THE LOGIC OF COMPARATIVE CARDINALITY

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Abstract. This paper investigates the principles that one must add to Boolean algebra to capture reasoning not only about intersection, union, and complementation of sets, but also about the relative size of sets. We completely axiomatize such reasoning under the Cantorian definition of relative size in terms of injections.

§1. Introduction. Reasoning about the relative size of infinite sets has been a source of puzzles since at least Galileo [8]. Any consistent extension of the notion of relative size from finite to infinite sets must give us very different principles in the infinite compared to the finite. For two key principles that hold in the finite—that proper subsets are smaller than their supersets, and that sets in one-to-one correspondence have the same size—are inconsistent in the infinite. Cantor's theory of infinite cardinalities [3] maintains the latter principle at the expense of the former, while the more recent theory of infinite numerosities [12] does the reverse.

For logicians, a precise definition of relative size of sets raises an obvious question: can we completely axiomatize reasoning about the relative size of finite sets, of infinite sets, and of arbitrary sets in a formal set-theoretic language? Just as the laws for reasoning about intersection, union, and complementation of sets are captured by the laws of Boolean algebra, what are the laws one must add to Boolean algebra to capture reasoning about the relative size of sets according to the given definition?

In this paper, we answer this question for a particular language and definition of relative size. Our language (see Definition 2.1) allows us to build terms using the standard set-theoretic operations of intersection, union, and complementation, and to express that a set s is at least as big as a set t: $|s| \geq |t|$. Thus, we work with a comparative notion of size, prior to the reification of sizes as cardinal numbers. The semantics is given by the Cantorian definition: $|s| \geq |t|$ is true iff there is an injection from t into s.

This language has an alternative interpretation in terms of the relative *likelihood* of events, instead of the relative size of sets. We will exploit

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this connection to prove some of our main results. In essence, Cantorian reasoning about the relative size of *finite* size is the same as *probabilistic* reasoning about the relative likelihood of events, while Cantorian reasoning about the relative size of *infinite* sets is the same as what is called *possibilistic* reasoning [6] about the relative likelihood of events. Each type of likelihood reasoning has been axiomatized by itself [18, 9, 1, 5]. If we reinterpret these results in terms of cardinality, then reasoning about the comparative cardinality of finite sets and reasoning about the comparative cardinality of infinite sets have each been axiomatized by themselves.

The goal of this paper is to bridge the divide between finite and infinite and axiomatize reasoning about the cardinality of arbitrary sets. In § 2, we define our formal language and its interpretation in fields of sets. We then present our first axiomatization, which uses two extra predicates Fin and Inf to express that a set is finite or infinite. The axiomatization without these predicates is more complicated and saved for later. Both axiomatizations use the so-called finite cancellation axiom schema, which encodes an infinite sequence of axioms of exponentially growing length. In an appendix § A, we show how this schema can be replaced with the combination of a simple axiom and a simple rule. In § 3, we define models based on Boolean algebras to be used later and adapt to this context the representation theorem in the classic paper [11] by Kraft, Pratt, and Seidenberg. We also show the effective finite model property and as a corollary the decidability of our two logics (with or without the Fin and Inf predicates). In § 4, we construct canonical models from maximally consistent sets, as is common in proofs of completeness, leading in § 5 to the completeness of the system with extra predicates. In § 6, we first show in what sense finiteness and infiniteness of a set can be defined in the language with only cardinality comparisons between set terms. Then we finally define the axiomatic system mentioned in § 2 without the two extra predicates and prove its soundness and completeness. Lastly, we end with open problems in § 7.

Comparison to related work. Two strands of work related to ours are worth mentioning. The first is the study of computable fragments of set theory, as in [2, 7]. For example, consider the quantifier-free language with intersection and set difference as binary functions and membership, inclusion, and equality as binary predicates. When this language is interpreted on the universe of all sets in the obvious way, the satisfiability problem is decidable; in fact, more functions and predicates can be added without loss of decidability [2]. In particular, a cardinality comparison predicate can be added, resulting in a language very similar to ours [7]. However, the language is still different from ours, due to its lack of set-theoretic complementation. Moreover, the cited works do not provide any axiomatization.

Another strand is the work on extending syllogistic logic with cardinality comparison initiated by Lawrence Moss (see [14] for an introduction). In this setting, the language consists of sentences of the form "all x are y", "some x are y", "there are at least as many x as y", and "there are more x than y" with variables interpreted as subsets of an arbitrary set. In [13, 15], axiomatizations of the valid sentences (on finite or infinite domains) are provided. However, in this setting there are no sentential Boolean connectives, nor Boolean set operators except complementation. Thus, the expressivity of the syllogistic language with cardinality comparisons is much weaker than ours, though with the consequent advantage of having a tractable satisfiability problem.

§2. Formal setup and statement of main result.

Definition 2.1. Given a countably infinite set Φ of *set labels*, the *set terms* t and *formulas* φ of the language $\mathcal{L}$ are generated by the following grammar:

$$t ::= a \mid t^c \mid (t \cap t)$$
$$\varphi ::= |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi),$$

where $a \in \Phi$. The other sentential connectives $\vee$, $\to$, and $\leftrightarrow$ are defined as usual, and we use $\varphi \oplus \psi$ as an abbreviation for $(\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. Standard set-theoretic notation may be defined as follows:

- $\varnothing := t \cap t^c$;
- $t \subseteq s := |\varnothing| \geq |t \cap s^c|$;
- $t = s := (t \subseteq s \wedge s \subseteq t)$ and $t \neq s := \neg(t = s)$;
- $t \not\subseteq s := \neg(t \subseteq s)$ and $t \subsetneq s := (t \subseteq s \wedge s \not\subseteq t)$.

We also use $|s| \leq |t|$ for $|t| \geq |s|$, $|s| > |t|$ for $\neg|t| \geq |s|$, and $|s| = |t|$ for $|s| \geq |t| \wedge |t| \geq |s|$. For any $\Delta \subseteq \Phi$, let $\mathcal{L}(\Delta)$ be the fragment of $\mathcal{L}$ using only set labels in Δ, and let $T(\Delta)$ be the set of set terms generated by Δ.

Our models consist essentially of a collection of sets, some of which are assigned set labels from Φ.

Definition 2.2. A *field of sets* is a pair $\langle X, \mathcal{F}\rangle$ where X is a nonempty set and $\mathcal{F}$ is a collection of subsets of X closed under intersection and set-theoretic complementation. A *field of sets model* is a triple $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$ where $\langle X, \mathcal{F}\rangle$ is a field of sets and $V\colon \Phi \to \mathcal{F}$.

The satisfaction relation is defined in the obvious way.

Definition 2.3. Given a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$, we define a function $\widehat{V}$, which assigns to each set term a set in $\mathcal{F}$, by:

- $\widehat{V}(a) = V(a)$ for $a \in \Phi$;

- $\widehat{V}(t^c) = X \setminus \widehat{V}(t)$;
- $\widehat{V}(t \cap s) = \widehat{V}(t) \cap \widehat{V}(s)$.

We then define a satisfaction relation $\models$ as follows:

- $\mathcal{M} \models |t| \geq |s|$ iff there is an injection from $\widehat{V}(s)$ into $\widehat{V}(t)$;
- $\mathcal{M} \models \neg\varphi$ iff $\mathcal{M} \not\models \varphi$;
- $\mathcal{M} \models \varphi \wedge \psi$ iff $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$.

Given a class K of field of sets models, φ *is valid over* K iff $\mathcal{M} \models \varphi$ for all $\mathcal{M} \in \mathsf{K}$.

In Definition 6.12, we will define the cardinality comparison logic CardCompLogic. Our main result is that this logic is sound and complete.

THEOREM 2.4. *The cardinality comparison logic* CardCompLogic *is sound and complete with respect to field of sets models.*

The logic is somewhat complicated, so we will leave its definition for later. For now, it will be helpful to consider the expanded language $\mathcal{L}_{\text{Fin,Inf}}$ that adds predicates Fin and Inf that pick out the finite and infinite sets, respectively. Then the logic CardCompLogic can be obtained by eliminating Fin and Inf.

DEFINITION 2.5. Let $\mathcal{L}_{\text{Fin,Inf}}$ be the language extending $\mathcal{L}$ with two new unary predicates Fin and Inf using the following grammar:

$$t ::= a \mid t^c \mid (t \cap t)$$
$$\varphi ::= \text{Fin}(t) \mid \text{Inf}(t) \mid |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi),$$

where $a \in \Phi$.

Satisfaction can then be extended from $\mathcal{L}$ to $\mathcal{L}_{\text{Fin,Inf}}$ as follows.

DEFINITION 2.6. For any field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$, define the satisfaction relation $\models$ for $\mathcal{L}_{\text{Fin,Inf}}$ with the following two new clauses:

- $\mathcal{M} \models \text{Fin}(t)$ iff $\widehat{V}(t)$ is finite;
- $\mathcal{M} \models \text{Inf}(t)$ iff $\widehat{V}(t)$ is infinite.

It will be convenient for later use to divide the logic of cardinality comparison with Fin and Inf into two parts, the first of which gives basic comparison principles such as transitivity and the second of which involves additional principles such as that infinite sets are larger than finite sets.

DEFINITION 2.7. The basic comparison logic BasicCompLogic is the logic for $\mathcal{L}$ (or $\mathcal{L}_{\text{Fin,Inf}}$) with the following axiom schemas and rules:

(BC1) all substitution instances of classical propositional tautologies;
(BC2) $\neg\, |\varnothing| \geq |\varnothing^c|$;
(BC3) $|s| \geq |t| \vee |t| \geq |s|$;

(BC4) $(|s| \geq |t| \wedge |t| \geq |u|) \to |s| \geq |u|$;
(BC5) $|\varnothing| \geq |s \cap t^c| \to |t| \geq |s|$;
(BC6) $(|\varnothing| \geq |s| \wedge |\varnothing| \geq |t|) \to |\varnothing| \geq |s \cup t|$;
(BC7) if φ and $\varphi \to \psi$ are theorems, so is ψ;
(BC8) if $t = 0$ is provable in the equational theory of Boolean algebras, then $|\varnothing| \geq |t|$ is a theorem.

DEFINITION 2.8. The logic $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, the cardinality comparison logic with predicates Fin and Inf, consists of the axioms and rules of the basic comparison logic $\mathsf{BasicCompLogic}$ together with the following axiom schemas:

(A1) $\mathrm{Fin}(s) \oplus \mathrm{Inf}(s)$;
(A2) $\mathrm{Fin}(\varnothing) \wedge ((\mathrm{Fin}(s) \wedge \mathrm{Fin}(t)) \to \mathrm{Fin}(s \cup t))$;
(A3) $(\mathrm{Fin}(t) \wedge s \subseteq t) \to \mathrm{Fin}(s)$;
(A4) $(\mathrm{Fin}(s) \wedge \mathrm{Inf}(t)) \to |t| > |s|$;
(A5) $\bigwedge_{i=1}^{n} (\mathrm{Fin}(s_i) \wedge \mathrm{Fin}(t_i)) \to \mathsf{FC}_n(s_1, \cdots, s_n, t_1, \cdots, t_n)$ (for all $n \geq 1$);
(A6) $(\mathrm{Inf}(s) \wedge |s| \geq |t| \wedge |s| \geq |u|) \to |s| \geq |t \cup u|$;

Here $\mathsf{FC}_n(s_1, \cdots, s_n, t_1, \ldots, t_n)$ is what we call the *finite cancellation axiom.* To define this formula, first for each m such that $1 \leq m \leq n$, define the term S_m as the union of the terms of the form $s_1^{c_1} \cap s_2^{c_2} \cap \cdots \cap s_n^{c_n}$ where exactly m many c_i's are c and the rest are empty. Similarly define T_m with s replaced by t. Intuitively, S_n denotes the set of elements which are in exactly m many sets among the sets denoted by $s_1, s_2, \ldots, s_n$.

Then $\mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots, t_n)$ is defined by

$$\left(\bigwedge_{i=1}^{n} S_i = T_i \right) \to \left(\left(\bigwedge_{i=1}^{n-1} |s_i| \geq |t_i| \right) \to |t_n| \geq |s_n| \right).$$

The first four axioms set up the relations between finite sets and infinite sets—for example, that finite sets are smaller than infinite sets. Axioms (A5) and (A6) describe the distinct behavior of finite and infinite cardinal arithmetic. To understand (A5), suppose the condition expressed by $\bigwedge_{i=1}^{n} S_i = T_i$ is true, assuming that the sets denoted by the s_i's and t_i's are all finite. Note that to compute the sum K of the cardinalities of the sets denoted by the s_i's, instead of the most straightforward way of adding their cardinalities, we can consider how much each element contributes to K: if an element e is in k_e many sets denoted by the s_i's, then the contribution of this e is k_e, and then K is the sum of the k_e's. Thus,

$$\sum_{i=1}^{n} |V(s_i)| = \sum_{i=1}^{n} i \times |V(S_i)|$$

as S_i is precisely the set of elements that lie in exactly i many sets denoted by s_i's. The same holds for the t_i's. Then given that $\bigwedge_{i=1}^n S_i = T_i$ is true,

$$\sum_{i=1}^{n} |V(s_i)| = \sum_{i=1}^{n} i \times |V(S_i)| = \sum_{i=1}^{n} i \times |V(T_i)| = \sum_{i=1}^{n} |V(t_i)|.$$

Hence it is not hard to see that the consequent

$$\left(\bigwedge_{i=1}^{n-1} |s_i| \geq |t_i| \right) \to |t_n| \geq |s_n|$$

must be true in the same model.

Example 2.9. Let $A, B, C, D, E \subseteq X$ be disjoint and finite. Then it follows from

- $|A| \geq |B \cup C|$,
- $|B \cup E| \geq |A \cup C|$, and
- $|C \cup D| \geq |A \cup B|$,

that $|D \cup E| \geq |A \cup B \cup C|$. To see this, we only need to add the cardinalities of the inequalities on both sides, which leads to $|A|+|B|+|C|+|D|+|E| \geq |A|+|A|+|B|+|B|+|C|+|C|$. Hence by cancelling $|A|+|B|+|C|$ since they are finite, we get $|D|+|E| \geq |A|+|B|+|C|$. Thus $|D \cup E| \geq |A \cup B \cup C|$. In our system, this reasoning is captured by

$$\mathsf{FC}_4(a, b \cup e, c \cup d, a \cup b \cup c, b \cup c, a \cup c, a \cup b, d \cup e)$$

with $a, b, c, d, e \in \Phi$, as the antecedent of FC_4 follows from the assumption that these five sets are disjoint, which can be expressed by formulas like $|a \cap b| = \varnothing$.

Finally, (A6) captures the distinct absorption property (or non-additivity) of infinite sets. In terms of the analogy with relative likelihood, it is (A5) that matches probabilistic reasoning, as is shown in [11] and [17], while (A6) matches what is called possibilistic reasoning [6].

Theorem 2.10. $\mathsf{CardCompLogic}_{\mathrm{Fin},\mathrm{Inf}}$*, the cardinality comparison logic with predicates* Fin *and* Inf*, is sound and complete with respect to field of sets models.*

Remark 2.11. Admittedly, (A5) is an infinite sequence of axioms that are long and somewhat complicated. We remark here that (A5) can be replaced by the combination of the following axiom and rule:

(A7) $(\mathrm{Fin}(s) \wedge \mathrm{Fin}(t)) \to (|s| \geq |t| \leftrightarrow |s \cap t^c| \geq |t \cap s^c|)$;

(A8) where $a|t$ abbreviates $|t \cap a| = |t \cap a^c|$ for $a \in \Phi$, if $a|t \to \varphi$ is derivable, then φ is derivable, assuming that a does not occur in t or in φ.

Axiom (A7) is sometimes called the quasi-additivity axiom. Intuitively, it means that taking unions with a disjoint set does not change the ordering of sets by cardinality. Rule (A8) is slightly non-standard. Intuitively, $a|t$ says that set a splits t into two parts of the same size. In both [11] and [1], this is expressed by "a polarizes t". Hence (A8) is called the "polarizability rule" in [1] ((A8) also appeared, but not as a rule in a formal system, in [11]). Proof theoretically, (A8) allows us to assume without loss of generality that any set can be polarized with a fresh set when proving some formula φ. Semantically, the idea behind (A8) is the invariance of truth under duplication: for any sets a, b, $|a| \succeq |b|$ iff $|a \times \{0,1\}| \succeq |b \times \{0,1\}|$, and a is finite iff $a \times \{0,1\}$ is finite. This warrants the use of (A8), as to show that φ is true on all field-of-sets models, it is enough to focus on those models that come from duplication, in which every set in the field-of-sets can be polarized. The power of (A8) lies in the fact that with it, we can simulate the addition of overlapping sets so that we can count overlaps correctly. Hence all instances of (A5) are provable from the system with (A5) replaced by (A7) and (A8). This point is already made implicitly in [11] and explicitly in [1], but we give a direct syntactic proof in the appendix § A. To get a flavor of the strategy, see Figure 1 where the shaded area in the second picture has a cardinality equal to one fourth of the sum of the cardinalities of the three larger sets (say A, B, and C). To see this, first polarize all minimal regions like $A \cap B^c \cap C^c$ and $A \cap B \cap C$ into four parts, and then for regions that are contained in m of the sets A, B, and C, select m parts of those regions. For example, as shown in the diagram, three of the four parts of region $A \cap B \cap C$ are selected, while only one of the four in $A \cap B^c \cap C^c$ is selected. So we have replaced the non-disjoint union of the shapes A, B, and C by a disjoint union of the shaded areas, while keeping the same area up to a factor of $1/4$.

For the logic in the language $\mathcal{L}$, without the predicates Fin and Inf, it takes more work to capture the difference between finite and infinite cardinal arithmetic. The cardinality comparison logic CardCompLogic will also extend the same basic comparison logic BasicCompLogic. The key idea for the extra axioms is that of a set being "witnessed to be finite" or "witnessed to be infinite." For example, if

$$\mathcal{M} \models |s| \geq |s \cup t| \wedge |s \cap t| = |\varnothing|,$$

then s must denote an infinite set in $\mathcal{M}$. Since s denotes an infinite set, anything true of infinite sets must be true of s. One can think of the cardinality comparison logic CardCompLogic as being the same as the cardinality comparison logic $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ except with $\mathrm{Fin}(t)$ and $\mathrm{Inf}(t)$ being replaced by formulas that witness t to be finite or infinite, respectively.

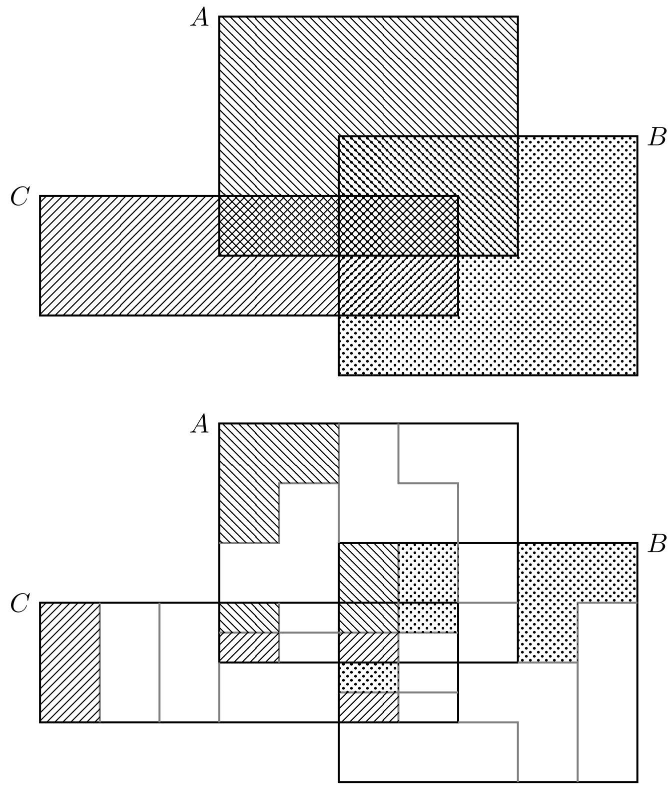

Figure 1. Polarization and set addition

Of course the definition of a set being finite or infinite would be superfluous if we were to restrict our models so that every set in the field of sets is finite (or infinite except for the empty set). In fact, letting FinCardCompLogic be the result of adding to BasicCompLogic all instances of $\mathsf{FC}_n(s_1, \dots, s_n, t_1, \dots, t_n)$, FinCardCompLogic is sound and complete with respect to all field of sets models where the underlying set is finite. Similarly, the system InfCardCompLogic defined by adding $(|s| \geq |t| \wedge |s| \geq |u|) \rightarrow |s| \geq |t \cup u|$ is sound and complete with respect to all field of sets models where all nonempty sets in the field are infinite. We will not formally prove these two completeness results since the strategy we use to prove the completeness of $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ can be readily adapted.

§3. Other types of models. While our interest is in field of sets models, it is convenient to think in terms of an abstract Boolean algebra rather than a concrete field of sets. For then we do not have to worry about which particular elements a set contains, but instead we only have to consider the cardinality of the set. This will help us focus on the structures related to the truth of formulas and to show the effective finite model property. In the following, we use $\wedge, \vee$, and $'$ for meet, join, and complementation in arbitrarily picked Boolean algebras. For specifically constructed Boolean algebras, the symbols for the operations may change and we will usualy specify only the complementation and meet operation. The lattice ordering of a Boolean algebra will be denoted by $\leq$ (below) and $\geq$ (above[1]), possibly with a subscript to show which Boolean algebra we are talking about. Since the models defined below are all Boolean algebras with extra structure, we call them *algebra-based models* and call them finite when the underlying Boolean algebra is finite. As a convenient notation, for any models $\mathcal{B}, \mathcal{C}$ and set L of formulas, we write $\mathcal{B} \equiv_L \mathcal{C}$ when for any $\varphi \in L$, $\mathcal{B} \models \varphi$ iff $\mathcal{C} \models \varphi$.

3.1. Measure algebra models. The first step is to forget the elements in sets and only keep their Boolean structure and their cardinality. This gives us the following definition.

DEFINITION 3.1. A *measure algebra* is a pair $\langle B, \mu \rangle$ where B is a Boolean algebra and μ is a function assigning a cardinal to each element of B such that

- if $a \wedge b = \bot$, then $\mu(a \vee b) = \mu(a) + \mu(b)$, and
- $\mu(b) = 0$ iff $b = \bot$.

We call such a cardinal-valued function μ with which $\langle B, \mu \rangle$ is a measure algebra a *cardinal measure* on B.

A *measure algebra model* is a triple $\mathcal{B} = \langle B, \mu, V \rangle$ where $\langle B, \mu \rangle$ is a measure algebra and V is a function from Φ to B. This V can be extended to a function $\widehat{V}$ from $T(\Phi)$ to B as in Definition 2.3 but using the Boolean complement and meet in place of set-theoretic complement and intersection.

Note that μ is only finitely additive, which is good enough because the language is finitary and unable to express countable additivity.

DEFINITION 3.2. Given a measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$, we define the satisfaction relation $\models$ as follows, where $\varphi, \psi \in \mathcal{L}$ and $s, t \in T(\Phi)$:

- $\mathcal{B} \models |t| \geq |s|$ iff $\mu(V(t)) \geq \mu(V(s))$;
- $\mathcal{B} \models \neg\varphi$ iff $\mathcal{B} \not\models \varphi$;
- $\mathcal{B} \models \varphi \wedge \psi$ iff $\mathcal{B} \models \varphi$ and $\mathcal{B} \models \psi$.

We also have the following two clauses for $\mathcal{L}_{\mathrm{Fin,Inf}}$ sentences:

- $\mathcal{B} \models \mathrm{Inf}(t)$ iff $\mu(\widehat{V}(t))$ is infinite;
- $\mathcal{B} \models \mathrm{Fin}(t)$ iff $\mu(\widehat{V}(t))$ is finite.

[1] We use "below" and "above" in the weak sense.

We can turn any field of sets $\langle X, \mathcal{F}\rangle$ into a measure algebra $\langle B, \mu\rangle$ by setting $B = \langle \mathcal{F}, X \setminus \cdot, \cap\rangle$ and $\mu(a) = |a|$ for $a \in B$. It is easy to check that this is a measure algebra. If we have a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$ then we get a measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$ using the same valuation V; moreover, $\mathcal{M} \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}$ by a simple induction.

On the other hand, given a *finite* measure algebra model $\mathcal{B}$, we can turn it into a field of sets model $\mathcal{M}$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}$. Since the cardinal measure functions in measure algebras are only finitely additive, the construction will fail for infinite measure algebra models.

Proposition 3.3. *For any finite measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$, there is a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V'\rangle$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}$.*

Proof. Since B is finite, let $a_1, \dots, a_n$ be the atoms of B. Let $S_1, \dots, S_n$ be disjoint sets with $|S_i| = \mu(a_i)$. Let $X = \bigcup_{i=1}^n S_i$ and let $\mathcal{F}$ be the field of sets generated by $S_1, \dots, S_n$ under complementation (in X) and intersection. The map $f(a_i) = S_i$ extends to an isomorphism between B and $\mathcal{F}$, which maps an element $a_{i_1} \vee \cdots \vee a_{i_\ell}$ to $S_{i_1} \cup \cdots \cup S_{i_\ell}$. We have

$$\begin{aligned}\mu(a_{i_1} \vee \cdots \vee a_{i_\ell}) &= \mu(a_{i_1}) + \cdots + \mu(a_{i_\ell}) \\ &= |S_{i_1}| + \cdots + |S_{i_\ell}| \\ &= |S_{i_1} \cup \cdots \cup S_{i_\ell}|.\end{aligned}$$

Thus, $|f(a)| = \mu(a)$. Let $V' = f \circ V$. Then it is easy to see that for any set term s, $|V(s)| = \mu(V'(s))$, since f is an isomorphism preserving cardinalities. A simple induction then shows that $\langle X, \mathcal{F}, V'\rangle \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}$. ⊣

Thus, there is no loss of generality when focusing on measure algebra models if we consider only finite models, and as we will see later in this section, there is also no loss of generality in restricting to finite models.

3.2. Comparison algebra models. As one often does to prove the completeness of some logic, we will use a canonical model construction. In building the canonical model, we start with a maximally consistent set of sentences in our language $\mathcal{L}$ or $\mathcal{L}_{\mathrm{Fin,Inf}}$, which encodes only comparisons between set terms and their being finite or infinite in the case of $\mathcal{L}_{\mathrm{Fin,Inf}}$. Hence it will be convenient to forget about the cardinals we assign to each element of the Boolean algebra and to remember only the comparisons between elements. When we need to work with $\mathcal{L}_{\mathrm{Fin,Inf}}$, we also need the model to contain a set of distinguished elements.

Definition 3.4. A *comparison algebra* is a pair $\langle B, \succeq\rangle$ where B is a Boolean algebra and $\succeq$ is a total preorder on B such that

- for all $a, b \in B$, $a \geq_B b$ implies $a \succeq b$, and
- $\bot_B \not\succeq b$ for all $b \in B \setminus \{\bot_B\}$.

A *labeled comparison algebra* is a triple $\langle B, \succeq, F\rangle$ where $\langle B, \succeq\rangle$ is a comparison algebra and $F \subseteq B$. A *comparison algebra model* is a triple $\mathcal{B} = \langle B, \succeq, V\rangle$ where $\langle B, \succeq\rangle$ is a comparison algebra and V a function from Φ to B. Similarly, a *labeled comparison algebra model* is a labeled comparison algebra model together with a valuation. The valuation V can be extended in the usual way to a valuation $\widehat{V}$ from $T(\Phi)$ to B.

Here the relation $\succeq$ is intended to interpret "at least as great in cardinality as" and F is intended to interpret "being a finite set". Hence we have the required constraints for $\succeq$ in the above definition: it is a total preorder, extends the Boolean lattice order (set inclusion relation), and makes the bottom element (the empty set) the strictly smallest set. Formally, the interpretation is given by the satisfaction relation.

Definition 3.5. Given a comparison algebra model $\mathcal{B} = \langle B, \succeq, V\rangle$, we define the satisfaction relation $\models$ for $\mathcal{L}$ as follows, where $s, t \in T(\Phi)$:

- $\mathcal{B} \models |s| \geq |t|$ iff $\widehat{V}(s) \succeq \widehat{V}(t)$;
- usual clauses for $\neg$ and $\wedge$.

Given a labeled comparison algebra model $\mathcal{B} = \langle B, \succeq, F, V\rangle$ the satisfaction relation $\models$ can be extended to $\mathcal{L}_{\mathrm{Fin,Inf}}$ with the extra clauses:

- $\mathcal{B} \models \mathrm{Fin}(s)$ iff $\widehat{V}(s) \in F$;
- $\mathcal{B} \models \mathrm{Inf}(s)$ iff $\widehat{V}(s) \notin F$.

While $\succeq$ is intended to compare cardinality and F is intended to include exactly finite elements, the requirements given above are not enough to let us know that it is cardinality that $\succeq$ is comparing and that elements in F are precisely those that are finite. Given a measure algebra $\langle B, \mu\rangle$, we can easily build a comparison algebra $\langle B, \succeq\rangle$ by taking $a \succeq b$ if and only if $\mu(a) \geq \mu(b)$ and further a labeled measure algebra $\langle B, \succeq, F\rangle$ by taking $F = \{b \in B \mid \mu(b) \text{ is finite}\}$. But for the other direction, to fix that $\succeq$ is comparing cardinality and F captures finiteness, we need some extra conditions. We state this in terms of a representation theorem.

Definition 3.6. A comparison algebra $\langle B, \succeq\rangle$ (labeled comparison algebra $\langle B, \succeq, F\rangle$) is *represented by* a cardinal measure μ on B if for all $a, b \in B$, we have $a \succeq b$ iff $\mu(a) \geq \mu(b)$ (and $F = \{b \in B \mid \mu(b) \text{ is finite}\}$). We also say $\langle B, \succeq\rangle$ or $\langle B, \succeq, F\rangle$ is represented by a measure algebra $\mathcal{B}'$ when it is represented by μ and $\mathcal{B}' = \langle B, \mu\rangle$. A (labeled) comparison algebra model is representable if its (labeled) comparison algebra part is representable.

Clearly if a finite (labeled) comparison algebra model $\langle \mathcal{B}, V\rangle$ is represented by a measure algebra $\langle \mathcal{B}', V\rangle$, then $\mathcal{B} \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}'$. Hence if φ is satisfiable on a finite representable (labeled) comparison algebra model, then, in light of Proposition 3.3, φ is satisfiable on a field of sets model.

Before proving the full representation, we recall the following classic theorem on when an ordering is representable by a probability measure.

THEOREM 3.7 (Kraft, Pratt, Seidenberg [11], Theorem 2). *For any finite Boolean algebra B with $\top$ as the top element and $\bot$ the bottom element and any binary relation $\succeq$ on B, there is a probability measure μ on B such that for all $a, b \in B$, $a \succeq b$ iff $\mu(a) \geq \mu(b)$, if and only if the following conditions are satisfied:*

- *not $\bot \succeq \top$;*
- *for all $b \in B$, $b \succeq \bot$;*
- *$\succeq$ is transitive, and for any $a, b \in B$, $a \succeq b$ or $b \succeq a$;*
- *for any two sequences of elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ from B of equal length, if every atom of B is below (in the order of the Boolean algebra) exactly as many a's as b's, and if $a_i \succeq b_i$ for all $i \in \{1, \ldots, n-1\}$, then $b_n \succeq a_n$.*

The fourth condition, known also as *finite cancellation*, is precisely the truth condition of FC_n. Put more algebraically, if we represent elements in B as their characteristic functions over the atoms of B and further identify those functions with vectors of 0's and 1's, then the vector sum of a's being the same as the vector sum of b's implies that the sum of the probabilities of a's is also equal to the sum of the probabilities of b's. This is of course because vector sums count overlaps properly, unlike unions. Dana Scott used this representation in [17] and provided a lucid proof of the above theorem. It can also be observed from the proof in [11] (see Corollary 2) or [17] that the probability measure μ can be turned into an additive function to non-negative rational numbers and then to natural numbers by scaling, since μ is obtained by solving a finite system of (possibly strict) linear inequalities with rational coefficients.

With this component dealing with finite elements, we can prove the representation theorem for both finite and infinite elements.

THEOREM 3.8. *A finite labeled comparison algebra $\mathcal{B} = \langle B, \succeq, F \rangle$ is represented by some cardinal measure μ on B if and only if the following conditions hold:*

(1) F is an ideal;
(2) elements in F satisfy the finite cancellation condition in Theorem 3.7: for any two sequences of elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ from F of equal length, if every atom of B is below (in the order of the Boolean algebra) exactly as many a's as b's, and if $a_i \succeq b_i$ for all $i \in \{1, \ldots, n-1\}$, then $b_n \succeq a_n$;
(3) for any $a, b, c \in B$ such that $a \notin F$, if $a \succeq b$ and $a \succeq c$, then $a \succeq b \vee_B c$;
(4) for any $a, b \in B$, if $a \in F$ and $b \notin F$, then $b \succeq a$ and not $a \succeq b$.

It is then easy to see that a finite comparison algebra $\mathcal{B} = \langle B, \succeq \rangle$ is represented by μ if and only if there exists an $F \subseteq B$ such that $\langle B, \succeq, F \rangle$ is represented by μ.

Proof. For the proof, we use the following definitions. For any Boolean algebra B and $b \in B$, let $At(B)$ be the set of all atoms in B and $At(b)$ the set of atoms below b. Given a preorder $\succeq$ on B and $b \in B$, let $[b]$ be $\{b' \in B \mid b' \succeq b \text{ and } b \succeq b'\}$. Then define $Rank(b)$ for $b \in B$ to be the number of atoms strictly smaller than b in the order $\succeq$, modulo equivalence, i.e., the cardinality of the set $\{[a] \mid a \in At(B), b \succ a\}$. Finally, let $\max At(b)$ be any one of the $\succeq$-maximal elements in $At(b)$, if there is such an element.

Since B is finite, F is a finite ideal and hence principal. Then the quotient $B|_F$ becomes a finite Boolean algebra with a binary relation that satisfies all the conditions required in Theorem 3.7 because we required that the finite cancellation condition holds for all elements in F. Hence there is an additive measure function μ_0 from $B|_F$ to $\mathbb{N}$ such that for any $b_1, b_2 \in B|_F$, we have $b_1 \succeq b_2$ iff $\mu_0(b_1) \geq \mu_0(b_2)$.

If $B = B|_F$ then we are done, so we now consider the case where $B \neq B|_F$. Consider an arbitrary element b outside $B|_F$. Because B is finite, it is atomic and hence $At(b)$ must contain an atom that is not in $B|_F$. Then by (4) this atom is strictly greater in $\succeq$ than all atoms in $B|_F$. Hence $\max At(b)$ is outside $B|_F$ (it exists because B is finite).

By definition, $\max At(b)$ is an atom under b. Thus $b \succeq \max At(b)$. But because $b \notin F$, we can also show that $\max At(b) \succeq b$ using condition (3). To see this, list $At(b)$ as $b_1, b_2, \dots, b_n$. Then we have the following inductive argument:

- $\max At(b) \succeq b_1$;
- supposing $\max At(b) \succeq \bigvee_{i=1}^{k} b_i$, then together with $\max At(b) \succeq b_{k+1}$ and condition (3), $\max At(b) \succeq \bigvee_{i=1}^{k} b_i \vee b_{k+1} = \bigvee_{i=1}^{k+1} b_i$.

Hence at the end of the induction we have $\max At(b) \succeq \bigvee_{i=1}^{n} b_i = b$.

Now define a measure μ on B as follows:

$$\mu(b) = \begin{cases} \mu_0(b) & b \in B|_F \\ \aleph_{Rank(\max At(b))} & b \notin B|_F. \end{cases}$$

It is not hard to see that this is indeed a measure function on B. Moreover, we now show that for any $b_1, b_2 \in B$, we have $b_1 \succeq b_2$ iff $\mu(b_1) \geq \mu(b_2)$:

- if both $b_1, b_2 \in B|_F$, then we can use μ_0;
- if both $b_1, b_2 \notin B|_F$, then $b_1 \succeq b_2$ iff $\max At(b_1) \succeq \max At(b_2)$ iff $Rank(\max At(b_1)) \geq Rank(\max At(b_2))$ iff $\mu(b_1) \geq \mu(b_2)$;
- if $b_1 \in B|_F$ and $b_2 \notin B|_F$, then $b_2 \succeq b_1$ by condition (4), but it is also trivially true that $\mu(b_2) \geq \mu(b_1)$.

⊣

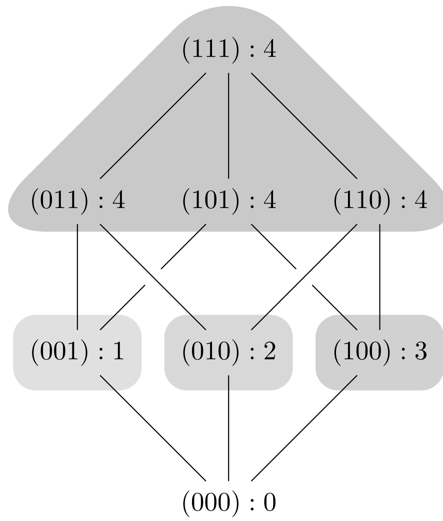

FIGURE 2. A non-representable comparison algebra

The above theorem again only works for finite models. Representation of infinite models requires different conditions and techniques to prove. We will review this as an open problem in § 7.

EXAMPLE 3.9. Figure 3.2 presents a comparison algebra that cannot be represented by any cardinal measure. The number after the colon in each node is the rank of that node in $\succeq$; this determines the preorder $\succeq$, as $x \succeq y$ iff the rank of x is at least that of y. We also group the nodes of the same rank into shaded areas. We can then see that if $\succeq$ were representable by a cardinal measure, all nodes would be finite. To see this, note that (110) is the join of (010) and (100), but (110) is also strictly greater than both of them, which implies that (110) is finite. Now (110) is of rank 4, which means it is as large as any other set, so all the sets would be finite. However, we can also see that, letting $a_1 = (101), a_2 = (010), b_1 = (110), b_2 = (001)$, the finite cancellation condition fails. First, every atom is below exactly one of a_1 and a_2 and also exactly one of b_1 or b_2. Thus, the antecedent of finite cancellation is true. But the consequent is false, as $a_1 \succeq a_2$ but not $b_2 \succeq b_1$. The non-representability of this comparison algebra also implies, by the previous theorem, that the required ideal does not exist.

3.3. Effective finite model property. In the previous two subsections, we saw that our representation theorems (Proposition 3.3 and Theorem 3.8) only work on finite models. However, to make a formula true, we only need finite models. In fact, we can effectively bound the size of satisfying models for any formula that is satisfiable by some (possibly infinite) field of sets model. Since the construction of a finite satisfying model will be used later, we provide a systematic treatment, starting with the following definition.

Definition 3.10. Let $\mathcal{B}$ be an algebra-based model and $\Delta \subseteq \Phi$. $\mathcal{B}$ is *adapted to* Δ if $\widehat{V}$ is surjective from $T(\Delta)$ to the underlying Boolean algebra B in $\mathcal{B}$.

The importance of this definition is that for any algebra-based model $\mathcal{B}$ that is adapted to Δ, every element $b \in \mathcal{B}$ is named in the sense that there exists $t \in T(\Delta)$ such that $\widehat{V}(t) = b$. It is easy to see that an algebra-based model adapted to a finite set Δ is finite. To be more precise, when $\Delta \subseteq \Phi$ is finite, let $T_0(\Delta)$ be the set of all distinct terms in Δ in disjunctive normal form with no repetition of conjuncts or disjuncts. $T_0(\Delta)$ is finite, and using Boolean identities, for every term $t \in T(\Delta)$, there is a term $t' \in T_0(\Delta)$ such that $\widehat{V}(t)$ is always the same as $\widehat{V}(t')$. Thus, in any model, $\widehat{V}(T(\Delta)) = \widehat{V}(T_0(\Delta))$ is finite.

Proposition 3.11. *Fix a finite set $\Delta \subseteq \Phi$. For any measure (resp. comparison, labeled comparison) algebra model $\mathcal{B}$, there is a measure (resp. comparison, labeled comparison) algebra model $\mathcal{B}_\Delta$ that is adapted to Δ and satisfies $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$.*

Proof. For any measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$, define $\mathcal{B}_\Delta = \langle B_\Delta, \mu_\Delta, V_\Delta\rangle$ where the B_Δ is the subalgebra of B with $\widehat{V}(T(\Delta))$ as the carrier set, μ_Δ is the restriction of μ to $\widehat{V}$, and V_Δ is defined as:

$$V_\Delta(a) = \begin{cases} V(a) & a \in \Delta \\ \bot_B & a \notin \Delta. \end{cases}$$

Similarly, for any comparison algebra model $\mathcal{B} = \langle B, \succeq, V\rangle$, we can define $\mathcal{B}_\Delta = \langle B_\Delta, \succeq_\Delta, V_\Delta\rangle$ where now $\succeq_\Delta$ is the restriction of $\succeq$ to $\widehat{V}(T(\Delta))$. It is not hard to see that $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$. For labeled comparison algebras, we just need to further define $F_\Delta = F \cap \widehat{V}(T(\Delta))$. ⊣

Now we prove the effective finite model property.

Theorem 3.12. *For any formula $\varphi \in \mathcal{L}_{\text{Fin,Inf}}$, φ is satisfied by some field of sets model if and only if it is satisfied by a labeled comparison algebra model $\langle B, \succeq, F, V\rangle$ such that:*

1. *B is finite with at most $2^{|\Delta|}$ many atoms where Δ is the set of set labels appearing in φ, and*
2. *$\langle B, \succeq, F\rangle$ is representable (i.e., satisfies the conditions listed in Theorem 3.8).*

The right-to-left direction does not require any bound on the size of B, so long as it is finite. In addition, it is decidable whether a finite labeled comparison algebra model is representable. The complexity of deciding whether a finite labeled comparison algebra model is representable is NP in the size of the underlying Boolean algebra of the labeled comparison algebra model.

Proof. The right-to-left direction is immediate by Theorem 3.8 and Proposition 3.3. For the left-to-right direction, suppose $\mathcal{M} = \langle X, \mathcal{F}, V\rangle \models \varphi$, and let Δ be the set of set labels appearing in φ. As we described above, the field of sets model can be naturally turned into a measure algebra model $\mathcal{B} = \langle \mathcal{F}, \mu, V\rangle$ and then into a labeled comparison algebra model $\mathcal{C} = \langle \mathcal{F}, \succeq, F, V\rangle$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}, \mathcal{C}$. By adapting $\mathcal{C}$ to Δ using Proposition 3.11, we obtain $\mathcal{C}_\Delta$ that satisfies the same formulas in $\mathcal{L}_{\text{Fin,Inf}}(\Delta)$, which includes φ. Note that $\mathcal{C}_\Delta$ is represented by $\mathcal{B}_\Delta$. Also, since $\mathcal{B}_\Delta$ is adapted to Δ, the size of the Boolean algebra base of $\mathcal{B}_\Delta$ is at most $2^{2^{|\Delta|}}$, and there are at most $2^{|\Delta|}$ many atoms.

To decide whether a finite labeled comparison algebra $\langle B, \succeq, F, V\rangle$ is representable, the only non-trivial part is to verify whether $B|_F$ satisfies the finite cancellation condition. However, rather than using this characterization, it is easier to naively check the definition of representability; this is an integer linear programming problem with at most $2^{|\Delta|}$ (the number of atoms in B) many variables and $(2^{2^{|\Delta|}})^2$ many inequalities, the coefficients of which are all in $\{0, 1, -1\}$. According to a standard result on integer linear programming (see [16]), the complexity is NP in the size of B. $\dashv$

As a simple corollary, the sets of sentences in $\mathcal{L}$ or $\mathcal{L}_{\text{Fin,Inf}}$ valid on all field of sets models are decidable, as to decide whether φ is satisifiable (or equivalently whether $\neg\varphi$ is valid) we can enumerate all labeled comparison algebra models of size up to $2^{2^{|\Delta|}}$ and for each one first check if it is representable and then check if φ is satisfied. If φ is never satisfied in this procedure, then φ is in fact unsatisfiable.

§4. Canonical comparison algebra models. This section is devoted to the construction of canonical comparison algebra models. In Definition 2.7, we formulated a logic that captures our basic intuitions about the comparison of "sizes" of sets. For example, (BC2) says that the complement of the empty set is strictly larger than the empty set, which we should assume as otherwise all set terms would simply be empty. (BC3) and (BC4) together provide the order structure of the "sizes" of sets: it is a total preorder. (BC5) and (BC6) gives two basic interactions between set construction and size comparison: (BC5) says that if a set is a subset of another, then the size of the subset should be no greater than that of the superset, and (BC6) says that the union of two empty sets (those with sizes no greater than the empty set) is still empty. (This is the union of the empty set with itself, though two different set terms denote it.)

Those axioms may capture some notion of "size" comparison, but they are not enough to completely capture the notion of "cardinality" comparison. Loosely speaking, we may treat "cardinality" as a special kind of "size", less general but perhaps more interesting, due to the distinct behaviors of finite and infinite cardinalities. The following theorem says that the basic comparison logic BasicCompLogic captures precisely the notion of "size" comparison in comparison algebra models defined in Definition 3.4. This will be useful, as to show that CardCompLogic captures the notion of "cardinality" comparison, we only need to show that the difference between the two logics captures the difference between the models: the extra properties identified in Theorem 3.8 that make a comparison algebra model *representable* by a cardinal measure.

LEMMA 4.1. BasicCompLogic *derives the following for terms* s, t, u, s', t'*:*

1. $s = t$ *if it is provable in the equational theory of Boolean algebras.*
2. $\subseteq$ *is a preorder:* $s \subseteq s$, $(s \subseteq t \wedge t \subseteq u) \to s \subseteq u$.
3. $=$ *is an equivalence relation:* $s = s$, $s = t \to t = s$, $(s = t \wedge t = u) \to s = u$.
4. $\subseteq$ *works as the subset relation:* $s \subseteq t \to t^c \subseteq s^c$, $s \subseteq t \to (s \cap u) \subseteq t$, $(s \subseteq t \wedge s \subseteq u) \to s \subseteq (t \cap u)$, *and* $(s \subseteq u \wedge t \subseteq u) \to (s \cup t) \subseteq u$.
5. $=$ *is a congruence relation:* $s = t \to s^c = t^c$ *and* $(s = s' \wedge t = t') \to (s \cap t) = (s' \cap t')$. *With axioms in* $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, $s = t \to (\mathrm{Fin}(s) \leftrightarrow \mathrm{Fin}(t))$ *and* $s = t \to (\mathrm{Inf}(s) \leftrightarrow \mathrm{Inf}(t))$ *are also derivable.*
6. *Substitution of equal terms:* $s = t \to (\varphi \leftrightarrow \psi)$ *for all formulas* φ *and* ψ *in* $\mathcal{L}$ *where* ψ *is obtained from* φ *by replacing one or more occurrences of* s *by* t*. When using* $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$*, this substitution schema is valid for all* $\varphi \in \mathcal{L}_{\mathrm{Fin,Inf}}$.

PROOF. If $s = t$ is provable in the equational theory of Boolean algebras, then so are $0 = s \cap t^c$ and $0 = t \cap s^c$. Then by axiom (BC8), both $|\varnothing| \geq |s \cap t^c|$ and $|\varnothing| \geq |t \cap s^c|$ are provable. But they are abbreviated by $s \subseteq t$ and $t \subseteq s$. Putting them together, we have that $s = t$ is provable.

Note that $s = s$ is obviously provable. So we have $s \subseteq s$ as well. Now assume $s \subseteq t$ and $t \subseteq u$. They abbreviate $|\varnothing| \geq |s \cap t^c|$ and $|\varnothing| \geq |t \cap u^c|$. By (BC6), we have $|\varnothing| \geq |(s \cap t^c) \cup (t \cap u^c)|$. Note that the following is in the equational theory of Boolean algebras by distinguishing cases t and t^c:

$$0 = (s \cap u^c) \cap ((s \cap t^c) \cup (t \cap u^c))^c.$$

Hence by (BC7), we have $(s \cap u^c) \subseteq ((s \cap t^c) \cup (t \cap u^c))$. Then by (BC5), we have $|s \cap u^c| \leq |(s \cap t^c) \cup (t \cap u^c)|$. Combining this with what we derived by (BC6) above, using (BC4) we have $|\varnothing| \geq |s \cap u^c|$, which is just $s \subseteq u$.

Part 3 follows directly from part 2 with just a few Boolean manipulations. Part 4 is also not hard using the same technique we used in part 2. The congruence over complementation and union in part 5 follows from part 4 by Boolean manipulations. The congruence over Fin is an easy consequence of (A3). Note also that using (A1), (A4), and (BC5), we have $(\mathrm{Inf}(s) \wedge s \subseteq t) \to \mathrm{Inf}(t)$. So we can also easily derive the congruence over Inf.

Finally, to show substitution, we need to use two inductions. First, an induction on terms using part 5 will show that for any four terms s, t, u_0, u_1 with u_1 being the result of replacing some occurrences of s in u_0 by t, we can derive $s = t \to u_0 = u_1$. Obviously for any terms s, t, we can derive $s = t \to |s| = |t|$. So we proved substitution for atomic sentences in $\mathcal{L}$. When we are in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, part 5 also provides substitution for the rest of the atomic sentences in $\mathcal{L}_{\mathrm{Fin,Inf}}$. Then a simple induction on formulas will do, since $\leftrightarrow$ is again congruential over $\neg$ and $\wedge$. ⊣

Theorem 4.2. *For any set X of $\mathcal{L}$-sentences that is maximally consistent relative to* BasicCompLogic*, there exists a comparison algebra model $\mathcal{C}^X$ such that $\mathcal{C}^X \models X$.*

Proof. For terms s and t, define $s \subseteq_X t$ iff $s \subseteq t \in X$, $s \succeq_X t$ iff $|s| \geq |t| \in X$, $s =_X t$ iff $s = t \in X$, and $s \simeq_X t$ iff $s \succeq_X t$ and $t \succeq_X s$, for all $s, t \in T(\Phi)$. We also write $s \equiv t$ when $s = t$ is provable in the equational theory of Boolean algebras.

Note that by the maximality of X and Lemma 4.1, $=_X$ is a congruence relation on $T(\Phi)$. Let $\mathcal{B}^X = \langle T(\Phi)/_{=_X}, \cdot^c, \sqcap \rangle$ be the homomorphic image of the term algebra $T(\Phi)$, with the homomorphism $[\cdot]_{=_X}$ that sends a term to its equivalence class under $=_X$, and $[s]^c_{=_X} = [s^c]_{=_X}$, $[s]_{=_X} \sqcap [t]_{=_X} = [s \cap t]_{=_X}$. Since $=_X$ extends $\equiv$ (by Lemma 4.1 again), $\mathcal{B}^X$ is a Boolean algebra. The bottom element $\bot_{\mathcal{B}^X}$ is obviously $[\varnothing]_{=_X}$ as it is the meet of $[t]_{=_X}$ and $[t^c]_{=_X}$, but the latter is just $[t]^c_{=_X}$. Regarding the Boolean lattice ordering in $\mathcal{B}^X$, note that:

$$\begin{aligned} [s]_{=_X} \leq_{\mathcal{B}^X} [t]_{=_X} &\Leftrightarrow [s]_{=_X} \sqcap [t]^c_{=_X} = \bot_{\mathcal{B}^X} \\ &\Leftrightarrow [s \cap t^c]_{=_X} = [\varnothing]_{=_X} \\ &\Leftrightarrow s \cap t^c =_X \varnothing. \end{aligned}$$

It is also not hard to see that $s \cap t^c =_X \varnothing$ iff $s \subseteq_X t$ since BasicCompLogic derives $s \cap t^c = \varnothing \leftrightarrow s \subseteq t$. Hence $\leq_{\mathcal{B}^X}$ is just $\subseteq_X$.

Now we add a comparison structure to $\mathcal{B}^X$. Note that by (BC5), $\simeq_X$ extends $=_X$. So we can take the quotient $\succeq_X/_{=_X}$, so that $[s]_{=_X} \succeq_X/_{=_X} [t]_{=_X}$ iff $s \succeq_X t$. Let $\mathcal{C}^X = \langle \mathcal{B}^X, \succeq_X/_{=_X}, [\cdot]_{=_X} \rangle$. Now we show that $\mathcal{C}^X$ is a comparison algebra model:

- We have just shown that $\leq_{\mathcal{B}^X}$ is identical to $\subseteq_X$. By (BC5), $\succeq_X$ extends $\supseteq_X$. So $\succeq_X$ extends $\geq_{\mathcal{B}^X}$.

- Suppose $\bot_{\mathcal{B}^X} \succeq_X/_{=_X} [s]_{=_X}$. Then by definition, $\varnothing \succeq_X s$. What we need is $[s]_{=_X} = \bot_{\mathcal{B}^X}$. To show this, we just need BasicCompLogic to derive $|\varnothing| \geq |s| \to s = \varnothing$ as the antecedent is given by $\varnothing \succeq_X s$. First, $\varnothing \subseteq s$ is derivable trivially. To derive $s \subseteq \varnothing$, we need $|\varnothing| \geq |s \cap \varnothing^c|$ by definition. Obviously $s \cap \varnothing \equiv s$. So we can substitute and then use the assumption that $|\varnothing| \geq |s|$.

Finally, we verify that $\varphi \in X$ iff $\mathcal{C}^X \models \varphi$. For the atomic case, consider a formula $|s| \geq |t|$ for arbitrary $s, t \in T(\Phi)$. Then $|s| \geq |t| \in X$ iff $s \succeq_X t$ iff $[s]_{=_X} \succeq_X/_{=_X} [t]_{=_X}$ iff $\mathcal{C}^X \models |s| \geq |t|$. The induction is trivial. ⊣

§5. Completeness with predicates for infinite and finite sets. In this short section, we will prove Theorem 2.10, which says that the cardinality comparison logic with Fin and Inf, $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, is sound and complete with respect to field of sets models. It is not hard to check that it is sound with respect to field of sets models as well as measure algebra models and labeled comparison algebra models. For completeness, we follow the standard strategy by starting with a consistent formula φ, building a canonical labeled comparison model satisfying φ, adapting it to the set labels appearing in φ so that we obtain a finite model, and finally using the fact that the canonical model must also satisfy all the axioms to show that it is representable. By Theorem 3.12, this means φ is satisfied by a field of sets model.

Theorem 2.10. $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ *is sound and complete with respect to the class of all measure algebra models and also the class of all field of sets models.*

Proof. Soundness is almost trivial. For completeness, we show that every formula φ that is consistent in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ is also satisfied by a measure alegbra model. Since φ is consistent, let X be a maximally consistent set that contains φ.

Since $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ includes BasicCompLogic $\subseteq \mathcal{L}$, $X|_{\mathcal{L}} = X \cap \mathcal{L}$ is a maximally consistent set for the logic BasicCompLogic in $\mathcal{L}$. By the canonical model theorem (Theorem 4.2),

$$\mathcal{C} = \langle \langle T(\Phi)/_{=_{X|_{\mathcal{L}}}}, \cdot^c, \sqcap \rangle, \succeq_{X|_{\mathcal{L}}}/_{=_{X|_{\mathcal{L}}}}, [\cdot]_{=_{X|_{\mathcal{L}}}} \rangle$$

is a comparison algebra model and $\mathcal{C} \models X|_{\mathcal{L}}$. Now we need to build an $F \subseteq T(\Phi)/_{X|_{\mathcal{L}}}$ to interpret Fin and Inf. Define $[s]_{=_{X|_{\mathcal{L}}}} \in F$ iff $\mathrm{Fin}(s) \in X$. For this F to be well defined, we need to show that if $s =_{X|_{\mathcal{L}}} t$, then $\mathrm{Fin}(s) \in X$ iff $\mathrm{Fin}(t) \in X$. As shown in the beginning of the proof of Theorem 4.2, $=_{X|_{\mathcal{L}}}$ is extended by $\simeq_{X|_{\mathcal{L}}}$. Thus, once $s =_{X|_{\mathcal{L}}} t$, both $|s| \geq |t|$ and $|t| \geq |s|$ are in $X|_{\mathcal{L}}$. By axiom (A3) in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ and the maximality of X, this implies $\mathrm{Fin}(s) \in X$ iff $\mathrm{Fin}(t) \in X$.

So we can define $F = \{[s]_{=_{X|_\mathcal{L}}} \mid \mathrm{Fin}(s) \in X\}$. Then

$$\mathcal{C}^F = \langle\langle T(\Phi)/_{=_{X|_\mathcal{L}}}, \cdot^c, \sqcap\rangle, \succeq_{X|_\mathcal{L}}/_{=_{X|_\mathcal{L}}}, F, [\cdot]_{=_{X|_\mathcal{L}}}\rangle$$

is a labeled comparison algebra model. For any $s, t \in T(\Phi)$, $\mathcal{C}^F \models |s| \geq |t|$ iff $\mathcal{C} \models |s| \geq |t|$ iff $|s| \geq |t| \in X$; and $\mathcal{C}^F \models \mathrm{Fin}(s)$ iff $\mathrm{Fin}(s) \in X$. Because of axiom (A1) in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, $\mathrm{Inf}(s) \in X$ iff $\mathrm{Fin}(s) \notin X$. So it follows that $\mathcal{C}^F \models \mathrm{Inf}(s)$ iff $s \notin F$ iff $\mathrm{Fin}(s) \notin X$ iff $\mathrm{Inf}(s) \in X$. Then a simple inductive argument on $\mathcal{L}_{\mathrm{Fin,Inf}}$ shows that $\mathcal{C}^F \models X$. Hence, in particular, $\mathcal{C}^F \models \varphi$. In other words, φ is satisfied by the labeled comparison algebra model $\mathcal{C}^F$ that also satisfies the axioms of $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$.

Now we adapt $\mathcal{C}^F$ to the finite set Δ of the set labels appearing in φ. By Proposition 3.11, the resulting model $\mathcal{C}^F_\Delta$ is finite and satisfies the same formulas in $\mathcal{L}_{\mathrm{Fin,Inf}}$ as $\mathcal{C}^F$ does. This implies that:

- $\mathcal{C}^F_\Delta \models \varphi$.
- Given that $\mathcal{C}^F_\Delta$ is adapted to Δ, every element $b \in \mathcal{C}^F_\Delta$ is equal to $\widehat{V}(t)$ for some $t \in T(\Delta)$, where V is the valuation in $\mathcal{C}^F_\Delta$.
- $\mathcal{C}^F_\Delta$ is representable, using Theorem 3.8. As we noted before Proposition 3.11, it is finite. Now we need to check the four conditions listed in Theorem 3.8. Condition (1) is guaranteed since all instances of (A2) and (A3) are theorems and hence true of $\mathcal{C}^F_\Delta$, which means we can apply (A2) and (A3) to every element as they are all named by terms. Hence it is clear that condition (1) is true. Similarly conditions (2), (3), and (4) are guaranteed by axioms (A5), (A6), and (A4), respectively.

Then by Theorem 3.12, φ is satisfied by a field of sets model. This completes the proof of completeness. ⊣

§6. Completeness without predicates for infinite and finite sets.

In this section, we define the logic $\mathsf{CardCompLogic}$ and prove Theorem 2.4.

THEOREM 2.4. *The cardinality comparison logic* $\mathsf{CardCompLogic}$ *is sound and complete with respect to field of sets models.*

By earlier results, it suffices to consider only measure algebra models. The key idea is to find two formulas Fin and Inf in $\mathcal{L}$ to replace the two primitive predicates, Fin and Inf, added in $\mathcal{L}_{\mathrm{Fin,Inf}}$. The general strategy is as follows:

1. Show that there is a way to define formulas $\mathsf{Fin}(u)$ and $\mathsf{Inf}(u)$ in $\mathcal{L}$, instead of adding the two extra predicates, to capture the finiteness or infiniteness of u on all adapted measure algebra models except a very special class of models, which we call *flexible* models in § 6.1. We use the name "flexible" because in those models we can change

the cardinality of an element to be anything finite or infinite without changing the comparative structure.

2. Then, in § 6.2, we give the axioms for CardCompLogic using the formulas Fin and Inf defined in § 6.1, and we show that any (adapted) comparison algebra model satisfying these axioms can be turned into a measure algebra model. This uses Theorem 3.8 and splits into two cases depending on whether the model is flexible or not.
3. Finally, in § 6.3, given a formula φ consistent with CardCompLogic, we use the canonical model construction of Theorem 4.2 to build a comparison algebra model satisfying φ. Then, using previous results in this section, we can assume that the model is adapted and hence turn it into a measure algebra model.

6.1. Flexible models. In this subsection, we define flexible models and show how they appear when we try to define Fin and Inf in $\mathcal{L}$. Essentially, flexible models are models where our definition, or in fact any definition to capture Fin and Inf in language $\mathcal{L}$, fails. This is because we can make the cardinality of an element in a flexible model anything we like, be it finite or infinite, without changing the formulas in $\mathcal{L}$ satisfied by that model.

DEFINITION 6.2. A finite measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$ is *flexible* if there is an atom a in B whose measure is strictly smaller than the measure of all other atoms in B, and a is the only atom in B with finite measure, if there is any atom with finite measure.

The following two propositions show why we call such models flexible.

PROPOSITION 6.3. *If $\mathcal{B} = \langle B, \mu, V\rangle$ is a flexible finite measure algebra model, then for any non-bottom element $b \in B$, we have*

$$\mu(b) = \max\{\mu(a) \mid a \in At(B), a \leq b\}.$$

PROOF. Write b as a finite join of the atoms below it: $b = a_1 \vee \cdots \vee a_n$. Then $\mu(b) = \mu(a_1) + \cdots + \mu(a_n)$. If b is an atom, then $\mu(b) = \mu(a_1)$; otherwise, $n \geq 2$ and at least one of $\mu(a_1), \ldots, \mu(a_n)$ is infinite, so

$$\mu(b) = \max\{\mu(a_1), \ldots, \mu(a_n)\} = \max\{\mu(a) \mid a \in At(B), a \leq b\}.$$

⊣

PROPOSITION 6.4. *For any flexible finite measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$ and cardinal κ, if a_0 is the atom of B with the smallest measure, then there is a flexible finite measure algebra model $\mathcal{C} = \langle B, \nu, V\rangle$ such that:*

1. $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$;
2. $\nu(a_0) = \kappa$.

In other words, for any flexible finite measure algebra model, the measure of the smallest atom does not matter, if we are only concerned with the truth of formulas in $\mathcal{L}(\Delta)$.

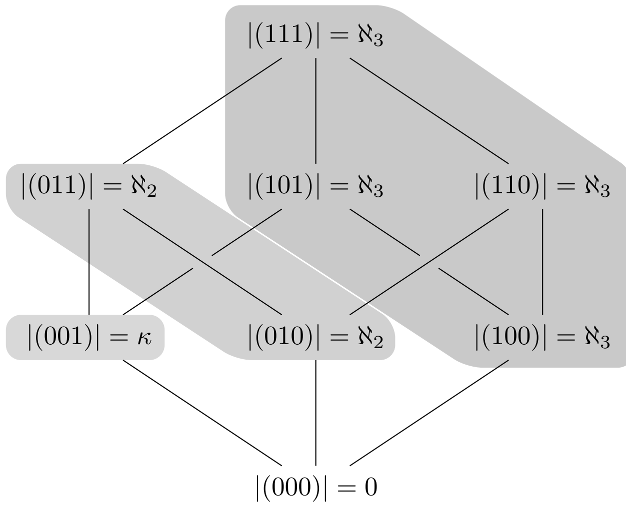

FIGURE 3. A flexible measure algebra

PROOF. First, $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$ when for any $b_1, b_2 \in B$, we have $\mu(b_1) \geq \mu(b_2)$ iff $\nu(b_1) \geq \nu(b_2)$.

But notice that by Proposition 6.3, both $\mu(b)$ and $\nu(b)$ are calculated by taking maximums of the measures of the atoms below b, given that the $\langle B, \nu, V\rangle$ is flexible. This means the condition for equivalence can be weakened to: for any two atoms $a_1, a_2 \in B$, we have $\mu(a_1) \geq \mu(a_2)$ iff $\nu(a_1) \geq \nu(a_2)$. So we only need to have an order-preserving map for the measures of all atoms in B while keeping the flexibility.

Thus, we can define ν on the atoms of B as follows:

$$\nu(a) = \begin{cases} \kappa & \text{if } a = a_0, \text{ the smallest atom} \\ \aleph_{\kappa+i} & \text{if } |\{\mu(a') \mid a' \in At(B), \mu(a') < \mu(a)\}| = i \end{cases}$$

where to compute $\aleph_{\kappa+i}$ we view κ as an ordinal, i.e., as the least order type of a well-order of size κ. It is not hard to verify that $\mathcal{C} = \langle B, \nu, V\rangle$ is still a flexible finite measure algebra model and that for any $a_1, a_2 \in At(b)$, we have $\mu(a_1) \leq \mu(a_2)$ iff $\nu(a_1) \leq \nu(a_2)$. ⊣

EXAMPLE 6.5. Figure 6.1 displays a particular flexible measure algebra (flexible model without the valuation). The comparison structure (illustrated by shaded areas in the same way as in Example 3.9) is the same regardless of what the cardinal κ is so long as $1 \leq \kappa \leq \aleph_1$.

Now we capture Fin and Inf in the language $\mathcal{L}$ by the following.

DEFINITION 6.6. When $\Delta \subseteq \Phi$ is finite, define the $\mathsf{Fin}_\Delta(u)$ for any set term $u \in T(\Delta)$ as:

$$\bigvee_{\substack{R \subseteq T_0(\Delta) \\ S,T \in T_0(\Delta)^{|R|}}} \left(u = \bigcup_{i=1}^{|R|} r_i \;\wedge\; \bigwedge_{i=1}^{|R|} \big(|s_i \cup t_i| > |s_i| \geq |t_i| \;\wedge\; |s_i \cup t_i| \geq |r_i| \big) \right)$$

and then define $\mathsf{Inf}_\Delta(u)$ for any set term $u \in T(\Delta)$ as

$$\mathsf{Inf}_\Delta(u) := \bigvee_{s,t \in T_0(\Delta)} \left(t \not\subseteq s \;\wedge\; |u| \geq |s| \geq |s \cup t| \right).$$

Here r_i ranges over elements in R, and s_i, t_i range over elements in sequences S and T, respectively. When no confusion arises, we may drop the subscript Δ.

To understand this definition, recall that by basic cardinal arithmetic, the distinct feature of infinite sets is so-called absorption: if a set X is infinite, then $|X| \geq |X \cup Y|$ whenever $|X| \geq |Y|$, even if Y is not a subset of X. On the other hand, when Y is not a subset of X and yet $|X| \geq |X \cup Y|$, then X must be infinite. Hence, we can witness X's finiteness by a set Y such that $|X \cup Y| > |X| \geq |Y|$. Note that this also shows that Y, and thus $X \cup Y$, is finite. Similarly, we can witness X's infiniteness by a set Y that is not a subset of X yet for which $|X| \geq |X \cup Y|$. Our definitions of Fin_Δ and Inf_Δ are based on these simple observations. However, since they are meant to capture as many finite/infinite sets as possible, they must be slightly more complicated than simple cardinal arithmetic, as the intuitions such as "a finite union of finite sets is still finite" must also be incorporated into the definition of Fin_Δ. Similarly, Inf_Δ also incorporates the intuition that if a set is no smaller than an infinite set, then they are both infinite. Intuitively, $\mathsf{Fin}_\Delta(u)$ says that u can be expressed as a union of finite sets, the r_i's, with the finiteness witnessed by s_i's and t_i's. On the other hand, intuitively $\mathsf{Inf}_\Delta(u)$ says that there exists a set s whose cardinality is infinite yet u's cardinality is no smaller, with the infiniteness of s witnessed by t.

The following two propositions tell us precisely to what extent these formulas capture Fin and Inf. In sum, their truth forces the respective properties (finiteness and infiniteness) on adapted models, but not vice versa. However, (in adapted models) the other direction fails only on the smallest atom of flexible models. This is the best we can do in $\mathcal{L}$, due to Proposition 6.4 and the existence of flexible models.

PROPOSITION 6.7. *Fix a finite $\Delta \subseteq \Phi$. For any adapted measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$:*

1. *if $\mathcal{B} \models \mathsf{Fin}_\Delta(u)$, then $\mu(\widehat{V}(u))$ is finite;*

2. if $\mathcal{B} \models \neg\mathsf{Fin}_\Delta(u)$ and yet $\mu(\widehat{V}(u))$ is finite, then $\mathcal{B}$ is flexible and $\widehat{V}(u)$ is the smallest atom in B.

Proof. Suppose that $\mathcal{B} \models \mathsf{Fin}_\Delta(u)$. Then there are sequences of terms r_i, s_i, and t_i in $T(\Delta)$ that make the disjunct true. Let $a = \widehat{V}(u)$, $b_i = \widehat{V}(r_i)$, $c_i = \widehat{V}(s_i)$, and $d_i = \widehat{V}(t_i)$. Then:

1. for each i, $\mu(c_i) \geq \mu(d_i)$ but $\mu(c_i) < \mu(c_i \cup d_i)$;
2. for each i, $\mu(b_i) \leq \mu(c_i \cup d_i)$;
3. $a = \bigvee_i b_i$.

The first item ensures that c_i and d_i have finite measure: d_i's cardinality is no greater than c_i's, and c_i cannot be infinite as otherwise the cardinality of $c_i \cup d_i$ will not be strictly greater than that of c_i. The second item says that b_i is smaller than $c_i \cup d_i$, which is finite in measure. The third item says that a is a finite union of finite elements. So we have shown that when $\mathcal{B} \models \mathsf{Fin}_\Delta(u)$, $\mu(\widehat{V}(u)) = \mu(a)$ is finite. (We have not yet used the fact that $\mathcal{B}$ is adapted: $\widehat{V}(T(\Delta)) = \widehat{V}(T_0(\Delta)) = B$.)

For the second claim, assume for contradiction that $\widehat{V}(u)$ is not an atom in B. $\widehat{V}(u)$ cannot be the bottom element, since then $\mathsf{Fin}_\Delta(u)$ is trivially true. So $\widehat{V}(u)$ is neither the bottom nor an atom. This means there are two non-bottom elements $a, b \in B$ that are below $\widehat{V}(u)$, whose join is $\widehat{V}(u)$, whose meet is bottom, and $\mu(a) \geq \mu(b)$. Since $\mu(\widehat{V}(u))$ is finite, $\mu(a)$ and $\mu(b)$ must also be finite. Then $\mu(a) \geq \mu(b)$ but $\mu(a) < \mu(a \vee b) = \mu(a) + \mu(b)$, since b is not bottom and $\mu(b) > 0$. From this, we have:

- $\mu(a \vee b) > \mu(a) \geq \mu(b)$;
- $\widehat{V}(u) = a \vee b$.

So $\mathcal{B} \models \mathsf{Fin}_\Delta(u)$, a contradiction. Here, we are using the fact that $\widehat{V}(T(\Delta)) = \widehat{V}(T_0(\Delta)) = B$ to get terms s and t with $\widehat{V}(s) = a$ and $\widehat{V}(t) = b$.

Now assume again for contradiction that $\widehat{V}(u)$ is not the only atom finite in measure and in particular that $a \in At(B)$, $a \neq \widehat{V}(u)$, and $\mu(a)$ is finite. Then $a \wedge \widehat{V}(u) = 0$ and $\mu(a \vee \widehat{V}(u)) = \mu(a) + \mu(\widehat{V}(u)) > \mu(\widehat{V}(u))$, since a is not bottom and $\mu(a) > 0$. Then we again have a witness for $\mathsf{Fin}_\Delta(u)$, depending on which of $\widehat{V}(u)$ and a is larger. For example, if a is larger, then $\widehat{V}(u)$ is smaller than a, a does not absorb a smaller element $\widehat{V}(u)$, and $\widehat{V}(u)$ is smaller than the join of $\widehat{V}(u)$ and a. Thus, we contradict $\mathcal{B} \models \neg\mathsf{Fin}_\Delta(u)$.

In summary, we have that $\widehat{V}(u)$ is the only finite atom in B, which immediately shows that $\widehat{V}(u)$ is the smallest atom in B and $\mathcal{B}$ is flexible. ⊣

Proposition 6.8. *Fix a finite $\Delta \subseteq \Phi$. For any adapted measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$:*

1. if $\mathcal{B} \models \mathsf{Inf}_\Delta(u)$, then $\mu(\widehat{V}(u))$ is infinite;

2. *if $\mathcal{B} \models \neg\mathsf{Inf}_\Delta(u)$ and yet $\mu(\widehat{V}(u))$ is infinite, then $\mathcal{B}$ is flexible and $\widehat{V}(u)$ is the smallest atom in B.*

Proof. After expanding the semantics, it is easy to see that $\mathsf{Inf}_\Delta(u)$ expresses that there is an element $b \in B$ such that b absorbs an element c not contained in b, and $\widehat{V}(u)$ is no smaller than b in measure. The elements b and c are obtained by first picking out the true disjunct $(t \not\subseteq s \wedge |u| \geq |s| \geq |s \cup t|)$ of $\mathsf{Inf}_\Delta(u)$ and then taking b and c to be just $\widehat{V}(s)$ and $\widehat{V}(t)$, respectively. Then $t \not\subseteq s$ being true means that c is not contained in b, and $|s| \geq |s \cup t|$ means c absorbs b. Hence $\mu(b)$ must be infinite, and since $|u| \geq |s|$, $\mu(\widehat{V}(u))$ is infinite too.

For the second claim, first suppose that there is an atom $a \in At(B)$ with $\mu(a)$ finite. Then for any atom $b \in At(B)$ that is infnite in measure, $a \not\subseteq b$ and $\mu(b) \geq \mu(b \vee a)$. Now $\mu(\widehat{V}(u))$ is infinite, so $\widehat{V}(u)$ must have an infnite atom b below it. Then $\mathsf{Inf}_\Delta(u)$ is witnessed by b and a, a contradiction. Thus, there is no finite atom in B.

Now suppose there is no strictly smallest atom—for any atom b, there is another atom a such that $\mu(b) \geq \mu(a)$. Then $a \not\subseteq b$, and because they are both infinite in measure, $\mu(b) \geq \max\{\mu(b), \mu(a)\} = \mu(b \vee a)$. So by the same reasoning as the previous case, $\mathsf{Inf}_\Delta(u)$ is witnessed, and we have a contradiction. Thus, there is also a strictly smallest infinite atom a_0 in B.

Finally, suppose $\widehat{V}(u) \neq a_0$. Then $\widehat{V}(u)$ must be above another atom b, as it is infinite in measure and cannot be bottom. But then $a_0 \not\subseteq b$ and $\mu(b) \geq \max\{\mu(a_0), \mu(b)\} = \mu(b \vee a_0)$. So $\mathsf{Inf}_\Delta(u)$ is still witnessed. In sum, $\widehat{V}(u)$ is the strictly smallest infinite atom in B, so $\mathcal{B}$ is flexible. $\dashv$

6.2. Representation using axioms of the language. It is now time to give the axioms for cardinality comparison in $\mathcal{L}$ that are not already in BasicCompLogic.

Definition 6.9. Where $\Delta \subseteq \Phi$ is finite, define $\mathsf{Axiom}(\Delta)$ as the set containing all of the following formulas for all $u, s, t \in T_0(\Delta)$:

(C1) $\neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$;
(C2) $(\neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)) \rightarrow \bigwedge_{t \in T_0(\Delta)}(|u| \geq |t| \rightarrow (t = \varnothing \vee t = u))$;
(C3) $\bigwedge_{i=1}^{n}(\mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)) \rightarrow \mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots t_n)$;
(C4) $\mathsf{Inf}_\Delta(u) \rightarrow ((|u| \geq |s| \wedge |u| \geq |t|) \rightarrow |u| \geq |s \cup t|)$;
(C5) $(\mathsf{Inf}_\Delta(s) \wedge \mathsf{Fin}_\Delta(t)) \rightarrow |s| > |t|$,

where $n \geq 1$, and $s_1, \ldots, s_n, t_1, \ldots, t_n \in T_0(\Delta)$ are also all arbitrary.

Given that Fin and Inf do not fully capture Fin and Inf, we cannot use $\mathsf{Fin}_\Delta(u) \oplus \mathsf{Inf}_\Delta(u)$ like axiom (A1) for Fin and Inf, since it is outright invalid among all adapted measure algebra models. Instead, we have (C1) and (C2) here. Put together, they ensure that the only case when $\mathsf{Fin}_\Delta(u)$

and $\mathsf{Inf}_\Delta(u)$ fail to capture Fin and Inf is when we are in a flexible model and u is the smallest atom.

As we have done for $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, we prove a representation theorem using $\mathsf{Axiom}(\Delta)$. In other words, we show that $\mathsf{Axiom}(\Delta)$ is enough to force the comparison relation $\succeq$ in an adapted comparison algebra model to be a comparison of cardinalities. To start, we need the following straightforward lemma.

LEMMA 6.10. *Fix a finite $\Delta \subseteq \Phi$. For any comparison algebra model $\mathcal{B}$ and $s, t \in T(\Delta)$:*

1. $\mathcal{B} \models |\varnothing| \geq |t|$ *iff* $V(t) = \bot_B$*;*
2. $\mathcal{B} \models t \subseteq s$ *iff* $V(t) \leq_B V(s)$*;*
3. $\mathcal{B} \models t = s$ *iff* $V(t) = V(s)$*;*
4. *if* $\mathcal{B} \models \mathsf{Fin}_\Delta(t) \wedge s \subseteq t$, *then* $\mathcal{B} \models \mathsf{Fin}_\Delta(s)$*;*
5. *if* $\mathcal{B} \models \mathsf{Fin}_\Delta(s) \wedge \mathsf{Fin}_\Delta(t)$, *then* $\mathcal{B} \models \mathsf{Fin}_\Delta(s \cup t)$*;*
6. *if* $\mathcal{B} \models \mathsf{Inf}_\Delta(t) \wedge |s| \geq |t|$, *then* $\mathcal{B} \models \mathsf{Inf}_\Delta(s)$*.*

PROOF. The first item follows from the requirement that $\bot_B \not\succeq b$ for all $b \in B \setminus \{\bot_B\}$. The second and third follow easily.

For the fourth item, suppose $\mathcal{B} \models \mathsf{Fin}_\Delta(t) \wedge s \subseteq t$. Then by definition, we have terms r_i as the witnesses of the finiteness of s. It is easy to see that $r_i \cap s$'s are witnesses of the finiteness of s. Similarly for the fifth item, the witnesses of $s \cup t$ are just the union of witnesses for s and witnesses for t. The sixth item is even easier, as the same witness works. ⊣

THEOREM 6.11. *Fix a finite $\Delta \subseteq \Phi$. Let $\mathcal{B}$ be an adapted comparison algebra model such that $\mathcal{B} \models \mathsf{Axiom}(\Delta)$. Then there is a μ such that $m(\mathcal{B}) = \langle B, \mu, V\rangle$ is a finite measure algebra model representing $\mathcal{B}$ and hence $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} m(\mathcal{B})$.*

PROOF. Since $\mathcal{B}$ is adapted to Δ, every element in B is named by some term in $T(\Delta)$. Thus, given $b \in B$ we may write $\varphi(b)$ for the sentence $\varphi(t_b)$ where $t_b \in T(\Delta)$ and $V(t_b) = b$. By axiom (C1) in $\mathsf{Axiom}(\Delta)$, there are two cases:

Case 1: there is a $b \in B$ such that $\mathcal{B} \models \neg(\mathsf{Fin}_\Delta(b) \vee \mathsf{Inf}_\Delta(b))$;
Case 2: for any $b \in B$, $\mathcal{B} \models \mathsf{Fin}_\Delta(b) \oplus \mathsf{Inf}_\Delta(b)$.

In both cases, we want to obtain an $F \subseteq B$, so that using F as the labeling set, $\langle B, \succeq, F, V\rangle$ is a labeled comparison algebra model satisfying all the conditions in Theorem 3.8.

Case 1: First, we show that there is a unique atom $a_0 \in At(B)$ such that $\mathcal{B} \models \neg(\mathsf{Fin}_\Delta(a_0) \vee \mathsf{Inf}_\Delta(a_0))$, and that for all other atoms b, we have $b \succ a_0$. Suppose for $a_0, a_1 \in B$ that we have both $\mathcal{B} \models \neg\mathsf{Fin}_\Delta(a_i) \wedge \neg\mathsf{Inf}_\Delta(a_i)$ for $i \in \{0, 1\}$. Then by axiom (C2) and Lemma 6.10, for any $b \in B$, if $a_0 \succeq b$, then $b = a_0$ or $b = \bot_B$. But for any $b \in B$ that is below a_0 in the Boolean algebra, i.e., $b \leq a_0$, it is also true that $b \preceq a_0$, by the definition of a comparison algebra. So whenever $b \leq a_0$, we have $b \preceq a_0$ and hence b is a_0 or $\bot_B$. This means that a_0 is an atom in B, as it must not be $\bot_B$, since $\mathcal{B} \models \mathsf{Fin}_\Delta(\bot_B)$. In exactly the same fashion we can show that a_1 is also an atom. Now $\succeq$ is a total preorder. So either $a_0 \succeq a_1$ or $a_1 \succeq a_0$. But in either case, they must be equal, as they cannot be $\bot_B$, but axiom (C2) says they are either equal or are bottom. Thus, there is a unique atom a_0 such that $\mathcal{B} \models \neg\mathsf{Fin}_\Delta(a_0) \wedge \neg\mathsf{Inf}_\Delta(a_0)$.

Building on the previous conclusion, for any $a \in At(B) \setminus \{a_0\}$, we have $\mathcal{B} \models \mathsf{Fin}_\Delta(a) \oplus \mathsf{Inf}_\Delta(a)$. The second step is to show that in fact $\mathcal{B} \models \mathsf{Inf}_\Delta(a)$. Consider $a \cup a_0$. If $\mathcal{B} \models (a \cup a_0) \succ a$, then we have $\mathcal{B} \models (a \cup a_0) \succ a \succeq a_0 \wedge (a \cup a_0) \succeq a_0$. It is not hard to see that then $\mathcal{B} \models \mathsf{Fin}_\Delta(a_0)$, a contradiction. Hence $\mathcal{B} \models a \succeq (a \cup a_0)$ instead. However, $\mathcal{B} \models a_0 \not\subseteq a$ since a_0 and a are distinct atoms. So $\mathcal{B} \models \mathsf{Inf}_\Delta(a)$.

Thus we see that there is no element in B satisfying Fin except the bottom element. So define $F = \{\bot_B\}$. We can show that Theorem 3.8 can be applied to $\langle B, \succeq, F, V\rangle$. In fact, the only nontrivial condition is the third condition: for any $a, b, c \in B$ with $a \notin F$, if $a \succeq b$ and $a \succeq c$, then $a \succeq b \vee c$. There are two cases:

- $a = a_0$: then a is the smallest atom, and b, c are either $\bot_B$ or a_0 and so is $b \vee c$; hence $a \succeq b \vee c$;
- $a \neq a_0$: as we have shown, now $\mathcal{B} \models \mathsf{Inf}_\Delta(a)$; thus by axiom (C4) in $\mathsf{Axioms}(\Delta)$, $a \succeq b \vee c$.

Hence we can invoke Theorem 3.8 to build μ.

Case 2: Define $F = \{b \in B \mid \mathcal{B} \models \mathsf{Fin}_\Delta(b)\}$. By Lemma 6.10, F is an ideal. Axioms (C3), (C4), and (C5) in $\mathsf{Axioms}(\Delta)$ ensure conditions (2), (3), and (4) in Theorem 3.8. Thus, Theorem 3.8 applies again. ⊣

6.3. Completeness. Finally we are in a position to present the logic of cardinal comparison CardCompLogic.

Definition 6.12. Let CardCompLogic be the logic for $\mathcal{L}$ with the following axioms and rules:

1. all axioms and rules in BasicCompLogic;
2. for any finite $\Delta \subseteq \Phi$, all formulas in $\mathsf{Axioms}(\Delta)$;

Now we show that CardCompLogic is sound and complete with respect to all measure algebra models and also field of sets models, completing our proof of Theorem 2.4.

Theorem 6.13 (Soundness). CardCompLogic *is sound with respect to the class of all measure algebra models. Since every field of sets model can be equivalently turned into a measure algebra model in an obvious way,* CardCompLogic *is also sound on the class of all field of sets models.*

PROOF. The only non-trivial axioms are in $\mathsf{Axioms}(\Delta)$ for an arbitrary finite Δ. Given an arbitrary measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$, by Proposition 3.11 there is a measure algebra model $\mathcal{B}_\Delta$ which is adapted to Δ and which satisfies $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$. So to show that $\mathcal{B} \models \mathsf{Axioms}(\Delta)$, it is enough to show $\mathcal{B}_\Delta \models \mathsf{Axioms}(\Delta)$.

Hence we only need to show that all sentences in $\mathsf{Axioms}(\Delta)$ are true on any adapted measure algebra model $\mathcal{B}$. Fix an arbitrary such $\mathcal{B} = \langle B, \mu, V\rangle$; we check that all the sentences in $\mathsf{Axioms}(\Delta)$ are true in $\mathcal{B}$:

- $\neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$: By Propositions 6.7 and 6.8, $\mathcal{B} \models \mathsf{Fin}_\Delta(u)$ implies that $\mu(\widehat{V}(u))$ is finite, and $\mathcal{B} \models \mathsf{Inf}_\Delta(u)$ implies that $\mu(\widehat{V}(u))$ is infinite. But $\mu(\widehat{V}(u))$ cannot be both finite and infinite. Therefore, $\mathcal{B} \models \neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$.
- $(\neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)) \to \bigwedge_{t\in T_0(\Delta)}(|u| \geq |t| \to (t = \varnothing \vee t = u))$: Suppose $\mathcal{B} \models \neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)$. Then by the second part of Proposition 6.7, if $\mu(\widehat{V}(u))$ is infinite, then $\widehat{V}(u)$ is the strictly smallest atom in B. Then it follows that
$$\mathcal{B} \models \bigwedge_{t\in T_0(\Delta)} (|u| \geq |t| \to (t = \varnothing \vee t = u)).$$
Similarly, by Proposition 6.8, if $\mu(\widehat{V}(u))$ is finite, the above statement holds as well. So indeed the formula is valid.
- $\bigwedge_{i=1}^{n}(\mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)) \to \mathsf{FC}_n(s_1,\dots,s_n,t_1,\dots t_n)$: By Proposition 6.7, when $\mathcal{B} \models \mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)$, $\widehat{V}(s_i)$ and $\widehat{V}(t_i)$ are indeed finite. But as we have explained both in § 2 and immediately after Theorem 3.7, the consequent (finite cancellation axiom) is clearly valid for elements of finite cardinality.
- $\mathsf{Inf}_\Delta(u) \to ((|u| \geq |s_1| \wedge |u| \geq |s_2|) \to |u| \geq |s_1 \cup s_2|)$: By Proposition 6.8, $\widehat{V}(u)$ is infinite, and the consequent expresses a simple property of elements of infinite cardinality.
- $(\mathsf{Inf}_\Delta(s_1) \wedge \mathsf{Fin}_\Delta(s_2)) \to |s_1| > |s_2|$: Using Propositions 6.7 and 6.8 again, this says that when $\mu(\widehat{V}(s_1))$ is infinite and $\mu(\widehat{V}(s_2))$ is finite, then $\mu(\widehat{V}(s_1))$ is greater than $\mu(\widehat{V}(s_2))$, which is trivial.

⊣

THEOREM 6.14 (Completeness). $\mathsf{CardCompLogic}$ *is complete with respect to the class of all field of sets models: every valid formulas is derivable in* $\mathsf{CardCompLogic}$.

PROOF. We show that any formula that is consistent is satisfied by a measure algebra model. By Proposition 3.3, it is also satisfied by a field of sets model.

Suppose φ is consistent. Then let X be a maximally consistent set of CardCompLogic containing φ. Using the canonical model theorem (Theorem 4.2), we obtain a comparison algebra model $\mathcal{C} \models X$.

Now let Δ be the set of all set labels appearing in φ. Then Δ is finite. By Proposition 3.11, there is a comparison algebra model $\mathcal{C}_\Delta$ that is adapted to Δ and that satisfies $\mathcal{C}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$. Since $\mathsf{Axioms}(\Delta) \subseteq X$, $\mathcal{C} \models \mathsf{Axioms}(\Delta)$, and since $\mathsf{Axioms}(\Delta) \cup \{\varphi\} \subseteq \mathcal{L}(\Delta)$, $\mathcal{C}_\Delta \models \mathsf{Axioms}(\Delta) \cup \{\varphi\}$. Now the representation theorem (Theorem 6.11) can be applied to $\mathcal{C}_\Delta$, and we obtain a measure algebra model $\mathcal{B}$ such that $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}_\Delta$. Thus $\mathcal{B} \models \varphi$. So φ is satisfied on a measure algebra model. ⊣

Thus, the question with which we began—what are the laws one must add to Boolean algebra to capture reasoning about the relative size of sets according to Cantor's definition?—is answered by the laws of CardCompLogic.

§7. Open problems. In our proofs, we quickly passed to finite models, that is, models with only finitely many sets (some of which may of course be infinite). For example, our representation theorem (Theorem 3.8) applies only to finite models, and in Theorems 2.10 and 2.4, we proved completeness rather than strong completeness.

Problem 7.1. Find a logic that is sound and strongly complete with respect to field of sets models.

Problem 7.2. Prove a representation theorem for infinite comparison algebras.

We will give some examples that show the difficulties that arise here. First, such a logic cannot be compact. Indeed (in the language $\mathcal{L}_{\mathrm{Fin,Inf}}$), with distinct set terms $\langle s_n \rangle_{n\in\omega}$ and t, the following set of formulas is finitely satisfiable in field of sets models, but not satisfiable:

$$\{|s_n| < |s_{n+1}| \mid n \in \omega\} \cup \{|s_n| \leq |t| \mid n \in \omega\} \cup \{\mathrm{Fin}(t)\}.$$

One can give similar examples in the language $\mathcal{L}$. Then to obtain a strongly complete logic, one might add an infinitary rule stating that if the sentences in

$$\{|s_n| < |s_{n+1}| \mid n \in \omega\} \cup \{|s_n| \leq |t| \mid n \in \omega\}$$

are derivable, then so is $\mathrm{Fin}(t)$.

Another interesting example comes from the fact that the relation of cardinality comparison must be well-founded (assuming the axiom of choice). Thus, if $\langle s_n \rangle_{n\in\omega}$ is a sequence of distinct set terms, then the set of sentences

$$\{|s_{n+1}| < |s_n| \mid n \in \omega\}$$

is not satisfiable in field of sets models, but it is again finitely satisfiable.

Note also that finite cardinalities are just natural numbers, whose ratios are all rational. However, with infinitely many formulas, we can express that the sizes of two sets are of irrational ratio. To do this, define the following set of formulas:

$$A = \{|a_i| = |a_j|, |b_i| = |b_j| \mid i, j \in \omega\} \cup \\ \{|a_i \cap a_j| = \varnothing \wedge |b_i \cap b_j| = \varnothing \mid i, j \in \omega, i \neq j\}.$$

Intuitively, this says that the a_i's are disjoint and of the same size, and the same holds for the b_i's. Then we can approximate any ratio by using a_i's and b_i's. For example, consider sequences $\langle l_i \rangle_{i\in\omega}, \langle r_i \rangle_{i\in\omega}, \langle n_i \rangle_{i\in\omega}$ of natural numbers such that l_i/n_i approaches $\sqrt{2}$ from below and r_i/n_i from above. Then let

$$B = \{|\bigcup_{k<l_i} a_k| < |\bigcup_{k<n_i} b_k| < |\bigcup_{k<r_i} a_k| \mid i \in \omega\}.$$

The set $A \cup B$ is then finitely satisfiable, but not satisfiable as it forces the ratio of $|a_0|$ and $|b_0|$ to be $\sqrt{2}$.

As the last example of non-compactness, suppose that we allow more than countably many set labels in the language. Let $|\Phi| = \aleph_1$. Then the set of sentences

$$\{|a| \neq |b| \mid a, b \in \Phi, a \neq b\} \cup \{\mathrm{Fin}(a) \mid a \in \Phi\}$$

is not satisfiable. However, any countable subset is satisfiable.

A natural extension of our language is to add the powerset operation. In this case, one must replace the complement operation with the relative complement operation $s \setminus t$. Then a field of sets model (with powerset) is a collection $\mathcal{F}$ of sets closed under intersection, union, relative complement, and powerset, together with a valuation of the set labels.

Problem 7.3. Axiomatize the logic of cardinality comparison with the powerset operation.

In this language, one can consider principles such as

$$|s| < |t| \rightarrow |\mathcal{P}(s)| < |\mathcal{P}(t)|,$$

which is true under GCH but is independent of ZFC [10]. It would be interesting to have a logic for comparing such principles.

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Appendix A. Polarizability rule and finite cancellation axiom schema. Let $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ be the system obtained by adding axiom schemas (A1)–(A4), (A6), (A7), and (A8) to $\mathsf{BasicCompLogic}$. In this appendix, we discuss how (A5) can be derived in $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. First, we verify that the rule is sound in the sense that if the premise $a|t \to \varphi$ is valid, then the conclusion φ is also valid.

PROPOSITION A.1. *The polarizability rule* (A8) *is sound on field of sets models.*

Proof. Suppose φ is not valid, so $\neg\varphi$ is satisfiable. Our goal is to show that $a|t \wedge \neg\varphi$ is satisfiable. Let Δ be the set of set labels in φ or t. Then by the constraint of (A7), $a \notin \Delta$. The strategy is simple: take a model $\mathcal{M}$ of $\neg\varphi$; construct the disjoint union $\mathcal{N}$ of two copies of $\mathcal{M}$ with a valued to exactly one copy of $\mathcal{M}$; then we have that both $a|t$ and $\neg\varphi$ are true in $\mathcal{N}$.

Formally, take a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$ that makes $\neg\varphi$ true. Let $X' = X \times \{0,1\}$. Define a duplication function $d : \wp(X) \to \wp(X')$ by $d(S) = S \times \{0,1\}$. Then $d \circ V$ is a valuation on $\langle X', \wp(X')\rangle$. Define V' such that if $a \in \Delta$, then $V'(a) = d(V(a))$, and otherwise $V'(a) = X \times \{0\}$. Let $\mathcal{N} = \langle X', \wp(X'), V'\rangle$. A simple induction shows that for any term $t \in T(\Delta)$, $\widehat{V'}(t) = d(\widehat{V}(t)) = \widehat{V}(t) \times \{0,1\}$. This implies that for any two terms $s, t \in T(\Delta)$, $\mathcal{M} \models |s| \geq |t|$ iff $\mathcal{N} \models |s| \geq |t|$. In addition, a set S is finite iff $S \times \{0,1\}$ is finite. Another simple induction then shows that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same formulas in $\mathcal{L}_{\mathrm{Fin,Inf}}$ using only labels in Δ. In particular, $\mathcal{N} \models \neg\varphi$. Since $a \notin \Delta$, we have $\widehat{V'}(a) = X \times \{0\}$ while $\widehat{V'}(t) = \widehat{V}(t) \times \{0,1\}$. Thus, $\mathcal{N} \models a|t$. ⊣

While it is not hard to understand the content of the polarizability rule itself, it is harder to see what it can prove and how it can be used.

Kraft, Pratt, and Seidenberg famously observed in [11] that without the polarizability rule (A8), the remaining system does not capture all valid reasoning patterns for finite sets, contrary to a conjecture of de Finetti [4].

For compact notation, we use the standard set theoretical definition of $n = \{0, 1, \ldots, n-1\}$ and do not distinguish a sequence of length n and a function with domain n. We let ${}^{n}2$ denote the set of such functions/sequences with codomain 2. Then FC_n can be defined under this notation by the following.

Definition A.2. For each $n, m \in \mathbb{N}$, sequence $\vec{s} = \langle s_0, \cdots, s_{n-1}\rangle$ of n terms, and $f \in {}^{n}2$, define the term

$$\vec{s}[f] = \bigcap\{s_i \mid f(i) = 1\} \cap \bigcap\{s_i^c \mid f(i) = 0\}$$

and the term

$$\mathsf{N}_m(\vec{s}) = \bigcup\{\vec{s}[f] \mid f : n \to 2 \text{ and } |f^{-1}(1)| = m\}.$$

For each $f \in {}^{n}2$, $\vec{s}[f]$ is intuitively a "definable" atom (in the Boolean algebra of terms constructible from $\vec{s}$), and $\mathsf{N}_m(\vec{s})$ is then the union of atoms that appear in exactly m terms in $\vec{s}$. Given two sequences $\vec{s}$ and $\vec{t}$ of n terms, we can then define the formula

$$\vec{s}\ \mathsf{E}\ \vec{t} = \bigwedge_{0 \leq i \leq n} (\mathsf{N}_i(\vec{s}) = \mathsf{N}_i(\vec{t})).$$

Recall that equality between terms is defined in Definition 2.1. Then

$$\mathsf{FC}_n(\vec{s},\vec{t}) = \vec{s}\ \mathsf{E}\ \vec{t} \to ((\bigwedge_{i<n-1} |s_i| \geq |t_i|) \to |t_{n-1}| \geq |s_{n-1}|).$$

Consequently (A5) is now

$$\text{(A5)} \qquad (\mathrm{Fin}(\vec{s}) \wedge \mathrm{Fin}(\vec{t})) \to \mathsf{FC}_n(\vec{s},\vec{t}),$$

where Fin is extended to sequences of terms in the obvious way.

Now we show that we can derive (A5) in $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. The main strategy is to repeatedly use (A8) so that (A7) can be applied. In fact, Kraft, Pratt, and Seidenberg already sketched a proof of this in [11] for their Theorem 5. More specifically, the idea is the following, assuming that we are dealing with only finite set terms (for convenience we often speak loosely of terms as sets, say that one set is a subset of another when the relevant formula involving terms is provable, etc.):

1. Given two sequences $\vec{s}$ and $\vec{t}$ of length n, use the polarizability rule (A8) to keep polarizing atomic terms (minimal regions in the Venn diagram) definable from the terms in $\vec{s}$ and $\vec{t}$ until each atomic term is split into $2^n \geq 2n$ pieces of equal size.
2. Now each s_i and t_i are unions of definable atomic terms. For each s_i, define s'_i to be the union of the ith piece of the definable atomic terms that are subsets of s_i (so for example, if there are just s_1 and s_2, then s'_1 is the union of the first piece of $s_1 \cap s_2$ and the first piece of $s_1 \cap s_2^c$). Similarly define t'_i by using the $(i+n)$th pieces.
3. Then intuitively s'_i and t'_i are disjoint representatives of s_i and t_i: for any $i \neq j$, s'_i, s'_j, t'_i, and t'_j are all disjoint, and for each i, $|s'_i| = \frac{1}{2^n}|s_i|$ and $|t'_i| = \frac{1}{2^n}|t_i|$.
4. Recall that intuitively, when $\vec{s}\ \mathsf{E}\ \vec{t}$, we have $\sum_{i<n}|s_i| = \sum_{i<n}|t_i|$. This means $\sum_{i<n}|s'_i| = \sum_{i<n}|t'_i|$ as we just need to scale both sides by $\frac{1}{2^n}$. Also, since now the primed versions of s_i and t_i are disjoint, the sum of the sizes is just the size of the union. So intuitively we should get $\left|\bigcup_{i<n} s'_i\right| = \left|\bigcup_{i<n} t'_i\right|$. Indeed, this is derivable from $\vec{s}\ \mathsf{E}\ \vec{t}$.
5. Using (A7), which deals with disjoint unions, we can then derive $(\bigwedge_{i<n-1}|s'_i| \geq |t'_i|) \to |t'_{n-1}| \geq |s'_{n-1}|$. But recall that intuitively $|s'_i|$ and $|t'_i|$ are just $\frac{1}{2^n}$ of $|s_i|$ and $|t_i|$. Formally, this means that $|s_i| \geq |t_i| \leftrightarrow |s'_i| \geq |t'_i|$ is derivable for any $i < n$. So the real consequent of $\mathsf{FC}_n(\vec{s},\vec{t})$ is derivable.

The rest of this section implements the sketch above in the formal system $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. Now we start with a lemma showing that for disjoint finite sets, cardinality comparison works as intended. Note that we have proved that theorems are closed under substitution in Lemma 4.1. Hence we will use substitution freely without explicit reference.

LEMMA A.3. *For any sequence $\vec{s}$ of n terms, define the disjointness of terms in $\vec{s}$ by*

$$\mathsf{D}(\vec{s}) := \bigwedge_{0\leq i<j<n} (s_i \cap s_j) = \varnothing.$$

Then $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ *derives the following with $\vec{s}$ a sequence of $2n$ terms:*

(1) $\big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})\big) \to \Big(\big(\bigwedge_{i<n} |s_i| \geq |s_{i+n}|\big) \to |\bigcup_{i<n} s_i| \geq |\bigcup_{i<n} s_{i+n}|\Big)$;

(2) $\big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})\big) \to \Big(\big(\bigwedge_{i<n} |s_i| = |s_{i+n}|\big) \to |\bigcup_{i<n} s_i| = |\bigcup_{i<n} s_{i+n}|\Big)$;

(3) $\Big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s}) \wedge |\bigcup_{i<n} s_i| = |\bigcup_{i<n} s_{i+n}|\Big) \to$
$$\Big(\big(\bigwedge_{i<n-1} |s_i| \geq |s_{i+n}|\big) \to |s_{n-1}| \leq |s_{2n-1}|\Big).$$

PROOF. Note that (2) follows directly from (1). Also, we need only prove the case for $n = 2$, as the general formula can then be derived inductively.

Suppose now that $\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})$ holds with $n = 2$. Then consider the following three terms: $s_{01} = s_0 \cup s_1$, $s_{12} = s_1 \cup s_2$, and $s_{23} = s_2 \cup s_3$. Using $\mathsf{BasicCompLogic}$, we have

$$s_{01} \cap s_{12}^c = s_0, \qquad s_{12} \cap s_{01}^c = s_2,$$
$$s_{12} \cap s_{23}^c = s_1, \qquad s_{23} \cap s_{12}^c = s_3.$$

So by (A7), we have

$$|s_{01}| \geq |s_{12}| \leftrightarrow |s_0| \geq |s_2|, \quad |s_{12}| \geq |s_{23}| \leftrightarrow |s_1| \geq |s_3|.$$

Hence, we get $(|s_0| \geq |s_2| \wedge |s_1| \geq |s_3|) \to |s_{01}| \geq |s_{12}|$. This shows (1). Also, when $|s_0| \geq |s_2|$, suppose further that $\neg|s_3| \geq |s_1|$, that is, $|s_1| > |s_3|$. Then $|s_{01}| \geq |s_{12}| > |s_{23}|$. Hence $|s_{01}| > |s_{23}|$, contradicting $|s_{01}| = |s_{23}|$.

For induction, we just need to consider the union of the first $n-1$ sets, the nth set, the next $n-1$ sets, and the last set as a four-set sequence. ⊣

PROPOSITION A.4. $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ *derives* (A5).

PROOF. Take an arbitrary sequence $\vec{s}$ of $2n$ terms. Let $\vec{s}_<$ be the sequence of the first n terms in $\vec{s}$ and $\vec{s}_>$ that of the last n terms. Similarly, for any function $f \in {}^{2n}2$, define $f_<$ to be the restriction of f on n and $f_>$ the restriction of f on $\{n, n+1, \cdots, 2n-1\}$. Our final goal is to derive

(4) $$(\mathrm{Fin}(\vec{s}) \wedge \vec{s}_< \mathrel{\mathsf{E}} \vec{s}_>) \to \Big(\big(\bigwedge_{i<n-1} |s_i| \geq |s_{i+n}|\big) \to |s_{n-1}| \leq |s_{2n-1}|\Big).$$

As we mentioned above, our strategy will be to "disjointify" $\vec{s}$ so that we can use (3) in Lemma A.3. This is done by constructing in each s_i a subset s'_i so that $\langle s'_i \rangle_{i<2n}$ is a sequence of pairwise disjoint sets while each s'_i is $\frac{1}{2^n}$ of s_i. Then Lemma A.3 can be applied.

More formally, our plan is to use the polarizability rule (A8) to construct a term s'_i for each $i < 2n$ so that the following three formulas are derivable:

$$(\bigwedge_{i<2n} s'_i \subseteq s_i) \wedge \mathsf{D}(\langle s'_i \rangle_{i<2n}); \tag{5}$$

$$\mathrm{Fin}(\vec{s}) \to (\bigwedge_{i<n} (|s'_i| \geq |s'_{i+n}| \leftrightarrow |s_i| \geq |s_{i+n}|)); \tag{6}$$

$$\vec{s}_< \mathsf{E}\, \vec{s}_> \to |\bigcup_{i<n} s'_i| = |\bigcup_{i<n} s'_{i+n}|. \tag{7}$$

Once the three formulas are derived, it is then quite obvious that the system can derive (4) with the help of (3).

Hence the rest of this proof is devoted to the construction of s'_i and s'_{i+n} and the derivation of (5)–(7) above. Indented passages marked with a vertical line give details that may be skipped on a first reading.

Polarization and construction. By repeated use of (A8), for any $f \in {}^{2n}2$, we can also assume that $\vec{s}[f]$ is polarized into 2^n many pieces. Let us enumerate the partitions of $\vec{s}[f]$ by $\vec{s}[f][i]$ with $i < 2^n$. Let us also generalize the notation of $\vec{s}[f][i]$ to $\vec{s}[F][I]$ where $F \subseteq {}^{2n}2, I \subseteq 2^n$, defined by

$$\bigcup\{\vec{s}[f][i] \mid f \in F, i \in I\}.$$

Then we abbreviate $\vec{s}[\{f\}][I]$ as $\vec{s}[f][I]$ and $\vec{s}[F][\{i\}]$ as $\vec{s}[F][i]$.

Now define $C_i = \{f \in {}^{2n}2 \mid f(i) = 1\}$ for $i < 2n$. The equation $\vec{s}[C_i] = s_i$ is in the equational theory of Boolean algebras and hence is derivable in our system. Then for any $i < 2n$, our s'_i used in the outline above is defined by $\vec{s}[C_i][i]$ (note that for any $n \geq 1$, $2^n \geq 2n$). In Figure 4, we use a grid to illustrate the partition resulting from polarization. Each column is an $\vec{s}[f]$ for some $f \in {}^{2n}2$. Each cell is then an $\vec{s}[f][i]$. We shade $\vec{s}[C_i][i]$ for $i = 0, 1, 2, 3$, each in its own row; note that they are disjoint, and each is $1/4$ the size of the corresponding $\vec{s_i}$. This is essentially Figure 1 but since there are 4 sets, we choose not to draw a Venn diagram in the usual way.

The indented passage provides more details on the construction of $\vec{s}[f][i]$:

> We can prepare for each $\vec{s}[f]$ and each natural number $l < n$ a set of 2^l many fresh set labels. For convenience, we can just use functions in ${}^l2$. Now, we can first assume that the empty function ε polarizes $\vec{s}[f]$: $\varepsilon|\vec{s}[f]$. This gives us two sets: $\vec{s}[f] \cap \varepsilon$ and $\vec{s}[f] \cap \varepsilon^c$. Then we can inductively polarize the generated sets.

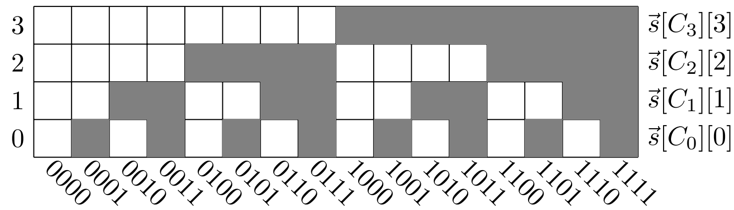

FIGURE 4. Polarization and construction when $n = 2$. Squares in the same row can be of different sizes. But squares in the same column must be of the same size.

In fact, for each function $g \in {}^{l}2$ with $l < n$, we can define

$$\vec{s}[f][g] = \vec{s}[f] \cap \bigcap_{k<l} g_{<k}{}^{c \cdot g(j)},$$

where $g_{<k}$ is g restricted to k and $c \cdot x$ is c if x is 1 and empty otherwise. Then (A8) allows us to assume that for all $0, 1$ sequences g of length at most $n-1$, $g|\vec{s}[f][g]$, or equivalently by our definition, $|\vec{s}[f][\langle g, 0\rangle]| = |\vec{s}[f][\langle g, 1\rangle]|$. Of course, due to the restriction of (A8), we need to arrange those formulas so that those with shorter variables come first. Fix an enumeration $\langle g_i \rangle$ of $\bigcup_{l<n} {}^{l}2$ so that if g_j extends g_i then $j \geq i$. Then formally we are using (A8), so that it suffices to prove

$$g_{2^n-1}|\vec{s}[f][g_{2^n-1}] \to (\cdots (g_i|\vec{s}[f][g_i] \to (\cdots (g_0|\vec{s}[f][g_0] \to \varphi) \cdots)) \cdots)$$

when we want to prove φ. Hence, from now on, we have that each $\vec{s}[f]$ is polarized into 2^n many pieces, enumerated by $\vec{s}[f][g]$ with $g \in {}^{n}2$.

Deriving formula (5). Since for all $f \in C_i$, $\vec{s}[f][i] \subseteq s_i$ is obviously derivable, we have $\vec{s}[C_i][i] \subseteq s_i$. Hence the first part of (5) can be derived.

Disjointness is slightly less trivial. Recall that by our definition of $\vec{s}[f][i]$, for any $f \in {}^{2n}2$, $\vec{s}[f][i] \cap \vec{s}[f][j] = \varnothing$ is derivable when $i \neq j$. Thus when relativized to each $\vec{s}[f]$, $\vec{s}[C_i][i]$ and $\vec{s}[C_j][j]$ are disjoint for $i \neq j$. Some simple Boolean equational theory will then show that $\vec{s}[C_i][i]$ and $\vec{s}[C_j][j]$ themselves are disjoint.

Deriving formula (6). Assume $\mathrm{Fin}(\vec{s})$. Note that for any $f \in {}^{2n}2$ that is not constantly 0, there is an $i < 2n$ such that $\vec{s}[f] \subseteq s_i$ is derivable: just pick i with $f(i) = 1$. Hence, using (A2) and (A3), for any $I \subseteq 2^n$ and $F \subseteq {}^{2n}2$ with the constantly 0 function not in F, the system derives $\mathrm{Fin}(\vec{s}[F][I])$. Then, by repeated use of Lemma A.3, the system derives that for any $i, j < 2^n$ and $f \in {}^{2n}2$ with f not constantly 0, $|\vec{s}[f][i]| = |\vec{s}[f][j]|$.

Recall how we defined $\vec{s}[f][i]$ by polarization. We can in fact use a simple induction on $0 < l < n$ to show that for each l and $g_0, g_1 \in {}^l 2$, $|\vec{s}[f][g_0]| = |\vec{s}[f][g_1]|$ is derivable. The base case is when $l = 1$ and $g_0 = \langle 0 \rangle, g_1 = \langle 1 \rangle$. Here what we need to show is already assumed when we apply (A8): $\varepsilon|\vec{s}[f]$, as this is defined precisely as $|\vec{s}[f][\langle 0 \rangle]| = |\vec{s}[f][\langle 1 \rangle]|$. To go from l to $l+1$, note that any function in ${}^{l+1}2$ is obtained by appending a 0 or 1 to functions in ${}^l 2$. So it is enough to show that for any $g_0, g_1 \in {}^l 2$, the four sets in the sequence

$$\vec{t} = \langle \vec{s}[f][\langle g_0, 0 \rangle], \vec{s}[f][\langle g_0, 1 \rangle], \vec{s}[f][\langle g_1, 0 \rangle], \vec{s}[f][\langle g_1, 1 \rangle] \rangle$$

are of equal size. In the previous (unindented) paragraph, we have derived $\mathrm{Fin}(\vec{s}[F][I])$ for any F and I, and hence we have derived $\mathrm{Fin}(\vec{t})$. It is also obvious that the system can derive $\mathsf{D}(\vec{t})$ using the equational theory of Boolean algebras. By the induction hypothesis, we also have that the union of the first two and the last two are of equal size. Hence we can apply (3) to $\vec{t}$ and obtain

$$\vec{s}[f][\langle g_0, 0 \rangle] \geq \vec{s}[f][\langle g_1, 0 \rangle] \to \vec{s}[f][\langle g_1, 1 \rangle] \geq \vec{s}[f][\langle g_0, 1 \rangle].$$

By switching the first two and the second two sets in $\vec{t}$ and applying (3) again, we get

$$\vec{s}[f][\langle g_0, 0 \rangle] \leq \vec{s}[f][\langle g_1, 0 \rangle] \to \vec{s}[f][\langle g_1, 1 \rangle] \leq \vec{s}[f][\langle g_0, 1 \rangle].$$

Now $|\vec{s}[f][\langle g_0, 0 \rangle]| = |\vec{s}[f][\langle g_0, 1 \rangle]|$ and $|\vec{s}[f][\langle g_1, 0 \rangle]| = |\vec{s}[f][\langle g_1, 1 \rangle]|$ are derivable since we have assumed when using the polarizability rule (A8) that $g_0|\vec{s}[f][g_0]$ and $g_1|\vec{s}[f][g_1]$. With the transitivity of $\geq$ encoded by axiom (BC3), we can derive that the four sets involved are all equal in size. This shows that the 2^n subsets of $\vec{s}[f]$ obtained by polarization are of equal size whenever f is not constantly 0.

Since C_i does not contain the constantly 0 function and $\vec{s}[C_i][j]$ is a disjoint union of $\vec{s}[f][j]$ with $f \in C_i$, using (2) we have $|\vec{s}[C_i][j]| = |\vec{s}[C_i][k]|$ for any $i < n$ and $j, k < 2^n$.

Now we can start to derive the consequent of (6). Fix an $i < n$. The idea is simple: $|\vec{s}[C_i][i]| > |\vec{s}[C_{i+n}][i+n]|$ iff for any j, $|\vec{s}[C_i][j]| > |\vec{s}[C_{i+n}][j]|$. Summing over j, this is equivalent to $|\vec{s}[C_i]| > |\vec{s}[C_{i+n}]|$. Of course, the equivalences must be derived by Lemma A.3 and in particular (3).

First, since both C_i and C_{i+n} do not include the constantly 0 function, we have $\mathrm{Fin}(\vec{s}[C_i][j])$ and $\mathrm{Fin}(\vec{s}[C_{i+n}][j])$. With (A7), we have for all $j < 2^n$, $|\vec{s}[C_i][j]| \geq |\vec{s}[C_{i+n}][j]| \leftrightarrow |\vec{s}[C_i \setminus C_{i+n}][j]| \geq |\vec{s}[C_{i+n} \setminus C_i][j]|$. Let $\vec{t}$ be the sequence of 2×2^n terms with the first 2^n terms being $\langle \vec{s}[C_i \setminus C_{i+n}][j] \rangle_{j<2^n}$ and the rest being $\langle \vec{s}[C_{i+n} \setminus C_i] \rangle_{j<2^n}$. Also let $\vec{t'}$

be the same as $\vec{t}$ except that the first 2^n terms and the last 2^n terms are switched.

Then $\mathsf{D}(\vec{t})$ and $\mathsf{D}(\vec{t'})$ are derivable. This is because for any two terms, if they do not share the same second coordinate, then they are certainly disjoint. But if they do share the same second coordinate, then they are of the form $\vec{s}[C_i \setminus C_{i+n}][j]$ and $\vec{s}[C_{i+n} \setminus C_i][j]$, which are disjoint. Obviously we also have $\mathrm{Fin}(\vec{t})$ and $\mathrm{Fin}(\vec{t'})$.

Now, from left to right, suppose $|\vec{s}[C_i][i]| \geq |\vec{s}[C_{i+n}][i+n]|$. Then, for any $j < 2^n$, we have $|\vec{s}[C_i][j]| \geq |\vec{s}[C_{i+n}][j]|$. By (A7), this implies $|\vec{s}[C_i \setminus C_{i+n}][j]| \geq |\vec{s}[C_{i+n} \setminus C_i][j]|$. Then we can apply (2) to $\vec{t}$ and obtain $|\vec{s}[C_i \setminus C_{i+n}]| = |\vec{s}[C_{i+n} \setminus C_i]|$. But by (A7) again, this gives us $|\vec{s}[C_i]| \geq |\vec{s}[C_{i+n}]|$.

From right to left, assume $|\vec{s}[C_i]| \geq |\vec{s}[C_{i+n}]|$ and suppose for contradiction that $|\vec{s}[C_{i+n}][i+n]| > |\vec{s}[C_i][i]|$. Then for any $j < 2^n$, we have $|\vec{s}[C_{i+n}][j]| > |\vec{s}[C_i][j]|$. By (A7), this implies $|\vec{s}[C_{i+n} \setminus C_i][j]| > |\vec{s}[C_i \setminus C_{i+n}][j]|$. Thus, in sequence $\vec{t'}$ the first 2^n terms are strictly larger than the last 2^n terms, respectively. By (BC3), $>$ implies $\geq$. Hence, by (1), $|\vec{s}[C_{i+n}]| \geq |\vec{s}[C_i]|$, as they are the unions of the first and last 2^n terms, respectively. Together with the assumption, we have $|\vec{s}[C_{i+n}]| = |\vec{s}[C_i]|$. At this point, we can apply (3) and obtain $|\vec{s}[C_{i+n} \setminus C_i][2^n-1]| \leq |\vec{s}[C_i \setminus C_{i+n}][2^n-1]|$. With (A7), this contradicts $|\vec{s}[C_{i+n}][2^n-1]| > |\vec{s}[C_i][2^n-1]|$, which is derived from $|\vec{s}[C_{i+n}][i+n]| > |\vec{s}[C_i][i]|$.

Deriving formula (7)*.* First, note that

$$\begin{aligned} \bigcup_{i<n} \vec{s}[C_i][i] &= \bigcup_{i<n} \bigcup_{f \in C_i} \vec{s}[f][i] = \bigcup_{f \in {}^{2n}2} \vec{s}[f][f_<^{-1}(1)], \\ \bigcup_{i<n} \vec{s}[C_{i+n}][i+n] &= \bigcup_{i=n}^{2n-1} \bigcup_{f \in C_i} \vec{s}[f][i] = \bigcup_{f \in {}^{2n}2} \vec{s}[f][f_>^{-1}(1)]. \end{aligned} \tag{8}$$

Now assume $\vec{s}_< \mathsf{E}\ \vec{s}_>$. Recall that our goal is to derive $|\bigcup_{i<n} \vec{s}[C_i][i]| = |\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]|$. Our strategy is the following. When we assume $\vec{s}_< \mathsf{E}\ \vec{s}_>$, we can show that for any $f \in {}^{2n}2$, treated as a sequence of 0's and 1's, if the number of 1's in the first n places of f and the number of 1's in the last n places of n are not equal, then $\vec{s}[f] = \varnothing$ can be derived. We can call f "balanced" when this condition is satisfied; when f is not balanced, $\vec{s}[f] = \varnothing$ can be derived. However, for those balanced f, when restricted to $\vec{s}[f]$, $\bigcup_{i<n} \vec{s}[C_i][i]$ and $\vec{s}[C_i][i]| = |\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]|$ are of the same size. For a simple illustration, see Figure 5. Then summing over all balanced f, we obtain the required formula.

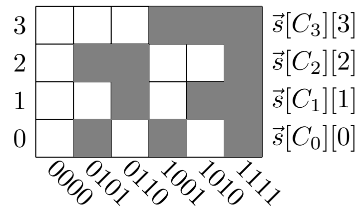

FIGURE 5. The grid with $n = 2$ when $\vec{s}_< \ \mathsf{E}\ \vec{s}_>$. Recall that squares in the same column are of the same size. It is not hard to see then that $|\vec{s}[C_0][0] \cup \vec{s}[C_1][1]| = |\vec{s}[C_2][2] \cup \vec{s}[C_3][3]|$ by comparing them in each column.

Pick an arbitrary $f \in {}^{2n}2$ and let $k_< = |f_<^{-1}(1)|, k_> = |f_>^{-1}(1)|$. Then it is easy to see that the system can derive the following through the equational theory of Boolean algebras:

$$\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<) \wedge \vec{s}[f] \subseteq \mathsf{N}_{k_>}(\vec{s}_>).$$

Also by the definition of N in Definition A.2 and by using the equational theory of Boolean algebras, $\mathsf{N}_i(\vec{s}_>) \cap \mathsf{N}_j(\vec{s}_>) = \varnothing$ and $\mathsf{N}_i(\vec{s}_>) \subseteq (\mathsf{N}_j(\vec{s}_>))^c$ are derivable when $i \neq j$. Hence, if $k_< \neq k_>$, then $\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<)$ and also $\vec{s}[f] \subseteq (\mathsf{N}_{k_<}(\vec{s}_>))^c$. Since we have assumed $\vec{s}_< \ \mathsf{E}\ \vec{s}_>$, we have $\mathsf{N}_{k_<}(\vec{s}_<) = \mathsf{N}_{k_<}(\vec{s}_>)$. This means that we can derive $\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<) \wedge \vec{s}[f] \subseteq (\mathsf{N}_{k_<}(\vec{s}_<))^c$ and then $\vec{s}[f] = \varnothing$.

Now we derive that $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$. When $k_< \neq k_>$, we derive $\vec{s}[f] = \varnothing$. Then trivially $|\vec{s}[f][f_<^{-1}(1)]| = |\varnothing| = |\vec{s}[f][f_>^{-1}(1)]|$.

If $k_< = k_>$, then let $k = k_< = k_>$ and consider the sequence $\vec{t}$ where the terms are $\langle \vec{s}[f][i] \rangle_{f \in C_i}$:

- $\mathsf{D}(\vec{t})$ is derivable using the equational theory of Boolean algebras.
- $\vec{t}$ has $2k$ terms; the union of the first k terms is $\vec{s}[f][f_<^{-1}(1)]$ and the union of the last k terms is $\vec{s}[f][f_>^{-1}(1)]$;
- we showed when we derived (6) that for any i, j, $|\vec{s}[f][i]| = |\vec{s}[f][j]|$; hence for any $i < k$, $|t_i| = |t_{i+k}|$.

Given these three points, we can apply (2) to $\vec{t}$ and derive the equation $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$.

In sum, we have derived for any $f \in {}^{2n}2$ the equation $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$. But then we can apply (2) to the sequence where the first 2^{2n} terms are $\langle \vec{s}[f][f_<^{-1}(1)] \rangle_{f \in {}^{2n}2}$ and the last 2^{2n} are $\langle \vec{s}[f][f_>^{-1}(1)] \rangle_{f \in {}^{2n}2}$. Hence it is derivable that the unions of each, which by (8) are just $\bigcup_{i<n} \vec{s}[C_i][i]$ and $\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]$, are of equal size.

This completes the whole proof. ⊣

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