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ABSTRACT

The relationship between nuclear reactions and nuclear decay derived by F. T. Smith is presented and illustrated by a simple example.

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I. INTRODUCTION

Consider a metastable nucleus C which decays by alpha emission to a daughter nucleus D.

$$C \rightarrow \alpha + D .$$
(1)

An alternative process which will also reflect the properties of the metastable nucleus C is any nuclear reaction resulting from the bombardment of D by an alpha particle of the appropriate energy. One is thus led to expect some sort of relationship between these two processes. The relationship can, in fact, be stated very simply:

$$\Gamma \tau = \hbar$$
(2)

where τ is the mean life of the metastable nucleus C and Γ is the width of the resonance in the cross section for the reaction induced by α on D.

This result was proved by G. Breit and F. L. Yost[1] for one-channel reactions. The general result was stated by R. G. Thomas[2] and proved by F. T. Smith.[3]   The purpose of this report is to describe the application of F. T. Smith's derivation to α-decay.

II.   The Mean Life for Alpha Decay

Suppose we have a metastable system described by the wave function $\psi(t)$. Let H be the Hamiltonian for the system and let the eigenfunctions of H be denoted by $\emptyset_n$. Then

$$H \psi(t) = i \frac{\partial}{\partial t} \psi(t) \tag{3}$$

and

$$H \emptyset_n = E_n \emptyset_n . \tag{4}$$

Expanding $\psi(t)$ in terms of the $\emptyset_n$'s gives

$$\psi(t) = \sum_n \emptyset_n \langle \emptyset_n | \psi(0) \rangle e^{-\frac{iE_n t}{\hbar}} . \tag{5}$$

The probability of finding the system still in the initial configuration after a time interval t is just

$$P(t) = \left| \langle \psi(0) | \psi(t) \rangle \right|^2 .$$

Substituting Eq. (5) into Eq. (6) gives

$$P(t) = \left| \sum_n \left| \langle \emptyset_n \psi(0) \rangle \right|^2 e^{-\frac{E_n t}{\hbar}} \right|^2 \tag{7}$$

Presuamably, $\langle \emptyset_n | \psi(0) \rangle$ will be appreciable only in a narrow range of energies about an energy, say E_0. Then we can write

$$P(t) \approx \left| N(E_0) \int dE \left| \langle \emptyset_E | \psi(0) \right|^2 e^{-\frac{iEt}{\hbar}} \right|^2 \tag{8}$$

where $N(E_0)$ is the density of states at energy E_0.

If $\psi(t)$ is to represent a nucleus experiencing α-decay, then $P(t)$ must be expected to be an exponential. To achieve this result we must have

$$\left|\langle\phi_E|\psi(0)\rangle\right|^2 \cong \frac{\Gamma}{2\pi}\frac{N^{-1}}{(E-E_0)^2 + \Gamma/4^2}. \tag{9}$$

in the vicinity of E_0. To verify this we substitute Eq. (9) into Eq. (8) to find

$$P(t) \approx \left|N(E_0)\right|^2 e^{-\frac{\Gamma t}{\hbar}} N^{-2} = e^{\frac{-\Gamma t}{\hbar}} = e^{-t/\tau} \tag{10}$$

The wave function ϕ_E describes the scattering of α-particles by by the daughter nucleus. The cross section for this process must have a resonance at energy E_0, and the width of this resonance must be Γ. This will be shown to be the case. We will first illustrate these ideas with a simple model and then proceed to a general proof.

III. The δ-Function Barrier Problem

For illustrative purposes we consider a particle moving in one dimension. This particle is initially confined to the region $0 \leqslant r \leqslant R$ by an infinite barrier at $r = 0$ and a finite barrier at $r = R$. In the course of time the particle will leak out through the finite barrier. We want to show how the decay of this state is related to a resonance in the elastic scattering by the barrier.

For the sake of simplicity we choose the barrier to have the shape of a delta function. The Schroedinger equation will be

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} - c^{-1}\delta(r-R)\right)\psi. \tag{11}$$

The eigenfunctions of the Hamiltonian will be

$$\phi_E = (e^{-ikr} \, \eta \, e^{ikr}) \quad r > R$$

(12)

$$\phi_E = A \, (e^{ikr} - e^{ikr}) \quad r < R$$

where

$$E = \hbar^2 k^2/2m \, .$$

(13)

The scattering amplitude $1-\eta$ will exhibit resonances at the energies $E_n = \hbar^2 k_n^2/2m$ where

$$k_n \approx n \, \pi/(R + C).$$

(14)

In the vicinity of these resonances

$$1-\eta = \frac{i\Gamma}{(E-E_n)+i\Gamma/2}$$

(15)

and

$$A = \left(\frac{\Gamma}{(E-E_n) + i\Gamma/2} \right) \frac{R}{2n\pi C}$$

(16)

where

$$\Gamma = - \frac{2\hbar^2}{m} \frac{(n\pi)^3}{R^4} C^2 \, .$$

(17)

By making the length C very small we increase the barrier at $r = R$ and in consequence make the resonances very narrow.

Now suppose $\psi(t)$ is the solution of Eq. (11) which describes the motion of a particle initially confined behind the barrier. We are interested in determining the energy dependence of the quantity

$$B(E) = \langle \phi_E | \psi(0) \rangle$$

$$= \frac{2A}{i} \int_0^R dr \, \sin kr \, \psi(0) = A \, b(E). \qquad (18)$$

The factor A will be of the order of magnitude $\frac{C}{R}$ except near resonances where it will rise to a maximum of $\frac{R}{n\pi C}$. The factor $b(E)$ will in general be an oscillitory function which has a relatively broad maximum at some energy E_0 and tends to vanish as the energy E moves away from E_0. For example, if we choose

$$\psi(0) = \sin k_0 r \qquad r < R$$

$$= \qquad\qquad r > R \qquad\qquad (19)$$

then

$$B(E) = \frac{A}{i} \left\{ \frac{\sin(k_0-k)R}{(k_0 - k)} - \frac{\sin(k_0 + k)R}{(k_0 + k)} \right\} \qquad (20)$$

It follows that $B(E)$ will in general be negligible except at the resonance E_n nearest to the principle maximum of $b(E)$ located at E_0. Thus

$$B(E) \approx b(E_n) \frac{R}{2n\pi C} \quad (E-E_n) + i\,\Gamma/2 \qquad (21)$$

We conclude that unless $\psi(0)$ has a very peculiar form, it will decay exponentially with a time constnat $\tau = \hbar/\Gamma$ where Γ is given by Eq. (17).

We see that that $B(E) = \langle \emptyset_E | \psi(0) \rangle$ shows resonances primarily because
the normalization of the wave function $\emptyset_E$ itself goes through sharp reson-
ances.   These resonances have the same widths as the corresponding resonances
in the cross section.

IV.   The Delay Time

We have seen that the exponential character of the decay of a
metastable nucleus is connected with a resonance in the amplitude of the
wave function describing the scattering of the emitted particle with the
daughter nucleus. At this point we want to relate the resonance behavior
of the wave function to the resonance behavior of the scattering amplitude.
The method we use is due to F. T. Smith.[3]

For nuclear reactions where the total energy is low enough so that no
three-or-more-body channels are open it is possible to divide space into
two regions: the inside or interaction region and the outside or asymptotic
region.   The boundary separating these two regions of space is such that
when the nucleons are found in the outside region they are bound into two
clusters whose separation is great enough so that there is no nuclear inter-
action between them.   Each such pair of nuclides that can be found in the
outside region represents a possible channel for the system.

Thus in the outside region the wave function for such a system is
a sum of contributions from the various channels.

$$\emptyset \rightarrow \sum_{aa'} A_a X_{a'} \; \emptyset_{aa'} (r_{a'}) \; , \tag{22}$$

The A_a are constant coefficients.   The X_a are each a sum of products of
three factors.   One factor is the internal wave function for one of the
clusters in that channel, the second is the internal wave function for the
other cluster in that channel, and the third factor is a spherical harmonic
of the direction of r_a , the displacement of the center of mass of the
first cluster from that of the second.   The sum results from coupling the
spins of the two clusters to their relative angular momentum to make each

X_a an eigenfunction of the total angular momentum. The radial wave functions $\emptyset_{aa'}$ $(r_{a'})$ will have the form

$$\emptyset_{aa'} = (\delta_{a'a} U_{a'}^{(2)} (r_{a'}) + S_{a'a} U_{a'}^{(1)} (r_{a'}))\,. \qquad (23)$$

$U_a^{(2)}$ and $U_a^{(1)}$ are unit current radial wave functions of the incoming and outgoing types respectively. $S_{a'a}$ is the scattering matrix.

We consider the situation resulting from unit incident current in channel a.

$$\emptyset_a \rightarrow \sum X_b\, \emptyset_{ab} \qquad (24)$$

$\emptyset_a$ will satisfy the Schroedinger equation

$$(E-H)\, \emptyset_a = 0\,. \qquad (25)$$

Taking the derivative with respect to energy gives

$$(E-H)\, \frac{\partial \emptyset_a}{\partial E} = -\,\emptyset_a\,. \qquad (26)$$

Multiplying both sides by $\emptyset_a^*$ and integrating over the inside region gives

$$\int^R d\tau\, \emptyset_a^*\, \emptyset_a = J = \int^R d\tau\, \emptyset_a^* (H-E)\, \frac{\partial \emptyset_a}{\partial E} \qquad (27)$$

Now using Eq. (25) and the fact that H is real allows us to write

$$
J = \int^R d\tau \left\{ \phi_a^* (H-E) \frac{\partial \phi_a}{\partial E} - \frac{\partial \phi_a}{\partial E} (H-E) \phi_a^* \right\}
$$

$$
\tag{28}
$$

$$
= \int^R d\tau \left\{ \phi_a^* T \frac{\partial \phi_a}{\partial E} - \frac{\partial \phi_a}{\partial E} T \phi_a^* \right\}
$$

where T, the non-multiplicative part of H, is taken to be the kinetic energy. Next we apply Green's theorem to the right side of Eq. (28) to find

$$
J = \sum_b \frac{\hbar^2 R_b^2}{2M_b} \left\{ \frac{\partial \phi_{ab}}{\partial E} \frac{\partial}{\partial r_b} \phi_{ab}^* - \phi_{ab}^* \frac{\partial}{\partial r_b} \frac{\partial}{\partial E} \phi_{ab} \right\}_{r_b = R_b}
$$

$$
\tag{29}
$$

where the channel radii R_a, R_b, define the boundary of the inside region and M_a is the reduced mass in channel a. Using Eq. (23) and the properties of $U_a^{(2)}$ and $U_a^{(1)}$ it is easy to show that Eq. (29) reduces to

$$
J = \frac{2R_a}{v_a} + \frac{\hbar}{i} \sum_b S_{ab}^+ \frac{\partial}{\partial E} S_{ba} + O\left(\frac{1}{kR}\right)
$$

$$
\tag{30}
$$

where

$$
v_a = \frac{\hbar k_a}{M_a} = \sqrt{\frac{2E}{M_a}}
$$

is the asymptotic relative velocity in channel a. In order to derive Eq. (30) it was necessary to choose the channel radii so that

$$\frac{R_a}{v_a} = \frac{R_b}{v_b} = \frac{R_c}{v_c} = ----$$ (31)

We will let the inside region be large enough so that we can neglect the terms of order $\frac{1}{kR}$.

Since the wave function ϕ_a is normalized to unit incident current, $J = \int^R d\tau\, \phi_a^* \phi_a$ can be interpreted as the time per collision spent by the system in the inside region. $\frac{2R_a}{v_a}$ represents the time that would be spent in the inside region if there were no interaction between the incident particles.

$\frac{\hbar}{i} \sum_b S_{ab}^+ \frac{\partial}{\partial E} S_{ba}$ then represents the time delay due to the interaction. It follows that in the vicinity of a resonance, the quantity

$\frac{\hbar}{i} \sum_b S_{ab}^+ \frac{\partial}{\partial E} S_{ba}$ will represent, to a good approximation the value of

$\int d\tau\, \phi_a^* \phi_a$ where integration is limited to the interior of the compound nucleus C.

If $\phi(0)$ is the wave function for the initial configuration of the metastable nucleus C, then in the vicinity of a resonance the quantity $|\langle \phi_a \phi \langle (0) \rangle|^2$ will have the energy dependence of

$\frac{\hbar}{i} \sum_b S_{ab}^+ \frac{\partial}{\partial E} S_{ba}$. In the vicinity of an isolated resonance

$$S_{ab} = - e^{-i(h_a + h_b)} \left\{ \delta_{ab} - i \ \frac{\sqrt{\Gamma_a \Gamma_b}}{E - E_0 + i\frac{\Gamma}{2}} \right\}$$

$$\Gamma = \sum_a \Gamma_a \tag{32}$$

$$e^{-ih_a} = \sqrt{\frac{U_a^{(2)}(R_a)}{U_a^{(1)}(R_a)}}$$

With the help of this expression we find

$$\left| \langle \phi_a | \psi(0) \rangle \right|^2 \sim \frac{\hbar}{i} \sum_b S_{ab}^+ \frac{\partial}{\partial E} S_{ba} = \frac{\hbar \Gamma_a}{(E-E_0)^2 + \Gamma 2/4} \tag{33}$$

Comparison of Eq. (33) with Eqs. (9) and (10) shows that, indeed, the width for decay is equal to the width of the corresponding reaction cross section. The importance of this result lies in the fact that nuclear reaction theory provides an explicit expression for the width:

$$\Gamma = \sum \Gamma_a$$

$$F_a = 2 \gamma_a^2 s_a$$

$$s_a = \ell m \left(ra \frac{\partial}{\partial r_a} \ell n \ r_a U_a^{(1)}(r_a) \right)_{r_a = R_a}$$

$$\gamma_a^2 = \phi_{ab}^{\ 2} \frac{\hbar^2 R_a}{2 M_a} \bigg/ \langle \phi_b \ \phi_b \rangle_R \tag{34}$$

where Γ_a is the partial width for channel $\underline{a}$, s_a is the penetration factor for channel $\underline{a}$, γ_a^2 is the reduced width for channel $\underline{a}$, and $\emptyset_{ab}$ and $\emptyset_b$ correspond to the quantities that appear in Eqs (23) and (24) respectively. The normalization integral $\langle \emptyset_b \ \emptyset_b \rangle_R$ has the subscript R to indicate that the integration is confined to the inside region.

V. Acknowledgments

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3. F. T. Smith, Phys. Rev. <u>118</u> (1960) 349.