Scarcity of Ideas and R&D Options: Use it, Lose it, or Bank it[1]

Nisvan Erkal
University of Melbourne

Suzanne Scotchmer
University of California, Berkeley and NBER

April 15, 2009

[1]This paper expands and supersedes the first part of our 2007 paper "Scarcity of Ideas and Options to Invest in R&D." We especially thank Jennifer Reinganum for her comments at the Fifth Summer Workshop in Industrial Organization in Auckland. We also thank Michael Katz, Stephen Maurer, Robert Merges, Deborah Minehart, Paul Seabright, Brian Wright and participants of the 2008 Research Symposium on the Economics and Law of the Entrepreneur organized by the Searle Center on Law, Regulation, and Economic Growth for useful discussion. We thank the Toulouse Network on Information Technology, NSF Grants SES 0531184, 0830186, and Australian Research Council Grant DP0987070 for financial support. Nisvan Erkal thanks the University of California, Berkeley (ARE), for hosting her as a visiting scholar. Email: n.erkal@unimelb.edu.au, scotch@berkeley.edu.

Abstract

We investigate optimal rewards in an R&D model where substitute ideas for innovation arrive to random recipients at random times. By foregoing investment in a current idea, society as a whole preserves an option to invest in a better idea for the same market niche, but with delay. Because successive ideas may occur to different people, there is a conflict between private and social optimality. We investigate the optimal policy when the social planner learns over time about the arrival rate of ideas, and when private recipients of ideas can bank their ideas for future use. We argue that private incentives to create socially valuable options can be achieved by giving higher rewards where "ideas are scarce."

JEL Classifications: O34, K00, L00

Keywords: Scarce ideas; imagination; innovation; real options; search models; rewards to R&D; unknown hazard rate

1 Introduction

Optimal rewards for R&D depend on how the innovative environment is conceptualized. We consider an innovative environment where different agents have substitute ideas for how to fill a given market niche. The purpose of the reward system is to choose among the substitute ideas.

Many models of R&D begin from a concept that opportunities are common knowledge and eternally present, but progress can nevertheless be slow because R&D is costly and resources are scarce.[1] However, there is another reason that progress can be slow: Ideas for investment are themselves scarce, not only from an individual's point of view, but also for society as a whole. Even if a market niche or economic need is known, there may be considerable delay before someone realizes how to fill it at reasonable cost. In addition to the scarcity of resources, the scarcity of ideas is another constraint on progress.

In the innovative environment we study, investment opportunities are not common knowledge. Ideas for how to fill the market niche have random cost, and arrive at random times to random individuals. If the recipient of an idea invests the cost, the idea becomes an innovation that fills the market niche. Most importantly, innovative environments are distinguished by the arrival rate (scarcity) of ideas.

Our objective is to show how rewards should reflect the scarcity of ideas. The social planner does not know the ideas that have arrived, and does not know who received them. If all ideas were available at the same time, the goal of the social planner would be to find the minimum cost idea. However, that is not possible because the ideas arrive at random times. The planner can weed out high-cost ideas by offering limited rewards, but he still faces a trade-off between cost and delay. To ensure that the market niche is filled at low cost, he may have to endure a costly delay.

We show that

- optimal rewards should increase with the scarcity of ideas;

[1] This is implicitly the premise of a large literature on patent races that builds on models surveyed by Reinganum (1989).

- optimal rewards should increase with delay in filling the market niche; and
- the profit on R&D investments will be positive in equilibrium due to the scarcity of ideas.

Our conclusions about the optimal reward structure are tied to the notion that ideas are scarce, so that the reward policy must mediate between cost and delay. Society should be willing to tolerate higher cost in environments where ideas are scarce (the arrival rate is low). The second conclusion arises when the social planner does not know the scarcity (arrival rate) of ideas, but must make inferences about it as time passes. As time passes with no innovation, the planner becomes more pessimistic about the arrival rate, and will tolerate higher cost in order to reduce delay. We do not know of other papers where the reward policy changes dynamically.[2]

As in O'Donoghue, Scotchmer and Thisse (OST, 1998) and Scotchmer (1999), our model distinguishes between ideas and innovations. However, delay is never optimal in these papers. OST (1998) address environments where the ideas are complements in the sense that each idea builds on previous ideas, and Scotchmer (1999) addresses environments where ideas serve different market niches, but there are no substitute ideas for a given market niche. In the model we discuss here, it is because ideas are substitutes that a certain amount of delay should be tolerated. One of the ideas that arrives during the delay may have low cost.

Our model is a real options model in the spirit of MacDonald and Siegel (1986) and Dixit and Pindyck (1994). An investment is irreversible and could turn out to be a mistake. To avoid mistakes, there is a value to delay. In many real options models, the value of the option is internalized by the firm. In our model, ideas (investment opportunities) accrue to random firms, which means that although waiting is valuable to society, the value of

[2] A dynamically changing reward policy would presumably be optimal in any model where learning takes place about something relevant to rewards. Although the authors do not analyze this aspect, two papers where that might be true are Choi (1991) and Malueg and Tsutsui (1997). In those papers, there is an unknown parameter that governs the hazard rate of success in a production function for R&D that is common knowledge among the firms in a race. In our model below, the planner is learning about the hazard rate at which the population as a whole receives ideas for investment. There is no commonly known but uncertain production function for R&D.

waiting is not internalized by any potential innovator. The problem of the social planner is to ensure that private recipients of ideas preserve socially optimal options.

Our modeling apparatus is reminiscent of search models (see McCall and McCall, 2008), although we do not interpret our random process as search. In search models, all opportunities arrive to a single searcher who sets an optimal stopping policy. In our model, ideas are so scarce that no individual is likely to receive more than one idea. The planner knows that the population as a whole is receiving ideas, and knows something about the stochastic process, but does not know who receives ideas or when. Nevertheless, the planner must set the reward policy. Despite this fundamental difference, the planner's reward policy in the case of a known arrival rate is similar to the stopping rule that emerges in search models. Although some search models involve learning, we do not know of results in the search literature that are analogous to our results for the case that the planner is learning about the arrival rate.

The paper is structured as follows. In section 2, we set forth a simple model of scarce ideas. In sections 3 and 4, we characterize the optimal cost threshold that the planner would like to implement at each date. The optimal cost threshold is a cost such that the possessor of a lower-cost idea should invest. The planner realizes that, if someone invests, society is giving up an option, namely, the option to wait for a better idea. The option that is preserved by not investing is a social option, not internalized by any potential investor.

The stochastic process that determines the option value depends on whether rejected ideas are lost forever ("use it or lose it") or banked for future use ("use it or bank it"). We treat these two cases in sections 3 and 4. Banking is attractive to the social planner when he does not know the arrival rate of ideas. As time passes, the planner becomes more pessimistic about the arrival rate. An idea that seemed too costly a year ago will seem more attractive at present because more delay is predicted. The planner will therefore want access to the banked idea with lowest cost.

In section 5, we show how the planner can implement the optimal cost threshold with and without banking of ideas. The optimal reward policy plays on the fact that banking is attractive to the recipient of an idea whenever the reward function is increasing. The recipient of the idea may be willing to forego the profit available by investing at present in order to gamble on a higher reward in the future. Of course, the recipient may be preempted in the meantime.

We conclude in section 6 by mentioning some ways that the optimal reward policy corresponds to legal institutions.

2 A Model of Scarce Ideas

We assume there is market niche that may be filled with an innovation. The social value of filling the market niche is v/r, where r is the discount rate.

There is an exogenous process by which the potential innovators receive ideas for filling this market niche. To innovate, the inventor must first have an idea, which we interpret as an act of imagination, and then have an incentive to invest in it. Each idea occurs at a random time, to a random recipient. Each idea has associated to it an R&D cost that is drawn independently from a common distribution F with support in $[0, \infty)$ and density f. To create an innovation, the recipient of an idea must invest the cost. We assume that the ideas rain down on the population as a whole according to a Poisson process with parameter λ, and we take the parameter λ as a measure of scarcity. If the hit rate λ is low, ideas are scarce.

The recipient of an idea can invest in it, discard it, or bank it, which means to remember it for future use. If the recipient of an idea invests in it, the process stops because the market niche has been filled. The optimal policy will therefore operate by getting the population of potential innovators to screen their ideas and then to discard or bank those with costs that are too high. The value of the social option created by not investing is that another idea might entail a lower cost. There is thus a social trade-off between cost and delay. The policy objective is to manage this trade-off in a way that is socially optimal.

We assume that each agent receives at most one idea. This is an intentionally extreme assumption that highlights the main premise of the paper. Ideas are scarce, not only for society as a whole, but especially from the perspective of any individual.

The social policy is described by a *threshold function* $c : \mathbf{R}_+ \to \mathbf{R}_+$ such that the recipient of an idea at time t invests if the cost of the idea is less than $c(t)$. A threshold function is *stationary* if there exists $\bar{c}$ in $\mathbf{R}_+$ such that $c(t) = \bar{c}$ for all arrival times t. We say an idea at time t is *viable* if it has cost less than $c(t)$. The expected cost of a random viable idea that arrives at time t is

$$E_F(c(t)) = \int_0^{c(t)} \hat{c} \frac{f(\hat{c})}{F(c(t))} d\hat{c}. \tag{1}$$

We say that the investment process *survives* to t if there is no viable idea before t.

We now consider two versions of the ideas process. In the "use it or lose it" model, an idea that is not used immediately is lost. For example, an idea may be lost or forgotten if the recipient moves on to other projects. However, not all ideas will be lost, especially if there is an incentive to remember them. We consider this in the "use it or bank it" model. The truth is probably somewhere between these two models for most R&D environments. We study the two extreme cases in order to show their implications for optimal rewards.

3 Use it or lose it

In this section, we assume that, if the recipient of an idea decides not to invest, the idea is lost to everyone, including the recipient, and cannot be reclaimed later.

Let $P(t|\lambda, c)$ be the probability of surviving to time t, as seen from time 0, when the threshold function is c and the arrival rate of ideas is λ. The survival probability P differs according to whether ideas can be banked, but in both models, the probability distribution on survival times is stochastically larger at smaller arrival rates.

When recipients either use their ideas or forget them immediately, the instantaneous arrival rate of viable ideas at time t is $\lambda F(c(t))$. As seen from time $t = 0$, the probability of survival to time t with no viable idea is $P(t|\lambda, c)$, defined as follows.

$$P(t|\lambda, c) = e^{-\Lambda(t,c)} \quad \text{where} \quad \Lambda(t, c) = \int_0^t \lambda F\left(c\left(\tilde{t}\right)\right) d\tilde{t} \tag{2}$$

(See, for example, Snyder and Miller, 1991, p. 51.) The probability of surviving to $\hat{t}$, conditional on surviving to an earlier time t, is $P\left(\hat{t}|\lambda, c\right)/P\left(t|\lambda, c\right)$. As seen from time t, the probability that the first viable idea arrives at $\hat{t} > t$ is

$$\frac{d}{d\hat{t}}\left[1 - \frac{P\left(\hat{t}|\lambda, c\right)}{P\left(t|\lambda, c\right)}\right] = \lambda F\left(c\left(\hat{t}\right)\right) e^{-\Lambda\left(\hat{t},c\right)+\Lambda(t,c)} \tag{3}$$

We study both the case that the arrival rate λ is known, and the case that the arrival rate is unknown. We now show for the case when λ is known, the optimal threshold function is stationary, and further, that the optimal stationary value decreases with the arrival rate of ideas, λ.

Conditional on an arbitrary threshold function c, and assuming that no viable idea has occurred before t, social welfare measured from time t is V, defined by

$$\begin{aligned} V\left(t, c, \lambda\right) &= \int_t^\infty e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right)\lambda F\left(c\left(\hat{t}\right)\right) e^{-\Lambda\left(\hat{t},c\right)+\Lambda(t,c)} d\hat{t} \\ &= \int_t^\infty e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right)\lambda F\left(c\left(\hat{t}\right)\right)\frac{P\left(\hat{t}|\lambda, c\right)}{P\left(t|\lambda, c\right)} d\hat{t} \end{aligned} \tag{4}$$

If the threshold function c is optimal, the following condition holds at each t.

$$\left(\frac{v}{r} - c\left(t\right)\right) = V\left(t, c, \lambda\right) \tag{5}$$

The left hand side is the net social value of investing in the threshold idea at time t. The right hand side is the expected, discounted value of waiting for a better idea. If the left hand side were greater than the right hand side, then social welfare could be improved by increasing the threshold cost. If the right hand side were greater than the left hand side, then social welfare could be increased by decreasing the threshold cost.

It is well known in the search literature that, because this is a stationary problem, the optimal threshold is a stationary value, say $\bar{c}$. Welfare as a function of the stationary threshold $\bar{c}$ can be written as

$$\begin{aligned} \bar{V}\left(t, \bar{c}, \lambda\right) &= \left(\frac{v}{r} - E_F\left(\bar{c}\right)\right)\int_t^\infty e^{-r\left(\hat{t}-t\right)}\lambda F\left(\bar{c}\right) e^{-\lambda F(\bar{c})\left(\hat{t}-t\right)} d\hat{t} \\ &= \left(\frac{v}{r} - E_F\left(\bar{c}\right)\right)\frac{\lambda F\left(\bar{c}\right)}{\lambda F\left(\bar{c}\right) + r} \end{aligned} \tag{6}$$

This expression shows the trade-off faced by the policy maker. If a higher stationary cost threshold $\bar{c}$ is tolerated, the innovation will arrive sooner since the hit rate of viable ideas, $\lambda F(\bar{c})$, is then higher, and the discounting expression, $\frac{\lambda F(\bar{c})}{(\lambda F(\bar{c})+r)}$, is larger.

Since the optimal threshold function is stationary, we can conceive of the optimal policy as a value $c^*(\lambda) \in \mathbf{R}_+$, where $c(t) = c^*(\lambda)$ for each t. The first order condition for maximizing (6) can be written for each λ as

$$\left(\frac{v}{r} - c^*(\lambda)\right) - \frac{\lambda F(c^*(\lambda))}{(\lambda F(c^*(\lambda)) + r)}\left(\frac{v}{r} - E_F(c^*(\lambda))\right) = 0. \tag{7}$$

The (unique) solution $c^*(\lambda)$ has the property that investing in the marginal innovation today, and receiving net value $\left(\frac{v}{r} - c^*(\lambda)\right)$, is as valuable as waiting for the next viable idea. The next viable idea will arrive with delay, but may have a lower cost. With a higher arrival rate λ, the cost of waiting is reduced, and it is optimal to be more selective in choosing an idea for investment.

We summarize these conclusions in the following proposition. Part (a) is proved in the appendix in a different way than in the search literature. Part (b) follows from differentiating (7) implicitly.

Proposition 1 *Suppose that the recipient of an idea must use it or lose it. Suppose that the arrival rate of ideas, λ, is fixed and known. Then (a) given λ, the optimal cost threshold is stationary; and (b) the optimal stationary threshold $c^*(\lambda)$ is decreasing with λ.*

We now turn to the more realistic case that λ is unknown. Like all contracts, R&D incentives must depend on things that are verifiable. A prize or patent authority knows whether the market niche has been filled, but does not observe the hypothetical distribution of arrival times, and does not observe the arrival of ideas that are rejected.

The length of time without arrival of a viable idea is a signal of λ. A long period with no arrival should make the observer more pessimistic about λ – it shifts the posterior distribution on λ toward lower values. However, the posterior distribution on λ must also account for the fact that some ideas are rejected. Thus, the threshold function for accepting or rejecting ideas is an ingredient to forming a posterior belief on λ.

We show that, when the posterior distribution on λ is changing as time passes, neither the optimized value function nor the optimal investment strategy is stationary. Because the posterior distribution on λ shifts toward lower values as time passes with no viable idea, the (optimized) value of waiting for a better idea decreases with time. This implies that society should optimally be less discriminating about which idea is accepted. In particular, the socially optimal cost threshold is increasing instead of being stationary.

Let $\tilde{h}$ be the prior density function for the distribution of λ with support $[0, \infty)$. Then the posterior density, conditional on a threshold function c, and conditional on no viable hit having arrived by time $\hat{t}$, is $h\left(\cdot|\hat{t}, c\right)$ with cumulative distribution $H\left(\cdot|\hat{t}, c\right)$, where $h\left(\cdot|\hat{t}, c\right)$ satisfies

$$h\left(\lambda|\hat{t}, c\right) = \frac{\tilde{h}(\lambda) P\left(\hat{t}|\lambda, c\right)}{\int \tilde{h}(\lambda) P\left(\hat{t}|\lambda, c\right) d\lambda} \text{ for each } \lambda \in (0, \infty) \tag{8}$$

The posterior depends on the threshold function c up to time $\hat{t}$, through the values of $P(\cdot|\lambda, c)$, and more specifically in the "use it or lose it" model, through $\Lambda\left(\hat{t}, c\right)$. Let $E\left(\lambda|\hat{t}, c\right)$ be the expected value of λ:

$$E\left(\lambda|\hat{t}, c\right) = \int_0^\infty \lambda h\left(\lambda|\hat{t}, c\right) d\lambda$$

Lemma 1 *If $t_1 < t_2$, the distribution $H(\cdot|t_1, c)$ stochastically dominates $H(\cdot|t_2, c)$. Moreover, $E(\lambda|t, c)$ decreases with t.*

Seen from $t = 0$, and accounting for the uncertainty on λ, the probability of survival to time t is

$$\tilde{P}\left(\hat{t}|c\right) = \int_0^\infty P\left(\hat{t}|\lambda, c\right) \tilde{h}(\lambda) d\lambda = \int_0^\infty e^{-\Lambda\left(\hat{t}, c\right)} \tilde{h}(\lambda) d\lambda \tag{9}$$

Similarly, the probability of surviving to $\hat{t}$, conditional on surviving to t is

$$\frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}(t|c)} = \int_0^\infty \frac{P\left(\hat{t}|\lambda, c\right)}{P(t|\lambda, c)} h(\lambda|t, c) d\lambda = \int_0^\infty e^{-\Lambda\left(\hat{t}, c\right) + \Lambda(t, c)} h(\lambda|t, c) d\lambda$$

The probability that the first viable idea arrives at $\hat{t}$, conditional on surviving to $t < \hat{t}$, is given by

$$\frac{d}{d\hat{t}}\left[1 - \frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}(t|c)}\right] = \frac{F\left(c\left(\hat{t}\right)\right) \int \lambda e^{-\Lambda\left(\hat{t}, c\right)} \tilde{h}(\lambda) d\lambda}{\tilde{P}(t|c)} = \frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}(t|c)} F\left(c\left(\hat{t}\right)\right) E\left(\lambda|\hat{t}, c\right)$$

The social value of continuing from time t is given by a function $\tilde{V}$ defined in the first line of (10). Substituting for $V(t,c,\lambda)$ from (4) gives the expression in the second line, which shows more explicitly the probabilities of investing at each time $\hat{t}$, as they depend on the underlying λ. The rewriting in the third line focuses on the fact that it is the beliefs $E\left(\lambda|\hat{t},c\right)$ that matter at each $\hat{t}$. The instantaneous probability of receiving a viable idea is $E\left(\lambda|\hat{t},c\right)F\left(c\left(\hat{t}\right)\right)$. As seen from t, the probability of arriving at time $\hat{t}$ in the first place is $\frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}(t|c)}$.

$$
\begin{aligned}
\tilde{V}\left(t,c,\tilde{h}\right) &= \int_0^\infty V(t,c,\lambda)\,h(\lambda|t,c)\;d\lambda \qquad (10)\\
&= \int_t^\infty e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r}-E_F\left(c\left(\hat{t}\right)\right)\right)\int_0^\infty \lambda F\left(c\left(\hat{t}\right)\right)e^{-\left[\Lambda\left(\hat{t},c\right)-\Lambda(t,c)\right]}h(\lambda|t,c)\;d\lambda d\hat{t}\\
&= \int_t^\infty e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r}-E_F\left(c\left(\hat{t}\right)\right)\right)E\left(\lambda|\hat{t},c\right)F\left(c\left(\hat{t}\right)\right)\frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}(t|c)}d\hat{t}
\end{aligned}
$$

Let $c:\mathbf{R}_+\to\mathbf{R}_+$ be the threshold function that maximizes $\tilde{V}\left(0,\cdot,\tilde{h}\right)$. Then, analogous to (5), the optimal threshold function c satisfies the following at each t:

$$
\frac{v}{r}-c(t)=\tilde{V}\left(t,c,\tilde{h}\right) \qquad (11)
$$

To show that the optimal c is increasing, it is enough to show that $\tilde{V}\left(\cdot,c,\tilde{h}\right)$ is decreasing. Intuitively, $\tilde{V}$ is decreasing because the observer becomes more and more pessimistic about the arrival rate of ideas as time continues without a viable hit. Because of this pessimism, more delay is expected. To mitigate delay, it is optimal to tolerate higher cost.

In the appendix we prove the following result.

Proposition 2 *Suppose that the recipient of an idea must use it or lose it. Suppose that the arrival rate of ideas, λ, has a prior distribution $\tilde{h}$ with support $[0,\infty)$. Let c be the threshold function that maximizes $\tilde{V}\left(0,\cdot,\tilde{h}\right)$. Then c is increasing.*

In section 5, we discuss how the optimal cost threshold can be implemented. In the "use it or lose it" model, it is easy to implement the optimal cost threshold by setting the reward equal to the cost threshold. This is because each recipient of an idea has a single opportunity to invest. He will not receive another idea (ideas are scarce), and he must either invest in the idea immediately or lose it forever.

4 Use it or Bank it

When the reward is equal to the cost threshold, and therefore (with unknown λ) increasing, the possessor of an idea may have an incentive to delay investment to get a higher reward. If the recipient can bank his idea for later use, the social planner needs to take this into account in choosing the optimal cost threshold as well as a reward policy to implement it.

How should the planner view banking? The social planner does not want to delay investments that should be viable under his optimal cost threshold. His reward function should ensure that this does not happen. At the same time, banking ideas for future use is tantamount to increasing the arrival rate of ideas in the future. Since this is valuable, the optimal cost threshold should take it into account.

With banking, the social policy is again a threshold function c. Ideas accumulate over time and are banked by the recipients. An idea that is converted to an innovation can either be a banked idea or a new idea.

The marginal probability of investment at time t must be described differently according to whether the threshold function c is increasing or decreasing at that t. If decreasing, the banked ideas are irrelevant. Any banked idea that would be chosen at t would also have been chosen at $t - dt$. If there is investment at t, it is because a viable idea materializes at that moment. On the other hand, if c is increasing at t, then banked ideas may become viable. If c is increasing, both the banked ideas and the increasing cost threshold affect the probability of investment at time t.

We will describe the probabilities of survival at each t by reference to Figure 1, which shows an arbitrary cost threshold function. To describe the probability of investing in an idea at any t where c is decreasing, such as t_1 in Figure 1, let t_m be the largest value smaller than t where c is nonincreasing. On the domain $[t_m, t]$, there are no viable banked ideas. Let $Q(t_m|\lambda, c)$ be the probability of surviving to t_m. Then the probability of survival to t,

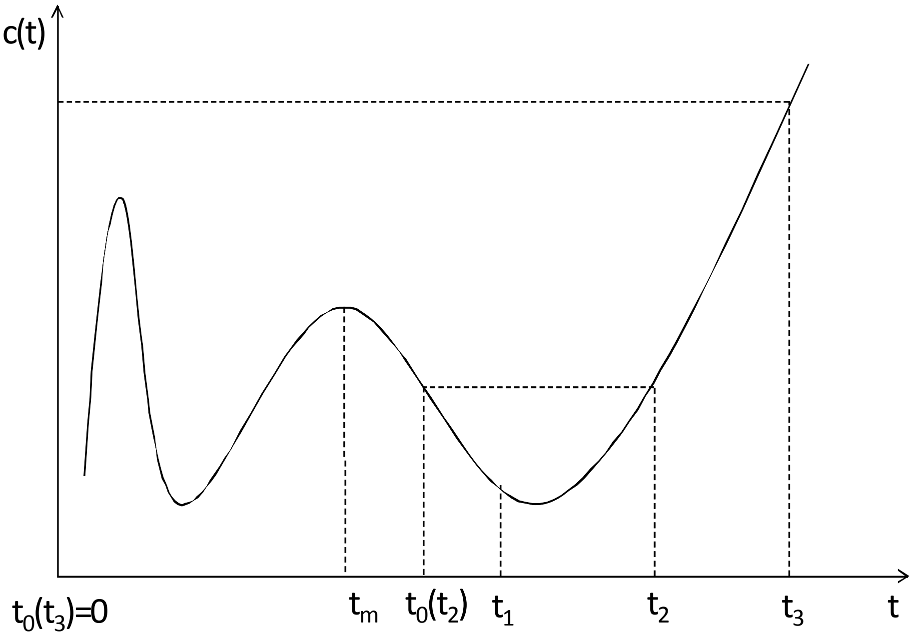

Figure 1: The stochastic process with banking

(that is, the probability that there is no viable idea by time t) is similar to the case without banking:

$$Q\left(t_m|\lambda, c\right) e^{-\Lambda(t,c)+\Lambda(t_m,c)}$$

where

$$\Lambda\left(t, c\right) = \int_0^t \lambda F\left(c\left(\tilde{t}\right)\right) d\tilde{t}$$

This is the probability of survival to t_m times the probability that no viable idea arrives in the interval $[t_m, t]$.

To describe the probabilities of survival at t where c is increasing, let $t_0\left(t\right)$ be the largest value smaller than t such that $c\left(t_0\left(t\right)\right) = c\left(t\right)$. If there is no such value, let $t_0\left(t\right) = 0$.

Two such points are t_2 and t_3 in Figure 1. At t_2, the relevant banked ideas have been accumulating for a shorter period of time than at t_3. At t_2, any ideas below the cost threshold $c\left(t_2\right)$ that were received before $t_0\left(t_2\right)$ would have been used before $t_0\left(t_2\right)$. Therefore the relevant banked ideas are those which accumulated between $t_0\left(t_2\right)$ and t_2. At t_3, there may be relevant ideas with cost near $c(t_3)$ that accumulated very early, since there was never a time when such high-cost ideas were below the cost threshold.

The probability of survival to t (that is, the probability that there is no viable idea by time t) is

$$Q(t_0(t)|\lambda, c)e^{-\lambda F(c(t))[t-t_0(t)]}$$

This is the probability of survival to $t_0(t)$ times the probability that no viable idea arrives in the interval $[t_0(t), t]$.

Thus, if ideas are banked, the stochastic process that determines the probability of survival until t satisfies

$$Q(t|\lambda, c) = \begin{cases} Q(t_m|\lambda, c)\, e^{-\Lambda(t,c)+\Lambda(t_m,c)} & \text{if} \quad c \text{ is decreasing in } [t_m, t] \\ Q(t_0(t)|\lambda, c)e^{-\lambda F(c(t))[t-t_0(t)]} & \text{if} \quad c \text{ is increasing at } t \end{cases} \tag{12}$$

It is useful in the following analysis to define

$$\mathcal{F}(t|c) = \begin{cases} F(c(t)) & \text{if } c'(t) \leq 0 \\ [F(c(t)) + f(c(t))(t - t_0(t))\, c'(t)] & \text{if } c'(t) > 0 \end{cases}$$

As seen from time t, the probability of arriving at $\hat{t}$ is $\frac{Q(\hat{t}|\lambda,c)}{Q(t|\lambda,c)}$. The probability that the first viable idea becomes available at $\hat{t}$ is the probability of arriving there, times the instantaneous probability that a viable idea arrives at time $\hat{t}$, namely,

$$\frac{d}{d\hat{t}}\left[1 - \frac{Q(\hat{t}|\lambda, c)}{Q(t|\lambda, c)}\right] = \lambda\mathcal{F}(\hat{t}|c)\frac{Q(\hat{t}|\lambda, c)}{Q(t|\lambda, c)}$$

When c is decreasing, the instantaneous probability of an innovation, $\lambda\mathcal{F}(\hat{t}|c) = \lambda F(c(\hat{t}))$, is the same as in the "use it or lose it" model, namely, the probability that a viable idea occurs in the interval $(\hat{t}, \hat{t} + d\hat{t})$. But when c is increasing, the innovation may result from a banked idea rather than from an idea that occurs in the interval $(\hat{t}, \hat{t} + d\hat{t})$. The instantaneous probability of innovation is therefore larger, namely, $\lambda\mathcal{F}(\hat{t}|c) > \lambda F(c(\hat{t}))$. The instantaneous probability of innovation has two parts. First is the probability that a viable idea arrives to someone in $d\hat{t}$, namely $\lambda F(c(\hat{t}))\, d\hat{t}$. Second is the probability that a banked idea is called into play. When the threshold rises by $c'\left(\hat{t}\right)d\hat{t}$, the probability that there is a banked idea in the cost band $c'\left(\hat{t}\right)d\hat{t}$ is $\lambda f\left(c\left(\hat{t}\right)\right)\left(\hat{t}-t_0\left(\hat{t}\right)\right)$.

Conditional on surviving to time t, social welfare measured from time t is B defined by

$$B(t,c,\lambda)=\int_t^{\infty}e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r}-E_F\left(c\left(\hat{t}\right)\right)\right)\lambda\mathcal{F}\left(\hat{t}|c\right)\frac{Q\left(\hat{t}|\lambda,c\right)}{Q\left(t|\lambda,c\right)}d\hat{t}$$

If the threshold function c is optimal, the following condition holds at each t.

$$\left(\frac{v}{r}-c(t)\right)=B(t,c,\lambda) \tag{13}$$

When ideas can be banked, it remains true that the optimal cost threshold is stationary. If the threshold is stationary, then the probability distribution $Q(\cdot|\lambda,\bar{c})$ is the same as $P(\cdot|\lambda,\bar{c})$, so $\bar{B}(t,\bar{c},\lambda)=\bar{V}(t,\bar{c},\lambda)$. Thus, the first order condition is the same as in the "use it or lose it" model, namely (7). Therefore, as recorded in the next proposition, the stationary cost thresholds are the same in both cases.[3]

Proposition 3 *Suppose that the recipient of an idea can use it or bank it. Suppose that the arrival rate of ideas, λ, is fixed and known. Then, given λ, the optimal cost threshold is stationary, has the same value $c^*(\lambda)$ as in the "use it or lose it" model, and is thus decreasing with λ.*

We now turn to the case that ideas can be banked, and the hit rate of ideas, λ, is unknown. The prior is again $\tilde{h}$, and using the survival probabilities described in (12), the posterior distribution on λ is again described by (8), substituting Q for P. The analog to Lemma 1 holds for the distribution Q, by the same proof as for the distribution P.[4] $E\left(\lambda|\hat{t},c\right)$ decreases with $\hat{t}$.

[3] Banking is the same as recall in the search literature. See McCall and McCall (2008) for similar results in search theory with and without recall. As in the "use it or lose it" model, we give a different proof, using our social welfare function. The social welfare function is useful for understanding the case of unknown λ.

[4] Lemma 1 is proved by using Claim 2 in the proof. Claim 2 applies here because, for the distribution Q,

$$\frac{d}{dt}h(\lambda|t,c)=\mathcal{F}(c(t))\,h(\lambda|t,c)\left[E(\lambda|t,c)-\lambda\right].$$

We define $\tilde{Q}\left(\hat{t}|c\right)$ analogously to (9), except that the stochastic process underlying $Q\left(\hat{t}|\lambda, c\right)$ is defined by (12) instead of (2). The social value of continuing from time t is given by a function $\tilde{B}$, which we write in two ways. The first line is the definition, and the second line is equivalent, emphasizing that the belief on λ is updated at each $\hat{t}$.

$$\begin{aligned} \tilde{B}\left(t, c, \tilde{h}\right) &= \int_0^\infty B\left(t, c, \lambda\right) h\left(\lambda|t, c\right) d\lambda \\ &= \int_t^\infty e^{-r\left(\hat{t}-t\right)} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) E\left(\lambda|\hat{t}, c\right) \mathcal{F}\left(\hat{t}|c\right) \frac{\tilde{Q}\left(\hat{t}|c\right)}{\tilde{Q}\left(t|c\right)}\, d\hat{t} \end{aligned} \tag{14}$$

The optimal c satisfies

$$\frac{v}{r} - c\left(t\right) = \tilde{B}\left(t, c, \tilde{h}\right) \tag{15}$$

The following proposition shows that the optimal cost threshold with banking is still increasing when the social planner is continuously updating his posterior about λ.

Proposition 4 *Suppose that the recipient of an idea can use it or bank it. Suppose that the arrival rate of ideas, λ, has a prior distribution $\tilde{h}$ with support* $[0, \infty)$. *Let c be the threshold function that maximizes $\tilde{B}\left(0, \cdot, \tilde{h}\right)$. Then c is increasing.*

We conclude from Propositions 1 and 2, together with Propositions 3 and 4, that it is learning about λ that causes the optimal threshold to be increasing. It is not the banking of ideas *per se.*

Finally we show that the optimized social welfare is higher with banking than without. When ideas are banked, the social planner is more pessimistic about λ at each t for a given c. At the same time, the arrival rate of viable ideas is higher when some of ideas may come from the idea bank. The next proposition shows that the latter effect dominates.

Proposition 5 *Let $\mathcal{C}$ be the set of threshold functions c that are increasing. Then for each* $t > 0$, $\max_{c\in\mathcal{C}} \tilde{B}\left(t, c, \tilde{h}\right) > \max_{c\in\mathcal{C}} \tilde{V}\left(t, c, \tilde{h}\right)$.

From (11) and (15), this proposition implies the following:

Corollary 1 *Let c^V be the optimal threshold function in the "use it or lose it" model, and let c^B be the optimal threshold function in the "use it or bank it" model. Then $c^V > c^B$.*

The social planner prefers to be more selective when he can rely on banked ideas, even if he is more pessimistic about λ at each t.

5 Implementing the Optimal Cost Threshold

The social planner cannot implement the optimal cost threshold directly, because the social planner is not the recipient of the ideas. Ideas for R&D are widely dispersed within the population of potential innovators. At best the social planner can try to implement the optimal threshold by setting rewards.

We suppose that the social planner sets a *reward function* $\rho : \mathbf{R}_+ \rightarrow \mathbf{R}_+$. For example, the reward function can represent patent policy or a prize system. The reward function ρ implements the threshold function c if the possessor of an idea with cost c_0 at time t invests in the idea if and only if his idea satisfies $c_0 \leq c(t)$. In the "use it or lose it" model, the only relevant ideas are those that just arrived, but in the "lose it or bank it" model, the relevant idea might previously have been banked.

We already pointed out how to implement the optimal cost threshold in the "use it or lose it" model. We record it formally here.

Proposition 6 *Suppose that c is the optimal cost threshold in the "use it or lose it" model. Then c can be implemented by setting $\rho(t) = c(t)$ for all t.*

Implementation is also easy whenever the optimal cost threshold is stationary. This occurs in both models when λ is known:

Proposition 7 *Let c be the optimal threshold function when the hit rate of ideas, λ, is fixed and known. Then c is stationary, and can be implemented by a reward function that satisfies $\rho(t) = c(t)$ for all t. This applies in both the "use it or lose it" model and the "use it or bank it" model. In both cases, the optimal stationary reward decreases with λ.*

We interpret this proposition to mean that rewards should be higher when ideas are scarce.

Implementation is not as easy in the "use it or bank it" model when the hit rate of ideas is unknown. The optimal cost threshold is increasing, which implies that the reward must be greater than the implemented cost threshold at each t. If equal, the recipient of a marginal idea (with cost equal to $c(t)$) would not invest as intended, since investing in the marginal idea would lead to zero profit. Since the reward function is increasing, the possessor of the marginal idea might make positive profit by waiting for some period until the reward is higher. Even if the possessor of the idea might be preempted during the delay, the expected profit with delay is still larger than zero.[5]

We suppose that the social planner chooses a reward function ρ, and the recipients of ideas choose investment strategies. Each recipient's investment strategy is a threshold function that indicates whether, when the opportunity arises, the possessor of the idea will invest in the idea or bank it. The planner's objective is to make sure that the privately chosen threshold functions correspond to the threshold function that is optimal.

Each idea recipient's incentive to bank or invest depends on his belief about λ, and also on his belief about the other agents' investment strategies. If there is a large accumulation of banked ideas, the probability of being preempted is high.

Further, the social planner must predict these beliefs. If the social planner does not know the beliefs of the idea recipients, he cannot predict their investment strategies, and thus cannot predict the cost threshold that will be implemented by his reward function.

We solve the problem of beliefs in a familiar way. We require that beliefs must be correct in equilibrium. For the threshold function c that will be implemented, a recipient must believe in equilibrium that other agents invest according to c, and the recipient must find it most profitable to invest according to c himself. The point is to find a reward function with this result.

The planner's belief about λ is irrelevant in the following discussion. The planner can implement any nondecreasing threshold he wants, provided he knows the beliefs of the idea recipients. The idea recipients have more information than the planner, and will have

[5] A similar type of trade-off exists in Weeds (2002), who considers a model of R&D competition where delay is undermined by the fear of pre-emption.

different beliefs than the planner. This is because, when a nonviable idea arrives to a recipient, the arrival contains information about λ even if the idea is banked instead of used.

Let $\hat{h}(\cdot|t,c)$ represent the belief of each recipient about λ, with expected value $\hat{E}(\lambda|t,c)$. The argument c is a belief, namely, the recipient's belief about the investment strategy (cost threshold) of the other recipients. Our notation $\hat{h}$ incorporates an assumption about equilibrium: that in equilibrium all other recipients of ideas obey the same investment strategy, c. We justify this assumption after defining the idea recipients' profit function.

A recipient's belief $\mathcal{P}$ on the probability of arriving at $\hat{t}$ is given by

$$\mathcal{P}\left(\hat{t}|c\right) = \int_0^\infty Q\left(\hat{t}|\lambda, c\right)\tilde{h}(\lambda)\, d\lambda$$

where c is the threshold function that other recipients are assumed to obey, and $\hat{h}$ is the recipients' posterior belief on λ. The probability of arriving at $\hat{t}$, having already arrived at t, is given by

$$\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)} = \int_0^\infty \frac{Q\left(\hat{t}|\lambda, c\right)}{Q(t|\lambda, c)}\hat{h}(\lambda|t,c)\, d\lambda$$

To define the idea recipient's profit function, suppose that he possesses an idea with cost c_0 at time t. The innovator's profit, as a function of the time $\hat{t}$ at which he will invest, is given by (16).

$$\pi\left(\hat{t}, c_0|t, c\right) = \left(\rho\left(\hat{t}\right) - c_0\right) e^{-r\left(\hat{t}-t\right)}\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)} \tag{16}$$

The derivative of the profit function is

$$\begin{aligned}
\frac{d}{d\hat{t}}\pi\left(\hat{t}, c_0|t, c\right) &= \rho'\left(\hat{t}\right) e^{-r\left(\hat{t}-t\right)}\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)} - \left(\rho\left(\hat{t}\right) - c_0\right) e^{-r\left(\hat{t}-t\right)}\left[r\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)} + \frac{d}{d\hat{t}}\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)}\right] \\
&= e^{-r\left(\hat{t}-t\right)}\frac{\mathcal{P}\left(\hat{t}|c\right)}{\mathcal{P}(t|c)} \times \left(\rho'\left(\hat{t}\right) - \left(\rho\left(\hat{t}\right) - c_0\right)\left[r + \mathcal{F}\left(\hat{t}|c\right)\hat{E}\left(\lambda|\hat{t}, c\right)\right]\right)
\end{aligned}$$

where

$$\hat{E}\left(\lambda|\hat{t}, c\right) = \int_0^\infty \lambda\hat{h}\left(\lambda|\hat{t}, c\right) d\lambda$$

The optimal investment decision at $\hat{t}$ is

$$\begin{aligned} \text{delay if } \rho'\left(\hat{t}\right) &> \left(r + \hat{E}\left(\lambda|\hat{t}, c\right)\mathcal{F}\left(\hat{t}|c\right)\right)\left[\rho\left(\hat{t}\right) - c_0\right] \\ \text{invest if } \rho'\left(\hat{t}\right) &\leq \left(r + \hat{E}\left(\lambda|\hat{t}, c\right)\mathcal{F}\left(\hat{t}|c\right)\right)\left[\rho\left(\hat{t}\right) - c_0\right] \end{aligned} \tag{17}$$

In (17), ρ' on the left hand side is the benefit of delay. The right hand side is the cost of delay, namely, the interest cost r on the foregone profit $\rho\left(\hat{t}\right) - c_0$, and the perceived probability $\hat{E}\left(\lambda|\hat{t}, c\right)\mathcal{F}\left(\hat{t}|c\right)$ of being preempted.

The investment strategy derived in (17) is the same for all idea recipients. However, the derivation is based on the prior assumption that all idea recipients have the same belief about λ. This is justified in the following Remark.

We write the following as a remark instead of a lemma because the proof in the appendix elaborates the model, assuming that the population of idea recipients is finite instead of infinite, and taking limits. We do this in order to derive limit beliefs as the population becomes large. We show that, in the limit, the beliefs of idea recipients do not depend on when a recipient received his idea. In the limit, the probability of receiving an idea is zero, and the timing of the idea has negligible impact beyond the impact of receiving one. Nevertheless, the limit beliefs are more optimistic than those of the planner because the planner has not observed the arrival of any idea at all.

Remark 1 *Let c be an arbitrary nondecreasing investment strategy (cost threshold). (a) If all recipients of ideas believe that c is the investment strategy of every other recipient, then at a given time t, every recipient of an idea has the same belief on arrival rates, which we call $\hat{h}(\cdot|t, c)$. (b) The recipients' belief $\hat{h}(\cdot|t, c)$ stochastically dominates the planner's belief $h(\cdot|t, c)$ at every t, and as a consequence, $\hat{E}(\lambda|t, c) > E(\lambda|t, c)$. (c) Every recipient of an idea with a given cost, say c_0, has the same optimal investment strategy given by (17).*

The optimal investment behavior (17) should guide the planner in choosing his reward function. For an arbitrary nondecreasing cost threshold c, let the reward function satisfy

$$\rho'(t) = \left(r + \hat{E}(\lambda|t, c)\mathcal{F}(t|c)\right)\left[\rho(t) - c(t)\right] \tag{18}$$

The following lemma says that, with the reward function defined in (18), idea recipients will indeed obey the threshold function c that determines beliefs (represented by $\hat{E}(\lambda|t,c)\mathcal{F}(t|c))$.

Lemma 2 *Let c be a nondecreasing cost threshold, and suppose that the belief of each idea recipient is c. Suppose that the reward function ρ solves (18) at every t. Then for each t, a recipient's most profitable investment strategy is to invest if he has an idea with cost $c_0 \leq c(t)$, and not otherwise.*

Proof: The condition (17) is clearly optimal at the margin, for choosing whether to invest at t or delay for a length of time dt. We must also show that if (17) holds, a longer delay is also not profitable.

If $c_0 \leq c(t) \leq c(\tilde{t})$, then

$$\rho'(t) = \left(r + \hat{E}(\lambda|t,c)\mathcal{F}(t|c)\right)[\rho(t) - c(t)] \leq \left(r + \hat{E}(\lambda|t,c)\mathcal{F}(t|c)\right)[\rho(t) - c_0]$$

and

$$\rho'(\tilde{t}) = \left(r + \hat{E}(\lambda|\tilde{t},c)\mathcal{F}(\tilde{t}|c)\right)[\rho(\tilde{t}) - c(\tilde{t})] \leq \left(r + \hat{E}(\lambda|\tilde{t},c)\mathcal{F}(\tilde{t}|c)\right)[\rho(\tilde{t}) - c_0]$$

The first line (respectively, second line) means that it is more profitable to invest at t rather than $t+dt$ (respectively, $\tilde{t}$ rather than $\tilde{t}+dt$) because the additional profit from delay (the left hand term) is no greater than the cost of delay (the right hand term). The first line (respectively, second line) holds because $c_0 \leq c(t)$ (respectively, $c_0 \leq c(\tilde{t})$). If the idea was not available at t (or if the possessor of the idea made a mistake by not investing), he will invest at the earliest next time, such as at $\tilde{t}$, since $c_0 \leq c(t)$ implies $c_0 \leq c(\tilde{t})$ whenever $t < \tilde{t}$. At any time after t, the possessor of an idea with cost less than $c(t)$ prefers to invest rather than bank. $\square$

Thus, if c is nondecreasing and the reward function ρ is chosen to satisfy (18), recipients of ideas will invest according to the investment strategy c. We therefore say that ρ *implements* c if ρ satisfies (18) and also satisfies $\rho(t) \geq c(t)$.

Proposition 8 *Suppose that c is nondecreasing. There exists a reward function ρ that implements c. The function ρ satisfies $\rho > c$ if c is increasing. Further, $\rho(t) - c(t) \to 0$ as $t \to \infty$.*

Proof: Let

$$\begin{aligned} \rho(t) &= c(t) + \frac{k - \int_0^t e^{-D(\hat{t})} c'\left(\hat{t}\right) d\hat{t}}{e^{-D(t)}} \qquad (19) \\ \text{where } D\left(\hat{t}\right) &= \int_0^{\hat{t}} \left(r + \hat{E}\left(\lambda|\tilde{t}, c\right) \mathcal{F}\left(\tilde{t}|c\right)\right) d\tilde{t} \\ k &= \lim_{t\to\infty} \int_0^t e^{-D(\hat{t})} c'\left(\hat{t}\right) d\hat{t}. \end{aligned}$$

The function ρ defined by (19) is a solution to (18). Since $\int_0^t e^{-D(\hat{t})} c'\left(\hat{t}\right) d\hat{t}$ is nondecreasing with t, the choice of k ensures that $\rho(t) - c(t) \geq 0$ for all t, with strict inequality if c is increasing on some domain. The definition of k can be satisfied because at each t, $\int_0^t e^{-D(\hat{t})} c'\left(\hat{t}\right) d\hat{t} \leq \int_0^t c'\left(\hat{t}\right) d\hat{t} = c(t) - c(0)$. Since c is nondecreasing and bounded by (v/r), there exists $\bar{c} \leq (v/r)$ such that $c(t) \to \bar{c}$.

It holds that $\rho(t) - c(t) \to 0$ because

$$\frac{k - \int_0^t e^{-D(\hat{t})} c'\left(\hat{t}\right) d\hat{t}}{e^{-D(t)}} \to 0$$

by L'Hopital's rule:

$$\frac{e^{-D(t)} c'(t)}{e^{-D(t)} \left(r + \hat{E}(\lambda|t, c) \mathcal{F}(t|c)\right)} = \frac{c'(t)}{\left(r + \hat{E}(\lambda|t, c) \mathcal{F}(t|c)\right)} \to 0$$

□

We close this section with a comment on the profitability of R&D in aggregate. Due to the scarcity of ideas, innovators make positive profit on average. This is because the cost of an implemented idea will generally be lower than the threshold. The recipient of a low-cost idea is in a favored position, and everyone would like to have such an idea, but there is little that one can do to create the investment opportunity. In fact, we have taken the extreme assumption that investment opportunities arrive entirely by chance. We have done this to emphasize our key departure from the more standard R&D literature, where all firms have access to an investment opportunity, and profit may be dissipated in a patent race or through preemptive strategies. As a consequence, one would not expect to observe in equilibrium that the return to R&D investments is the same as the return to capital. On average, it should be higher.

Our argument that the optimal cost threshold can be implemented seems to overlook the possibility that an innovator might be able to keep his innovation secret while charging a proprietary price. This would presumably subvert the objective of the reward, and possibly not implement the optimal cost threshold. However, as we show in our (2007) paper, at least in the case of known λ, secrecy is never optimal.[6] This assumes that with secrecy, another innovator can claim the reward and end the prior innovator's proprietary profit stream.

6 Economic Concepts and Legal Concepts

We interpret ideas, and the fact that ideas are private, as a model of imagination or creativity. Ideas have economic value because they are scarce. We have argued that rewards should be higher in environments where ideas are scarce. If ideas are scarce, higher cost should be tolerated in order to reduce delay. We have also argued that rewards should be increasing as time passes without an innovation. Longer delay leads to expectation of an even longer delay. The delay can be mitigated with higher rewards, since higher rewards encourage investment in higher-cost ideas. Because ideas are not common knowledge, innovators make positive profit in expectation.

These arguments apply equally well to patents and prizes, and any other way of giving rewards.[7] Patents raise the issue of whether our prescriptions can be implemented under existing patent doctrine. They also raise the question of how deadweight loss incurred in collecting the reward money changes the optimal cost threshold.

[6] This result contrasts with previous treatments of secrecy in the literature. See, for example, Denicolo and Franzoni (2004) and Erkal (2005).

[7] For a sample of the many ways, other than patents, that economists have thought about incentives in R&D, see Wright (1983), chapters 2 and 8 of Scotchmer (2004), and Hopenhayn, Llobet and Mitchell (2006).

The main requirements for obtaining a patent are novelty, nonobviousness, utility and enablement. Together, these requirements govern the breadth of claims that are granted. When the statutory patent life is the same for all patentable innovations, breadth is the main lever to differentiate rewards. Our prescription is therefore that the patent office and courts should grant generous claims (broad patents) when ideas are scarce, or more particularly, when the innovation arrives after long delay.

Patent law doctrine also has a threshold standard for granting a patent, namely, the nonobviousness requirement. Our arguments can be interpreted to mean that this threshold standard should be interpreted more leniently when ideas are scarce. In fact, patent doctrine has its own term for this circumstance, namely, "long felt need." Long-felt need is one of the secondary considerations for patentability.

7 Appendix

7.1 Proof of Proposition 1

(a) We first show that the optimized value of V is stationary. Stationarity of c follows from (5).[8]

Claim 1 *Given $t_1 < t_2$, let $c_1 : (t_1, \infty) \to \mathbf{R}_+$ be the function that maximizes $V(t_1, c, \lambda)$, and let $c_2 : (t_2, \infty) \to \mathbf{R}_+$ be the function that maximizes $V(t_2, c, \lambda)$. Then $V(t_1, c_1, \lambda) = V(t_2, c_2, \lambda)$.*

Proof: Define a function $\tilde{c}_1 : (t_1, \infty) \to \mathbf{R}_+$ by

$$\tilde{c}_1\left(\hat{t}\right) = c_2\left(\hat{t} + t_2 - t_1\right) \tag{20}$$

The function $\tilde{c}_1$ is the same function as c_2, except shifted to begin at t_1 instead of t_2. Then by definition, $V(t_1, c_1, \lambda) \geq V(t_1, \tilde{c}_1, \lambda)$, and by construction, $V(t_1, \tilde{c}_1, \lambda) = V(t_2, c_2, \lambda)$. Hence, $V(t_1, c_1, \lambda) \geq V(t_2, c_2, \lambda)$.

[8] Stationarity is proved in the search literature by using a value function and the Bellman equation. We take a different approach because V is useful when we discuss the social welfare function $\tilde{V}$ for unknown λ.

Now reverse the roles and define a threshold function $\tilde{c}_2 : (t_2, \infty) \rightarrow \mathbf{R}_+$ by

$$\tilde{c}_2\left(\hat{t}\right) = c_1\left(\hat{t} - t_2 + t_1\right)$$

Then by definition, $V(t_2, c_2, \lambda) \geq V(t_2, \tilde{c}_2, \lambda)$, and by construction, $V(t_2, \tilde{c}_2, \lambda) = V(t_1, c_1, \lambda)$. Hence, $V(t_2, c_2, \lambda) \geq V(t_1, c_1, \lambda)$. Together with $V(t_1, c_1, \lambda) \geq V(t_2, c_2, \lambda)$, this proves the result. □

Claim 1 implies that V has a constant value. Using (5), this implies that c is also stationary.

Part (b) follows by differentiating (7) implicitly.

7.2 Proof of Lemma 1

The lemma follows from Claim 2. When the stochastic process is given by (2) as in section 3, and h is given by (8), the hypothesis of the Claim is satisfied because

$$\frac{d}{dt}h(\lambda|t,c) = F(c(t))\,h(\lambda|t,c)\left[E(\lambda|t,c) - \lambda\right] \tag{21}$$

Claim 2 *Suppose there exists $\hat{\lambda}$ such that $\frac{d}{dt}h(\lambda|t,c) > 0$ for $\lambda < \hat{\lambda}$ and $\frac{d}{dt}h(\lambda|t,c) < 0$ for $\lambda > \hat{\lambda}$. Then $\frac{d}{dt}H(\lambda|t,c) > 0$ at each $\lambda \in [0,\infty)$.*

Proof: For each $\lambda \in [0, \infty)$,

$$0 = \frac{d}{dt}\int_0^\infty h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} = \frac{d}{dt}\int_0^\lambda h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} + \frac{d}{dt}\int_\lambda^\infty h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda}$$

For $\lambda \leq \hat{\lambda}$,

$$\frac{d}{dt}H(\lambda|t,c) = \int_0^\lambda \frac{d}{dt}h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} > 0$$

For $\lambda > \hat{\lambda}$,

$$\begin{aligned}\frac{d}{dt}H(\lambda|t,c) &= \int_0^\lambda \frac{d}{dt}h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} = \int_0^\infty \frac{d}{dt}h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} - \int_\lambda^\infty \frac{d}{dt}h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} \\ &= 0 - \int_\lambda^\infty \frac{d}{dt}h\left(\tilde{\lambda}|t,c\right)d\tilde{\lambda} > 0\end{aligned}$$

Therefore, $H(\cdot|t_1,c)$ stochastically dominates $H(\cdot|t_2,c)$ and $(d/dt)\,E(\lambda|t,c) < 0$.

7.3 Proof of Proposition 2

The conclusion that c is increasing follows from (11) since we can show that $\tilde{V}$ is decreasing. For the derivative of $\tilde{V}$, we need the derivative of the conditional density function at $\hat{t} > t$,

$$\frac{d}{dt} E\left(\lambda|\hat{t},c\right) F\left(c\left(\hat{t}\right)\right) \frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}\left(t|c\right)} = F\left(c\left(t\right)\right) E\left(\lambda|t,c\right) \gamma\left(\hat{t}|t,c,\lambda\right) h\left(\lambda|t,c\right)$$

Differentiating $\tilde{V}$ with respect to t gives

$$\begin{aligned}
& \frac{d}{dt}\tilde{V}\left(t,c,\tilde{h}\right) \\
= & -\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) E\left(\lambda|t,c\right) F\left(c\left(t\right)\right) + r\tilde{V}\left(t,c,\tilde{h}\right) \\
& + \int_t^\infty e^{-r\left(\hat{t}-t\right)} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) E\left(\lambda|\hat{t},c\right) F\left(c\left(\hat{t}\right)\right) \frac{d}{dt}\left[\frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}\left(t|c\right)}\right] d\hat{t} \\
= & -\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) E\left(\lambda|t,c\right) F\left(c\left(t\right)\right) + r\tilde{V}\left(t,c,\tilde{h}\right) \\
& + E\left(\lambda|t,c\right) F\left(c\left(t\right)\right) \int_t^\infty e^{-r\left(\hat{t}-t\right)} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) E\left(\lambda|\hat{t},c\right) F\left(c\left(\hat{t}\right)\right) \frac{\tilde{P}\left(\hat{t}|c\right)}{\tilde{P}\left(t|c\right)} d\hat{t} \\
= & -\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) E\left(\lambda|t,c\right) F\left(c\left(t\right)\right) + \left(r + E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)\right) \tilde{V}\left(t,c,\tilde{h}\right) \\
= & \left(r + E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)\right) \left[-\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) \frac{E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)}{r + E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)} + \tilde{V}\left(t,c,\tilde{h}\right)\right] \\
= & \left(r + E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)\right) \left[-\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) \frac{E\left(\lambda|t,c\right) F\left(c\left(t\right)\right)}{\left(r + E\left(\lambda|t,c\right)\right) F\left(c\left(t\right)\right)} + \left(\frac{v}{r} - c\left(t\right)\right)\right] \quad (22)
\end{aligned}$$

where the last line follows from (11).

First, the optimizing function c cannot be "U-shaped" on any domain. If the function c is "U-shaped" on some domain, there exist t_1 and t_2 such that $t_1 < t_2$, $c\left(t_1\right) = c\left(t_2\right)$, and $c'\left(t_1\right) < 0 < c'\left(t_2\right)$. However, this generates a contradiction. It holds that $\left(\frac{v}{r} - E_F\left(c\left(t_1\right)\right)\right) = \left(\frac{v}{r} - E_F\left(c\left(t_2\right)\right)\right)$, $\left(\frac{v}{r} - c\left(t_1\right)\right) = \left(\frac{v}{r} - c\left(t_1\right)\right)$, $F\left(c\left(t_1\right)\right) = F\left(c\left(t_2\right)\right)$, and (using Lemma 1) $E\left(\lambda|t_1,c\right) > E\left(\lambda|t_2,c\right)$. Hence, using (22), $\frac{d}{dt}\tilde{V}\left(t_1,c,\tilde{h}\right) < \frac{d}{dt}\tilde{V}\left(t_2,c,\tilde{h}\right)$. Together with $c'\left(t_1\right) < 0 < c'\left(t_2\right)$, this contradicts (11).

Proposition 2 then follows from Claim 3 and Claim 4 below. By Claim 4, if c is the optimal threshold function, $\tilde{V}\left(t,c,\tilde{h}\right)$ is decreasing with t on a domain $\left(\bar{t},\infty\right)$. Therefore, using (11), it also holds that c is increasing on that domain. But it then follows that the entire function c is nondecreasing, since c cannot be U-shaped on any domain. And, in fact, c is increasing because the derivative (22) is not constant on any interval.

Claim 3 *Let c be the threshold function that maximizes $V\left(0,\cdot,\tilde{h}\right)$. Then there exists $\bar{t}$ such that the function $t \rightarrow e^{-rt}\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right)$ is decreasing on the domain $\left(\bar{t},\infty\right)$.*

Proof of Claim 3: Because the optimal c cannot be U-shaped, it is either nonincreasing or nondecreasing for sufficiently large t. Further, because c is bounded above and below, it holds that $c'\left(t\right) \rightarrow 0,\ c\left(t\right) \rightarrow c^*,\ E_F\left(c\left(t\right)\right) \rightarrow E_F\left(c^*\right)$ for some $c^* \in \left[0,\frac{v}{r}\right]$. The result follows because

$$\begin{aligned} \frac{d}{dt}e^{-rt}\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) &= e^{-rt}\left[-r\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right) - \frac{dE_F\left(c\left(t\right)\right)}{dc\left(t\right)}c'\left(t\right)\right] \\ &\rightarrow -re^{-rt}\left(\frac{v}{r} - E_F\left(c^*\right)\right) \end{aligned}$$

□

Claim 4 *Let c be the threshold function that maximizes $V\left(0,\cdot,\tilde{h}\right)$. Then there exists a domain $\left(\bar{t},\infty\right)$ for which*

$$\tilde{V}\left(t_1,c,\tilde{h}\right) > \tilde{V}\left(t_2,c,\tilde{h}\right) \qquad \textit{if} \qquad \bar{t} \leq t_1 < t_2$$

Proof of Claim 4: We will take the domain $\left(\bar{t},\infty\right)$ as the domain on which $e^{-rt}\left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right)$ is decreasing, by Claim 3. We will show that

$$\tilde{V}\left(t_1,c,\tilde{h}\right) \geq \tilde{V}\left(t_1,\tilde{c},\tilde{h}\right) > \tilde{V}\left(t_2,c,\tilde{h}\right) \quad \text{if} \quad \bar{t} \leq t_1 < t_2 \tag{23}$$

where $\tilde{c}$ is defined by $\tilde{c}\left(t\right) = c\left(t\right)$ for $t \leq t_1$ and $\tilde{c}\left(t\right) = c\left(t + t_2 - t_1\right)$ for $t > t_1$.

The first inequality in (23) is true by the principle of optimality. Beginning from time t_1, the optimizing function is still c, as it was when optimized from the beginning. If $\bar{c}$ satisfies $\tilde{V}\left(t_1,\bar{c},\tilde{h}\right) \geq \tilde{V}\left(t_1,\hat{c},\tilde{h}\right)$ for all threshold functions $\hat{c}$, then $\bar{c}\left(t\right) = c\left(t\right)$ for every $t \geq t_1$.

It is the second inequality in (23) that we must show. The function $\tilde{c}$ in $\tilde{V}\left(t_1,\tilde{c},\tilde{h}\right)$ is defined by the function c restricted to $\left(t_2,\infty\right)$ and shifted back in time to t_1. For a fixed

λ, it therefore holds that $V(t_1, \tilde{c}, \lambda) = V(t_2, c, \lambda)$. Further, $h(\lambda|t_1, \tilde{c}) = h(\lambda|t_1, c)$ because $c = \tilde{c}$ for $t \leq t_1$. Therefore,

$$\tilde{V}\left(t_1, \tilde{c}, \tilde{h}\right) = \int_0^\infty V(t_1, \tilde{c}, \lambda)\, h(\lambda|t_1, \tilde{c}) \ d\lambda = \int_0^\infty V(t_2, c, \lambda)\, h(\lambda|t_1, c) \ d\lambda,$$

and to show the second inequality in (23), it is enough to show that

$$\int_0^\infty V(t_2, c, \lambda)\, h(\lambda|t_1, c)\, d\lambda > \int_0^\infty V(t_2, c, \lambda)\, h(\lambda|t_2, c)\, d\lambda = \tilde{V}\left(t_2, c, \tilde{h}\right). \qquad (24)$$

Since $e^{-r(t-t_2)}\left(\frac{v}{r} - E_F(c(t))\right)$ is decreasing with t for $t \in (\bar{t}, \infty)$, $V(t_2, c, \cdot)$ increases with λ. Then (24) follows because, by Lemma 1, the distribution $h(\cdot|t_1, c)$ stochastically dominates $h(\cdot|t_2, c)$. This means that $h(\cdot|t_2, c)$ puts relatively more weight on low values of λ, where the value of $V(t_2, c, \lambda)$ is low, and $h(\cdot|t_1, c)$ puts relatively more weight on high values of λ, where the value of $V(t_2, c, \lambda)$ is high. $\square$

7.4 Proof of Proposition 3

We show this in two claims.

Claim 5 *Let $c : (t_1, \infty) \to \mathbf{R}_+$ be the function that maximizes $B(0, c, \lambda)$. Then c is not increasing on any interval.*

Proof: Suppose to the contrary that c is increasing on a domain $[0, t_2]$. (The same argument works for any domain where c is increasing.) Define a threshold function $\tilde{c} : (0, \infty) \to \mathbf{R}_+$ by

$$\tilde{c}\left(\hat{t}\right) = \begin{cases} c\left(\hat{t}\right) & \text{for} \quad \hat{t} \leq t_2 \\ c\left(\hat{t} - t_2\right) & \text{for} \quad \hat{t} > t_2 \end{cases}$$

Thus, the function $\tilde{c}$ is identical to the optimizing function c until t_2, but then the function c repeats, so that $c(0) = \tilde{c}(t_2)$. During the period $[0, t_2]$, ideas are being banked. When the threshold function is $\tilde{c}$, the banked ideas may be useful at times $\tilde{t} > 2t_2$, when it holds that $\tilde{c}(t) > c(t_2) = \tilde{c}(2t_2)$.

We will show that $B(t_2, c, \lambda) < B(0, c, \lambda) < B(t_2, \tilde{c}, \lambda)$, which contradicts the fact that c is optimal from time t_2. Using (13), the first inequality holds because c is assumed optimal and increasing, so B is decreasing. We show the second inequality.

$$\begin{aligned}
B\left(t_2,\tilde{c},\lambda\right) &= \int_{t_2}^{2t_2} e^{-r(t-t_2)} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(\tilde{c}\left(t\right)\right)\right) \times \\ \lambda\left[F\left(\tilde{c}\left(t\right)\right) + f\left(\tilde{c}\left(t\right)\right)\tilde{c}'\left(t\right)\left(t-t_2\right)\right] \times \\ e^{-\lambda F\left(\tilde{c}(t)\right)(t-t_2)} \end{array} \right\} dt \\
&+ \int_{2t_2}^{\infty} e^{-r(t-t_2)} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(\tilde{c}\left(t\right)\right)\right) \times \\ \lambda\left[F\left(\tilde{c}\left(t\right)\right) + f\left(\tilde{c}\left(t\right)\right)\tilde{c}'\left(t\right)t\right] \times \\ e^{-\lambda F\left(\tilde{c}(t)\right)(t-t_2)} \end{array} \right\} dt
\end{aligned}$$

$$\begin{aligned}
&= \int_{t_2}^{2t_2} e^{-r(t-t_2)} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(c\left(t-t_2\right)\right)\right) \times \\ \lambda\left[F\left(c\left(t-t_2\right)\right) + f\left(c\left(t-t_2\right)\right)c'\left(t-t_2\right)\left(t-t_2\right)\right] \times \\ e^{-\lambda F\left(c(t-t_2)\right)(t-t_2)} \end{array} \right\} dt \\
&+ \int_{2t_2}^{\infty} e^{-r(t-t_2)} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(c\left(t-t_2\right)\right)\right) \times \\ \lambda\left[F\left(c\left(t-t_2\right)\right) + f\left(c\left(t-t_2\right)\right)c'\left(t-t_2\right)t\right] \times \\ e^{-\lambda F\left(c(t-t_2)\right)(t-t_2)} \end{array} \right\} dt
\end{aligned}$$

$$\begin{aligned}
&= \int_{0}^{t_2} e^{-r\hat{t}} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) \times \\ \lambda\left[F\left(c\left(\hat{t}\right)\right) + f\left(c\left(\hat{t}\right)\right)c'\left(\hat{t}\right)\hat{t}\right] \times \\ e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}} \end{array} \right\} d\hat{t} \\
&+ \int_{t_2}^{\infty} e^{-r\hat{t}} \left\{ \begin{array}{l} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) \times \\ \lambda\left[F\left(c\left(\hat{t}\right)\right) + f\left(c\left(\hat{t}\right)\right)c'\left(\hat{t}\right)\left(\hat{t}+t_2\right)\right] \times \\ e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}} \end{array} \right\} d\hat{t} \\
&= \int_{0}^{\infty} \left(\frac{v}{r} - E_F\left(c\left(t\right)\right)\right)\tilde{G}'\left(t|t_2\right)dt
\end{aligned}$$

where

$$\tilde{G}'\left(\hat{t}|t_2\right) = \left\{ \begin{array}{ll} \lambda\left[F\left(c\left(\hat{t}\right)\right) + f\left(c\left(\hat{t}\right)\right)c'\left(\hat{t}\right)\hat{t}\right] e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}} & \text{if} \quad \hat{t} \in [0, t_2] \\ \\ \lambda\left[F\left(c\left(\hat{t}\right)\right) + f\left(c\left(\hat{t}\right)\right)c'\left(\hat{t}\right)\left(\hat{t}+t_2\right)\right] e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}} & \text{if} \quad \hat{t} \in [t_2, \infty] \end{array} \right\}$$

Similarly, write

$$B\left(0,c,\lambda\right) = \int_0^\infty \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) G'\left(\hat{t}\right) d\hat{t}$$

where

$$G'\left(\hat{t}\right) = \lambda\left[F\left(c\left(\hat{t}\right)\right) + f\left(c\left(\hat{t}\right)\right)c'\left(\hat{t}\right)\hat{t}\right] e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}} \text{ for each } \hat{t} \in [0,\infty)$$

Then $\tilde{G}\left(t|t_2\right) \geq G\left(t\right)$ for all t, with strict inequality for $t > t_2$, that is, the distribution $\tilde{G}$ stochastically dominates the distribution G. Since $\left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right)$ is decreasing with $\hat{t}$, this implies that $B\left(0,c,\lambda\right) < B\left(t_2,\tilde{c},\lambda\right)$. □

Claim 6 *Let $c : (t_1, \infty) \to \mathbf{R}_+$ be the function that maximizes $B(0, c, \lambda)$. Then c is not decreasing on any interval.*

Proof: Suppose to the contrary that c is decreasing on a domain $[0, t_2]$. (The same argument works for any domain where c is decreasing.) Define a threshold function $\tilde{c} : (0, \infty) \to \mathbf{R}_+$ by

$$\tilde{c}\left(\hat{t}\right) = c\left(\hat{t} + t_2\right) \text{ for all } t \in [0, \infty)$$

Thus, the function $\tilde{c}$ is identical to the optimizing function c as c is defined from t_2 forward, but it is shifted back to start at 0 instead of t_2. We will show that $B(0, c, \lambda) < B(t_2, c, \lambda) = B(0, \tilde{c}, \lambda)$, which contradicts the fact that c is optimal from time 0. Using (13), the first inequality holds because c is assumed optimal and decreasing, so B is increasing. We show the equality.

$$\begin{aligned}
B(t_2, c, \lambda) &= \int_{t_2}^{\infty} e^{-r\left(\hat{t}-t_2\right)} \left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right) \lambda F\left(c\left(\hat{t}\right)\right) e^{-\int_{t_2}^{\hat{t}} \lambda F(c(t))dt} d\hat{t} \\
&= \int_{t_2}^{\infty} e^{-r\left(\hat{t}-t_2\right)} \left(\frac{v}{r} - E_F\left(\tilde{c}\left(\hat{t}-t_2\right)\right)\right) \lambda F\left(\tilde{c}\left(\hat{t}-t_2\right)\right) e^{-\int_{0}^{\hat{t}-t_2} \lambda F(\tilde{c}(x))dx} d\hat{t} \\
&= \int_{0}^{\infty} e^{-rt} \left(\frac{v}{r} - E_F(\tilde{c}(t))\right) \lambda F(\tilde{c}(t)) e^{-\int_0^t \lambda F(\tilde{c}(x))dx} dt \\
&= B(0, \tilde{c}, \lambda) \qquad \square
\end{aligned}$$

7.5 Proof of Proposition 4

We first show in Claim 7 that the optimal c is either increasing everywhere, as we wish to show, or there exists $t_1 \geq 0$ such that c is increasing for $t < t_1$ and decreasing for $t > t_1$.

Claim 7 *If c maximizes $\tilde{B}\left(0, \cdot, \tilde{h}\right)$, and if $c'(t) = 0$ at some t, then $c''(t) < 0$. (The optimal c can have at most one point where $c' = 0$, and at that point, c is concave.)*

Proof: We will use the derivative of $\tilde{B}$.

$$\frac{d}{dt}\tilde{B}\left(t, c, \tilde{h}\right) = \left[\frac{v}{r} - c(t)\right] r + \left[E_F(\tilde{c}(t)) - c(t)\right] E(\lambda|t, \tilde{c}) \mathcal{F}(t|\tilde{c}) \tag{25}$$

If c is optimal, it follows from (15) that $c'(t) = 0 \iff \frac{d}{dt}\tilde{B}\left(t, c, \tilde{h}\right) = 0$ and $c''(t) < 0 \iff \frac{d^2}{dt^2}\tilde{B}\left(t, c, \tilde{h}\right) > 0$. If $c'(t) = 0$ then $(d/dt)\mathcal{F}(t|\tilde{c}) = 0$ and $(d/dt)\left[E_F(\tilde{c}(t)) - c(t)\right] = 0$.

But by Lemma 1, $(d/dt)\,E(\lambda|t,\tilde{c}) < 0$. Because $[E_F(\tilde{c}(t)) - c(t)] < 0$, this proves that $\frac{d^2}{dt^2}\tilde{B}\left(t,c,\tilde{h}\right) > 0$ and $c''(t) < 0$. □

Suppose, then, that there exists t_1 such that for $t > t_1$, c is decreasing and $\tilde{B}$ is increasing. There are no relevant banked ideas at t_1. An idea with cost $c < c(t_1)$ would already have been used. An idea with cost $c > c(t_1)$ will never be used at any $t > t_1$ because $c > c(t_1) > c(t)$. Therefore, banking is irrelevant after t_1, and the stochastic process beginning at t_1 is exactly the same as when ideas are not banked. Let the "initial" beliefs at t_1 be $\tilde{h}_1$. Then, maintaining the hypothesis that c is decreasing for $t > t_1$ (and $\tilde{B}$ is increasing), it holds that $\tilde{B}\left(t,c,\tilde{h}\right) = \tilde{V}\left(t,c,\tilde{h}_1\right)$ for $t > t_1$. But we already showed in Proposition 2 that $\tilde{V}$ is decreasing, not increasing, which is a contradiction. This completes the proof.

7.6 Proof of Proposition 5

It is enough to show that $\tilde{B}\left(t,c,\tilde{h}\right) > \tilde{V}\left(t,c,\tilde{h}\right)$ for every increasing threshold function c. Define functions g, $\tilde{Q}_V$ and $\tilde{Q}_B$ by

$$g\left(\hat{t}|c\right) = e^{-r\left(\hat{t}-t\right)}\left(\frac{v}{r} - E_F\left(c\left(\hat{t}\right)\right)\right)$$

$$\begin{aligned}
\tilde{Q}_V\left(\hat{t}|c\right) &= \int_0^\infty \tilde{h}(\lambda)\,e^{-\Lambda\left(\hat{t},c\right)}\,d\lambda \\
\tilde{Q}_B\left(\hat{t}|c\right) &= \int_0^\infty \tilde{h}(\lambda)\,e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}}\,d\lambda
\end{aligned}$$

$$\begin{aligned}
\tilde{V}\left(t,c,\tilde{h}\right) &= \int_t^\infty \int_0^\infty g\left(\hat{t}|c\right)\lambda F\left(c\left(\hat{t}\right)\right)e^{-\left[\Lambda\left(\hat{t},c\right)-\Lambda(t,c)\right]}h(\lambda|t,c)\;d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_V(t|c)}\int_t^\infty \int_0^\infty g\left(\hat{t}|c\right)\lambda F\left(c\left(\hat{t}\right)\right)\tilde{h}(\lambda)\,e^{-\Lambda\left(\hat{t},c\right)}\;d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_V(t|c)}\int_t^\infty g\left(\hat{t}|c\right)F\left(c\left(\hat{t}\right)\right)\int_0^\infty \lambda\tilde{h}(\lambda)\,e^{-\Lambda\left(\hat{t},c\right)}\;d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_V(t|c)}\int_t^\infty g\left(\hat{t}|c\right)\frac{d}{d\hat{t}}\left[1-\tilde{Q}_V\left(\hat{t}|c\right)\right]d\hat{t}
\end{aligned}$$

$$
\begin{aligned}
\tilde{B}\left(t,c,\tilde{h}\right) &= \int_t^\infty \int_0^\infty g\left(\hat{t}|c\right)\lambda\mathcal{F}\left(\hat{t}|c\right)e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}+\lambda F(c(t))t}h\left(\lambda|t,c\right)\ d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_B\left(t|c\right)}\int_t^\infty \int_0^\infty g\left(\hat{t}|c\right)\lambda\mathcal{F}\left(\hat{t}|c\right)\tilde{h}\left(\lambda\right)e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}}\ d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_B\left(t|c\right)}\int_t^\infty g\left(\hat{t}|c\right)\mathcal{F}\left(\hat{t}|c\right)\int_0^\infty \lambda\tilde{h}\left(\lambda\right)e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}}\ d\lambda d\hat{t} \\
&= \frac{1}{\tilde{Q}_B\left(t|c\right)}\int_t^\infty g\left(\hat{t}|c\right)\frac{d}{d\hat{t}}\left[1-\tilde{Q}_B\left(\hat{t}|c\right)\right]d\hat{t}
\end{aligned}
$$

At each $\hat{t}>0$, $e^{-\Lambda\left(\hat{t},c\right)}>e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}}$. Therefore,

$$
\tilde{Q}_V\left(\hat{t}|c\right)=\int_0^\infty \lambda\tilde{h}\left(\lambda\right)e^{-\Lambda\left(\hat{t},c\right)}d\lambda>\int_0^\infty \lambda\tilde{h}\left(\lambda\right)e^{-\lambda F\left(c\left(\hat{t}\right)\right)\hat{t}}d\lambda=\tilde{Q}_B\left(\hat{t}|c\right)
$$

and $1-\tilde{Q}_V\left(\cdot|c\right)$ stochastically dominates $1-\tilde{Q}_B\left(\cdot|c\right)$. Since $g\left(\cdot\right)$ is decreasing when c is increasing, it follows that $\tilde{Q}_B\left(t|c\right)\tilde{B}\left(t,c,\tilde{h}\right)>\tilde{Q}_V\left(t|c\right)\tilde{V}\left(t,c,\tilde{h}\right)$.

But since $\tilde{Q}_B\left(t|c\right)/\tilde{Q}_V\left(t|c\right)<1$, it follows that $\tilde{B}\left(t,c,\tilde{h}\right)>\tilde{V}\left(t,c,\tilde{h}\right)$. □

7.7 Proof of Remark 1

(a) Let n be the number of potential recipients. Suppose that each recipient believes that each other recipient follows an investment strategy described by a cost threshold function c. At date t, some recipients have received ideas. If a recipient received a single idea at, say $\tilde{t}\leq t$, the recipient takes this into account in forming his belief on λ. In a population of size n, with individual arrival rate λ/n, the agent's belief on λ is given by the following posterior density:

$$
g_n\left(\hat{\lambda}|\tilde{t},t,c\right)=\frac{\frac{\hat{\lambda}}{n}e^{-\frac{\hat{\lambda}}{n}\tilde{t}}e^{-\left(\frac{n-1}{n}\right)\hat{\lambda}\int F(c(t))dt}\tilde{h}\left(\hat{\lambda}\right)}{\int\frac{\lambda}{n}e^{-\frac{\lambda}{n}\tilde{t}}e^{-\left(\frac{n-1}{n}\right)\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda}=\frac{\hat{\lambda}e^{-\left(\frac{n-1}{n}\right)\hat{\lambda}\int F(c(t))dt}\tilde{h}\left(\hat{\lambda}\right)}{\int\lambda e^{-\frac{\left(\lambda-\hat{\lambda}\right)}{n}\tilde{t}}e^{-\left(\frac{n-1}{n}\right)\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda}
$$

Let r be the limit density function as $n\to\infty$:

$$
r\left(\hat{\lambda}|t,c\right)=\frac{\hat{\lambda}e^{-\hat{\lambda}\int F(c(t))dt}\tilde{h}\left(\hat{\lambda}\right)}{\int\lambda e^{-\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda}
$$

The limit distribution does not depend on $\tilde{t}$, as asserted in part (a).

Further, the probability of receiving more than one idea has a second-order effect, and we therefore ignore it. The numerator in the following expression is the probability of receiving two or more ideas by time t, and the denominator is the probability of receiving one or more ideas by time t. Using L'Hopital's rule, the ratio converges to zero.

$$\lim_{n\to\infty} \frac{\left[1 - e^{-\frac{\lambda}{n}t} - \left(\frac{\lambda}{n}t\right) e^{-\frac{\lambda}{n}t}\right]}{\left[1 - e^{-\frac{\lambda}{n}t}\right]} \to 0$$

This concludes part (a).

(b) Nevertheless, the recipients are more optimistic about λ than the planner. When the planner believes that c describes the recipients' investment behavior, the density of the planner's posterior is

$$h\left(\hat{\lambda}|t,c\right) = \frac{e^{-\hat{\lambda}\int F(c(t))dt}\tilde{h}\left(\hat{\lambda}\right)}{\int e^{-\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda}$$

The ratio of the densities, $r\left(\hat{\lambda}|t,c\right)/h\left(\hat{\lambda}|t,c\right)$, is proportional to λ. This implies that the recipients, as opposed to the planner, place higher weight on higher λ, that r stochastically dominates h, and that r has a higher expected value. It is instructive to show the latter directly.

Divide both numerator and denominator of r by $\int e^{-\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda$:

$$r\left(\hat{\lambda}|t,c\right) = \frac{\frac{\hat{\lambda}e^{-\hat{\lambda}\int F(c(t))dt}\tilde{h}(\hat{\lambda})}{\int e^{-\lambda\int F(c(t))dt}\tilde{h}(\lambda)d\lambda}}{\frac{\int \lambda e^{-\lambda\int F(c(t))dt}\tilde{h}(\lambda)d\lambda}{\int e^{-\lambda\int F(c(t))dt}\tilde{h}(\lambda)d\lambda}} = \frac{\hat{\lambda}e^{-\hat{\lambda}\int F(c(t))dt}\tilde{h}\left(\hat{\lambda}\right)}{E_h\left(\lambda|t,c\right)\int e^{-\lambda\int F(c(t))dt}\tilde{h}\left(\lambda\right)d\lambda}$$

Let $E_h\left(\lambda|t,c\right)$ and $E_h\left(\lambda^2|t,c\right)$ be the expected values with respect to the planner's posterior belief, and let $Var_h\left(\lambda\right)$ be the variance of the planner's belief. Now consider the expected value of λ with respect to the recipients' belief instead of the planner's belief:

$$\begin{aligned} E_r\left(\lambda|t,c\right) &= \int \hat{\lambda} r\left(\hat{\lambda}|t,c\right)d\hat{\lambda} = \frac{\int \hat{\lambda}^2 \frac{e^{-\hat{\lambda}\int F(c(t))dt}\tilde{h}(\hat{\lambda})}{\int e^{-\lambda\int F(c(t))dt}\tilde{h}(\lambda)d\lambda}d\hat{\lambda}}{E_h\left(\lambda|t,c\right)} = \frac{E_h\left(\lambda^2|t,c\right)}{E_h\left(\lambda|t,c\right)} \\ &= \left(\frac{1}{E_h\left(\lambda|t,c\right)}\right)\left(Var_h\left(\lambda\right) + E_h\left(\lambda|t,c\right)^2\right) = \frac{Var_h\left(\lambda\right)}{E_h\left(\lambda|t,c\right)} + E_h\left(\lambda|t,c\right) \end{aligned}$$

Thus, $E_r\left(\lambda|t,c\right) > E_h\left(\lambda|t,c\right)$.

Part (c) follows from (17). $\square$

References

[1] Choi, J. P. 1991. "Dynamic R&D Competition under 'Hazard Rate' Uncertainty," *RAND Journal of Economics,* 22, 596-610

[2] Denicolo, V. and L. Franzoni. 2004. "Patents, Secrets, and the First Inventor Defense," *Journal of Economics and Management Strategy*, 13, 517-538.

[3] Dixit, A. and R. Pindyck. 1994. *Investment under Uncertainty.* Princeton, NJ: Princeton University Press.

[4] Erkal, N. 2005. "The Decision to Patent, Cumulative Innovation, and Optimal Policy," *International Journal of Industrial Organization*, 23(7-8), 535-562.

[5] Erkal, N. and S. Scotchmer. 2007. "Scarcity of Ideas and Options to Invest in R&D," University of California, Berkeley, Department of Economics, Working Paper 07-348.

[6] Hopenhayn, H., G. Llobet, and M. Mitchell. 2006. "Rewarding Sequential Innovators: Patents, Prizes, and Buyouts." *Journal of Political Economy* 114:1041-1068.

[7] McCall, B. P. and J. J. McCall. 2008 *Economics of Search.* New York: Routledge Press.

[8] MacDonald, R. and D. Siegel. 1986. "The Value of Waiting to Invest," *Quarterly Journal of Economics*, 101, 707-728.

[9] Malueg, D. A. and S. O. Tsutsui. 1997. "Dynamic R&D Competition with Learning," *RAND Journal of Economics* 28, 751-772.

[10] O'Donoghue, T., Scotchmer, S. and Thisse, J.-F. 1998. "Patent Breadth, Patent Life and the Pace of Technological Progress," *Journal of Economics and Management Strategy*, 7, 1-32.

[11] Reinganum, J. 1989. "The Timing of Innovation: Research, Development and Diffusion," in R. Schmalensee and R. D. Willig, eds., *Handbook of Industrial Organization*, Amsterdam: Elsevier.

[12] Scotchmer, S. 1999. "On the Optimality of the Patent Renewal System," *RAND Journal of Economics,* 30, 131-196.

[13] Scotchmer, S. 2004. *Innovation and Incentives.* Cambridge, MA: MIT Press.

[14] Snyder, D. L. and M. I. Miller. 1991. *Random Point Processes in Time and Space.* New York: Springer-Verlag.

[15] Weeds, H. 2002. "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies*, 69, 729-747.

[16] Wright, B. D. 1983. "The Economics of Invention Incentives: Patents, Prizes and Research Contracts." *American Economic Review* 73:691-707.