulus of Elasticity), f_t (Modulus of Rupture)

Beam No.	f_c^l psi	E psi	f_t psi	Cross Section	Initial Prestress Condition
19	5100	4.0×10^6, 4.0×10^6	659	2 No. 9 bars (1.5", 1.5", 3", 3")	No prestress
20	5100	3.8×10^6, 4.1×10^6	678	2 No. 6 bars (1.5", 1.5", 3", 3")	No prestress
21	4087	3.6×10^6, 3.6×10^6	605	No. 9 bar (3", 3")	1140 psi, 41^k
22	4087	3.9×10^6, 3.9×10^6	616	2 No. 6 bars (1.5", 1.5", 3", 3")	No prestress
23	6360	5.0×10^6, 5.0×10^6	693	No. 9 bar (3", 3")	2640 psi, 95^k
24	5983	5.1×10^6, 5.5×10^6	686	2 No. 6 bars (1.5", 1.5", 3", 3")	No prestress

Beam No.	f_c^l psi	E psi	f_t psi	Cross Section	Initial Prestress Condition
13	5243	4.1×10^6, 4.5×10^6	649	No. 6 bar (1.65", 3", 3")	220 psi, 12.0^k, 890 psi
14	5510	4.0×10^6, 5.0×10^6	709	No. 6 bar (1.65", 3", 3")	450 psi, 24.2^k, 1790 psi
15	5330	4.2×10^6, 4.2×10^6	580	No. 9 bar (1.65", 3", 3")	36.3^k, 2990 psi
16	5390	4.3×10^6, 4.3×10^6	654	No. 9 bar (1.65", 3", 3")	48.5^k, 4000 psi
17	5230	4.0×10^6, 4.0×10^6	619	No. 9 bar (3", 3")	No prestress
18	5250	4.2×10^6, 4.6×10^6	535	No. 6 bar (3", 3")	No prestress
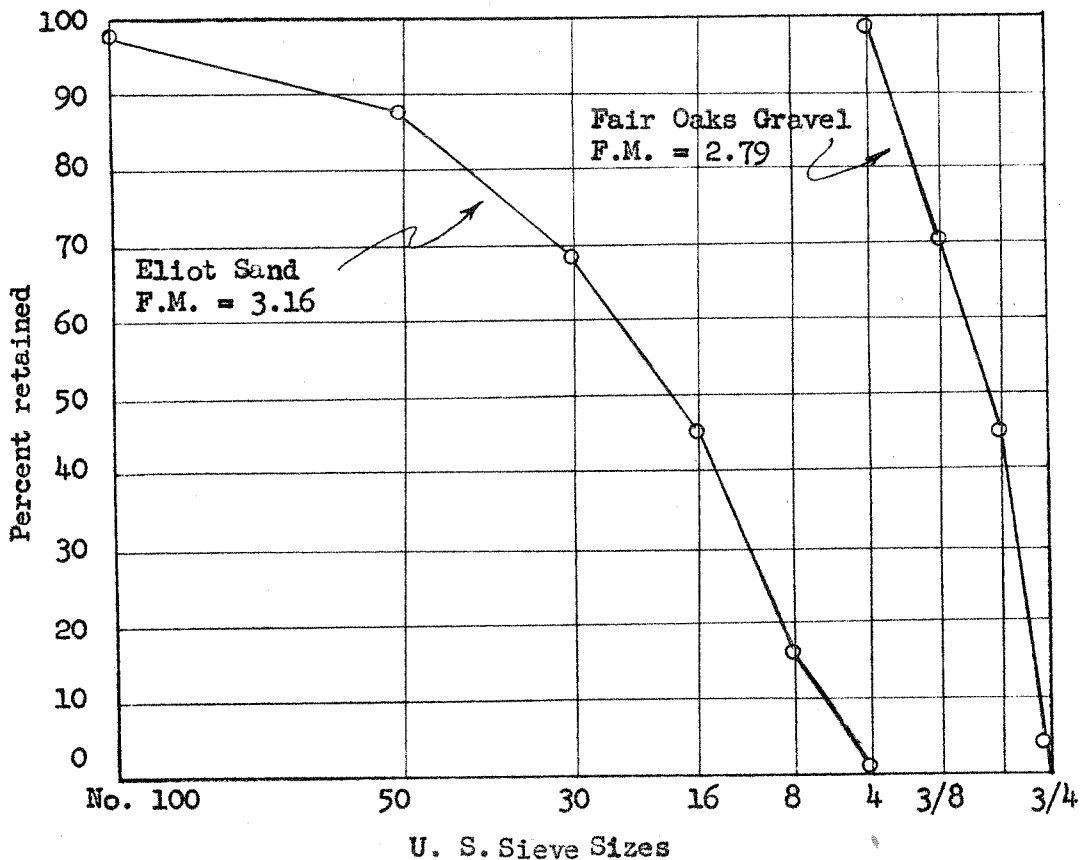
Gradation curves for aggregates

Figure 1
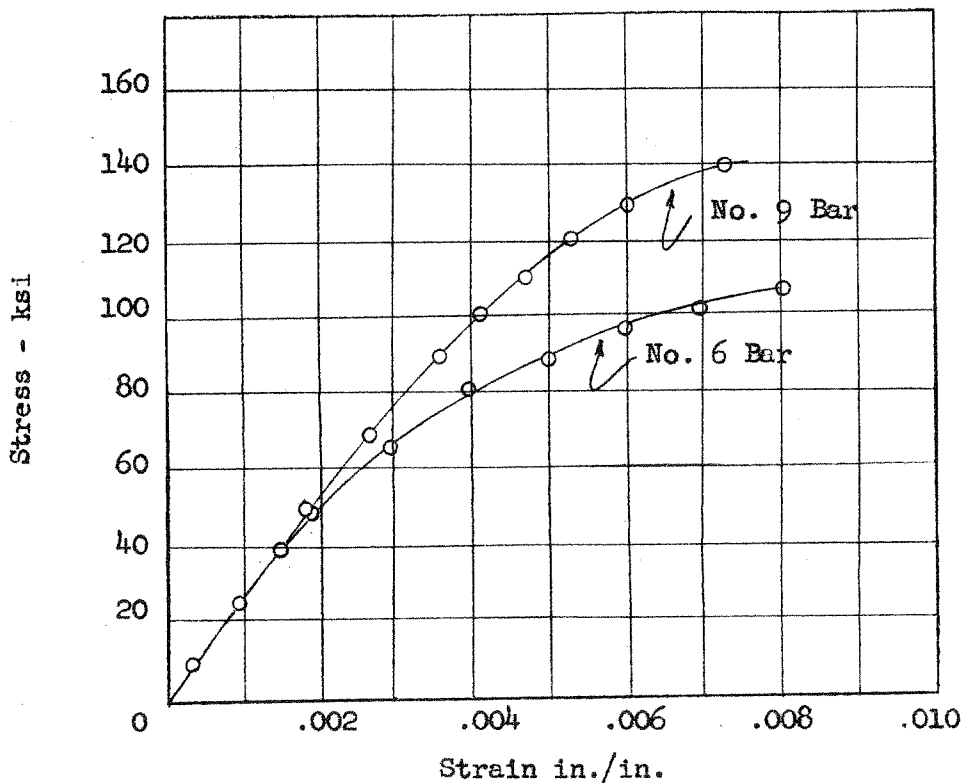
Stress-strain relationships for reinforcing bars

Figure 2
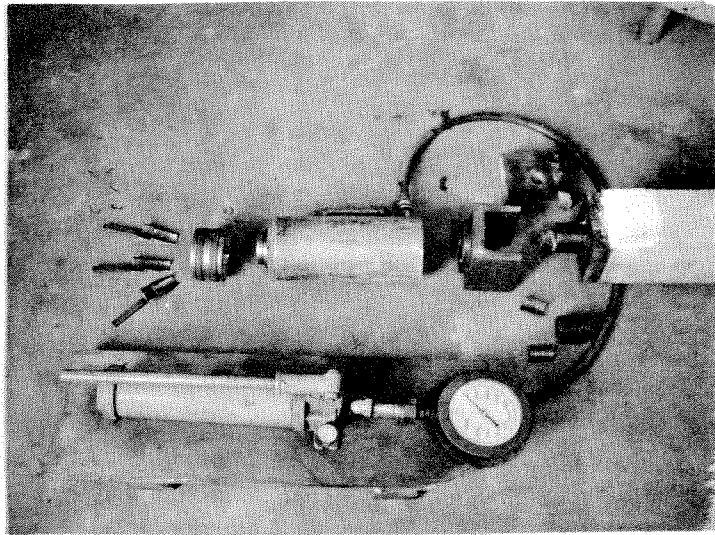
Prestressing equipment

Figure 3
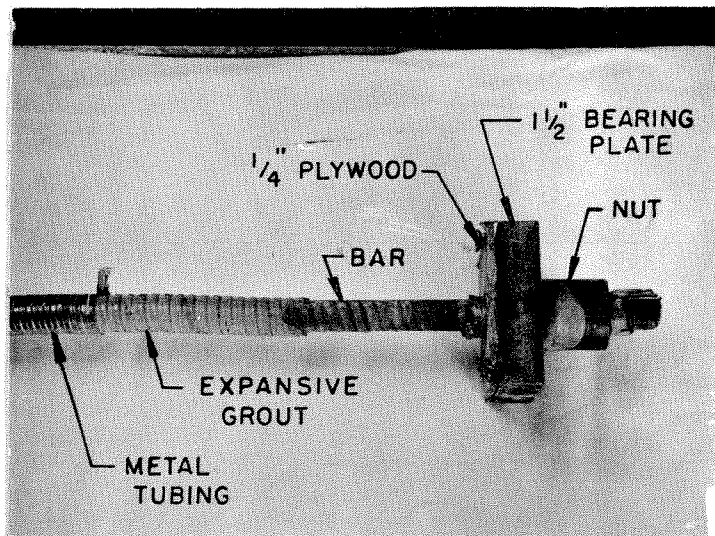
Prestressing bar after removal from beam

Figure 4
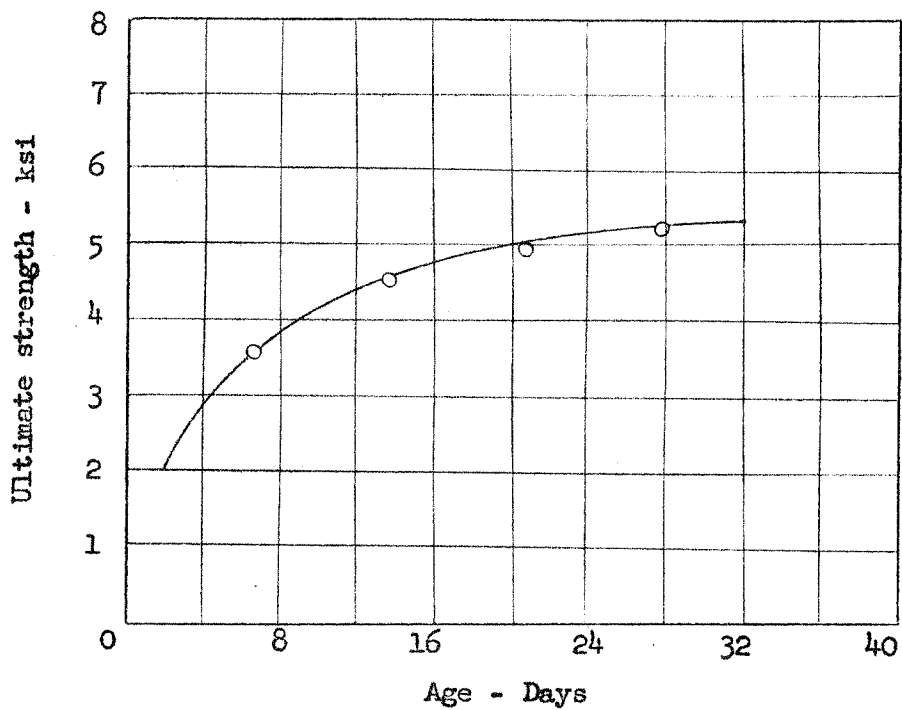
Ultimate compressive strength versus age for Mix No. 2

Figure 5
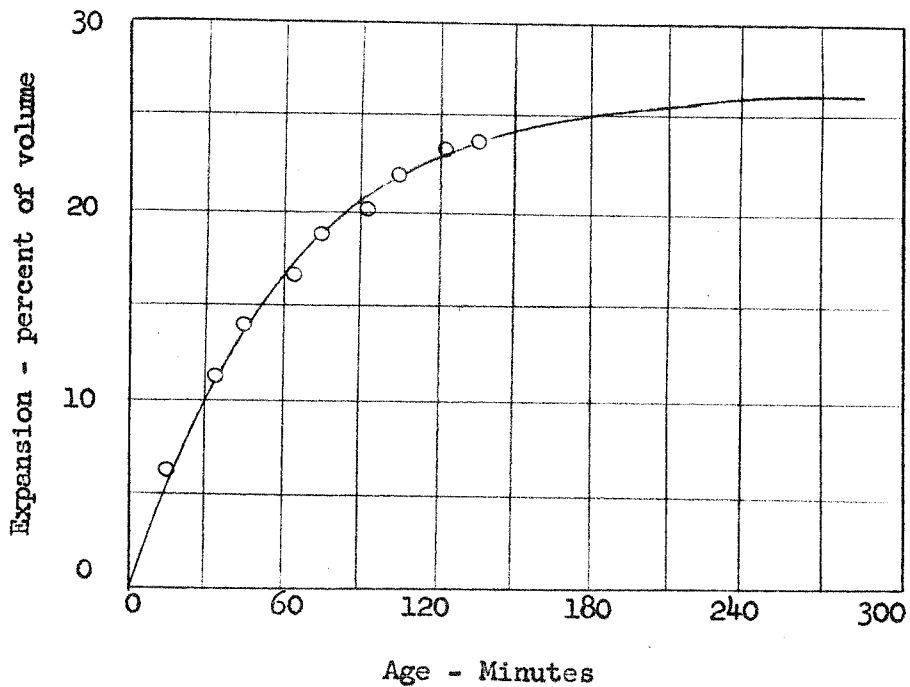
Unrestrained expansion of grout versus age

Figure 6

Figure 7
Beam No. 1

E = 4.0 x 10⁶ psi

E = 3.1 x 10⁶ psi

Stress - ksi

Strain   (10⁻⁴ in./in.)

Figure 8
Beam No. 2

E = 4.7 x 10⁶ psi

Stress - ksi

Strain   (10⁻⁴ in./in.)

Figure 9
Beam No. 3

E = 4.3 x 10⁶ psi

Stress - ksi

Strain   (10⁻⁴ in./in.)

Figure 10
Beam No. 4

E = 3.6 x 10⁶ psi

Stress - ksi

Strain   (10⁻⁴ in./in.)

Cylinder stress-strain curves for Beams 1-4

Figure 11
Beam No. 5

E = 4.0 x 10^6 psi

Figure 12
Beam No. 6

E = 4.6 x 10^6 psi

E = 4.2 x 10^6 psi

Figure 13
Beam No. 7

E = 4.3 x 10^6 psi

Figure 14
Beam No. 8

E = 4.8 x 10^6 psi

E = 4.2 x 10^6 psi

Cylinder stress-strain curves for Beams 5-8

Figure 15
Beam No. 9

E = 4.1 x 10^6 psi

E = 3.6 x 10^6 psi

Figure 16
Beam No. 10

E = 3.3 x 10^6 psi

Figure 17
Beam No. 11

E = 3.6 x 10^6 psi

Figure 18
Beam No. 12

E = 4.0 x 10^6 psi

Cylinder stress-strain curves for Beams 9-12

Figure 19
Beam No. 13

E = 4.5 x 10⁶ psi

E = 4.1 x 10⁶ psi

Figure 20
Beam No. 14

E = 5.0 x 10⁶ psi

E = 4.0 x 10⁶ psi

Figure 21
Beam No. 15

E = 4.2 x 10⁶ psi

Figure 22
Beam No. 16

E = 4.3 x 10⁶ psi

Cylinder stress-strain curves for Beams 13-16

Figure 23
Beam No. 17

E = 4.0 x 10⁶ psi

Figure 24
Beam No. 18

E = 4.6 x 10⁶ psi

E = 4.2 x 10⁶ psi

Figure 25
Beam No. 19

E = 4.0 x 10⁶ psi

Figure 26
Beam No. 20

E = 4.1 x 10⁶ psi

E = 3.8 x 10⁶ psi

Cylinder stress-strain curves for Beams 17-20

Figure 27
Beam No. 21

E = 3.6 x 10^6 psi

Figure 28
Beam No. 22

E = 3.9 x 10^6 psi

Figure 29
Beam No. 23

E = 5.0 x 10^6 psi

Figure 30
Beam No. 24

E = 5.5 x 10^6 psi

E = 5.1 x 10^6 psi

Cylinder stress-strain curves for Beams 21-24

Free-free beam attached to exciter

Figure 31
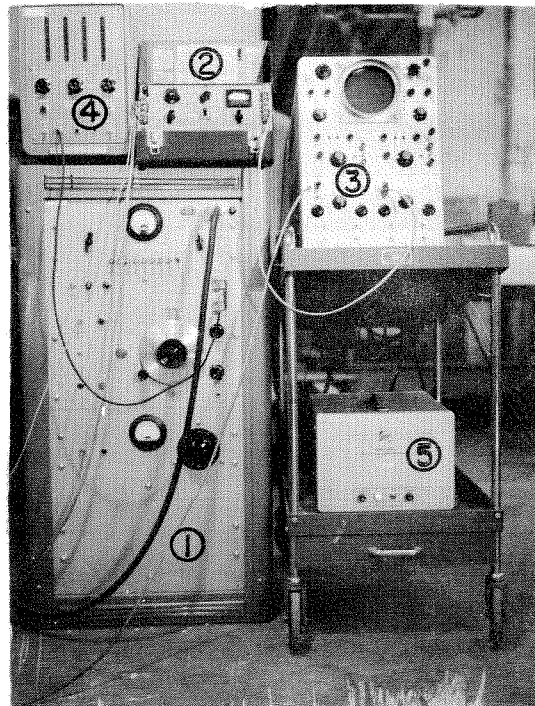
Exciter power supply unit and electronic
recording equipment for steady state tests

Figure 32
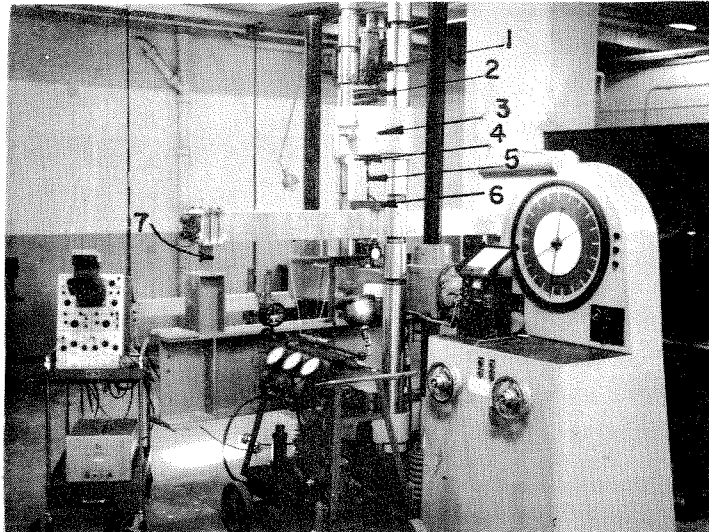
1. Loading jack.
2. Railroad car spring.
3. Upper head Baldwin machine.
4. Lower end 1" bolt.
5. Fractured load bar.
6. Rigid bracket.
7. Flexible support plates.

Concrete beam mounted and
ready for free-vibration testing

Figure 33
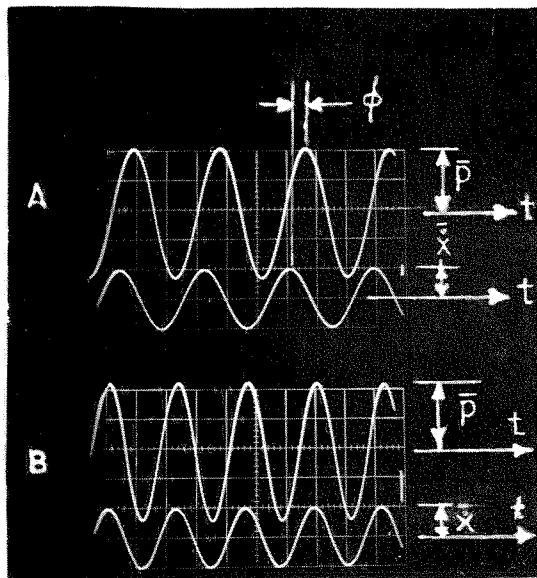
ϕ - PHASE ANGLE
$\bar{P}$ - FORCE AMPLITUDE
$\bar{\dot{x}}$ - VELOCITY AMPLITUDE
t - TIME

Exciter force and vertical velocity of
center of beam under steady state vibration

Figure 34
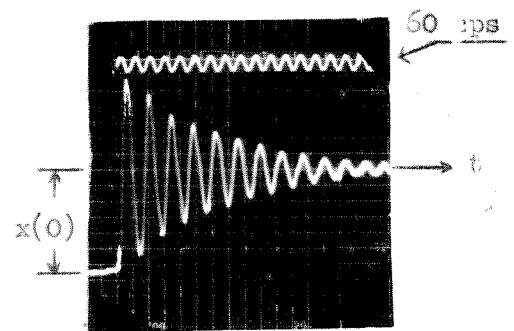
Beam deflection versus time
under free vibration

Figure 35

Figure 36
Beam No. 1
$\bar{p}$ = 25 lbs.
DMF = 74
λ = .0067

$\bar{p}$ = 10 lbs.
DMF = 93
λ = .0054

Figure 37
Beam No. 2
$\bar{p}$ = 25 lbs.
DMF = 52
λ = .0096

$\bar{p}$ = 10 lbs.
DMF = 72
λ = .0070

Figure 38
Beam No. 4
$\bar{p}$ = 50 lbs.
DMF = 60
λ = .0083

$\bar{p}$ = 25 lbs.
DMF = 66
λ = .0076

$\bar{p}$ = 12.5 lbs.
DMF = 75
λ = .0067

Figure 39
Beam No. 5
$\bar{p}$ = 25 lbs.
DMF = 69
λ = .0073

$\bar{p}$ = 12.5 lbs.
DMF = 76
λ = .0066
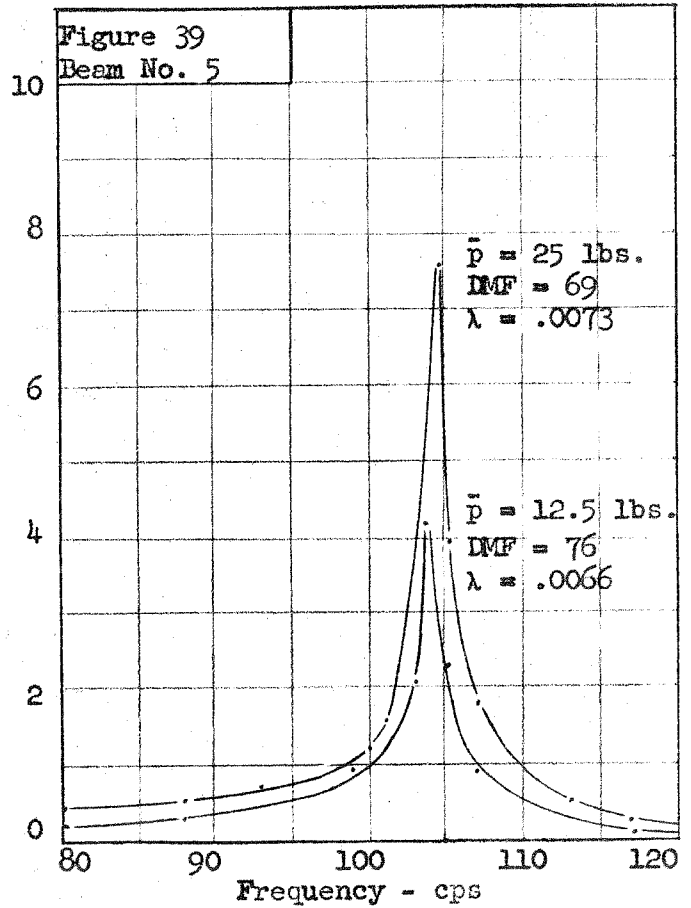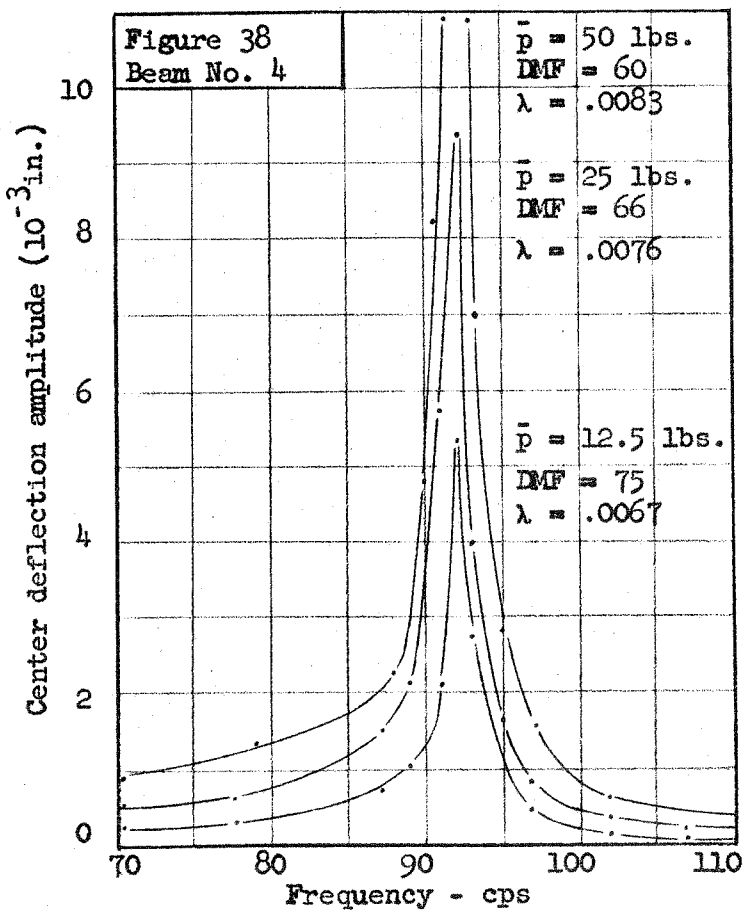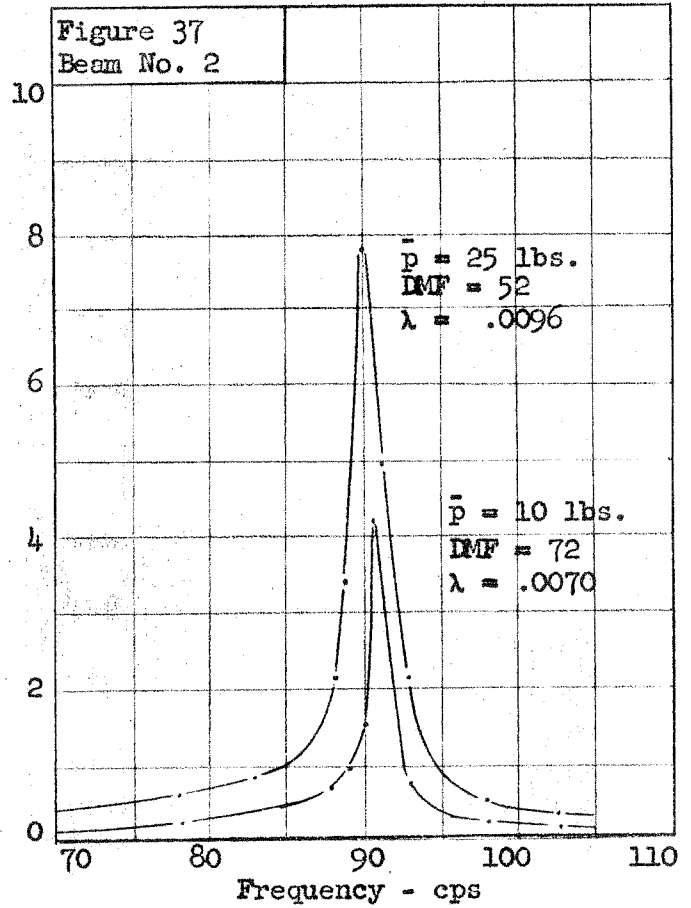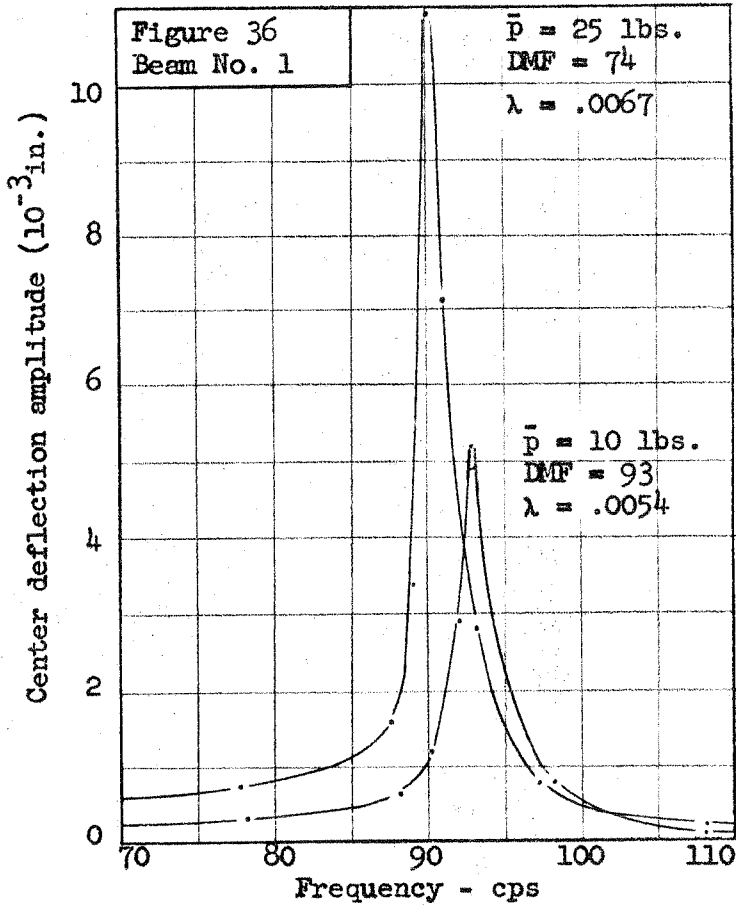
Resonance curves - Beam Nos. 1, 2, 4, and 5
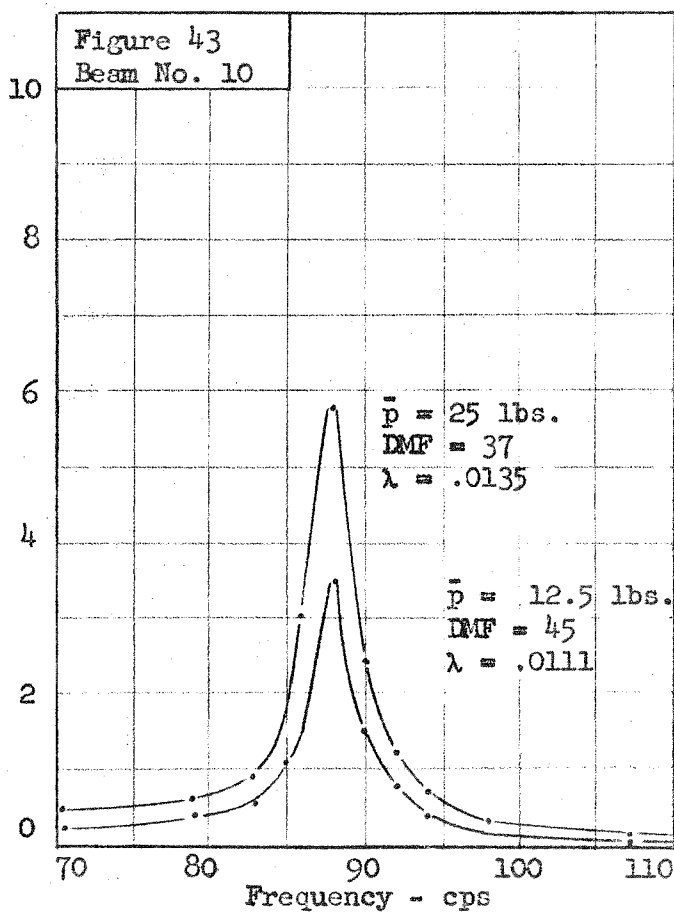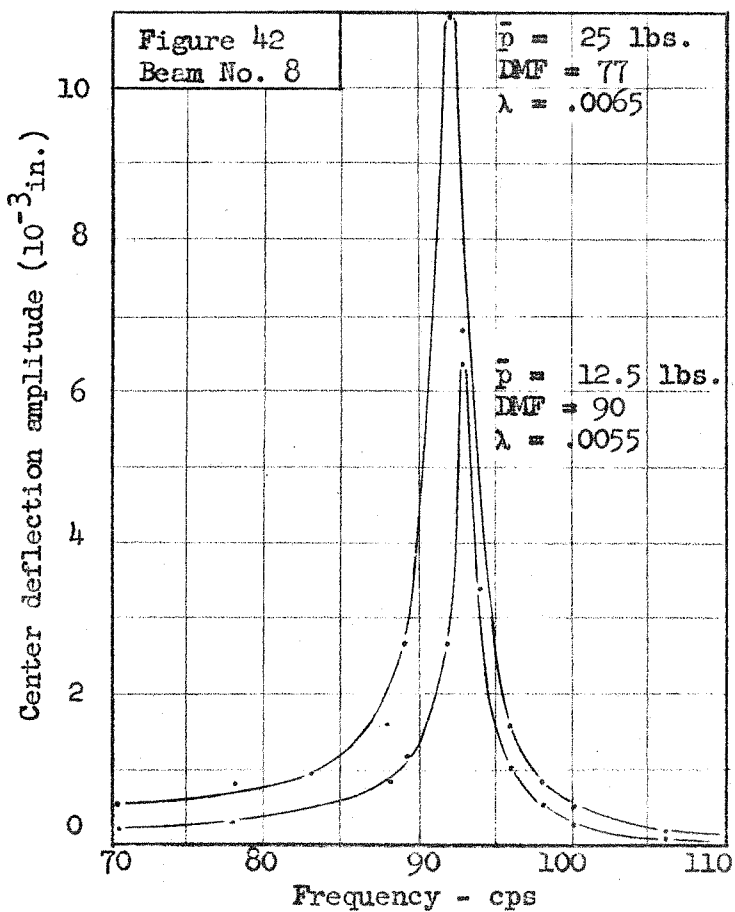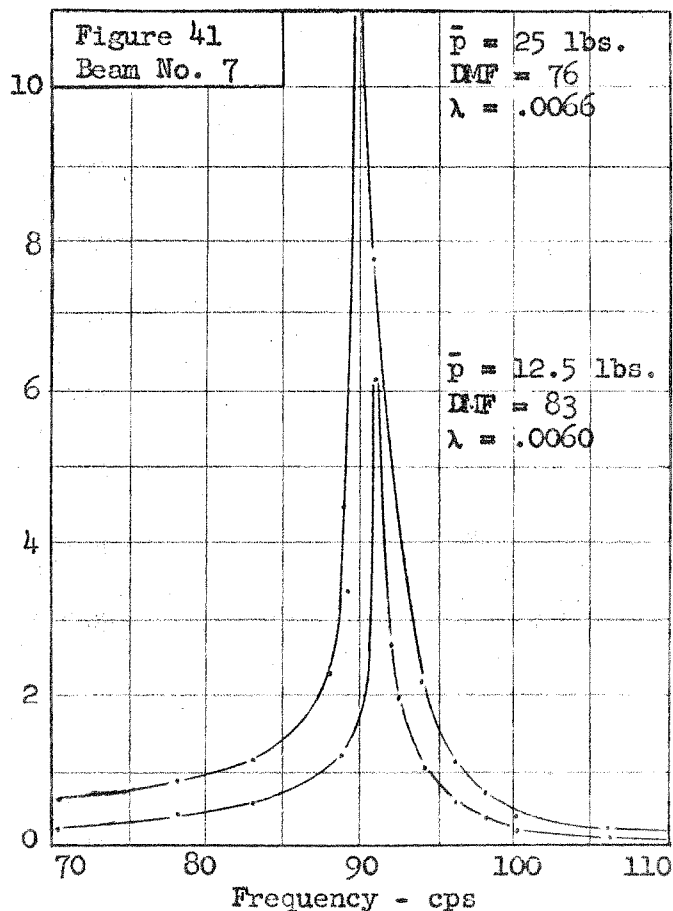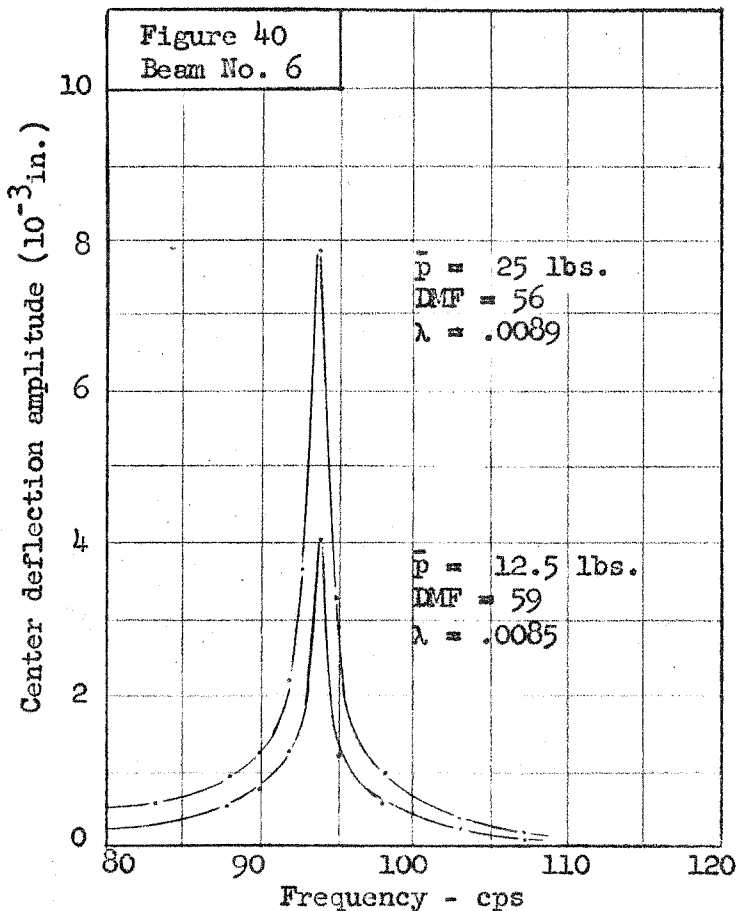
Resonance curves - Beam Nos. 6, 7, 8, and 10

Figure 44
Beam No. 11

$\bar{p} = 25$ lbs.
DMF = 32
$\lambda = .0156$

Figure 45
Beam No. 12

$\bar{p} = 25$ lbs.
DMF = 59
$\lambda = .0085$

$\bar{p} = 12.5$ lbs.
DMF = 68
$\lambda = .0074$

Figure 46
Beam No. 13

$\bar{p} = 50$ lbs.
DMF = 24
$\lambda = .0208$

Figure 47
Beam No. 14

$\bar{p} = 25$ lbs.
DMF = 36
$\lambda = .0139$

$\bar{p} = 12.5$ lbs.
DMF = 46
$\lambda = .0109$

Figure 48
Beam No. 15

$\bar{p}$ = 25 lbs.
DMF = 27
λ = .0185

$\bar{p}$ = 12.5 lbs.
DMF = 32
λ = .0156

Figure 49
Beam No. 16

$\bar{p}$ = 25 lbs.
DMF = 34
λ = .0147

$\bar{p}$ = 12.5 lbs.
DMF = 36
λ = .0138

Figure 50
Beam No. 17

$\bar{p}$ = 25 lbs.
DMF = 22
λ = .0228

$\bar{p}$ = 12.5 lbs.
DMF = 24
λ = .0208

Figure 51
Beam No. 18

$\bar{p}$ = 25 lbs.
DMF = 43
λ = .0116

$\bar{p}$ = 12.5 lbs.
DMF = 52
λ = .0096

Resonance curves — Beam Nos. 15 - 18

Figure 52
Beam No. 19

$\bar{p}$ = 25 lbs.
DMF = 38
λ = .0131

$\bar{p}$ = 12.5 lbs.
DMF = 53
λ = .0094

Figure 53
Beam No. 20

$\bar{p}$ = 25 lbs.
DMF = 45
λ = .0111

$\bar{p}$ = 12.5 lbs.
DMF = 52
λ = .0096

Figure 54
Beam No. 21

$\bar{p}$ = 50 lbs.
DMF = 52
λ = .0096

$\bar{p}$ = 25 lbs.
DMF = 79
λ = .0063

Figure 55
Beam No. 22

$\bar{p}$ = 50 lbs.
DMF = 20
λ = .0250

$\bar{p}$ = 25 lbs.
DMF = 40
λ = .0125

$\bar{p}$ = 12.5 lbs.
DMF = 46
λ = .0108

Figure 56
Beam No. 23

$\bar{p} = 50$ lbs.
DMF = 32
$\lambda = .0156$

$\bar{p} = 25$ lbs.
DMF = 38
$\lambda = .0131$

$\bar{p} = 12.5$ lbs.
DMF = 60
$\lambda = .0083$

Center deflection amplitude (10^{-3}in.)

Frequency - cps

Figure 57
Beam No. 24

$\bar{p} = 50$ lbs.
DMF = 22
$\lambda = .0226$

$\bar{p} = 25$ lbs.
DMF = 36
$\lambda = .0138$

$\bar{p} = 12.5$ lbs.
DMF = 46
$\lambda = .0108$

Center deflection amplitude (10^{-3}in.)

Frequency - cps

Resonance curves - Beam Nos. 23-24

Figure 58
Beam No. 1

Figure 59
Beam No. 2

Figure 60
Beam No. 3

Figure 61
Beam No. 4

Static load – deflection curves – Beams 1-4

Figure 62
Beam No. 5

Figure 63
Beam No. 6

Figure 64
Beam No. 7

Figure 65
Beam No. 8

Static load - deflection curves - Beams 5-8

Figure 66
Beam No. 9

Figure 67
Beam No. 10

Figure 68
Beam No. 11

Figure 69
Beam No. 12

Figure 70
Beam No. 13

Figure 71
Beam No. 14

Figure 72
Beam No. 15

Figure 73
Beam No. 16

Static load – deflection curves – Beams 13-16

Figure 74
Beam No. 17

Figure 75
Beam No. 18

Figure 76
Beam No. 19

Figure 77
Beam No. 20

Static load - deflection curves, Beams 17-20

Static load - kips

Center deflection - inches

Figure 78
Beam No. 21

Figure 79
Beam No. 22

Figure 80
Beam No. 23

Figure 81
Beam No. 24

Static load - deflection curves - Beams 21-24

Test 1-1D $\quad \lambda = 0.047$
$x(0) = 0.047"$ $\quad p_o = 2.00k$

Test 1-2D $\quad \lambda = 0.049$
$x(0) = 0.178"$ $\quad p_o = 3.30k$

Test 1-3D $\quad \lambda = 0.064$
$x(0) = 0.330"$ $\quad p_o = 4.25k$

Test 2-1D $\quad \lambda = 0.038$
$x(0) = 0.050"$ $\quad p_o = 2.00k$

Test 2-2D $\quad \lambda = 0.033$
$x(0) = 0.115"$ $\quad p_o = 3.93k$

Test 2-3D $\quad \lambda = 0.041$
$x(0) = 0.240"$ $\quad p_o = 5.82k$
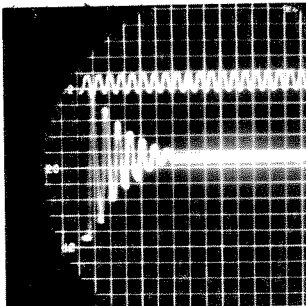
Test 3-1D $\quad \lambda = 0.052$
$x(0) = 0.030"$ $\quad p_o = 1.20k$
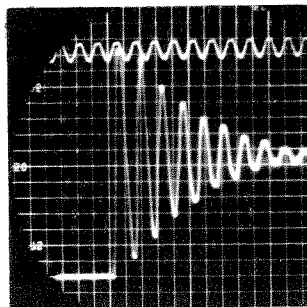
Test 3-2D $\quad \lambda = 0.043$
$x(0) = 0.060"$ $\quad p_o = 2.32k$
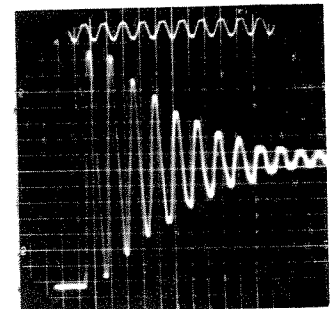
Test 3-3D $\quad \lambda = 0.035$
$x(0) = 0.085"$ $\quad p_o = 3.30k$
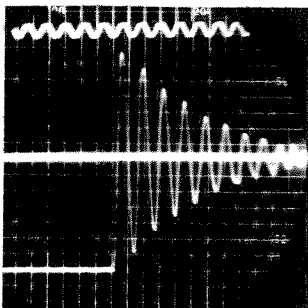
Test 3-4D $\quad \lambda = 0.037$
$x(0) = 0.125"$ $\quad p_o = 4.60k$
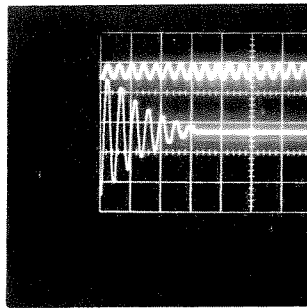
Test 4-1D $\quad \lambda = 0.047$
$x(0) = 0.033"$ $\quad p_o = 1.40k$
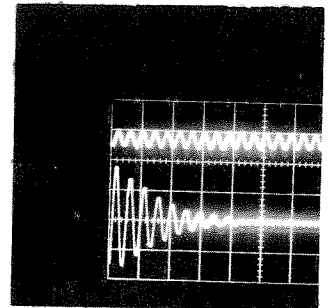
Test 4-2D $\quad \lambda = 0.045$
$x(0) = 0.045"$ $\quad p_o = 2.00k$

Figure 82 $\qquad$ Free vibration response - Beam Nos. 1-4

Test 4-3D λ = 0.040
x(0) = 0.100" p_o = 3.92k

Test 4-4D λ = 0.039
x(0) = 0.210" p_o = 7.10k

Test 5-1D λ = 0.043
x(0) = 0.050" p_o = 2.00k

Test 5-2D λ = 0.066
x(0) = 0.405" p_o = 3.50k

Test 5-3D λ = 0.037
x(0) = 0.435" p_o = 3.55k

Test 6-1D λ = 0.032
x(0) = 0.050" p_o = 2.00k

Test 6-2D λ = 0.044
x(0) = 0.195" p_o = 4.00k

Test 6-3D λ = 0.063
x(0) = 0.405" p_o = 5.20k

Test 7-1D λ = 0.031
x(0) = 0.050" p_o = 2.00k

Test 7-2D λ = 0.032
x(0) = 0.115" p_o = 4.00k

Test 7-3D λ = 0.050
x(0) = 0.380" p_o = 6.00k
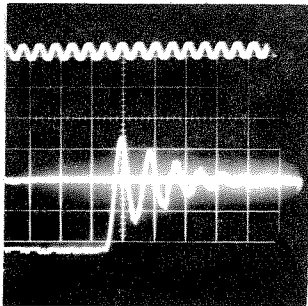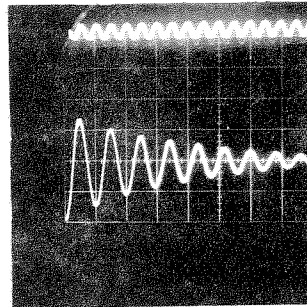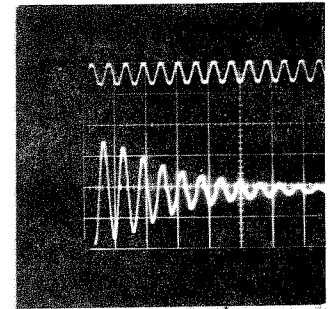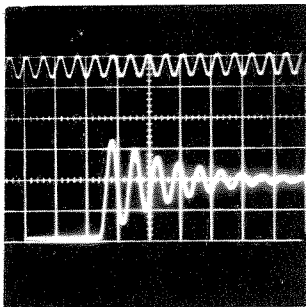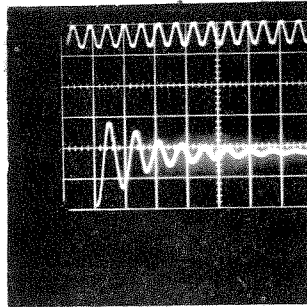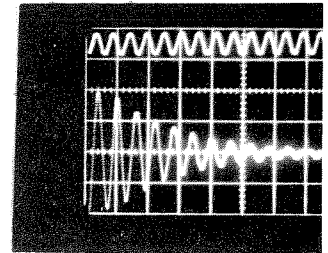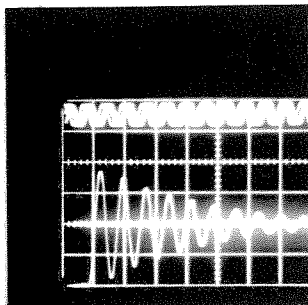
Test 8-1D λ = 0.034
x(0) = 0.049" p_o = 2.00k

Figure 83   Free vibration response - Beam Nos. 4-8

Test 8-2D λ = 0.034
x(0) = 0.100" p_o = 4.00k

Test 8-3D λ = 0.040
x(0) = 0.235" p_o = 5.80k

Test 9-1D λ = 0.056
x(0) = 0.030" p_o = 1.00k

Test 9-2D λ = 0.043
x(0) = 0.20" p_o = 1.45k

Test 9-3D λ = 0.040
x(0) = 0.333" p_o = 1.70k

Test 9-4D λ = 0.030
x(0) = 0.375" p_o = 1.75k
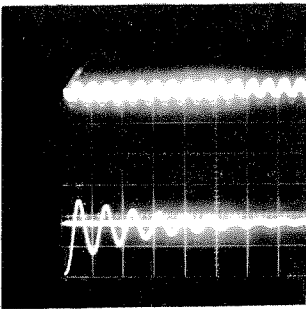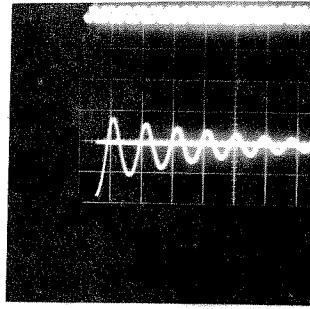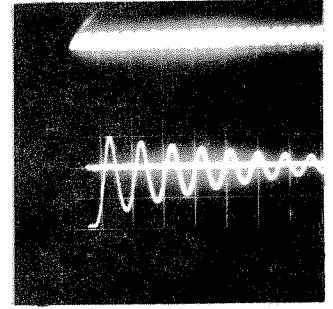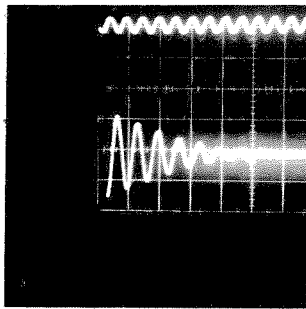
Test 10-1D λ = 0.052
x(0) = 0.032" p_o = 1.00k
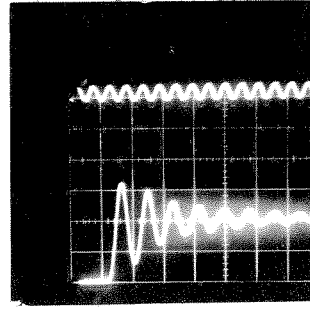
Test 10-2D λ = 0.060
x(0) = 0.145" p_o = 2.00k
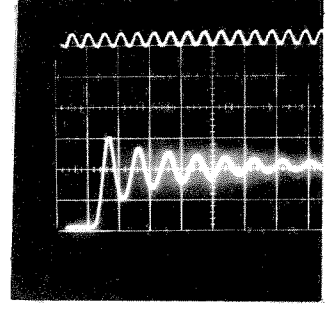
Test 10-3D λ = 0.050
x(0) = 0.335" p_o = 2.40k
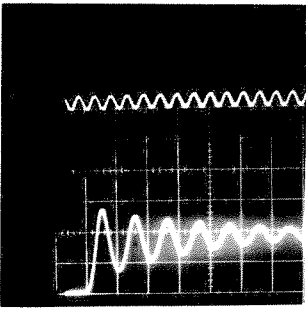
Test 10-4D λ = 0.050
x(0) = 0.435" p_o = 2.55k

Test 11-1D λ = 0.047
x(0) = 0.032" p_o = 1.00k

Test 11-2D λ = 0.054
x(0) = 0.120" p_o = 2.00k

Figure 84   Free vibration response - Beam Nos. 8-11

Test 11-3D λ = 0.049
x(0) = 0.245" p_o = 2.45k

Test 11-4D λ = 0.042
x(0) = 0.355" p_o = 2.80k

Test 12-1D λ = 0.046
x(0) = 0.032" p_o = 1.00k
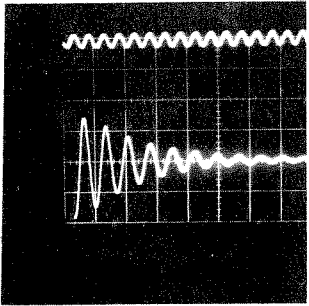
Test 12-2D λ = 0.048
x(0) = 0.084" p_o = 2.00k
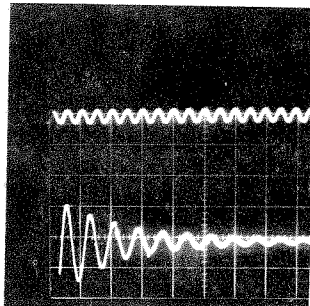
Test 12-3D λ = 0.049
x(0) = 0.212" p_o = 2.70k
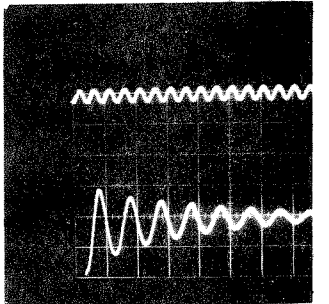
Test 12-4D λ = 0.045
x(0) = 0.330" p_o = 3.05k
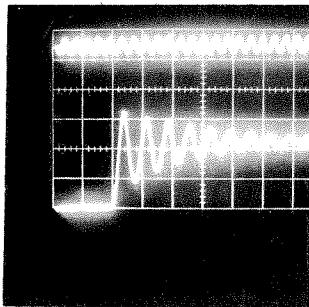
Test 13-1D λ = 0.038
x(0) = 0.145" p_o = 0.78k
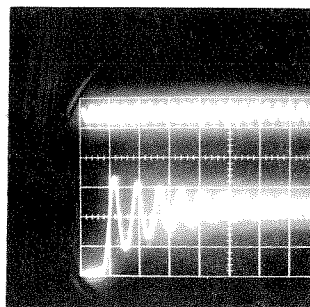
Test 13-2D λ = 0.041
x(0) = 0.170" p_o = 0.78k
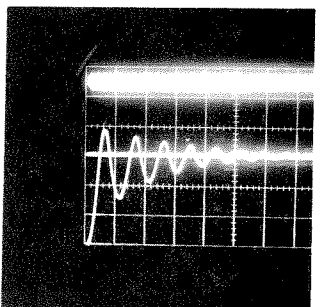
Test 13-3D λ = 0.058
x(0) = 0.430" p_o = 0.87k
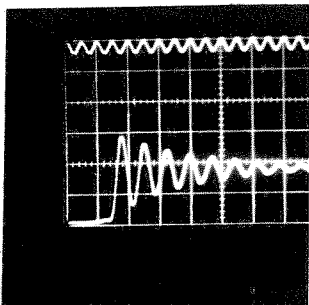
Test 14-1D λ = 0.041
x(0) = 0.050" p_o = 0.83k

Test 14-2D λ = 0.040
x(0) = 0.100" p_o = 1.10k

Test 14-3D λ = 0.046
x(0) = 0.253" p_o = 1.27k

Figure 85   Free vibration response - Beam Nos. 11-14
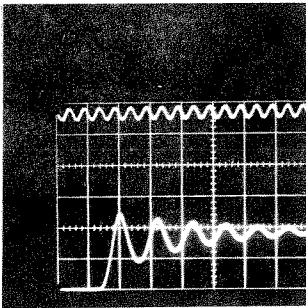
Test 14-4D λ = 0.050
x(0) = 0.500" p_o = 1.50k
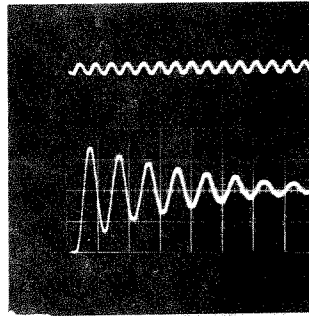
Test 15-1D λ = 0.043
x(0) = 0.065" p_o = 0.64k
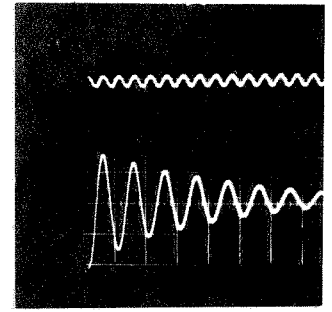
Test 15-2D λ = 0.038
x(0) = 0.115" p_o = 0.98k
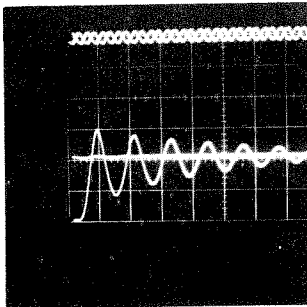
Test 15-3D λ = 0.047
x(0) = 0.300" p_o = 1.35k
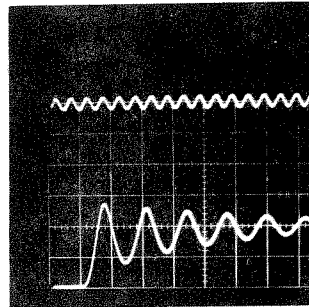
Test 15-4D λ = 0.041
x(0) = 0.525" p_o = 1.60k
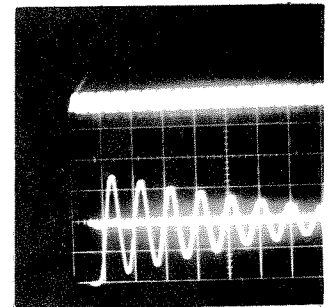
Test 16-1D λ = 0.027
x(0) = 0.050" p_o = 0.61k
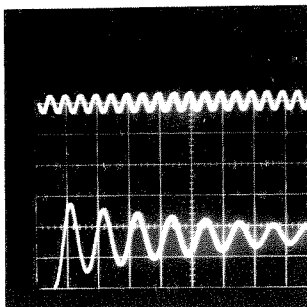
Test 16-2D λ = 0.030
x(0) = 0.100" p_o = 1.00k
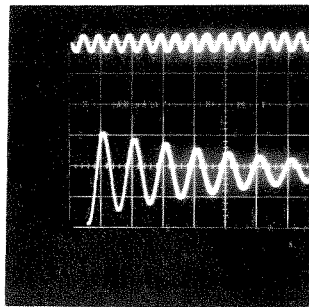
Test 16-3D λ = 0.030
x(0) = 0.200" p_o = 1.50k
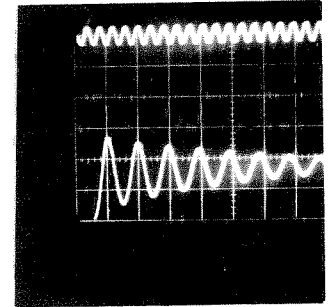
Test 16-4D λ = 0.031
x(0) = 0.312" p_o = 1.68k
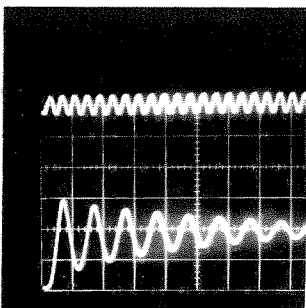
Test 16-5D λ = 0.035
x(0) = 0.400" p_o = 1.75k
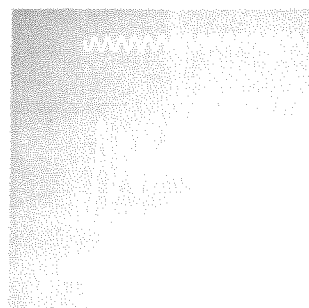
Test 17-1D λ = 0.051
x(0) = 0.044" p_o = 0.78k
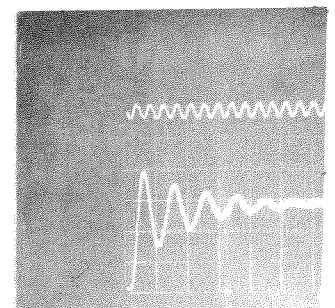
Test 17-2D λ = 0.064
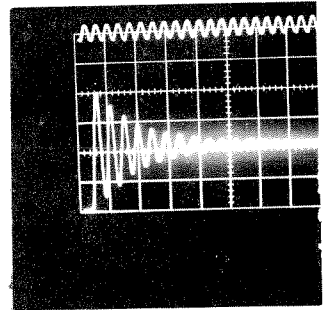x(0) = 0.114" p_o = 1.10k

Figure 86   Free vibration response - Beam Nos. 14-17
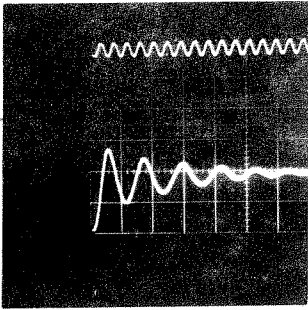
Test 17-3D $\lambda = 0.060$
$x(0) = 0.204"$ $p_o = 1.65k$
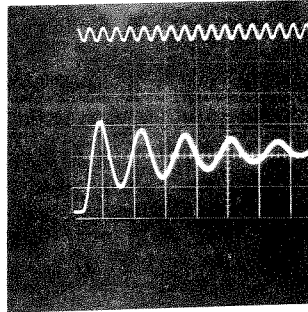
Test 17-4D $\lambda = 0.050$
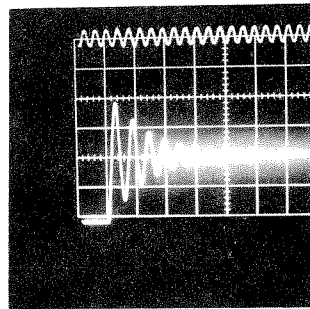$x(0) = 0.302"$ $p_o = 2.28k$
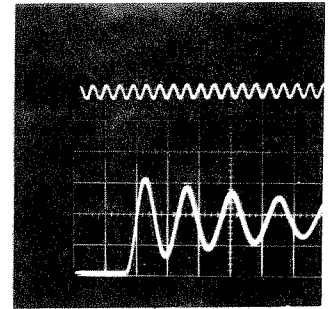
Test 17-5D $\lambda = 0.034$
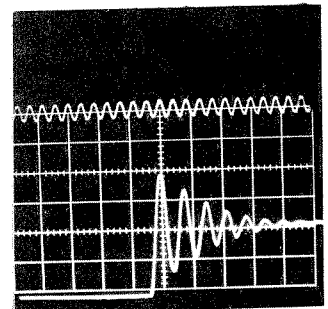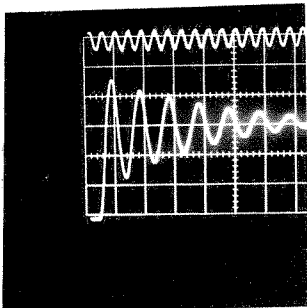$x(0) = 0.410"$ $p_o = 2.90k$
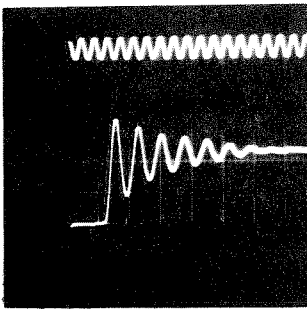
Test 18-1D $\lambda = 0.051$
$x(0) = 0.070"$ $p_o = 0.70k$
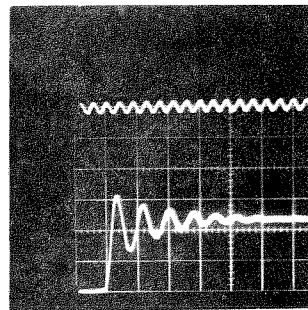
Test 18-2D $\lambda = 0.063$
$x(0) = 0.111"$ $p_o = 0.95k$
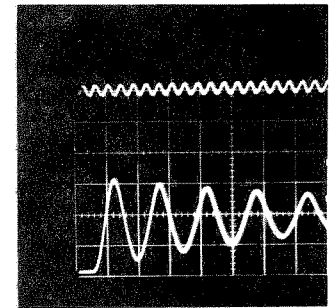
Test 18-3D $\lambda = 0.029$
$x(0) = 0.500"$ $p_o = 1.90k$
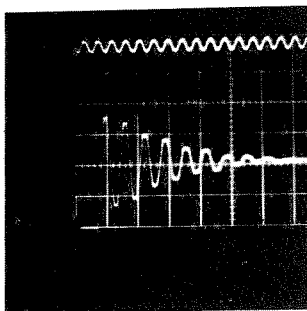
Test 19-1D $\lambda = 0.042$
$x(0) = 0.050"$ $p_o = 1.48k$
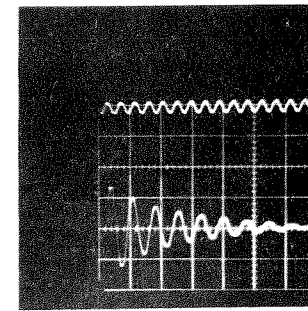
Test 19-2D $\lambda = 0.047$
$x(0) = 0.109"$ $p_o = 2.30k$
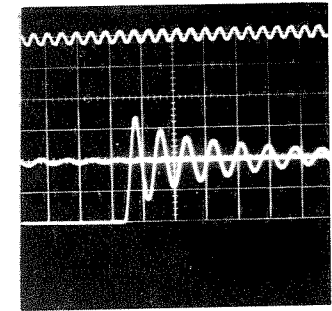
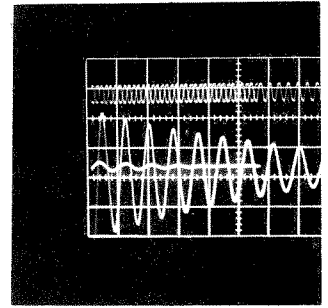Test 19-3D $\lambda = 0.044$
$x(0) = 0.210"$ $p_o = 4.10k$
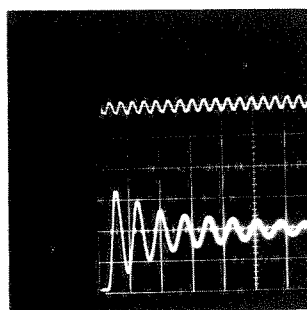
Test 19-4D $\lambda = 0.044$
$x(0) = 0.530"$ $p_o = 8.43k$
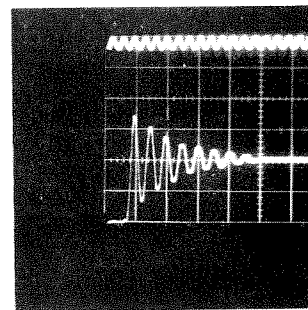
Test 20-1D $\lambda = 0.048$
$x(0) = 0.060'$ $p_o = 1.28k$
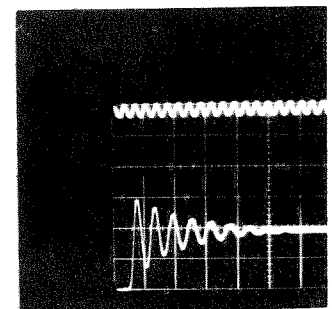
Test 20-2D $\lambda = 0.051$
$x(0) = 0.102"$ $p_o = 1.52k$

Figure 87   Free vibration response - Beam Nos. 17-20

Test 20-3D   λ = 0.046
x(0) = 0.205"   p_o = 2.40k

Test 20-4D   λ = 0.032
x(0) = 0.420"   p_o = 4.20k

Test 21-1D   λ = 0.042
x(0) = 0.039"   p_o = 1.40k
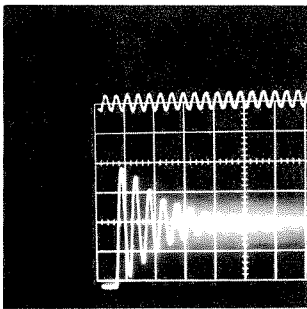
Test 21-2D   λ = 0.043
x(0) = 0.94"   p_o = 3.30k

Test 21-3D   λ = 0.052
x(0) = 0.274"   p_o = 5.20k

Test 21-4D   λ = 0.060
x(0) = 0.500"   p_o = 6.00k

Test 22-1D   λ = 0.050
x(0) = 0.200'   p_o = 2.00k
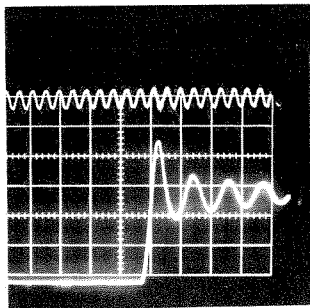
Test 22-2D
x(0) = 0.369"   p_o = 3.30k

Test 22-3D   λ = 0.020
x(0) = 0.369"   p_o = 3.30k
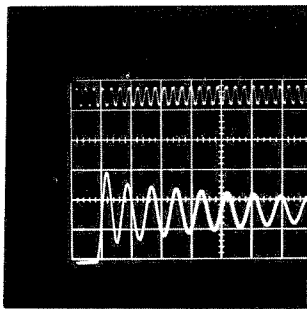
Test 22-4D   λ = 0.024
x(0) = 0.560'   p_o = 5.20k

Test 23-1D   λ = 0.030
x(0) = 0.060"   p_o = 3.28k
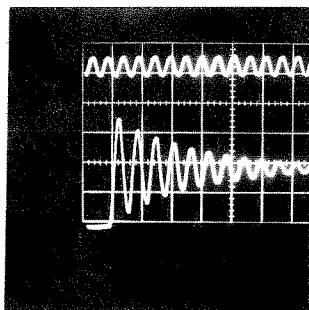
Test 23-2D   λ = 0.024
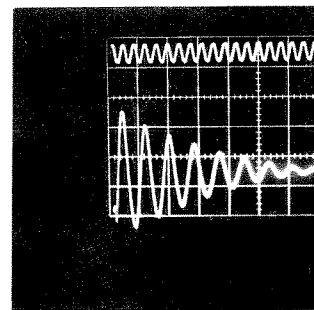x(0) = 0.095"   p_o = 5.20k

Figure 88   Free vibration response - Beam Nos. 20-23
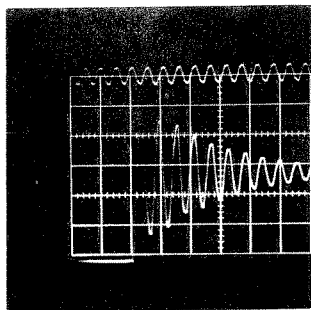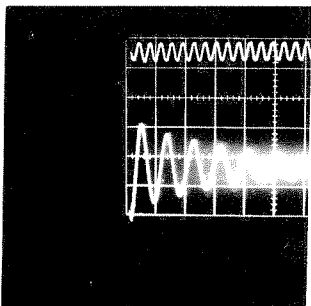
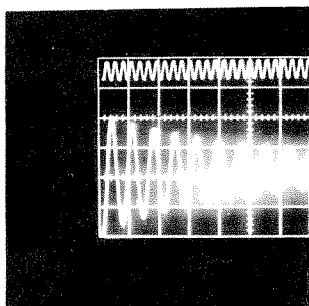Test 23-3D $\lambda = 0.029$
$x(0) = 0.104'$ $p_o = 5.05k$

Test 23-4D $\lambda = 0.035$
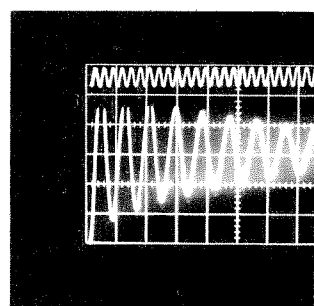$x(0) = 0.250'$ $p_o = 5.00k$

Test 24-1D $\lambda = 0.044$
$x(0) = 0.140'$ $p_o = 2.00k$

Test 24-2D $\lambda = 0.051$
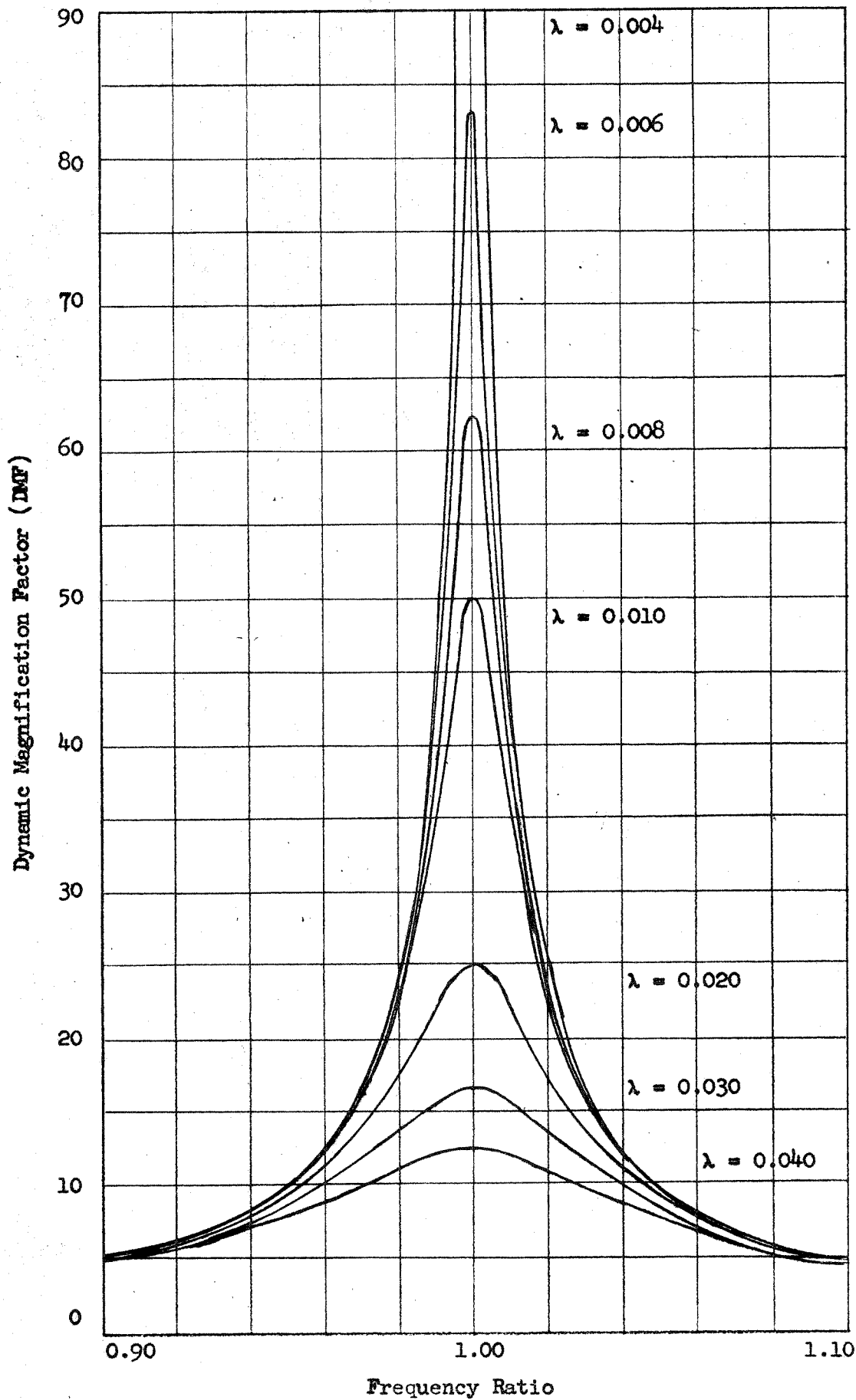$x(0) = 0.316'$ $p_o = 3.21k$

Test 24-3D $\lambda = 0.061$
$x(0) = 0.477'$ $p_o = 4.59k$

Test 24-4D $\lambda = 0.025$
$x(0) = 0.730'$ $p_o = 5.60k$

Figure 89   Free vibration response - Beam Nos. 23-24

Dynamic Magnification Factor vs. Frequency Ratio
for a single degree system

Figure 90