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Journal

American journal of physics, 59(12)

Author

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Publication Date

1991

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Materials & Chemical Sciences Division

Submitted to American Journal of Physics

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January 1991

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

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Berry's Geometric Phase and Motional Fields

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This work was supported by the U.S. Department of Energy under Contract DE-AC03-76F00098, through the Lawrence Berkeley Laboratory's Exploratory Research and Development Program; and by the Director, Office of Energy Research, Office of Basic Energy Sciences, Chemical Sciences Division; and in part by the Office of High Energy and Nuclear Physics, Division of High Energy Physics and Division of Nuclear Physics.

BERRY'S GEOMETRIC PHASE AND MOTIONAL FIELDS

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ABSTRACT

An experiment is described that provides an example of how Berry's geometric phase arises from motion of a particle with a magnetic moment in an electric field and an inhomogeneous magnetic field.

Considerable attention has been paid in recent years to Berry's phase¹⁻⁴. In the simplest circumstances this quantity arises in the following way. Suppose that a quantum mechanical system possesses a Hamiltonian $H[R(t)]$ that depends on a set of parameters, (represented by vector R in parameter space) which vary continuously and slowly with time. Suppose that at some initial time $t=0$, $R=R_0$ and the system is described by a wave function ψ_0 . Now suppose that after a certain time T , $R(T)$ returns to its original values R_0 . Then the wave function at time T is not in general equal to

$\psi_0 \exp[-i/\hbar \int_0^T E(t) dt]$, but differs from it (even in the adiabatic limit) by a phase factor

$e^{i\alpha}$, where α is Berry's phase. In particular, we consider the Hamiltonian $H = -\underline{\mu} \cdot \underline{B}$, where $\underline{\mu}$ is a magnetic moment, and $\underline{B}(t)$ is a time-dependent magnetic field. If $\underline{B}$ varies continuously and slowly with time, and at time $t=T$ returns to its initial value, then it can be shown¹ that Berry's phase is

$$\alpha_m = -m\Omega \quad (1)$$

where m is the magnetic quantum number of state ψ , and Ω is the solid angle subtended by the tip of the magnetic field vector with respect to its point of origin as it traces out its closed path between $t=0$ and $t=T$.

In this note I would like to describe an experiment and its interpretation that provide an example of the foregoing, in which at least a part of the time-dependent magnetic field is motional, which means that it arises from motion of the particle of interest in an electric field. There is really nothing fundamentally new or startling about this situation, but it is described here because it may be of some pedagogical interest.

The observations were carried out in the course of a continuing experiment to search for the electric dipole moment of the electron, an experiment that actually makes use of atomic beams of neutral thallium-205 in the $F=1$ hyperfine component of the

$6^2P_{1/2}$ ground state⁵. However the point of interest to us in the present note can be demonstrated most simply by considering instead a hypothetical experiment with a mono-energetic beam of spin 1/2 particles, for example neutrons.

However, since this hypothetical experiment involves the interaction of a moving neutron with electric and magnetic fields, it is appropriate to recall in general terms how this interaction should be described, before we consider the experiment itself. Thus, we start with the Dirac equation for a neutral particle of spin 1/2 and magnetic moment μ . This moment is entirely anomalous and enters the Dirac equation only in a Pauli moment term. The equation is:

$$m\psi + \gamma_\mu \partial_\mu \psi - \frac{\mu}{2} \sigma_{\mu\nu} F_{\mu\nu} \psi = 0 \quad (2)$$

where we set $\hbar = c = 1$ and employ the "Pauli metric"⁶. Let

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

where ψ_A and ψ_B are the "large" and "small" two-component spinors in the standard representation. Then (2) can be written:

$$\begin{aligned} \sigma \cdot \hat{\mathbf{p}} \psi_B + i\mu \sigma \cdot \mathbf{E} \psi_B &= (W + \mu \sigma \cdot \mathbf{B}) \psi_A \\ \sigma \cdot \hat{\mathbf{p}} \psi_A - i\mu \sigma \cdot \mathbf{E} \psi_A &= (W - \mu \sigma \cdot \mathbf{B} + 2m) \psi_B \end{aligned} \quad (3)$$

where $\hat{\mathbf{p}}$ is the momentum operator, and $W = E - m$, where E is the total energy. For a non-relativistic particle and for physically attainable magnetic fields $\mathbf{B}$, $W - \mu \sigma \cdot \mathbf{B} \ll 2m$. Thus in the last equation, we replace $W - \mu \sigma \cdot \mathbf{B} + 2m$ by $2m$ to obtain:

$$\psi_B = \frac{\sigma \cdot \hat{\mathbf{p}} - i\mu \sigma \cdot \mathbf{E}}{2m} \psi_A$$

The first of equations (3) then yields:

$$W \psi_A = -\mu \sigma \cdot \mathbf{B} \psi_A + \frac{1}{2m} (\sigma \cdot \hat{\mathbf{p}} + i\mu \sigma \cdot \mathbf{E})(\sigma \cdot \hat{\mathbf{p}} - i\mu \sigma \cdot \mathbf{E}) \psi_A \quad (4)$$

In the last term of (4) we ignore the contribution proportional to μ^2 , employ the identity $\sigma \cdot \mathbf{a} \sigma \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i \sigma \cdot \mathbf{a} \times \mathbf{b}$ and also assume that $\mathbf{E}$ is independent of spatial coordinates, so that $\mathbf{E} \cdot \hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot \mathbf{E}$. Then making use of:

$$i\sigma \cdot (\hat{\mathbf{p}} + i\mu\mathbf{E}) \times (\hat{\mathbf{p}} - i\mu\mathbf{E}) = 2\mu \sigma \cdot \hat{\mathbf{p}} \times \mathbf{E}$$

we find that (4) becomes:

$$W\psi_A = \frac{\hat{\mathbf{p}}^2}{2m} \psi_A - \mu \sigma \cdot \mathbf{B} \psi_A - \frac{\mu}{mc} \sigma \cdot \mathbf{E} \times \hat{\mathbf{p}} \psi_A \quad (5)$$

where in (5) c has been written explicitly. Now we consider a plane wave solution corresponding to a particle with definite momentum $\mathbf{p} = m\mathbf{v}$. Then (5) becomes:

$$W = \frac{p^2}{2m} - \mu \sigma \cdot \mathbf{B} - \mu \sigma \cdot \frac{\mathbf{E} \times \mathbf{v}}{c} \quad (6)$$

The meaning of (6) is of course quite obvious: in the laboratory frame the energy of the particle, apart from its rest energy, is given by $p^2/2m$, and two other contributions: one arising from interaction of the magnetic moment with the actual magnetic field $\mathbf{B}$, and another arising from interaction with the motional magnetic field $\mathbf{E} \times \mathbf{v}/c$.

We are now ready to describe our hypothetical experiment with neutrons. Imagine, then, a beam of non-relativistic neutrons, all with velocity $\mathbf{v}$ ($\ll c$) in the x direction, (see Fig. 1) and with spins prepared initially along the $+z$ direction:

$$\psi_{\text{initial}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (7)$$

This beam traverses an apparatus of the "Ramsey" type⁷ with separated oscillating fields for magnetic resonance, the principal features of which are as follows:

- a) A uniform magnetic field $\mathbf{B}_0$ in the z direction is imposed throughout.
- b) An inhomogeneous magnetic field $\mathbf{B}' = B' \hat{\mathbf{i}}$ is applied in the x direction by a pair of coils C, C' with oppositely directed currents. $\mathbf{B}'$ is odd with respect to reflection about the midpoint (x_5) of the apparatus (see Fig.2), and $|\mathbf{B}'| \ll |\mathbf{B}_0|$.
- c) An electric field $\mathbf{E}$ in the z direction is applied with a pair of parallel plates, and exists between x_3 and x_7 . The neutrons enter and leave the electric field "rapidly" but "adiabatically". By rapid we mean that over the short lengths in which $\mathbf{E}$ varies between zero and full strength, the variation in $\mathbf{B}'$ can be neglected. By adiabatic we mean that the neutron magnetic moment precesses many cycles in $\mathbf{B}_0$ in the time it takes for a neutron to traverse either of these short lengths. It is assumed that $|\mathbf{B}_0| \gg |\mathbf{E} \times \mathbf{v}/c|$.
- d) There is a pair of radio-frequency regions RF1, RF2 within which exist rf magnetic fields that rotate in the xy plane and are tuned to the magnetic resonance frequency:

$$\omega_0 = \gamma B_0 \quad (8)$$

These fields are of equal magnitude, but differ in phase by $\pi/2$.

e) There exists a detector that is sensitive only to neutrons with spin up along z.

Let us explain the function of this apparatus by following a neutron as it travels through the apparatus starting from an initial point x_0 . First it passes through RF1, the effect of which may be described in the rotating frame by a spin-flipping rotation matrix:

$$M_1 = \begin{pmatrix} \cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\ \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} \quad (9)$$

We choose the rf field intensity and neutron transit time through RF1 so that $\beta = \pi/2$. Hence, immediately after RF1 the spinor becomes:

$$\psi = M_1 \psi_{\text{initial}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (10)$$

which means that the spin now points in the x direction in the rotating frame. The spinor of (10) evolves further as the neutron travels through the region of finite B' and finite E :

$$\psi \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\delta} \\ e^{i\delta} \end{pmatrix} \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i\delta \\ 1 + i\delta \end{pmatrix} \quad (11)$$

Here δ , the quantity of interest to us in this paper, is a small phase that depends on v , B' , E , and B_0 in a way that we shall shortly explain. But before we do, let us complete our description of the experiment, by accounting for RF2 and the detector. These components are included to make observation of the small quantity δ possible. The effect of RF2 in the rotating frame is also described by a 2x2 rotation matrix M_2 , which differs from M_1 because the two rf fields differ in phase by $\pi/2$:

$$M_2 = \begin{pmatrix} \cos \frac{\beta}{2} & i \sin \frac{\beta}{2} \\ i \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (12)$$

We apply matrix M_2 to the spinor of (11) to find the final state:

$$\Psi_{\text{final}} = \frac{1}{2}(1+i) \begin{pmatrix} 1-\delta \\ 1+\delta \end{pmatrix} \quad (13)$$

The signal S in the detector is proportional to the probability of finding a neutron, described by expression (13), with spin up:

$$S = |1-\delta|^2 \approx 1-2\delta \quad (14)$$

Now let us return to the question of what δ is, and how it depends on B' , B_0 , E , and v . We can see that δ is the extra phase acquired in the rotating frame by the $m=1/2$ component of the neutron spinor, because of its adiabatic evolution through a magnetic field (actual plus motional) that begins and ends as B_0 but is something else in-between. Hence it is clear that $-\delta$ is just Berry's phase. From (1) and (13) we have:

$$\delta = \frac{\Omega}{2} \quad (15)$$

As for Ω , it can be obtained very easily in the limit where $B_0 \gg B'$, $B_0 \gg vE/c$, by the following pictorial argument. We once again follow a neutron as it emerges from RF1 in Fig.1. At point x_1 on its path B' is negligible and the resultant magnetic field is just B_0 and in the z direction (see Fig. 1b). The neutron proceeds along its path to point x_2 where B' is non-zero. The resultant magnetic field vector now points to "2" in Fig.1b. The neutron proceeds further and enters the electric field rapidly but adiabatically at x_3 . The magnetic field thus acquires a y component, which is, to order v/c :

$$B_y = vE/c$$

As a result, immediately after point x_3 , the resultant magnetic field vector now points to "3" in Fig. 1b. As the neutron moves further along, B' increases for a time, but then decreases to zero at point x_5 , then reverses, and is negative at point x_6 . When the

neutron emerges from the electric field at x_7 , the component B_y rapidly but adiabatically goes to zero. Thus finally at point x_9 , the magnetic field is once again B_0 . It can be seen from Fig. 1b that the resultant magnetic field vector has traced out a closed curve, which subtends the solid angle:

$$\Omega = 2 B'(x_3) \frac{v}{c} \frac{E}{B_0^2} \quad (16)$$

Of course, Berry's phase, given in this example by (15) with (16), is a purely geometric phase. Hence it should not, and does not, contain any mention of Planck's constant, or of the neutron magnetic moment. However, there is a restriction: the derivation we have just given is valid only if the magnetic moment is sufficiently large, so that the neutron spin precesses many times in the regions where E is turning on and off (adiabatic approximation).

We have presented the foregoing discussion in terms of a beam of polarized neutrons. A closely analogous description applies for the experiment we have actually carried out, with ^{205}Tl atoms in the ground $6^2P_{1/2}$ state, with total angular momentum (including nuclear spin $1/2$) given by $F=1$. In the thallium experiment the rf fields and state selection procedure are set up in such a way that (13) is replaced by:

$$\psi = \begin{pmatrix} -\frac{1}{2} e^{i\pi/4} (1 - \delta) \\ -\frac{1}{\sqrt{2}} (1 + \delta) \\ \frac{1}{2} e^{-i\pi/4} (1 - \delta) \end{pmatrix} \quad (17)$$

Also the detector is set up to yield a signal S proportional to the sum of the probabilities that the atom is found in states $m=1$, $m=-1$:

$$S = 1 - 2\delta$$

Finally, since with thallium we are dealing with a system of angular momentum unity instead of one half, (15) is replaced by:

$$\delta = \Omega \quad (18)$$

with Ω given by (16) as before. We have made observations of δ , and the results are in excellent accord with (18).

The foregoing analysis is easily generalized to cover the case where B_0 is not necessarily much larger than B' or vE/c , and where the resultant magnetic field does not describe a closed loop (as in Fig. 1b), for example because B' does not quite return to zero at x_g . By employing an "instantaneous frame" with quantization axis along the resultant magnetic field (actual plus motional) at any point of the particle trajectory, we can calculate δ , as follows:

Let the magnetic field at a point $x=vt$ be $B_x=B'$, $B_y=vE/c$, $B_z=B_0$, and define the angles ϕ , θ by the expressions:

$$\phi = \text{Arctan} \frac{B_y}{B_x}, \quad \theta = \text{Arcos} \frac{B_z}{\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

Returning to the spin-1/2 case, let us also define the "instantaneous" basis states as:

$$u_+ = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix} \quad u_- = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix} \quad (19)$$

These represent neutron spin up, down with respect to $\mathbf{B}(x(t))$, respectively.

The Hamiltonian is:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = |\mu|g \mathbf{S} \cdot \mathbf{B}$$

where $\mathbf{S}$ is the spin operator. We have

$$H\psi = i\hbar\dot{\psi} \quad (20)$$

and $Hu_+ = E_+u_+$, $Hu_- = E_-u_-$, where $E_{\pm} = \pm\gamma [B_x^2 + B_y^2 + B_z^2]^{1/2}$. Let us express ψ in terms of the basis states $u_{\pm}$ as follows:

$$\psi = a_+u_+ \exp[-\frac{i}{\hbar} \int_0^t E_+(\tau)d\tau] + a_-u_- \exp[-\frac{i}{\hbar} \int_0^t E_-(\tau)d\tau] \quad (21)$$

Substituting (21) into (20), and taking into account that $a_{\pm}$, $u_{\pm}$, and $E_{\pm}$ all depend on the time, we obtain:

$$[\dot{a}_+u_+ + a_+\dot{u}_+] \exp(-\frac{i}{\hbar} \int_0^t E_+d\tau) + [\dot{a}_-u_- + a_-\dot{u}_-] \exp(-\frac{i}{\hbar} \int_0^t E_-d\tau) = 0 \quad (22)$$

We now multiply (22) on the left by $u_+^\dagger$. Noting that $u_+^\dagger u_+ = 1$, $u_+^\dagger u_- = 0$, we find :

$$\dot{a}_+ + a_+(u_+^\dagger \dot{u}_+) + a_-(u_+^\dagger \dot{u}_-) \exp\left(\frac{i}{\hbar} \int_0^t [E_+ - E_-] d\tau\right) = 0 \quad (23)$$

Dropping the third (rapidly oscillating) term on the left hand side of (23) in the adiabatic approximation, and employing (19) to calculate the second term, we obtain:

$$\dot{a}_+ + i\dot{\phi} \sin^2\left(\frac{\theta}{2}\right) a_+ = 0 \quad (24)$$

A similar calculation leads to the following result for a_- :

$$\dot{a}_- - i\dot{\phi} \sin^2\left(\frac{\theta}{2}\right) a_- = 0 \quad (25)$$

It is then clear that

$$\delta = \int_0^T \dot{\phi} \sin^2\left(\frac{\theta}{2}\right) dt \quad (26)$$

It is easy to show that

$$\dot{\phi} = \frac{B_x \dot{B}_y - B_y \dot{B}_x}{B_x^2 + B_y^2} \quad (27)$$

and

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} \left(1 - \frac{B_z}{B}\right) \quad (28)$$

where $B = (B_x^2 + B_y^2 + B_z^2)^{1/2}$.

Finally, in the limit where $B_z \gg B_{x,y}$ one obtains from (27) and (28):

$$\dot{\phi} \sin^2\left(\frac{\theta}{2}\right) \approx \frac{(B_x \dot{B}_y - B_y \dot{B}_x)}{4B_z^2} = \frac{v}{4B_z^2} (B_x B'_y - B_y B'_x) \quad (29)$$

where here the prime denotes differentiation with respect to x . Consequently, we have:

$$\delta = \frac{1}{4B_z^2} \frac{v}{c} \int_0^L \left(B_x \frac{\partial E}{\partial x} - E \frac{\partial B_x}{\partial x} \right) dx \quad (29)$$

The second term in (29) may be integrated by parts, yielding:

$$\int_0^L E \frac{\partial B_x}{\partial x} dx = [EB_x]_0^L - \int_0^L B_x \frac{\partial E}{\partial x} dx \quad (30)$$

However, the first term on the right hand side of (30) vanishes. Thus, we find that (29) becomes:

$$\delta = \frac{1}{2B_z^2} \frac{v}{c} \int_0^L \left(B_x \frac{\partial E}{\partial x} \right) dx \quad (31)$$

As we have already stated, the regions over which E varies are sufficiently short that we may neglect the variation of B_x in those regions. Consequently B_x may be extracted from the integral in (31). It is then easy to see that we obtain result (15) with (16) once again.

I would like to thank Stephen Ross and Conny Carlberg for many useful discussions. The experiment on which this note is based was supported by the U.S. Dept. of Energy under Contract DE-AC03-76SF00098, through the Lawrence Berkeley Laboratory's Exploratory Research and Development Program; and by the Director, Office of Energy Research, Office of Basic Energy Sciences, Chemical Sciences Division; and in part by the Office of High Energy and Nuclear Physics, Division of High Energy Physics and Division of Nuclear Physics.

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FIGURE CAPTIONS

1a. Schematic diagram of experimental setup used to observe Berry's phase arising from a motional field. The numbers 0,...,9 correspond to points $x_0, \dots, x_9$ referred to in text. Coils C,C' have currents $I, -I$ respectively that generate the magnetic field B' , (see Fig. 2).

1b.(Insert) Diagram showing how tip of resultant magnetic field (actual plus motional) depends on $x(t)$. The numbers 1,...,9 correspond to the points of Fig. 1a (see also Fig. 2).

2. Dependence of the magnetic field B' on x .

Figure 1a

Figure 1b

Figure 2

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