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ON RECIPROCITY IN GENERAL LINEAR VISCOELASTICITY

BY

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SUMMARY

A reciprocal theorem is developed for a time-variable, linear viscoelastic body whose mechanical properties are governed by a single aging time function. The reciprocal relation obtained is of the Maxwell-Betti type, but it involves a convolution in time over a <u>family</u> of histories of mechanical states, rather than over a single history. A simple application of the theorem is made, to deduce a formula for the total volume change of a time-variable viscoelastic body.

INTRODUCTION

The Maxwell-Betti reciprocal theorem of statics for linear elastic bodies is very well known, and has found wide application in the classical theory of elasticity and the classical theory of structures. Dynamic reciprocal theorems for linear elastic bodies have also been developed and applied[1,2,3,4]*, although such theorems do not appear to be quite so well known.

Reciprocal theorems of this type have been extended to linear viscoelastic bodies. Utilizing the Laplace transform, as Graffi[3,4] had done in developing his dynamic reciprocal theorems for the elastic body, Predeleanu[5] established a dynamic reciprocal theorem for a non-aging, isotropic, linear viscoelastic body (including the dilation due to temperature change)**. The body was taken to be constrained at certain points of its surface by fixed supports, and was subjected to tractions over the remainder of its surface and to body forces throughout its volume. Although not done so by Predeleanu, his theorem may be directly extended to the case of mixed-mixed boundary conditions.

Actually, V. Volterra[6] apparently developed the first reciprocal theorem for linear viscoelastic bodies very early in the development of his

* Numbers in brackets designate References at end of paper.

** Predeleanu assumed the viscoelastic properties to be independent of temperature. Since, in fact, these properties are highly temperature dependent, it would appear that the thermal effects are not realistically accounted for in Predeleanu's theorem.

theory of "hereditary elasticity". Restricting attention to quasi-static motions of the medium, he considered a more general class of linear visco-elastic materials than did Predeleanu: an anisotropic material with time-variable relaxation properties, but with time-invariable instantaneous elastic response. However, this theorem is of an entirely different type than the usual kind of reciprocal theorem, for - as was pointed out by Gurtin and Sternberg[7] - the theorem involves "...two states only one of which is viscoelastic". That is, one of the sets of displacement, strain and stress histories appearing in the theorem is not that which could occur in a viscoelastic body. For this reason, applications of Volterra's reciprocal theorem would appear to be very limited in comparison to applications of the usual Maxwell-Betti type of reciprocal theorem.

A reciprocal theorem of the usual Maxwell-Betti type for linear visco-elastic bodies was also established recently by Gurtin and Sternberg[7]. They restricted attention to quasi-static motions of time-invariable, isotropic, linear viscoelastic bodies, and by use of the properties of ordinary Stieltjes convolution established a reciprocal theorem which has the same form as Predeleanu 's theorem. However, the theorem as given by Predeleanu[5] is more general, since he stated that it holds also for dynamic cases.

Here we establish a quasi-static reciprocal theorem for a subclass of the <u>general</u> linear viscoelastic body. By a general linear viscoelastic body, we mean one in which the instantaneous elastic response, as well as the hereditary properties, may change with time. (Aging viscoelastic materials, such as concrete, are of this type. Some time-invariable viscoelastic materials subjected to transient temperature distributions also may be effectively considered of this type.) In contradistinction to Volterra's theorem both of the states involved in our theorem are viscoelastic. Neither the Laplace transform nor ordinary convolution are found to be particularly useful in the development of the theorem. Instead a generalization of ordinary convolution to a convolution over a family of histories of mechanical states is used.

PRELIMINARIES

Consider a viscoelastic body occupying a closed region R in three-dimensional space, the boundary of R being B. We are concerned with infinitesimal deformations, i.e., deformations so small that the body may be regarded as occupying R throughout its history. If we use cartesian coordinates x_i (i = 1,2,3) to denote a point x of R, and t to denote time (restricting attention to $t \geq 0$, $t = 0$ being our arbitrary origin of time), then the mechanical state of the body is specified by the displacement vector $\underline{u}$ (x,t) with components u_i (x,t), the strain tensor $\underline{\epsilon}(x,t)$ with components $\epsilon_{ij}(x,t)$ and the symmetric stress tensor $\underline{\sigma}(x,t)$ with components $\sigma_{ij}(x,t)$; the indices i, j range over 1,2,3.

The strain components are related to those of the displacement vector by

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \qquad (1)$$

where $(\ )_{,i} \equiv \frac{\partial (\ )}{\partial x_i}$. The stress satisfies the equation of equilibrium,

$$\sigma_{ij,j} + f_i = 0 \tag{2}$$

where $\underline{f}(x,t)$ is the body force vector (per unit of volume) having components $f_i(x,t)$, and where we employ the usual summation convention over repeated indices. At a point x of B, if $\underline{n}$ denotes the outward unit normal vector with components n_i, then the traction vector $\underline{t}(x,t)$, with components $t_i(x,t)$, is defined by

$$t_i = n_j \sigma_{ij} \tag{3}$$

We will consider boundary-value problems in which a body is subjected to specified body forces $\underline{f}(x,t)$ in R, to specified tractions $\underline{T}(x,t)$ over B_I, part of its boundary surface, and is supported "conservatively" over B_{II}, the remaining portion of its boundary surface, with $B = B_I + B_{II}$. Thus on B_I we have

$$\underline{t}(x,t) = \underline{T}(x,t) \qquad x \in B_I,\ t \geq 0 \tag{4}$$

As a matter of convenience, we restrict attention to $t \geq 0$, and assume the body to be in a quiescent state for $t < 0$.

By "conservative" supports we mean those which give rise to displacement and traction vectors in the surface of support such that for all t and for all $x \in B_{II}$, either

$$
\begin{aligned}
t_1 (x,t) &= t_2 (x,t) = t_3 (x,t) = 0, \text{ or} \\
t_1 (x,t) &= t_2 (x,t) = u_3 (x,t) = 0, \text{ or} \\
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u_1 (x,t) &= u_2 (x,t) = u_3 (x,t) = 0
\end{aligned}
\tag{5}
$$

where, in the above, $t_i (x,t)$ and $u_i (x,t)$ are components referred to a suitable time-invariable local cartesian base at x, $x \in B_{II}$. The local base may vary with x, $x \in B_{II}$. In the above, it is to be understood that at any point x_1, $x_1 \in B_{II}$, only <u>one</u> of the eight relations given in Equation (5) holds, and it holds there for <u>all</u> times. At any other point x_2, $x_2 \in B_{II}$, the particular one of the above eight relations holding there for all times may be different than that holding at x_1. Such supports include not only all of the conventional types of supports ordinarily encountered in the classical theory of structures (the "fixed" support, the "pinned" support, the "roller" support and the "free" support), but generalizations of them.

We will also consider boundary-value problems (for $t \geq 0$, and starting in an initially quiescent state) in which $\underline{f} (x,t) \equiv 0$ and the displacement is specified over the whole of B_I as the vector function $\underline{U} (x,t)$. Then we would have in place of Equations (2) and (4) the equations

$$\sigma_{ij,j} = 0 \tag{2'}$$

$$\underline{u}(x,t) = \underline{U}(x,t) \qquad x \in B_I, \; t \geq 0 \tag{4'}$$

Attention will be restricted to time-variable linear viscoelastic bodies all of whose viscoelastic moduli are governed by a single aging time function.* Thus we assume the stress-strain relation in the relaxation-integral form

$$\sigma_{ij}(x,t) = \int_o^t \left[- \frac{\partial}{\partial\tau} E_{ijk\ell}(x;\, t,\tau) \right] \epsilon_{k\ell}(x,\tau)\, d\tau \tag{6a}$$

with

$$- \frac{\partial}{\partial\tau} E_{ijk\ell}(x;t,\tau) = G_{ijk\ell}(x)\, \phi(t,\tau) \tag{6b}$$

where we have admitted anisotropy and inhomogeniety, and have assumed quiescence for $t < 0$. $E_{ijk\ell}$, the components of a fourth rank tensor, represents the aging relaxation moduli of the body. It is understood that $\phi(t,\tau) \equiv 0$ when $t < \tau$. $\phi(t,\tau)$ is in general a distribution, so that the integral in Equation (6a) includes instantaenous elastic response.

In dealing with the relationship between stress and strain, it is convenient to represent them (by employing the standard reduced indical

* For an isotropic medium this would imply that in a standard creep test, Poisson's ratio would be a constant, rather than a function of time. On the basis of the limited experimental data available [8,9,10], it appears (at least to a first approximation) that such a relationship holds for concrete- probably the most important structural material exhibiting aging, linear viscoelastic behavior. We also note that <u>incompressible</u>, isotropic, aging linear viscoelastic materials fall into this category.

notation[11]) as vectors $\underline{\sigma}$ and $\underline{\epsilon}$ in a 6-space with components σ_α and ϵ_α, respectively ($\alpha = 1,2,\ldots,6$). The fourth rank tensor with components $G_{ijk\ell}$ then goes over to a matrix $\underline{G}$ with elements $G_{\alpha\beta}$ ($\alpha,\beta = 1,2,\ldots,6$) so that employing matrix multiplication, the relaxation integral form of the stress-strain relation may be written as

$$\underline{\sigma}(x,t) = \underline{G}(x) \cdot \int_0^t \underline{\epsilon}(x,\tau)\ \phi\ (t,\tau)\ d\tau \tag{7a}$$

where the $\cdot$ indicates matrix multiplication.*

It is useful to have this relation in inverted form - that is, in creep-integral form:

$$\underline{\epsilon}(x,t) = \underline{J}\ (x) \cdot \int_0^t \underline{\sigma}(x,\tau)\ \ \psi\ (t,\tau)\ d\tau \tag{7b}$$

In the above, $\underline{J}$ is the matrix inverse to $\underline{G}$, and $\psi(t,\tau)$ (in general, also a distribution, with $\psi(t,\tau) \equiv 0$ when $t < \tau$) is the kernel inverse to $\phi(t,\tau)$, so that

$$\underline{G} \cdot \underline{J} = \underline{J} \cdot \underline{G} = \underline{I} \tag{8a}$$

$$\int_\tau^t \phi(t,\xi)\ \psi(\xi,\tau)\ d\xi = \int_\tau^t \psi(t,\xi)\ \phi\ (\xi,\tau) = \delta\ (t-\tau) \tag{8b}$$

where $\underline{I}$ is the identity matrix (i.e., the Kronecker delta in 6-space) and δ is the Dirac delta function.

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* Generally, $\underline{G}$ is taken to be symmetric. The symmetry of $\underline{G}$ is established by means of either microscopic [12] or macroscopic [13] thermodynamic arguments.

In summary then, the "traction" boundary-value problem of interest here is governed by the field Equations (1), (2) and (7a) or (7b) in the region R, and by the boundary conditions (4) on B_I and (5) on B_{II}, with the traction defined by Equation (3). The "displacement" boundary-value problem of interest here is governed by the field Equations (1), (2') and (7a) or (7b) in R, and by the boundary conditions (4') on B_I and (5) on B_{II} with the traction defined by Equation (3). In actuality, both problems represent a special class of the so-called "mixed-mixed" boundary-value problem.[14]

Before considering the solutions to these viscoelastic problems, it is convenient to first define associated elastic problems and to consider their solutions. The associated elastic problem for the "traction" boundary value problem under consideration here, is defined by the field Equations (1) and (2) in R, but in place of Equations (7a) or (7b), we have instead the elastic stress-strain relation in R given by

$$\underline{\sigma}(x,t) = \underline{G}(x) \cdot \underline{\epsilon}(x,t) \tag{9a}$$

or

$$\underline{\epsilon}(x,t) = \underline{J}(x) \cdot \underline{\sigma}(x,t) \tag{9b}$$

The same boundary conditions (4) on B_I and (5) on B_{II} hold, along with Equation (3). We call the solution to this elastic boundary-value problem (which vanishes by definition for $t < 0$) the associated elastic solution.

Denoting the viscoelastic solution by unprimed symbols, and the associated elastic solution by symbols carrying a superscript prime, then in terms of the associated elastic solution, the viscoelastic solution is given by

$$\underline{\sigma}(x,t) = \underline{\sigma}'(x,t) \tag{10a}$$

$$\underline{u}(x,t) = \int_0^t \underline{u}'(x,\tau)\, \psi(t,\tau)\, d\tau \tag{10b}$$

$$\underline{\epsilon}(x,t) = \int_0^t \underline{\epsilon}'(x,\tau)\, \psi(t,\tau)\, d\tau \tag{10c}$$

It is a simple matter to verify by direct substitution that if $\underline{u}'$, $\underline{\epsilon}'$, $\underline{\sigma}'$ is the associated elastic solution (i.e., if it satisfies Equations (1), (2), (3), (4), (5) and (9)) then $\underline{u}$, $\underline{\epsilon}$, $\underline{\sigma}$, as defined by Equations (10), is the viscoelastic solution (i.e., it satisfies Equations (1), (2), (3), (4), (5) and (7)). Assuming that $E_{ijk\ell}(x;t,t)\,\epsilon_{ij}\,\epsilon_{k\ell}$ is a positive definite quadratic form for $x \in R$, $t \geq 0$, it then follows, from a uniqueness theorem recently established by Lubliner and Sackman[15], that the above solution is the unique solution to the posed viscoelastic boundary-value problem.

For the "displacement" boundary-value problem of interest here, the associated elastic solution is defined to be the solution of Equations (1), (2'), (3), (4'), (5) and (9). As before, denoting this solution by $\underline{u}'$, $\underline{\epsilon}'$, $\underline{\sigma}'$, then by direct substitution it is easy to verify that the viscoelastic solution $\underline{u}$, $\underline{\epsilon}$, $\underline{\sigma}$ (governed by Equations (1), (2'), (3), (4'), (5) and (7)) is given by

$$\underline{u}(x,t) = \underline{u}'(x,t) \tag{11a}$$

$$\underline{\epsilon}(x,t) = \underline{\epsilon}'(x,t) \tag{11b}$$

$$\underline{\sigma}(x,t) = \int_o^t \underline{\sigma}'(x,\tau)\ \phi\ (t,\tau)\ d\tau \tag{11c}$$

RECIPROCAL THEOREM

We shall first construct a reciprocal theorem appropriate for the "traction" boundary-value problem previously described.

Consider two linear viscoelastic bodies, body (1) and body (2), each occupying a closed region R, with boundary $B = B_I + B_{II}$. The stress-strain relationships for the bodies are given by Equations (7a) or (7b), with $\underline{G}$ replaced by $\underline{G}_{(1)}$ and $\underline{G}_{(2)}$, and $\underline{J}$ by $\underline{J}_{(1)}$ and $\underline{J}_{(2)}$, respectively, for bodies (1) and (2). Let $\underline{G}_{(2)} = \underline{G}^T_{(1)}$ and $\underline{J}_{(2)} = \underline{J}^T_{(1)}$, where the superscript T denotes the transpose of a matrix. Each of the bodies is taken to be "conservatively" supported, in the same manner, over the boundary region B_{II}.

Body (1) is subjected to the specified family of loadings consisting of the traction distribution $\underline{t}^{(1)} = \underline{T}^{(1)}(x,t-\xi)$ over B_I and the body force distributions $\underline{f}^{(1)}(x,t-\xi)$ in R, whereas body (2) is subjected to the specified family $\underline{t}^{(2)} = \underline{T}^{(2)}(x,t-\xi)$ over B_I and $\underline{f}^{(2)}(x,t-\xi)$ in R. We restrict attention to $t \geq 0$ and assume each body to be in a quiescent state for $t < 0$. It is to be understood that $\xi \geq 0$, and that $\underline{T}^{(q)} \equiv \underline{f}^{(q)} \equiv 0$ for $t < \xi$, $q = 1,2$.

Denote the displacement, strain and stress fields for the associated elastic problem produced in bodies (1) and (2) by these loadings as $\underline{u}'^{(1)}(x,t-\xi)$, $\underline{\epsilon}'^{(1)}(x,t-\xi)$, $\underline{\sigma}'^{(1)}(x,t-\xi)$ and $\underline{u}'^{(2)}(x,t-\xi)$, $\underline{\epsilon}'^{(2)}(x,t-\xi)$, $\underline{\sigma}'^{(2)}(x,t-\xi)$, respectively. (Obviously these solutions are identically zero for $t < 0$). Then from Equations (10) we have for the viscoelastic solution $\underline{u}^{(q)}$, $\underline{\epsilon}^{(q)}$, $\underline{\sigma}^{(q)}$ in body (q), q = 1,2:

$$\underline{\sigma}^{(q)}(x,t-\xi) = \underline{\sigma}'^{(q)}(x,t-\xi) \tag{12a}$$

$$\underline{u}^{(q)}(x,t,\xi) = \int_o^t \underline{u}'^{(q)}(x,\tau-\xi)\,\psi(t,\tau)\,d\tau \tag{12b}$$

$$\underline{\epsilon}^{(q)}(x,t,\xi) = \int_o^t \underline{\epsilon}'^{(q)}(x,\tau-\xi)\,\psi(t,\tau)\,d\tau \tag{12c}$$

We shall now prove that the following reciprocal relation holds:

$$\int_R dV \int_o^t \underline{f}^{(1)}(x,\xi)\cdot\underline{u}^{(2)}(x,t,\xi)\,d\xi + \int_B dA \int_o^t \underline{t}^{(1)}(x,\xi)\cdot\underline{u}^{(2)}(x,t,\xi)\,d\xi =$$
$$\int_R dV \int_o^t \underline{f}^{(2)}(x,\xi)\cdot\underline{u}^{(1)}(x,t,\xi)\,d\xi + \int_B dA \int_o^t \underline{t}^{(2)}(x,\xi)\cdot\underline{u}^{(1)}(x,t,\xi)\,d\xi \tag{13}$$

It may be noted that in Equation (13), the integrals carried out over B are identical to the same integrals carried out over B_I. This follows from our definition of "conservative" supports and the fact that both bodies (1) and (2) are supported in an identical manner.

By use of the definition of the traction vector, the divergence theorem, the symmetry of the stress tensor, the strain-displacement relations, and the stress equation of equilibrium, the left-hand side of Equation (13), which we denote as L, may be reduced to

$$L = \int_R dV \int_o^t \sigma_{ij}^{(1)}(x,\xi)\, \epsilon_{ij}^{(2)}(x,t,\xi)\; d\xi \tag{14}$$

From Equation (12) we may rewrite this as

$$L = \int_R dV \int_o^t \sigma'^{(1)}_{ij}(x,\xi) \left[\int_o^t \epsilon'^{(2)}_{ij}(x,\tau-\xi)\psi(t,\tau)\; d\tau\right] d\xi \tag{15}$$

But, recalling that $\epsilon'^{(2)}_{ij}(x,\tau-\xi) \equiv 0$ for $\tau < \xi$,

$$\int_o^t \epsilon'^{(2)}_{ij}(x,\tau-\xi)\psi(t,\tau)\; d\tau = \int_\xi^t \epsilon'^{(2)}_{ij}(x,\tau-\xi)\psi(t,\tau)\; d\tau =$$

$$= \int_o^{t-\xi} \epsilon'^{(2)}_{ij}(x,\eta)\; \psi(t,\xi+\eta)\; d\eta \tag{16}$$

where we have used the substitution $\tau - \xi = \eta$. Thus Equation (15) becomes

$$L = \int_R dV \int_o^t \sigma'^{(1)}_{ij}(x,\xi)\left[\int_o^{t-\xi} \epsilon'^{(2)}_{ij}(x,\eta)\psi(t,\xi+\eta)d\eta\right] d\xi \tag{17}$$

We may rewrite the iterated integral on the time variable in Equation (17) as a double integral:

$$L = \int_R dV \iint_D \sigma'^{(1)}_{ij} (x,\xi)\, \epsilon'^{(2)}_{ij} (x,\eta)\, \psi\,(t,\xi+\eta)\, d\xi\, d\eta \tag{18}$$

where D is the closed region in the (ξ,η) plane bounded by the triangle consisting of the three straight line segments connecting the points (0,0), (t,0) and (0,t).

Utilizing the elastic stress-strain relationship, Equation (9a), for the associated elastic solution, and returning again to our matrix-vector notation, Equation (18) may be rewritten as

$$L = \int_R dV \iint_D \underline{\epsilon}'^{(2)} (x,\eta) \cdot \underline{G}_{(1)}(x) \cdot \underline{\epsilon}'^{(1)}(x,\xi)\psi(t,\xi+\eta)\, d\xi\, d\eta \tag{19}$$

In a similar manner, it may be shown that the right-hand side of Equation (13), which we denote by M, may be rewritten as

$$M = \int_R dV \iint_D \underline{\epsilon}'^{(1)} (x,\eta) \cdot \underline{G}_{(2)}(x) \cdot \underline{\epsilon}'^{(2)}(x,\xi)\psi(t,\xi+\eta)\, d\xi\, d\eta \tag{20}$$

Recalling that we chose $\underline{G}_{(2)} = \underline{G}^T_{(1)}$, and that the region D is symmetric with respect to interchange of the variables ξ and η, it then follows that L = M, and the reciprocal theorem is proved.

A similar reciprocal relation holds for the "displacement" boundary-value problem previously described. For that case, the body force $\underline{f}$ is taken to be zero throughout R for both bodies (1) and (2). Instead of two families of traction histories, $\underline{T}^{(q)}(x,\ \tau-\xi)$, being prescribed on B_I, two families of displacement histories, $\underline{U}^{(q)}\ (x,\tau-\xi)$, q = 1, 2, are prescribed (with $t \geq 0$, $\underline{U}^{(q)} \equiv 0$ for $t < \xi$, and quiescence for $t < 0$).

The reciprocal relation then takes the form

$$\int_B dA \int_o^t \underline{t}^{(1)}(x,t,\xi) \cdot \underline{u}^{(2)}(x,\xi)\, d\xi = \int_B dA \int_o^t \underline{t}^{(2)}(x,t,\xi) \cdot \underline{u}^{(1)}(x,\xi)\, d\xi \tag{21}$$

The proof of this theorem is similar to that given above to establish the validity of Equation (13), and is not repeated here.

EXAMPLE

As a simple application of the reciprocal relation just developed, we deduce a formula for the total volume change of a viscoelastic body of the type under consideration here. The body is assumed to be homogeneous and isotropic, subjected to a body force distribution $\underline{f}(x,t)$, and subjected to a traction distribution $\underline{t}(x,t)$ over its boundary surface.

In dealing with the homogeneous, isotropic body, it is convenient to rewrite the stress-strain relationship (given for the general anisotropic case by Equations (7)) in the form

$$e_{ij}(x,t) = \frac{1}{2\mu} \int_o^t s_{ij}(x,\tau)\, \psi(t,\tau)\, d\tau \tag{22a}$$

$$\epsilon_{mm}(x,t) = \frac{1}{3k} \int_o^t \sigma_{mm}(x,\tau)\, \psi(t,\tau)\, d\tau \tag{22b}$$

with μ and k constant. e_{ij} and s_{ij} are the deviatoric components of the strain and stress tensors, respectively, defined by

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{mm} \delta_{ij} \tag{23a}$$

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{mm} \delta_{ij} \tag{23b}$$

δ_{ij} being the Kronecker delta.

Now, in Equation (13), set

$$\underline{f}^{(2)}(x,t) = \underline{f}(x,t) \ , \ \underline{t}^{(2)}(x,t) = \underline{t}(x,t) \tag{24}$$

and let state (1) (with $\underline{f}^{(1)} \equiv 0$) be given by

$$\sigma_{ij}^{(1)}(x,t-\xi) = \delta(t-\xi)\ \delta_{ij} \tag{25a}$$

$$u_i^{(1)}(x,t,\xi) = \frac{1}{3k} \psi(t,\xi)\ x_i \tag{25b}$$

$$\epsilon_{ij}^{(1)}(x,t,\xi) = \frac{1}{3k} \psi(t,\xi)\ \delta_{ij} \tag{25c}$$

It is readily verified that state (1) satisfies all of the governing field equations for a viscoelastic body of the type being considered here. $\sigma_{ij}^{(1)}$ gives rise to the traction vector

$$\underline{t}^{(1)}(x,t-\xi) = \underline{n}\ \delta(t-\xi) \tag{26}$$

on the boundary surface B of the body, where $\underline{n}$ is the outward unit vector normal to B.

Application of Equation (13) yields

$$\Delta V(t) = \frac{1}{3k} \int_R dV \int_o^t \underline{f}(x,\xi) \cdot \underline{r}\,\psi(t,\xi)\, d\xi + \frac{1}{3k} \int_B dA \int_o^t \underline{t}(x,\xi) \cdot \underline{r}\,\psi(t,\xi)\, d\xi \tag{27}$$

where $\Delta V(t)$ is the total volume change of the body, and $\underline{r}$ is the position vector, with components x_i, from our arbitrary choice of origin to a generic point in the closed region R. In arriving at Equation (27) we have used the fact that

$$\Delta V = \int_B \underline{n} \cdot \underline{u}\, dA = \int_R \nabla \cdot \underline{u}\, dV = \int_R \epsilon_{mm}\, dV \tag{28}$$

where $\nabla \cdot \underline{u} = u_{i,i}$.

Formulas similar to (27) were obtained by Gurtin and Sternberg[7] and by Predeleanu[16] for the time-invariable viscoelastic body.

REFERENCES

1. Lord Rayleigh, "Some General Theorems Relating to Vibration", Proc. London Math. Soc., 4, 1873, p. 366.

2. Lamb, H., "On Reciprocal Theorems in Dynamics", Proc. London Math Soc., 19, 1888, p. 144.

3. Graffi, D., "Sui Teoremi di Reciprocità nei Fenomeni Dipendenti dal Tempo," Ann. di Mat. (4) 18, 1939, p. 173.

4. Graffi, D., "Sul Teorema di Reciprocità nella Dinamica dei Corpi Elastici," Mem. Acc. Scienze, Bologna (10) 4, 1946/47, p. 103.

5. Predeleanu, M., "On Thermal Stresses in Visco-Elastic Bodies," Bull. Math. de la Soc. Sci, Math. Phys. de la R.P.R., Tome 3 (51), No. 2, 1959, p. 223.

6. Volterra, V., "Sulle Equazioni Integro--Differenziali della Teoria dell'Elasticita," Atti della Reale Acc. Dei Lincei, 18 (2), 1909, p. 295.

7. Gurtin, M.E. and Sternberg, E., "On the Linear Theory of Viscoelasticity," Arch. Rat. Mech. Anal., 11 (4), 1962, p. 291.

8. McHenry, D., "A New Aspect of Creep in Concrete and its Application to Design," Proc. ASTM, 43, 1943, p. 1069.

9. Duke, C. M. and Davis, H. E., "Some Properties of Concrete under Sustained Combined Stresses," Proc. ASTM, 44, 1944, p. 888.

10. Arutiunian, N. Kh., "Applications de la Theorie du Fluage," Editions Eyroles, Paris, 1957.

11. Sokolnikoff, I.S., "Mathematical Theory of Elasticity," 2nd Ed., McGraw Hill, New York, 1956.

12. DeVault, G.P. and McLennan, J.A., "Statistical Mechanics of Viscoelasticity," Phys. Rev., 137, No. 3A, p. A724, 1965.

13. Gurtin, M.E. and Herrera, I., "On Dissipative Inequalities and Linear Viscoelasticity", Quart. App. Math., 23, 1965, p. 235.

14. Boley, B.A. and Weiner, J.H., "Theory of Thermal Stresses," John Wiley & Sons, New York, 1960.

15. Lubliner, J. and Sackman, J. L., "A Uniqueness Theorem for Aging Viscoelastic Bodies," Rpt. No. 65-15, Proj. No. 4547-E, Struct. Eng. Lab., Univ. of Calif., Berkeley, Dec. 1965.

16. Predeleanu, M., "An Application of a Reciprocal Theorem in the Linear Viscoelasticity Theory," Bull. Math. de la Soc. Sci. Math. Phys., de la R.P.R., Tome 6 (54) nr. 3-4, 1962, p. 231.